Estimation and Portfolio Optimization with Expert Opinions in Discrete-time Financial Markets

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Dissertation

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Abstract

In this thesis, we mainly discuss the problem of parameter estimation and portfolio optimization with partial information in discrete-time. In the portfolio optimization problem, we specifically aim at maximizing the utility of terminal wealth. We focus on the logarithmic and power utility functions. We consider expert opinions as another observation in addition to stock returns to improve estimation of drift and volatility parameters at different times and for the purpose of asset optimization.

In the first part, we assume that the drift term has a fixed distribution, and the volatility term is constant. We use the Kalman filter to combine the two types of observations. Moreover, we discuss how to transform this problem into a non-linear problem of Gaussian noise when the expert opinion is uniformly distributed. The generalized Kalman filter is used to estimate the parameters in this problem.

In the second part, we assume that drift and volatility of asset returns are both driven by a Markov chain. We mainly use the change-of-measure technique to estimate various values required by the EM algorithm. In addition, we focus on different ways to combine the two observations, expert opinions and asset returns. First, we use the linear combination method. At the same time, we discuss how to use a logistic regression model to quantify expert opinions. Second, we consider that expert opinions follow a mixed Dirichlet distribution. Under this assumption, we use another probability measure to estimate the unnormalized filters, needed for the EM algorithm.

In the third part, we assume that expert opinions follow a mixed Dirichlet distribution and focus on how we can obtain approximate optimal portfolio strategies in different observation settings. We claim the approximate strategies from the dynamic programming equations in different settings and analyze the dependence on the discretization step. Finally we compute different observation settings in a simulation study.

Zusammenfassung

In dieser Dissertation diskutieren wir hauptsächlich in zeit-diskreten Finanzmärkten das Problem der Parameterschätzung und der Optimierung eines Portfolios bei partieller Informationen. Beim Problem der Optimierung des Portfolios zielen wir speziell darauf ab, den erwarteten Nutzen zu maximieren. Unter anderem konzentrierten wir uns auf die logarithmische und auf Power-Nutzenfunktionen. Wir betrachten Expertenmeinungen als eine weitere Beobachtung neben den Aktienrenditen, um die Drift- und Volatilitätsparameter in unterschiedlichen Zeitpunkte als Grundlage der Optimierung zu schätzen.
Im ersten Teil nehmen wir an, dass der Drift einer festen Verteilung folgt und der Volatilitätsparameter konstant ist. Wir verwenden den Kalman-Filter, um die beiden Arten von Beobachtungen zu kombinieren. Darüber hinaus zeigen wir, wie dieses Problem in ein nichtlineares Problem des Gaußschen Rauschens umgewandelt werden kann, wenn das Rauschen uniform verteilt ist. Der verallgemeinerte Kalman-Filter wird verwendet, um die Parameter in diesem Problem zu schätzen, die wir in EM-Algorithmus benötigen.
Im zweiten Teil gehen wir davon aus, dass sowohl die Drift als auch die Volatilität der Anlagenrenditen von einer Markov-Kette abhängen. Wir verwenden hauptsächlich die Change-of-Measure-Technologie, um verschiedene Werte zu schätzen, die im EM-Algorithmus erforderlich sind. Wir betrachten verschiedene Arten, die beiden Beobachtungen, Expertenmeinungen und Aktienrenditen, zu kombinieren. Zunächst verwenden wir eine lineare Kombination. Dabei diskutierten wir, wie das logistische Regressionsmodell zur Quantifizierung von Expertenmeinungen verwendet werden kann. Ferner untersuchen wir den Fall, dass Expertenmeinungen einer gemischten Dirichlet-Verteilung folgen. Unter dieser Annahme verwenden wir ein anderes Wahrscheinlichkeitsmaß, um die nicht normalisierten Filter zu schätzen.
Im dritten Teil gehen wir davon aus, dass Expertenmeinungen einer gemischten Dirichlet-Verteilung folgen und untersuchen approximative optimale Portfoliostrategien in verschiedenen Beobachtungsszenarien. Wir erhalten die approximativen Lösungen aus den entsprechenden Dynamic Programming Equations. In Simulationsstudien untersuchen wir die Abhängigkeit vom Diskretisierungsschritt und vergleichen die verschiedenen Szenarien.

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I couldn't have done it without you!

Contents

Chapter 1

Introduction

1.1 Background

Portfolio Optimization under Partial Information

Portfolio optimization problems with partial information, in particular with unknown drift process, have be considered in many aspects over the last two decades. Partial information means: agents have access only to the history of interest rates and stocks (risky assets or securities) prices. These prices are driven by an exogenous Brownian motion whose dimension can be larger than the number of stocks. In other words, the market can be dynamically incomplete. The unobservability is always modelled as the unobservable drift rate of returns. [Lak95], [Lak98] treats the case in which the drift rate follows a linear Gaussian model and solves the utility maximization problem by using the martingale approach. [Bre06] focuses on the case when the utility function is power utility by the Bellman equation. Moreover, this model is also called Kim-Omberg model which leads to well-known Kalman filter. [KX91], [KZ01] model the drift as a time-independent random variable, which leads to the Bayesian case. Another popular assumption for the drift term is modelling it as a continuous-time Markov chain, which leads to Wonham filter in Hidden Markov models (HMMs). [SH04] derive an explicit solution of the portfolio optimization problem using a martingale approach. [RB05] solves the same problem with the help of stochastic control methods. It turns out that the value function is a classical solution of the corresponding HJB equation. Apart from the drift models, [PQ01] introduces a stochastic volatility model under partial information.

Previous Work: Drift Model with Expert Opinions

Trading decisions in financial markets are always made based on very limited information on stock developments available to the investors. Such information may comprise historical observed security prices in terms of partial information. It is notoriously difficult to estimate drift parameter only from historical asset prices. Hence, it is natural to include expert opinions or investors' views as additional source of information. In the context of well-known one-period Markowitz model, [BL92] incorporate investors' information to forecast the updated expected return of stocks. This model can be viewed as the Bayesian updating of a prior distribution of risk premia (without views) into a posterior distribution limited by views. The Black-Litterman model is a typically static (one-period) model, and it has been extended to a continuous-time setting. [GKSW14] investigate discrete-time expert opinions for linear Gaussian drift. They make the assumption that investors want to maximize expected logarithmic utility of terminal wealth obtained by trading in a financial market consisting of one risk-less asset and one stock. The authors prove that the conditional variance, which is for combined observation of return and expert opinions, is smaller than the one from observed return or expert opinions. Besides, the paper shows that the optimal logarithmic values of terminal wealth have the following quantitative relationship: the one with full information is larger than the one with combined information. And they are both larger than the one with information only coming from returns or coming from expert opinions. These findings are extended in [SWW17] to the multivariate case. Consequently, investor views will certainly improve the accuracy of estimators of drift. However, it is not realistic to expect that investor's information can always improve the performance of portfolio, as the information may be not correct, in other words the views may be biased. In this thesis, general settings of expert opinions are related to the partial information, rather than absolute and relative view. Furthermore, different distributions of expert opinions are considered. [FGW12], [FGW14] model drift as a function of a Markov chain with finitely many states and expert opinions are modeled in the form of signals at random discrete time points. The optimal strategy under partial information is derived by replacing the unknown drift by the filter estimated. [Mac12] also extend the Black-Litterman approach to continuous time settings, while in his model the expert opinions are also continuous.

Discrete-Time: Markov Switching Model with Expert Opinions

Markov Switching Models (MSMs), sometimes also called regime-switching models, are widely applied in the field of mathematical finance to describe return processes with time-varying drift or volatility. The drift or volatility terms are driven by an unobservable Markov chain in MSMs instead of constant ones in Black-Scholes model. In continuous time one should distinguish MSMs and HMMs, in the former the volatility jumps with Markov chain, while the latter has a constant volatility. Empirical evidence illustrates that the volatility is not constant, e.g. [Fam65]. Once the volatility is not constant, the well-known martingale method is not applicable easily. That is the reason why we use dynamic programming when we try to solve the optimization problem. In addition, It would be of special interest to the portfolio optimization problem with expert opinions in MSMs. For this study it is also of importance to investigate the model of expert opinions and the impact of them. We will mainly focus on the discrete-time Markov Switching model, since in continuous setting the volatility is observable via the quadratic variation as shown in [KLS18]. [TZ07] has the same setting as we. They also consider the multi-dimensional cases, but not with expert opinions.

1.2 Outline and Methodology

The structure of this thesis is as follows: In the first chapter, we start with the linear Gaussian model. And we use the simple assumption that the drift is an unobservable random variable. We use Bayesian methods to estimate its value to determine the optimal strategy that can be used at each time. Through this model, we will have a simple understanding of the application of expert opinions in continuous and discrete time. In that section, we also consider that expert opinions follow a uniform distribution. By transforming non-Gaussian noise into Gaussian noise, we talk about the relationship between linear Gaussian model and nonlinear Gaussian model. At the same time, this problem can be solved approximately with a generalised Kalman filter.
The second chapter is about estimating parameters for MSMs and includes different models for expert opinions. A technique used throughout this section is a change-of-probability measure. This is a discrete time version of

Girsanov's Theorem. By this technique, we calculate filters from non-linear relationships to linear one. To estimate parameters and the filter, the EM algorithm is applied. Moreover, we utilize different assumptions for expert opinions. For instance, expert opinions follow mixture Dirichlet distribution. Additionally, a logistic regression model is used to formulated expert opinions from exogenous factors.
The last chapter is on optimizing utility of terminal wealth in the discrete-time MSM with expert opinions. The dynamic programming approach is applied and approximated optimal strategies of order Δt are given. We also use Monte Carlo method and function approximation method in terms of utility function. We use that to analyze the dependence on Δt and we compare different observation settings in a simulation study.

1.3 Market Settings

Most of the results in the following chapters aim at solving a class of filtering problems and portfolio optimization problem. Thus, at the beginning we classify the basic settings of the market that we want to work with and discuss the assumptions we make throughout the following chapters.
An investor has a prevailing objective, which is to maximize her initial wealth by investing risky asset or non-risky one.

- One dimensional stock market:
  In order to investigate the effect of expert opinions, we typically focus on the one-dimensional stock market, in which only one risky asset and one non-risky asset can be invested. Naturally, the result can be extended to a multiple stock market, e.g. Black-Litterman model. The main reason for the one-dimensional stock market is to make the influence of expert opinions on estimation of one stock visible. And the computing cost of increasing dimension is not considered in our thesis.

- Discrete time:
  Compared to the continuous setting, the discrete one is realistic when investment is decided. There is always no analytical results in discrete setting while portfolio optimization problems often allow for explicit solutions in continuous time. That is also the reason we use numerical examples to illustrate our results. In addition, existence can be shown.

- Partial observable information:
  We are typically interested in the assumption that knowledge of the investor is restricted. It is fairly realistic for the investor to be able to observe the market daily and see the stock returns. We will specify the investor's knowledge in detail below when introducing the market in different chapters. In particular, for investors who can observe stock returns, only the resulting portfolio strategy has to be adapted to some observation filtrations, e.g. $\mathcal{F}^R$ generated by the returns.

- Expert opinions:
  Another important assumption in our thesis is to allow the use of external input other than observations of stock returns. The expert opinion support another estimation of parameters with some uncertainty. They are often necessary since investors cannot base their future investment just on historical observations from stock returns in practice. Expert opinions are formulated in different ways in our thesis. For example, expert opinions can be given as a Gaussian variable and expert opinions can also follow a mixture Dirichlet distribution.

- The risk-aversion: α
  The investors need to know their risk-aversion, described by a utility function U. We will especially concentrate on the case of logarithmic utility $U(x) = \log(x)$ and power utility $U(x) = \frac{x^{1-\alpha}}{1-\alpha}, \alpha \neq 1, \alpha > 0$. The case of logarithmic utility is almost always the limiting case of power utility for $\alpha \to 1$. Although there are good reasons for using logarithmic utility like the easy solvability of most portfolio optimization problems, there are limitations of it, for instance when looking at different risk aversion values.
  We compute both the cases that $\alpha < 1$ and $\alpha > 1$, in which the former is less risk averse than logarithmic utility while the latter is more more averse.

- Deterministic market parameters: r
  Of course the market risk-free rates change over time. However it changes slightly in a short period of time. Moreover, we assume the risky-free rate can be observable. Hence a constant approximation is often sufficient.

- No transaction cost:
  We assume that our investment will not influence the asset price and also all other fees are not considered. By restricting the admissibility set significantly, since in in discrete time no short selling position and no borrowing is optimal in our model, and hence reducing the amount of wealth that can be shifted with each trade, we reduce the impact of transaction costs anyway.

The assumptions of drift term μ and volatility term σ are different in different chapters. In Chapter 2, we assume that drift, μ, is normally distributed and volatility, σ, is constant and observable, while in Chapter 3 and Chapter 4 both drift term and volatility term are driven by a homogeneous Markov chain. For more details including how to discretize the continuous market model we refer to the single chapters.

Chapter 2

Kalman Filter and Portfolio Optimization

To start with, we have some simple assumptions. For a fixed date $T > 0$ representing the investment horizon, we work on a filtered probability space $(\Omega, \mathcal{F}_T, \mathcal{F}, \mathbb{P})$, with filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. All processes are assumed to be $\mathcal{F}$-adapted. We consider to discretize the continuous financial market with one stock with prices

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

with $S_0 > 0$, and one bond with prices

$$dB_t = B_t r dt.$$

with $B_0 = 1$. The initial values of stocks are available to the investor, the market parameters r and σ are time-independent, while μ is time-independent, but stochastic and not observable. W_t is a standard $\mathcal{F}$-Brownian motion which is independent of μ.
The drift term μ is assumed to be normally distributed as

$$\mu \sim \mathcal{N}(\mu_0, \sigma_0)$$

where the value of μ is available when investors have full information, while one needs to estimate it when the situation is under partial information. Partial information here means that the investor or agent can observe stock returns only or has other sources of information, we say, expert opinions.

The information available to an investor is described by the investor filtration $\mathcal{F}^H = (\mathcal{F}_t)_{t\in[0,T]}$, for which we consider three cases $H \in \{R, C, F\}$.

The investor's aim is to maximize her expected utility of terminal wealth according to the available information:

$$\pi^* = \arg \max_{\pi \in \mathcal{A}^H(x_0)} \mathbb{E}[U(X_T^\pi)]$$

where $\mathcal{A}^H(x_0)$ is the set of $\mathcal{F}^H-$admissible strategies, $U : \mathbb{R}^+ \to \mathbb{R} \cup [-\infty]$ is a utility function, and X_t^π is the wealth process when investing according to portfolio strategies $\pi \in \mathcal{A}^H(x_0)$. Note that π is only $\mathcal{F}^H$-adapted for all the cases $H \in \{R, C, F\}$.

2.1 Portfolio Optimization in Discrete-time

In discrete time, investment over the time horizon $[0, T]$ is considered, where $T = N\Delta t$, and the investor can only make investment decisions (or rebalance the portfolio) at times $t_k = k\Delta t$. The set $\mathcal{T} = \{t_0, t_1, \ldots, t_{N-1}\}$ describes the time points, where investment decisions are made. At the terminal time T, there is no decision made. In the following, we will denote t_k as k to simplify the notation.
At any time $k \in \mathcal{T}$, the investor can rebalance in two assets, non-risky asset and risky asset respectively. For $B_0 = 1$, the dynamics of the non-risky asset B_k at time k is

$$B_k = e^{\rho\Delta t} B_{k-1} \text{ for } k = 1, \ldots, N. \tag{2.1}$$

Sometimes B_k can be formulated as

$$B_k = (1 + r\Delta t)B_{k-1}, r\Delta t > -1,$$

where r is called nominal interest rate, while ρ represents continuous compounding. r can be approximated by Taylor expansion from (2.1) of order Δt.

The discrete return on risky assets over the interval from k to $k+1$ is defined as

$$R^D_{k+1} := \frac{S_{k+1} - S_k}{S_k}.$$

For the purpose of discretization of a continuous model, the log-return R^L_{k+1} over the time interval from k to $k+1$, R^L_{k+1} , is defined as

$$R^L_{k+1} := \log\left(\frac{S_{k+1}}{S_k}\right) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_{k+1} \tag{2.2}$$

where $\varepsilon_{k+1} := \frac{W_{(k+1)\Delta t} - W_{k\Delta t}}{\sqrt{\Delta t}}$. Thus $\varepsilon_1, \ldots, \varepsilon_N$ are independent and standard normally distributed. Note that $R^D_{k+1} = e^{R^L_{k+1}} - 1$ and thus R^D_{k+1} can be approximated of order Δt around 0:

$$R^D_{k+1} \approx \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_{k+1} + \frac{\sigma^2}{2}\Delta t \varepsilon^2_{k+1}. \tag{2.3}$$

The drift term μ is assumed to be identically normally distributed as

$$\mu \sim \mathcal{N}(\mu_0, \sigma_0)$$

where the value of μ is available when investors have full information, while one needs to estimate them if the situation is under partial information. The partial information here means the investor or agent can observe stock returns only and may have other sources of information, we say, expert opinions. [Von17] considered the case that the parameters μ, σ are not observable under partial information, otherwise one could use the Merton plug-in strategy.

A trading strategy is a stochastic process $(\varphi_k^0, \varphi_k^1)_{k\in\mathcal{T}}$, where φ_k^0 and φ_k^1 denote the number of units of non-risky asset and risky asset held in the period from k to $k+1$.
The wealth, $X_k^{\varphi^0,\varphi^1}$ is the value of the portfolio at time k, if we follow the trading strategy (φ^0, φ^1). So for initial capital $x_0 > 0$, the investor enters date $k \in \mathcal{T}$ with a wealth of

$$X_k^{\varphi^0,\varphi^1} := \varphi_{k-1}^0 B_k + \varphi_{k-1}^1 S_k.$$

Definition 2.1.1. *The trading strategy is self-financing if for $k \in \mathcal{T}$, we have $X_k^{\varphi^0,\varphi^1} = \varphi_k^0 B_k + \varphi_k^1 S_k$.*

We assume self-financing trading.
We denote the fraction of total wealth we invest in the risky assets by the portfolio vector π_k for all $k \in \mathcal{T}$

$$\pi_k := \frac{\varphi_k^1 S_k}{X_k^{\varphi^0,\varphi^1}}$$

if $X_k^{\varphi^0,\varphi^1} > 0$. Because of the self-financing assumption, the fraction of wealth we invest in the non-risky asset is $1 - \pi_k$. Due to the fact that (φ^0, φ^1) and π for given x_0 can be represented by each other, the wealth can be expressed as X_k^π for $X_k^{(\varphi^0,\varphi^1)}$.

Definition 2.1.2. *A stochastic process $\pi = (\pi_k)_{k=0,\dots,N-1}$ is called $\mathcal{F}-$admissible if it is $\mathcal{F}$-adapted, self-financing and $P(X_k^\pi > 0) = 1$ for $k = 0, \dots, N$.*

The increment of wealth over the interval from k to $k+1$ can be represented

$$X_{k+1}^\pi - X_k^\pi = (1-\pi_k) X_k^\pi r\Delta t + \pi_k X_k^\pi (e^{R_{k+1}} - 1).$$

So we get a transition function for X_k^π, $f : \mathbb{R} \to \mathbb{R}$,

$$X_{k+1}^\pi := f(X_k^\pi) = \Big(1 + (1-\pi_k) r\Delta t + \pi_k (e^{R_{k+1}} - 1)\Big) X_k^\pi. \tag{2.4}$$

Lemma 2.1.3. *(No Short Selling in Discrete-Time) For $x_0 > 0$ the risky fraction process $(\pi_k)_{k=0,\dots,N-1}$ is admissible if and only if*

$$P(\pi_k \in [0,1]) = 1 \text{ and } P(X^\pi_{k+1} > 0) = 1 \text{ for } k = 0,\dots,N-1.$$

Proof. Suppose for time $u < k$, $X^\pi_u > 0$ and $(\pi_u)_{u=0}^{k-1}$ is admissible. Showing by induction,

$$P(X^\pi_k > 0 | X^\pi_{k-1} > 0) = 1,$$

we need

$$P(1 + (1-\pi_k)r\Delta t + \pi_k(e^{R_k} - 1) > 0) = 1$$

since $r\Delta t \in (-1,\infty)$ and $(e^{R_k} - 1) \in (-1,\infty)$, we can get $\pi_k \in [0,1]$ and as a result $X^\pi_k > 0$.
For the other direction, it is obvious. □

The terminal wealth with respect to strategy π is

$$X^\pi_N = x_0 \prod_{i=1}^{N} \Big(1 + r\Delta t + \pi_{i-1}(e^{R_i} - 1 - r\Delta t)\Big)$$

where $x_0 \in \mathbb{R}^+$ is the initial wealth, which is known.

The information available to an investor is described by the investor filtration $\mathcal{F}^H = (\mathcal{F}_t)_{t\in[0,T]}$, for which we consider three cases $H \in \{R, C, F\}$, where

$$\begin{aligned}
&\mathcal{F}^R = (\mathcal{F}^R_t)_{t\in[0,T]} \text{ with } \mathcal{F}^R_t \text{ generated by } \{R_s, s \le t\},\\
&\mathcal{F}^C = (\mathcal{F}^C_t)_{t\in[0,T]} \text{ with } \mathcal{F}^C_t \text{ generated by } \{R_s, E_s, s \le t\},\\
&\mathcal{F}^F = \mathcal{F},
\end{aligned}$$

where we assume that $\sigma-$algebras $\mathcal{F}^H_t$, $H \in \{R, C, F\}$ are augmented by the null sets $\mathcal{N}$ of P. $\mathcal{F}^R$ correspond to an investor who observes only returns. $\mathcal{F}^C$ indicates the information deriving from the combination of returns and expert opinions. Finally, $\mathcal{F}^F$ expresses an investor who has full information on the parameters of models.

The investor's aim is to maximize her expected utility of terminal wealth according to the available information:

$$\pi^* = \arg\max_{\pi\in\mathcal{A}^H} \mathbb{E}[U(X^\pi_T)] \tag{2.5}$$

where $\mathcal{A}^H(x_0)$ is the set of $\mathcal{F}^H-$admissible strategies, $U : \mathbb{R}^+ \to \mathbb{R} \cup [-\infty]$ is a utility function, and X_t^π is the wealth process when investing according to portfolio strategies $\pi \in \mathcal{A}^H(x_0)$. Note that π is only $\mathcal{F}^H$-adapted for all the cases $H \in \{R, C, F\}$.

2.1.1 Approximated Optimal Strategies in Discrete-time

As a motivation, we will first approximate the optimal portfolio strategies in case of $H = F$. This can be regarded as the benchmark to the restricted strategies. We can then compute how much information is lost under partial information. We consider the wealth process in (2.4) for a self-financing admissible strategy π_k:

$$X^{\pi}_{k+1} = \Big(1 + (1-\pi_k)r\Delta t + \pi_k(e^{R_{k+1}} - 1)\Big)X^{\pi}_k, \ X_0 = x_0.$$

Logarithmic utility function

For logarithmic utility we maximize the following expected utility of terminal wealth:

$$\mathbb{E}[\log(X^{\pi}_T)] = \log x_0 + \sum_{k=1}^{N} \mathbb{E}\Big[\log\Big(1 + r\Delta t + \pi_{k-1}(e^{R_{k+1}} - r\Delta t - 1)\Big)\Big]$$

where $R_{k+1} \in \mathbb{R}$ is discretized log-return in (2.2). Hence we want to maximize the summation over $\mathbb{E}\Big[\log\Big(1 + r\Delta t + \pi_{k-1}(e^{R_{k+1}} - r\Delta t - 1)\Big)\Big]$. Point-wise maximization of this term already leads to an admissible solution. We could derive optimal strategy at time k

$$\pi^*_k = \arg\sup_{\pi_k \in \mathcal{A}} \mathbb{E}\Big[\log\Big(1 + r\Delta t + \pi_k(e^{R_{k+1}} - r\Delta t - 1)\Big)\Big].$$

There is no explicit solution for the optimal strategy in discrete-time even if we know the parameter μ. One may only solve by numerical methods, e.g. Monte Carlo method.

However one can get an approximate strategy instead. Applying second order Taylor expansion for $\log(1+x)$ and e^x around 0, we get of order Δt

$$\begin{aligned}\mathbb{E}[\log(X^{\pi}_T)|X^{\pi}_0 = x_0] &\approx \log x_0 + rT + \sum_{k=1}^{N} \mathbb{E}\Big[\pi_{k-1}(\mu - r)\Delta t - \frac{1}{2}\pi^2_{k-1}\sigma^2\Delta t\varepsilon^2_k\Big] \\ &= \log x_0 + rT + \sum_{k=0}^{N-1}\Big(\pi_k(\mu - r)\Delta t - \frac{1}{2}\pi^2_k\sigma^2\Delta t\Big).\end{aligned} \tag{2.6}$$

Hence, in order to maximize the sum in (2.6) pointwise, for any $k \in \mathcal{T}$, the optimal strategy is

$$\pi_k^{log,F} = \frac{\mu - r}{\sigma^2} \vee 0 \wedge 1.$$

Here, π_k^* is a stochastic strategy since we assume investors can observe μ, which is random, in the case of $H = F$. To solve the optimization problem using logarithmic utility function is consistent with using mean-variance approach as shown in [BC10]. However, in our setting the distribution of returns are not constant. The dynamic mean-variance optimization problem is

$$\pi^* = \arg\max_{\pi_k \in \mathcal{A}} \sum_{k=1}^{N} \mathbb{E}[W_k|\mu] - \frac{1}{2}\operatorname{Var}[W_k|\mu], \tag{2.7}$$

where

$$W_k := 1 + r\Delta t + \pi_{k-1}(R_k^D - r\Delta t)$$

and R_k^D is discrete stock return. From (2.3), one could derive of order Δt

$$\begin{aligned} \mathbb{E}[W_k|\mu] &= 1 + r\Delta t + \pi_{k-1}(\mu - r)\Delta t, \\ \operatorname{Var}[W_k|\mu] &= \pi_{k-1}^2 \sigma^2 \Delta t. \end{aligned}$$

When maximizing the summation in (2.7) point-wise, we get the same formula for the optimal strategy, π_t^*.

In addition, we could get a deterministic optimal strategy if only μ_0 and σ_0 are available to investors. In this scenario, μ is not observable. From the Taylor expansion for expected utility of terminal wealth in (2.6). For assuming deterministic π

$$\begin{aligned} \mathbb{E}[\ln(X_T^\pi)] &\approx \log x_0 + rT + \sum_{k=1}^{N} \mathbb{E}\Big[\pi_{k-1}(\mu - r)\Delta t - \frac{1}{2}\pi_{k-1}^2\sigma^2\Delta t \varepsilon_k^2\Big] \\ &= \log x_0 + rT + \sum_{k=1}^{N} \Big(\pi_{k-1}(\mu_0 - r)\Delta t - \frac{1}{2}\pi_{k-1}^2\sigma^2\Delta t\Big). \end{aligned}$$

To maximize the above expected utility, one could get a deterministic admissible optimal strategy, known as Merton strategy in discrete-time:

$$\pi_{log,Merton}^* = \frac{\mu_0 - r}{\sigma^2} \vee 0 \wedge 1.$$

Power utility function

For power utility $U(x) = \frac{x^{1-\alpha}}{1-\alpha}, \alpha > 0, \alpha \neq 1$, we need to maximize the following expected utility of terminal wealth:

$$\mathbb{E}\Big[\frac{1}{1-\alpha}(X_T^\pi)^{1-\alpha}\Big] = \frac{x_0^{1-\alpha}}{1-\alpha}\mathbb{E}\Big[\prod_{k=1}^{N}\Big(1 + r\Delta t + \pi_{k-1}(e^{R_k} - 1 - r\Delta t)\Big)^{1-\alpha}\Big].$$

Since μ is time-independent and $\varepsilon_1, \ldots, \varepsilon_N$ are independent of μ, one can further derive conditional on μ

$$\mathbb{E}\Big[\frac{1}{1-\alpha}(X_T^\pi)^{1-\alpha}|\mu\Big] = \frac{x_0^{1-\alpha}}{1-\alpha}\prod_{k=1}^{N}\mathbb{E}\Big[\Big(1 + r\Delta t + \pi_{k-1}(e^{R_k} - 1 - r\Delta t)\Big)^{1-\alpha}|\mu\Big].$$

Hence, point-wise maximization can lead to an admissible solution. We could derive optimal strategy at time k:

$$\pi_k^* = \arg\max_{\pi_k \in \mathcal{A}^H(x_0)} \mathbb{E}\Big[\Big(1 + r\Delta t + \pi_k(e^{R_{k+1}} - 1 - r\Delta t)\Big)^{1-\alpha}\Big|\mu\Big]$$

which does not depend on time, since $\varepsilon_1, \ldots, \varepsilon_N \sim \mathcal{N}(0,1)$. Like the case for logarithmic utility function, there is no explicit solution for the optimal strategy in discrete-time even if we know the value of μ. However one can also get an approximate strategy instead. Applying second order of Taylor expansion for $x^{1-\alpha}$ around 1 and e^x around 0, we get of order Δt:

$$\begin{aligned}\mathbb{E}\Big[\frac{1}{1-\alpha}(X_T^\pi)^{1-\alpha}|\mu\Big] \approx \frac{x_0^{1-\alpha}}{1-\alpha}\prod_{k=0}^{N-1}\Big(1 &+ \frac{1}{1-\alpha}(r\Delta t + \pi_k(\mu - r)\Delta t) \\ &+ \frac{\pi_k^2}{2(1-\alpha)(-\alpha)}\sigma^2\Delta t\Big).\end{aligned} \tag{2.8}$$

Hence, in order to maximize the product in (2.8) point-wise, for any $k \in \mathcal{T}$, the optimal strategy is

$$\pi_k^{power,F} = \frac{1}{\alpha}\frac{\mu - r}{\sigma^2} \vee 0 \wedge 1.$$

The representations of approximated optimal strategies for both logarithmic and power utility in discrete-time settings are identical to the ones in continuous settings.

2.1.2 Partial Information with Expert Opinions

In this section, we are interested in the case under partial information. The expert opinions are given by E_s at time s. The partial information available to an investor can be described by the investor filtration $\mathcal{F}^H = (\mathcal{F}_k^H)_{k\in\mathcal{T}}$ with $\mathcal{F}_k^H$ generated by $\{R_s, s \leq k\}$ when $H = R$ and $\{R_s, E_s, s \leq k\}$ for expert opinions E_s when $H = C$, where we assume that the $\sigma-$algebras $\mathcal{F}_t^H$ are augmented by the null sets $\mathcal{N}$ of $\mathbb{P}$.

Logarithmic utility function

We could also get an approximate solution of the portfolio optimization problem in discrete-time setting as $\Delta t \to 0$. From (2.6), using the tower property of conditional expectation, linearity operator of conditional expectation and that π_t is $\mathcal{F}_t^H$-measurable, we get

$$\begin{aligned}\mathbb{E}[\log(X_T^\pi)] &= \mathbb{E}[\mathbb{E}[\ln(X_T^\pi)|\mathcal{F}_t^H]]\\ &= \log x_0 + rT + \sum_{k=0}^{N-1}\mathbb{E}\Big[\pi_k^H(\mathbb{E}[\mu_k|\mathcal{F}_k^H] - r)\Delta t - \frac{1}{2}(\pi_k^H)^2\sigma^2\Delta t\Big].\end{aligned}$$

Hence, the approximated optimal strategy for problem 2.5 is given by

$$\hat{\pi}_k^{log,H} = \frac{\hat{\mu}_k^H - r}{\sigma^2},\ k \in \mathcal{T}, \tag{2.9}$$

where $\hat{\mu_k} = \mathbb{E}[\mu_k|\mathcal{F}_k^H],\ k \in \mathcal{T}$. In practice, this equality represents the best estimator for the mean rate of returns.

Power utility function

We typically focus on the problem:

$$\begin{aligned}\pi^{power,H} &= \arg\max_{\pi^H\in\mathcal{A}}\mathbb{E}\Big[\frac{(X_0^{\pi^H})^{1-\alpha}}{1-\alpha}\prod_{k=0}^{N-1}\Big(1 + r\Delta t + \pi_k^H(e^{R_{k+1}} - 1 - r\Delta t)\Big)^{1-\alpha}\Big]\\ &= \arg\max_{\pi^H\in\mathcal{A}}\frac{x_0^{1-\alpha}}{1-\alpha}\mathbb{E}\Big[\mathbb{E}\Big[\prod_{k=0}^{N-1}\Big(1 + r\Delta t + \pi_k^H(e^{R_{k+1}} - 1 - r\Delta t)\Big)^{1-\alpha}\Big|\mathcal{F}_k^H\Big]\Big].\end{aligned}$$

It is not achievable to get an approximated optimal strategy in this scenario by maximizing point-wise since for two random variables A and B

$\mathbb{E}[AB|\mathcal{F}_k^H] \neq \mathbb{E}[A|\mathcal{F}_k^H] * \mathbb{E}[B|\mathcal{F}_k^H]$ even if A is independent of B. [FGW14] proposed a dynamic programming method in continuous settings.

We put forward a method by applying multivariate version of the Taylor Theorem in discrete setting. Let $h : \mathbb{R}^n \to \mathbb{R}$ be the second order Taylor polynomial $h(y)$ for y near a

$$h(y) \approx h(a) + Dh(a)(y-a) + \frac{1}{2}(y-a)^T Hh(y)(y-a), \tag{2.10}$$

where $Dh(y)$ is $1 \times n$ matrix of partial derivatives, and $Hh(y)$ is Hessian matrix of h.

One can represent the maximal expected utility $V_0(x_0)$ as a multivariate function $h : \mathbb{R}^N \to \mathbb{R},\ y \mapsto y_1 \cdot y_2 \ldots y_N$

$$\begin{aligned} V_0(x_0) &:= \sup_{\pi \in \mathcal{A}} \mathbb{E}\Big[\frac{(X_0^\pi)^{1-\alpha}}{1-\alpha} \prod_{k=0}^{N-1} \Big(1 + r\Delta t + \pi_k(e^{R_{k+1}} - 1 - r\Delta t)\Big)^{1-\alpha}\Big] \\ &= \frac{1}{1-\alpha} x_0^{1-\alpha} \sup_{\pi \in \mathcal{A}} \mathbb{E}[h(Y)^{1-\alpha}]. \end{aligned}$$

Let $a = (1, \ldots, 1)^T$. Applying second order Taylor polynomial approximation (2.10), one gets

$$\begin{aligned} V_0(x_0) &= \frac{1}{1-\alpha} x_0^{1-\alpha} \sup_{\pi \in \mathcal{A}} \mathbb{E}[h(Y)^{1-\alpha}] \\ &\approx \frac{1}{1-\alpha} x_0^{1-\alpha} \sup_{\pi \in \mathcal{A}} \mathbb{E}\Big[1 + (1-\alpha) \sum_{i=1}^N \Big((X_i - 1) + \frac{(-\alpha)}{2}(X_i - 1)^2\Big) \\ &+ (1-\alpha)^2 \sum_{j \neq k} (X_j - 1)(X_k - 1)\Big]. \end{aligned}$$

Here, maximizing terminal wealth Y can be approximated as maximizing the above function with respect to X.

Moreover, one can get an approximation for V_0 of order Δt

$$V_0(x_0) \approx \frac{1}{1-\alpha} x_0^{1-\alpha} \sup_{\pi \in \mathcal{A}} \mathbb{E}\Big[1 + (1-\alpha) \sum_{i=1}^{N} \Big((X_i - 1) + \frac{(-\alpha)}{2}(X_i - 1)^2\Big)$$
$$+ (1-\alpha)^2 \sum_{j \neq k} (X_j - 1)(X_k - 1)\Big]$$
$$= \frac{1}{1-\alpha} x_0^{1-\alpha} \sup_{\pi^H \in \mathcal{A}} \mathbb{E}\Big[1 + (1-\alpha) \sum_{i=1}^{N} \Big(r\Delta t + \pi_k^H (\mathbb{E}[\mu|\mathcal{F}_k^H] - r)\Delta t + (\pi_k^H)^2 \frac{\sigma^2}{2} \Delta t\Big)\Big].$$

From above, we can get an approximate optimal strategy by maximizing point-wise in terms of power utility function.

$$\pi_k^{power,H} = \frac{1}{\alpha} \frac{\hat{\mu}_k - r}{\sigma^2} \tag{2.11}$$

where $\hat{\mu}_k = \mathbb{E}[\mu_k|\mathcal{F}_k^H]$. It can be interpreted as a myopic strategy for power utility function.

2.1.3 Discrete Kalman Filter for Drift Term

Our next task is to find the best estimator for μ. In the case $H = R$, [CLMZ09] stated that the filter for μ can be interpreted as the well-known Kalman-Bucy filter (Kalman filter). That the filter $\hat{\mu}_k = \mathbb{E}[\mu_k|\mathcal{F}_k^R]$ is the best estimate of μ_k in L^2 at time k, was illustrated in [BUV12]. We could reformulate it in discrete-time as

$$\begin{aligned} \hat{\mu}_k = \mathbb{E}[\mu|\mathcal{F}_k^R] &= \frac{\bar{\sigma}_k^2}{\sigma^2 \Delta t + \bar{\sigma}_k^2}(R_k + \frac{\sigma^2}{2}\Delta t) + \frac{\sigma^2 \Delta t}{\sigma^2 \Delta t + \bar{\sigma}_k^2} \hat{\mu}_{k-1}, \\ \mathrm{Var}[\mu|\mathcal{F}_k^R] &= \frac{\sigma^2 \Delta t \cdot \bar{\sigma}_k^2}{\sigma^2 \Delta t + \bar{\sigma}_k^2}. \end{aligned}$$

Here, $\mathcal{N}(\bar{\mu}_{k-1}, \bar{\sigma}_{k-1}^2)$ for $\bar{\mu}_{k-1} = \hat{\mu}_{k-1}$ and $\bar{\mu} = \hat{\mu}$ is the prior normal distribution where $\hat{\sigma}_{k-1}^2 = \mathrm{Var}[\mu_{k-1}|\mathcal{F}_{k-1}^R]$. We start with $(\hat{\mu}_0, \hat{\sigma}_0^2) = (\mu_0, \sigma_0^2)$.

In the case $H = C$, we propose the following theorem to get the best estimator in L^2.

Theorem 2.1.4. *(One-dimensional version: Kalman Filter with two observations) The hidden state $x_k \in \mathbb{R}$ at time k of a discrete-time controlled* *process that is governed by the linear stochastic difference equation (thinking of u_k as control)*

$$x_k = a_k x_{k-1} + b_k u_k + \epsilon_k$$

with two measurements:

$$z_k^1 = c_k^1 x_k + \zeta_k^1$$
$$z_k^2 = c_k^2 x_k + \zeta_k^2$$

where $a_k \in \mathbb{R}$ is the state transition model which is applied to the previous state x_{k-1}; $b_k \in \mathbb{R}$ describes how the control u_k changes the state from $k-1$ to k; c_k^1 and c_k^2 describe how to map the state x_k to observation z_k^1 and z_k^2 respectively; ϵ_k represents the process noise, which is assumed independent and $\epsilon_k \sim \mathcal{N}(0, w_k^2)$; ζ_k^1 and ζ_k^2 are measurement noises, which are all independent of ϵ_k, time-independent, and $\zeta_k^1 \sim \mathcal{N}(0, m_k^2)$, $\zeta_k^2 \sim \mathcal{N}(0, n_k^2)$.
Denote

$$\begin{aligned}\hat{x}_k^* &:= \mathbb{E}[x_k | z_k^1, z_k^2, x_{k-1} = \hat{x}_{k-1}] \\ &= \Big(1 - K_k(\lambda_k c_k^1 + (1-\lambda_k)c_k^2)\Big)\mathbb{E}[x_k | x_{k-1} = \hat{x}_{k-1}] + K_k\Big(\lambda_k z_k^1 + (1-\lambda_k) z_k^2\Big)\end{aligned}$$

and K_k and λ_k are the solutions of equations set

$$\frac{\partial \operatorname{Var}(x_k - \hat{x}_k^*)}{\partial K_k} = 0,$$

$$\frac{\partial \operatorname{Var}(x_k - \hat{x}_k^*)}{\partial \lambda_k} = 0;$$

and $K_k \stackrel{!}{=} K_k \vee 0 \wedge \frac{1}{\lambda_k c_k^1 + (1-\lambda_k)c_k^2}$, $\lambda_k \stackrel{!}{=} \lambda_k \vee 0 \wedge 1$;
Here $\hat{x}_k^*$ is the best estimator in the L^2, i.e.

$$\hat{x}_k^* = \arg\inf_{\hat{x}_k} \mathbb{E}[(x_k - \hat{x}_k)^2].$$

Proof. See Appendix A.1 □

Corollary 2.1.5. *If $c_k^1 = c_k^2 = c_k$, Theorem 2.1.4 has an explicit solution as follows:*

$$\lambda_k = \frac{n_k^2 - \rho m_k n_k}{m_k^2 + n_k^2 - 2\rho m_k n_k} \vee 0 \wedge 1$$

$$K_k = \frac{c_k(a_k^2 Var(x - \hat{x}_{k-1}) + w_k^2)}{c_k^2(a_k^2 \operatorname{Var}(x - \hat{x}_{k-1}) + w_k^2) + (\lambda_k^2 m_k^2 + (1-\lambda_k)^2 n_k^2 + 2\lambda_k(1-\lambda_k)\rho m_k n_k)}$$

where $K_k \stackrel{!}{=} K_k \vee 0 \wedge \frac{1}{\lambda_k c_k^1 + (1-\lambda_k)c_k^2}$.

Proof. Substituting $c_k^1 = c_k^2 = c_k$. □

Remark 2.1.6. *W.l.o.g, we assume $m_k > n_k$. If $\rho \leq \frac{n_k}{m_k}$, $0 \leq \lambda_k \leq 1$, and else if $\frac{n_k}{m_k} \leq \rho \leq 1$, $\lambda_k = 0$. As a conclusion, if $\rho_k \leq \frac{n_k}{m_k}$,$n_k$ representing lower variance and m_k higher one here, we choose weight $\frac{n_k^2 - \rho m_k n_k}{m_k^2 + n_k^2 - 2\rho m_k n_k}$ on the observation with larger variance and $\frac{m_k^2 - \rho m_k n_k}{m_k^2 + n_k^2 - 2\rho m_k n_k}$ on the observation with less variance. If $\frac{n_k}{m_k} \leq \rho \leq 1$ we weight 0 on the observation with larger variance and 1 on the observation with less variance. In the former case, we can use both observations, which will decrease the expected square error of estimator for state. However, in the latter we utilize the observation with less variance as the only observation.*

Proposition 2.1.7. *Consider the case*

$$\begin{aligned}
\mu_k &= \bar{\mu}_k + \epsilon_k, \\
E_{k+1} &= \mu_k \Delta t + \zeta_k, \\
R_{k+1} &= (\mu_k - \frac{\sigma^2}{2})\Delta t + \varepsilon_k,
\end{aligned}$$

where noisy terms $\epsilon_k \sim \mathcal{N}(0, \bar{\sigma}_k^2), \zeta_k \sim \mathcal{N}(0, m^2\Delta t), \varepsilon_k \sim \mathcal{N}(0, \sigma^2 \Delta t)$. These noisy terms are all independent with each other. The prior distribution for μ_k is given by $\mathcal{N}(\bar{\mu}_k, \bar{\sigma}_k^2)$. Then the best estimator for μ_k is

$$\begin{aligned}
\hat{\mu}_k &:= \mathbb{E}[\mu_k | E_{k+1}, R_{k+1}, \bar{\mu}_k, \bar{\sigma}_k^2] \\
&= (1 - K_k \Delta t)\bar{\mu}_k + K_k(\lambda_k E_{k+1} + (1 - \lambda_k)(R_{k+1} + \frac{\sigma^2}{2}\Delta t)) \\
&= \frac{\sigma^2 m^2 \bar{\mu}_k + \sigma^2 \bar{\sigma}_k^2 E_{k+1} + \bar{\sigma}_k^2 m^2 (R_{k+1} + \frac{\sigma^2}{2}\Delta t)}{(m^2 + \sigma^2)\Delta t \bar{\sigma}_k^2 + \sigma^2 m^2}
\end{aligned}$$

Proof. Applying Theorem 2.1.4, and letting

$$\begin{aligned}
\rho &= 0 \\
m_k &= m\sqrt{\Delta t}, n_k = \sigma\sqrt{\Delta t} \\
c_k^1 &= c_k^2 = \Delta t \\
z_k^1 &= E_{k+1} \\
z_k^2 &= R_{k+1} + \frac{\sigma^2}{2}\Delta t
\end{aligned}$$

we can derive

$$\lambda = \frac{\sigma^2}{m^2 + \sigma^2} \in [0, 1]$$
$$K_k = \frac{\bar{\sigma}_k^2}{\Delta t \bar{\sigma}_k^2 + \frac{\sigma^2 m^2}{m^2+\sigma^2}} \in [0, \Delta t]$$

Substituted in

$$\begin{aligned}&\mathbb{E}[\mu_k | E_{k+1}, R_{k+1}, \bar{\mu}_k^2, \bar{\sigma}_k^2] \\ &= (1 - K_k \Delta t)\bar{\mu}_k + K_k(\lambda_k E_{k+1} + (1 - \lambda_k)(R_{k+1} + \frac{\sigma^2}{2}\Delta t))\end{aligned}$$

we get the final result. □

Remark 2.1.8. *In our case, we have $\mu_k = \mu$, $\bar{\mu}_k = \hat{\mu}_{k-1}$, $\bar{\sigma}_k = \hat{\sigma}_{k-1}$, so the prior is the filter from the step before. The prior assumption $\bar{\mu}$, the observation and expert opinions can be understood as three observations. Two of them could combined with using the discrete Kalman filter, and the resulting filter can be combined with remaining observation again. The resulting filter coming from these two-step process is the same as what we defined in Proposition 2.1.7.*

2.2 Generalized Kalman Filter and Uniform Expert Opinions

In the remaining part of this chapter we shall discuss estimation of μ if we have uniform expert opinions, i.e., if an expert provides a range for the real parameter. We discuss only the estimation. Then the estimate (approximate filter) can be put in the approximate optimal strategy for logarithmic or power utility as in (2.9) or in (2.11).
Investors sometimes will have the idea that the drift term is located in an interval with no additional weight in the middle. This assumption will give a uniform expert opinion. We start with the univariate setting.
Let $Y \in \mathbb{R}$ be defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The prior assumption for is $Y \sim \mathcal{N}(\mu, \sigma^2)$. Now we have additional observation (or expert opinion) which gives $Y \in (a, b), -\infty \leq a \leq b \leq +\infty$. The Y conditional on $a \leq Y \leq b$ has a truncated normal distribution. Its probability density function is given by

$$f(y; \mu, \sigma, a, b) = \frac{\phi(\frac{y-\mu}{\sigma})}{\sigma(\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma}))}.$$

Here $\phi(.)$ is PDF of standard normal distribution and $\Phi(.)$ is CDF of standard normal distribution. Note that $\Phi'(x) = \phi(x)$. The denominator of f represents the probability that Y can lie in the interval of $[a, b]$. Thus f can be interpreted that normal distribution density function uniformly scatters in the targeted range $[a, b]$.

