Variational Methods for Elliptic Boundary Value Problems

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Abstract

The mathematical modelling of problems in science and engineering leads often to partial differential equations in time and space with boundary and initial conditions. The boundary value problems can be written as extremal problems(principle of minimal potential energy), as variational equations (principle of virtual power) or as classical boundary value problems. There are connections concerning existence and uniqueness results between these formulations, which will be investigated using the powerful tools of functional analysis. The first part of the lecture is devoted to the analysis of linear elliptic boundary value problems given in a variational form. The second part deals with the numerical approximation of the solutions of the variational problems. Galerkin methods as FEM and BEM are the main tools. The h-version will be discussed, and an error analysis will be done. Examples, especially from the elasticity theory, demonstrate the methods.

Contents

0.1 Introduction

Let us start with some historical remarks [23] . In 1782 Laplace remarked, in a study on the space of planets that the potential

$$u(y) = \frac{1}{4\pi}\int_\Omega \frac{\varrho(x)}{|x-y|}dx$$

satisfies the partial differential equation (Laplace equation)

$$\triangle u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} = 0 \text{ in } \mathbb{R}^3 \setminus \bar{\Omega}\,.$$

1813 Poisson found that for a constant density ϱ it holds $-\triangle u = \varrho$ in $K_R(0) = \{x \in \mathbb{R}^3 : |x| < R\}$. This equation is called Poissson equation. 1839 Gauss has proved the validity of the Poisson equation for more general cases. 1846 until 1847 Riemann studied under Gauss in Göttingen. Then (1847-1849) he was a student in Berlin and attented lectures by Dirichlet on potential theory. Riemanns idea was to give a foundation to the theory of complex analytical functions by means of partial differential equations. To explain this let $f(z) = u+iv$ be an analytic function. The real functions u and v satisfy the so called Cauchy-Riemann differential equations $u_x = v_y, u_y = -v_x$, where $z = x+iy$. By differentiation we get $\triangle u = \triangle v = 0$. The Dirichlet-principle reads: The solution $u \in M = \{u \in C^2(\bar{\Omega}) : u = g \text{ on } \partial\Omega\}$ of the extremal problem

$$I(u) = \int_\Omega \sum_{i=1}^{2}\left(\frac{\partial u}{\partial x_i}\right)^2 dx = min! \tag{1}$$

is harmonic in Ω; or more general (Riemann 1857): For given sufficiently smooth functions v and g, the minimum problem

$$I(u) = \int_\Omega (u_x - v_y)^2 + (u_x + v_y)^2 dx = min!, \quad u = g \text{ on } \partial\Omega,$$

always has a solution in the set of piecewise continuously differentiable functions. Note that $I(u) \geq 0$ is bounded from below. Here, Ω is a bounded region in $\mathbb{R}^2$. Like Dirichlet, Riemann made no attempt to prove this existence principle. In studying Riemann's work Weierstrass found the Dirichlet principle unsatisfactory. In 1870 he constructed a counterexample, that means he found a functional, where the minimum is not realized by a function from $C^1[-1,+1]$:

$$I(u) = \int_{-1}^{+1} [xu'(x)]^2 dx = min! \tag{2}$$

with $u(-1) = 0, u(1) = 1$. [22, p. 176] For the sequence of functions

$$u_n(x) = \frac{1}{2} + \frac{\arctan(nx)}{2\arctan(n)} \qquad n = 1,2,\ldots$$

satisfying the boundary conditions, it holds $I(u_n) \to 0$ for $n \to \infty$, i.e. (u_n) is a minimal sequence. Suppose $u \in C^1[-1,1]$ is a solution of (0.2), then $I(u) = 0$ and $xu'(x) = 0$ for all $x \in [-1,+1]$. It follows $u'(x) = 0$ in $[-1,+1]$ and $u(x) = const$. This contradicts the boundary conditions. In 1900 D. Hilbert has formulated (Paris, 23 open problems) " ... one sensibly generalizes the concept of solution in order to save the Dirichlet principle." In 1928 Courant, Friedrichs and Levy have used "generalized" solutions of partial differential equations. The introduction of generalized derivatives leads to the definition of Sobolev spaces. In these spaces (and not in $C^2(\bar{\Omega})$) the Dirichlet principle can be justified.
Therefore extremal problems like (1) or equivalent variational (or weak) formulations of boundary value problems can be taken as basis for numerical computations. The algorithms developed work mostly in appropriate Sobolev spaces and the error analysis is done in corresponding norms. The lecture will not follow the historical paths.
In Section 1.1 we start with a short introduction into the theory of Sobolev spaces, mostly refering to different books for proofs of the statements. In Section 1.2 we discuss the equivalence of operator equations (classical boundary value problems) with variational and extremal problems in Hilbert spaces (Sobolev spaces). In this context we use some main principles and theorems from functional analysis. In Section 1.3 we introduce elliptic boundary value problems and give corresponding examples especially from mechanics of solid bodies. We show, how we can find reasonable variational formulations, which allow to apply the results of section 1.2. These three sections are the contents of Chapter 1. In Chapter 2 we start with Galerkin methods. Besides general results about the convergence, we discuss error estimates for linear elements as example and the influence of the regularity of solutions on the convergence rate. Section 2.2 deals with finite element methods whereas in Section 2.3 boundary element methods are studied.

Chapter 1

The Analysis

The first part of the lecture is devoted to the analysis of variational problems for elliptic boundary value problems. The theory is governed by the concept of generalized weak solutions which belong to Sobolev spaces.

1.1 Sobolev Spaces

Here we give a short introduction and refer to [1], [21], [8], [6] for further studies.

1.1.1 The space $L_2(\Omega)$

Let Ω be a nonempty open set in $\mathbb{R}^N$, $N \geq 1$.

Definition 1. *$L_2(\Omega)$ denotes the set of classes of in Ω measurable functions $u : \Omega \to \mathbb{R}$(or $\mathbb{C}$) with $\int_\Omega |u(x)|^2 dx < \infty$. Two functions u and v belong to the same class if $\int_\Omega |u-v|^2 dx = 0$; that means $u(x)=v(x)$ for almost all $x \in \Omega$.*

$L_2(\Omega)$ is endowed with the scalar product

$$(u,v)_0 = (u,v)_{L_2(\Omega)} = \int_\Omega u(x)\bar{v}(x)dx.$$

Lemma 1. *$L_2(\Omega)$ is a Hilbert space with respect to the norm*

$$\|u\|_0 = \|u\|_{L_2(\Omega)} := \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

Proof. [22, p.110] for $\Omega = (a,b)$ and [22, p.114]: there Ω is given as a measurable subset of $\mathbb{R}^N$. □

The elements of $L_2(\Omega)$ can be approximated by smoother functions, which we describe:

Definition 2.

1. *$C^k(\Omega)$ is the set of all real (or complex) valued functions $u : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) which have continuous partial derivatives of orders $m = 0, 1, \ldots, k$.*

2. *$C^k(\bar{\Omega})$ is the set of all $u \in C^k(\Omega)$ for which all partial derivatives of order $m = 0, 1, \dots, k$ can be extended continuously to the closure $\bar{\Omega}$ of Ω.*

3. *If $u \in C^k(\Omega)$ or $(C^k(\bar{\Omega}))$ for all $k = 0, 1, 2, \dots$, then we write $u \in C^\infty(\Omega)$ (or $C^\infty(\bar{\Omega})$).*

4. *$C_0^\infty(\Omega)$ is the set of all functions $u \in C^\infty(\Omega)$ which vanish outside a compact subset of Ω, i.e.* $\operatorname{supp} u$ *is compact,* $\operatorname{supp} u \subset \Omega$.