Lemma 2.2.1. *The conditional expectation and conditional variance for $Y_t \in [a, b]$ is given by*

$$\mathbb{E}[Y|a \leq Y \leq b] = \mu + \sigma \frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}$$

$$Var[Y|a \leq Y \leq b]$$

$$= \sigma^2 \left\{ 1 + \frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} - \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}\right]^2 \right\}.$$

Proof. See Appendix A.1. □

Remark 2.2.2. *From the definition of* ϕ *and* Φ*, we know that* $\lim_{a\to-\infty}\phi(\frac{a-\mu}{\sigma}) = \lim_{b\to+\infty}\phi(\frac{b-\mu}{\sigma}) = 0$*,* $\lim_{a\to-\infty}\Phi(\frac{a-\mu}{\sigma}) = 0$ *and* $\lim_{b\to+\infty}\phi(\frac{b-\mu}{\sigma}) = 1$*. Such that* $\lim_{a\to-\infty,b\to+\infty}\mathbb{E}[Y|a \le Y \le b] = \mu$*,* $\lim_{a\to-\infty,b\to+\infty}Var[Y|a \le Y \le b] = \sigma^2$*. Actually the distribution of* Y *conditional on* $[a,b]$ *is approaching the normal distribution whose expected value is* μ *when* $a \to -\infty$ *and* $b \to +\infty$*.*

Remark 2.2.3. *Suppose* $-\infty \le a \le b \le +\infty$ *and* $a, b \in \mathbb{R}$*, we have* $\lim_{b\to a}\mathbb{E}[Y|a \le Y \le b] = a$*, and* $\lim_{b\to a}Var[Y|a \le Y \le b] = 0$*. This leads to a Dirac delta distribution* $\delta_a(y)$ *with variance 0.*

2.2.1 Multivariate Case and Black-Litterman Model

Let $Y = (Y_1, Y_2, \cdots, Y_n)^T$ denote n-dimensional random variables of stock returns, all defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Y has a non-singular n-variate normal distribution with mean vector $\mu = (\mu_1, \mu_2, \cdots, \mu_n)^T$ and $(n \times n)$ positive definite correlation matrix $\Sigma = (\sigma_{i,j})$, such that Y has density:

$$f_Y(y, \mu, \Sigma) = (2\pi)^{-\frac{n}{2}}(\det\Sigma)^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right\}; Y \in \mathbb{R}^n.$$

Suppose partial information at time $t > 0$ is given as

$$Y_1 \in B_1, Y_2 \in B_2, \cdots, Y_n \in B_n,$$

where $B_i = [a_i, b_i]$. The notation of information can be represented as $y \in B = \{y \in \mathbb{R}^n | y_i \in B_i, i \in [1, n]\}$. If the information is not given, the range can be represented as $(-\infty, +\infty)$. Letting $a = \{a_1, a_2, \cdots, a_n\}$ and $b = \{b_1, b_2, \cdots, b_n\}$, the density function of conditional probability is given by

$$\varphi_{\mu\Sigma B} = f_{Y|X}(y, \mu, \Sigma, B) = \begin{cases} \frac{\exp\{-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\}}{\beta} & y \in B, \\ 0 & \text{otherwise;} \end{cases} \tag{2.12}$$

where, $\beta = \int_B \exp\left\{-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right\}dy$.

Equation ((2.12)) illustrates distribution of conditional probability is doubly truncated multivariate normal distributed. [Wih12] proposed explicit expression for the truncated mean and variance for the multivariate normal distribution with arbitrary rectangular double truncation. Mean and covariance matrix can be derived by computing moment generating function.

Lemma 2.2.4. *The expected value and covariance matrix for Y, which is normally distributed and truncated by B, are*

$$\mathbb{E}(Y_i) = \sum_{k=1}^{n} \sigma_{i,k}(F_k(a_k) - F_k(b_k)) + \mu_i.$$
$$\mathrm{Cov}(Y_i, Y_j) = \mathbb{E}(Y_iY_j) - \mathbb{E}(Y_i)\mathbb{E}(Y_j)$$

with

$$\begin{aligned}\mathbb{E}(Y_iY_j) =&\sigma_{i,j} + \sum_{k=1}^{n} \sigma_{i,k} \frac{\sigma_{j,k}(a_kF_k(a_k) - b_kF_k(b_k))}{\sigma_{k,k}} \\ &+ \sum_{k=1}^{n} \sigma_{i,k} \sum_{q\neq k} (\sigma_{j,q} - \frac{\sigma_{k,q}\sigma_{j,k}}{\sigma_{k,k}})[(F_{k,q}(a_k, a_q) - F_{k,q}(a_k, b_q) \\ &- (F_{k,q}(b_k, a_q) - F_{k,q}(b_k, b_q))],\end{aligned}$$

where $F_k(x)$ represents the marginal density function of $\varphi_{\mathbf{0}\Sigma B^*}$, here $B^* = \{y \in \mathbb{R}^n | a_i - \gamma_i \leq y_i \leq b_i - \gamma_i\}_{i\in[0,n]}$ with $\gamma_i = \sum_{k=1}^{n} \sigma_{i,k}t_k$. $F_{k,q}(x, y)$ is the bivariate marginal density, which is given by

$$F_{k,q}(x, y) = \int_{a_1}^{b_1} \cdots \int_{a_{k-1}}^{b_{k-1}} \int_{a_{k+1}}^{b_{k+1}} \cdots \int_{a_{q-1}}^{b_{q-1}} \int_{a_{q+1}}^{b_{q+1}} \int_{a_n}^{b_n} \varphi_{\mu\Sigma B}(x, y, \mathbf{x}_{-k,-q}) d\mathbf{x}_{-k,-q},$$

Proof. See Appendix A.1 □

[Tal61] stated another formula for $F_k(x)$. It was represented by the product of density of $a - \gamma$ or $b - \gamma$ and another multiple integral.

Next, we apply the assumption of uniform distribution to the Black-Litterman model, which has one-period setting. [Wal14] illustrated how Black-Litterman model derived by Bayesian network and derived explicit formula with expert's confidence for conditional expectation of stock return. Posterior distribution is given by

$$\mathbb{P}[Y|X] \sim \mathcal{N}\left([\Omega^{-1}\mu + \Sigma^{-1}\Pi]^T[\Omega^{-1} + \Sigma^{-1}]^{-1}, [\Omega^{-1} + \Sigma^{-1}]^{-1}\right),$$

where X represents the expert opinion, which is Gaussian given by

$$\mathbb{P}[X|Y] \sim \mathcal{N}(\mu, \Omega),$$

and the prior distribution, whose information comes from observing stock returns, is also normal distribution given by

$$\mathbb{P}[Y] \sim \mathcal{N}(\Pi, \Sigma),$$

i.e. $\hat{\Pi} = \mathbb{E}[Y|X] = [\Omega^{-1}\mu + \Sigma^{-1}\Pi]^T[\Omega^{-1} + \Sigma^{-1}]^{-1}$ and $\mathrm{Cov}[Y|X] = [\Omega^{-1} + \Sigma^{-1}]^{-1}$.

The inverse of the covariance, which is illustrated as investors' confidence, can be also interpreted as precision. [Wal14] described the posterior mean as the weighted mean of the prior and conditional means, and weighting factors are respective precision. Moreover, the posterior precision can be described as the sum of the prior and conditional precision. Note that $\lim_{\Omega \to +\infty} \hat{\Pi} = \Pi$. Expert opinions will not be functional if the opinions are not sufficiently believable.

Example 2.2.5. *Suppose three risky assets (Stock A Stock B and Stock C)are considered. The current means of stock returns are* $\mu = \{1, 1, 1\}$ *and the covariance matrix is assumed to be*

$$\begin{bmatrix} 9.1 & 3.0 & 6.0 \\ 3.0 & 1.1 & 2.0 \\ 6.0 & 2.0 & 4.1 \end{bmatrix}.$$

The expert opinion is that return of stock A is assumed to be 3 *and the variance* Ω *of expert opinion is to be* 1*. Using [Wal14]'s formula, we can derive* $\hat{\mu}_{BL} = \{2.80, 1.60, 2.19\}$ *and updating covariance becomes*

$$\begin{bmatrix} 0.90 & 0.30 & 0.59 \\ 0.30 & 0.21 & 0.22 \\ 0.59 & 0.22 & 0.54 \end{bmatrix}.$$

If Ω *increases to* 9*, the result is significantly different as follows* $\hat{\mu}_{BL} = \{2.01, 1.33, 1.66\}$ *and updating covariance becomes*

$$\begin{bmatrix} 4.52 & 1.49 & 2.98 \\ 1.49 & 0.60 & 1.01 \\ 2.98 & 1.01 & 2.11 \end{bmatrix}.$$

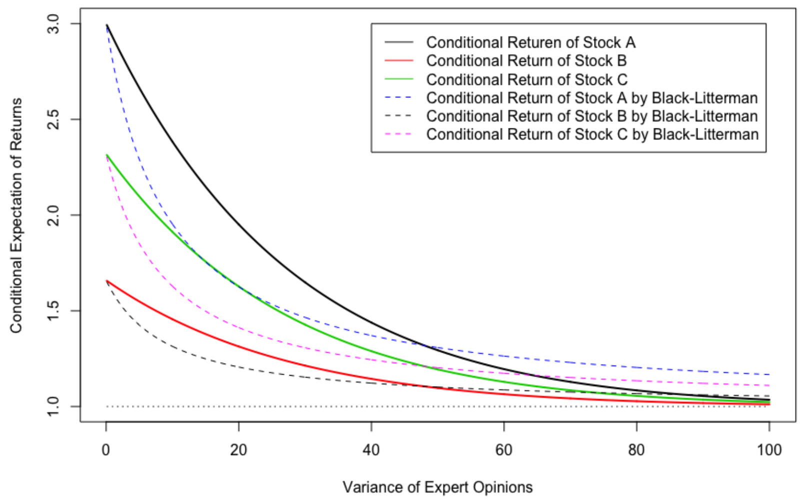
The posterior distribution approaches to prior distribution if Ω *increases.*
By contrast, we set lower limits as $\boldsymbol{a} = \{2, -\infty, -\infty\}$ *and upper limits as* $\boldsymbol{b} = \{4, +\infty, +\infty\}$*. The bounds are done to the range of mean of condition* X *between* $\left[\mu - \sqrt{\Omega}, \mu + \sqrt{\Omega}\right]$*. The updated mean is* $\hat{\mu} = \{2.93, 1.64, 2.27\}$ *and the updated covariance is*

$$\begin{bmatrix} 0.33 & 0.11 & 0.21 \\ 0.11 & 0.15 & 0.09 \\ 0.21 & 0.09 & 0.29 \end{bmatrix}.$$

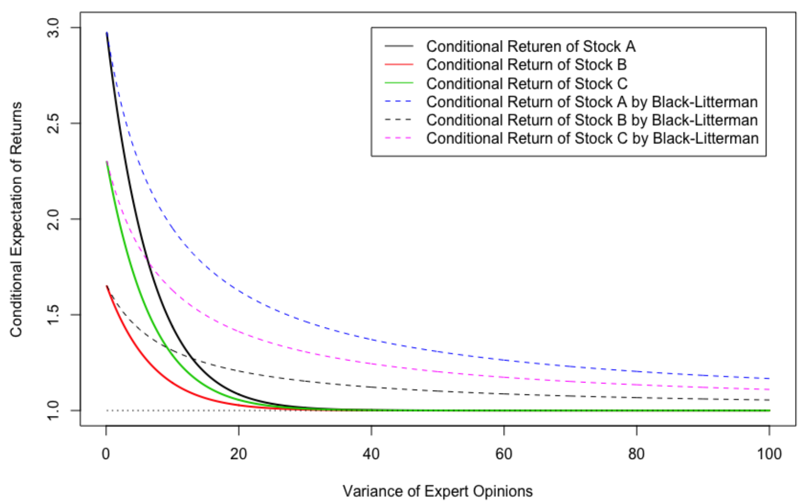
Then the range of X *can be extended. The lower boundary is* $\boldsymbol{a} = \{0, -\infty, -\infty\}$ *and upper one is* $\boldsymbol{b} = \{6, +\infty, +\infty\}$*.*
The updated mean is now $\hat{\mu} = \{2.44, 1.47, 1.95\}$ *and the updated covariance is*

$$\begin{bmatrix} 2.46 & 0.81 & 1.62 \\ 0.81 & 0.38 & 0.56 \\ 1.62 & 0.56 & 1.21 \end{bmatrix}.$$

Figure 2.1 and Figure 2.2 show the comparison with Black-Litterman Model with Gaussian expert opinions. The setting of ranges of condition X is considered to compare with Black-Litterman model, in which the expert opinion is not 100% reliable. Two factors are taken into account. Firstly, how the conditional expectation of stock returns acts when the ranges of condition X increases. Secondly, whether the actions of conditional expectation will differ when the targeted μ locates in different intervals of prior distribution.
Figure 2.1 and Figure 2.2 illustrate that the conditional expectation of both models will approach 1 if the ranges extended. Moreover, the mean value of prior distribution is easier to touch in our model when the range of condition extend.
However, when the condition of X is equivalent to $[\mu - \sqrt{\Omega_A}, \mu + \sqrt{\Omega_A}]$, the conditional expectations of all three stocks in our model will be larger than that in Black-litterman model with Gaussian expert until Ω reaches a value. When the condition of X is equivalent to $[\mu - 2\sqrt{\Omega_A}, \mu + 2\sqrt{\Omega_A}]$, the conditional expectations of all three stocks in our model are throughout smaller than the ones in Black-Litterman model with Gaussian. What's more, no matter what intervals of the prior distribution the targeted μ locates in , the above properties keep similar.
We can apply our model when experts can only identify the range of several

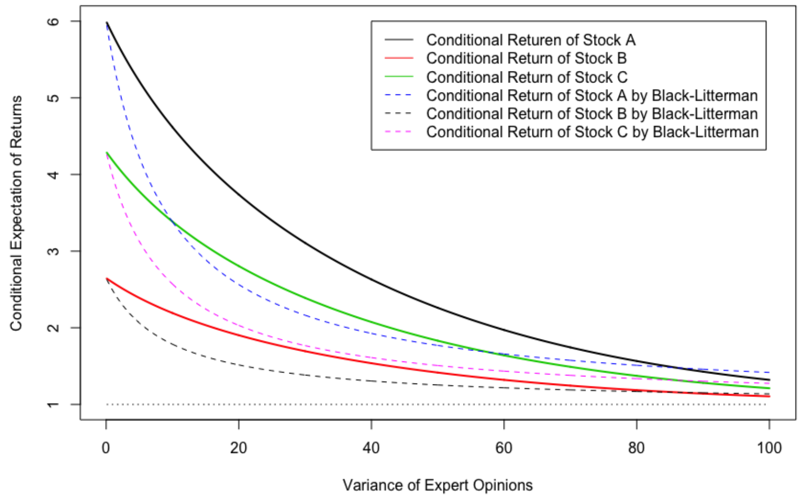


(a) The lower limit and upper limit of X is equal to $\mu - \sqrt{\Omega_A}$ and $\mu + \sqrt{\Omega_A}$, when targeted $\mu = 3$, which is between $\Pi - \sqrt{\Sigma_A}$ and $\Pi + \sqrt{\Sigma_A}$

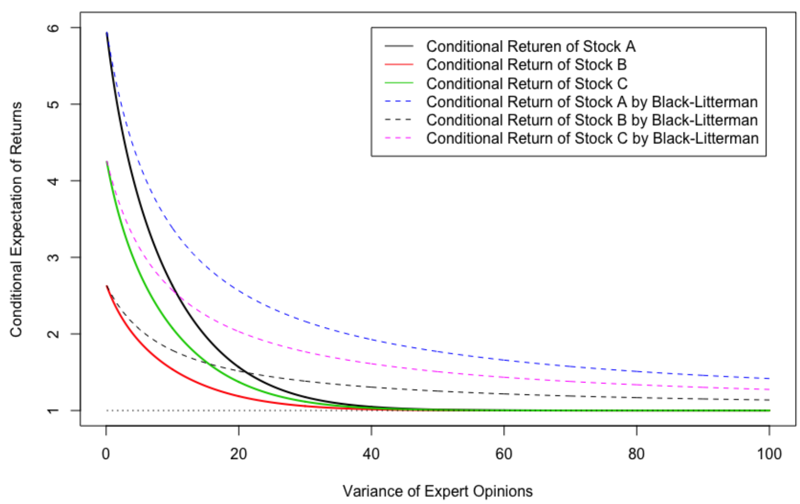


(b) The lower limit and upper limit of X is equal to $\mu - 2\sqrt{\Omega_A}$ and $\mu + 2\sqrt{\Omega_A}$, when targeted $\mu = 3$, which is between $\Pi - \sqrt{\Sigma_A}$ and $\Pi + \sqrt{\Sigma_A}$

Figure 2.1



(a) The lower limit and upper limit of X is equal to $\mu - \sqrt{\Omega_A}$ and $\mu + \sqrt{\Omega_A}$, when targeted $\mu = 6$, which is between $\Pi - 2\sqrt{\Sigma_A}$ and $\Pi + 2\sqrt{\Sigma_A}$



(b) The lower limit and upper limit of X is equal to $\mu - 2\sqrt{\Omega_A}$ and $\mu + 2\sqrt{\Omega_A}$, when targeted $\mu = 6$, which is between $\Pi - 2\sqrt{\Sigma_A}$ and $\Pi + 2\sqrt{\Sigma_A}$

Figure 2.2

stocks but don't have sufficient confidence to identify these stocks' targeted returns.

2.2.2 Dynamic Model and Generalized Kalman Filter

In this section, we try to use generalized Kalman filter to calculate the filter for μ. To start with, we will discuss the relationship between Bayesian Network and Kalman filter.
To define the generalised Kalman Filter, we suppose that the state $\{x_k, k \in \mathcal{N}\}$ evolve by

$$x_k = f_k(x_{k-1}, w_k) \tag{2.13}$$

where $f_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function (can be linear or non-linear) of the states x_{k-1} and i.i.d. tracking error term w_k.
The observations or measurements of states are also given recursively as

$$z_k = h_k(x_k, v_k) \tag{2.14}$$

where $h_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function (can be linear or non-linear) of the state x_k and i.i.d. measurement error term v_k.
In particular, we estimate the filter of x_k based on the set of all available information from observations $z_{1:k} = \{z_i, i = 1 : k\}$. For linear Kalman Filter, f_k and h_k are assumed linear and the noise term w_k and n_k are assumed Gaussian. In this assumption the estimation of x_k is a optimal estimation by minimizing the variance of the difference of true state and estimated state. However, in the model of linear Kalman Filter, only mean and covariance of the state is considered rather that the whole distribution.
From a Bayesian perspective, the state x_k is estimated in the sense of probability at time k. So it is required to calculate the PDF $p(x_k|z_{1:k})$. Suppose the initial PDF $p(x_0|z_0) = p(x_0)$ is known, the PDF $p(x_k|z_{1:k})$ can be obtained recursively.
Now, we only focus on the step that we estimate the state x_k at time k from the state x_{k-1} at time $k-1$. Suppose the PDF $p(x_{k-1}|z_{1:k-1})$ is know by the recursive scheme. Firstly, the function (2.14) is applied to derive the prior PDF of the state at time k via the law of total probability (or Chapman-Kolmogorov equation)

$$p(x_k|z_{1:k-1}) = \int p(x_k|x_{k-1})p(x_{k-1}|z_{1:k-1})dx_{k-1}. \tag{2.15}$$

To derive the above equation, we use the Markov property of the state as (2.13) describes, i.e. $p(x_k|x_{k-1}, z_{1:k-1}) = p(x_k|x_{k-1})$. Moreover, the PDF $p(x_k|x_{k-1})$ can be obtained from the system equation (2.13) and the known probability of w_k.
For the classical Kalman filter, at time $t, t = 1, \ldots, k-1$, the noise term $w_{1:k-1}$ is Gaussian and independent of $x_{1:k-1}$. f_k is a linear function. The function f_k is

$$f_k(x_{k-1}, w_k) = F_k x_{k-1} + w_k, F_k \in \mathbb{R}^+ \text{ and } w_k \sim \mathcal{N}(0, \Gamma_1^2). \tag{2.16}$$

The conditional state $x_k|x_{k-1}$ is also Gaussian and its distribution is given by $\mathcal{N}(F_k x_{k-1}, \Gamma_1^2)$. Note that we eliminate the control variable of classical Kalman filter here.
Suppose that we have estimated the state x_{k-1} and it follows $\mathcal{N}(\vartheta, \Gamma_2^2)$. Then we can obtain the density function of conditional state $x_k|z_{1:k-1}$ via function (2.15)

$$\begin{aligned} p(x_k|z_{1:k-1}) &= \frac{1}{2\pi\Gamma_1\Gamma_2}\int_{-\infty}^{\infty} e^{-\frac{(x_k - F_k x_{k-1})^2}{2\Gamma_1^2} - \frac{(x_{k-1}-\vartheta)^2}{2\Gamma_2^2}} dx_{k-1} \\ &= \frac{1}{\sqrt{2\pi(F_k^2\Gamma_2^2 + \Gamma_1^2)}} e^{-\frac{(x_k - F_k\vartheta)}{2(F_k^2\Gamma_2^2+\Gamma_1^2)}} \\ &\quad \left(\text{by Gaussian intergral} \int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}\right). \end{aligned}$$

Hence, for the conditional state we have $x_k|z_{1:k-1} \sim \mathcal{N}(F_k\vartheta, \sqrt{F_k^2\Gamma_2^2 + \Gamma_1^2})$. Moreover, another approach can be applied to the case of general prior distribution. It is called the arithmetic of random variables.

We define the continuous random variables X, Y, Z having the relationship $Z = AX + BY$ $A, B \in \mathbb{R}^+$. And $f_Z(z)$, $f_X(x)$, $f_Y(y)$ are the PDF of Z, X and Y respectively. $f_{XY}(x, y)$ is the joint density function of X and Y. $F_Z(z)$

is the CDF of Z. Then, the PDF of Z can be obtained, as

$$\begin{aligned}
f_Z(z) &= \frac{dF_Z(z)}{dz} \\
&= \frac{d}{dz}\int_{-\infty}^{\infty}\int_{-\infty}^{\frac{z-Ax}{B}} f_{XY}(x,y)dydx \\
&= \int_{-\infty}^{\infty}\frac{d}{dz}\int_{-\infty}^{\frac{z-Ax}{B}} f_{XY}(x,y)dydx \\
&= \frac{1}{B}\int_{-\infty}^{\infty} f_{XY}\Big(x, \frac{z-Ax}{B}\Big)dx, \text{ (by Leibniz rule)}
\end{aligned}$$

if X, Y are independent, the above function is

$$f_Z(z) = \frac{1}{B}\int_{-\infty}^{\infty} f_X(x) f_Y(\frac{z-Ax}{B})dx. \tag{2.17}$$

If we have the conditions that

$$f_X(x) = p(x_{k-1}|z_{1:k-1}) = \frac{1}{\sqrt{2\pi}\Gamma_2}e^{-\frac{(x-\vartheta)^2}{2\Gamma_2^2}},$$

$$f_Y(y) = p(w_k) = \frac{1}{\sqrt{2\pi}\Gamma_1}e^{-\frac{y^2}{2\Gamma_1^2}},$$

$A = F_k$, and $B = 1$, the PDF of Z representing state $x_k|z_{1:k-1}$ can be attained:

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z - F_k x)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\Gamma_2}e^{-\frac{(x-\vartheta)^2}{2\Gamma_2^2}} \cdot \frac{1}{\sqrt{2\pi}\Gamma_1}e^{-\frac{(z-F_k x)^2}{2\Gamma_1^2}} dx \\
&= \frac{1}{\sqrt{2\pi(F_k^2\Gamma_2^2+\Gamma_1^2)}}e^{-\frac{(z-F_k\vartheta)}{2(F_k^2\Gamma_2^2+\Gamma_1^2)}}. \\
&\text{(by Gassian intergral)}
\end{aligned}$$

Note that the result is the same as the one derived via the Chapman-Kolmogorov equation.

At time k, observations z_k is available and we can update the prior distribution $p(x_k|z_{1:k-1})$ by Bayes' rule

$$p(x_k|z_{1:k}) = \frac{p(z_k|x_k)p(x_k|z_{1:k-1})}{\int p(z_k|x_k)p(x_k|z_{1:k-1})dx_k}, \tag{2.18}$$

where the PDF $p(z_k|x_k)$ is defined by the function (2.14) and the statistics of v_k. [AMGC02] explain that in some situations the posterior density cannot be determined analytically, however there is some other solutions, for instance the Extended Kalman Filter and some particle filters, which are not optimal.

Observations with uniform distribution

We have the same assumptions at time t, $t = 1, \ldots, k-1$. Typically at time t, $t = 1, \ldots, k-1$, the noise term w_t is Gaussian and independent of x_t. f_k is a linear function. The function f_k is given in function (2.16). Such that the state $x_{1:k-1}$ is also Gaussian.
However, at time k observation is modelled as random variable with uniform distribution, i.e. h_k is a linear function and v_k follows uniform distribution. Under these assumptions, the state $p(x_k|z_{1:k})$ can be estimated by the function (2.18) and its distribution will be truncated normal.
However, the state x_{k+1} at time $k+1$ is problematic. The reason is that the distribution of state x_k is truncated normally distributed rather than Gaussian and this property transfers the problem to non-Gaussian state or non-linear system function, which may lead to intractable analytical solutions.

Firstly, we try to derive the optimal Bayesian solution via function (2.13) and (2.14). The PDF of $p(x_{k+1}|z_{1:k})$ can be obtained from the arithmetic sum of $x_k|z_{1:k}$ and w_{k+1}.
Suppose $x_k|z_{1:k}$ follows truncated normal distribution $TN(\vartheta_2, \Gamma_2^2, a, b)$ and w_{k+1} follows $\mathcal{N}(\vartheta_1, \Gamma_1^2)$. Then the PDF of $x_{k+1}|z_{1:k}$, represented by $f_Z(z)$, can be obtained by function (2.17),

$$\begin{aligned} f_Z(z) &= \int_a^b \frac{e^{-\frac{(x-\vartheta_2)^2}{2\Gamma_2^2}}}{\sqrt{2\pi}\Gamma_2 K} \cdot \frac{e^{-\frac{(z-F_{k+1}x-\vartheta_1)^2}{2\Gamma_1^2}}}{\sqrt{2\pi}\Gamma_1} dx + 0(x < a) + 0(x > b) \\ &= \frac{1}{2\pi K \Gamma_1 \Gamma_2} \int_a^b e^{-\frac{(x-\vartheta_2)^2}{2\Gamma_2^2} - \frac{(z-F_{k+1}x-\vartheta_1)^2}{2\Gamma_1^2}} dx \end{aligned}$$

where $K = \Phi\Big(\frac{b-\vartheta_2}{\Gamma_2}\Big) - \Phi\Big(\frac{a-\vartheta_2}{\Gamma_2}\Big)$.
Further, $x_{k+1}|z_{1:k+1}$ can be derived via the Bayesian rule since the PDF of $z_{k+1}|x_{k+1}$ and $x_{k+1}|z_{1:k}$ are known. Here, we want to check whether Z follows a well-known distribution (e.g. Gaussian or truncated Gaussian). $f_Z(z)$ can be reformulated as

$$f_Z(z) = \frac{1}{2\pi K \Gamma_1 \Gamma_2} \cdot \exp\Bigg\{ - \frac{(z-\vartheta_1 - F_{k+1}\vartheta_2)^2}{2(\Gamma_1^2 + \Gamma_2^2 F_{k+1}^2)^2} \Bigg\}.$$

$$\int_a^b \exp\Bigg\{ - \frac{(\Gamma_1^2 + \Gamma_2^2 F_{k+1}^2)}{2\Gamma_1^2\Gamma_2^2}\Bigg(x - \frac{\Gamma_1^2\vartheta_2 + \Gamma_2^2 F_{k+1}(z-\vartheta_1)}{\Gamma_1^2 + \Gamma_2^2 F_{k+1}^2}\Bigg)^2 \Bigg\} dx,$$

where

$$\int_a^b \exp\{-c(x-h)^2\}dx = \frac{\sqrt{\pi}}{2\sqrt{c}}\Bigg[\operatorname{erf}\Big(\sqrt{c}(b-h)\Big) - \operatorname{erf}\Big(\sqrt{c}(a-h)\Big)\Bigg].$$

Let $c = \frac{(\Gamma_1^2+\Gamma_2^2F_{k+1}^2)}{2\Gamma_1^2\Gamma_2^2}$ and $h = \frac{\Gamma_1^2\vartheta_2 + \Gamma_2^2F_{k+1}(z-\vartheta_1)}{\Gamma_1^2+\Gamma_2^2F_{k+1}^2}$, $f_Z(z)$ can be rewrite as

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sqrt{\Gamma_1^2 + \Gamma_2^2 F_{k+1}^2}} \cdot e^{-\frac{(z-\vartheta_1-F_{k+1}\vartheta_2)^2}{2(\Gamma_1^2+\Gamma_2^2F_{k+1}^2)}} \cdot \frac{h(z)}{2K}$$

where

$$\begin{aligned} h(z) = erf\Bigg(&\sqrt{\frac{(\Gamma_1^2+\Gamma_2^2F_{k+1}^2)}{2\Gamma_1^2\Gamma_2^2}}(b - \frac{\Gamma_1^2\vartheta_2 + \Gamma_2^2F_{k+1}(z-\vartheta_1)}{\Gamma_1^2+\Gamma_2^2F_{k+1}^2})\Bigg) \\ &- \operatorname{erf}\Bigg(\sqrt{\frac{(\Gamma_1^2+\Gamma_2^2F_{k+1}^2)}{2\Gamma_1^2\Gamma_2^2}}(a - \frac{\Gamma_1^2\vartheta_2 + \Gamma_2^2F_{k+1}(z-\vartheta_1)}{\Gamma_1^2+\Gamma_2^2F_{k+1}^2})\Bigg). \end{aligned}$$

Note that $h(z) \to 2$ and $K \to 1$ when $a \to -\infty$ and $b \to +\infty$, such that $f_Z(z)$ is the PDF of $\mathcal{N}(\vartheta_1 + F_{k+1}\vartheta_2, \Gamma_1^2 + \Gamma_2^2F_{k+1}^2)$. Otherwise, $f_Z(z)$ should not be a PDF of a well-known distribution.

This formula can be written in terms of the cumulative distribution function of standard normal distribution $\Phi(\cdot)$

$$h(z) = 2\Bigg(\Phi\Big(\frac{z - \frac{\Gamma_1^2}{\Gamma_2^2F_{k+1}^2}(b-\vartheta_2) - (b-\vartheta_1)}{\frac{\sqrt{(\Gamma_1^2+\Gamma_2^2F_{k+1}^2)}\Gamma_1\Gamma_2}{\Gamma_2^2F_{k+1}^2}}\Big) - \Phi\Big(\frac{z - \frac{\Gamma_1^2}{\Gamma_2^2F_{k+1}^2}(a-\vartheta_2) - (a-\vartheta_1)}{\frac{\sqrt{(\Gamma_1^2+\Gamma_2^2F_{k+1}^2)}\Gamma_1\Gamma_2}{\Gamma_2^2F_{k+1}^2}}\Big)\Bigg).$$

In the next step we aim at finding a sub-optimal solution to estimate the state x_{k+1}. The target is to find a method by which state x_{k+1} is estimated and has a similar distribution as given by Bayesian solution above.
The problem can be transferred from non-Gaussian state to a non-linear system and hence we can apply non-linear Kalman filter, for instance extended Kalman filter, to solve it.
For a scalar measurement, if the CDF F of the distribution of state x_k is invertible, the state can be transferred from a new state $\hat{x}_k$ with Gaussian distribution by the probability integral transform

$$x_k = F^{-1}(G(\hat{x}_k))$$

where, G is the CDF of distribution of the new state $\hat{x}_k$. If $\hat{x}_k \sim \mathcal{N}(\nu, \zeta^2)$ and x_k follows a truncated normal distribution $TN(\vartheta_2, \mathit{\Gamma}_2^2, a, b)$. Then, x_k can be expressed via $\hat{x}_k$

$$x_k = \Phi^{-1}\left(K\Phi\left(\frac{\hat{x}_k - \nu}{\zeta}\right) + \Phi\Big(\frac{a - \vartheta_2}{\mathit{\Gamma}_2}\Big)\right) \cdot \mathit{\Gamma}_2 + \vartheta_2.$$

Hence,

$$\begin{aligned} f_{k+1}(\hat{x}_k, w_{k+1}) &= F_{k+1}\left[\Phi^{-1}\left(K\Phi\left(\frac{\hat{x}_k - \nu}{\zeta}\right) + \Phi\Big(\frac{a - \vartheta_2}{\mathit{\Gamma}_2}\Big)\right) \cdot \mathit{\Gamma}_2 + \vartheta_2\right] + w_{k+1} \\ &= F_{k+1}\left[\operatorname{erf}^{-1}\left(2K\Phi\left(\frac{\hat{x}_k - \nu}{\zeta}\right) + 2\Phi\Big(\frac{a - \vartheta_2}{\mathit{\Gamma}_2}\Big) - 1\right) \cdot \sqrt{2}\mathit{\Gamma}_2 + \vartheta_2\right] + w_{k+1}. \end{aligned} \tag{2.19}$$

Note F_{k+1} is only a constant coefficient, i.e. $F_{k+1} \in \mathbb{R}^+$, $w_{k+1} \sim \mathcal{N}(0, \mathit{\Gamma}_1^2)$. And the error function erf is used since the property of inverse error function and its derivative will be discussed later.

Extended Kalman Filter

The extended Kalman filter (EKF) is the non-linear version of the Kalman filter which linearizes the estimate of the current mean and covariance. It utilizes the first order Taylor expansion to approximate the function f_{k+1}. We only approximate the function f_{k+1} here, because we assume that h_{k+1} is already linear.

The first order derivative $W_{k+1}(\hat{x}_k)$ of $f_{k+1}(\hat{x}_k, w_{k+1})$ with respect to $\hat{x}_k$ is

$$\begin{aligned} W_{k+1}(\hat{x}_k) &= \frac{\partial f_{k+1}(\hat{x}_k, w_{k+1})}{\partial \hat{x}_k} \\ &= \frac{2KF_{k+1}\mathit{\Gamma}_2}{\sqrt{\pi}\zeta} \cdot e^{-\frac{(\hat{x}_k-\nu)^2}{2\zeta^2}} \cdot \frac{\partial \operatorname{erf}^{-1}(z)}{\partial z}\Bigg|_{w=2K\Phi\left(\frac{\hat{x}_k-\nu}{\zeta}\right)+2\Phi\left(\frac{a-\vartheta_2}{\mathit{\Gamma}_2}\right)-1}. \end{aligned}$$

The intuition is that the inverse error function can be approximated by the Maclaurin series (usually the inverse error function is approximated by polynomial function or rational function)

$$\operatorname{erf}^{-1}(w) = \sum_{k=0}^{\infty} \frac{c_k}{2k+1}\left(\frac{\sqrt{\pi}}{2}w\right)^{2k+1}$$

where

$$c_k = \sum_{m=0}^{k-1} \frac{c_m c_{k-1-m}}{(m+1)(2m+1)}.$$

If the first two terms are left

$$\operatorname{erf}^{-1}(w) = \frac{\sqrt{\pi}}{2}(w + \frac{\pi}{12}w^3 + O(w^5)),$$

$W_{k+1}(\hat{x}_{k+1})$ can be approximated as

$$W_{k+1}(\hat{x}_k) = \frac{KF_{k+1}\mathit{\Gamma}_2}{\zeta} \cdot e^{-\frac{(\hat{x}_k-\nu)^2}{2\zeta^2}} \left[1 + \frac{\pi}{4}\left(2K\Phi\left(\frac{\hat{x}_k-\nu}{\zeta}\right) + 2\Phi\left(\frac{a-\vartheta_2}{\mathit{\Gamma}_2}\right) - 1\right)^2\right].$$

Note that $w = K + 2\Phi\left(\frac{a-\vartheta_2}{\mathit{\Gamma}_2}\right) - 1 = \Phi\left(\frac{b-\vartheta_2}{\mathit{\Gamma}_2}\right) + \Phi\left(\frac{a-\vartheta_2}{\mathit{\Gamma}_2}\right) - 1$ such that $w \in (-1, 1)$ here. The approximation will be accurate around $w = 0$, i.e. $\vartheta_2 = \frac{a+b}{2}$.

Then if $\hat{x}_k \sim \mathcal{N}(\nu, \zeta^2)$, we can use extended Kalman filter to derive a Gaussian state $\hat{x}_{k+1|z=1:k}$,

$$\hat{x}_{k+1|z=1:k} \sim \mathcal{N}\Big(f_{k+1}(\nu, 0), W_{k+1}(\hat{x}_k = \nu)^2\zeta^2 + \mathit{\Gamma}_1^2\Big).$$

Here,

$$f_{k+1}(\nu,0) = F_{k+1}\left[\sqrt{2}\varGamma_2 \cdot \mathrm{erf}^{-1}\left(\Phi\Big(\frac{b-\vartheta_2}{\varGamma_2}\Big) + \Phi\Big(\frac{a-\vartheta_2}{\varGamma_2}\Big) - 1\right) + \vartheta_2\right],$$

$$W_{k+1}(\hat{x}_k = \nu) = \frac{2KF_{k+1}\varGamma_2}{\sqrt{\pi}\zeta} \cdot \frac{\partial\, \mathrm{erf}^{-1}(w)}{\partial w}\bigg|_{w=\Phi\left(\frac{b-\nu}{\zeta}\right)+\Phi\left(\frac{a-\vartheta_2}{\varGamma_2}\right)-1}.$$

Note that the mean of new state ν is irrelevant to the approximated Gaussian distribution.

Unscented Kalman filter

Unscented Kalman filter (UKF) computes a set of so-called sigma points and transforms each sigma point through the non-linear function f_{k+1}, and then computes a Gaussian distribution from the transformed and weighted sigma points. The unscented Kalman filter avoids to linearize around the mean as Taylor expansion and EKF does. Next we typically focus on the one-dimensional case. The sigma points are choose as

$$\begin{aligned}\chi^0 &= \nu,\\ \chi^1 &= \nu + \sqrt{1+\lambda}\zeta,\\ \chi^2 &= \nu - \sqrt{1+\lambda}\zeta.\end{aligned}$$

The weights are given by

$$\begin{aligned}w_m^0 &= \frac{\lambda}{1+\lambda},\\ w_c^0 &= w_m^0 + (1-\alpha^2+\beta),\\ w_m^i &= w_c^i = \frac{1}{2(1+\lambda)} \text{ for } i = 1,2.\end{aligned}$$

We have some free parameters here as there is no unique solution. The scaled unscented transform suggests $\kappa \geq 0, \alpha \in (0,1]$ and $\lambda = \alpha^2(1+\kappa) - 1$. $\beta = 2$ is optimal the choice for Gaussian (however our model is not Gaussian). A typical recommendation is $\alpha = 0.001, \kappa = 0$. However, a larger value of α may be beneficial in order to better capture the spread of the distribution and possible non-linearities as discussed in [Bit16]. Next, we can predict the Gaussian state $\hat{x}_{k+1}|z_{1:k}$. Suppose

$$\hat{\chi}_k^i = f_{k+1}(\chi_k^i, w_{k+1} = 0),$$

then the expected value and variance of state $\hat{x}_{k+1}|z_{1:k}$ is

$$\mathbb{E}(\hat{x}_{k+1}|z_{1:k}) = \sum_{i=0}^{2} w_m^i \hat{\chi}_k^i,$$
$$Var(\hat{x}_{k+1}|z_{1:k}) = \sum_{i=0}^{2} w_c^i (\hat{\chi}_k^i - \nu)^2 + \Gamma_1^2.$$

Note that the mean of new state ν is irrelevant to the mean of approximated Gaussian distribution, but it can influence the variance of the Gaussian distribution.

Accuracy of inverse error function

In some settings of values for the parameters, e.g. $(a = 3, b = 6, \Gamma_2 = 2)$, the methods of EKF and UKF both have strong ability to approximate the distribution of $f_Z(z)$ to be a Gaussian distribution (see Figure 2.3a). Also from Table 1, the total variation distance of true distribution and approximated Gaussian is not significant. However, in other settings of parameters $(a = 17, b = 20, \Gamma_2 = 2)$, these two methods will give distinguished result (see Figure 2.3b).

Table 2.1: Total variation distance of true distribution and approximated Gaussian (10^{-3})

Expert Opinion with Uniform Distribution	**EKF**	**UKF**
$[3, 6]$	0.1	0.07
$[17, 20]$	3.4	3.5

In Table 2.1, we compute the total variation distance between true distribution and the approximated Gaussian in these settings.

Table 2.2: Total variation distance of true distribution and approximated Gaussian (10^{-3})

Expert opinion with uniform distribution $\mathcal{U}(17, 20)$	**EKF**	**UKF**
$O(x^5)$	3.4	3.5
$O(x^9)$	1.4	1.4
More accurate inverse error function	0.1	0.07

Because of the fact that as $|w| = \left|\Phi\left(\frac{b-\vartheta_2}{\mathit{\Gamma}_2}\right) + \Phi\left(\frac{a-\vartheta_2}{\mathit{\Gamma}_2}\right) - 1\right| \to 1$, the speed of convergence of Maclaurin series will decrease, one needs higher order Maclaurin series to approximate when w is closed to -1 or 1.
The higher order of Maclaurin series is expressed as

$$\operatorname{erf}^{-1}(w) = \frac{\sqrt{\pi}}{2}(w + \frac{\pi}{12}w^3 + \frac{7\pi^2}{480}w^5 + \frac{127\pi^3}{40320}w^7 + O(w^9)).$$

Then, the formula of $W_{k+1}(w)$ can be rewritten as

$$W_{k+1}(w) = \frac{KF_{k+1}\mathit{\Gamma}_2}{\zeta} \cdot e^{-\frac{(\hat{x}_k-\nu)^2}{2\zeta^2}} \left[1 + \frac{\pi}{4}w^2 + \frac{7\pi^2}{96}w^4 + \frac{127\pi^3}{5760}w^6\right]$$

where, $w(\hat{x}_k) = 2K\Phi\left(\frac{\hat{x}_k-\nu}{\zeta}\right) + 2\Phi\left(\frac{a-\vartheta_2}{\mathit{\Gamma}_2}\right) - 1.$

For the function $f_{k+1}(w, w_{k+1})$, the higher order Maclaurin series are applied

$$f_{k+1}(w, w_{k+1}) = F_{k+1}\left[\frac{\sqrt{2\pi}}{2}(w + \frac{\pi}{12}w^3 + \frac{7\pi^2}{480}w^5 + \frac{127\pi^3}{40320}w^7)\mathit{\Gamma}_2 + \vartheta_2\right] + w_{k+1}.$$

From the Figure 2.4a, the mean of approximated Gaussian via higher order Maclaurin series is closer to the true distribution compared to that via lower order Maclaurin series. Due to the fact that the last term of $W_{k+1}(w)$ is $O(x^8)$, converging more slowly than that of $f_{k+1}(w)$ with last term $O(x^9)$, the variance of approximated Gaussian via higher order Maclaurin series is not estimated well in this case. From Figure 2.4b, the inverse error function is approximated more accurately. Hence the distortion of both the mean and variance is revised.

From Table 2.2, the accuracy of approximation of inverse error function can affect the total variation distance between true distribution and approximated Gaussian. In conclusion, how to approximate the inverse error function is crucial especially when the point we want to approximate is far away from zero.

New Gaussian State

To determine the new Gaussian state $\hat{x}_k$ is important, because it influences the non-linear function and the moment of new state will also affect the result of prediction. Since our goal is to approximate a truncated Gaussian state $TN(\vartheta_2, \Gamma_2^2, a, b)$ to a Gaussian one, we utilize the different methods as mentioned in the above chapter. However, these methods can be only used numerically.
The first approach is to construct an optimization problem, in which uniform distribution $\mathcal{U}(a, b)$ with density function f_U is approximated with a Gaussian distribution with density function f_N^θ by statistic distance $d(\cdot)$.

The parameters of density function $f_N^{\theta^*}$ of approximated Gaussian can be obtained by minimizing the statistical distance of the class of normal distribution and targeted uniform distribution, i.e.

$$\theta^* = arg \inf_{\theta \in \Theta} d(f_U, f_N^\theta)$$

where, Θ is the parameter space of normal distributions and f_N^θ is the density function of normal distribution parametrized by θ. Suppose

$$f_N^{\theta^*} = \frac{1}{\sqrt{2\pi}\sigma_U} e^{-\frac{(x-\mu_U)^2}{2\sigma_U^2}}.$$

Then these parameters can be represented by uniform distribution $\mathcal{U}(a, b)$ numerically,

$$\mu_U = \frac{a+b}{2},$$

$$\sigma_U = \frac{b-a}{m},$$

where m can vary from different statistical distance (e.g. 3.9598 by Hellinger distance and 2.9728 by total variance distance). Further the pdf of the new

Gaussian state $\hat{x}_k$ can be derived by Bayesian rule,

$$f_{\hat{T}}(x) = \frac{1}{\sqrt{2\pi}\sigma_{\hat{T}}} e^{-\frac{(x-\mu_{\hat{T}})^2}{2\sigma_{\hat{T}}^2}}$$

where,

$$\mu_{\hat{T}} = \frac{\Gamma_2^2}{\sigma_U^2 + \Gamma_2^2}\mu_U,$$
$$\sigma_{\hat{T}}^2 = \frac{\sigma_U^2 \Gamma_2^2}{\sigma_U^2 + \Gamma_2^2}.$$

We use the notation $\hat{T}$ here because the truncated normal distribution is not approximated with Gaussian directly. Instead we approximate a uniform distribution first.

Another approach is to approximate truncated Gaussian with normal distribution directly. The optimization problem can be constructed in the mathematical form as

$$\theta^* = \arg\min_{\theta \in \Theta} d(f_T, f_N^\theta)$$

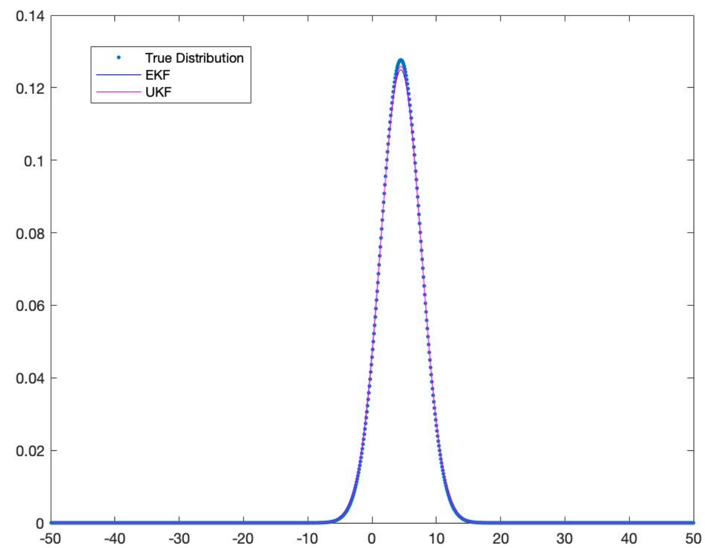
where, Θ is the parameter space of normal distributions and f_N^θ is the density function of normal distribution parametrized by θ. And importantly, f_T is the truncated normal distribution here.
From this optimization problem, a normal distribution with different parameters is given by

$$f_T(x) = \frac{1}{\sqrt{2\pi}\sigma_T} e^{-\frac{(x-\mu_T)^2}{2\sigma_T^2}}.$$

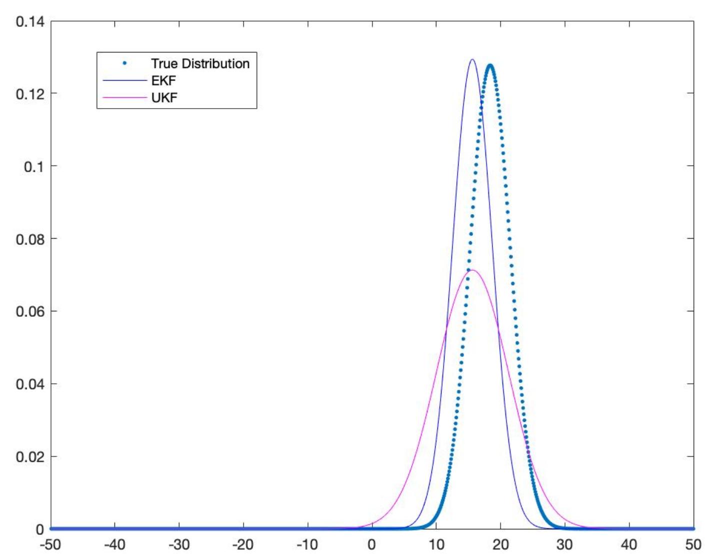
Table 2.3: Total variation distance of true distribution and Approximated Gaussian (10^{-3})

Expert Opinion with Uniform Distribution with $\mathcal{U}(17, 20)$	**EKF**	**UKF**
Uniform distribution is approximated by TV	0.12	0.074
Truncated Normal distribution is approximated by TV	0.079	0.090
Uniform distribution is approximated by Hellinger	0.078	0.077
Truncated Normal distribution is approximated by Hellinger	0.078	0.077

Different approaches have different results, one needs to compare what is close to the true distribution of $x_{k+1|z_{1:k}}$. Moreover, whether the choice of the new state affects the result needs to be checked. From Table 2.3, for extended Kalman filter it is better to approximate truncated normal distribution directly if total variation distance is utilized, while for the unscented Kalman filter, approximating uniform distribution is preferred. However, there is no big difference when using Hellinger distance.



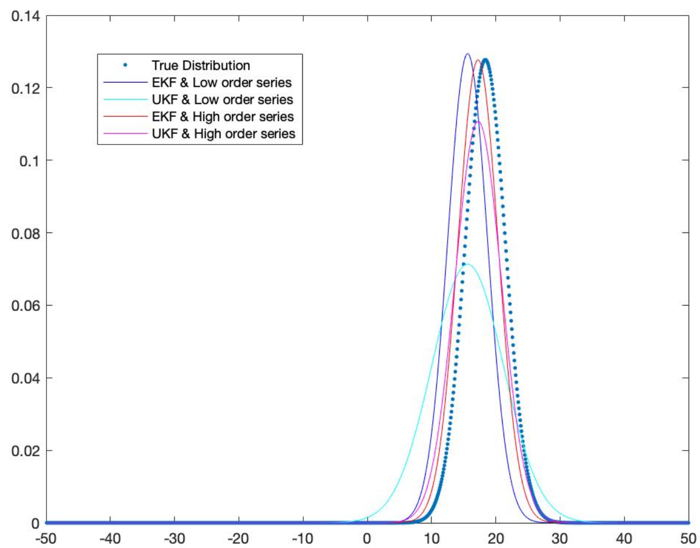
(a) $w_{k+1} \sim \mathcal{N}(0, 9)$, $x_k \sim TN(2, 100, 3, 6)$. $\alpha = 1, \kappa = 0, \beta = 2$ for UKF and $F_{k+1} = 1$. Non-linear function (2.19) is approximated by Extended Kalman filter and Unscented Kalman filter.

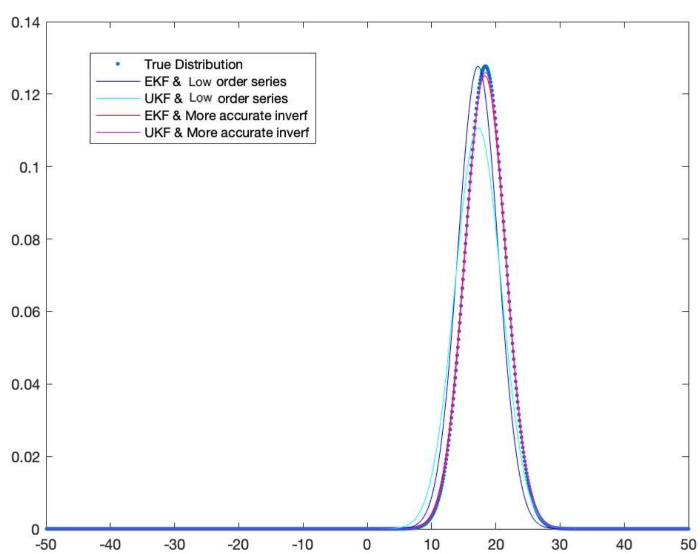


(b) $w_{k+1} \sim \mathcal{N}(0, 9)$, $x_k \sim TN(2, 100, 17, 20)$. $\alpha = 1, \kappa = 0, \beta = 2$ for UKF and $F_{k+1} = 1$. Non-linear function is approximated by Extended Kalman filter and Unscented Kalman filter respectively.

Figure 2.3: EKF and UKF with different parameters

(a) $w_{k+1} \sim \mathcal{N}(0,9)$, $x_k \sim TN(2,100,17,20)$. $\alpha = 1, \kappa = 0, \beta = 2$ for UKF and $F_{k+1} = 1$. Non-linear function is approximated by Extended Kalman filter and Unscented Kalman filter, however the non-linear function (2.19) is expressed by lower Maclaurin series with $O(x^5)$ and higher Maclaurin series with $O(x^9)$.





(b) $w_{k+1} \sim \mathcal{N}(0,9)$, $x_k \sim TN(2,100,17,20)$. $\alpha = 1, \kappa = 0, \beta = 2$ for UKF and $F_{k+1} = 1$. Non-linear function is approximated by Extended Kalman filter and Unscented Kalman filter, however the non-linear function is expressed by high Maclaurin series with $O(x^9)$ and more accurate inverse error function.

Figure 2.4: Higher order Maclaurin series for inverse error function

Chapter 3

Estimation for MSMs with Expert Opinions

In this chapter, we consider a discrete-time market model, where both drift and volatility are driven by a Markov chain:

$$R_k = b^T Y_{k-1} + a^T Y_{k-1} \varepsilon_k.$$

Expert Opinions provide additional information on each state of Markov chain. We combine information from returns and expert opinions by Kalman filter and Bayesian filter. We also consider the case that, expert opinions are assumed to be conditional Dirichlet distributed.

The term estimation is used to cover observations (log-returns) filters, model parameter identification, state estimation, observation smoothing, and observation prediction. A basic technique used in this chapter is a change-of-measure. A new probability measure is defined such that under this probability measure, called reference measure, observations are independent and identically distributed random variables.

3.1 Markov Switching Model for Log-Returns

Let $E \neq \varnothing$ be finite and $(\Omega, \mathcal{A}, P)$ be a suitable probability space.
Let $Y = (Y_n)_{n \in \mathbb{N}_0}$ be a homogeneous Markov Chain, which means that for all $i, j \in E$

$$p_{ij} := P(Y_{n+1} = j | Y_n = i)$$

is independent of n. Then,

$$\Pi := (p_{ij})_{i,j \in E}$$

is called transition matrix.

Definition 3.1.1. *By $\pi = (\pi_i)_{i \in E}$, $\pi_i = P(Y_0 = i)$, we denote the initial distribution. We denote further*

$$P_i(A) := P(A | Y_0 = i) \text{ for all } A \in \mathcal{A}$$

Definition 3.1.2. *We call $\Pi^h := \Pi \cdots \Pi$, the $h-$th power of the transition matrix.*

The following proposition shows that this determines the long-run behaviour of the Markov chain in terms of the h-step ahead predictive distribution.

Proposition 3.1.3. *We have*

$$P(Y_{k+h} = i | Y_k = j) = p_{ji}^{(h)}$$

where $p_{ji}^{(h)}$ is the element (j, i) of Π^h.

Proof. We can prove this by induction. It is trivial for $h = 1$ from the definition of Π.
For $h > 1$

$$\begin{aligned} P(Y_{k+h} = l | Y_k = m) &= \sum_{i=1}^{d} P(Y_{k+h} = l | Y_{k+h-1} = i) P(Y_{k+h-1} = i | P_k = m) \\ &= \sum_{i=1}^{d} p_{il} p_{mi}^{(h-1)} = p_{ml}^{(h)} \end{aligned}$$

□

This result is also referred as the Chapman-Kolmogorov equation.

Definition 3.1.4. *Any probability vector* $\eta = (\eta_1, \ldots, \eta_d)^T$ *that satisfies the invariance property*

$$\Pi^T \eta = \eta \tag{3.1}$$

is called an invariant distribution (or stationary) distribution of Y_t*.*

Remark 3.1.5. *The invariant distribution* η *is left-hand eigenvector of* Π*, associated with eigenvalue* 1*.*
For invariant distribution η *and initial distribution* $\pi = \eta$*, we have*

$$P(Y_n = j) = \eta_j \;\, j \in E.$$

Several numerical methods can be used for solving equation (3.1). [Ham94] proposed a closed-form expression for the invariant probability distribution η in terms of transition matrix Π. Define by

$$A = \begin{bmatrix} \boldsymbol{I}_d - \Pi^T \\ \mathbf{1}_d \end{bmatrix}$$

with $\boldsymbol{I}_d$ being the identity matrix with d rows and $\mathbf{1}_d$ being a row vector of ones. Then η is given as the $(d+1)-$th column of the matrix $(A^T A)^{-1} A^T$:

$$\eta = \left((A^T A)^{-1} A^T \right)_{\cdot, d+1}.$$

Definition 3.1.6. *Irreducibility means that starting* Y_k *from an arbitrary state* $i \in E$*, any state* $j \in E$ *must be reached in finite time, i.e., for all* $i, j \in E$

$$\exists h(i,j) : p_{ij}^{(h(i,j))} > 0.$$

The irreducibility of transition matrix will make sure the uniqueness of invariant distribution [FS06].

Definition 3.1.7. *The absence of periodicity is called aperiodicity. A Markov chain is aperiodic if the period of each state is equal to one, i.e.,*

$$GCD\{n \geq 1 : p_{ii}^{(n)} > 0\} = 1, \; \textit{for all } i \in E.$$

Thus, a Markov chain is aperiodic if all diagonal elements of Π are positive.

Definition 3.1.8. *Ergodicity of a Markov chain implies the distribution $P(Y_h|Y_0 = i)$, which is equal to the $i-$th row of Π^h converges to the invariant distribution (or called ergodic distribution here), regardless of the state i of Y_0:*

$$\lim_{h\to\infty} (\Pi^h)_{i,\cdot} = \eta^T.$$

where $(\Pi^h)_{i,\cdot}$ is the $i-$th row of Π^h.

Irreducibility and aperiodicity are sufficient conditions for the ergodicity of a Markov chain [Sas18]. Let $Y = (Y_k)_{k\in\mathbb{N}_0}$ be a homogeneous Markov chain with states space $S = \{e_1, \ldots, e_d\}$ where e_i denotes the $i-$th unit vector in $\mathbb{R}^d$, and with transition matrix $\Pi = (p_{ij})_{i,j\in E}$ which is irreducible and aperiodic.

Definition 3.1.9. *The Markov switching model in discrete time is given by*

$$R_k = b^T Y_{k-1} + a^T Y_{k-1}\varepsilon_k$$

where $b \in \mathbb{R}^d$ is state vector and $a \in (0,\infty)^d$ volatility vector, and ε_k are iid. standard normal random variables, which are also independent of Y.

Remark 3.1.10. *We write R_k to represent log-return. For each state e_i, $b_i = (\mu_i - \frac{\sigma_i^2}{2})\Delta t$ and $a_i = \sigma_i\sqrt{\Delta t}$.*

Proposition 3.1.11. *R_k is conditional Gaussian and the distribution of R_k is a mixture of normal distributions.*

Proof. For $k \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$, the conditional distribution of R_k given $Y_{k-1} = e_i$

$$P_{R_k|Y_{k-1}=e_i} : \mathcal{B} \to [0,1],\ B \mapsto P_{R_k|Y_{k-1}=e_i}(B) := P(R_k \in B|Y_{k-1} = e_i)$$

of R_k given $Y_{k-1} = e_i$ is $\mathcal{N}(b_i, a_i^2)$. i.e. R_k is conditional Gaussian.

The joint distribution of (R_k, Y_{k-1}) is

$$P(R_k \in B, Y_{k-1} = e_i) = P(Y_{k-1} = e_i)P_{R_k|Y_{k-1}=e_i}(B)$$

for $B \in \mathcal{B}, i = 1, \ldots, d$.
Thus,

$$P(R_k \in B) = \sum_{i=1}^{d} P(Y_{k-1} = e_i)P_{R_k|Y_{k-1}=e_i}(B)$$

i.e. the distribution of R_k is a mixture of Gaussians. □

Remark 3.1.12. *$P_{R_k|Y_{k-1}=e_i}(B)$ is set to be 0 if $P(Y_{k-1} = e_i) = 0$ due to the concept of general conditional probabilities. Since $P(R_k \in B|Y_{k-1})$ can be understood as some function with respect to Y_{k-1}, $P(R_k \in \cdot|Y_{k-1}) = \mathcal{N}(b_{Y_{k-1}}, a^2_{Y_{k-1}})$, where we identify b_{e_i} with b_i and a_{e_i} with a_i.*

3.2 Filter for State under Partial Information

Partial information means that investors can not observe Y_k and ε_{k+1} but asset prices only, maybe some additional information is available which we call expert opinions. We are interested in the expectation of $\mathbb{E}[Y_k|\mathcal{F}_k^H]$, $H \in \{R, C\}$. In the scenario $H = R$, investors can observe asset prices or returns. Investors have in addition expert opinions in $H = C$. To start with, we give the definition of expert opinions.

Definition 3.2.1. *The expert opinions at time t are given by E_t with values in $\mathcal{E}$, where $\mathcal{E}$ is expert opinions space, satisfying*

$$\mathcal{E} = \Big\{E \in [0,1]^d | \sum_{i=1}^{d} E^i = 1\Big\}.$$

We say E_u is a certain expert opinion, if $E_k = Y_u,\ k < u$.

Note that below $\mathcal{E}$ can not only represent the expert opinions, but also the space of estimators of state Y.
Then, $E^{(n)} = (E_{u_i})_{i=1,\dots,n}$, $u_1 < \dots < u_n$, denotes the set of all n expert opinions. We denote by $\mathcal{F}^E = (\mathcal{F}_k^E)_{k\geq 0}$ the filtration generated by E, augmented by the null sets, i.e.

$$\mathcal{F}_k^E = \sigma(\{E_{u_i} : u_i \leq k\} \cup \mathcal{N}_P).$$

We denote by $\mathcal{F}^C = (\mathcal{F}_k^C)_{k\geq 0}$ the filtrations generated by R and expert opinions up to time k, augmented by null sets, i.e. $\mathcal{F}_k^C = \mathcal{F}_k^R \vee \mathcal{F}_k^E$.
In addition, we denote by $\mathcal{F}^{C_0} = (\mathcal{F}_k^{C_0})_{k\geq 0}$ the filtrations generated by R and all expert opinions, augmented by null sets, i.e. $\mathcal{F}_k^{C_0} = \mathcal{F}_k^R \vee \mathcal{F}_K^E$. Moreover, we denote by $\mathcal{G} = (\mathcal{G}_k)_{k\geq 0}$, $\mathcal{G}_k = \mathcal{F}_k \vee \mathcal{F}_K^E$, and $\mathcal{H} = (\mathcal{H}_k)_{k\geq 0}$, $\mathcal{H}_k = \mathcal{F}_k \vee \mathcal{F}_k^E$,where $\mathcal{F} = (\mathcal{F}_k)_{k\geq 0}$ denotes the filtration generated by Y and ε. These two filtrations are typically used in parameter estimation in the following chapter.

The best estimate for Y_k given information $\mathcal{F}_k^R$ is in an L^2-sense given conditional expectation $\mathbb{E}[Y_k|R_1, \dots, R_k]$. And the filter for Y_k given $\mathcal{F}_k^R$ is defined as $\hat{Y}_k := \mathbb{E}[Y_k|R_1, \dots, R_k]$. Once the initial estimate for state, transition matrix and observation up to k are know, the state at time k can be estimated.

Theorem 3.2.2. *For $k = 1, \dots, K$,*

$$\hat{Y}_k^i = \frac{\sum_{j=1}^d p_{ji}\varphi_j(R_k)\hat{Y}_{k-1}^j}{\sum_{j=1}^d \varphi_j(R_k)\hat{Y}_{k-1}^j},$$

where $\varphi_j = \varphi_{b_j,a_j^2}$ is the pdf of a normal distribution with mean b_j and standard deviation a_j. To vectorize, we get

$$\hat{Y}_k = \frac{\Pi^T Diag(\varphi(R_k))\hat{Y}_{k-1}}{\varphi(R_k)^T\hat{Y}_{k-1}},$$

where $\varphi(r) = (\varphi_1(r), \dots, \varphi_d(r))^T$ and $Diag((\varphi(r))$ is the diagonal matrix with diagonal $\varphi(r)$.

Proof. The proof is given on Page 59 of [EAM95]. □

Corollary 3.2.3. *For $k \in \mathbb{N}$*

$$\hat{Y}_k = \frac{(\prod_{l=0}^{k-1}\Pi^T Diag(\varphi(R_{k-l})))\hat{Y}_0}{\mathbf{1}_d^T\prod_{l=0}^{k-1}\Pi^T Diag(\varphi(R_{k-l})))\hat{Y}_0} = \frac{\left(\hat{Y}_0^T\prod_{l=1}^k\Big(Diag(\varphi(R_l))\Pi\Big)\right)^T}{\hat{Y}_0^T\prod_{l=1}^k\Big(Diag(\varphi(R_l))\Pi\Big)\mathbf{1}_d}$$

where $\mathbf{1}_d = (1, 1, \dots, 1)^T$.

Proof. Note that we have $\mathbf{1}_d^T\Pi^T = (\Pi\mathbf{1}_d)^T = \mathbf{1}_d^T$ and $\varphi(R_k)^T = \mathbf{1}_d^T Diag(\varphi(R_k))$. Firstly, we substitute for $\hat{Y}_{k-1}$ by $\hat{Y}_{k-2}$

$$\begin{aligned}
\hat{Y}_k &= \frac{\Pi^T Diag(\varphi(R_k))\hat{Y}_{k-1}}{\varphi(R_k)^T\hat{Y}_{k-1}}. \\
&= \frac{\Pi^T Diag(\varphi(R_k))\frac{\Pi^T Diag(\varphi(R_{k-1}))\hat{Y}_{k-2}}{\varphi(R_{k-1})^T\hat{Y}_{k-2}}}{\varphi(R_k)^T\frac{\Pi^T Diag(\varphi(R_k))\hat{Y}_{k-2}}{\varphi(R_{k-1})^T\hat{Y}_{k-2}}} \\
&= \frac{\Pi^T Diag(\varphi(R_k))\Pi^T Diag(\varphi(R_{k-1}))\hat{Y}_{k-2}}{\varphi(R_k)^T\Pi^T Diag(\varphi(R_k))\hat{Y}_{k-2}} \\
&= \frac{\Pi^T Diag(\varphi(R_k))\Pi^T Diag(\varphi(R_{k-1}))\hat{Y}_{k-2}}{\mathbf{1}_d^T\Pi^T\varphi(R_k)^T\Pi^T Diag(\varphi(R_k))\hat{Y}_{k-2}}.
\end{aligned}$$

By iteration we can finish this proof. □

Suppose, at time k, the expert opinion is given by $E_k = Y_u$, $u > k$. Then we try to estimate $\hat{Y}_k$, where we define $\hat{Y}_k := \mathbb{E}[Y_k|R_{1:k}, E_k]$, where $R_{1:k} = \{R_1, \dots, R_k\}$ and $E_k = Y_u$. This scenario is that the expert opinions have further information of the state Y_k.

Lemma 3.2.4. *(i)*$\hat{Y}_k^i = P(Y_k = e_i|\mathcal{F}_k^H)$, $i = 1, \dots, d$
(ii)For any function $f : E \to \mathbb{R}$,

$$\mathbb{E}[f(Y_k)|\mathcal{F}_k^H] = \sum_{i=1}^{d} f(e_i)\hat{Y}_k^i.$$

Proof. (i)

$$\begin{aligned}\hat{Y}_k^i =&\mathbb{E}[Y_k^i|\mathcal{F}_k^H]\\ &= \mathbb{E}[\mathbb{1}_{\{Y_k=e_i\}}|\mathcal{F}_k^H] = P(Y_k = e_i|\mathcal{F}_k^H)\end{aligned}$$

(ii)

$$\mathbb{E}[f(Y_k)|\mathcal{F}_k^H] = \mathbb{E}\Big[\sum_{i=1}^{d} \mathbb{1}_{\{Y_k=e_i\}} f(e_i)|\mathcal{F}_k^H\Big]$$

since $f(e_i)$ is constant, we get

$$\begin{aligned}\mathbb{E}[f(Y_k)|\mathcal{F}_k^H] &= \mathbb{E}\Big[\sum_{i=1}^{d} \mathbb{1}_{\{Y_k=e_i\}} f(e_i)|\mathcal{F}_k^H\Big]\\ &= \mathbb{E}\Big[\sum_{i=1}^{d} \mathbb{E}[\mathbb{1}_{\{Y_k=e_i\}}|\mathcal{F}_k^H] f(e_i)|\mathcal{F}_k^H\Big]\\ &= \sum_{i=1}^{d} f(e_i)\hat{Y}_k^i.\end{aligned}$$

□

To find all conditional expectations of functions of Y_k, it thus suffices to find $\hat{Y}_k = \mathbb{E}[Y_k|\mathcal{F}_k^H]$. Without proof we remind of the formula for conditional probabilities in settings with one discrete and one continuous random variable:

Lemma 3.2.5. *(Bayes for one discrete and one continuous r.v.) If Y has a density f_Y, X is discrete, and g is given by*

$$P(X = x, Y \leq t) = \int_{-\infty}^{t} g(x, z)dz \text{ for all } t,$$

then

$$P(X = x|Y = y) = \frac{g(x, y)}{f_Y(y)}.$$

The best estimator for Y_k, if we can observe the stock returns $R_1, \ldots, R_k$ and a future state Y_u $u > k$, is given by the following theorem.

Theorem 3.2.6. *The filter $\hat{Y}_k := \mathbb{E}[Y_k|R_{1:k}, Y_u]$ at time k conditional on observations $R_{1:k}$ and expert opinion $E_k = Y_u$, which gives the state at time $u, u > k$, can be derived iteratively by*

$$\hat{Y}_k^i = \frac{\sum_{w=1}^d \sum_{j=1}^d \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} \varphi_j(R_k) \hat{Y}_{k-1}^j Y_u^w}{\sum_{j=1}^d \varphi_j(R_k) \hat{Y}_{k-1}^j}$$

where $p_{iw}^{(u-k)} := P(Y_u = e_w|Y_k = e_i)$.
Chapman-Kolmogorov equation implies $p_{iw}^{(u-k)} = \Pi_{i,w}^{u-k}$, where Π_{iw}^{u-k} represents the $i-$th row and $w-$th column element of matrix Π^{u-k}.
To vectorize the filter $\hat{Y}_k$,

$$\hat{Y}_k = \frac{\sum_{w=1}^d Y_u^w \Pi^T Diag\Big(\varphi(R_k) \oslash (p_{.w}^{(u-k+1)})\Big) \hat{Y}_{k-1} \odot (p_{.w}^{(u-k)})}{\varphi(R_k)^T \hat{Y}_{k-1}},$$

where $p_{.w}^{(u-k)} = (p_{1w}^{(u-k)}, \ldots, p_{dw}^{(u-k)})^T$. $\odot, \oslash$ are Hadamard product and Hadamard division respectively given by $\boldsymbol{A} \odot \boldsymbol{B} = Diag(a_1, \ldots, a_n)\boldsymbol{B}$. And $\boldsymbol{C} = \boldsymbol{A} \oslash \boldsymbol{B}$ represents $C_i = \frac{A_i}{B_i}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times 1$ matrix.

Proof.

$$\begin{aligned}
&P(Y_k = e_i, R_k \le t|R_{1:k-1}, Y_u) \\
&= \sum_{j=1}^d P(Y_k = e_i, Y_{k-1} = e_j, R_k \le t|R_{1:k-1}, Y_u) \\
&= \sum_{j=1}^d P(Y_k = e_i, R_k \le t|Y_{k-1} = e_j, R_{1:k-1}, Y_u) P(Y_{k-1} = e_j|R_{1:k-1}, Y_u) \\
&= \sum_{j=1}^d P(Y_k = e_i, R_k \le t|Y_{k-1} = e_j, R_{1:k-1}, Y_u) \hat{Y}_{k-1}^j,
\end{aligned}$$

where, we can derive by conditional independence

$$\begin{aligned}
&P(Y_k = e_i, R_k \leq t | Y_{k-1} = e_j, R_{1:k-1}, Y_u) \\
&= P(Y_k = e_i | Y_{k-1} = e_j, R_{1:k}, Y_u) P(R_k \leq t | Y_{k-1} = e_j) \\
&= \sum_{w=1}^{d} \frac{P(Y_u = e_w | Y_k = e_i) P(Y_k = e_i | Y_{k-1} = e_j)}{P(Y_u = e_w | Y_{k-1} = e_j, R_{1:k})} P(R_k \leq t | Y_{k-1} = e_j) Y_u^w \\
&= \sum_{w=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} \int_{-\infty}^{t} \varphi_j(x) dx Y_u^w.
\end{aligned}$$

Putting this together we have

$$P(Y_k = e_i, R_k \leq t | R_{1:k-1}, Y_u) = \sum_{j=1}^{d} \sum_{w=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} \int_{-\infty}^{t} \varphi_j(x) dx Y_u^w \hat{Y}_{k-1}^j.$$

By Bayes rule in Lemma 3.2.5 we then have that

$$P(Y_k = e_i | R_{1:k-1}, Y_u, R_k = r) = \sum_{j=1}^{d} \sum_{w=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} \varphi_j(x) Y_u^w \hat{Y}_{k-1}^j.$$

Moreover, by Proposition 3.1.3

$$\begin{aligned}
&P(R_k \leq t | R_{1:k-1}, Y_u) \\
&= \sum_{i=1}^{d} P(Y_k = e_i, R_k \leq t | R_{1:k-1}, Y_u) \\
&= \sum_{w=1}^{d} Y_u^w \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} \int_{-\infty}^{t} \varphi_j(x) dx \hat{Y}_{k-1}^j \\
&= \sum_{j=1}^{d} \int_{-\infty}^{t} \varphi_j(x) dx \hat{Y}_{k-1}^j.
\end{aligned}$$

Therefore

$$f_{R_k | R_{1:k-1}, Y_u}(r) = \sum_{j=1}^{d} \varphi_j(r) \hat{Y}_{k-1}^j,$$

where $f_{R_k|R_{1:k-1},Y_u}$ is the conditional pdf of R_k.
By Lemma 3.2.4, we then get the final result:

$$\begin{aligned}\hat{Y}_k^i &= P(Y_k = e_i|R_{1:k}, Y_u) = P(Y_k = e_i|R_{1:k-1}, Y_u, R_k)\\ &= \frac{\sum_{w=1}^d \sum_{j=1}^d \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} \varphi_j(R_k) \hat{Y}_{k-1}^j Y_u^w}{\sum_{j=1}^d \varphi_j(R_k)\hat{Y}_{k-1}^j}\end{aligned}$$

□

Corollary 3.2.7. *For* $k \in \mathbb{N}$,

$$\hat{Y}_k =$$
$$\sum_{w=1}^d Y_u^w \frac{\prod_{l=0}^{k-1} \Pi^T Diag(\varphi(R_{k-l}) \oslash (p_{.w}^{u-k+l+1}))(\hat{Y}_0 \odot p_{.w}^{(u-1)} \odot \cdots \odot p_{.w}^{(u-k)})}{\mathbf{1}_d^T \prod_{l=0}^{k-1} \Pi^T Diag(\varphi(R_{k-l}) \oslash (p_{.w}^{u-k+l+1}))(\hat{Y}_0 \odot p_{.w}^{(u-1)} \odot \cdots \odot p_{.w}^{(u-k)})}$$
$$= \sum_{w=1}^d Y_u^w \frac{(\hat{Y}_0^T \odot (p_{.w}^{(u-1)})^T \odot \cdots \odot (p_{.w}^{(u-k)})^T) \prod_{l=1}^k Diag(\varphi(R_l) \oslash (p_{.w}^{u-l+1}))\Pi)^T}{(\hat{Y}_0^T \odot (p_{.w}^{(u-1)})^T \odot \cdots \odot (p_{.w}^{(u-k)})^T) \prod_{l=1}^k Diag(\varphi(R_l) \oslash (p_{.w}^{u-l+1}))\Pi)^T \mathbf{1}_d}$$

Proof. Similar as in the proof in Corollary 3.2.3, we have

$$\mathbf{1}_d^T \cdot \left(\Pi^T \odot (p_{.w}^{(u-k)}, p_{.w}^{(u-k)}, \dots, p_{.w}^{(u-k)}) Diag\Big(\mathbf{1}_d \oslash (p_{.w}^{(u-k+1)})\Big)\right) = \mathbf{1}_d^T,$$

and $\varphi(R_k)^T = \mathbf{1}_d^T Diag(\varphi(R_k))$. We substitute $\hat{Y}_{k-1}$ by $\hat{Y}_{k-2}$,

$$\hat{Y}_k = \sum_{w=1}^d Y_u^w \cdot$$
$$\frac{\Pi^T Diag(\varphi(R_k) \oslash (p_{.w}^{(u-k+1)}))\Pi^T Diag(\varphi(R_{k-1}) \oslash (p_{.w}^{(u-k+2)}))\hat{Y}_{k-2} \odot (p_{.w}^{(u-k+1)}) \odot (p_{.w}^{(u-k)})}{\mathbf{1}_d^T \Pi^T Diag(\varphi(R_k) \oslash (p_{.w}^{(u-k+1)}))\Pi^T Diag(\varphi(R_{k-1}) \oslash (p_{.w}^{(u-k+2)}))\hat{Y}_{k-2} \odot (p_{.w}^{(u-k+1)}) \odot (p_{.w}^{(u-k)})}$$

By iteration we can finish this proof. □

3.3 Reference Measure Approach and Expert Opinions Models

As mentioned before, one of the fundamental techniques employed throughout this section is the discrete-time version of Girsanov's Theorem (see Appendix A.1.2 in [EAM95]). In this section, we also use Kalman filter to combine information from returns and expert opinions. We look at expert opnions again as certain expert opinions (Section 3.3.2), uncertian expert opinions providing probabilities for the state (Section 3.3.3) and logistic regression (Section 3.3.5). We also discuss estimation by the EM algorithm.

3.3.1 Reference Measure

A reference measure is introduced by the following definition.

Definition 3.3.1. *Set $Z_0 = 1$ and $Z_k = Z_{k-1}L_k$ for $k \geq 1$, where*

$$L_k := \frac{\varphi_{0,1}(R_k)}{\varphi_{b^T Y_{k-1},(a^T Y_{k-1})^2}(R_k)}.$$

The probability measure given by $\frac{d\tilde{P}}{dP}|_{\mathcal{G}_k} = Z_k$ is the reference measure for filtering,
where $\mathcal{G}_k = \mathcal{F}_k \vee \mathcal{F}^E_K$. The existence of $\tilde{P}$ follows from Kolmogorov's Extension Theorem. Z_k is known as Radon Nikodym derivative of $\tilde{P}$ w.r.t. P.

Next, we show some good properties of measure $\tilde{P}$.

Lemma 3.3.2. *(i) $Z = (Z_k)_{k\in\mathbb{N}_0}$ is a $\mathcal{G}$-martingale under measure P.*
$Z^{-1} = (Z_k^{-1})_{k\in\mathbb{N}_0}$ is a $\mathcal{G}$-martingale under measure $\tilde{P}$.
(ii) For all $u \leq k$,

$$\tilde{\mathbb{E}}[Z_k^{-1} Y_u | \mathcal{F}_u^{C_0}] = \tilde{\mathbb{E}}[Z_u^{-1} Y_u | \mathcal{F}_u^{C_0}].$$

(iii) Under $\tilde{P}$, $R_1, R_2, \ldots$ are iid. standard normal distributed and independent of Y.

Proof. See proof in Appendix A.2. □

The next lemma shows the same property of reference measure $\tilde{P}$ with the one of original measure P.

Lemma 3.3.3. *(i)* $\tilde{P}(Y_k = e_i|Y_k) = P(Y_k = e_i|Y_k) = Y_k^i$.
(ii) The Markov chain Y has the same transition matrix under reference probability measure, i.e.

$$\tilde{P}(Y_k = e_j|Y_{k-1} = e_i) = P(Y_k = e_j|Y_{k-1} = e_i)$$

(iii) Expert opinions can be given under either probability measures P or $\tilde{P}$, i.e. for all $w = 1, \dots, d$

$$\tilde{P}(Y_k = e_w|\mathcal{F}_K^E) = P(Y_k = e_w|\mathcal{F}_K^E).$$

if $E_k \in E^{(n)}$.

Proof. See proof in Appendix A.2. □

As there are some good properties in reference measure, we could compute filters under $\tilde{P}$ and use Bayes' formula to get filters under P.
Since for $A \in \mathcal{G}_k$,

$$\tilde{\mathbb{E}}[\mathbb{1}_A Z_k^{-1}] = \mathbb{E}[\mathbb{1}_A Z_k^{-1} Z_k] = \mathbb{E}[\mathbb{1}_A] = P(A)$$

we have $\frac{dP}{d\tilde{P}}|_{\mathcal{G}_k} = Z_k^{-1}$.

Definition 3.3.4. *For any $\mathcal{G}$-adapted $H = (H_k)_{k\in\mathbb{N}_0}$ such that the following expectations exist,*

$$\rho_k(H) := \tilde{\mathbb{E}}[Z_k^{-1} H_k|\mathcal{F}_k^{C_0}]$$

is the unnormalized filter of H under $\tilde{P}$ at time k.

Lemma 3.3.5. *For any $\mathcal{G}$-adapted real-valued $H = (H_k)_{k\in\mathbb{N}_0}$ such that the following expectations exist,*

$$\hat{H}_k := \mathbb{E}[H_k|\mathcal{F}_k^{C_0}] = \frac{\rho_k(H)}{\rho_k(1)} = \frac{\mathbf{1}_d^T \rho_k(HY)}{\mathbf{1}_d^T \rho_k(Y)}.$$

Proof. By Bayes' formula of conditional expectation

$$\mathbb{E}[H_k|\mathcal{F}_k^{C_0}] = \frac{\tilde{\mathbb{E}}[H_k Z_k^{-1}|\mathcal{F}_k^{C_0}]}{\tilde{\mathbb{E}}[Z_k^{-1}|\mathcal{F}_k^{C_0}]} = \frac{\rho_k(H)}{\rho_k(1)}$$

Further,

$$\rho_k(1) = \tilde{\mathbb{E}}[Z_k^{-1}\mathbf{1}_d^T Y_k|\mathcal{F}_k^R] = \mathbf{1}_d^T \tilde{\mathbb{E}}[Z_k^{-1} Y_k|\mathcal{F}_k^R] = \mathbf{1}_d^T \rho_k(Y),$$

$$\rho_k(H) = \tilde{\mathbb{E}}[H_k Z_k^{-1}|\mathcal{F}_k^{C_0}] = \tilde{\mathbb{E}}[H_k \mathbf{1}_d^T Y_k Z_k^{-1}|\mathcal{F}_k^{C_0}] = \mathbf{1}_d^T \tilde{\mathbb{E}}[H_k Y_k Z_k^{-1}|\mathcal{F}_k^{C_0}] = \mathbf{1}_d^T \rho_k(HY)$$

which concludes the proof. □

We compute the unnormalized filter of some suitable class under $\tilde{P}$ by the following theorem:

Theorem 3.3.6. *Let $H = (H_k)_{k\in\mathbb{N}_0}$ be $\mathcal{G}$-adapted, $H_k : \Omega \to \mathbb{R}$, with*

$$H_k = H_{k-1} + \alpha_{k-1} + \beta_{k-1}^T Y_k + \gamma_{k-1} f(R_k),$$

where α, β, γ are $\mathcal{F}$-adapted and $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}$-valued, respectively, and $f : \mathbb{R} \to \mathbb{R}$ is measurable, s.t. H is integrable. For $\Gamma_i(r) := \frac{\varphi_i(r)}{\varphi_{0,1}(r)}$, where $\varphi_i = \varphi_{b_i,a_i^2}$. Define by $\ddot{Y}_k^{(i)} := \tilde{\mathbb{E}}[Y_k|\mathcal{G}_{k-1}, Y_{k-1} = e_i]$, Then we have

$$\begin{aligned}\rho_k(HY) = \sum_{i=1}^{d} \Big\{ &\rho_{k-1}(HY^i)\Gamma_i(R_k)\ddot{Y}_k^{(i)} \\ &+ \rho_{k-1}(\alpha Y^i)\Gamma_i(R_k)\ddot{Y}_k^{(i)} \\ &+ \rho_{k-1}(\gamma Y^i)\Gamma_i(R_k) f(R_k)\ddot{Y}_k^{(i)} \\ &+ (Diag(\ddot{Y}_k^{(i)}))\rho_{k-1}(\beta Y^i)\Gamma_i(R_k)\Big\}\end{aligned}$$

where $\rho_k(HY) = (\rho(HY^1), \rho(HY^2), \dots, \rho(HY^d))^T$, and Y^i is the $i-$th element of vector Y.

Proof. See proof in Appendix A.2. □

Remark 3.3.7. *If we have no expert opinions, then $\ddot{Y}_k^{(i)} = \Pi^T e_i$ which leads to the corresponding result without expert opinions.*

3.3.2 Certain Expert Opinions

The definition of certain expert opinions is given in Definition 3.2.1. We say some states are observable from expert opinions rather than hidden, i.e. $E_{u_i} = Y_{u_i}$ for all $i = 1, \dots, n$, where $0 \le u_1 < \dots < u_n \le K$.
The homogeneous Markov chain has the characteristics as shown in [AHE89], for $s \le t \le u$

$$\mathbb{E}[Y_t|\mathcal{F}_s^Y \vee \mathcal{F}_u^E] = \mathbb{E}[Y_t|Y_s, Y_u].$$

Actually, the filter in Markov Switching Model can be estimated by the following lemma if future state is observable.

Lemma 3.3.8. *For $k \le u$, and $u = \min\{u_i | u_i \ge k, i = 1, \ldots, n\}$,*

$$\begin{aligned}\tilde{\mathbb{E}}[Y_k|\mathcal{G}_{k-1}] &= \sum_{w=1}^{d} Y_u^w \Pi^T \odot (p_{.w}^{(u-k)}, p_{.w}^{(u-k)}, \ldots, p_{.w}^{(u-k)}) Diag\Big(\mathbf{1}_d \oslash (p_{jw}^{(u-k+1)})\Big) Y_{k-1} \\ &= \sum_{w=1}^{d} Y_u^w \Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{jw}^{(u-k+1)})\Big) Y_{k-1} \odot (p_{.w}^{(u-k)}).\end{aligned}$$

Proof. We have

$$\begin{aligned}&\tilde{\mathbb{E}}[Y_k^i|\mathcal{G}_{k-1}] \\ &= \sum_{w=1}^{d}\sum_{j=1}^{d} \tilde{P}(Y_k = e_i | Y_u = e_w, Y_{k-1} = e_j)\tilde{P}(Y_{k-1} = e_j | Y_{k-1})\tilde{P}(Y_u = e_w | Y_u) \\ &= \sum_{w=1}^{d}\sum_{j=1}^{d} \tilde{P}(Y_k = e_i | Y_u = e_w, Y_{k-1} = e_j) Y_{k-1}^j Y_u^w \\ &= \sum_{w=1}^{d}\sum_{j=1}^{d} \frac{\tilde{P}(Y_u = e_w | Y_k = e_i)\tilde{P}(Y_k = e_i | Y_{k-1} = e_j)}{\tilde{P}(Y_u = e_w | Y_{k-1} = e_j)} Y_{k-1}^j Y_u^w \\ &= \sum_{w=1}^{d}\sum_{j=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} Y_{k-1}^j Y_u^w.\end{aligned}$$

The second equation is due to Lemma 3.3.3. By vectorization, we finish this proof. □

Remark 3.3.9. *When $u = k$, since $\Pi^0 = \boldsymbol{I}_d$, we compute from Lemma 3.3.8,*

$$\begin{aligned}\tilde{\mathbb{E}}[Y_k^i|\mathcal{G}_{k-1}] &= \sum_{w=1}^{d}\sum_{j=1}^{d} \frac{p_{iw}^0 p_{ji}}{p_{jw}} Y_{k-1}^j Y_u^w \\ &= \sum_{w=1}^{d} p_{iw}^0 \sum_{j=1}^{d} \frac{p_{ji}}{p_{jw}} Y_{k-1}^j Y_u^w \\ &= Y_u^i,\end{aligned}$$

where the last equation is due to the fact that only when $i = w$, $p_{iw}^0 = 1$. This result is consistent with $\tilde{\mathbb{E}}[Y_k|Y_k] = Y_k$. Thus Lemma 3.3.8 can be applied when $k \le u_n$.

Corollary 3.3.10. *Let $H = (H_k)_{k\in\mathbb{N}_0}$ be $\mathcal{G}$-adapted, $H_k : \Omega \to \mathbb{R}$, with*

$$H_k = H_{k-1} + \alpha_{k-1} + \beta_{k-1}^T Y_k + \gamma_{k-1} f(R_k),$$

where α, β, γ are $\mathcal{F}$-adapted and $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}$-valued respectively. And let $f : \mathbb{R} \to \mathbb{R}$ be measurable, s.t. H is integrable. For $\Gamma_i(r) := \frac{\varphi_i(r)}{\varphi_{0,1}(r)}$, where $\varphi_i = \varphi_{b_i,a_i^2}$. If $E_{u_i} = Y_{u_i}$ for all $i = 1,\ldots,n$, and $k \le u$, where $u = \min\{u_i | u_i \ge k, i = 1,\ldots,n\}$, we have

$$\begin{aligned}
&\rho_k(HY)\\
&= \sum_{i=1}^{d} \Big\{ \rho_{k-1}(HY^i)\Gamma_i(R_k) \sum_{w=1}^{d} Y_u^w \Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{jw}^{(u-k+1)})\Big) e_i \odot (p_{.w}^{(u-k)})\\
&+ \rho_{k-1}(\alpha Y^i)\Gamma_i(R_k) \sum_{w=1}^{d} Y_u^w \Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{jw}^{(u-k+1)})\Big) e_i \odot (p_{.w}^{(u-k)})\\
&+ \rho_{k-1}(\gamma Y^i)\Gamma_i(R_k) f(R_k) \sum_{w=1}^{d} Y_u^w \Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{jw}^{(u-k+1)})\Big) e_i \odot (p_{.w}^{(u-k)})\\
&+ (Diag(\sum_{w=1}^{d} Y_u^w \Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{jw}^{(u-k+1)})\Big) e_i \odot (p_{.w}^{(u-k)})))\rho_{k-1}(\beta Y^i)\Gamma_i(R_k) \Big\}
\end{aligned}$$

else if $k > u_n$

$$\begin{aligned}
\rho_k(HY) = \sum_{i=1}^{d} \Big\{ &\rho_{k-1}(HY^i)\Gamma_i(R_k)\Pi^T e_i + \rho_{k-1}(\alpha Y^i)\Gamma_i(R_k)\Pi^T e_i\\
&+ \rho_{k-1}(\gamma Y^i)\Gamma_i(R_k) f(R_k)\Pi^T e_i + (Diag(\Pi^T e_i))\rho_{k-1}(\beta Y^i)\Gamma_i(R_k) \Big\}
\end{aligned}$$

where $\rho_k(HY) = (\rho(HY^1), \rho(HY^2), \ldots, \rho(HY^d))^T$, and Y^i is the $i-$th element of vector Y.

Proof. Substituted by $\ddot{Y}_k^{(i)} = \tilde{\mathbb{E}}[Y_k|\mathcal{G}_{k-1}, Y_{k-1} = e_i]$ in Theorem 3.3.6, we can get the corollary. $\square$

Remark 3.3.11. *In particular, for $H = 1$, $H_k Y_k = Y_k$, choose $H = 1$, $\alpha = 0, \beta = 0_d, \gamma = 0$, for $k \le u$, where $u = \min\{u_i | u_i \ge k, i = 1,\ldots,n\}$, this* *yields*

$$\begin{aligned}\rho_k(Y) &= \sum_{w=1}^{d} Y_u^w \sum_{i=1}^{d} \rho_{k-1}^E(HY^i)\Gamma_i(R_k)\Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{.w}^{(u-k+1)})\Big)e_i \odot (p_{.w}^{(u-k)}) \\ &= \sum_{w=1}^{d} Y_u^w \Pi^T Diag\Big(\Gamma(R_k) \oslash (p_{.w}^{(u-k+1)})\Big)\rho_{k-1}^E(Y) \odot (p_{.w}^{(u-k)})\end{aligned}$$

$$\begin{aligned}\hat{Y}_k &= \mathbb{E}[Y_k|\mathcal{F}_k^{C_0}] = \frac{\rho_k(Y)}{\mathbf{1}_d^T \rho_k(Y)} \\ &= \frac{\Pi^T Diag\Big(\Gamma(R_k) \oslash (p_{.w}^{(u-k+1)})\Big)\rho_{k-1}(Y) \odot (p_{.w}^{(u-k)})}{\Gamma(R_k)^T \rho_{k-1}(Y)} \\ &= \frac{\Pi^T Diag\Big(\varphi(R_k) \oslash (p_{.w}^{(u-k+1)})\Big)\hat{Y}_{k-1} \odot (p_{.w}^{(u-k)})}{\varphi(R_k^T)\hat{Y}_{k-1}}\end{aligned}$$

The second equation is because that

$$\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{\sum_{z=1}^{d} p_{zw}^{(u-k)} p_{jz}} = 1$$

and this equation can be vectorized as

$$\mathbf{1}_d^T \cdot \left(\Pi^T \odot (p_{.w}^{(u-k)}, p_{.w}^{(u-k)}, \ldots, p_{.w}^{(u-k)}) Diag\Big(\mathbf{1}_d \oslash (\Pi \cdot p_{.w}^{(u-k)})\Big)\right) = \mathbf{1}_d^T$$

This formula is as the same as Lemma 3.2.6.

3.3.3 Linear Combination of Both Information

Again we are interested in $\tilde{\mathbb{E}}[Y_k|\mathcal{G}_{k-1}]$. However the state Y_u at $t = u$ is not observable, that is E_u^i provides only the probability for $Y_u = e_i$. We will have the following lemma to estimate the state.

Lemma 3.3.12. *For* $\tilde{\mathbb{E}}[Y_u^i|E_u] = E_u^i \in [0, 1]$, *where* $u = \min\{u_i|u_i \geq k, i = 1, \ldots, n\}$.

$$\tilde{\mathbb{E}}[Y_k^i|\mathcal{G}_{k-1}] = \sum_{w=1}^{d}\sum_{j=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} Y_{k-1}^j \tilde{P}(Y_u = e_w|Y_{k-1,} E_u).$$

Proof. Proof see Appendix A.2. □

Afterwards, in order to compute $\tilde{P}(Y_u = e_w|Y_{k-1}, E_u)$, the linear combination approach motived by discrete Kalman filter is proposed. Then both information from previous state and expert opinions are exploited.