Lemma 2. *The sets $C_0^\infty(\Omega)$ and $C(\bar{\Omega}) = C^0(\bar{\Omega})$ are dense in $L_2(\Omega)$.*

Proof. [22, p.117/186] There is used a smoothing technique. □

The space $C_0^\infty(\Omega)$ plays an important role in the following. Here we formulate the variational lemma (Lemma of Du Bois-Reymond):

Lemma 3. *Let $u \in L_2(\Omega)$ be and*

$$\int_\Omega uv\,dx = 0 \qquad \forall v \in C_0^\infty(\Omega), \tag{1.1}$$

then $u(x) = 0$ for almost all $x \in \Omega$.

Proof. The relation (1.1) means that

$$(u, v)_0 = 0 \qquad \forall v \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$, we get

$$(u, v)_0 = 0 \qquad \forall v \in L_2(\Omega).$$

(For an arbitrary $\varepsilon > 0$ and any $\tilde{v} \in L_2(\Omega)$ we choose a function $v \in C_0^\infty(\Omega)$ with $||v - \tilde{v}|| < \varepsilon$. Then $(u, \tilde{v})_0 = (u, \tilde{v} - v)_0 + (u, v)_0 = (u, \tilde{v} - v)_0$ and $|(u, \tilde{v})_0| \le ||u||_0 ||\tilde{v} - v||_0 \le \varepsilon ||u||_0 = \varepsilon'$. Consequently $(u, \tilde{v})_0 = 0$). For $u = v$ we get $||u||^2 = \int_\Omega |u|^2 dx = 0$ and hence $u(x) = 0$ for almost all $x \in \Omega$. If u is continuous in Ω, than $u(x) = 0$ for all $x \in \Omega$. □

Remark. Lemmata 2 and 3 are equivalent.

1.1.2 Weak (generalized) derivatives

The integration by parts formula is the key for the definition of (global) generalized derivatives:

$$\int_a^b u'v dx = -\int_a^b uv' dx + uv|_a^b \quad \text{for } N = 1,$$

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = -\int u \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} uvn_i d\sigma \quad \text{for } N > 1.$$

Here n_i are the components of the exterior unit normal vector n. If $v \in C_0^\infty(\Omega)$, then the boundary term vanishes. For shortness we use the notation $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1 \dots \alpha_n)$, $|\alpha| = \sum_{i=1}^n \alpha_i$, $\alpha_i \in \{0, 1, 2, \dots\}$.

Definition 3. *Let be $u \in L_2(\Omega)$. The function $w = D^\alpha u \in L_2(\Omega)$ is called the weak derivative of u in the domain Ω if*

$$(w, v)_0 = (-1)^{|\alpha|}(u, D^\alpha v)_0 \qquad \forall v \in C_0^\infty(\Omega).$$

The following properties hold:

- The weak derivative is uniquely determined in $L_2(\Omega)$. This follows from Lemma 3.
- If a classical derivative $D^\alpha u \in C(\bar{\Omega})$ exists, then it coincides with the weak derivative (a.e.).
- $D^{\alpha+\beta}u = D^\beta(D^\alpha u) = D^\alpha(D^\beta u)$.
- If $u_n \in C^\infty(\bar{\Omega}), u_n \to u$ in $L_2(\Omega)$ and $D^\alpha u_n \to w$ in $L_2(\Omega)$ then $w = D^\alpha u$.

Exercise
1^0 Let be $\Omega = (-1, +1), u(x) = |x|$. Show that $u'(x) = w(x) = sgn(x)$.
2^0 Step functions (piecewise constant) have no weak derivatives.