Theorem 3.3.13. *(Kalman filter for MSMs) $\tilde{Y}$ and E are two estimators of Markov chain Y.The updated filter of Y, $\tilde{Y}^{new}$, is constructed by*

$$\tilde{Y}^{new} = \lambda\tilde{Y} + (1-\lambda)E$$

with

$$\lambda = \Big(\frac{\sum_{i=1}^{d}\mathbb{E}[(E^i - Y^i)^2] - Corr}{\sum_{i=1}^{d}\mathbb{E}[(\tilde{Y}^i - Y^i)^2] + \sum_{i=1}^{d}\mathbb{E}[(E^i - Y^i)^2] - 2Corr} \vee 0\Big) \wedge 1$$

where $Corr = \sum_{i=1}^{d}\mathbb{E}[(E^i - Y^i)(\tilde{Y}^i - Y^i)]$. Then we have

$$\sum_{i=1}^{d}\mathbb{E}[(Y^i - \tilde{Y}^{new,i})^2] \leq \sum_{i=1}^{d}\mathbb{E}[(Y^i - \tilde{Y}^i)^2].$$

Proof. The new filter can be constructed linearly by

$$\tilde{Y}^{new} = \lambda\tilde{Y} + (1-\lambda)E \ \lambda \in [0,1].$$

$$\begin{aligned}\sum_{i=1}^{d}\mathbb{E}[(Y^i - \tilde{Y}^{new,i})^2] &= \sum_{i=1}^{d}\mathbb{E}[(\lambda\tilde{Y}^i + (1-\lambda)E^i - Y^i)^2] \\ &= \sum_{i=1}^{d}\lambda^2\mathbb{E}[(\tilde{Y}^i - Y^i)^2] \\ &+ (1-\lambda)^2\mathbb{E}[(E^i - Y^i)^2] + 2\lambda(1-\lambda)\mathbb{E}[(\tilde{Y}^i - Y^i)(E^i - Y^i)]\end{aligned} \tag{3.2}$$

taking the first derivative with respect to λ equal to zero, we get the corresponding λ:

$$\lambda = \frac{\sum_{i=1}^{d}\mathbb{E}[(E^i - Y^i)^2] - \sum_{i=1}^{d}\mathbb{E}[(E^i - Y^i)(\tilde{Y}^i - Y^i)]}{\sum_{i=1}^{d}\mathbb{E}[(\tilde{Y}^i - Y^i)^2] + \sum_{i=1}^{d}\mathbb{E}[(E^i - Y^i)^2] - 2\sum_{i=1}^{d}\mathbb{E}[(E^i - Y^i)(\tilde{Y}^i - Y^i)]}$$

Putting this λ into equation (3.2), we get

$$\sum_{i=1}^{d}\mathbb{E}[(Y^i - \tilde{Y}^{new,i})^2] \leq \sum_{i=1}^{d}\mathbb{E}[(Y^i - \tilde{Y}^i)^2].$$

□

Proposition 3.3.14. *For $k \leq u$, $u = \min\{u_i | u_i \geq k, i = 1, \ldots, n\}$, the $\mathcal{G}$-measurable Kalman filter is*

$$\tilde{\mathbb{E}}[Y_u|\mathcal{G}_{k-1}] = \lambda_{k-1}^u \tilde{\mathbb{E}}[Y_u|Y_{k-1}] + (1-\lambda_{k-1}^u)\tilde{\mathbb{E}}[Y_u|E_u],$$

where $\lambda_{k-1}^u := \lambda(k-1,u), \lambda : \mathbb{R}^+ \times \mathbb{R}^{\rightarrow}[0,1]$, and $\lambda : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow [0,1]$ then we have

$$\begin{aligned}\tilde{\mathbb{E}}[Y_k|\mathcal{G}_{k-1}] &= \lambda_{k-1}^u \Pi^T Y_{k-1} \\ &+ (1-\lambda_{k-1}^u)\sum_{w=1}^{d} E_u^w \Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{.w}^{(u-k+1)})\Big) Y_{k-1} \odot (p_{.w}^{(u-k)}).\end{aligned}$$

Proof. See proof in Appendix A.2. □

Corollary 3.3.15. *Let $H = (H_k)_{k\in\mathbb{N}_0}$ be $\mathcal{G}$-adapted, $H_k : \Omega \rightarrow \mathbb{R}$, with*

$$H_k = H_{k-1} + \alpha_{k-1} + \beta_{k-1}^T Y_k + \gamma_{k-1} f(R_k),$$

where α, β, γ are $\mathcal{F}$-adapted and $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}$-valued respectively. And $f : \mathbb{R} \rightarrow \mathbb{R}$ is measurable, s.t. H is integrable. $\Gamma_i(r) := \frac{\varphi_i(r)}{\varphi_{0,1}(r)}$, where $\varphi_i = \varphi_{b_i,a_i^2}$. If the $\mathcal{G}$-measurable filter is estimated based on the method from Lemma 3.3.13, and $k \leq u \leq u_n$, where $u = \min\{t \in \mathbb{N}^+ | E_t \in E^{(n)}\}$, we have

$$\begin{aligned}\rho_k(HY) &= \lambda_{k-1}^u \sum_{i=1}^{d} \Big\{ \rho_{k-1}(HY^i)\Gamma_i(R_k)\Pi^T e_i + \rho_{k-1}(\alpha Y^i)\Gamma_i(R_k)\Pi^T e_i \\ &+ \rho_{k-1}(\gamma Y^i)\Gamma_i(R_k) f(R_k)\Pi^T e_i + (Diag(\Pi^T e_i))\rho_{k-1}(\beta Y^i)\Gamma_i(R_k)\Big\} \\ &+ (1-\lambda_{k-1}^u)\sum_{i=1}^{d} \Big\{ \rho_{k-1}(HY^i)\Gamma_i(R_k)\sum_{w=1}^{d} Y_u^w \Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{jw}^{(u-k+1)})\Big) e_i \odot (p_{.w}^{(u-k)}) \\ &+ \rho_{k-1}(\alpha Y^i)\Gamma_i(R_k)\sum_{w=1}^{d} Y_u^w \Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{jw}^{(u-k+1)})\Big) e_i \odot (p_{.w}^{(u-k)}) \\ &+ \rho_{k-1}(\gamma Y^i)\Gamma_i(R_k) f(R_k)\sum_{w=1}^{d} Y_u^w \Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{jw}^{(u-k+1)})\Big) e_i \odot (p_{.w}^{(u-k)}) \\ &+ (Diag(\sum_{w=1}^{d} Y_u^w \Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{jw}^{(u-k+1)})\Big) e_i \odot (p_{.w}^{(u-k)})))\rho_{k-1}(\beta Y^i)\Gamma_i(R_k)\Big\}\end{aligned}$$

else if $k > u_n$

$$\rho_k(HY) = \sum_{i=1}^{d} \Big\{ \rho_{k-1}(HY^i)\Gamma_i(R_k)\Pi^T e_i + \rho_{k-1}(\alpha Y^i)\Gamma_i(R_k)\Pi^T e_i \\ + \rho_{k-1}(\gamma Y^i)\Gamma_i(R_k) f(R_k)\Pi^T e_i + (Diag(\Pi^T e_i))\rho_{k-1}(\beta Y^i)\Gamma_i(R_k) \Big\}$$

where $\rho_k(HY) = (\rho(HY^1), \rho(HY^2), \dots, \rho(HY^d))^T$, *and* Y^i *is the* $i-$*th element of vector* Y.

Proof. From Theorem 3.3.6 and Proposition 3.3.14, we can carry out the proof. □

3.3.4 Expectation Maximization Algorithm

We want to determine a set of parameters $\hat{\theta} = \{b, a, \Pi\}$ given the arrival of new information. We proceed by using the Expectation Maximization(EM) algorithm.
Let $(P_\theta)_{\theta \in \Theta}$ a family of probability measures, all absolute continuous w.r.t. a fixed probability measure P_0. The likelihood function for computing an estimate of θ based on observations $R_1, \dots, R_K$ is

$$L(\theta) = E_0 \Big[\log \frac{dP_\theta}{dP_0} | \mathcal{F}_K^{C_0} \Big]$$

The maximum likelihood estimator (MLE) of θ is

$$\hat{\theta} \in \arg_{\theta \in \Theta} \max L(\theta).$$

This is difficult to compute and that's the reason EM algorithm is applied to approximate alternatively.

EM algorithm
Step 1:Set p=0 and choose $\hat{\theta}_0$.
Step 2:(E-Step) Set $\theta^* = \hat{\theta}_p$ and compute

$$Q(\theta, \theta^*) = E_{\theta^*} \Big[\log \frac{dP_\theta}{dP_{\theta^*}} | \mathcal{F}_K^{C_0} \Big].$$

Step 3:(M-Step) Find

$$\hat{\theta}_{p+1} \in \arg_{\theta\in\Theta} \max Q(\theta, \theta^*).$$

Step 4: Replace p by $p+1$ and repeat from Step 2 until a stopping criterion is satisfied.

Suppose there are $N \leq K$ expert opinions at different time $u \in U$, where $U = \{(u_n)_{n=1,\ldots,N} | Y_{u_n} = e_{w_n}\}$.
For known $Y_1, \ldots, Y_K$, $R_1, \ldots, R_K$, we had

$$\Lambda_K = \frac{dP_\theta}{dP_{\theta'}} = \prod_{k=1}^{K} \prod_{i,j=1}^{d} \Big(\frac{p_{ij}}{p'_{ij}}\Big)^{Y^i_{k-1} Y^j_k} \sum_{i=1}^{d} Y^i_{k-1} \frac{\varphi_{b_i, a_i^2}(R_k)}{\varphi_{b'_i, a'^2_i}(R_k)}$$

This leads to log-likelihood function to be

$$\begin{aligned}\log \Lambda_K = &\sum_{i,j=1}^{d} N^{ij}_K \log(p_{ij}) + \sum_{i=1}^{d} \left(- O^i_K \log a_i - \frac{1}{2}\Big(\frac{b_i}{a_i}\Big)^2 O^i_K \right) \\ &+ \sum_{i=1}^{d} \left(\frac{b_i}{a_i^2} T^i_K(f_1) - \frac{1}{2a_i^2} T^i_K(f_2) \right) + h(\Pi', a', b'),\end{aligned}$$

where h is some functions of the parameters Π', a', b' and

$$\begin{aligned} N^{ij}_K &= \sum_{k=1}^{K} Y^i_{k-1} Y^j_k, \\ O^i_K &= \sum_{k=1}^{K} Y^i_{k-1}, \\ T^i_K(f) &= \sum_{k=1}^{K} f_i(R_k) Y^i_{k-1} \end{aligned}$$

where $f_1(x) = x, f_2(x) = x^2$.

Remark 3.3.16. *These items ($N^{ij}_K, O^i_K, T^i_K(f)$) can be understood as numbers of jumps, occupation time and observation transitions respectively.*

Taking conditional expectation w.r.t. $\mathcal{F}^{C_0}_K$ and maximizing in Π, a, b, we yield

Theorem 3.3.17. *The updates in the EM algorithm with expert opinions for the MSM are*

$$\hat{b}_l = \frac{\hat{T}^l_K(f_1)}{\hat{O}^l_K}$$

$$(\hat{a}_l)^2 = \frac{\hat{T}^l_K(f_2) - 2\hat{b}_l\hat{T}^l_K(f_1) + \hat{b}_l^2 O^l_K}{O^l_K}$$

$$\hat{p}_{lm} = \frac{N^{lm}_K}{O^l_K}$$

where, the filters $N^{lm}_K, O^l_K, T^l_K(f)$ can be computed based on parameters Π', b', a' which can be computed by the following proposition.

Proof. See proof in Appendix A.2. □

Proposition 3.3.18. *The filters for $H \in N^{lm}_K, O^l_K, T^l_K(f)$ can be computed by*

$$\hat{H}_K = \frac{\mathbf{1}_d^T \rho_K(HY)}{\mathbf{1}_d^T \rho_K(Y)}.$$

with

$$\rho_k(O^lY) = \sum_{i=1}^d \rho_{k-1}(O^lY^i)\Gamma_i(R_k)\ddot{Y}_k + \rho_{k-1}(Y^l)\Gamma_l(R_k)\Pi^T e_l$$

$$\rho_k(T^l(f)Y) = \sum_{i=1}^d \rho_{k-1}(T^l(f)Y^i)\Gamma_i(R_k)\ddot{Y}_k + \rho_{k-1}(Y^l)\Gamma_l(R_k)f(R_k)\Pi^T e_l$$

$$\rho_k(N^{lm}Y) = \sum_{i=1}^d \rho_{k-1}(N^{lm}Y^i)\Gamma_i(R_k)\ddot{Y}_k + \rho_{k-1}(Y^l)\Gamma_l(R_k)p_{lm}e_m$$

for $k = 1, \dots, K$ with initial values 0 and

$$\rho_k(Y) = \sum_{i=1}^d \rho_{k-1}(Y^i)\Gamma_i(R_k)\ddot{Y}_k$$

for $k = 1, \dots, K$ with initial value Y_0.

Proof. This proof is just the application of Theorem 3.3.6

- For O^l_k, choose $H_0 = 0, \alpha_{k-1} = Y^l_{k-1}, \beta = 0_d, \gamma = 0$;

- For $T^l(f)$, choose $H_0 = 0, \gamma_{k-1} = Y^l_{k-1}, \alpha_{k-1} = 0, \beta = 0_d$;
- For N^{lm}, choose $H_0 = 0, \beta_{k-1} = Y^l_{k-1} e_m, \alpha_{k-1} = 0, \gamma = 0$.

$\square$

Remark 3.3.19. *We can derive this forward algorithm rather than forward and backward one since $\tilde{\mathbb{E}}[Y_k|\mathcal{F}^{C_0}_k] = \tilde{\mathbb{E}}[Y_k|\mathcal{F}^{C_0}_{k-1}]$. As the poof of Theorem 3.3.6 illustrates that R and Y are conditional independent under $\tilde{P}$ and the information of expert opinions are pre-established. These two conditions ensure $\tilde{\mathbb{E}}[Y_k|\mathcal{F}^{C_0}_k] = \tilde{\mathbb{E}}[Y_k|\mathcal{F}^{C_0}_{k-1}]$.*

The following theorem illustrates the expected square error of a filter with more information is less than one with less information. And it is also our motivation to combine more information to estimate the state.

Theorem 3.3.20. *For $k = 1, \dots, K$, $\forall i \in \{1, \dots, d\}$,*

$$\mathbb{E}\Big[(Y^i_k - \mathbb{E}[Y^i_k|\mathcal{F}^{C_0}_k])^2\Big] \le \mathbb{E}\Big[(Y^i_k - \mathbb{E}[Y^i_k|\mathcal{F}^R_k])^2\Big]$$

Proof.

$$\begin{aligned}\mathbb{E}\Big[(Y^i_k - \mathbb{E}[Y^i_k|\mathcal{F}^{C_0}_k])^2\Big] &= \mathbb{E}[Var(Y^i_k|\mathcal{F}^{C_0}_k)] \\ &= \mathbb{E}\Big[\mathbb{E}[(Y^i_k)^2|\mathcal{F}^{C_0}_k] - (\mathbb{E}[Y^i_k|\mathcal{F}^{C_0}_k])^2\Big] \\ &= \mathbb{E}\Big[\mathbb{E}[(Y^i_k)^2|\mathcal{F}^R_k]\Big] - \mathbb{E}\Big[(\mathbb{E}[Y^i_k|\mathcal{F}^{C_0}_k])^2\Big]\end{aligned}$$

The last equality is due to the tower property, and by Jensen's inequality we have,

$$\begin{aligned}\mathbb{E}\Big[(\mathbb{E}[Y^i_k|\mathcal{F}^{C_0}_k])^2|\mathcal{F}^R_k\Big] &\ge \Big(\mathbb{E}\Big[\mathbb{E}[Y^i_k|\mathcal{F}^{C_0}_k]|\mathcal{F}^R_k\Big]\Big)^2 \\ &= \Big(\mathbb{E}[Y^i_k|\mathcal{F}^R_k]\Big)^2\end{aligned}$$

Taking expectation on both sides and by tower property, we get

$$\begin{aligned}\mathbb{E}\Big[(Y^i_k - \mathbb{E}[Y^i_k|\mathcal{F}^{C_0}_k])^2\Big] &= \mathbb{E}\Big[\mathbb{E}[(Y^i_k)^2|\mathcal{F}^R_k]\Big] - \mathbb{E}\Big[(\mathbb{E}[Y^i_k|\mathcal{F}^{C_0}_k])^2\Big] \\ &\le \mathbb{E}\Big[\mathbb{E}[(Y^i_k)^2|\mathcal{F}^R_k]\Big] - \mathbb{E}\Big[(\mathbb{E}[Y^i_k|\mathcal{F}^R_k])^2\Big] \\ &= \mathbb{E}[Var(Y^i_k|\mathcal{F}^R_k)] \\ &= \mathbb{E}\Big[(Y^i_k - \mathbb{E}[Y^i_k|\mathcal{F}^R_k])^2\Big]\end{aligned}$$

$\square$

3.3.5 Logistic Regression Model for Expert Opinions

Let us look at a different way to model expert opinions. The intuitive idea is that expert opinions model is constructed as

$$Y_{k-1} = \mathcal{C}(\boldsymbol{X}_k) + \boldsymbol{\zeta}_k$$

where $\boldsymbol{\zeta}_k$ is r,v. and $\mathbb{E}\boldsymbol{\zeta}_k^i = 0$ for $i = 1, \ldots, d$, they are independent of ε_k and $\sum_{i=1}^d \zeta^i = 1$. Here $\mathcal{C}(\boldsymbol{X}_k)$ is proposed as expert opinions E_k.
$\mathcal{C}$ is one family of classifier e.g. naive Bayes, SVM, Decision tree or artificial neural network. Each setting of family can be understood as the prior information.
Thus, the elements of Y_{k-1} will remain in $[0, 1]$ and the sum of them will be 1. $\boldsymbol{X}_k$ is an exogenous observations while Y_{k-1} is not observable in hidden Markov chain. Hence this issue is different from supervised learning or function approximation, in which both Y and $\boldsymbol{X}$ are observable. $\boldsymbol{X}_k$ varies homogeneously with some fixed probability, which is not specified now.

The logistic regression model is applicable since it arises from the idea to model the posterior probabilities of d states via linear functions in $\boldsymbol{x}$. $\boldsymbol{x}$ is an $(M + 1)$-dimensional vector where M is the number of external factors, and we set $\boldsymbol{x}^0 = 1$.
At the same time, logistic regression model ensures its sum to be one and remains in $[0, 1]$. The model has the form

$$P_i(\boldsymbol{X}_k; \boldsymbol{\beta}) = LG(\boldsymbol{X}\boldsymbol{;}\boldsymbol{\beta}) = \frac{\exp(\beta_i^T \boldsymbol{X})}{\sum_{l=1}^d \exp(\beta_l^T \boldsymbol{X})}$$

where $\beta \in \mathbb{R}^{(M+1)\times d}$ is parameter matrix describing the behaviour of this model. And β_i is the $i-$th column of β. We denote the estimated probabilities $P(\boldsymbol{X}_k \in \cdot | Y_{k-1} = e_i)$ parameterized by β, $P_\beta(\boldsymbol{X}_k \in \cdot | Y_{k-1} = e_i)$. Then,

$$P_\beta(\boldsymbol{X}_k \in \cdot | Y_k = e_i) = \frac{P_i(\boldsymbol{X}_k; \boldsymbol{\beta}) P(\boldsymbol{X}_k \in \cdot)}{P(Y_k = e_i)}.$$

In hidden Markov chain, the state Y is not observable. Hence we need to find one scheme to estimate the parameter β the logistic regression model. We want to maximize below the likelihood with respect to $\theta = \{b, a, \Pi, \beta\}$.

For observed $Y_1, \dots, Y_K,\ R_1, \dots, R_K,\ \boldsymbol{X}_1, \dots, \boldsymbol{X}_K$

$$
\begin{aligned}
\Lambda_K &= \frac{dP_\theta}{dP_{\theta'}} \\
&= \prod_{k=1}^{K} \prod_{i,j=1}^{d} \left(\frac{p_{ij}}{p'_{ij}}\right)^{Y^i_{k-1} Y^j_k} \sum_{i=1}^{d} Y^i_{k-1} \frac{\varphi_{b_i, a_i^2}(R_k)}{\varphi_{b'_i, a_i'^2}(R_k)} \sum_{j=1}^{d} Y^j_k \frac{P_\beta(\boldsymbol{X}_k \in \cdot | Y_{k-1} = e_i)}{P_{\beta'}(\boldsymbol{X}_k \in \cdot | Y_{k-1} = e_i)} \\
&= \prod_{k=1}^{K} \prod_{i,j=1}^{d} \left(\frac{p_{ij}}{p'_{ij}}\right)^{Y^i_{k-1} Y^j_k} \sum_{i=1}^{d} Y^i_{k-1} \frac{\varphi_{b_i, a_i^2}(R_k)}{\varphi_{b'_i, a_i'^2}(R_k)} \sum_{j=1}^{d} Y^j_k \frac{P_i(\boldsymbol{X}_k; \boldsymbol{\beta})}{P_i(\boldsymbol{X}_k; \boldsymbol{\beta'})}
\end{aligned}
$$

since $\frac{P(\boldsymbol{X}_k \in \cdot)}{P(Y_{k-1} = e_i)}$ cancelled out. which leads to $\mathbb{E}[\log \Lambda_K | \mathcal{F}^{C_0}_K]$

$$
\begin{aligned}
&= \sum_{i,j=1}^{d} \hat{N}^{ij}_K \log(p_{ij}) + \sum_{i=1}^{d} \left(- \hat{O}^i_K \log a_i - \frac{1}{2} \left(\frac{b_i}{a_i}\right)^2 \hat{O}^i_K \right) \\
&+ \sum_{i=1}^{d} \left(\frac{b_i}{a_i^2} \hat{T}^i_K(f_1) - \frac{1}{2a_i^2} \hat{T}^i_K(f_2) \right) + \sum_{k=1}^{K} \sum_{i=1}^{d} \hat{Y}^i_k \log P_i(\boldsymbol{X}_k; \boldsymbol{\beta}) \\
&+ \hat{h}(\Pi', a', b') + \hat{g}(\beta').
\end{aligned} \tag{3.3}
$$

From equation (3.3), we see that parameters Π, b, a can be estimated by Theorem 3.3.6 as usual. Similarly, once we have the estimated for Y_k, β can be estimated by some numerical methods, e.g. Newton-Raphson.

It is convenient to illustrate the method when there are only two states of Y. More states method is referred in [Cze02].
Denote $l(\beta) := \sum_{k=1}^{K} \sum_{i=1}^{d} Y^i_k \log P_i(\boldsymbol{X}_k; \boldsymbol{\beta})$, referring to log-likelihood function for the logistic regression model. Note that maximizing $l(\beta)$ is equivalent to maximize $\log \Lambda_K$ with respect to β.
For a two state Markov chain,

$$
l(\beta) = \sum_{k=1}^{K} \left(Y^1_k \beta_1^T \boldsymbol{X}_k - \log(1 + \exp(\beta_1^T \boldsymbol{X}_k)) \right).
$$

To maximize the log-likelihood function , we set its derivatives to zero. These score equations are

$$
\frac{\partial l(\beta_1)}{\partial \beta_1} = \sum_{k=1}^{K} \boldsymbol{X}_k \Big(Y^1_k - P_1(\boldsymbol{X}_k; \beta_1) \Big) = 0. \tag{3.4}
$$

To solve the above equation (3.4), the Newton-Raphson algorithm can be applied. The second derivative is

$$\frac{\partial^2 l(\beta_1)}{\partial \beta_1 \partial \beta_1^T} = -\sum_{k=1}^{K} \boldsymbol{X}_k \boldsymbol{X}_k^T P_1(\boldsymbol{X}_k;\beta_1)(1 - P_1(\boldsymbol{X}_k;\beta_1)).$$

Starting with β_1^p, a single Newton update is

$$\beta_1^{p+1} = \beta_1^p - \Big(\frac{\partial^2 l(\beta_1)}{\partial \beta_1 \partial \beta_1^T}\Big)^{-1} \frac{\partial l(\beta_1)}{\partial \beta_1},$$

where the derivatives are evaluated at β_1^p.
It is convenient to write the first and second derivatives and Hessian in matrix notation. Let $\boldsymbol{Y}$ denote the vector of Y_k^1, $\boldsymbol{X}$ the $K \times (M+1)$ matrix of $\boldsymbol{X}_k$, $\boldsymbol{p}$ the vector of probabilities with $k-$th element $P_1(\boldsymbol{X}_k;\beta_1^p)$ and $\boldsymbol{W}$ a $K \times K$ diagonal matrix of weights with $k-$th diagonal element $P_1(\boldsymbol{X}_k;\beta_1^p)(1-P_1(\boldsymbol{X}_k;\beta_1^p))$. Then we have

$$\frac{\partial l(\beta_1)}{\partial \beta_1} = \boldsymbol{X}^T(\boldsymbol{Y} - \boldsymbol{p})$$

$$\frac{\partial^2 l(\beta_1)}{\partial \beta_1 \partial \beta_1^T} = -\boldsymbol{X}^T \boldsymbol{W} \boldsymbol{X}.$$

Thus the Newton step is

$$\begin{aligned} \beta_1^{p+1} &= \beta_1^p + (\boldsymbol{X}^T\boldsymbol{W}\boldsymbol{X})^{-1}\boldsymbol{X}^T(\boldsymbol{y} - \boldsymbol{p}) \\ &= (\boldsymbol{X}^T\boldsymbol{W}\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{W}(\boldsymbol{X}\beta_1^p + \boldsymbol{W}^{-1}(\boldsymbol{y} - \boldsymbol{p})) \\ &= (\boldsymbol{X}^T\boldsymbol{W}\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{W}\boldsymbol{z}, \end{aligned}$$

where, $\boldsymbol{z} := \boldsymbol{X}\beta_1^p + \boldsymbol{W}^{-1}(\boldsymbol{y} - \boldsymbol{p})$. In the last line the Newton step is re-expressed as a weighted least squares step, since

$$\beta_1^{p+1} = \arg\min_{\beta} (\boldsymbol{z} - \boldsymbol{X}\beta_1)^T \boldsymbol{W} (\boldsymbol{z} - \boldsymbol{X}\beta_1). \tag{3.5}$$

Remark 3.3.21. *From equation (3.5), β_0 can be set to be zero for the iterative procedure. And the algorithm does convergence once log-likelihood is concave.*

The EM algorithm for the Markov switching model with expert opinions driven by logistic regression (LG) model is as follows:

EM Algorithm:
Inputs: $R_1, \dots, R_K$, $\boldsymbol{X}_1, \dots, \boldsymbol{X}_K$
Initialization: Y_0, b_0, a_0, Π_0, Θ_0 (Parameters of LG)
Step 1: Calculate uncertain expert opinion by $E_k^{(p)} = LG_{\Theta_p}(\boldsymbol{X}_k)$.
Step2: Use $E_0^{(p)}, \dots, E_{K-1}^{(p)}$ and $R_1, \dots, R_K$ and then construct filtrations $\mathcal{F}^{C_p}$ as inputs in EM algorithm with expert opinions and estimate the parameters b_p, a_p, Π_p.
Step3: Estimate $Y^p := \mathbb{E}[Y|\mathcal{F}_K^{C_p}]$ using parameters b_p, a_p, Π_p.
Step4: Update $\mathcal{C}$ by

$$\Theta_{p+1} = \arg\max_{\Theta} \mathbb{E}[\log \Lambda_K | \mathcal{F}_K^{C_p}]$$

return to Step 1 and repeat until some stopping criteria is satisfied.

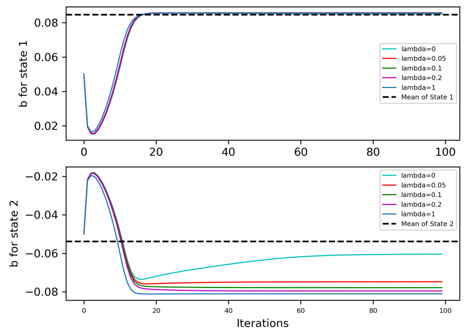
3.3.6 Numerical Results

The data is simulated by the following settings.

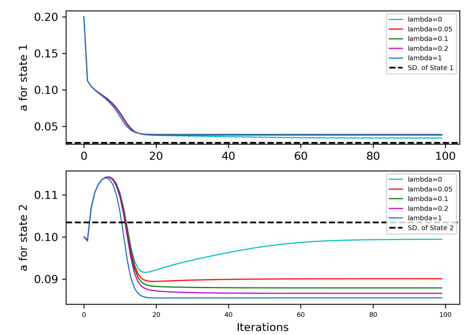
- $d = 2$;
- $b = (0.08, -005)^T, a = (0.03.0.1)^T$, $\Pi = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$;
- $Y_0 = e_1$.

Figure 3.1 shows for different λ (see Section 3.3.3), which can be understood as the trust of observations, how the estimated parameters vary. The expert opinion is given by $E^U = \{E_{81} = Y_{81} = e_2\}$. Note that $\lambda = 1$ is corresponding to EM with only observations and $\lambda = 0$ corresponds to the EM with certain expert opinion in our setting.

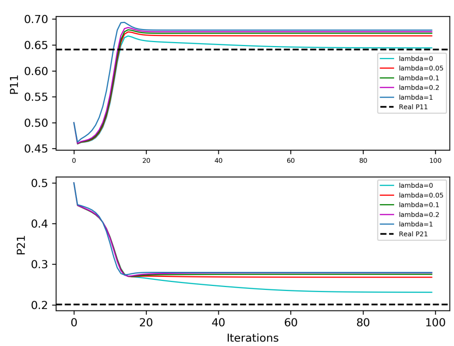
For another example, λ is fixed to be 0.01, then the effect of different uncertain experts can be significant. Figure 3.2 shows with different uncertain expert opinions, $\{E_{81} = (0,1)^T\}, \{E_{81} = (0.1, 0.9)^T\}, \{E_{81} = (0.2, 0.8)^T\}, \{E_{81} = (1,0)^T\}$ respectively, how the estimated parameters vary. The figures show



(a) Mean value of distributions for two states

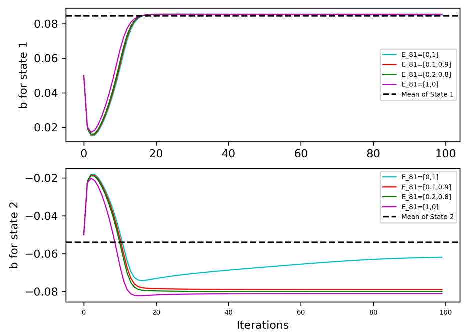


(b) Standard deviation of distributions for two states

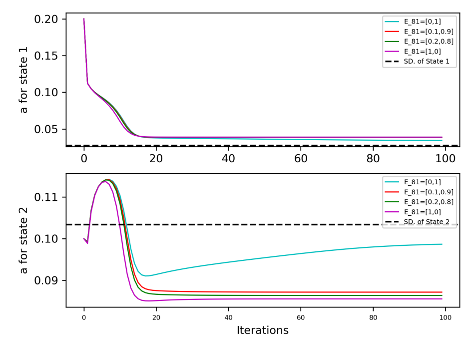


(c) p_{11} and p_{12} of transition matrix

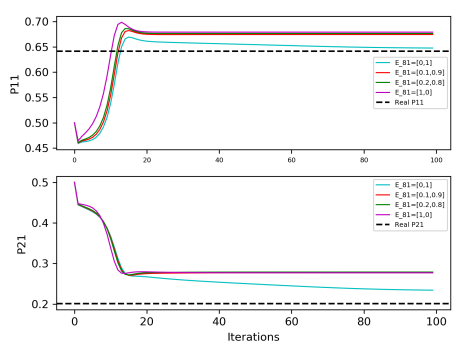
Figure 3.1: Estimated Parameters by EM algorithm with different λ



(a) Mean value of distributions for two states



(b) Standard deviation of distributions for two states



(c) p_{11} and p_{12} of transition matrix

Figure 3.2: Estimated parameters by EM algorithm with different uncertain expert opinions

that with more accurate expert opinions the EM algorithm has a better performance.

For the case of logistic regression model, one needs to simulate the exogenous factors $\boldsymbol{X}$ first.
Suppose $Y_{k-1} = LG(\boldsymbol{X}_k) + \boldsymbol{\varepsilon}_k$, implying

$$P(Y = e_1|\boldsymbol{X} = \boldsymbol{x}) = \frac{\exp(\beta^T \boldsymbol{x})}{1 + \exp(\beta^T \boldsymbol{x})}$$

where $\beta \in \mathbb{R}^d$, $\boldsymbol{x} \in \mathbb{R}^{K\times(M+1)}$. The expression for $\boldsymbol{x}$ is

$$\boldsymbol{x} = \arg\min_{\boldsymbol{x}}(logitY - \boldsymbol{x}\beta)^T(logitY - \boldsymbol{x}\beta),$$

and the solution is

$$\boldsymbol{x} = (\beta^T\beta)^{-1} logitY \beta^T.$$

where

$$logitY = \log(|\boldsymbol{Y}^1 - \boldsymbol{\varepsilon}|) - \log(1 - |\boldsymbol{Y}^1 - \boldsymbol{\varepsilon}|)$$

$\boldsymbol{Y}^1 = (Y_0^1, \ldots, Y_{K-1}^1)^T$, $\boldsymbol{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_{k-1})$ and $\varepsilon_i \sim U(0, \frac{1}{10})$.

Figure 3.3 illustrates that the log-likelihood function conditional on $R_1, \ldots, R_K$ and $\boldsymbol{X}$ increases significantly as iteration goes from 0 to 5 and converges fast, when λ is fixed to be 0.1.

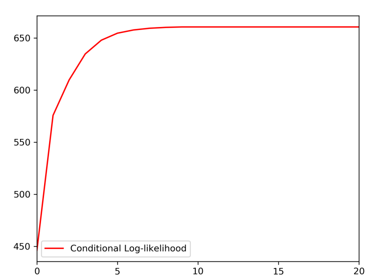
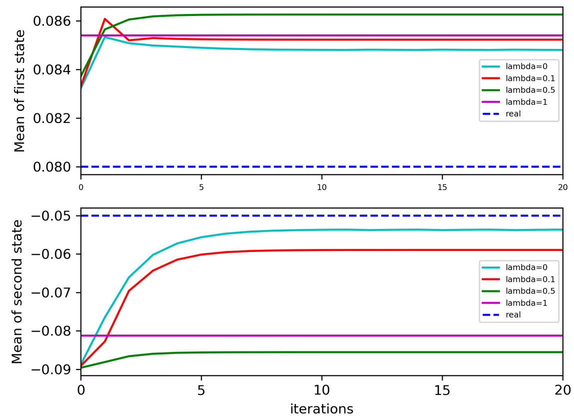


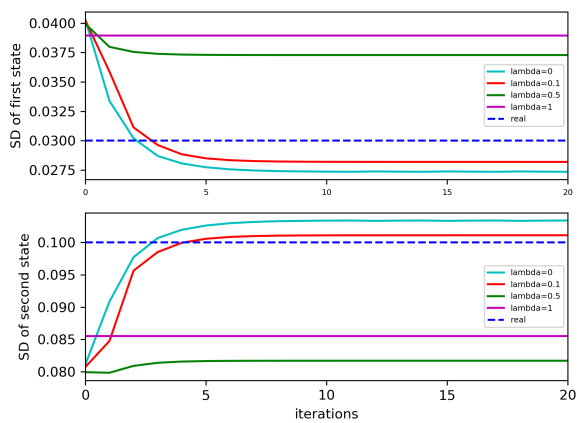
Figure 3.3: Pseudo conditional log-likelihood based on $R_1, \ldots, R_K$ and $\boldsymbol{X}$

Figure 3.4 shows the intuition that as iterations go up, the expert model can be calibrated and gives a more accurate expert opinion which as response gives a more accurate estimation of parameters of Markov Switching

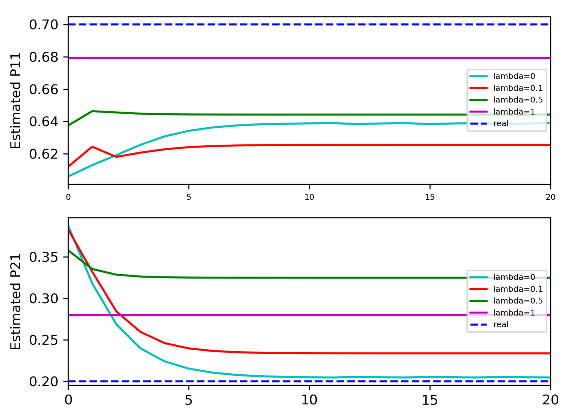
Model. These figures also show a good convergence except for the case of $\lambda = 0$. However one can choose the parameters which contribute larger log-likelihood.
As before λ is a pre-defined variable, which affects the estimation of parameters. The closer λ approaches to 1, the more weights we give on returns R. The estimation of parameters have different performance when λ varies. For estimation of mean of returns, b, $\lambda = 0$ works best, next $\lambda = 0.1$, $\lambda = 1$ and $\lambda = 0.5$. For estimation of a, also $\lambda = 0$ and $\lambda = 0.1$ have better performance than the other two. For estimation of transition matrix, $\lambda = 0$ and $\lambda = 0.1$ have contributions of good estimation of p_{21}, while worse estimation of p_{11}.



(a) Mean of Returns



(b) Standard Deviation of Returns



(c) Transition Matrix

Figure 3.4: Estimated Parameters by EM algorithm

Moreover, the EM algorithm also works well to estimate the parameters of logistic regression model, β (See Figure 3.5). It is not fair to rank the performance of different λ just by inspection. Hence we define a better estimation by minimizing the cross-entropy loss function. The Cross-Entropy loss function with respect to λ is given by

$$loss(\lambda) = -\sum_{k=1}^{K}\sum_{i=1}^{d} Y_k^i \log \hat{Y}_k^i(\lambda)$$

We have $loss(0) = 0.67$, $loss(0.1) = 10.41$, $loss(0.5) = 72.21$, $loss(1) = 87.74$. Because the $\boldsymbol{X}$ have better ability to estimate Y, the cross-entropy loss function decrease significantly as λ approaches to 0.

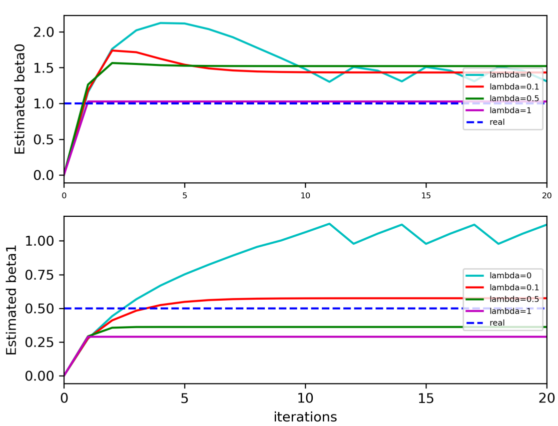


Figure 3.5: Pseudo conditional log-likelihood based on $R_1, \ldots, R_K$ and $\boldsymbol{X}$ with different lambda

3.4 Dirichlet Distribution for Expert Opinions

In this section, expert opinions are assumed to follow a specific continuous distributions. The first proposal is that all expert opinions are independent and identically distributed in Dirichlet distribution, which is known as distribution of distributions.

Definition 3.4.1. *The d-dimensional radom vector $Z = (Z_1, \dots, Z_d)$ is Dirichlet-distributed with parameter $\zeta = (\zeta_1, \dots, \zeta_d)$, $\zeta_1, \dots, \zeta_d > 0$, if Z has pdf*

$$f(z_1, \dots, z_d) = \frac{1}{B(\zeta)} \prod_{i=1}^{d} z_i^{\zeta_i - 1}$$

where $B(\zeta) = \frac{\prod_{i=1}^{d} \Gamma(\zeta_i)}{\Gamma(\zeta_0)}$ for $\zeta_0 = \sum_{i=1}^{d} \zeta_i$. We write $Z \sim D(\zeta)$.

Further

$$\begin{aligned}
\mathbb{E}[Z_i] &= \frac{\zeta_i}{\zeta_0}, \\
\mathrm{Var}(Z_i) &= \frac{\zeta_i(\zeta_0 - \zeta_i)}{\zeta_0^2(\zeta_0 + 1)}, \\
\mathrm{Cov}(Z_i, Z_i) &= \frac{-\zeta_i \zeta_j}{\zeta_0^2(\zeta_0 + 1)}.
\end{aligned}$$

We have the following assumptions:

Assumption 3.4.2. *Expert opinions E_u give the information on state Y_u by $\mathbb{E}[Y_u | E_u] = E_u$, where $E_{u_1}, E_{u_2}, \dots, E_{u_n}$ are i.i.d. from Dirichlet distribution, i.e. $P(E_{u_1}) \sim \mathcal{D}(\zeta)$ where ζ is a d-dimensional vector. Here we have $u \in \{u_1 \dots, u_n\}, u_1 < \dots, < u_n$.*

Under Assumption 3.4.2, we can have the recursive scheme.

Lemma 3.4.3. *For each time k, and all $i = 1, \dots, d$*

$$\mathbb{E}[Y_k^i | \mathcal{F}_{k-1}^Y \vee E_k] = \frac{E_k^i}{\zeta_i} \sum_{j=1}^{d} \frac{1}{\sum_{z=1}^{d} \frac{E_k^z}{\zeta_z} p_{jz}} p_{ji} Y_{k-1}^j$$

Proof. The proof starts with the $i-$th element of Y_k

$$\mathbb{E}[Y_k|Y_{k-1}, E_k] = \sum_{j=1}^{d} P(Y_k = e_i|Y_{k-1} = e_j, E_k)Y_{k-1}^j$$

Then we can show

$$\begin{aligned}
P(Y_k = e_i|Y_{k-1} = e_j, E_k) &= \frac{P(E_k|Y_k = e_i, Y_{k-1} = e_j)P(Y_k = e_i|Y_{k-1} = e_j)}{P(E_k|Y_{k-1} = e_j)} \\
&= \frac{P(E_k|Y_k = e_i)p_{ji}}{P(E_k|Y_{k-1} = e_j)} \\
&= \frac{P(E_k|Y_k = e_i)p_{ji}}{\sum_{z=1}^{d} P(E_k|Y_k = e_z, Y_{k-1} = e_j)p_{jz}} \\
&= \frac{P(E_k|Y_k = e_i)p_{ji}}{\sum_{z=1}^{d} P(E_k|Y_k = e_z)p_{jz}} \\
&= \frac{\frac{P(Y_k=e_i|E_k)P(E_k)}{P(Y_k=e_i)}p_{ji}}{\sum_{z=1}^{d} \frac{P(Y_k=e_z|E_k)P(E_k)}{P(Y_k=e_z)}p_{jz}} \\
&= \frac{\frac{E_k^i}{P(Y_k=e_i)}p_{ji}}{\sum_{z=1}^{d} \frac{E_k^z}{P(Y_k=e_z)}p_{jz}}.
\end{aligned}$$

In fact we have for $i = 1, \ldots, d$

$$\begin{aligned}
P(Y_k = e_i) &= \mathbb{E}[\mathbb{E}[Y_k^i|E_k]] \\
&= \mathbb{E}[E_k^i] \\
&= \frac{\zeta^i}{\zeta_0}.
\end{aligned}$$

In conclusion, we get

$$\mathbb{E}[Y_k^i|\mathcal{F}_{k-1}^Y \vee E_k] = \frac{E_k^i}{\zeta_i} \sum_{j=1}^{d} \frac{1}{\sum_{z=1}^{d} \frac{E_k^z}{\zeta_z} p_{jz}} p_{ji} Y_{k-1}^j.$$

□

From the above proof, we can have the intuition that Assumption 3.4.2 gives additional information of prior distribution of $P(Y_k = e_i)$. However this assumption does not help to improve the estimation of state since the parameters of Dirichlet distribution are not driven by Y. Hence we propose another idea to combine both information, which is know as Bayesian filter.

Assumption 3.4.4. *As expert opinions lag time Δt, we assume from now on that E_k represents the information of Y_{k-1}, i.e.*

$$\mathbb{E}[Y_{k-1}|E_k] = E_k,$$

where $E_k \sim \mathcal{D}(\boldsymbol{\gamma}^T Y_{k-1})$, and $\boldsymbol{\gamma}$ is a $d \times d$ parameter matrix for Dirichlet distribution.

Then, we will have the following characteristics of E_k.

Lemma 3.4.5. *Under Assumption 3.4.4, E_k is conditional Dirichlet distributed, and the distribution of E_k is a mixture of Dirichlet distributions. In addition, E_k is conditionally independent of R_k.*

Proof. At time k, for $i \in \{1, \dots, d\}$, the conditional distribution of E_k given $Y_{k-1} = e_i$ is

$$P_{E_k|Y_{k-1}=e_i} : \mathcal{B} \to \mathcal{E}, B \mapsto P_{E_k|Y_{k-1}=e_i}(B) := P(E_k \in B|Y_{k-1} = e_i)$$

is $\mathcal{D}(\gamma^{(i)})$, where $\gamma^{(i)}$ is the $i-$th row of $\boldsymbol{\gamma}$. Thus, E_k is conditional Dirichlet. The joint distribution of (E_k, Y_{k-1}) is

$$P(E_k \in B, Y_{k-1} = e_i) = P(Y_{k-1} = e_i)P_{E_k|Y_{k-1}=e_i}(B)$$

for $B \in \mathcal{B}, i = 1, \dots, d$. Thus,

$$P(E_k \in B) = \sum_{i=1}^{d} P(Y_{k-1} = e_i)P_{E_k|Y_{k-1}=e_i}(B)$$

i.e. distribution of E_k is a mixture of Dirichlet distributions.
For $B_1, B_2 \in \mathcal{B}$,

$$P(E_k \in B_1, R_k \in B_2|Y_{k-1}) = P(E_k \in B_1|Y_{k-1})P(R_k \in B_2|Y_{k-1}),$$

i.e. E_k is conditional independent of R_k. $\square$

Next step is using the technique of change-of-measure again, but we will need another reference measure.

3.4.1 Reference Measure including Expert Opinions

A new reference measure is introduced by the following definition.

Definition 3.4.6. *Set* $Z_0 = 1$ *and* $Z_k = Z_{k-1}L_k$ *for* $k \geq 1$, *where*

$$L_k := \frac{\varphi_{0,1}(R_k)}{\varphi_{b^T Y_{k-1},(a^T Y_{k-1})^2}(R_k)} \frac{f^D_{1^d}(E_k)}{f^D_{\boldsymbol{\gamma}^T Y_{k-1}}(E_k)}$$

where f^D *represents the density function of Dirichlet distribution whose parameters are given by subscript. The probability measure given by*

$$\frac{d\tilde{P}}{dP}|_{\mathcal{H}_K} = Z_K$$

is the reference measure for filtering, where $\mathcal{H}_K = \mathcal{F}_K \vee \mathcal{F}^E_K$.

Next, we show some good properties of measure $\tilde{P}$.

Lemma 3.4.7. *(i)* $Z = (Z_k)_{k=0,\dots,K}$ *is a* $\mathcal{H}$*-martingale under measure* P. $Z^{-1} = (Z_k^{-1})_{k=0,\dots,K}$ *is a* $\mathcal{H}$*-martingale under measure* $\tilde{P}$.
(ii) For all $u \leq k$,

$$\tilde{\mathbb{E}}[Z_k^{-1} Y_u | \mathcal{F}^C_u] = \tilde{\mathbb{E}}[Z_u^{-1} Y_u | \mathcal{F}^C_u].$$

(iii) Under $\tilde{P}$, $R_1, R_2, \dots$ *are i.id. standard normally distributed and independent of* Y, *and* $E_1, E_2, \dots$ *are i.i.d.,* $\mathcal{D}_{1^d}$*-distributed, and independent of* Y.

Proof. See proof in Appendix A.2. □

The next lemma shows some properties which are the same under the reference measure $\tilde{P}$ and under the original measure P. Obviously we have:

Lemma 3.4.8. *(i)* $\tilde{P}(Y_k = e_i | Y_k) = P(Y_k = e_i | Y_k) = Y_k^i$.
(ii) The Markov chain Y *has the same transition matrix under reference probability measure, i.e.*

$$\tilde{P}(Y_k = e_j | Y_{k-1} = e_i) = P(Y_k = e_j | Y_{k-1} = e_i).$$

Proof. Proof are the same as in Lemma 3.3.3. □

As there are some good properties in reference measure, we could compute filters under $\tilde{P}$ and use Bayes' formula to get filters under P.
Since for $A \in \mathcal{H}_K$,

$$\tilde{\mathbb{E}}[\mathbb{1}_A Z_K^{-1}] = \mathbb{E}[\mathbb{1}_A Z_K^{-1} Z_K] = \mathbb{E}[\mathbb{1}_A] = P(A)$$

we have $\frac{dP}{d\tilde{P}}|_{\mathcal{H}_K} = Z_K^{-1}$.

Definition 3.4.9. *For any $\mathcal{H}$-adapted $H = (H_k)_{k\in\mathbb{N}_0}$ such that the following expectations exist,*

$$\rho_k(H) := \tilde{\mathbb{E}}[Z_k^{-1} H_k | \mathcal{F}_k^C]$$

is unnormalized filter of H under $\tilde{P}$ at time k.

Lemma 3.4.10. *For any $\mathcal{H}$-adapted $H = (H_k)_{k\in\mathbb{N}_0}$ such that the following expectations exist,*

$$\hat{H}_k := \mathbb{E}[H_k|\mathcal{F}_k^C] = \frac{\rho_k(H)}{\rho_k(1)} = \frac{\mathbf{1}_d^T \rho_k(HY)}{\mathbf{1}_d^T \rho_k(Y)}.$$

Proof. By Bayes' formula of conditional expectation

$$\begin{aligned}
\mathbb{E}[H_k|\mathcal{F}_k^C] &= \frac{\tilde{\mathbb{E}}[H_k Z_K^{-1}|\mathcal{F}_k^C]}{\tilde{\mathbb{E}}[Z_K^{-1}|\mathcal{F}_k^C]} \\
&= \frac{\tilde{\mathbb{E}}[\tilde{\mathbb{E}}[H_k Z_K^{-1}|\mathcal{H}_k]|\mathcal{F}_k^C]}{\tilde{\mathbb{E}}[\tilde{\mathbb{E}}[Z_K^{-1}|\mathcal{H}_k]|\mathcal{F}_k^C]} \\
&= \frac{\tilde{\mathbb{E}}[\tilde{\mathbb{E}}[H_k Z_k^{-1}|\mathcal{H}_k]|\mathcal{F}_k^C]}{\tilde{\mathbb{E}}[\tilde{\mathbb{E}}[Z_k^{-1}|\mathcal{H}_k]|\mathcal{F}_k^C]} \\
&= \frac{\rho_K(H)}{\rho_K(1)}
\end{aligned}$$

Further,

$$\rho_k(1) = \tilde{\mathbb{E}}[Z_k^{-1}\mathbf{1}_d^T Y_k|\mathcal{F}_k^C] = \mathbf{1}_d^T\tilde{\mathbb{E}}[Z_k^{-1} Y_k|\mathcal{F}_k^C] = \mathbf{1}_d^T\rho_k(Y),$$

$$\rho_k(H) = \tilde{\mathbb{E}}[H_k Z_k^{-1}|\mathcal{F}_k^C] = \tilde{\mathbb{E}}[H_k \mathbf{1}_d^T Y_k Z_k^{-1}|\mathcal{F}_k^C] = \mathbf{1}_d^T\tilde{\mathbb{E}}[H_k Y_k Z_k^{-1}|\mathcal{F}_k^C] = \mathbf{1}_d^T\rho_k(HY)$$

which concludes the proof. □

Then we computer unnormalized filter of some suitable class under $\tilde{P}$ by the following theorem:

Theorem 3.4.11. *Let $H = (H_k)_{k\in\mathbb{N}_0}$ be $\mathcal{H}$-adapted, $H_k : \Omega \to \mathbb{R}$, with*

$$H_k = H_{k-1} + \alpha_{k-1} + \beta_{k-1}^T Y_k + \kappa_{k-1} f(R_k) + \delta_{k-1} g(E_k),$$

where α, β, κ, δ are $\mathcal{F}$-adapted and $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}$, $\mathbb{R}$-valued respectively. And $f : \mathbb{R} \to \mathbb{R}, g : \mathcal{E} \to \mathbb{R}$ are both measurable, s.t. H is integrable. For $\Gamma_i(r,e) := \frac{\varphi_i(r)}{\varphi_{0,1}(r)} \frac{f_i^D(e)}{f_{1^d}^D(e)}$, where $\varphi_i = \varphi_{b_i,a_i^2}$ and $f_i^D = f_{\gamma^{(i)}}^D$. Then we have

$$\begin{aligned}\rho_k(HY) = \sum_{i=1}^d \Big\{ & \rho_{k-1}(HY^i)\Gamma_i(R_k,E_k)\Pi^T e_i \\ & + \rho_{k-1}(\alpha Y^i)\Gamma_i(R_k,E_k)\Pi^T e_i \\ & + \rho_{k-1}(\kappa Y^i)\Gamma_i(R_k,E_k) f(R_k)\Pi^T e_i \\ & + \rho_{k-1}(\delta Y^i)\Gamma_i(R_k,E_k) g(E_k)\Pi^T e_i \\ & + (Diag(\Pi^T e_i))\rho_{k-1}(\beta Y^i)\Gamma_i(R_k,E_k) \Big\}\end{aligned}$$

where $\rho_k(HY) = (\rho(HY^1), \rho(HY^2), \ldots, \rho(HY^d))^T$, and Y^i is the $i-th$ element of vector Y.

Proof. See proof in Appendix A.2. □

Theorem 3.4.12. *The recursive scheme for $\hat{Y}_k^i$ is*

$$\hat{Y}_k^i = \frac{\sum_{j=1}^d p_{ji}\varphi_j(R_k) f_j^E(E_k)\hat{Y}_{k-1}^j}{\sum_{z=1}^d \varphi_z(R_k) f_z^E(E_k)\hat{Y}_{k-1}^z},$$

where $\hat{Y}_k = \mathbb{E}[Y_k|\mathcal{F}_k^C]$, $f_j^E = f_{\alpha^{(j)}}^{\mathcal{D}}$ is a density function of Dirichlet distribution parameterized by $\gamma^{(j)}$. Vectorizing, we get

$$\hat{Y}_k = \frac{\Pi^T \Big(Diag(\varphi(R_k) \odot f^E(E_k)) \Big) \hat{Y}_{k-1}}{(Diag(\varphi(\tilde{R})) f^E(E_k))^T \hat{Y}_{k-1}},$$

where $f^E(e) = (f_1^E(e), \ldots, f_d^E(e))^T$ and $Diag(\varphi(r) \odot (f^E(e))$ is the diagonal matrix with diagonal $\varphi(r) \odot f^E(e)$.

Proof.

$$
\begin{aligned}
&P(Y_k = e_i, R_k \leq r, E_k \leq e | R_{1:1-k}, E_{1:1-k}) \\
&= \sum_{j=1}^{i} P(Y_k = e_i, R_k \leq r, E_k \leq e | Y_{k-1} = e_j) P(Y_{k-1} = e_j | R_{1:1-k}, E_{1:1-k}) \\
&= \sum_{j=1}^{d} p_{ji} P(R_k \leq r | Y_{k-1} = e_j) P(E_k \leq e | Y_{k-1} = e_j) \hat{Y}_{k-1}^j \\
&= \sum_{j=1}^{d} p_{ji} \int_{\mathbb{R}} \varphi_j(r) dr \int_{\mathcal{E}} f_j^E(e) de \hat{Y}_{k-1}^j
\end{aligned}
$$

In addition

$$
\begin{aligned}
&P(R_k \leq r, E_k \leq e | R_{1:1-k}, E_{1:1-k}) \\
&= \sum_{i=1}^{d} \sum_{j=1}^{d} p_{ji} \int_{\mathbb{R}} \varphi_j(r) dr \int_{\mathcal{E}} f_j^E(e) de \hat{Y}_{k-1}^j \\
&= \sum_{j=1}^{d} \int_{\mathbb{R}} \varphi_j(r) dr \int_{\mathcal{E}} f_j^E(e) de \hat{Y}_{k-1}^j
\end{aligned}
$$

Then we can use Bayes rules and get

$$
\begin{aligned}
\mathbb{E}[Y_k^i | \mathcal{F}_k^R \vee \mathcal{F}_k^E] &= P(Y_k = e_i | R_{1:k}, E_{1:k}) \\
&= \frac{\sum_{j=1}^{d} p_{ji} \varphi_j(R_k) f_j^E(E_k) \hat{Y}_{k-1}^j}{\sum_{z=1}^{d} \varphi_z(R_k) f_z^E(E_k) \hat{Y}_{k-1}^z}
\end{aligned}
$$

After vectorizing, we get the final result. □

Remark 3.4.13. *Using change-of-measure, we can get the same filter as the recursive one in Theorem 3.4.12. From Theorem 3.4.11, we set $H = 1, \alpha = \beta = \gamma = \delta = 0$ and get*

$$
\rho_k(Y^i) = \sum_{j=1}^{d} p_{ji} \rho_{k-1}(Y^j) \Gamma_j(R_k, E_k).
$$

Then we derive

$$
\mathbf{1}_d \rho_k(Y) = \sum_{j=1}^{d} \rho_{k-1}(Y^j) \Gamma_j(R_k, E_k).
$$

From above, we can compute for $\mathbb{E}[Y_k^i|\mathcal{F}_k^C]$*, which is given by*

$$\mathbb{E}[Y_k^i|\mathcal{F}_k^C] = \frac{\rho_k(Y^i)}{\mathbf{1}_d\rho_k(Y)} = \frac{\sum_{j=1}^d p_{ji}\varphi_j(R_k)f_j^E(E_k)\hat{Y}_{k-1}^j}{\sum_{z=1}^d \varphi_z(R_k)f_z^E(E_k)\hat{Y}_{k-1}^z}.$$

3.4.2 Posterior Probability Maximization

In EM algorithm, the following posterior likelihood is maximized. For observed $Y_1,\dots,Y_K$, $R_1,\dots,R_K$, $E_1,\dots,E_K$

$$\Lambda_K = \frac{dP_\theta}{dP_{\theta'}} = \prod_{k=1}^K\prod_{i,j=1}^d \Big(\frac{p_{ij}}{p'_{ij}}\Big)^{Y_{k-1}^iY_k^j}\sum_{i=1}^d Y_{k-1}^i \frac{\varphi_{b_i,a_i^2}(R_k)}{\varphi_{b'_i,a_i'^2}(R_k)}\frac{f^{\mathcal{D}}_{\gamma^{(j)}}(E_u)}{f^{\mathcal{D}}_{\gamma^{(j)\prime}}(E_u)}$$

which leads to

$$\begin{aligned}\log\Lambda_K &= \sum_{i,j=1}^d N_K^{ij}\log(p_{ij}) + \sum_{i=1}^d\bigg(-O_K^i\log a_i - \frac{1}{2}\Big(\frac{b_i}{a_i}\Big)^2 O_K^i\bigg)\\ &+\sum_{i=1}^d\bigg(\frac{b_i}{a_i^2}T_K^i(f_1) - \frac{1}{2a_i^2}T_K^i(f_2)\bigg) + \sum_{k=1}^K\sum_{j=1}^d Y_{k-1}^j\log f^{\mathcal{D}}_{\gamma^{(j)}}(E_k)\\ &+h(\Pi',a',b') + g(\gamma').\end{aligned}$$

Define

$$F(\gamma) := \sum_{k=1}^K\sum_{j=1}^d Y_{k-1}^j\log f^{\mathcal{D}}_{\gamma^{(j)}}(E_k)$$

where $\gamma = (\gamma^{(1)},\dots,\gamma^{(d)})^T \in \mathbb{R}^{d\times d}$. Thus,

$$F(\gamma) = \sum_{k=1}^K\sum_{j=1}^d Y_{k-1}^j\Big(\log\Gamma(\sum_{i=1}^d\gamma_i^{(j)}) - \sum_{i=1}^d\log\Gamma(\gamma_i^{(j)}) + \sum_{i=1}^d(\gamma_i^{(j)}-1)\log E_k^i\Big).$$

The goal is to derive γ, which can maximize F.
The first approach to try is Gradient Ascent, which iteratively steps along positive gradient directions of F until convergence. The gradient of the objective is given by differentiating F:

$$(\nabla F)_{ij} = \frac{\partial F}{\partial\gamma_i^{(j)}} = \sum_{k=1}^K Y_{k-1}^j\Big(\Psi(\sum_{i=1}^d\gamma_i^{(j)}) - \Psi(\gamma_i^{(j)}) + \log E_k^i\Big) \tag{3.6}$$

where $\Psi(x) = \frac{d\log\Gamma(x)}{dx}$ is known as digamma function. There is no closed-form solution to estimate Dirichlet distribution from above. One can always continue to step along a constant fraction of the gradient. Moreover one should always care about the constraints of this problem that γ_i is positive.

The second approach is the Newton-Raphson method, which provides a quadratically converging approach for parameter estimation. The general rule to update parameters iteratively is

$$\gamma^{new} = \gamma^{old} - H^{-1}(F)\nabla F$$

where H is Hessian matrix.
Here we need to express γ as an $d \times d$-dimensional vector, and hence we can have a $d^2 \times d^2$ Hessian matrix for F. In particular,

$$\begin{aligned}
\frac{\partial^2 F}{\partial\gamma_i^{(j)}\partial\gamma_i^{(j)}} &= \sum_{k=1}^{K} Y_{k-1}^{j}\Big(\Psi'(\sum_{i=1}^{d}\gamma_i^{(j)}) - \Psi'(\gamma_i^{(j)})\Big) \\
\frac{\partial^2 F}{\partial\gamma_i^{(j)}\partial\gamma_l^{(j)}} &= \sum_{k=1}^{n} Y_{k-1}^{j}\Big(\Psi'(\sum_{i=1}^{d}\gamma_i^{(j)})\Big)\ l \neq i \\
\frac{\partial^2 F}{\partial\gamma_i^{(j)}\partial\gamma_m^{(h)}} &= 0 \text{ for } h \neq j \text{ and all } i, m = 1, \ldots, d.
\end{aligned} \tag{3.7}$$

The estimator in equation (3.6) and equation (3.7) can be specified using Theorem 3.4.11.

3.4.3 Numerical Results

The data is simulated in the following setting:

- $d = 2$;
- $b = (0.08, -005)^T, a = (0.03.0.1)^T$, $\Pi = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$;
- $Y_0 = e_1$.
- $\gamma = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

From γ, we know $\gamma^{(1)} = (3, 1)$ and $\gamma^{(2)} = (1, 3)$. The first value represents the value of γ conditional on $Y = e_1$, and second one is accordingly conditional on $Y = e_2$.
Figure 3.6 illustrates the conditional log-likelihood once we know the real gamma. The convergence speed is faster than the scenario in which one can only observe asset returns.

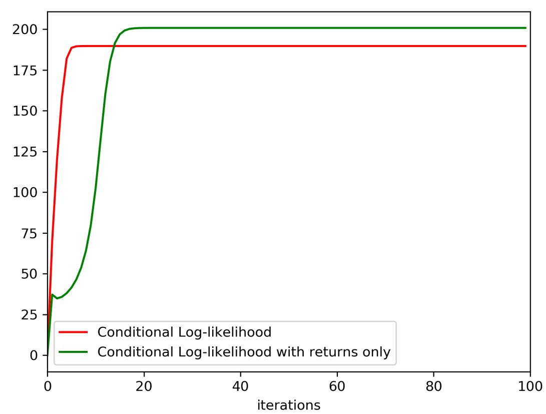
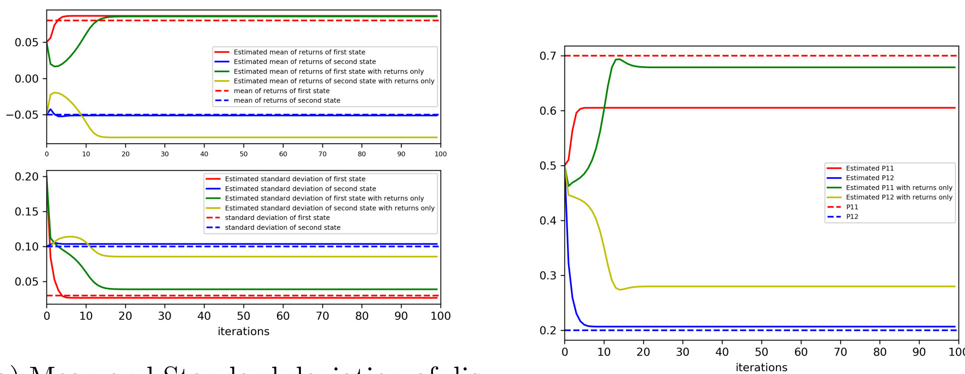


Figure 3.6: Conditional Log-likelihood

Once we know the exact value of gamma, we may combine the information of expert opinions and asset returns by Bayes rules. Figure 3.7a illustrates that with expert opinions the estimators of both b and a converge faster as EM algorithm iterates. In addition the estimators are closer to the real value of these parameters. To estimate the transition matrix, we can have a faster convergence speed than before. The estimator of p_{11} is closer to the real value, while the estimator of p_{11} is not.
We can combine the information of expert opinions and assets returns and have a better estimators of parameters once we know γ. Next, we will not assume that γ is known before. Instead, we have an initial guess of

$$\gamma_0 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Then, we use numerical methods of gradient ascent with constraint $\gamma_{ij} > 0$ to estimate the value of γ at the same time. The numerical method is not



(a) Mean and Standard deviation of distributions for two states

(b) p_{11} and p_{12} of transition matrix

Figure 3.7: Estimated Parameters by EM algorithm with Known γ

stable. As shown in Figure 3.8, the estimation for γ irregularly fluctuate at first, but converge closer to the real values in the end.

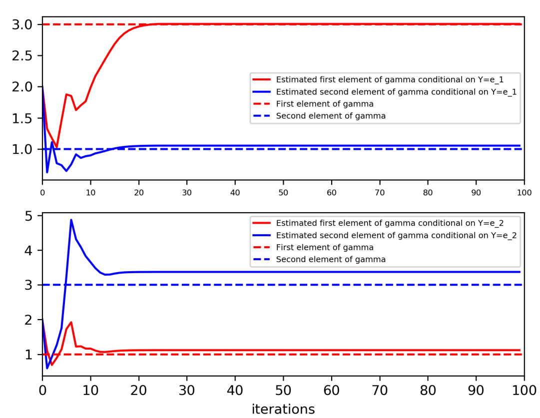
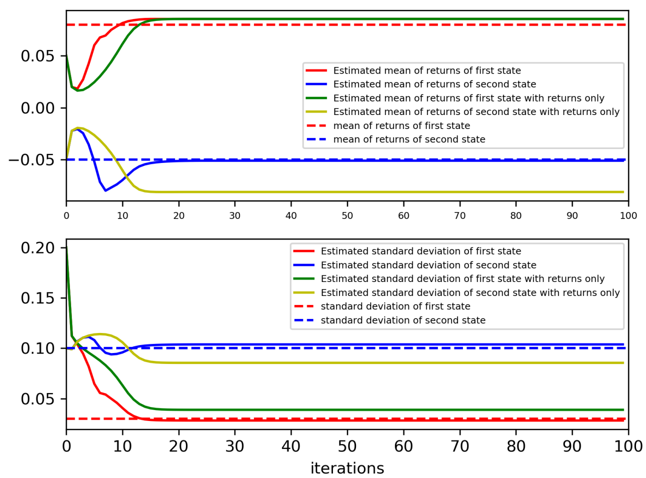
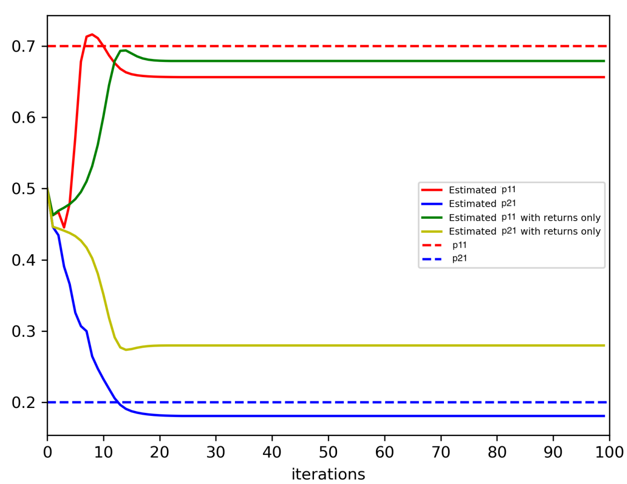


Figure 3.8: Estimators of γ

In the EM algorithm, we estimate both the parameters of MSMs and Dirichlet distributions at the same time. As shown in Figure 3.9 the parameters b, a, Π converge in the end with improvement of estimators of γ. The results are very similar to the scenario in which we know the value of γ, while the convergence speed is slower than before. The estimators of b and a are closer to real ones than when the only observation are the asset returns. The estimator of p_{21} is closer to the real value, while the estimator of p_{11} is not.



(a) Mean and Standard deviation of distributions for two states



(b) p_{11} and p_{12} of transition matrix

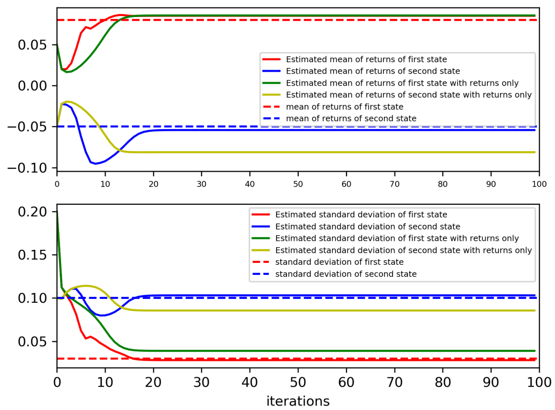
Figure 3.9: Estimated Parameters by EM algorithm with Unknown γ

To see the impact of γ, we have a different setting with

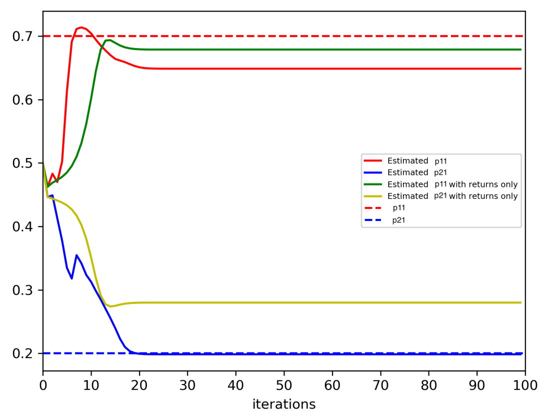
$$\gamma = \begin{bmatrix} 3 & 1 \\ 1 & 6 \end{bmatrix}.$$

Then we get similar results as shown in Figure 3.10.

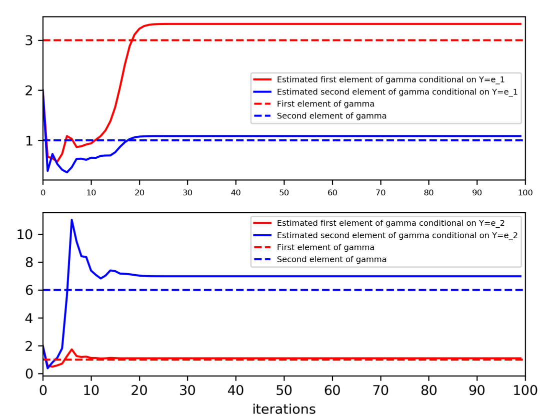
Remark 3.4.14. *An examples with 3 states are illustrated in A.3.*



(a) Mean and Standard deviation of distributions for two states



(b) p_{11} and p_{12} of transition matrix



(c) Estimators of γ

Figure 3.10: Estimated Parameters by EM algorithm with Unknown γ

Chapter 4

Portfolio Optimization with Expert Opinions in MSMs

In this chapter, our aim will be to maximize her expected utility of terminal wealth according to the available information:

$$\pi^* = \arg\max_{\pi \in \mathcal{A}^H} \mathbb{E}[U(X_T^\pi)]$$

where

$$\begin{aligned}\mathcal{A}^H(x) = \Big\{ \pi = (\pi_k)_{k=0,\dots,k} : & \pi_k \text{ is } \mathcal{F}_k^H - \text{measurable}, X_k^\pi > 0 \text{ for } k = 0, \dots, N \\ & \text{and } \mathbb{E}[U(X_N^\pi)^- < \infty], X_0^\pi = x \Big\}\end{aligned}$$

is the set of admissible strategies for all the cases $H \in \{R, E, C, F\}$. We first show existence of the solution by a dynamic programming approach.
We derive an order-Δt-approximation of the strategy in each case, and the results are the same as the ones when we use second order multivariate Taylor expansion.
We investigate the impact of the time step by using Monte Carlo simulation and interpolation methods.
At last we show the benefit of strategies using combined information by a numerical study.

4.1 Financial Market Model

For a fixed date $T > 0$ representing the investment horizon, we work on a filtered probability space $(\Omega, \mathcal{F}_T, \mathcal{F}, \mathbb{P})$, with filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. All processes are assumed to be $\mathcal{F}$-adapted. As before, we consider to discretize the continuous financial market with one stock with prices

$$dS_t = S_t(\mu_t dt + \sigma dW_t)$$

and one bond with prices

$$dB_t = B_t r dt.$$

We model the daily log-returns of risky assets prices, $R_k = \log \frac{S_k}{S_{k-1}}$, by MSMs

$$R_k = b^T Y_{k-1} + a^T Y_{k-1}\varepsilon_k \text{ for k=1,...,N},$$

where $Y_k \in \mathcal{Y}$, $\mathcal{Y}$ is state space $\{e_1, \dots, e_d\}$, the units vector in $\mathbb{R}^d$. The parameters are discretized from continuous model given by $b_i = (\mu_i - \sigma_i^2/2)\Delta t$, $a_i = \sigma_i\sqrt{\Delta t}$, $\varepsilon_k = \frac{W_{t_k} - W_{t_{k-1}}}{\sqrt{\Delta t}} \sim \mathcal{N}(0,1)$, $T = N\Delta t$, $t_k = k\Delta t$, $k = 0, \dots, N$. We model the dynamics of the bond prices B_k at time k as

$$B_k = (1 + r\Delta t)B_{k-1}, r\Delta t > -1.$$

The wealth process using strategies π at time k is given by

$$X^\pi_{k+1} = \Big(1 + (1-\pi_k)r\Delta t + \pi_k(e^{R_{k+1}-1}\Big)X^\pi_k$$

and x_0 denotes the initial wealth.

As in Lemma 2.1.3, the stock returns in MSM have full support too. Hence we can get the same condition for admissible strategy that $\pi \in (0,1)$.

The information available to an investor is described by the investor filtration $\mathcal{F}^H = (\mathcal{F}^H_t)_{t\in[0,T]}$, for which we consider four cases $H \in \{R, E, C, F\}$. The investor's aim will be to maximize her expected utility of terminal wealth according to the available information:

$$\pi^* = \arg\max_{\pi^H \in \mathcal{A}} \mathbb{E}[U(X^\pi_T)] \tag{4.1}$$

where

$$\mathcal{A}^H(x) = \Big\{\pi = (\pi_k)_{k=0,\dots,k} : \pi_k \text{ is } \mathcal{F}_k^H - \text{measurable}, X_k^\pi > 0 \text{ for } k = 0,\dots,N \\ \text{and } \mathbb{E}[U(X_N^\pi)^- < \infty], X_0^\pi = x\Big\}$$

is the set of admissible strategies for all the cases $H \in \{R, E, C, F\}$, $U : \mathbb{R}^+ \to \mathbb{R} \cup [-\infty]$ is a utility function, and X_t^π is the wealth process when investing according to portfolio strategies $\pi \in \mathcal{A}^H$.

The portfolio optimization problem 4.1 can be solved by the following Markov decision model.

Definition 4.1.1. *The Markov decision model is given by* $(\mathcal{X}, \mathcal{A}, \{\mathcal{A}(x) : x \in \mathcal{X}\}, Q, G)$.
(1) a Borel set $\mathcal{X}$*, the states,*
(2) a Borel set $\mathcal{A}$*, the actions or controls,*
(3) $\mathcal{A}(x)$*, the admissible controls in state* x*, where* $\mathcal{D} = \{(x,a)|x \in \mathcal{X}, a \in \mathcal{A}(x)\}$ *is measurable.*
(4) a stochastic kernel Q *on* $\mathcal{X}$ *given* $\mathcal{D}$*, i.e.* $Q(\cdot|(x,a))$ *is a probability measure for each* $(x,a) \in \mathcal{D}$.
(5) a terminal reward function $G : \mathcal{D} \to \mathbb{R}$ *which is measurable.*

Remark 4.1.2. *(1) If* $x \in \mathcal{X}$ *is* $\mathcal{F}^H$*-adapted and observable, then as a result* $a \in \mathcal{A}(x)$*,* $\mathcal{D}$ *and the terminal reward function* G *are all* $\mathcal{F}^H$*-measurable.*

Definition 4.1.3. *To solve the problem of Markov decision model, we consider the* $\mathcal{F}^H$*-performance criterion*

$$J_k^{\pi(H)}(x) := \mathbb{E}[G(X_N)|\mathcal{F}_k^H, X_k = x]$$

and the $\mathcal{F}^H$*-value function*

$$V_k^H(x) = \sup_{\pi \in \mathcal{A}^H} J_k^{\pi(H)}(x),\ k = 0,\dots,N,\ x \in \mathcal{X}.$$

Our aim is to find the value $V_0^H(x)$ *in each case* H *and optimal strategy, denoted by* $\pi^*(H)$ *with* $V_0^H = J_0^{\pi^*(H)}$.

Definition 4.1.4. *If there exist i.i.d.* $(Z_k)_{k=1,\dots,N}$ *with values in* $\mathcal{Z}$ *and measurable.*

$$f : (\mathcal{X}, \mathcal{A}, \mathcal{Z}) \to \mathcal{X},$$

s.t. $X_{k+1} = f(X_k, a_k, Z_k)$*, then the Markov decision model is called Markov decision model with transition law* f*.*

E.g., in Markov decision model with full information we have two variables for state X_k, wealth process X_k^π and state Y_k respectively. Then we have

$$f(x, e_i, \pi, \varepsilon) = x(1 + r\Delta t + \pi(e^{b_i + a_i\varepsilon} - 1 - r\Delta t)).$$

Later we have unobserved Y_k in partial information, then the state X_k have two variables, wealth process X_k^π and filter $\hat{Y}_k$ instead.

Remark 4.1.5. *In our setting of portfolio optimization, the utility function is regarded as reward function,*

$$V_k^H(x) = \sup_{\pi \in \mathcal{A}(x)^H} \mathbb{E}[U(X_N^\pi)|\mathcal{F}_k^H, X_k^\pi = x]$$

where $\mathcal{A}^H(x)$ *is the set of admissible strategies,* $U : \mathbb{R}^+ \mapsto \mathbb{R}$ *is a utility function.* $[0, N\Delta t]$ *is the investment horizon and* X_k^π *is the wealth process when investing according to portfolio strategy* $\pi \in \mathcal{A}^H(x)$*.*

Theorem 4.1.6. *(Dynamic Programming) Define* $W_N^H, \dots, W_0^H$ *backwards by*

$$W_N^H(x) = U(x)$$

and for $n = N-1, \dots, 0$ *by the dynamic programming equation(DPE)*

$$W_k^H(x) = \sup_{a \in \mathcal{A}^H(x)} \mathbb{E}[W_{k+1}^H(f(x, a, Z))|\mathcal{F}_k^H].$$

If $W_0^H, \dots, W_{N-1}^H$ *are* $\mathcal{F}^H$*-measurable and if there exist a* $\mathcal{F}^H$*-measurable* $\varphi_k^H : \mathcal{X} \to \mathcal{A}$ *with* $\varphi_k^H(x) \in \mathcal{A}^H(x)$ *and*

$$W_k^H(x) = \mathbb{E}[W_{k+1}^H(f(x, \varphi_k^H(x), Z))|\mathcal{F}_k^H],$$

then $\pi^*(H) := \varphi_k^H$ defines an optimal Markov decision and $V_k^H = W_k^H, k = 0, \dots, N$.

Proof. We show the statement by backward induction. For $n = N$ we have

$$V_N^H(x) = U(x) = W_N^H(x)$$

Now suppose that for $n \in \{0, \dots, N-1\}$ we have $W_k^H = V_k^H$ for $k \geq n+1$ and that $\psi_{n+1}^*, \dots, \psi_{N-1}^*$ is an optimal policy from $n+1$. Then on one hand, since an admissible maximizer exists,

$$\begin{aligned} V_n^H(x) &\geq \sup_a \mathbb{E}[V_{n+1}^H(X_{n+1}^\psi)|\mathcal{F}_n^H, \psi_n = a, X_n^\psi = x] \\ &= \sup_a \mathbb{E}[W_{n+1}^H(f(x, a, Z))|\mathcal{F}_k^H] = W_n^H(x). \end{aligned}$$

On the other hand, for optimal ψ_n^* and maximizer φ_n^H in the DPE,

$$\begin{aligned} W_n^H(x) &= \mathbb{E}[W_{n+1}^H(f(x, \varphi_n^H(x), Z))|\mathcal{F}_n^H] \\ &\geq \mathbb{E}[W_{n+1}^H(f(x, \psi_n^*(x), Z))|\mathcal{F}_n^H] \\ &= \mathbb{E}[W_{n+1}^H(X_{n+1}^{\psi^*})|\mathcal{F}_n^H, X_n^{\varphi^*} = x] \\ &= \mathbb{E}[V_{n+1}^H(X_{n+1}^{\psi^*})|\mathcal{F}_n^H, X_n^{\varphi^*} = x] \\ &= \mathbb{E}[\mathbb{E}[U(X_N^{\psi^*})|\mathcal{F}_{n+1}^H, X_{n+1}^{\psi^*}]|\mathcal{F}_n^H, X_n^{\varphi^*} = x] \\ &= V_n^H(x). \end{aligned}$$

□

4.2 Markov Decision Processes with Full Information

First, the scenario $H = F$ is discussed. We take (X_k^π, Y_k) as controlled process. Then we get a Markov control problem with transition kernel Q given by

$$\begin{aligned} Q(B \times \{e_j\}|(x, e_i, \eta)) &= P(X_{k+1}^\pi \in B, Y_{k+1} = e_j | X_k^\pi = x, Y_k = e_i, \pi_k = \eta) \\ &= P(f(x, e_i, \eta, \varepsilon_{k+1}) \in B) p_{ij} \end{aligned}$$

Theorem 4.2.1. *For logarithmic utility $U(x) = \log x$, the value function satisfy $V_N^F(x, e_i) = U(x)$, and for $k = N-1, \ldots, 0$, the DPEs*

$$V_k^F(x, e_i) = \sup_{\eta \in [0,1]} \mathbb{E}\Big[\sum_{j=1}^d V_{k+1}^F(f(x, e_i, \eta, \varepsilon_{k+1}), e_j) p_{ij} \Big| \mathcal{F}_k^F \Big]$$

where $\varepsilon_{k+1} \sim \mathcal{N}(0,1)$. An optimal strategy is given by $\pi_k^* = \eta^*(Y_k)$, where $\eta^*(e_i)$ is the optimizer in

$$v(e_i) = \sup_{\eta \in [0,1]} \mathbb{E}[\log(f(1, \eta, e_i, \varepsilon))]$$

where $\varepsilon \sim \mathcal{N}(0,1)$. Further,

$$V_k^F(x, e_i) = \log x + d_k(e_i)$$

where, $d_N(e_i) := 0$ and

$$d_k(e_i) := \sum_{j=1}^d p_{ij}(v(e_i) + d_{k+1}(e_j)).$$

Proof. We prove the form of V_k^F by backward induction. First we have $V_N^F(x, e_i) = U(x) = \log x$.

Suppose $V_k^F(x, e_i) = \log x + d_k(e_i)$ is satisfied for $k+1, (k < N)$, then

$$
\begin{aligned}
V_k^F(x, e_i) &= \sup_{\eta \in [0,1]} \mathbb{E}\Big[\sum_{j=1}^{d} V_{k+1}^F(f(x, e_i, \eta, \varepsilon_{k+1}), e_j) p_{ij} \Big| \mathcal{F}_k^F \Big] \\
&= \sup_{\eta \in [0,1]} \mathbb{E}\Big[\sum_{j=1}^{d} \Big(\log x + \log(f(1, \eta, e_i, \varepsilon_{t+1})) + d_{k+1}(e_j) \Big) p_{ij} \Big] \\
&= \log x + \sum_{j=1}^{d} p_{ij}(v(e_i) + d_{k+1}(e_j)) \\
&= \log x + d_k(e_i).
\end{aligned}
$$

□

If using Taylor expansions for e^x, and $\log(1+x)$ around 0, we have first

$$
e^{b_i + a_i \varepsilon} \approx 1 + (\mu_i - \sigma_i^2/2)\Delta t + \sigma \sqrt{\Delta t} \varepsilon + 1/2 \sigma_i^2 \Delta t \varepsilon^2
$$

and thus

$$
\begin{aligned}
v_k(e_i) &= \sup_{\eta \in [0,1]} \mathbb{E}[\log(f(1, \eta, e_i, \varepsilon_{k+1}))] \\
&= \sup_{\eta \in [0,1]} \mathbb{E}[\log(1 + r\Delta t + \eta((\mu_i - \sigma_i^2/2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon + 1/2\sigma_i^2 \Delta t \varepsilon^2 - r\Delta t))] \\
&\approx \sup_{\eta \in [0,1]} \mathbb{E}[r\Delta t + \eta((\mu_i - \sigma_i^2/2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon + 1/2\sigma_i^2 \Delta t \varepsilon^2 - r\Delta t) \\
&- (r\Delta t + \eta((\mu_i - \sigma_i^2/2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon + 1/2\sigma_i^2 \Delta t \varepsilon^2 - r\Delta t))^2/2] \\
&= \sup_{\eta \in [0,1]} \mathbb{E}\Big[r\Delta t + \eta(\mu_i \Delta t - r\Delta t) - \eta^2 \sigma_i \Delta t/2 \Big].
\end{aligned}
$$

Then we get $\eta^*(e_i) \approx \frac{\mu_i - r}{\sigma_i^2}$.

Proposition 4.2.2. *The log-optimal strategy at time k, π_k^* has values in $(0,1)$ if and only if*

$$
\frac{Y_k^T \mu - \rho}{(Y_k^T \sigma)^2} \in (0,1)
$$

where ρ is compound interest rate, whose discretized one of order Δt is r, i.e. $e^{\rho \Delta t} = 1 + r\Delta t$.

Proof. From Theorem 4.2.1, the optimal strategy at time k, is only dependent on Y_k via

$$\pi_k^* = \eta^*(Y_k)$$

where $\eta^*(e_i)$ is the maximizer of

$$\mathbb{E}[\log(f(1,\eta,e_i,\varepsilon_{k+1}))] = \mathbb{E}[\log(1 + r\Delta t + \eta(e^{b_i + a_i\varepsilon_{k+1}} - 1 - r\Delta t))] =: \delta(\eta)$$

Then we have

$$\begin{aligned}
\frac{d\delta(\eta)}{d\eta} &= \mathbb{E}\Big[\frac{e^{b_i+a_i\varepsilon_{k+1}} - 1 - r\Delta t}{1 + r\Delta t + \pi_k(e^{b_i+a_i\varepsilon_{k+1}} - 1 - r\Delta t)}\Big] \\
\frac{d^2\delta(\eta)}{d\eta^2} &= -\mathbb{E}\Big[\frac{(e^{b_i+a_i\varepsilon_{k+1}} - 1 - r\Delta t)^2}{(1 + r\Delta t + \pi_k(e^{b_i+a_i\varepsilon_{k+1}} - 1 - r\Delta t))^2}\Big] < 0
\end{aligned}$$

Therefore $0 < \pi_k^* < 1$, if and only if

$$\mathbb{E}\Big[\frac{e^{b_i+a_i\varepsilon_{k+1}} - 1 - r\Delta}{1 + r\Delta t}\Big] > 0;$$

$$\mathbb{E}\Big[\frac{e^{b_i+a_i\varepsilon_{k+1}} - 1 - r\Delta t}{1 + r\Delta t + (e^{b_i+a_i\varepsilon_{k+1}} - 1 - r\Delta t)}\Big] < 0$$

The condition is given by

$$(\mathbb{E}[e^{-(b_i+a_i\varepsilon_{k+1})}])^{-1} < 1 + r\Delta t < \mathbb{E}[e^{b_i+a_i\varepsilon_{k+1}}]$$

In addition,

$$\begin{aligned}
\mathbb{E}[e^{-b_i-a_i\varepsilon_{k+1}}] &= e^{(-\mu_i+\sigma_i^2)\Delta t} \\
\mathbb{E}[e^{b_i+a_i\varepsilon_{k+1}}] &= e^{\mu_i\Delta t}
\end{aligned}$$

Finally we have

$$\begin{aligned}
0 < \eta^*(e_i) < 1 &\Leftrightarrow e^{(-\mu_i+\sigma_i^2)\Delta t} < e^{\rho\Delta t} < e^{\mu_i\Delta t} \\
&\Leftrightarrow (-\mu_i + \sigma_i^2)\Delta t < \rho\Delta t < \mu_i\Delta t \\
&\Leftrightarrow \frac{\mu_i - \rho}{\sigma_i^2} \in (0,1).
\end{aligned}$$

In total, $\pi_k^* \in (0,1)$ iff $\frac{Y_k^T\mu - \rho}{(Y_k^T\sigma)^2} \in (0,1)$. □

Remark 4.2.3. *By Proposition 4.2.2 and the expansion before we see that* $\pi_k^* = 0$ *if* $\frac{Y_k^T\mu-\rho}{(Y_k^T\sigma)^2} \leq 0$, $\pi_k^* = 1$ *if* $\frac{Y_k^T\mu-\rho}{(Y_k^T\sigma)^2} \geq 1$ *and that* $\pi_k^* \approx \frac{\mu_i-\rho}{\sigma_i^2}$ *if* $\frac{\mu_i-\rho}{\sigma_i^2} \in (0,1)$.

Theorem 4.2.4. *For power utility $U(x) = \frac{x^{1-\alpha}}{1-\alpha}, \alpha > 0, \alpha \neq 1$, the value function satisfy $V_N^F(x, e_i) = U(x)$, and for $k = N-1, \dots, 0$, the DPEs*

$$V_k^F(x, e_i) = \sup_{\eta \in [0,1]} \mathbb{E}\Big[\sum_{j=1}^{d} V_{k+1}^F(f(x, e_i, \eta, \varepsilon_{k+1}), e_j) p_{ij} \Big| \mathcal{F}_k^F\Big]$$

where $\varepsilon_{k+1} \sim \mathcal{N}(0,1)$. An optimal strategy is given by $\pi_k^* = \eta^*(Y_k)$, where $\eta^*(e_i)$ is the optimizer in

$$v(e_i) = \begin{cases} \sup_{\eta \in [0,1]} \mathbb{E}[f(1, e_i, \eta, \varepsilon)^{1-\alpha}], & \alpha \in (0,1). \\ \inf_{\eta \in [0,1]} \mathbb{E}[f(1, e_i, \eta, \varepsilon)^{1-\alpha}], & \alpha > 1. \end{cases}$$

where $\varepsilon \sim \mathcal{N}(0,1)$. Further,

$$V_k^F(x, e_i) = \frac{1}{1-\alpha} x^{1-\alpha} d_k(e_i)$$

where, $d_N(e_i) := 1$ and

$$d_k(e_i) := v_k(e_i) \sum_{j=1}^{d} p_{ij} d_{k+1}(e_j).$$

Proof. We have V_k^F, and $V_N^F(x, e_i) = U(x) = \frac{x^{1-\alpha}}{1-\alpha}$.
Suppose $V_k^F(x, e_i) = V_k^F(x, e_i) = \frac{1}{1-\alpha} x^{1-\alpha} d_k(e_i)$ is satisfied for $k+1$, $(k < N)$, then

$$\begin{aligned} V_k^F(x, e_i) &= \sup_{\eta \in [0,1]} \mathbb{E}\Big[\sum_{j=1}^{d} V_{k+1}^F(f(x, e_i, \eta, \varepsilon_{k+1}), e_j) p_{ij} \Big| \mathcal{F}_k^F\Big] \\ &= \sup_{\eta \in [0,1]} \mathbb{E}\Big[\sum_{j=1}^{d} \Big(\frac{1}{1-\alpha} x^{1-\alpha} f(1, \pi, e_i, \varepsilon_{t+1})^{1-\alpha} \cdot d_{k+1}(e_j)\Big) p_{ij}\Big] \\ &= \frac{1}{1-\alpha} x^{1-\alpha} v(e_i) \sum_{j=1}^{d} p_{ij} d_{k+1}(e_j) \\ &= \frac{1}{1-\alpha} x^{1-\alpha} d_k(e_i). \end{aligned}$$

□

4.3 Filtered Markov Decision Processes

We say we have partial information in the Markov switching model if we cannot observe the state Y. Instead, the underlying state is estimated by other related observations, e.g. asset returns and expert opinions. Or we can assume that the Markov chain is unobservable $H \in \{R, E, C\}$, i.e. the investor has partial information that indicated by stock returns or expert opinions or both information. The idea is to enlarge the state space of the problem by adding the filtered probability distribution of state Y_k given the history of observations up to time k. The filtered probability can be computed recursively in different cases. It will be shown that the filtered probability contains exactly the relevant information in order to derive optimal strategies.

Definition 4.3.1. *The Filtered Markov decision model is given by*

$$((\mathcal{X}, \mathcal{Y}), \mathcal{A}, \{\mathcal{A}(x, \hat{y}) : x \in \mathcal{X}, \hat{y} \in \mathcal{Y}\}, Q, G) :$$

- *a Borel set $(\mathcal{X}, \mathcal{Y})$, the dual states,*
- *a Borel set $\mathcal{A}$, the actions or controls,*
- *$\mathcal{A}(x, \hat{y})$, the admissible controls in dual state $(x, \hat{y})$, where we need that $\mathcal{D} = \{((x, \hat{y}), a)|(x, \hat{y}) \in (\mathcal{X}, \mathcal{Y}), a \in \mathcal{A}(x, \hat{y})\}$ is measurable.*
- *a stochastic kernel Q on $(\mathcal{X}, \mathcal{Y})$ given $\mathcal{D}$, i.e. $Q(|((x, \hat{Y}), a))$ is a probability measure for each $((x, \hat{y}), a) \in \mathcal{D}$.*
- *a terminal reward function $G : \mathcal{D} \to \mathbb{R}$ which is measurable.*

Theorem 4.1.6 can be applied to this setting using value functions

$$\begin{aligned} V_k^H(x, \hat{y}) &= \sup_{\pi} \mathbb{E}[U(X_N^\pi)|\mathcal{F}_k^H, X_k^\pi = x, \hat{Y}_k = \hat{y}] \\ &= \sup_{\pi} \mathbb{E}[U(X_N^\pi)|X_k^\pi = x, \hat{Y}_k = \hat{y}] \end{aligned}$$

where π is $\mathcal{F}^H$-admissible. Next, we have two useful lemmas.

Lemma 4.3.2. *For all random variables Z, which are independent of $\mathcal{F}_k^H \vee \mathcal{F}_k^Y$, and measurable functions $f : S \times \mathbb{R} \to \mathbb{R}$, we have*

$$
\begin{aligned}
\mathbb{E}[f(Y_k, Z)|\mathcal{F}_k^H] &= \sum_{i=1}^{d} \mathbb{E}[\mathbb{1}_{\{Y_k=e_i\}} \cdot f(e_i, Z)|\mathcal{F}_k^H] \\
&= \sum_{i=1}^{d} \hat{Y}_k^i \mathbb{E}[f(e_i, Z)]
\end{aligned}
$$

where $\hat{Y}_k^i = \mathbb{E}[Y_k^i|\mathcal{F}_k^H]$, if the expectation exists.