1.1.3 The spaces $H^k(\Omega)$ and $\overset{\circ}{H}{}^k(\Omega)$, $k \in \mathbb{N} \cup \{0\}$

Definition 4. *Let be $k \in \mathbb{N} \cup \{0\}$. $H^k(\Omega) \subset L_2(\Omega)$ is the set of all functions which have weak derivatives $D^\alpha u \in L_2(\Omega)$ for all α with $|\alpha| \leq k$:*

$$H^k(\Omega) := \{u \in L_2(\Omega) : D^\alpha u \in L_2(\Omega) \text{ for } |\alpha| \leq k\},$$

endowed with the scalar product

$$(u, v)_k = (u, v)_{H^k(\Omega)} := \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)_0$$

and the Sobolev-norm

$$\|u\|_k = \|u\|_{H^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^2_{L_2(\Omega)} \right)^{\frac{1}{2}}. \tag{1.2}$$

$H^k(\Omega)$ is a Hilbert space.

We remark that $C^\infty(\bar{\Omega})$ is dense in $H^k(\Omega)$, if Ω is smooth enough. Lemma 4 justifies another definition of $H^k(\Omega)$: see [11, p.17]

Lemma 4. *$C^\infty(\Omega) \cap H^k(\Omega)$ is dense in $H^k(\Omega)$.*

Proof. [21, p.74] □

Definition 5. *Let be $k \in \mathbb{N} \cup \{0\}$. $H^k(\Omega)$ is the completion of the set $\{u \in C^\infty(\Omega) : ||u||_{H^k(\Omega)} < \infty\}$ with respect to the norm (1.2).*

We now define the space $\overset{\circ}{H}{}^k(\Omega)$:

Definition 6. *$\overset{\circ}{H}{}^k(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm (1.2).*

Lemma 5. $\overset{\circ}{H}{}^0(\Omega) = L_2(\Omega)$.

Proof. Apply Lemma 2. □

Let us remark, that for a bounded domain Ω and $k \geq 1$ $\overset{\circ}{H}{}^k(\Omega)$ is a proper subspace of $H^k(\Omega)$. This follows from Theorem 1 below. The space $\overset{\circ}{H}{}^k(\Omega)$ is a very important space in the theory of boundary value problems. The norm in $\overset{\circ}{H}{}^k(\Omega)$ is equivalent to a seminorm, which is given by the highest derivatives.

Theorem 1. *For a bounded domain Ω, the norm $\|\cdot\|_k$ and the seminorm $|\cdot|_k$*

$$|u|_k = |u|_{H^k(\Omega)} := \left[\sum_{|\alpha|=k} ||D^\alpha u||_0^2\right]^{\frac{1}{2}}$$

are equivalent in $\overset{\circ}{H}{}^k(\Omega)$.

Proof. a) It is evident that $|u|_k \leq \|u\|_k$.
b) It holds the Poincaré-Friedrichs inequality:

$$||u||_k \leq c|u|_k \quad \text{for all} \quad u \in \overset{\circ}{H}{}^k(\Omega) \tag{1.3}$$

for a positive constant c.
We proof b) firstly for elements u from $C_0^\infty(\Omega)$. Let α be an multiindex with $|\alpha| = k-1$. Then $w = D^\alpha u \in C_0^\infty(\Omega)$. Since Ω is bounded there is a ball $K_R(0)$ with $\Omega \subset K_R(0)$. For every point $x = (x_1, \ldots x_N) \in \Omega$ it holds, that $x_1 \in [-R, R]$ and

$$\begin{aligned} |w(x)|^2 &= \left|\int_{-R}^{x_1} \frac{\partial w}{\partial x_1}(\xi, x_2, \ldots x_N) d\xi\right|^2 \\ &\overset{\text{Schwarz inequality}}{\leq} (x_1 + R) \int_{-R}^{x_1} \left|\frac{\partial w}{\partial x_1}(\xi, \ldots)\right|^2 d\xi \\ &\leq 2R \int_{-R}^{+R} \left|\frac{\partial w}{\partial x_1}(\xi, \ldots)\right|^2 d\xi\,. \end{aligned}$$