Proof.

$$
\begin{aligned}
\mathbb{E}[f(Y_k, Z)|\mathcal{F}_k^H] &= \sum_{i=1}^{d} \mathbb{E}[\mathbb{1}_{\{Y_k=e_i\}} \cdot f(e_i, Z)|\mathcal{F}_k^H] \\
&= \sum_{i=1}^{d} \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{Y_k=e_i\}} \cdot f(e_i, Z)|\mathcal{F}_k^H \vee \mathcal{F}_k^Y]|\mathcal{F}_k^H] \\
&= \sum_{i=1}^{d} \mathbb{E}[\mathbb{1}_{\{Y_k=e_i\}} \mathbb{E}[f(e_i, Z)|\mathcal{F}_k^H \vee \mathcal{F}_k^Y]|\mathcal{F}_k^H] \\
&= \sum_{i=1}^{d} \mathbb{E}[\mathbb{1}_{\{Y_k=e_i\}} \mathbb{E}[f(e_i, Z)]|\mathcal{F}_k^H] \\
&= \sum_{i=1}^{d} \mathbb{E}[\mathbb{1}_{\{Y_k=e_i\}}|\mathcal{F}_k^H] \mathbb{E}[f(e_i, Z)] \\
&= \sum_{i=1}^{d} \hat{Y}_k^i \mathbb{E}[f(e_i, Z)].
\end{aligned}
$$

The second equation is due to the tower property of conditional expectation, the third equation is because $\mathbb{1}_{\{Y_k=e_i\}}$ is measurable in $\mathcal{F}_k^Y$, the forth one is because of the independence of Z, and the fifth one is because of the fact that $\mathbb{E}[f(e_i, Z)]$ is a constant. □

The lemma is extended to the following form, which we will use in our dynamic programming equations, in case of $H = R$.

Lemma 4.3.3. *For all random variables Z, which are independent of $\mathcal{F}_k^H \vee \mathcal{F}_k^R \vee \mathcal{F}_k^Y$, and measurable functions $h : S \times \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\hat{Y}_k$ and X_k both $\mathcal{F}_k^R$-measurable, we have*

$$\mathbb{E}[h(Y_k, \hat{Y}_k, Z, X_k)|\hat{Y}_k = \hat{y}, X_k = x] = \sum_{i=1}^{d} \hat{y}^i \mathbb{E}[h(e_i, \hat{y}, Z, x)].$$

Proof. by the property of Markov switching model, and tower property of expectation

$$\mathbb{E}[h(Y_k, \hat{Y}_k, Z, X_k)|\hat{Y}_k, X_k] = \mathbb{E}[\sum_{i=1}^{d} \mathbb{1}_{\{Y_k=e_i\}} h(e_i, \hat{Y}_k, Z, X_k)|\hat{Y}_k, X_k]$$
$$= \mathbb{E}\Big[\mathbb{E}[\mathbb{E}[\sum_{i=1}^{d} \mathbb{1}_{\{Y_k=e_i\}} h(e_i, \hat{Y}_k, Z, X_k)|\mathcal{F}_k^H \vee \mathcal{F}_k^R \vee \mathcal{F}_k^Y]|\mathcal{F}_k^H \vee \mathcal{F}_k^R]\Big|\hat{Y}_k, X_k\Big]$$

since $\sigma\{X_k^\pi, \hat{Y}_k\} \subseteq \mathcal{F}_k^H \vee \mathcal{F}_k^R$, moreover $\sum_{i=1}^{d} \mathbb{1}_{\{Y_k=e_i\}}$ is $\mathcal{F}_k^Y$-measurable, $\hat{Y}_k$ and X_k are both $\mathcal{F}_k^R$-measurable, by Lemma 4.3.2

$$\mathbb{E}\Big[\mathbb{E}[\mathbb{E}[\sum_{i=1}^{d} \mathbb{1}_{\{Y_k=e_i\}} h(e_i, \hat{Y}_k, Z, X_k)|\mathcal{F}_k^H \vee \mathcal{F}_k^R \vee \mathcal{F}_k^Y]|\mathcal{F}_k^H \vee \mathcal{F}_k^R]\Big|\hat{Y}_k, X_k\Big]$$
$$= \mathbb{E}\Big[\sum_{i=1}^{d} \hat{Y}_k^i h(e_i, \hat{Y}_k, Z, X_k)\Big|\hat{Y}_k, X_k\Big]$$
$$= \sum_{i=1}^{d} \hat{Y}_k^i h(e_i, \hat{Y}_k, Z, X_k)$$

As a result,

$$\mathbb{E}[h(Y_k, \hat{Y}_k, Z, X_k)|\hat{Y}_k = \hat{y}, X_k = x] = \sum_{i=1}^{d} \hat{y}^i \mathbb{E}[h(e_i, \hat{y}, Z, x)]$$

□

4.3.1 The Case $H = R$

In this section we focus on the case that

$$\mathcal{F}^H = \mathcal{F}^R = \sigma(R_1, \ldots, R_k).$$

In our setting, the transition law is given by $f(x, Y, \pi, Z)$ for x, and $g(\hat{y}, Y, Z)$ for $\hat{y}$, where the explicit formula for f is given by

$$f(x, Y_k, \pi, \varepsilon_{k+1}) = x \sum_{i=1}^{d} \mathbb{1}_{\{Y_k = e_i\}}(1 + r\Delta t + \pi(e^{b_i + a_i \varepsilon_{k+1}} - 1 - r\Delta t)).$$

Note that g has different formulas when $H \in \{R, E, C\}$, for the case $H = R$,

$$g(\hat{y}, Y_k, \varepsilon_{k+1}) = \sum_{i=1}^{d} \mathbb{1}_{\{Y_k = e_i\}} \frac{\Pi^T Diag(\varphi(b_i + a_i\varepsilon_{k+1}))\hat{y}}{\varphi(b_i + a_i\varepsilon_{k+1})^T \hat{y}}.$$

Recall that at $t = k$, $R_{k+1} = \sum_{i=1}^{d} \mathbb{1}_{\{Y_k = e_i\}}(b_i + a_i\varepsilon_{k+1})$, so that the transition law add up all the possibilities of R_{k+1} by indicator functions. Further remember also that $\varphi = (\varphi_1, \ldots, \varphi_d)^T$, where φ_j is the pdf of $\mathcal{N}(b_j, a_j^2)$.

The stochastic kernel Q on $(\mathcal{X}, \mathcal{Y})$ given $\mathcal{D}$, is

$$\begin{aligned} &Q(B_1 \times B_2, (x, \hat{y}, \eta)) = P((X^\pi_{k+1} \in B_1, \hat{Y}_{k+1} \in B_2) | X^\pi_k = x, \hat{Y}_k = \hat{y}, \pi_k = \eta) \\ &= \sum_{i=1}^{d} P((X^\pi_{k+1} \in B_1, \hat{Y}_{k+1} \in B_2) | X^\pi_k = x, \hat{Y}_k = \hat{y}, \pi_k = \eta, Y_k = e_i) P(Y_k = e_i) \end{aligned}$$

Theorem 4.3.4. *If $H = R$, we have for logarithmic utility $U(x) = \log x$, the value function satisfy $V_N(x, \hat{y}) = U(x)$, and for $k = N-1, \ldots, 0$, the DPEs*

$$\begin{aligned} V_k^R(x, \hat{y}) &= \sup_{\eta \in [0,1]} \mathbb{E}\Big[\sum_{i=1}^{d} \mathbb{1}_{\{Y_k = e_i\}} V^R_{k+1}\Big(f(X^\pi_k, e_i, \eta, \varepsilon_{k+1}), g(\hat{Y}_k, e_i, \varepsilon_{k+1})\Big)\Big| X^\pi_k = x, \hat{Y}_k = \hat{y}\Big] \\ &= \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[V^R_{k+1}\Big(f(x, e_i, \eta, \varepsilon_{k+1}), g(\hat{y}, e_i, \varepsilon_{k+1})\Big)\Big] \end{aligned}$$

where $\hat{y} = \mathbb{E}[Y_k | \mathcal{F}^R_k]$, $\varepsilon_{k+1} \in \mathcal{N}(0, 1)$. An optimal strategy is given by $\pi^*_k = \eta^*(\hat{y})$, were $\eta^*(\hat{y})$ is the optimizer in

$$v(\hat{y}) = \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}[\log(1 + r\Delta t + \eta(e^{b_i + a_i\varepsilon_{k+1}} - 1 - r\Delta t))].$$

Further,

$$V_k^R(x, \hat{y}) = \log x + d_k(\hat{y}) \quad (*)$$

where $d_N(\hat{y}) = 0$ *and*

$$d_k(\hat{y}) = v(\hat{y}) + \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}))\Big].$$

Proof. We have $V_N(x) = U(x) = \log x$.
Suppose $(*)$ is satisfied for $k+1, (k < N-1)$, then by Lemma 4.3.3

$$\begin{aligned}
&V_k^R(x, \hat{y}) \\
&= \sup_{\eta \in [0,1]} \mathbb{E}\Big[\sum_{i=1}^{d} \mathbb{1}_{\{Y_k = e_i\}} V_{k+1}^R\Big(f(x, e_i, \eta, \varepsilon_{k+1}), g(\hat{y}, e_i, \varepsilon_{k+1})\Big)\Big| X_k^\pi = x, \hat{Y}_k = \hat{y}\Big] \\
&= \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[V_{k+1}^R\Big(f(x, e_i, \eta, \varepsilon_{k+1}), g(\hat{y}, e_i, \varepsilon_{k+1})\Big)\Big] \\
&= \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[\log(f(x, e_i, \eta, \varepsilon_{k+1})) + d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}))\Big] \\
&= \log x + \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[\Big(\log(f(1, e_i, \eta, \varepsilon_{k+1}))\Big)\Big] + \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}))\Big]
\end{aligned}$$

□

Moreover, $\eta^*(\hat{y})$ in Theorem 4.3.4 can be derived by

$$\eta^*(\hat{y}) = \arg \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}[\log((1 + r\Delta t + \eta(e^{b_i + a_i \varepsilon_{k+1}} - 1 - r\Delta t)))].$$

Remark 4.3.5. *When the utility function is logarithmic, one can derive a optimal strategy directly from the estimator of state* $\hat{y}$ *at current time* k *by (4.3.1) regardless of* d_{k+1}*. We can get an approximate strategy. Applying second order of Taylor expansion for* $\log(1+x)$ *and* e^x *around 0, we get of order* Δt

$$\pi_k^{log}(\hat{y}) = \frac{\sum_{i=1}^{d} \hat{y}^i \mu_i - r}{\sum_{i=1}^{d} \hat{y}^i \sigma_i^2}.$$

Theorem 4.3.6. *If* $H = R$*, we have for power utility* $U(x) = \frac{x^{1-\alpha}}{1-\alpha}$*,* $\alpha \neq 1$ *and* $\alpha > 0$ *that the value function satisfy* $V_N(x, \hat{y}) = U(x)$*, and for* $k =$

$N-1,\dots,0$, the DPEs

$$\begin{aligned}
V_k^R(x,\hat{y}) &= \sup_{\eta\in[0,1]} \mathbb{E}\Big[\sum_{i=1}^{d} \mathbb{1}_{\{Y_k=e_i\}} V_{k+1}^R\Big(f(x,e_i,\eta,\varepsilon_{k+1}), g(\hat{y},e_i,\varepsilon_{k+1})\Big)\Big|\mathcal{F}_k^R\Big] \\
&= \sup_{\eta\in[0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[V_{k+1}^R\Big(f(x,e_i,\eta,\varepsilon_{k+1}), g(\hat{y},e_i,\varepsilon_{k+1})\Big)\Big]
\end{aligned}$$

where $\hat{y} = \mathbb{E}[Y_k|\mathcal{F}_k^R]$, $\varepsilon_{k+1} \in \mathcal{N}(0,1)$. An optimal strategy is given by $\pi_k^* = \eta^*(\hat{y})$, were $\eta^*(\hat{y})$ is the optimizer in

$$d_k(\hat{y}) = \begin{cases} \sup_{\eta\in[0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[(f(1,e_i,\eta,\varepsilon_{k+1}))^{1-\alpha} d_{k+1}(g(\hat{y},e_i,\varepsilon_{k+1}))\Big] & \alpha\in(0,1) \\ \inf_{\eta\in[0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[(f(1,e_i,\eta,\varepsilon_{k+1}))^{1-\alpha} d_{k+1}(g(\hat{y},e_i,\varepsilon_{k+1}))\Big] & \alpha>1 \end{cases}$$

where $d_N(\hat{y}) = 1$. Further,

$$V_k^R(x,\hat{y}) = \frac{x^{1-\alpha}}{1-\alpha} d_k(\hat{y}) \quad (*)$$

Proof. For the case $H=R$, the value function satisfies $V_N(x,\hat{y}) = U(x)$, and for $k = N-1,\dots,0$, the DPEs

$$\begin{aligned}
V_k^R(x,\hat{y}) &= \sup_{\eta\in[0,1]} \mathbb{E}\Big[\sum_{i=1}^{d} \mathbb{1}_{\{Y_k=e_i\}} V_{k+1}^R\Big(f(x,e_i,\eta,\varepsilon_{k+1}), g(\hat{y},e_i,\varepsilon_{k+1})\Big)\Big|\mathcal{F}_k^R\Big] \\
&= \sup_{\eta\in[0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[V_{k+1}^R\Big(f(x,e_i,\eta,\varepsilon_{k+1}), g(\hat{y},e_i,\varepsilon_{k+1})\Big)\Big|\mathcal{F}_k^R\Big]
\end{aligned}$$

One may have the ansatz that $(*)$ satisfy for $k+1, k<N-1$

$$V_k^R(x,\hat{y}) = \frac{x^{1-\alpha}}{1-\alpha} d_k(\hat{y}) \quad (*)$$

where $d_N(\hat{y}) = 1$ and w.l.o.g. assume $\alpha\in(0,1)$ (for $\alpha>1$ take the infimum)

$$d_k(\hat{y}) = \sup_{\eta\in[0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[(f(1,e_i,\eta,\varepsilon_{k+1}))^{1-\alpha} d_{k+1}(g(\hat{y},e_i,\varepsilon_{k+1}))\Big]$$

Thus, by Lemma 4.3.3

$$\begin{aligned}
V_k^R(x,\hat{y}) &= \sup_{\eta\in[0,1]} \mathbb{E}\Big[\sum_{i=1}^{d} \mathbb{1}_{\{Y_k=e_i\}} V_{k+1}^R\Big(f(x,e_i,\eta,\varepsilon_{k+1}), g(\hat{y},e_i,\varepsilon_{k+1})\Big)\Big|\mathcal{F}_k^R\Big] \\
&= \sup_{\eta\in[0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[V_{k+1}^R\Big(f(x,e_i,\eta,\varepsilon_{k+1}), g(\hat{y},e_i,\varepsilon_{k+1})\Big)\Big] \\
&= \sup_{\eta\in[0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[\frac{1}{1-\alpha}(f(x,e_i,\eta,\varepsilon_{k+1}))^{1-\alpha} d_{k+1}(g(\hat{y},e_i,\varepsilon_{k+1}))\Big] \\
&= \frac{x^{1-\alpha}}{1-\alpha} \sup_{\eta\in[0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[(f(1,e_i,\eta,\varepsilon_{k+1}))^{1-\alpha} d_{k+1}(g(\hat{y},e_i,\varepsilon_{k+1}))\Big] \\
&= \frac{x^{1-\alpha}}{1-\alpha} d_k(\hat{y})
\end{aligned}$$

The optimal strategy $\eta^*(\hat{y})$ can be derived by

$$\eta_k^*(\hat{y}) = \arg\sup_{\eta\in[0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[(f(1,e_i,\eta,\varepsilon_{k+1}))^{1-\alpha} d_{k+1}(g(\hat{y},e_i,\varepsilon_{k+1}))\Big].$$

When $\alpha > 1$

$$\eta_k^*(\hat{y}) = \arg\inf_{\eta\in[0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[(f(1,e_i,\eta,\varepsilon_{k+1}))^{1-\alpha} d_{k+1}(g(\hat{y},e_i,\varepsilon_{k+1}))\Big].$$

$\square$

Note that when the utility function is power function, we need to know the real-valued function d_{k+1} in order to derive the optimal strategy at time k. One can use Monte Carlo methods. [TZ07] mainly discussed the accuracy of value function if we approximate the power function by second order Taylor expansion. And showed some numerical results when d_k keeps constant.

Proposition 4.3.7. *At each time $k = 1,\ldots,N-1$, w.l.o.g. assuming $\alpha \in (0,1)$ we have*

$$\sup_{\hat{y}} d_k(g(\hat{y},e_i,\varepsilon_k)) = 1 + O(\Delta t) \text{ for all } i = 1\ldots,d.$$

and the optimal strategy $\eta_k^*(\hat{y})$ can be of order Δt approximated by

$$\eta_k^*(\hat{y}) \approx \frac{1}{\alpha}\frac{\sum_{i=1}^d \hat{y}^i\mu_i - r}{\sum_{i=1}^d \hat{y}^i\sigma_i^2} \vee 0 \wedge 1.$$

Proof. We prove this by induction. At time $k = N-1$

$$d_{N-1}(\hat{y}) = \sup_{\eta\in[0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[(f(1, e_i, \eta, \varepsilon_{k+1}))^{1-\alpha}\Big]$$

$$= \sup_{\eta\in[0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[1 + (1-\alpha)\Big(r\Delta t + \eta((\mu_i - r + \frac{\varepsilon_{k+1}^2 - 1}{2}\sigma_i^2)\Delta t + \sigma_i\sqrt{\Delta t}\varepsilon_{k+1})\Big)$$

$$+ \frac{(1-\alpha)(-\alpha)}{2}\eta^2\sigma_i^2\Delta t\varepsilon_{k+1}^2 + O(\Delta t^2)\Big]$$

The maximization yields the approximation of $\eta_{N-1}^*(\hat{y})$, and we get

$$d_{N-1}(\hat{y}) = 1 + O(\Delta t).$$

Then we compute iteratively. Suppose at time $k+1, \ldots, N-1$, we have $\sup_{\hat{y}} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1})) = 1 + O(\Delta t)$, we have

$$d_k(\hat{y}) = \sup_{\eta\in[0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[(f(1, e_i, \eta, \varepsilon_{k+1}))^{1-\alpha} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}))\Big]$$

$$= \sup_{\eta\in[0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[\Big(1 + (1-\alpha)\Big(r\Delta t + \eta((\mu_i - r + \frac{\varepsilon_{k+1}^2 - 1}{2}\sigma_i^2)\Delta t + \sigma_i\sqrt{\Delta t}\varepsilon_{k+1})\Big)$$

$$+ \frac{(1-\alpha)(-\alpha)}{2}\eta^2\sigma_i^2\Delta t\varepsilon_{k+1}^2 + O(\Delta t^2)\Big)(1 + O(\Delta t))\Big]$$

We can have an optimal strategy of order Δt by

$$\eta_k^*(\hat{y}) = \frac{1}{\alpha}\frac{\sum_{i=1}^d \hat{y}^i\mu_i - r}{\sum_{i=1}^d \hat{y}^i\sigma_i^2} \vee 0 \wedge 1.$$

Substituting this optimal strategy, we get the expression for d_k, and

$$d_k = 1 + O(\Delta t)$$

□

4.3.2 The Case $H = E$

In this section, the scenario, $H = E$, is discussed. In this case, one estimates the state by expert opinions only. Expert opinions are assumed that they follow the Dirichlet distribution as

$$E_k \sim \mathcal{D}(\gamma^T Y_{k-1}) \tag{4.2}$$

where $\gamma \in \mathbb{R}^{d\times d}$ is a parameter matrix. The $i-$th row of γ is denoted by the $\gamma^{(i)}$.

Here, we discussed the model when E lags Y, i.e. Experts Opinions are related to the state at the previous time point. Hence we still need to use transition matrix to estimate the state now.

Remark 4.3.8. *(4.2) illustrates E_k can be also conditional on Y_k, while in Theorem 4.3.13 f_i^E represents the density function conditional on Y_{k-1}. This is no conflict, since there is the relationship that*

$$P(E_k \in \cdot | Y_{k-1} = e_i) = \sum_{i=1}^{d} P(E_k \in \cdot | Y_k = e_j) p_{ij}.$$

Compared to the case $H = R$, the stochastic kernel and transition law is different in Definition 4.3.1. In this case, the transition law $f(x, Y, \pi, Z)$ does not vary. While g has different formulas when $H = E$.

Lemma 4.3.9. *The recursive scheme for $\hat{Y}_k^i$ is*

$$\hat{Y}_k^i = \frac{\sum_{j=1}^d p_{ji} f_j^E(E_k) \hat{Y}_{k-1}^j}{\sum_{z=1}^d f_z^E(E_k) \hat{Y}_{k-1}^z},$$

where $\hat{Y}_k = \mathbb{E}[Y_k | \mathcal{F}_k^E]$, $f_j^E = f_{\gamma^{(j)}}^{\mathcal{D}}$ is a density function of Dirichlet distribution parameterized by $\gamma^{(j)}$. To vectorize, we get

$$\hat{Y}_k = \frac{\Pi^T Diag(f^E(E_k)) \hat{Y}_{k-1}}{f^E(E_k)^T \hat{Y}_{k-1}},$$

where $f^E(e) = (f_1^E(e), \dots, f_d^E(e))^T$ and $Diag((f^E(e))$ is the diagonal matrix with diagonal $f^E(e)$.

Proof.

$$
\begin{aligned}
\mathbb{E}[Y_k^i|\mathcal{F}_k^E] &= P(Y_k = e_i|\mathcal{F}_k^E) \\
&= \sum_{j=1}^{d} P(Y_k = e_i|Y_{k-1} = e_j, \mathcal{F}_k^E)P(Y_{k-1} = e_j|\mathcal{F}_k^E) \\
&= \sum_{j=1}^{d} p_{ji}\frac{P(E_k|Y_{k-1} = e_j)P(Y_{k-1} = e_j|\mathcal{F}_{k-1}^E)}{\sum_{z=1}^{d} P(E_k|Y_{k-1} = e_z)P(Y_{k-1} = e_z|\mathcal{F}_{k-1}^E)} \\
&= \frac{\sum_{j=1}^{d} p_{ji}f_j^E(E_k)\hat{Y}_{k-1}^j}{\sum_{z=1}^{d} f_z^E(E_k)\hat{Y}_{k-1}^z}
\end{aligned}
$$

□

This lemma gives us the formula for $g(\hat{Y}, E)$. We use it to derive our transition function for g. Denote E_{k+1}^i is the lag expert opinion conditional on $Y_k = e_i$. Then we have

$$
\hat{Y}_{k+1} = g(\hat{Y}_k, Y_k, \tilde{E}_{k+1}) = \sum_{i=1}^{d} \mathbb{1}_{\{Y_k=e_i\}}\frac{\Pi^T Diag(f^E(\tilde{E}_{k+1}^i))\hat{Y}_k}{f^E(\tilde{E}_{k+1}^i)^T\hat{Y}_k}.
$$

Therefore, the stochastic kernel Q on $(\mathcal{X}, \mathcal{Y})$ is given $\mathcal{D}$:

$$
\begin{aligned}
&Q(B_1 \times B_2, (x, \hat{y}, \eta)) = P((X_{k+1}^\pi \in B_1, \hat{Y}_{k+1} \in B_2)|X_k^\pi = x, \hat{Y}_k = \hat{y}, \pi_k = \eta) \\
&= \sum_{i=1}^{d} P((X_{k+1}^\pi \in B_1, \hat{Y}_{k+1} \in B_2)|X_k^\pi = x, \hat{Y}_k = \hat{y}, \pi_k = \eta, Y_k = e_i)P(Y_k = e_i) \\
&= \sum_{i=1}^{d} P(X_{k+1}^\pi \in B_1|X_k^\pi = x, \pi_k = \eta, Y_k = e_i)P(\hat{Y}_{k+1} \in B_2|\hat{Y}_k = \hat{y}, Y_k = e_i)P(Y_k = e_i) \\
&= \sum_{i=1}^{d} \int_{\mathcal{E}} P(f(x, e_i, \eta, \varepsilon_{k+1}) \in B_1)P(g(\hat{y}, e_i, \tilde{e}) \in B_2)dP(E_{k+1} = \tilde{e}|Y_k = e_i)P(Y_k = e_i).
\end{aligned}
$$

Theorem 4.3.10. *If $H = E$, we have for logarithmic utility $U(x) = \log x$, the value function $V_N^E(x, \hat{y}) = U(x)$, and for $k = N-1, \dots, 0$, the DPEs*

$$
V_k^E(x, \hat{y}) = \sup_{\eta\in[0,1]} \sum_{i=1}^{d} \int_{\mathcal{E}} \hat{y}^i \mathbb{E}\Big[V_{k+1}^E\Big(f(x, e_i, \eta, \varepsilon_{k+1}), g(\hat{y}, e_i, \tilde{e})\Big)\Big] f_i^E(\tilde{e})d\tilde{e}
$$

where $\hat{y} = \mathbb{E}[Y_k|\mathcal{F}_k^E]$*,* $\tilde{e} \in \mathcal{E}$*.* f_i^E *is density function conditional on* $Y_k = e_i$*,* $\varepsilon_{k+1} \in \mathcal{N}(0,1)$*. An optimal strategy is given by* $\pi_k^* = \eta^*(\hat{y})$*, were* $\eta^*(\hat{y})$ *is the optimizer in*

$$v(\hat{y}) = \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}[\log(1 + r\Delta t + \eta(e^{b_i + a_i \varepsilon_{k+1}} - 1 - r\Delta t))]$$

Further,

$$V_k^E(x, \hat{y}) = \log x + d_k(\hat{y}) \quad (*)$$

where $d_N(\hat{y}) = 0$ *and*

$$d_k(\hat{y}) = v(\hat{y}) + \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[d_{k+1}(g(\hat{y}, e_i, \tilde{E}_{k+1}))\Big].$$

Proof. We have $V_N^E(x) = U(x) = \log x$.
Suppose $(*)$ is satisfied for $k+1, (k < N-1)$, then by Lemma 4.3.3 and independence of ε_{k+1} and $\tilde{E}$

$$V_k^E(x, \hat{y}) = \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \int_{\mathcal{E}} \mathbb{E}\Big[V_{k+1}^E\Big(f(x, e_i, \eta, \varepsilon_{k+1}), g(\hat{y}, e_i, \tilde{e})\Big)\Big] f_i^E(\tilde{e}) d\tilde{e}$$

$$= \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \int_{\mathcal{E}} \mathbb{E}\Big[\log(f(x, e_i, \eta, \varepsilon_{k+1})) + d_{k+1}(g(\hat{y}, e_i, \tilde{e}))\Big] f_i^E(\tilde{e}) d\tilde{e}$$

$$= \log x + \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[\Big(\log(f(1, e_i, \eta, \varepsilon_{k+1}))\Big)\Big] + \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[d_{k+1}(g(\hat{y}, e_i, \tilde{E}))\Big].$$

□

Moreover, $\eta^*(\hat{y})$ can be derived by

$$\eta^*(\hat{y}) = \arg \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}[\log((1 + r\Delta t + \eta(e^{b_i + a_i \varepsilon_{k+1}} - 1 - r\Delta t)))]. \quad (4.3)$$

Note that when the utility function is logarithmic, one can derive an optimal strategy directly from the estimator of state $\hat{y}$ at current time k by (4.3) regardless of d_{k+1}.

Theorem 4.3.11. *If $H = E$, we have for power utility $U(x) = \frac{x^{1-\alpha}}{1-\alpha}$, $\alpha \neq 1$ and $\alpha > 0$ that the value function satisfy $V_N(x, \hat{y}) = U(x)$, and for $k = N-1, \dots, 0$, the DPEs*

$$V_k^E(x, \hat{y}) = \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \int_{\mathcal{E}} \mathbb{E}\Big[V_{k+1}^E\Big(f(x, e_i, \eta, \varepsilon_{k+1}), g(\hat{y}, e_i, \tilde{e})\Big)\Big] f_i^E(\tilde{e}) d\tilde{e}$$

where $\hat{y} = \mathbb{E}[Y_k|\mathcal{F}_k^E]$, $\tilde{e} \in \mathcal{E}$. f_i^E is density function conditional on $Y_k = e_i$, $\varepsilon_{k+1} \in \mathcal{N}(0, 1)$. An optimal strategy is given by $\pi_k^* = \eta^*(\hat{y})$, were $\eta^*(\hat{y})$ is the optimizer in

$$d_k(\hat{y}) = \begin{cases} \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha}\Big] \mathbb{E}\Big[d_{k+1}(g(\hat{y}, e_i, \tilde{E}))\Big] & \alpha \in (0, 1) \\ \inf_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha}\Big] \mathbb{E}\Big[d_{k+1}(g(\hat{y}, e_i, \tilde{E}))\Big] & \alpha > 1 \end{cases}$$

where $d_N(\hat{y}) = 1$. Further we have

$$V_k^R(x, \hat{y}) = \frac{x^{1-\alpha}}{1-\alpha} d_k(\hat{y}). \quad (*)$$

Proof. For the case $H = E$, the value function satisfy $V_N^E(x, \hat{y}) = U(x)$. W.o.l.g. we assume $\alpha \in (0, 1)$, suppose $(*)$ is satisfied for $k+1$, $(k < N-1)$, then by Lemma 4.3.3 and the independence of ε_{k+1} and $\tilde{E}$, the DPEs

$$\begin{aligned} V_k^E(x, \hat{y}) &= \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \int_{\mathcal{E}} \mathbb{E}\Big[V_{k+1}^E\Big(f(x, e_i, \eta, \varepsilon_{k+1}), g(\hat{y}, e_i, \tilde{e})\Big)\Big] f_i^E(\tilde{e}) d\tilde{e} \\ &= \frac{x^{1-\alpha}}{1-\alpha} \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \int_{\mathcal{E}} \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha} d_{k+1}(g(\hat{y}, e_i, \tilde{e}))\Big] f_i^E(\tilde{e}) d\tilde{e} \\ &= \frac{x^{1-\alpha}}{1-\alpha} \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha}\Big] \mathbb{E}\Big[d_{k+1}(g(\hat{y}, e_i, \tilde{E}))\Big] \\ &= \frac{x^{1-\alpha}}{1-\alpha} d_k(\hat{y}). \end{aligned}$$

□

The optimal strategy $\eta^*(\hat{y})$ can be derived by

$$\eta_k^*(\hat{y}) = \arg \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha}\Big] \mathbb{E}\Big[d_{k+1}(g(\hat{y}, e_i, \tilde{E}))\Big]$$

when $\alpha > 1$

$$\eta_k^*(\hat{y}) = \arg \inf_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha}\Big]\mathbb{E}\Big[d_{k+1}(g(\hat{y}, e_i, \tilde{E}))\Big].$$

Proposition 4.3.12. *At each time $k = 1, \dots, N-1$, we have*

$$\sup_{\hat{y}} d_k(g(\hat{y}, e_i, \varepsilon_k)) = 1 + O(\Delta t) \textit{ for all } i = 1 \dots, d$$

and the optimal strategy $\eta^*(\hat{y})$ can be of order Δt approximated by

$$\eta_k^*(\hat{y}) \approx \frac{1}{\alpha} \frac{\sum_{i=1}^{d} \hat{y}^i \mu_i - r}{\sum_{i=1}^{d} \hat{y}^i \sigma_i^2} \vee 0 \wedge 1.$$

Proof. We prove this by induction. W.o.l.g. we assume $\alpha \in (0, 1)$.At time $k = N - 1$

$$\begin{aligned} d_{N-1}(\hat{y}) &= \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[(f(1, e_i, \eta, \varepsilon_{k+1}))^{1-\alpha}\Big] \\ &= \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[1 + (1-\alpha)\Big(r\Delta t + \eta((\mu_i - r + \frac{\varepsilon_{k+1}^2 - 1}{2}\sigma_i^2)\Delta t + \sigma_i \sqrt{\Delta t}\varepsilon_{k+1})\Big) \\ &+ \frac{(1-\alpha)(-\alpha)}{2}\eta^2 \sigma_i^2 \Delta t \varepsilon_{k+1}^2 + O(\Delta t^2)\Big] \end{aligned}$$

Substituting the approximate strategy for $\eta_{N-1}^*(\hat{y})$, could maintain the first part, we get

$$d_{N-1}(\hat{y}) = 1 + O(\Delta t).$$

Then we compute iteratively. Suppose at time $k + 1, \dots, N - 1$, we have $\sup_{\hat{y}} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1})) = 1 + O(\Delta t)$, we have

$$\begin{aligned} d_k(\hat{y}) &= \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[(f(1, e_i, \eta, \varepsilon_{k+1}))^{1-\alpha} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}))\Big] \\ &= \sup_{\eta \in [0,1]} \sum_{i=1}^{d} \hat{y}^i \mathbb{E}\Big[\Big(1 + (1-\alpha)\Big(r\Delta t + \eta((\mu_i - r + \frac{\varepsilon_{k+1}^2 - 1}{2}\sigma_i^2)\Delta t + \sigma_i \sqrt{\Delta t}\varepsilon_{k+1})\Big) \\ &+ \frac{(1-\alpha)(-\alpha)}{2}\eta^2 \sigma_i^2 \Delta t \varepsilon_{k+1}^2 + O(\Delta t^2)\Big)\Big]\mathbb{E}[1 + O(\Delta t)] \end{aligned}$$

We can have an optimal strategy of order Δt by

$$\eta_k^*(\hat{y}) = \frac{1}{\alpha}\frac{\sum_{i=1}^d \hat{y}^i \mu_i - r}{\sum_{i=1}^d \hat{y}^i \sigma_i^2} \vee 0 \wedge 1.$$

Substituting this, we get the expression for d_k and

$$d_k = 1 + O(\Delta t).$$

□

4.3.3 The Case $H = C$

In this section, the scenario, $H = C$, is discussed. In this case, we estimate the state by both stock returns and expert opinions.
Expert opinions are modeled same as in the case $H = E$ in Section 4.3.2. Compared to the case $H = R$ or $H = E$, the transition law $f(x, Y, \pi, Z)$ does not vary while g has different form. From Theorem 3.4.12 we know

$$\hat{Y}_{k+1} = g(\hat{Y}_k, Y_k, \varepsilon_{k+1}, \tilde{E}_{k+1}) = \sum_{i=1}^d \mathbb{1}_{\{Y_k = e_i\}} \frac{\Pi^T \Big(Diag(\varphi(b_i + a_i\varepsilon_{k+1}) \odot f^E(\tilde{E}^i_{k+1}))\Big)\hat{Y}_k}{(Diag(\varphi(b_i + a_i\varepsilon_{k+1}))f^E(\tilde{E}^i_{k+1}))^T \hat{Y}_k},$$

where $\hat{Y}_k = \mathbb{E}[Y_k|\mathcal{F}_k^C]$, $f_j^E = f^{\mathcal{D}}_{\alpha^{(j)}}$ is a density function of Dirichlet distribution parameterized by $\alpha^{(j)}$. $f^E(e) = (f_1^E(e), \ldots, f_d^E(e))^T$ and $Diag((f^E(e))$ is the diagonal matrix with diagonal $f^E(e)$.

Therefore, the stochastic kernel Q on $(\mathcal{X}, \mathcal{Y})$ is given $\mathcal{D}$:

$$\begin{aligned} Q(B_1 \times B_2, (x, \hat{y}, \eta)) &= P((X^\pi_{k+1} \in B_1, \hat{Y}_{k+1} \in B_2)|X_k^\pi = x, \hat{Y}_k = \hat{y}, \pi_k = \eta) \\ &= \sum_{i=1}^d P((X^\pi_{k+1} \in B_1, \hat{Y}_{k+1} \in B_2)|X_k^\pi = x, \hat{Y}_k = \hat{y}, \pi_k = \eta, Y_k = e_i)P(Y_k = e_i). \end{aligned}$$

Theorem 4.3.13. *If $H = C$, we have for logarithmic utility $U(x) = \log x$, the value function satisfy $V_N^C(x, \hat{y}) = U(x)$, and for $k = N-1, \ldots, 0$, the DPEs*

$$V_k^C(x, \hat{y}) = \sup_{\eta \in [0,1]} \sum_{i=1}^d \hat{y}^i \int_{\mathcal{E}} \mathbb{E}\Big[V_{k+1}^C\Big(f(x, e_i, \eta, \varepsilon_{k+1}), g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{e})\Big)\Big] f_i^E(\tilde{e}) d\tilde{e}$$

where $\hat{y} = \mathbb{E}[Y_k|\mathcal{F}_k^C]$, $\tilde{e} \in \mathcal{E}$. f_i^E *is density function conditional on* $Y_k = e_i$, $\varepsilon_{k+1} \in \mathcal{N}(0,1)$. *An optimal strategy is given by* $\pi_k^* = \eta^*(\hat{y})$, *were* $\eta^*(\hat{y})$ *is the optimizer in*

$$v(\hat{y}) = \sup_{\eta \in [0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}[\log(1 + r\Delta t + \eta(e^{b_i + a_i \varepsilon_{k+1}} - 1 - r\Delta t))]$$

Further,

$$V_k^C(x, \hat{y}) = \log x + d_k(\hat{y}), \quad (*)$$

where $d_N(\hat{y}) = 0$ *and*

$$d_k(\hat{y}) = v(\hat{y}) + \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{E}_{k+1}))\Big].$$

Proof. We form of V_k^C, and $V_N^C(x) = U(x) = \log x$.
Suppose $(*)$ satisfy for $k+1, (k < N-1)$, then by Lemma 4.3.3 and independence

$$V_k^C(x, \hat{y}) = \sup_{\eta \in [0,1]} \sum_{i=1}^d \hat{y}^i \int_{\mathcal{E}} \mathbb{E}\Big[V_{k+1}^E\Big(f(x, e_i, \eta, \varepsilon_{k+1}), g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{e})\Big)\Big] f_i^E(\tilde{e}) d\tilde{e}$$

$$= \sup_{\eta \in [0,1]} \sum_{i=1}^d \hat{y}^i \int_{\mathcal{E}} \mathbb{E}\Big[\log(f(x, e_i, \eta, \varepsilon_{k+1})) + d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{e}))\Big] f_i^E(\tilde{e}) d\tilde{e}$$

$$= \log x + \sup_{\eta \in [0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[\Big(\log(f(1, e_i, \eta, \varepsilon_{k+1}))\Big)\Big] + \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{E}_{k+1}))\Big]$$

□

Moreover, $\eta^*(\hat{y})$ can be derived by

$$\eta^*(\hat{y}) = \arg \sup_{\eta \in [0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}[\log((1 + r\Delta t + \eta(e^{b_i + a_i \varepsilon_{k+1}} - 1 - r\Delta t)))].$$

Theorem 4.3.14. *If* $H = C$, *we have for power utility* $U(x) = \frac{x^{1-\alpha}}{1-\alpha}$, $\alpha \neq 1$ *and* $\alpha > 0$ *the value function satisfies* $V_N^C(x, \hat{y}) = U(x)$, *and for* $k = N-1, \ldots, 0$, *the DPEs*

$$V_k^C(x, \hat{y}) = \sup_{\eta \in [0,1]} \sum_{i=1}^d \hat{y}^i \int_{\mathcal{E}} \mathbb{E}\Big[V_{k+1}^C\Big(f(x, e_i, \eta, \varepsilon_{k+1}), g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{e})\Big)\Big] f_i^E(\tilde{e}) d\tilde{e}$$

where $\hat{y} = \mathbb{E}[Y_k|\mathcal{F}_k^C]$, $\tilde{e} \in \mathcal{E}$. f_i^E *is density function conditional on* $Y_k = e_i$, $\varepsilon_{k+1} \in \mathcal{N}(0,1)$. *An optimal strategy is given by* $\pi_k^* = \eta^*(\hat{y})$, *were* $\eta^*(\hat{y})$ *is the optimizer in*

$$d_k(\hat{y}) = \begin{cases} \sup_{\eta\in[0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{E}_{k+1}))\Big] & \alpha \in (0,1), \\ \inf_{\eta\in[0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{E}_{k+1}))\Big] & \alpha > 1, \end{cases}$$

where $d_N(\hat{y}) = 1$. *Further,*

$$V_k^R(x, \hat{y}) = \frac{x^{1-\alpha}}{1-\alpha} d_k(\hat{y}). \quad (*)$$

Proof. For the case $H = C$, the value function satisfies $V_N^C(x, \hat{y}) = U(x)$. W.o.l.g. we assume $\alpha \in (0,1)$. Suppose $(*)$ is satisfied for $k+1, (k < N-1)$, then by Lemma 4.3.3, the DPEs

$$\begin{aligned} V_k^C(x, \hat{y}) &= \sup_{\eta\in[0,1]} \sum_{i=1}^d \hat{y}^i \int_{\mathcal{E}} \mathbb{E}\Big[V_{k+1}^C\Big(f(x, e_i, \eta, \varepsilon_{k+1}), g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{e})\Big)\Big] f_i^E(\tilde{e}) d\tilde{e} \\ &= \frac{x^{1-\alpha}}{1-\alpha} \sup_{\eta\in[0,1]} \sum_{i=1}^d \hat{y}^i \int_{\mathcal{E}} \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{e}))\Big] f_i^E(\tilde{e}) d\tilde{e} \\ &= \frac{x^{1-\alpha}}{1-\alpha} \sup_{\eta\in[0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{E}_{k+1}))\Big] \\ &= \frac{x^{1-\alpha}}{1-\alpha} d_k(\hat{y}). \end{aligned}$$

□

The optimal strategy $\eta^*(\hat{y})$ can be derived by

$$\eta_k^*(\hat{y}) = \arg \sup_{\eta\in[0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{E}_{k+1}))\Big].$$

When $\alpha > 1$

$$\eta_k^*(\hat{y}) = \arg \inf_{\eta\in[0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[f(1, e_i, \eta, \varepsilon_{k+1})^{1-\alpha} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{E}_{k+1}))\Big]$$

Proposition 4.3.15. *At each time $k = 1, \dots, N-1$, we have*

$$\sup_{\hat{y}} d_k(g(\hat{y}, e_i, \varepsilon_k)) = 1 + O(\Delta t) \text{ for all } i = 1 \dots, d,$$

and the optimal strategy $\eta^*(\hat{y})$ can be of order Δt approximated by

$$\eta_k^*(\hat{y}) \approx \frac{1}{\alpha} \frac{\sum_{i=1}^d \hat{y}^i \mu_i - r}{\sum_{i=1}^d \hat{y}^i \sigma_i^2} \vee 0 \wedge 1.$$

Proof. We prove this by induction. W.o.l.g, we assume $\alpha \in (0, 1)$. At time $k = N-1$

$$\begin{aligned}
d_{N-1}(\hat{y}) &= \sup_{\eta \in [0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[(f(1, e_i, \eta, \varepsilon_{k+1}))^{1-\alpha}\Big] \\
&= \sup_{\eta \in [0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[1 + (1-\alpha)\Big(r\Delta t + \eta((\mu_i - r + \frac{\varepsilon_{k+1}^2 - 1}{2}\sigma_i^2)\Delta t + \sigma_i \sqrt{\Delta t}\varepsilon_{k+1})\Big) \\
&+ \frac{(1-\alpha)(-\alpha)}{2}\eta^2 \sigma_i^2 \Delta t \varepsilon_{k+1}^2 + O(\Delta t^2)\Big]
\end{aligned}$$

Maximizing the first part, we get

$$d_{N-1}(\hat{y}) = 1 + O(\Delta t)$$

and approximate $\eta_{N-1}^*(\hat{y})$. Then we compute iteratively. Suppose at time $k+1, \dots, N-1$, we have $\sup_{\hat{y}} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1})) = 1 + O(\Delta t)$, we have

$$\begin{aligned}
d_k(\hat{y}) &= \sup_{\eta \in [0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[(f(1, e_i, \eta, \varepsilon_{k+1}))^{1-\alpha} d_{k+1}(g(\hat{y}, e_i, \varepsilon_{k+1}, \tilde{E}_{k+1}))\Big] \\
&= \sup_{\eta \in [0,1]} \sum_{i=1}^d \hat{y}^i \mathbb{E}\Big[\Big(1 + (1-\alpha)\Big(r\Delta t + \eta((\mu_i - r + \frac{\varepsilon_{k+1}^2 - 1}{2}\sigma_i^2)\Delta t + \sigma_i \sqrt{\Delta t}\varepsilon_{k+1})\Big) \\
&+ \frac{(1-\alpha)(-\alpha)}{2}\eta^2 \sigma_i^2 \Delta t \varepsilon_{k+1}^2 + O(\Delta t^2)\Big)(1 + O(\Delta t))\Big]
\end{aligned}$$

We can have an optimal strategy of order Δt by

$$\eta_k^*(\hat{y}) = \frac{1}{\alpha} \frac{\sum_{i=1}^d \hat{y}^i \mu_i - r}{\sum_{i=1}^d \hat{y}^i \sigma_i^2} \vee 0 \wedge 1.$$

Substituting this optimal strategy, we get the expression for d_k, and

$$d_k = 1 + O(\Delta t).$$

□

4.3.4 Utility Assignment for Power Utility Function

In terms of logarithmic utility function, one need to solve

$$\begin{aligned}
\pi &= \arg\sup_{\pi\in\mathcal{A}} \mathbb{E}[\log(X_N^\pi)|X_0^\pi = x_0] \\
&= \arg\sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[\log\Big(x_0 \prod_{i=1}^{N}\Big(1 + r\Delta t + \pi_{i-1}(e^{R_i} - 1 - r\Delta t)\Big)\Big)\Big] \\
&= \arg\sup_{\pi\in\mathcal{A}} \sum_{i=1}^{N} \mathbb{E}\Big[\log\Big(1 + r\Delta t + \pi_{i-1}(e^{R_i} - 1 - r\Delta t)\Big)\Big] \\
&= \sum_{i=1}^{N} \arg\sup_{\pi_{i-1}} \mathbb{E}\Big[\log\Big(1 + r\Delta t + \pi_{i-1}(e^{R_i} - 1 - r\Delta t)\Big)\Big].
\end{aligned}$$

where π_i is $\mathcal{F}_i^H$-measurable. Hence, the task to maximize the expected utility function is equivalent to maximize the utility of next state of x in terms of logarithmic utility function.
That is the reason why the optimal strategies π_k at time k in terms of logarithmic utility function can be obtained regardless of d_{k+1} in different cases of H. However it is not the case in terms of power utility function, since the power utility of some multiplied variables can not be written as the form of sum of utility of single variable. Moreover, the target maximizing conditional expectation can not be broken down the task into maximizing the product of conditional expectations.

Our idea is to approximate power utility function by cumulative functions and then we can assign the task of maximize utility of terminal wealth into the task in different time steps.

By multivariate version of Taylor's Theorems, let $h : \mathbb{R}^n \to \mathbb{R}$, the second order Taylor polynomial approximates $h(\boldsymbol{x})$ as well for $\boldsymbol{x}$ near $\boldsymbol{a}$ is

$$h(\boldsymbol{x}) \approx h(\boldsymbol{a}) + Dh(\boldsymbol{a})(\boldsymbol{x} - \boldsymbol{a}) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{a})^T Hh(\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{a}), \tag{4.4}$$

where $Dh(\boldsymbol{x})$ is $1 \times n$ matrix of partial derivatives, and $Hh(\boldsymbol{x})$ is Hessian matrix of h.

To apply this to the target of maximization of expected power utility function, we first derive that

$$\begin{aligned}V_0(x_0) &= \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[\frac{(X_N^\pi)^{1-\alpha}}{1-\alpha} | X_0^\pi = x_0\Big] \\ &= \frac{1}{1-\alpha} x_0^{1-\alpha} \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[\prod_{i=1}^N \Big(1 + r\Delta t + \pi_{i-1}(e^{R_i} - 1 - r\Delta t)\Big)^{1-\alpha}\Big].\end{aligned}$$

Denote by $\boldsymbol{x} = (x_1, \ldots, x_N)^T$, where $x_i = 1 + r\Delta t + \pi_{i-1}(e^{R_i} - 1 - r\Delta t)$. One can represent $V_0(x_0)$ as a scalar function $h : \boldsymbol{x} \mapsto x_1 \cdot x_2 \ldots x_N, \mathbb{R}^N \to \mathbb{R}$

$$V_0(x_0) = \frac{1}{1-\alpha} x_0^{1-\alpha} \sup_{\pi\in\mathcal{A}} \mathbb{E}[h(\boldsymbol{x})^{1-\alpha}].$$

Let $\boldsymbol{a} = (1, \ldots, 1)^T$, Applying second order Taylor polynomial approximations (4.4), one gets

$$\begin{aligned}V_0(x_0) &= \frac{1}{1-\alpha} x_0^{1-\alpha} \sup_{\pi\in\mathcal{A}} \mathbb{E}[h(\boldsymbol{x})^{1-\alpha}] \\ &\approx \frac{1}{1-\alpha} x_0^{1-\alpha} \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[1 + (1-\alpha) \sum_{i=1}^N \Big((x_i - 1) + \frac{(-\alpha)}{2}(x_i - 1)^2\Big) \\ &+ (1-\alpha)^2 \sum_{j\neq k} (x_j - 1)(x_k - 1)\Big].\end{aligned}$$

Moreover, one can get an approximation for V_0 of order Δt

$$\begin{aligned}&V_0(x_0) = \frac{1}{1-\alpha} x_0^{1-\alpha} \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[1 + (1-\alpha) \sum_{i=1}^N \Big((x_i - 1) + \frac{(-\alpha)}{2}(x_i - 1)^2\Big) \\ &+ (1-\alpha)^2 \sum_{j\neq k} (x_j - 1)(x_k - 1)\Big] \\ &= \frac{x_0^{1-\alpha}}{1-\alpha} + x_0^{1-\alpha} \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[\sum_{k=0}^{N-1} \sum_{i=1}^d \mathbb{1}_{\{Y_k = e_i\}} \Big(r\Delta t + \pi_k((\mu_i - r + \frac{\varepsilon_{k+1}^2 - 1}{2}\sigma_i^2)\Delta t \\ &+ \sigma_i \sqrt{\Delta t}\varepsilon_{k+1}) + \pi_i^2 \sigma_i^2 \Delta t\Big)\Big] \\ &= \frac{x_0^{1-\alpha}}{1-\alpha} + x_0^{1-\alpha} \sum_{k=0}^{N-1} \sup_{\pi_i\in\mathcal{A}} \mathbb{E}\Big[\sum_{i=1}^d \mathbb{1}_{\{Y_k = e_i\}} \Big(r\Delta t + \pi_k((\mu_i - r + \frac{\varepsilon_{k+1}^2 - 1}{2}\sigma_i^2)\Delta t \\ &+ \sigma_i \sqrt{\Delta t}\varepsilon_{k+1}) + \pi_i^2 \sigma_i^2 \Delta t\Big)\Big].\end{aligned}$$

Using tower property of conditional expectation, we have

$$V_0(x_0) = \frac{x_0^{1-\alpha}}{1-\alpha} + x_0^{1-\alpha}\sum_{k=0}^{N-1}\sup_{\pi_i\in\mathcal{A}}\mathbb{E}\Big[\mathbb{E}\Big[\sum_{i=1}^{d}\mathbb{1}_{\{Y_k=e_i\}}\Big(r\Delta t+\pi_k((\mu_i-r+\frac{\varepsilon_{k+1}^2-1}{2}\sigma_i^2)\Delta t$$
$$+\sigma_i\sqrt{\Delta t}\varepsilon_{k+1})+\pi_i^2\sigma_i^2\Delta t\Big)\Big|\mathcal{F}_k^H\Big]\Big]$$
$$=\frac{x_0^{1-\alpha}}{1-\alpha} + x_0^{1-\alpha}\sum_{k=0}^{N-1}\sup_{\pi_i\in\mathcal{A}}\mathbb{E}\Big[\mathbb{E}\Big[\sum_{i=1}^{d}\mathbb{1}_{\{Y_k=e_i\}}\Big(r\Delta t+\pi_k((\mu_i-r+\frac{\varepsilon_{k+1}^2-1}{2}\sigma_i^2)\Delta t$$
$$+\sigma_i\sqrt{\Delta t}\varepsilon_{k+1})+\pi_i^2\sigma_i^2\Delta t\Big)\Big|\mathcal{F}_k^H\Big]\Big]$$
$$=\frac{x_0^{1-\alpha}}{1-\alpha} + x_0^{1-\alpha}\sum_{k=0}^{N-1}\sup_{\pi_i\in\mathcal{A}}\sum_{i=1}^{d}\hat{Y}_k^i\mathbb{E}\Big[\Big(r\Delta t+\pi_k((\mu_i-r+\frac{\varepsilon_{k+1}^2-1}{2}\sigma_i^2)\Delta t$$
$$+\sigma_i\sqrt{\Delta t}\varepsilon_{k+1})+\pi_i^2\sigma_i^2\Delta t\Big)\Big].$$

From above, one can get an order-Δt-approximation of optimal strategies for power utility function by

$$\eta^*(\hat{y}) = \frac{1}{\alpha}\frac{\sum_{i=1}^d \mu_i\hat{y}^i - r}{\sum_{i=1}^d \sigma_i^2\hat{y}^i}\vee 0\wedge 1.$$

This confirms the results in Propositions 4.3.7, 4.3.12, 4.3.15.

4.3.5 Impact of the Time Step

In the case of power utility, the approximated optimal strategy is given by

$$\pi_k^*(\hat{y}) \approx \frac{1}{\alpha}\frac{\sum_{i=1}^d \mu_i\hat{y}^i - r}{\sum_{i=1}^d \sigma_i^2\hat{y}^i}\vee 0\wedge 1 \tag{4.5}$$

where, $\hat{y} = \mathbb{E}[Y_k|\mathcal{F}_k^H]$ for all $H \in \{R, E, C\}$. This approximated optimal strategies are of order Δt. In other words,the optimal strategy in (4.5) works well if $\Delta t \to 0$.
The approximated optimal strategies are derived by approaching several functions of order Δt. Then first proposal is to approach second order Taylor expansion of functions $\frac{x^{1-\alpha}}{1-\alpha}$ and e^x. The second is to approximate $d_k = 1 + O(\Delta t)$. One can use numerical method, e.g. gradient ascent, to

get a maximizer for $\frac{x^{1-\alpha}}{1-\alpha}$. But, it is not our main goal. Next, we keep using second Taylor expansion to approximate $\frac{x^{1-\alpha}}{1-\alpha}$ and e^x, and we are more interested in the effect of d_k in terms of magnitude of Δt. Monte Carlo method and interpolation methods are used to approximate the value functions.
Next, we mainly focus on the case $H = E$, since Monte Carlo methods can be applied separately. We can get an expression for optimal strategy from Theorem 4.3.11 , if power utility is approached by second order Taylor series:

$$\eta_k^*(\hat{y}) \approx \frac{1}{\alpha} \frac{\sum_{i=1}^d \hat{y}^i \mathbb{E}[d_{k+1}(g(\hat{y}, e_i, \tilde{E}))]\mu_i - r}{\sum_{i=1}^d \hat{y}^i \mathbb{E}[d_{k+1}(g(\hat{y}, e_i, \tilde{E}))]\sigma_i^2} \vee 0 \wedge 1.$$

The method uses the following property for sampling Dirichlet random variables: let Y be a random vector which has components that follow a standard gamma distribution, then $E = \frac{1}{\sum_{i=1}^k Z_i} Z$ is Dirichlet-distributed, see [Mac12].
We approximate $\mathbb{E}[d_{k+1}(g(\hat{Y}_k, e_i, \tilde{E}))|\hat{Y}_k = \hat{y}]$ by the arithmetic mean

$$\mu := \frac{1}{M} \sum_{i=1}^{M} d_{k+1}(g(\hat{y}, e_i, \tilde{E}_i(w))$$

for some $M \in \mathbb{N}$. Here $\tilde{E}_i(w)$ are the results of M independent experiments that have the same Dirichlet density function f_i^E. From the Central Limit Theorem we know that the asymptotic distribution of the Monte Carlo estimator is approximately normal.
In our simulations, we set the other parameters as follows:

- Time horizon $T = 1$, $N = 252$;
- The parameters for Markov switching model are
  $\mu = (8, -5)^T, \sigma = (5, 4)^T$,(such extreme for illustration)
  $\Pi = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$;
- The parameters for expert opinions are $\gamma = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.
- Risk aversion parameter is $\alpha = 6$;
- Simulations of Monte Carlo are $M = 1000$.

Interpolation and dichotomy method

In order to approximate the value function at each time step, the ideal method is to simulate as much data, the corresponding value function with respect to filter $\hat{y}$, as possible. However, as the growing amounts of data set will result in exponentially increasing computational costs in the Monte Carlo method, this conflict is a dilemma.

For $d-$dimensional states, the problem can be regarded as the partition of the interval $[0,1]$ and the distance between adjacent dividing points are the probability of each states. In order to get d intervals, $d-1$ points are needed.

The first proposal is to divide the space of $\hat{y}$ into equal parts. At each breakpoint, we use Monte Carlo method to estimate the real expectation of d_{k+1},$\mathbb{E}[d_{k+1}(g(\hat{Y}_k, e_i, \tilde{E}))|\hat{Y}_k = \hat{y}]$. And we interpolate between two adjacent dividing points. The interpolation methods we consider are the linear, quadratic, cubic splines.

Figure 4.1a shows the farther away from the two endpoints of the interval

(a) [0,1] (b) [0,0.1]

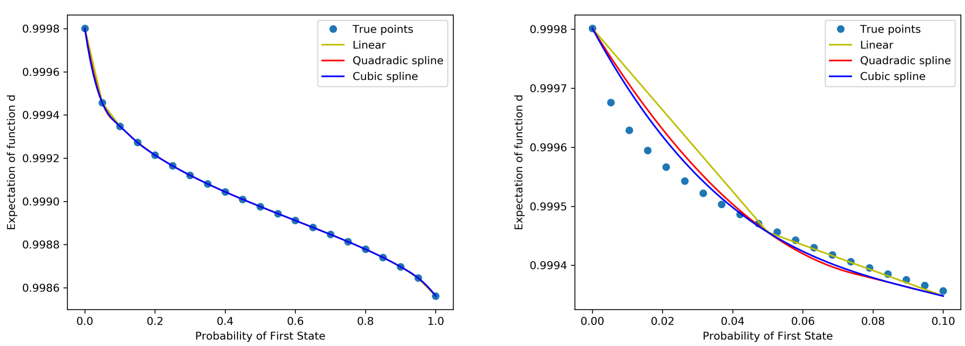


Figure 4.1: The expectation of d_{N-1} with respect to the probability of first state when $d = 2$

on the x-axis, the more gentle and linear the y-axis data are. Therefore, we subdivide the data closer to the two end points of the interval finer to observe which interpolation method is more in line with the data characteristics.

If we further look at the interval $[0, 0.1]$ on the x-axis (shown in Figure 4.1b), we can find that the interpolation method of cubic spline is better than other methods, but there is still a certain bias.

Figure 4.1b explains why we need more data near 0. In other words, we need more data to approximate the original function (here is conditional expectation) near the start-point and endpoint of the interval $[0, 1]$ because of the behaviour of the original function. The function away from endpoints is nearly linear. Therefore, we use a method called dichotomy to divide the interval $[0, 1]$.
If we have n subintervals, we divide the interval $[0, 1]$ by the following points

$$2^{-n/2}, 2^{-n/2+1}, \ldots, 2^{-1}, \ldots, 1-2^{-n/2+1}, 1-2^{-n/2}, \text{if } n \text{ is even}$$

For example, we divide $[0, 1]$ into 10 equal parts, nine points, $0.1, 0.2, \ldots, 0.9$ are derived. Now, rather than dividing $[0, 1]$ uniformly, we also derived 9 points by

$$2^{-5}, 2^{-4}, 2^{-3}, 2^{-2}, 2^{-1}, 1-2^{-2}, 1-2^{-3}, 1-2^{-4}, 1-2^{-5}$$

This method helps to possess dense points close to 0 and 1.

(a) [0,1] (b) [0,0.1]

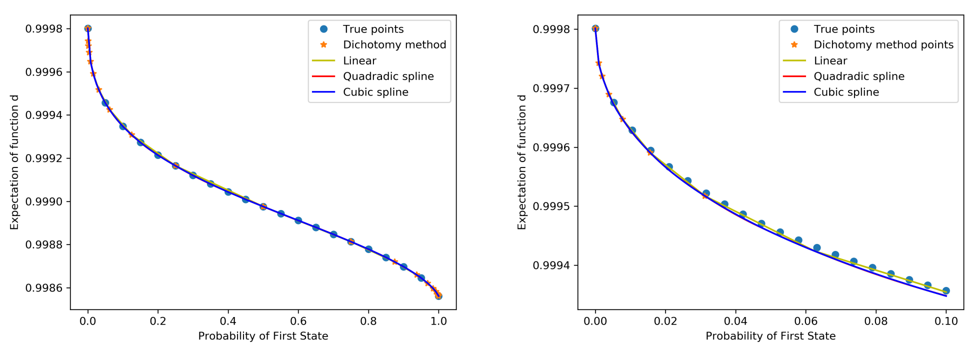


Figure 4.2: The expectation of d_{N-1} with respect to the probability of first state when $d = 2$

In Figure 4.2a, the marker $*$ represents the points we sampled from $[0, 1]$ by dichotomy method. More samples are closed 0 and 1 compared to the uniformly sampling.
Four interpolation methods are applied to approximate function with the new samples, and the blue points are the uniform samples, which are comparable. The approximated function with linear, quadratic and cubic spline

interpolation methods can fit the previous samples well.
Figure 4.2b reveals value of functions with different interpolation methods in $[0, 1]$ if we use the samples from dichotomy methods. It is obvious that these approximations have better performance than those in Figure 4.1b. What is surprising is that linear spline interpolation works best in $[0, 0.1]$.

Influence of cumulative time intervals

Because of the good result of fitting by dichotomy method, we combine both the uniform and dichotomy method to divide the space of $\hat{y}$, and use cubic spline to interpolate. Finally, we could get functions d_k, for all $k = 1, \ldots, N$. Figure 4.3a shows three functions of d at different time, $d_1, d_{50}, d_{150}, d_{250}$ with respect to $\hat{y}^1$ respectively. It reveals that d_t is detectably different with increasing t. And the further t is away from the terminal time, the more d_t differs from value 1.

(a) Approximation for function d at different time

(b) Ratio of conditional expectation respect to time$(N-k)$ with different $\hat{y}$

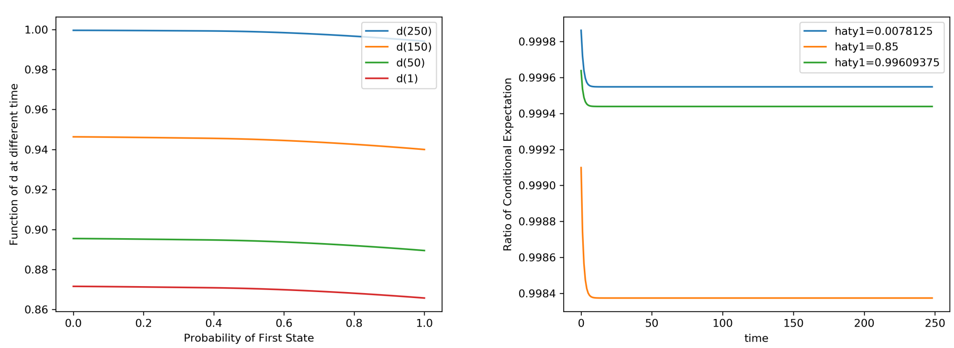


Figure 4.3: N=252

However, even if d_t differs from 1, one can still use the approximated strategy

$$\eta_k^*(\hat{y}) = \frac{1}{\alpha} \frac{\sum_{i=1}^{d} \hat{y}^i \mu_i - r}{\sum_{i=1}^{d} \hat{y}^i \sigma_i^2} \vee 0 \wedge 1.$$

From Figure 4.3b, we find that the with the increasing of time, the ratios $\frac{\mathbb{E}[d_{k+1}(g(\hat{Y}_k,e_1,\tilde{E}))]}{\mathbb{E}[d_{k+1}(g(\hat{Y}_k,e_2,\tilde{E}))]}$ are all closed to 1 no matter what the value of $\hat{Y}_k$ is. As a result, $\hat{y}^i \cdot \mathbb{E}[d_{k+1}(g(\hat{Y}_k, e_i, \tilde{E}))] \approx \hat{y}^i$ for all $i = 1, \ldots d$.

However, it is only the case when $\Delta \to 0$. In order to explore the effect of magnitudes of parameters. We make the same computation of dynamic programming, Monte Carlo method, and cubic spline interpolation for $N = 12$, (other parameters are remaining).
With large value of Δt, one can observe that the difference of function d changes over time as shown in Figure 4.4a. The lines are not as flat as the ones in Figure 4.3a. Also the ratios in Figure 4.4b are significantly different than 1.

(a) Approximation for function d at different time

(b) Ratio of conditional expectation respect to time$(N-k)$ with different $\hat{y}$

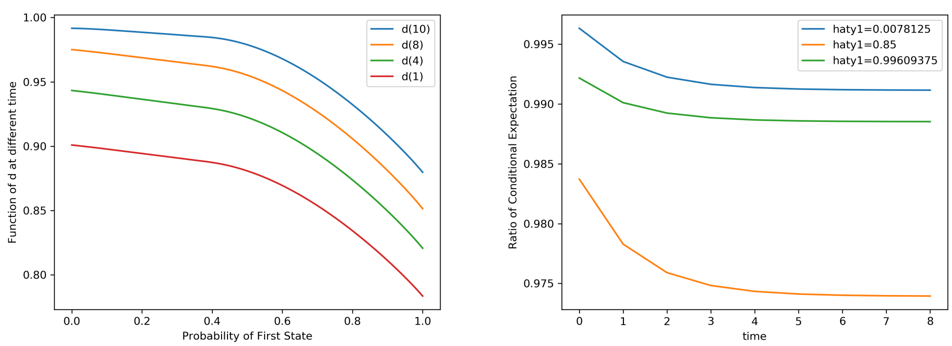


Figure 4.4: N=12

As a result, investors need to be careful with large Δt or equivalently with large parameters. If Δt is large, Monte Carlo methods should be used to get different strategies rather than by the time-independent expression (4.3.5).

4.4 Numerical Simulation

In this section, we mainly focus on the value function $V_0^H(x_0)$ with the initial wealth x_0, when $H = \{R, E, C, F\}$. In different cases of H, optimal strategies are determined based on available observations.

Moreover, the strategies from the Merton model and buy and hold strategies are both compared with above strategies. Since the Markov chain has several states, we use the Merton strategy, replacing the drift and volatility in it with those obtained from taking the average of the drift and volatility respectively, i.e. $\tilde{\mu} = \frac{1}{d}\sum_{i=1}^{d} \mu^T e_i, \tilde{\sigma} = \frac{1}{d}\sum_{i=1}^{d} \sigma^T e_i$. We also compared our optimal strategies with the buy-and-hold (b/h) strategy, which means to buy the stock using all cash available and hold the stock until the end, i.e. $\pi = 1$.

We generate the wealth process 1000 times and calculate the average of the utilities from the terminal wealth.

In our simulations, the default value of x_0 is 1. We set other parameters as follows:

- Time horizon $T = 1$, $N = 100$ and thus $\Delta t = 10^{-2}$;
- The parameters for Markov switching model are
  $\mu = (0.8, -0.5)^T, \sigma = (0.4, 0.7)^T$,
  $\Pi = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}$;
- The parameters for expert opinions are $\gamma = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$;
- Simulation times are $M = 1000$.

Typically, logarithmic and power utility function will be utilized.

Logarithmic Utility Function

When logarithmic utility function is applied, optimal strategies at time k can be derived approximately by

$$\pi_k(\hat{y}) = \frac{\sum_{i=1}^{d} \hat{y}^i \mu_i - r}{\sum_{i=1}^{d} \hat{y}^i \sigma_i^2} \vee 0 \wedge 1,$$

where $\hat{y} = \mathbb{E}[Y_k|\mathcal{F}_k^H]$, $H \in \{R, E, C, F\}$.

The results in Table 4.1 illustrate that the optimal strategies of full information have best performance without doubts, since the utility of terminal wealth in terms of the strategies of full information has largest average and medians. Moreover it has less standard deviation than that of strategies R, E, C. Besides, the strategy with combined information has a better performance than those which have only have expert opinions or only stock return observations (also with larger average and median and less standard deviation). This is intuitive.

Table 4.1

Logarithmic utility function $\log(x)$						
$U(X_T)$	R	E	C	F	Merton	b/h
avg	0.2770	0.3429	0.3463	0.4020	0.0233	0.0678
med	0.2752	0.3459	0.3477	0.3932	0.0274	0.1047
std	0.2271	0.2058	0.2053	0.1939	0.0516	0.4497

Figure 4.5 illustrates the average wealth process with respect to different strategies $H \in \{F, R, E, C\}$. The shaded area represents the 95%- confidence interval, reveals during investment horizon, the wealth with respect to strategy F is always above the others. Besides, wealth with respect to strategies C is always slightly larger than the one of strategy E. The impact of return observations is only small compared to that of expert opinions since expert opinions may carry out more information about Y than return observations. And the wealth process of strategy R is always at the bottom. Thus one can also rank different strategies and get the same result as shown in Table 4.1.

Power Utility Function

When power utility function is considered, the optimal strategy cannot be computed directly by $\hat{Y}_k$. One needs to know the function d_{k+1}, which can be calculated backwards from d_N.
However, as we mentioned before, the differences of $\mathbb{E}[d_{k+1}|Y_k = e_i]$, $i = 1, \ldots, d$, are very small. One can eliminate the influence and use approxi-

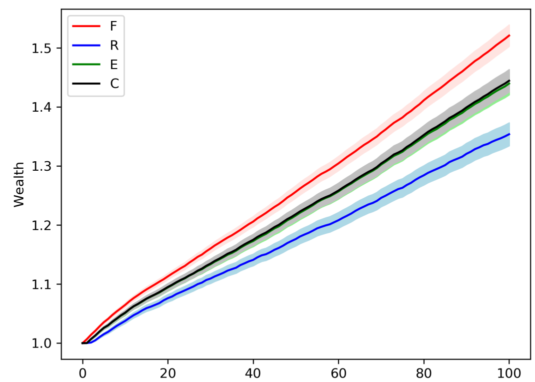


Figure 4.5: Wealth Processes with respect to different strategies (logarithmic utility function)

mated strategies

$$\eta^*(\hat{y}) = \frac{1}{\alpha}\frac{\sum_{i=1}^{d}\mu_i\hat{y}^i - r}{\sum_{i=1}^{d}\sigma_i^2\hat{y}^i},$$

where $\hat{y} = \mathbb{E}[Y_k|\mathcal{F}_k^H]$, $H \in \{R, E, C, F\}$.
We specify two values for α, 0.1 and 6 respectively.

Table 4.2

Power utility function $\frac{x^{1-\alpha}}{1-\alpha}$, $\alpha = 0.1$						
$U(X_T)$	R	E	C	F	Merton	b/h
avg	1.4657	1.5399	1.5450	1.6178	1.1359	1.2761
med	1.4327	1.5163	1.5189	1.5828	1.1389	0.1047
std	0.3168	0.2974	0.2969	0.2772	0.0518	1.2208

The results in Table 4.2 illustrate the power utility, when $\alpha = 0.1$, of terminal wealth with respect to different strategies. It also reveals that the strategy for full information has best performance. Besides that, the strategy with combined information has better performance than the remaining strategies but is only slightly better than E, since it has lager average, larger median and less standard deviation.

The results in Table 4.3 show the power utility, when $\alpha = 6$, of terminal wealth with respect to different strategies. And although $\alpha > 1$, the result is quite similar as in Table 4.2. The strategy of full information works best, and besides strategy of C have slightly better performance than that of E, and these strategies are all better than the one only with R available.

Table 4.3

Power utility function $\frac{x^{1-\alpha}}{1-\alpha}, \alpha = 6$						
$U(X_T)$	R	E	C	F	Merton	b/h
avg	-0.1078	-0.0773	-0.0758	-0.0490	-0.1841	-3.8213
med	-0.0929	-0.0633	-0.0620	-0.03101	-0.1744	-0.1185
std	0.0627	0.0518	0.0517	0.0361	0.0505	46.9416

The wealth process with respect to different strategies, when α is 0.1 and 6, are shown in Figure 4.6a and Figure 4.6b, respectively. In general, the two figures display similar results. The curve of F is above the others, E is at the bottom and C stays slightly above the one of E. An observable phenomena is that the lines in Figure 4.6b are more dispersed than that in in Figure 4.6a. This is because $\alpha = 6$ corresponds to less extreme strategies.

(a) $\alpha = 0.1$ (b) $\alpha = 6$

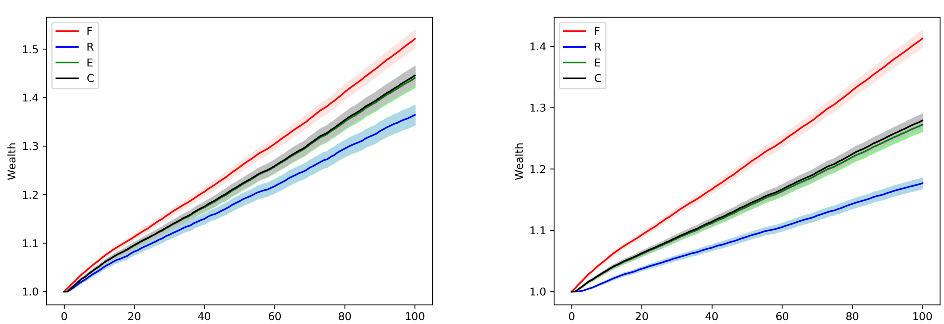


Figure 4.6: Wealth Processes with respect to different strategies (power utility function)

Chapter 5

Conclusion and Outlook on Further Research

In the thesis, we focused on how to combine additional information (expert opinions) to optimize parameter estimators under different assumptions and used the parameters in portfolio optimization.
In Chapter 2, we used the Kalman filter of two observation sources to combine two Gaussian observations. When one of the observations were non-Gaussian (e.g. uniformly distributed), we transformed the problem into a non-linear system with Gaussian observations. We used the generalized Kalman filter to estimate the hidden state, which represents the drift term of the stock return here.
In Chapter 3, we considered that both the drift term and the volatility term of the stock return are driven by a hidden Markov chain. Experts can estimate the state of the Markov chain based on their own information. First, the linear combination method was used to combine expert opinions and observations on stock returns. And using the logistic regression model, the multi-dimensional linear information was compressed, and the dimension reduced to the dimension of the Markov chain. In addition, the Bayes rule was used to incorporate expert opinions which are Dirichlet distributed. These two combined methods can effectively increase the accuracy of parameter estimation.
In Chapter 4, maximizing the utility of wealth at maturity was our goal. We derived the results of using the strategy under four different observation conditions, and we could conclude that the strategy can be improved by combining return observations with expert opinions. At the same time, we

discussed problems of the approximated optimal strategies under the conditions for discretization. If the adjusted frequency is too low, or the real parameters are too large, using the Monte Carlo simulation method will be closer to the real optimal strategy under various conditions.

One can think of several extensions of our results. Therefore in the following we present some further ideas and generalizations that can and should be considered in future research.

Remark 5.0.1. *(Multivariate cases)*
The one-dimensional-asset assumption, that is, the assumption that only in one risky and one non-risky assets can be invested is quite limiting. This assumption can be used to model an index in actual capital markets. However, an index is always very difficult for expert to estimate. Therefore, it makes sense to generalize the model to multiple risk assets. In particular, expert opinions can change from absolute to relative. For example, the estimation of the difference between drift terms of different stocks. In addition, multi-dimensional data will dramatically increase the amount of calculations in existing models, and more effective numerical methods may become valuable.

Remark 5.0.2. *(Statistical models for expert opinions)*
In Chapter 3, the logistic regression model is used to quantify expert opinions. By maximizing the posterior probability, we derive the parameters of this model. The logistic regression model combines external factors through a linear method. However, in reality, other non-linear combinations may be used between factors, such as neural networks, random forests, etc. If we use these methods, we can change the goal from maximizing the posterior probability to minimizing a loss function. Among them, the loss function can be the negative value of the posterior probability. The purpose of changing the target is the non-linearilty in the model above. However, under what circumstances these methods can optimize parameter estimates and whether they are stable is also worth studying.

Remark 5.0.3. *(No time-lag expert opinions)*
In Chapter 3 and 4, we assume that

$$E_k \sim D(\gamma^T Y_{k-1}).$$

From this assumption, we know that expert opinions at time k reflect the state Y_{k-1}. Even if the expert has very accurate estimator of Y_{k-1}, she still *needs to consider the effect of the transition matrix, which determines the next state Y_k. Once one has a deep insight of Y_k, investors can have a better strategy, since R_{k+1} is determined by Y_k. In practice, expert opinions showing current state show stronger ability to help investors. Therefore, one also need to consider the scenario that*

$$E_k \sim D(\gamma^T Y_k).$$

One needs to find a new probability measure to calculate different unnormalized terms in Chapter 3. One can not simply multiply both observations together. Moreover, one needs to be careful about the transition function of the filter of Y when computing value functions in Chapter 4.

Remark 5.0.4. *(Transaction cost, market impact and infinite investment horizon)*
In order to make the assumptions of our model realistic, we may consider the impact of transaction cost, market impact and infinite investment horizon. In such settings, our model would be more complex and not solvable. Some numerical methods rather than analytical results could be applicable in these complex scenarios, for instance online learning. One needs to use simulation method and function approximation methods to construct non-linear value functions or Q-functions instead. By dynamic learning and minimizing Q-functions at each time, one can obtain optimal strategies numerically.

Appendix A

Additional Proofs and Figures

A.1 Additional Proofs for Chapter 2

Theorem 2.1.4. *(One-dimensional version: Kalman Filter with two observations) The hidden state $x_k \in \mathbb{R}$ at time k of a discrete-time controlled process that is governed by the linear stochastic difference equation*

$$x_k = a_k x_{k-1} + b_k u_k + \epsilon_k$$

with two measurements:

$$z_k^1 = c_k^1 x_k + \zeta_k^1$$
$$z_k^2 = c_k^2 x_k + \zeta_k^2$$

where $a_k \in \mathbb{R}$ is the state transition model which is applied to the previous state x_{k-1}; $b_k \in \mathbb{R}$ describes how the control u_k changes from k to $k-1$; c_k^1 and c_k^2 describe how to map the state x_k to observation z_k^1 and z_k^2 respectively; ϵ_k represents the process noise, which is assumed independent and $\epsilon_k \sim \mathcal{N}(0, w_k^2)$; ζ_k^1 and ζ_k^2 are measurement noises, which are all independent of ϵ_k, time-independent, and $\zeta_k^1 \sim \mathcal{N}(0, m_k^2)$, $\zeta_k^2 \sim \mathcal{N}(0, n_k^2)$.
Denote

$$\begin{aligned}\hat{x}_k^* &:= \mathbb{E}[x_k | z_k^1, z_k^2, x_{k-1} = \hat{x}_{k-1}] \\ &= \Big(1 - K_k(\lambda_k c_k^1 + (1-\lambda_k) c_k^2)\Big)\mathbb{E}[x_k | x_{k-1} = \hat{x}_{k-1}] + K_k\Big(\lambda_k z_k^1 + (1-\lambda_k) z_k^2\Big)\end{aligned}$$

and K_k and λ_k are the solutions of equations set

$$\frac{\partial \operatorname{Var}(x_k - \hat{x}_k^*)}{\partial K_k} = 0$$

$$\frac{\partial\,\mathrm{Var}(x_k - \hat{x}_k^*)}{\partial \lambda_k} = 0;$$

and $K_k \overset{!}{=} K_k \vee 0 \wedge \frac{1}{\lambda_k c_k^1 + (1-\lambda_k) c_k^2}$, $\lambda_k \overset{!}{=} \lambda_k \vee 0 \wedge 1$*;*
Here $\hat{x}_k^*$ *is the best estimator in the* L^2*, i.e.*

$$\hat{x}_k^* = \arg\inf_{\hat{x}_k} \mathbb{E}[(x_k - \hat{x}_k)^2] \tag{A.1}$$

Proof. Denote by $\hat{x}_{k|k-1} := \mathbb{E}[x_k | x_{k-1} = \hat{x}_{k-1}] = a_k \hat{x}_{k-1} + b_k u_k$ The variance of $\hat{x}_{k|k-1}$ is given by

$$\mathrm{Var}(x_k - \hat{x}_{k|k-1}) = a_k^2 Var(x_k - \hat{x}_{k-1}) + w_k^2$$

Now, we are trying to combine both observations. The posteriori estimator is constructed linearly by

$$\hat{x}_k = \hat{x}_{k|k-1} + K_k(\lambda_k \hat{y}_k^1 + (1-\lambda_k)\hat{y}_k^2) \tag{A.2}$$

where

$$\begin{aligned} \hat{y}_k^1 &= z_k^1 - c_k^1 \hat{x}_{k|k-1} \\ \hat{y}_k^2 &= z_k^2 - c_k^2 \hat{x}_{k|k-1} \end{aligned} \tag{A.3}$$

and $\lambda \in [0,1]$. Our aim is to find the solution of (A.1). By substituting (A.2) and (A.3) and because of the independence between ϵ_k and ζ_k^1, ζ_k^2 we get

$$\begin{aligned} \mathbb{E}[(x_k - \hat{x}_k)^2] &= \mathrm{Var}(x_k - \hat{x}_k) \\ &= \Big(1 - K_k(\lambda_k c_k^1 + (1-\lambda_k)c_k^2)\Big)^2 Var(x_k - \hat{x}_{k|k-1}) \\ &+ K_k^2(\lambda_k^2 m_k^2 + (1-\lambda_k)^2 n_k^2 + 2\lambda_k(1-\lambda_k)\rho m_k n_k). \end{aligned} \tag{A.4}$$

$\rho_k = \frac{\mathrm{Cov}(\zeta_k^1, \zeta_k^2)}{m_k}$ represents correlation coefficient, $\mathrm{Var}(x_k - \hat{x}_{k-1})$ is the variance of previous state estimator.
Since (A.4) is a convex function with respective to K_k and λ_k, one can derive the optimal by letting

$$\frac{\partial\,\mathrm{Var}(x_k - \hat{x}_k)}{\partial K_k} = 0$$

and

$$\frac{\partial\,\mathrm{Var}(x_k - \hat{x}_k)}{\partial \lambda_k} = 0.$$

Since $1 - K_k(\lambda_k c_k^1 + (1-\lambda_k)c_k^2) \in [0,1]$ and $\lambda_k \in [0,1]$, the solutions K_k should be in $[0, \frac{1}{\lambda_k c_k^1 + (1-\lambda_k)c_k^2}]$ and λ_k in $[0,1]$.

$\square$

Lemma 2.2.1. *The conditional expectation and conditional variance for* $Y_t \in [a, b]$ *is given by*

$$\mathbb{E}[Y|a \leq Y \leq b] = \mu + \sigma \frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}$$

$$\begin{aligned}&\mathrm{Var}[Y|a \leq Y \leq b] \\ &= \sigma^2 \left\{ 1 + \frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} - \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}\right]^2 \right\}\end{aligned}$$

Proof. In order to derive the mean and variance of Y, we calculate moment generating function of Y firstly. The moment generating function is given by

$$M(t) = \mathbb{E}[e^{tY}|a \leq Y \leq b] = \frac{\frac{1}{\sigma}\int_a^b e^{ty}\phi(\frac{y-\mu}{\sigma})dy}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}.$$

The terms of numerator can be simplified as

$$\begin{aligned}\frac{1}{\sigma}\int_a^b e^{ty}\phi(\frac{y-\mu}{\sigma})dy &= \frac{1}{\sqrt{2\pi}\sigma}\int_a^b e^{ty-\frac{(y-\mu)^2}{2\sigma^2}}dy \\ &= \frac{1}{\sqrt{2\pi}\sigma}e^{\mu t+\frac{\sigma^2 t^2}{2}}\int_a^b e^{-\frac{1}{2\sigma^2}(y-\mu-\sigma^2 t)}dy \\ &= e^{\mu t+\frac{\sigma^2 t^2}{2}}\left[\Phi\left(\frac{b-\mu-\sigma^2 t}{\sigma}\right) - \Phi\left(\frac{a-\mu-\sigma^2 t}{\sigma}\right)\right]\end{aligned}$$

The first-order derivative of MGF is given by

$$\begin{aligned}&\frac{\partial M(t)}{\partial t} \\ &= \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}(\sigma^2 t+\mu)\left[\Phi\left(\frac{b-\mu-\sigma^2 t}{\sigma}\right) - \Phi\left(\frac{a-\mu-\sigma^2 t}{\sigma}\right)\right] - e^{\mu t+\frac{\sigma^2 t^2}{2}}\sigma\left[\phi\left(\frac{b-\mu-\sigma^2 t}{\sigma}\right) - \phi\left(\frac{a-\mu-\sigma^2 t}{\sigma}\right)\right]}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}.\end{aligned}$$

Thus, the expectation of Y can be given by

$$\mathbb{E}[Y|a \leq Y \leq b] = \left.\frac{\partial M(t)}{\partial t}\right|_{t=0} = \mu + \sigma\frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}$$

The second-order derivative of MGF is given by

$$
\begin{aligned}
&\frac{\partial^2 M(t)}{\partial t^2} \\
&= \frac{1}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}\left\{ e^{\mu t + \frac{\sigma^2 t^2}{2}}[(\sigma^2 t + \mu)^2 + \sigma^2]\left[\Phi\left(\frac{b-\mu-\sigma^2 t}{\sigma}\right) - \Phi\left(\frac{a-\mu-\sigma^2 t}{\sigma}\right)\right]\right. \\
&+ 2e^{\mu t + \frac{\sigma^2 t^2}{2}}(\sigma^2 t + \mu)\sigma\left[\phi\left(\frac{a-\mu-\sigma^2 t}{\sigma}\right) - \phi\left(\frac{b-\mu-\sigma^2 t}{\sigma}\right)\right] \\
&\left.+ e^{\mu t + \frac{\sigma^2 t^2}{2}}\sigma^2\left[\phi\left(\frac{a-\mu-\sigma^2 t}{\sigma}\right)\left(\frac{a-\mu-\sigma^2 t}{\sigma}\right) - \phi\left(\frac{b-\mu-\sigma^2 t}{\sigma}\right)\left(\frac{b-\mu-\sigma^2 t}{\sigma}\right)\right]\right\}.
\end{aligned}
$$

Then the expectation of Y^2 is given by

$$
\begin{aligned}
&\mathbb{E}[Y^2|a \le Y \le b] = \left.\frac{\partial^2 M(t)}{\partial t^2}\right|_{t=0} \\
&= \mu^2 + \sigma^2 + \frac{2\mu\sigma\left[\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)\right] + \sigma^2\left[\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)\right]}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})},
\end{aligned}
$$

hence the variance of Y is given by

$$
\begin{aligned}
&\mathrm{Var}[Y|a \le Y \le b] = \mathbb{E}[Y^2|a \le Y \le b] - \mathbb{E}^2[Y|a \le Y \le b] \\
&= \sigma^2\left\{1 + \frac{\phi\left(\frac{a-\mu}{\sigma}\right)\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)\left(\frac{b-\mu}{\sigma}\right)}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} - \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}\right]^2\right\}.
\end{aligned}
$$

□

Lemma 2.2.4. *The expected value and covariance matrix for $\boldsymbol{Y}$, which is normally distributed and truncated by B, are*

$$\mathbb{E}(Y_i) = \sum_{k=1}^{n} \sigma_{i,k}(F_k(a_k) - F_k(b_k)) + \mu_i.$$
$$\mathrm{Cov}(Y_i, Y_j) = \mathbb{E}(Y_iY_j) - \mathbb{E}(Y_i)\mathbb{E}(Y_j)$$

and

$$\begin{aligned}\mathbb{E}(Y_iY_j) =&\sigma_{i,j} + \sum_{k=1}^{n} \sigma_{i,k} \frac{\sigma_{j,k}(a_kF_k(a_k) - b_kF_k(b_k))}{\sigma_{k,k}} \\ &+ \sum_{k=1}^{n} \sigma_{i,k} \sum_{q \neq k} (\sigma_{j,q} - \frac{\sigma_{k,q}\sigma_{j,k}}{\sigma_{k,k}})[(F_{k,q}(a_k, a_q) - F_{k,q}(a_k, b_q) \\ &- (F_{k,q}(b_k, a_q) - F_{k,q}(b_k, b_q))],\end{aligned}$$

where

$$F_k(x) = \int_{a_1-\gamma_1}^{b_1-\gamma_1} \cdots \int_{a_{k-1}-\gamma_{k-1}}^{b_{k-1}-\gamma_{k-1}} \int_{a_{k+1}-\gamma_{k+1}}^{b_{k+1}-\gamma_{k+1}}$$
$$\cdots \int_{a_n-\gamma_n}^{b_n-\gamma_n} \varphi_{\mathbf{0}\Sigma B^*}(x_1, \cdots, x_{k-1}, x, x_{k+1}, \cdots, x_n) dx_n \cdots dx_{k+1} dx_{k-1} \cdots dx_1.$$

and $F_{k,q}(x, y)$ is the bivariate marginal density, which is given by

$$F_{k,q}(x, y) = \int_{a_1}^{b_1} \cdots \int_{a_{k-1}}^{b_{k-1}} \int_{a_{k+1}}^{b_{k+1}} \cdots \int_{a_{q-1}}^{b_{q-1}} \int_{a_{q+1}}^{b_{q+1}} \int_{a_n}^{b_n} \varphi_{\mu\Sigma B}(x, y, x_{-k,-q}) dx_{-k,-q},$$

Proof. we need to compute the moment generating function firstly.
The moment generating function(MGF) of a n-dimensional random variables Y, truncated at $\mathbf{a}$ and $\mathbf{b}$, having the density function $\varphi_{\mu\Sigma B}$, is defined as the n-fold integral of the form

$$m(t) = \mathbb{E}(e^{t'Y}) = \int_a^b e^{t'y} \varphi_{\mu\Sigma B}(y) dy$$

Hence, the moment generating function is given by

$$m(\mathbf{t}) = \frac{1}{\beta} \int_a^b \exp\Big\{\frac{1}{2}\big[(y-\mu)'\Sigma^{-1}(y-\mu) - 2t'y\big]\Big\} dy$$

For the convenience of calculation, in the following μ is set equal to zero firstly. Subsequently, μ can be generalized by location transformation. Now, the special case that $\mu = 0$ is considered first. Then we find that the term

$$-\frac{1}{2}[y'\Sigma^{-1}y - 2t'y]$$

can be rewrite as

$$T - \frac{1}{2}\Big[(y-\boldsymbol{\gamma})'\Sigma^{-1}(y-\boldsymbol{\gamma})\Big]$$

where, $T = \frac{1}{2}t'\Sigma t$, and $\boldsymbol{\gamma} = \Sigma t$. Consequently, the MGF can be formulated as

$$\begin{aligned} m(t) &= \frac{e^T}{\beta}\int_a^b \exp\Big\{\frac{1}{2}\big[(y-\mu)'\Sigma^{-1}(y-\mu)\big]\Big\}dy \\ &= \frac{e^T}{\beta}\int_{a-\gamma}^{b-\gamma} \exp\Big\{\frac{1}{2}\big[y'\Sigma^{-1}y\big]\Big\}dy \\ &= e^T \Phi_{\mathbf{0}\Sigma B^*}. \end{aligned} \tag{A.5}$$

Here,

$$\Phi_{\mathbf{0}\Sigma B^*} = \int_{B^*} \varphi_{\mathbf{0}\Sigma B^*}$$

and $B^* = \{x \in \mathbb{R}^n | a_i - \gamma_i \le x_i \le b_i - \gamma_i\}_{i\in[0,n]}$ with $\gamma_i = \sum_{k=1}^{n} \sigma_{i,k} t_k$.

The mean can be calculated by taking the first derivative of MGF when $\mathbf{t} = 0$. By taking partial derivative of equation (A.5) respect to t_i, we have

$$\frac{\partial m(t)}{\partial t_i} = e^T \frac{\partial \Phi_{\mathbf{0}\Sigma B^*}}{\partial t_i} + \Phi_{\mathbf{0}\Sigma B^*} \frac{\partial e^T}{\partial t_i}. \tag{A.6}$$

In above equation, the following essential term can be simplified as

$$\frac{\partial e^T}{\partial t_i} = e^T \sum_{k=1}^{n} \sigma_{i,k} t_k$$

and we may also get the derivative of $\Phi_{\mathbf{0}\Sigma B^*}$ respect to t_i

$$\begin{aligned}\frac{\partial \Phi_{\mathbf{0}\Sigma B^*}}{\partial t_i} &= \frac{\partial}{\partial t_i}\int_{a_1-\gamma_1}^{b_1-\gamma_1}\cdots\int_{a_n-\gamma_n}^{b_n-\gamma_n}\varphi_{\mathbf{0}\Sigma B^*}dy_n\ldots dy_1\\ &= \sum_{k=1}^{n}\left(\frac{\partial \varphi_{\mathbf{0}\Sigma B^*}}{\partial y_k}\frac{\partial y_k}{\partial \gamma_k}\frac{\gamma_k}{\partial t_i}\Bigg|_{y_k=a_k-\gamma_k}^{b_k-\gamma_k}\right)\\ &= \sum_{k=1}^{n}\sigma_{i,k}\Big[(F_k(a_k-\gamma_k)-F_k(b_k-\gamma_k)\Big]\end{aligned}$$

where,

$$\begin{aligned}F_k(x) = &\int_{a_1-\gamma_1}^{b_1-\gamma_1}\cdots\int_{a_{k-1}-\gamma_{k-1}}^{b_{k-1}-\gamma_{k-1}}\int_{a_{k+1}-\gamma_{k+1}}^{b_{k+1}-\gamma_{k+1}}\\ &\cdots\int_{a_n-\gamma_n}^{b_n-\gamma_n}\varphi_{\mathbf{0}\Sigma B^*}(x_1,\cdots,x_{k-1},x,x_{k+1},\cdots,x_n)dx_n\cdots dx_{k+1}dx_{k-1}\cdots dx_1.\end{aligned} \tag{A.7}$$

Now, we set $t_k = 0$ for all $k = 1, 2, \ldots, n$, then we have $\gamma = \mathbf{0}$. Thus, $F_i(x)$ will be $i-$th marginal density of doubly truncated multivariate normal distribution. From (A.6) to (A.7) for all $k = 1, 2, \ldots, n$ when $t_k = 0$, the expected value of Y is given by

$$\mathbb{E}(Y_i) = \frac{\partial m(t)}{\partial t_i}\Bigg|_{t=0} = \sum_{k=1}^{n}\sigma_{i,k}(F_k(a_k)-F_k(b_k)).$$

Subsequently, we will generalised the case of $\mu = \mathbf{0}$ to all μ. If $Y \sim \mathcal{N}(\mu, \Sigma)$, with $a \leq y \leq b$, then $Z = Y - \mu \sim \mathcal{N}(\mathbf{0}, \Sigma)$, with $a - \mu \leq z \leq b - \mu$. And $\mathbb{E}(Y) = \mathbb{E}(Z) + \mu$ and $\mathrm{Cov}(Y) = \mathrm{Cov}(Z)$. Hence, for all μ, the expected value of Y is given by

$$\mathbb{E}(Y_i) = \sum_{k=1}^{n}\sigma_{i,k}(F_k(a_k)-F_k(b_k)) + \mu_i.$$

Next, we will calculate the covariance matrix by deriving the second second moment. The covariance matrix can be calculated by

$$\mathrm{Cov}(Y_i, Y_j) = \mathbb{E}(Y_iY_j) - \mathbb{E}(Y_i)\mathbb{E}(Y_j)$$

and $\mathbb{E}(Y_iY_j)$ can be calculated as the second moment when $t = 0$, which is given by

$$\begin{aligned}
\mathbb{E}(Y_iY_j) =& \sigma_{i,j} + \sum_{k=1}^{n} \sigma_{i,k} \frac{\sigma_{j,k}(a_kF_k(a_k) - b_kF_k(b_k))}{\sigma_{k,k}} \\
&+ \sum_{k=1}^{n} \sigma_{i,k} \sum_{q \neq k} (\sigma_{j,q} - \frac{\sigma_{k,q}\sigma_{j,k}}{\sigma_{k,k}})[(F_{k,q}(a_k, a_q) - F_{k,q}(a_k, b_q) \\
&- (F_{k,q}(b_k, a_q) - F_{k,q}(b_k, b_q))],
\end{aligned}$$

where $F_{k,q}(x, y)$ is the bivariate marginal density, which is given by

$$F_{k,q}(x, y) = \int_{a_1}^{b_1} \cdots \int_{a_{k-1}}^{b_{k-1}} \int_{a_{k+1}}^{b_{k+1}} \cdots \int_{a_{q-1}}^{b_{q-1}} \int_{a_{q+1}}^{b_{q+1}} \int_{a_n}^{b_n} \varphi_{\mu\Sigma B}(x, y, x_{-k,-q}) dx_{-k,-q},$$

and the term $\mathbf{x}_{-k,-q}$ denotes the $(n-2)-$dimensional vector $(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{q-1}, x_{q+1}, \cdots, x_n)$ for $k \neq q$. $\square$

A.2 Additional Proofs for Chapter 3

Lemma 3.3.2. *(i) $Z = (Z_k)_{k\in\mathbb{N}_0}$ is a $\mathcal{G}$-martingale under measure P.*
$Z^{-1} = (Z_k^{-1})_{k\in\mathbb{N}_0}$ is a $\mathcal{G}$-martingale under measure $\tilde{P}$.
(ii) For all $u \leq k$,

$$\tilde{\mathbb{E}}[Z_k^{-1}Y_u|\mathcal{F}_u^{C_0}] = \tilde{\mathbb{E}}[Z_u^{-1}Y_u|\mathcal{F}_u^{C_0}].$$

(iii) Under $\tilde{P}$, $R_1, R_2, \ldots$ are iid. standard normal distributed and independent of Y.