Integrating on Ω we get

$$\begin{aligned} \int_\Omega |w(x)|^2 dx = ||w||_0^2 &\leq 2R \int_\Omega \int_{-R}^{+R} \left|\frac{\partial w}{\partial x_1}(\xi, x_2, \ldots x_N) d\xi\right|^2 d\xi dx \\ &= 4R^2 \left\|\frac{\partial w}{\partial x_1}\right\|_0^2 \leq 4R^2 |u|_k^2\,. \end{aligned}$$

It follows that

$$|u|_{k-1}^2 = \sum_{|\alpha|=k-1} \|D^\alpha u\|_0^2 \leq 4R^2 M(k)|u|_k^2 = C_{k-1}|u|_k^2,$$

and more general

$$|u|_{j-1}^2 \leq C_{j-1}|u|_j^2 \quad \text{for} \quad 1 \leq j \leq k,$$

and

$$|u|_j^2 \leq C_j|u|_{j+1}^2 \leq C_j C_{j+1}|u|_{j+2}^2 \leq \ldots \leq \tilde{C}_j|u|_k^2 \quad \text{for} \quad 0 \leq j \leq k.$$

Finally we have

$$\|u\|_k^2 = \sum_{j=0}^{k} |u|_j^2 \leq c^2|u|_k^2.$$

We now consider $u \in \overset{\circ}{H}{}^k(\Omega)$. For any $\varepsilon > 0$ we find an element $\tilde{u} \in C_0^\infty(\Omega)$ with $\|u - \tilde{u}\|_k < \varepsilon$. It is

$$\begin{aligned} \|u\|_k &\leq \|u-\tilde{u}\|_k + \|\tilde{u}\|_k \\ &\leq \varepsilon + c|\tilde{u}|_k \\ &\leq \varepsilon + c|\tilde{u}-u|_k + c|u|_k \\ &< \varepsilon(1+c) + c|u|_k, \end{aligned}$$

and consequently $\|u\|_k \leq c|u|_k$. □

Remark. Theorem 1 remains valid, if Ω is bounded in one direction, that means $\Omega \subset \{x \in \mathbb{R}^N : |x_i| < R, i \in \{1, \ldots, N\}\}$. The Poincaré-Friedrichs inequality can be generalized.

Theorem 2. [20, p.385,386] *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz-continuous boundary, let $\Gamma \in \partial\Omega$ be with mes $\Gamma \neq 0$.*
Then

$$\begin{aligned} ||u||_1 &\leq C\left[\left|\int_{\partial\Omega} u d\sigma\right| + \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^2 dx\right]^{\frac{1}{2}} \\ \textit{and} \quad ||u||_1 &\leq C\left[\int_\Gamma |u|^2 d\sigma + \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^2 dx\right]^{\frac{1}{2}} \end{aligned}$$

for all $u \in H^1(\Omega)$.

The case $p \neq 2$
For a finer analysis, especially of nonlinear problems, it is useful to enlarge the class of Sobolev spaces $H^k(\Omega)$.
Let $p \in [1, \infty)$. Analogously to the space $L_2(\Omega)$ we introduce the space $L_p(\Omega)$ as the set of classes of measurable functions $u : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) with $\int_\Omega |u(x)|^p dx < \infty$. The

functions u and v belong to the same class if $u(x) = v(x)$ a.e. in Ω. $L_p(\Omega)$ is equipped with the norm

$$||u||_{L_p(\Omega)} = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$$

and is a Banach space [8, p.74]. We remark that $L_\infty(\Omega)$ is also defined [8, p.82], and plays a role in connection with error estimates.
We define the space $H_p^k(\Omega)$ analogously to H^k and $\overset{\circ}{H}{}_p^k(\Omega)$ analogously to $\overset{\circ}{H}{}^k(\Omega) = \overset{\circ}{H}{}_2^k(\Omega)$ The norm in $H_p^k(\Omega)$ is defined as follows:

$$||u||_{H_p^k(\Omega)} = ||u||_{k,p} = \left[\sum_{|\alpha|\leq k} ||D^\alpha u||^p_{L_p(\Omega)}\right]^{\frac{1}{p}}. \tag{1.4}$$