Proof. (i)

$$\begin{aligned}\mathbb{E}[Z_k|\mathcal{G}_{k-1}] &= Z_{k-1}\mathbb{E}[L_k|\mathcal{G}_{k-1}] \\ &= Z_{k-1}\mathbb{E}[L_k|Y_{k-1}].\end{aligned}$$

The last equation is due to the fact that R_k is only conditional on Y_{k-1}. And we have

$$\begin{aligned}\mathbb{E}[L_k|Y_{k-1} = e_i] &= \int_{\mathbb{R}} \frac{\varphi_{0,1}(x)}{\varphi_{b_i,a_i^2}(x)}\varphi_{b_i,a_i^2}(x)dx \\ &= 1\end{aligned}$$

hence, $Z = (Z_k)_{k\in\mathbb{N}_0}$ is a $\mathcal{G}$-martingale under measure P.
Similarly, $Z^{-1} = (Z_k^{-1})_{k\in\mathbb{N}_0}$ is a $\mathcal{G}$-martingale under measure $\tilde{P}$.
(ii) For all $u \leq k$, since $\mathcal{F}_u^{C_0} \subset \mathcal{G}_u$

$$\begin{aligned}\tilde{\mathbb{E}}[Z_k^{-1}Y_u|\mathcal{F}_u^{C_0}] &= \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[Z_k^{-1}Y_u|\mathcal{G}_u]|\mathcal{F}_u^{C_0}] \\ &= \tilde{\mathbb{E}}[Z_u^{-1}\tilde{\mathbb{E}}[Y_u|\mathcal{G}_u]|\mathcal{F}_u^{C_0}] \\ &= \tilde{\mathbb{E}}[Z_u^{-1}Y_u|\mathcal{F}_u^{C_0}].\end{aligned}$$

This holds because of tower property of conditional expectation, part(i) and the reason that Y_u is $\mathcal{G}_u$-measurable.
(iii) By calculating the cumulative density function conditional R_k, we get

$$\begin{aligned}\tilde{P}(R_k \leq t|\mathcal{G}_{k-1}) &= \tilde{\mathbb{E}}[\mathbb{1}_{\{R_k\leq t\}}|\mathcal{G}_{k-1}] \\ &= \frac{\mathbb{E}[\mathbb{1}_{\{R_k\leq t\}}Z_k|\mathcal{G}_{k-1}]}{\mathbb{E}[Z_k|\mathcal{G}_{k-1}]} \\ &= \frac{Z_{k-1}\mathbb{E}[\mathbb{1}_{\{R_k\leq t\}}L_k|\mathcal{G}_{k-1}]}{Z_{k-1}} \\ &= \mathbb{E}[\mathbb{1}_{\{R_k\leq t\}}L_k|\mathcal{G}_{k-1}] \\ &= \mathbb{E}[\mathbb{1}_{\{R_k\leq t\}}L_k|Y_{k-1}].\end{aligned}$$

The second equation is due to the Bayes' formula in Lemma 3.2.5 and the third one is because Z_k is a martingale. Then as above

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{R_k \le t\}} L_k | Y_{k-1} = e_i] &= \int_{\mathbb{R}} \mathbb{1}_{(-\infty,t]}(x) \frac{\varphi_{0,1}(x)}{\varphi_{b_i,a_i^2}(x)} \varphi_{b_i,a_i^2}(x) dx \\ &= \Phi(t)\end{aligned}$$

where, Φ is cumulative density function of standard normally distribution. Hence R_k is standard normal distributed and there is no dependence on Y_{k-1}. Actually,

$$\begin{aligned}\tilde{P}(R_k \le t, Y_{k-1} = e_i) &= \tilde{P}(R_k \le t | Y_{k-1} = e_i) \tilde{P}(Y_{k-1} = e_i) \\ &= \tilde{P}(R_k \le t) \tilde{P}(Y_{k-1} = e_i).\end{aligned}$$

Hence, R is independent of Y under $\tilde{P}$.

□

Lemma 3.3.3. *(i)* $\tilde{P}(Y_k = e_i | Y_k) = Y_k^i$.
(ii) The Markov chain Y has the same transition matrix under the reference probability measure, i.e.

$$\tilde{P}(Y_k = e_j | Y_{k-1} = e_i) = P(Y_k = e_j | Y_{k-1} = e_i).$$

(iii) Expert opinions can be given under either probability measures P or $\tilde{P}$, i.e. for all $w = 1, \dots, d$

$$\tilde{P}(Y_k = e_w | \mathcal{F}_K^E) = P(Y_k = e_w | \mathcal{F}_K^E).$$

if $E_k \in E^{(n)}$.

Proof. (i)

$$\begin{aligned}\tilde{P}(Y_k = e_i | Y_k) &= \tilde{\mathbb{E}}[\mathbb{1}_{\{Y_k = e_i\}} | Y_k] \\ &= \frac{\mathbb{E}[Z_k \mathbb{1}_{\{Y_k = e_i\}} | Y_k]}{\mathbb{E}[Z_k | Y_k]} \\ &= \frac{Y_k^i \mathbb{E}[Z_k | Y_k]}{\mathbb{E}[Z_k | Y_k]} \\ &= Y_k^i.\end{aligned}$$

(ii)

$$
\begin{aligned}
\tilde{P}(Y_k = e_j | Y_{k-1} = e_i) =& \tilde{\mathbb{E}}[\mathbb{1}_{\{Y_k = e_j\}} | Y_{k-1} = e_i] \\
=& \frac{\mathbb{E}[Z_k \mathbb{1}_{\{Y_k = e_j\}} | Y_{k-1} = e_i]}{\mathbb{E}[Z_k | Y_{k-1} = e_i]} \\
=& \frac{\mathbb{E}[\mathbb{E}[Z_k \mathbb{1}_{\{Y_k = e_j\}} | \mathcal{F}_{k-1}] | Y_{k-1} = e_i]}{\mathbb{E}[\mathbb{E}[Z_k | \mathcal{F}_{k-1}] | Y_{k-1} = e_i]} \\
=& \frac{\mathbb{E}[Z_{k-1} \mathbb{E}[L_k \mathbb{1}_{\{Y_k = e_j\}} | \mathcal{F}_{k-1}] | Y_{k-1} = e_i]}{\mathbb{E}[Z_{k-1} | Y_{k-1} = e_i]}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
P(Y_k = e_j, L_k \leq l_k | \mathcal{F}_{k-1}) &= P(L_k \leq l_k | \mathcal{F}_{k-1}, Y_k = e_j) P(Y_k = e_j | \mathcal{F}_{k-1}) \\
&= P(L_k \leq l_k | \mathcal{F}_{k-1}) P(Y_k = e_j | \mathcal{F}_{k-1}),
\end{aligned}
$$

where the second equation is due to the fact that L_k is only dependent on Y_{k-1}. Thus, L_k and Y_k are $\mathcal{F}_{k-1}$-independent.
With $\mathbb{E}[L_k | \mathcal{F}_{k-1}] = 1$, we have

$$
\begin{aligned}
\tilde{P}(Y_k = e_j | Y_{k-1} = e_i) &= \frac{\mathbb{E}[Z_{k-1} \mathbb{E}[\mathbb{1}_{\{Y_k = e_j\}} | Y_{k-1}] | Y_{k-1} = e_i]}{\mathbb{E}[Z_{k-1} | Y_{k-1} = e_i]} \\
&= \frac{\mathbb{E}[Z_{k-1} P(Y_k = e_j | Y_{k-1}) | Y_{k-1} = e_i]}{\mathbb{E}[Z_{k-1} | Y_{k-1} = e_i]} \\
&= P(Y_k = e_j | Y_{k-1} = e_i).
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\tilde{P}(Y_k = e_w | \mathcal{F}_K^E) &= \tilde{P}(Y_k = e_w | E_k) \\
&= \tilde{\mathbb{E}}[\mathbb{1}_{\{Y_k = e_w\}} | E_k] \\
&= \frac{\mathbb{E}[Z_k \mathbb{1}_{\{Y_k = e_w\}} | E_k]}{\mathbb{E}[Z_k | E_k]} \\
&= \frac{\mathbb{E}[\mathbb{E}[Z_k \mathbb{1}_{\{Y_k = e_w\}} | \mathcal{G}_{k-1}] | E_k]}{\mathbb{E}[\mathbb{E}[Z_k | \mathcal{G}_{k-1}] | E_k]} \\
&= \frac{\mathbb{E}[Z_{k-1} \mathbb{E}[L_k \mathbb{1}_{\{Y_k = e_w\}} | \mathcal{G}_{k-1}] | E_k]}{\mathbb{E}[Z_{k-1} | E_k]}.
\end{aligned}
$$

The random variable L_k and $\mathbb{1}_{\{Y_k = e_w\}}$ are $\mathcal{G}_{k-1}$-independent since

$$
\begin{aligned}
P(L_k \leq l_k, \mathbb{1}_{\{Y_k = e_w\}} = c | \mathcal{G}_{k-1}) &= P(\mathbb{1}_{\{Y_k = e_w\}} = c | \mathcal{G}_{k-1}, L_k \leq l_k) P(L_k \leq l_k | \mathcal{G}_{k-1}) \\
&= P(\mathbb{1}_{\{Y_k = e_w\}} = c | \mathcal{G}_{k-1}) P(L_k \leq l_k | \mathcal{G}_{k-1}).
\end{aligned}
$$

Thus

$$\tilde{P}(Y_k = e_w|\mathcal{F}_K^E) = \frac{\mathbb{E}[Z_{k-1}\mathbb{E}[L_k|\mathcal{G}_{k-1}]\mathbb{E}[\mathbb{1}_{\{Y_k=e_w\}}|\mathcal{G}_{k-1}]|E_k]}{\mathbb{E}[Z_{k-1}|E_k]}$$
$$= \frac{\mathbb{E}[Z_{k-1}\mathbb{E}[\mathbb{1}_{\{Y_k=e_w\}}|\mathcal{G}_{k-1}]|E_k]}{\mathbb{E}[Z_{k-1}|E_k]}.$$

Actually

$$P(P(Y_k = e_w|\mathcal{G}_{k-1}) \le p, Z_{k-1} \le z|E_k)$$
$$= P(P(Y_k = e_w|\mathcal{G}_{k-1}) \le p|E_k, Z_{k-1} \le z)P(Z_{k-1} = z|E_k)$$
$$= P(P(Y_k = e_w|\mathcal{G}_{k-1}) \le p|E_k)P(Z_{k-1} \le z|E_k)$$

$\mathbb{E}[\mathbb{1}_{\{Y_k=e_w\}}|\mathcal{G}_{k-1}]$ and Z_{k-1} are $\mathcal{F}_K^E$-conditional independent. Hence

$$\tilde{P}(Y_k = e_w|\mathcal{F}_K^E) = \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{Y_k=e_w\}}|\mathcal{G}_{k-1}]|E_k]$$
$$= \mathbb{E}[\mathbb{1}_{\{Y_k=e_w\}}|E_k]$$
$$= P(Y_k = e_w|E_k).$$

□

Theorem 3.3.6. *Let $H = (H_k)_{k\in\mathbb{N}_0}$ be $\mathcal{G}$-adapted, $H_k : \Omega \to \mathbb{R}$, with*

$$H_k = H_{k-1} + \alpha_{k-1} + \beta_{k-1}^T Y_k + \gamma_{k-1} f(R_k),$$

where α, β, γ are $\mathcal{F}$-adapted and $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}$-valued respectively. And $f : \mathbb{R} \to \mathbb{R}$ is measurable, s.t. H is integrable. For $\Gamma_i(r) := \frac{\varphi_i(r)}{\varphi_{0,1}(r)}$, where $\varphi_i = \varphi_{b_i,a_i^2}$. Define by $\ddot{Y}_k^{(i)} := \tilde{\mathbb{E}}[Y_k|\mathcal{G}_{k-1}, Y_{k-1} = e_i]$, Then we have

$$\rho_k(HY) = \sum_{i=1}^{d} \Big\{ \rho_{k-1}(HY^i)\Gamma_i(R_k)\ddot{Y}_k^{(i)}$$
$$+ \rho_{k-1}(\alpha Y^i)\Gamma_i(R_k)\ddot{Y}_k^{(i)}$$
$$+ \rho_{k-1}(\gamma Y^i)\Gamma_i(R_k)f(R_k)\ddot{Y}_k^{(i)}$$
$$+ (Diag(\ddot{Y}_k^{(i)}))\rho_{k-1}(\beta Y^i)\Gamma_i(R_k) \Big\}$$

where $\rho_k(HY) = (\rho(HY^1), \rho(HY^2), \ldots, \rho(HY^d))^T$, and Y^i is the $i-$th element of vector Y.

Proof. For

$$H_kY_k = H_{k-1}Y_k + \alpha_{k-1}Y_k + \beta_{k-1}^T Y_k Y_k + \gamma_{k-1} f(R_k) Y_k$$

we compute

$$\rho_k(HY) = \tilde{\mathbb{E}}[Z_k^{-1} H_k Y_k | \mathcal{F}_k^{C_0}].$$

Using

$$Z_k^{-1} = Z_{k-1}^{-1} \sum_{i=1}^{d} Y_{k-1}^i \Gamma_i(R_k),$$

Thus based on Lemma 3.3.2 and defining $\ddot{Y}_k := \tilde{\mathbb{E}}[Y_k|\mathcal{G}_{k-1}]$ we get

$$\begin{aligned}
\rho_k(HY) = &\sum_{i=1}^{d} \Gamma_i(R_k) \tilde{\mathbb{E}}\Big[Z_{k-1}^{-1} Y_{k-1}^i H_{k-1} Y_k + Z_{k-1}^{-1} Y_{k-1}^i \alpha_{k-1} Y_k \\
&+ Z_{k-1}^{-1} Y_{k-1}^i \beta_{k-1}^T Y_k Y_k + Z_{k-1}^{-1} Y_{k-1}^i \gamma_{k-1} f(R_k) Y_k \Big| \mathcal{F}_k^{C_0} \Big] \\
&= \sum_{i=1}^{d} \Gamma_i(R_k) \Big(\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1} Y_{k-1}^i H_{k-1} Y_k + Z_{k-1}^{-1} Y_{k-1}^i \alpha_{k-1} Y_k \\
&+ Z_{k-1}^{-1} Y_{k-1}^i \beta_{k-1}^T Y_k Y_k \Big| \mathcal{F}_k^{C_0} \Big] + f(R_k) \tilde{\mathbb{E}}\Big[Z_{k-1}^{-1} Y_{k-1}^i \gamma_{k-1} Y_k \Big| \mathcal{F}_k^{C_0} \Big] \Big). \\
&= \sum_{i=1}^{d} \Gamma_i(R_k) \Big(\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1} Y_{k-1}^i H_{k-1} Y_k + Z_{k-1}^{-1} Y_{k-1}^i \alpha_{k-1} Y_k \\
&+ Z_{k-1}^{-1} Y_{k-1}^i \beta_{k-1}^T Y_k Y_k \Big| \mathcal{F}_{k-1}^{C_0} \Big] + f(R_k) \tilde{\mathbb{E}}\Big[Z_{k-1}^{-1} Y_{k-1}^i \gamma_{k-1} Y_k \Big| \mathcal{F}_{k-1}^{C_0} \Big] \Big).
\end{aligned}$$

Since $\mathcal{F}^{C_0} \subset \mathcal{G}$, by tower property of conditional expectation we get

$$\begin{aligned}
\rho_k(HY) = &\sum_{i=1}^{d} \Gamma_i(R_k) \Big(\tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1} Y_{k-1}^i H_{k-1} Y_k + Z_{k-1}^{-1} Y_{k-1}^i \alpha_{k-1} Y_k \\
&+ Z_{k-1}^{-1} Y_{k-1}^i \beta_{k-1}^T Y_k Y_k \Big| \mathcal{G}_{k-1} \Big] \Big| \mathcal{F}_{k-1}^{C_0} \Big] + f(R_k) \tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1} Y_{k-1}^i \gamma_{k-1} Y_k \Big| \mathcal{G}_{k-1} \Big] \Big| \mathcal{F}_{k-1}^{C_0} \Big] \Big) \\
&= \sum_{i=1}^{d} \Gamma_i(R_k) \Big(\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1} Y_{k-1}^i H_{k-1} \ddot{Y}_k + Z_{k-1}^{-1} Y_{k-1}^i \alpha_{k-1} \ddot{Y}_k \\
&+ Z_{k-1}^{-1} \sum_{i=1}^{d} Y_{k-1}^i \beta_{k-1}^T \ddot{Y}_k \ddot{Y}_k \Big| \mathcal{F}_{k-1}^{C_0} \Big] + f(R_k) \tilde{\mathbb{E}}\Big[Z_{k-1}^{-1} \sum_{i=1}^{d} Y_{k-1}^i \gamma_{k-1} \ddot{Y}_k \Big| \mathcal{F}_{k-1}^{C_0} \Big] \Big)
\end{aligned}$$

Since $Y^i_{k-1} = 0$ iff $Y_{k-1} \neq e_i$, we can replace $\ddot{Y}_k$ by $\ddot{Y}^{(i)}_k$ and thus $\ddot{Y}^{(i)}_k$ is $\mathcal{F}^{C_0}_{k-1}$-measurable, and we can pull it out of the conditional expectation. Using

$$\beta^T_{k-1} Y_k Y_k = Diag(Y_k)\beta_{k-1},$$

we get

$$\begin{aligned}\tilde{\mathbb{E}}\Big[Z^{-1}_{k-1}\sum_{i=1}^{d} Y^i_{k-1}\Gamma_i(R_k)\beta^T_{k-1}\ddot{Y}^{(i)}_k\ddot{Y}^{(i)}_k|\mathcal{F}^{C_0}_{k-1}\Big]\\ = \sum_{i=1}^{d}\left(Diag\Big(\ddot{Y}^{(i)}_k\Big)\right)\rho^E_{k-1}(\beta Y^i)\Gamma_i(R_k),\end{aligned}$$

which finishes the proof. □

Lemma 3.3.8. *For $\tilde{\mathbb{E}}[Y^i_u|E_u] = E^i_u \in [0,1]$, where $u = \min\{u_i|u_i \geq k, i = 1,\ldots,n\}$.*

$$\tilde{\mathbb{E}}[Y^i_k|\mathcal{G}_{k-1}] = \sum_{w=1}^{d}\sum_{j=1}^{d}\frac{p^{(u-k)}_{iw}p_{ji}}{p^{(u-k+1)}_{jw}}Y^j_{k-1}\tilde{P}(Y_u = e_w|Y_{k-1,}E_u).$$

Proof.

$$\tilde{\mathbb{E}}[Y^i_k|\mathcal{G}_{k-1}] = \tilde{P}(Y_k = e_i|Y_{k-1},E_u) = \sum_{w=1}^{d}\sum_{j=1}^{d}\tilde{P}(Y_k = e_i, Y_{k-1} = e_j, Y_u = e_w|Y_{k-1},E_u)$$

$$= \sum_{w=1}^{d}\sum_{j=1}^{d}\tilde{P}(Y_k = e_i|Y_{k-1} = e_j, Y_u = e_w, Y_{k-1},E_u)\tilde{P}(Y_{k-1} = e_j, Y_u = e_w|Y_{k-1},E_u)$$

$$= \sum_{w=1}^{d}\sum_{j=1}^{d}\tilde{P}(Y_k = e_i|Y_u = e_w, Y_{k-1} = e_j)\tilde{P}(Y_{k-1} = e_j|Y_{k-1})\tilde{P}(Y_u = e_w|Y_{k-1,}E_u)$$

$$= \sum_{w=1}^{d}\sum_{j=1}^{d}\tilde{P}(Y_k = e_i|Y_u = e_w, Y_{k-1} = e_j)Y^j_{k-1}\tilde{P}(Y_u = e_w|Y_{k-1,}E_u)$$

$$= \sum_{w=1}^{d}\sum_{j=1}^{d}\frac{\tilde{P}(Y_u = e_w|Y_k = e_i)\tilde{P}(Y_k = e_i|Y_{k-1} = e_j)}{\tilde{P}(Y_u = e_w|Y_{k-1} = e_j)}Y^j_{k-1}\tilde{P}(Y_u = e_w|Y_{k-1,}E_u)$$

$$= \sum_{w=1}^{d}\sum_{j=1}^{d}\frac{p^{(u-k)}_{iw}p_{ji}}{p^{(u-k+1)}_{jw}}Y^j_{k-1}\tilde{P}(Y_u = e_w|Y_{k-1,}E_u)$$

□

Proposition 3.3.14. *For $k \leq u$, $u = \min\{u_i | u_i \geq k, i = 1, \dots, n\}$, the $\mathcal{G}$-measurable Kalman filter is*

$$\tilde{\mathbb{E}}[Y_u|\mathcal{G}_{k-1}] = \lambda_{k-1}^u \tilde{\mathbb{E}}[Y_u|Y_{k-1}] + (1-\lambda_{k-1}^u)\tilde{\mathbb{E}}[Y_u|E_u],$$

where $\lambda_{k-1}^u := \lambda(k-1, u)$, and $\lambda : \mathbb{R}^+ \times \mathbb{R}^+ \to [0,1]$ then we have

$$\tilde{\mathbb{E}}[Y_k|\mathcal{G}_{k-1}] = \lambda_{k-1}^u \Pi^T Y_{k-1} + (1-\lambda_{k-1}^u)\sum_{w=1}^{d} E_u^w \Pi^T Diag\Big(\mathbf{1}_d \oslash (p_{.w}^{(u-k+1)})\Big) Y_{k-1} \odot (p_{.w}^{(u-k)})$$

Proof. From Lemma 3.3.12 and Theorem 3.3.13, we can compute

$$\begin{aligned}
\tilde{\mathbb{E}}[Y_k^i|\mathcal{G}_{k-1}] &= \sum_{w=1}^{d}\sum_{j=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} Y_{k-1}^j \tilde{P}(Y_u = e_w | Y_{k-1}, E_u) \\
&= \sum_{w=1}^{d}\sum_{j=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} Y_{k-1}^j \Big(\lambda_{k-1}^u \tilde{\mathbb{E}}[Y_u^w|Y_{k-1}] + (1-\lambda_{k-1}^u)\tilde{\mathbb{E}}[Y_u^w|E_u]\Big) \\
&= \sum_{w=1}^{d}\sum_{j=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} Y_{k-1}^j \Big(\lambda_{k-1}^u \sum_{z=1}^{d} p_{zw}^{(u-k+1)} Y_{k-1}^z + (1-\lambda_{k-1}^u) E_u^w\Big) \\
&= \lambda_{k-1}^u \sum_{w=1}^{d} p_{iw}^{(u-k)} \sum_{j=1}^{d} p_{ji} Y_{k-1}^j + (1-\lambda_{k-1}^u)\sum_{w=1}^{d} E_u^w \sum_{j=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} Y_{k-1}^j \\
&= \lambda_{k-1}^u \sum_{j=1}^{d} p_{ji} Y_{k-1}^j + (1-\lambda_{k-1}^u)\sum_{w=1}^{d} E_u^w \sum_{j=1}^{d} \frac{p_{iw}^{(u-k)} p_{ji}}{p_{jw}^{(u-k+1)}} Y_{k-1}^j
\end{aligned}$$

After vectorizing, we finish this proof. □

Theorem 3.3.17. *The updates in the EM algorithm with expert opinions for the MSM are*

$$\begin{aligned}
\hat{b}_l &= \frac{\hat{T}_K^l(f_1)}{\hat{O}_K^l} \\
(\hat{a}_l)^2 &= \frac{\hat{T}_K^l(f_2) - 2\hat{b}_l \hat{T}_K^l(f_1) + \hat{b}_l^2 O_K^l}{O_K^l} \\
\hat{p}_{lm} &= \frac{N_K^{lm}}{O_K^l}
\end{aligned}$$

where, the filters $N_K^{lm}, O_K^l, T_K^l(f)$ can be computed based on parameters Π', b', a' which can be computed by the following proposition.

Proof.

$$\mathbb{E}[\log \Lambda_K|\mathcal{F}_K^{C_0}] = \hat{h}(\Pi', a', b') + \sum_{i,j=1}^{d} \hat{N}_K^{ij} \log(p_{ij}) + \sum_{i=1}^{d} \Bigg(- \hat{O}_K^i \log a_i - \frac{1}{2}\Big(\frac{b_i}{a_i}\Big)^2 \hat{O}_K^i \Bigg)$$
$$+ \sum_{i=1}^{d} \Bigg(\frac{b_i}{a_i^2} \hat{T}_K^i(f_1) - \frac{1}{2a_i^2} \hat{T}_K^i(f_2) \Bigg)$$

where, $\hat{N}_k^{ij} = \mathbb{E}[N_k^{ij}|\mathcal{F}_k^{C_0}], \hat{O}_k^i = \mathbb{E}[O_k^i|\mathcal{F}_k^{C_0}], \hat{T}_k^i(f) = \mathbb{E}[T_k^i(f)|\mathcal{F}_k^{C_0}]$.
Define $h(\Pi, a, b) := \mathbb{E}[\log \Lambda_K|\mathcal{F}_K^{C_0}]$, such that

$$\frac{\partial h(\Pi, a, b)}{\partial a_l} = \Big(- \frac{1}{a_l} + \frac{b_l^2}{a_l^3}\Big) \hat{O}_K^l - \frac{2b_l}{a_l^3} \hat{T}_K^l(f_1) + \frac{1}{a_l^3} \hat{T}_K^l(f_2) \stackrel{!}{=} 0$$
$$(a_l)^2 = \frac{b_l^2 \hat{O}_K^l - 2b_l \hat{T}_K^l(f_1) + \hat{T}_K^l(f_2)}{\hat{O}_K^l}$$
$$\frac{\partial h(\Pi, a, b)}{\partial b_l} = -\frac{b_l}{a_l^2} \hat{O}_K^l + \frac{1}{a_l^2} \hat{T}_K^l(f_1) \stackrel{!}{=} 0$$
$$b_l = \frac{\hat{T}_K^l(f_1)}{\hat{O}_K^l}$$

For maximization in p_{lm} need $\sum_{j=1}^{d} p_{lj} = 1$. Lagrange multiplier approach applied, define

$$L_l(p_{l1}, p_{l2}, \dots, p_{ld}, \lambda) := \sum_{j=1}^{d} \log(p_{lj}) \hat{N}_K^{lj} + \lambda(\sum_{j=1}^{d} p_{lj} - 1),$$
$$\frac{\partial L_l}{\partial p_{lm}} = \frac{1}{p_{lm}} \hat{N}_K^{lm} + \lambda \stackrel{!}{=} 0.$$

Thus,

$$p_{lm} = -\frac{1}{\lambda} \hat{N}_K^{lm}$$

Since $\sum_{j=1}^d p_{lj} = -\frac{1}{\lambda}\sum_{j=1}^d \hat{N}_K^{lj} = 1$,

$$\begin{aligned}
\lambda &= -\sum_{j=1}^d \hat{N}_K^{lj} \\
&= -\sum_{j=1}^d \mathbb{E}[\sum_{k=1}^K Y_{k-1}^l Y_k^j | \mathcal{F}_k^{C_0}] \\
&= -\hat{O}_K^l
\end{aligned}$$

Hence, $p_{lm} = \frac{\hat{N}_K^{lm}}{\hat{O}_K^l}$ finishes proof. □

Lemma 3.4.7. *(i) $Z = (Z_k)_{k\in\mathbb{N}_0}$ is a $\mathcal{H}$-martingale under measure P.*
$Z^{-1} = (Z_k^{-1})_{k\in\mathbb{N}_0}$ is a $\mathcal{H}$-martingale under measure $\tilde{P}$.
(ii) For all $u \le k$,

$$\tilde{\mathbb{E}}[Z_k^{-1} Y_u | \mathcal{F}_u^C] = \tilde{\mathbb{E}}[Z_u^{-1} Y_u | \mathcal{F}_u^C].$$

(iii) Under $\tilde{P}$, $R_1, R_2, \ldots$ are i.id. standard normal distributed and independent of Y. $E_1, E_2, \ldots$ are i.i.d. from $\mathcal{D}_{1^d}$ and independent of Y.

Proof. (i)

$$\begin{aligned}
\mathbb{E}[Z_k|\mathcal{H}_{k-1}] &= Z_{k-1}\mathbb{E}[L_k|\mathcal{H}_{k-1}] \\
&= Z_{k-1}\mathbb{E}[L_k|Y_{k-1}]
\end{aligned}$$

The last equation is due to the fact that R_k and E_k are only conditional on Y_{k-1}. We have by conditional independence

$$\begin{aligned}
&\mathbb{E}[L_k|Y_{k-1} = e_i] \\
&= \int_{\mathcal{E}}\int_{\mathbb{R}} \frac{\varphi_{0,1}(x)}{\varphi_{b^T Y_{k-1},(a^T Y_{k-1})^2}(x)} \frac{f_{1^d}^D(y)}{f_{\gamma^T Y_{k-1}}^D(y)} \varphi_{b^T Y_{k-1},(a^T Y_{k-1})^2}(x) f_{\gamma^T Y_{k-1}}^D(y)\, dx dy \\
&= 1
\end{aligned}$$

hence, $Z = (Z_k)_{k\in\mathbb{N}_0}$ is a $\mathcal{H}$-martingale under measure P.
Similarly, $Z^{-1} = (Z_k^{-1})_{k\in\mathbb{N}_0}$ is a $\mathcal{H}$-martingale under measure $\tilde{P}$.
(ii) For all $u \le k$, since $\mathcal{F}_u^C \subset \mathcal{H}_u$

$$\begin{aligned}
\tilde{\mathbb{E}}[Z_k^{-1} Y_u | \mathcal{F}_u^C] &= \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[Z_k^{-1} Y_u | \mathcal{H}_u] | \mathcal{F}_u^C] \\
&= \tilde{\mathbb{E}}[Z_u^{-1}\tilde{\mathbb{E}}[Y_u | \mathcal{H}_u] | \mathcal{F}_u^C] \\
&= \tilde{\mathbb{E}}[Z_u^{-1} Y_u | \mathcal{F}_u^C]
\end{aligned}$$

The equation holds because of tower property of conditional expectation, the fact that $Z^{-1} = (Z_k^{-1})_{k\in\mathbb{N}_0}$ is a $\mathcal{H}$-martingale under measure $\tilde{P}$ and the reason that Y_u is $\mathcal{H}_u$-measurable.
(iii) By calculating the cumulative density function of R_k conditional $\mathcal{H}_{k-1}$, we get

$$\begin{aligned}\tilde{P}(R_k \le t|\mathcal{H}_{k-1}) &= \tilde{\mathbb{E}}[\mathbb{1}_{\{R_k\le t\}}|\mathcal{H}_{k-1}] \\ &= \frac{\mathbb{E}[\mathbb{1}_{\{R_k\le t\}}Z_k|\mathcal{H}_{k-1}]}{\mathbb{E}[Z_k|\mathcal{H}_{k-1}]} \\ &= \frac{Z_{k-1}\mathbb{E}[\mathbb{1}_{\{R_k\le t\}}L_k|\mathcal{H}_{k-1}]}{Z_{k-1}} \\ &= \mathbb{E}[\mathbb{1}_{\{R_k\le t\}}L_k|\mathcal{H}_{k-1}] \\ &= \mathbb{E}[\mathbb{1}_{\{R_k\le t\}}L_k|Y_{k-1}]\end{aligned}$$

The second equation is due to the Bayes' formula of conditional expectation and the third one is because Z_k is a $\mathcal{H}$-martingale under P. Then we have

$$\begin{aligned}&\mathbb{E}[\mathbb{1}_{\{R_k\le t\}}L_k|Y_{k-1}=e_i] \\ &= \int_{\mathcal{E}}\int_{\mathbb{R}} \mathbb{1}_{(-\infty,t]}(x)\frac{\varphi_{0,1}(x)}{\varphi_{b^TY_{k-1},(a^TY_{k-1})^2}(x)}\frac{f^D_{1^d}(y)}{f^D_{\boldsymbol{\gamma}^TY_{k-1}}(y)}\varphi_{b^TY_{k-1},(a^TY_{k-1})^2}(x)f^D_{\boldsymbol{\gamma}^TY_{k-1}}(y)dxdy \\ &= \Phi(t)\end{aligned}$$

where, Φ is cumulative density function of standard normal distribution. Hence R_k is standard normal distributed and no matter what Y_{k-1} is. Actually,

$$\begin{aligned}\tilde{P}(R_k\le t, Y_{k-1}=e_i) &= \tilde{P}(R_k\le t|Y_{k-1}=e_i)\tilde{P}(Y_{k-1}=e_i) \\ &= \tilde{P}(R_k\le t)\tilde{P}(Y_{k-1}=e_i).\end{aligned}$$

Hence, R is independent of Y under $\tilde{P}$.
Likewise by calculating the cumulative density function conditional E_k, we get

$$\tilde{P}(E_k\in B|\mathcal{H}_{k-1}) = \mathbb{E}[\mathbb{1}_{\{E_k\in B\}}L_k|Y_{k-1}]$$

Then we have

$$\begin{aligned}&\mathbb{E}[\mathbb{1}_{\{E_k\in B\}}L_k|Y_{k-1}=e_i] \\ &= \int_B\int_{\mathbb{R}} \frac{\varphi_{0,1}(x)}{\varphi_{b^TY_{k-1},(a^TY_{k-1})^2}(x)}\frac{f^D_{1^d}(y)}{f^D_{\boldsymbol{\gamma}^TY_{k-1}}(y)}\varphi_{b^TY_{k-1},(a^TY_{k-1})^2}(x)f^D_{\boldsymbol{\gamma}^TY_{k-1}}(y)dxdy \\ &= \int_B f^D_{1^d}(y)dy\end{aligned}$$

Note that it is the probability of $E_k \in B$ under probability measure $\tilde{P}$. And this probability is also independent of Y. Hence, E is independent of Y under $\tilde{P}$.

$\square$

Theorem 3.4.11. *Let $H = (H_k)_{k\in\mathbb{N}_0}$ be $\mathcal{H}$-adapted, $H_k : \Omega \to \mathbb{R}$, with*

$$H_k = H_{k-1} + \alpha_{k-1} + \beta_{k-1}^T Y_k + \kappa_{k-1} f(R_k) + \delta_{k-1} g(E_k),$$

where α, β, κ, δ are $\mathcal{F}$-adapted and $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}$, $\mathbb{R}$-valued respectively. And $f : \mathbb{R} \to \mathbb{R}, g : \mathcal{E} \to \mathbb{R}$ are both measurable, s.t. H is integrable. For $\Gamma_i(r,e) := \frac{\varphi_i(r)}{\varphi_{0,1}(r)} \frac{f_i^D(e)}{f_{1^d}^D(e)}$, where $\varphi_i = \varphi_{b_i,a_i^2}$ and $f_i^D = f_{\gamma^{(i)}}^D$. Then we have

$$\begin{aligned}\rho_k(HY) = \sum_{i=1}^d \Big\{&\rho_{k-1}(HY^i)\Gamma_i(R_k,E_k)\Pi^T e_i \\ &+ \rho_{k-1}(\alpha Y^i)\Gamma_i(R_k,E_k)\Pi^T e_i \\ &+ \rho_{k-1}(\kappa Y^i)\Gamma_i(R_k,E_k) f(R_k)\Pi^T e_i \\ &+ \rho_{k-1}(\delta Y^i)\Gamma_i(R_k,E_k) g(E_k)\Pi^T e_i \\ &+ (Diag(\Pi^T e_i))\rho_{k-1}(\beta Y^i)\Gamma_i(R_k,E_k)\Big\}\end{aligned}$$

where $\rho_k(HY) = (\rho(HY^1), \rho(HY^2), \ldots, \rho(HY^d))^T$, and Y^i is the $i-th$ element of vector Y.

Proof.

$$H_k Y_k = H_{k-1} Y_k + \alpha_{k-1} Y_k + \beta_{k-1}^T Y_k Y_k + \kappa_{k-1} f(R_k) Y_k + \delta_{k-1} g(E_k)$$

Then we compute

$$\rho_k(HY) = \tilde{\mathbb{E}}[Z_k^{-1} H_k Y_k | \mathcal{F}_k^C].$$

Set

$$Z_k^{-1} = Z_{k-1}^{-1} \sum_{i=1}^d Y_{k-1}^i \Gamma_i(R_k, E_k),$$

Thus based on Lemma(3.4.7) R,E are independent of Y under $\tilde{P}$

$$\rho_k(HY) = \sum_{i=1}^{d} \Gamma_i(R_k, E_k)\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1}Y_{k-1}^i H_{k-1}Y_k + Z_{k-1}^{-1}Y_{k-1}^i\alpha_{k-1}Y_k$$
$$+ Z_{k-1}^{-1}Y_{k-1}^i\beta_{k-1}^T Y_k Y_k + Z_{k-1}^{-1}Y_{k-1}^i\kappa_{k-1}f(R_k)Y_k + Z_{k-1}^{-1}Y_{k-1}^i\delta_{k-1}g(E_k)Y_k\Big|\mathcal{F}_k^C\Big]$$
$$= \sum_{i=1}^{d} \Gamma_i(R_k, E_k)\Big(\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1}Y_{k-1}^i H_{k-1}Y_k + Z_{k-1}^{-1}Y_{k-1}^i\alpha_{k-1}Y_k + Z_{k-1}^{-1}Y_{k-1}^i\beta_{k-1}^T Y_k Y_k\Big|\mathcal{F}_k^C\Big]$$
$$+ f(R_k)\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1}Y_{k-1}^i\kappa_{k-1}Y_k\Big|\mathcal{F}_k^C\Big] + g(E_k)\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1}Y_{k-1}^i\delta_{k-1}Y_k\Big|\mathcal{F}_k^C\Big]\Big)$$
$$= \sum_{i=1}^{d} \Gamma_i(R_k, E_k)\Big(\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1}Y_{k-1}^i H_{k-1}Y_k + Z_{k-1}^{-1}Y_{k-1}^i\alpha_{k-1}Y_k + Z_{k-1}^{-1}Y_{k-1}^i\beta_{k-1}^T Y_k Y_k\Big|\mathcal{F}_{k-1}^C\Big]$$
$$+ f(R_k)\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1}Y_{k-1}^i\kappa_{k-1}Y_k\Big|\mathcal{F}_{k-1}^C\Big] + g(E_k)\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1}Y_{k-1}^i\delta_{k-1}Y_k\Big|\mathcal{F}_{k-1}^C\Big]\Big)$$

Since $\mathcal{F} \subset \mathcal{H}$, we get

$$\rho_k(HY) = \sum_{i=1}^{d} \Gamma_i(R_k, E_k)\tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1}Y_{k-1}^i H_{k-1}Y_k + Z_{k-1}^{-1}Y_{k-1}^i\alpha_{k-1}Y_k$$
$$+ Z_{k-1}^{-1}Y_{k-1}^i\beta_{k-1}^T Y_k Y_k + Z_{k-1}^{-1}Y_{k-1}^i\kappa_{k-1}f(R_k)Y_k + Z_{k-1}^{-1}Y_{k-1}^i\delta_{k-1}g(E_k)Y_k\Big|\mathcal{H}_{k-1}\Big]\Big|\mathcal{F}_{k-1}^C\Big]$$
$$= \sum_{i=1}^{d} \Gamma_i(R_k, E_k)\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1}Y_{k-1}^i H_{k-1}\Pi^T e_i + Z_{k-1}^{-1}Y_{k-1}^i\alpha_{k-1}\Pi^T e_i$$
$$+ Z_{k-1}^{-1}\sum_{i=1}^{d} Y_{k-1}^i\beta_{k-1}^T(\Pi^T Y_{k-1})(\Pi^T Y_{k-1}) + Z_{k-1}^{-1}\sum_{i=1}^{d} Y_{k-1}^i\kappa_{k-1}f(R_k)\Pi^T e_i\Big|\mathcal{F}_{k-1}^C\Big]$$

Note the fact that

$$\beta_{k-1}^T Y_k Y_k = Diag(Y_k)\beta_{k-1}.$$

By this we get

$$\tilde{\mathbb{E}}\Big[Z_{k-1}^{-1}\sum_{i=1}^{d} Y_{k-1}^i\Gamma_i(R_k, E_k)\beta_{k-1}^T(\Pi^T Y_{k-1})(\Pi^T Y_{k-1})|\mathcal{F}_{k-1}^C\Big]$$
$$= \sum_{i=1}^{d}\Bigg(Diag\Big(\Pi^T e_i\Big)\Bigg)\rho_{k-1}^E(\beta Y^i)\Gamma_i(R_k, E_k),$$

which finishes the proof. $\square$

A.3 Numerical Simulations of Three States

The data is simulated by the following settings.

- $d = 3$;
- $b = (0.08, 0.01, -005)^T$, $a = (0.05, 0.03, 0.08)^T$, $\Pi = \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.05 & 0.9 & 0.05 \\ 0.05 & 0.05 & 0.9 \end{bmatrix}$;
- $Y_0 = e_1$;
- $\gamma = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

If we have an initial guess of $\gamma_0 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. The results of EM algorithm are shown below. Figure A.1 shows all a_i, b_i for $i = 1, 2, 3$, except a_1, has a better estimation if we combine Dirichlet distributed expert opinions.

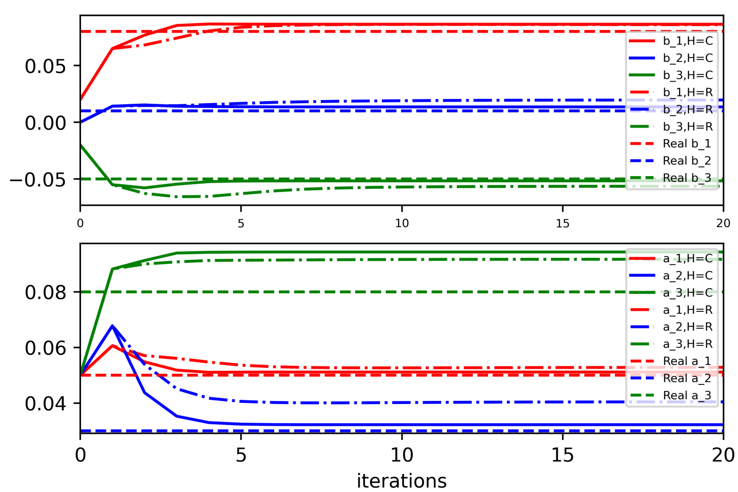


Figure A.1: Mean and standard deviation of distributions for three states

From Figure A.2, we find the results of $H = C$ and $H = R$ are similar, while $\Pi_{3,\cdot}$ has a better estimation when $H = C$.

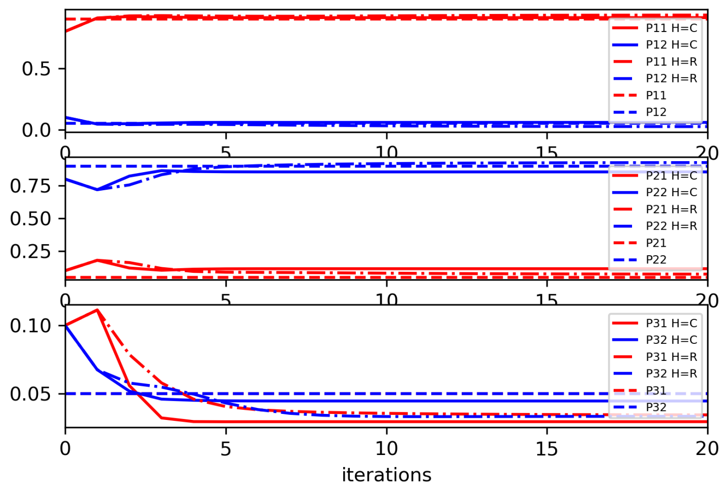


Figure A.2: Transition matrix

FigureA.3 reveals that the γ can be estimated accurately and all elements converge at end.

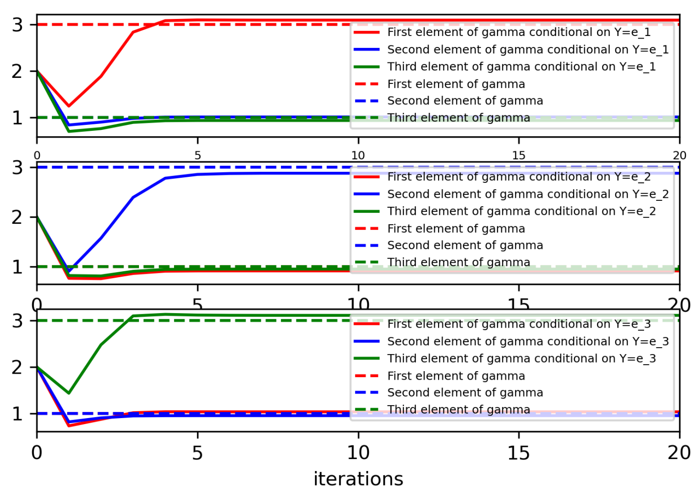


Figure A.3: γ

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