Besides the notation $H_p^k(\Omega)$ one will find also the notation $W_p^k(\Omega)$ for a slightly different Sobolev space [8, p.255]. For a bounded domain Ω $W_p^k(\Omega)$ is the completion of $C^\infty(R^N)|_{\bar{\Omega}}$ with the respect to the norm (1.4). $W_p^k(\Omega)$ and $H_p^k(\Omega)$ coincide for a large class of domains, e.g. for elements with piecewise Lipschitz-continuous boundary.
We cite now some embedding theorems, which describe the relations both between different Sobolev spaces and spaces of continuously differentiable functions.

Theorem 3 (Embedding theorems). [8, p.300], [1] *Let Ω be a bounded domain in $\mathbb{R}^N$ with cone property, i.e. every point $x \in \bar{\Omega}$ is the vertex of a finite cone Cx which is congruent to a cone C. The cone C is the intersection of an open ball in $\mathbb{R}^N$ with the set*

$$\{\lambda x; \lambda > 0, x \in \mathbb{R}^N, |x-y| < r\},$$

where y is a fixed point in $\mathbb{R}^N$ with $|y| > r$, $r > 0$.
It holds:

(i). If $kp < N$ and $p \leq q \leq \frac{Np}{N-kp}$, then $H_p^k(\Omega) \subset L_q(\Omega)$.

(ii). If $kp = N$ and $p \leq q \leq \infty$, then $H_p^k(\Omega) \subset L_q(\Omega)$, moreover, $H_1^N(\Omega) \subset C^0(\bar{\Omega})$.

(iii). If $(k-j)p > N$ for $j = 0, 1, 2, \dots$, then $H_p^k(\Omega) \subset C^j(\bar{\Omega})$.

Corollary.

$$\begin{aligned} H_2^1(\Omega) &\not\subset C(\bar{\Omega}), \text{ but } H_2^2(\Omega) \subset C(\bar{\Omega}) \text{ for } N = 2, 3, \\ H_2^k(\Omega) &\subset C(\bar{\Omega}) \text{ if } k > \frac{N}{2}, \\ H_p^1(\Omega) &\subset C(\bar{\Omega}) \text{ for } N = 2, \;\; p > 2. \end{aligned}$$

1.1.4 The space $H^s(\Omega)$ with real $s \geq 0$ (Sobolev-Slobodeckij space)

Let be $\Omega \in \mathbb{R}^N$, $s \geq 0$, $s = k + \lambda$ with $k \in \mathbb{N} \cup \{0\}$ and $0 < \lambda < 1$.
We define:

$$H^s(\Omega) := \Big\{ u \in H^k(\Omega) : ||u||_s^2 = = \|u\|_k^2 + \sum_{|\alpha| \leq k} \int_\Omega \int_\Omega \frac{[D^\alpha u(x) - D^\alpha u(y)]^2}{|x-y|^{N+2\lambda}} dxdy < \infty \Big\}.$$

$H^s(\Omega)$ is a Hilbert space with the scalar product

$$(u,v)_s = \sum_{|\alpha| \leq k} \Big[\int_\Omega D^\alpha u(x) D^\alpha v(x) dx + \int_{\Omega \times \Omega} \frac{[D^\alpha u(x) - D^\alpha u(y)][D^\alpha v(x) - D^\alpha v(y)]}{|x-y|^{N+2\lambda}} dxdy \Big].$$

Analogously we define:

$$H_p^s(\Omega) := \Big\{ u \in H_p^k(\Omega) : \|u\|_{s,p}^p = ||u||_{k,p}^p + \sum_{|\alpha| \leq k} \int_{\Omega \times \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+p\lambda}} dxdy < \infty \Big\}.$$

Example. The function

$$u(x) = \begin{cases} 1 & \text{for } x \in [-1,1], \\ 0 & \text{else} \end{cases}$$

does not belong to $H^1(-2,2)$, since $u' = \delta(-1) - \delta(+1)$ in the distributional sense, δ is the Dirac distribution. But: $u \in H^s(-2,2)$ for $0 \leq s < \frac{1}{2}$. Since $|u(x) - u(y)|$ vanishes in the hatched rectangles in Fig. 1 we have namely:

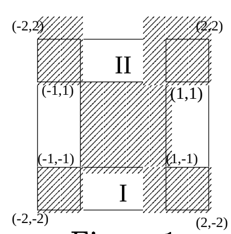


Figure 1

$$\int_{-2}^{2} \int_{-2}^{2} \frac{|u(x) - u(y)|}{|x-y|^{1+2\lambda}} dxdy = 2\Big(\int_{-2}^{-1} \int_{-1}^{1} \frac{1}{|x-y|^{1+2\lambda}} dxdy + \int_{1}^{2} \int_{-1}^{1} \frac{1}{|x-y|^{1+2\lambda}} dxdy \Big) = (I) + (II).$$

It is

$$\begin{aligned}
(I) &= \int_{-2}^{-1}\left(\int_{-1}^{+1}\frac{1}{|x-y|^{1+2\lambda}}dx\right)dy = \int_{-2}^{-1}\left(\int_{-1}^{+1}\frac{1}{(x-y)^{1+2\lambda}}dx\right)dy \\
&= \int_{-2}^{-1} -\frac{1}{2\lambda}\left[\frac{1}{(x-y)^{2\lambda}}\right]_{-1}^{+1} dy = \int_{-2}^{-1} -\frac{1}{2\lambda}\left[\frac{1}{(1-y)^{2\lambda}} - \left(\frac{1}{(1+y)^{2}\lambda}\right)\right] dy \\
&= -\frac{1}{2\lambda}\left[-\frac{(1-y)^{-2\lambda+1}}{1-2\lambda} - \frac{(1+y)^{-2\lambda+1}}{1-2\lambda}\right]_{-2}^{-1} < \infty
\end{aligned}$$

for $2\lambda < 1$, that means $\lambda < \frac{1}{2}$.
Similar we get

$$(II) = \int_{1}^{2}\left(\int_{-1}^{+1}\frac{1}{(y-x)^{1+2\lambda}}dx\right)dy < \infty \quad \text{for} \quad \lambda < \frac{1}{2}.$$

1.1.5 Trace spaces

It is natural to study the boundary values or so called "traces" of elements from Sobolev spaces in connection with boundary value problems.
A main tool is to use a diffeomorphism which maps locally a neighborhood of the boundary intersected with Ω into the half space $\mathbb{R}^N_+$ and to study the restrictions of the functions onto $\mathbb{R}^{N-1}$. For this we need a certain smoothness of the boundary.

Definition 7.
Ω is from the class $C^{k,\lambda}$ $(k \in \mathbb{N}_0, \lambda \in (0,1])$ if

1. *There exist open subsets $U_1, \ldots, U_m \in \mathbb{R}^N$ with*

$$\bigcup_{r=1}^{m} U_r \supset \partial\Omega, \quad \Lambda_r = U_r \cap \partial\Omega \neq 0.$$

2. *For any subset there is a local coordinate system, that $\Lambda_r = \{x_{r_1}, \ldots, x_{r_{N-1}}, a_r(x'_r)\}$ with $a_r \in C^{k,\lambda}(\Delta_r)$ and $\Delta_r = \{x'_r : |x_{r_i}| < \alpha, i = 1, \ldots, N-1\}$. Here is*

$$C^{k,\lambda}(\Omega) = \left\{u \in C^k(\Omega) : \sup_{x,y\in\Omega, x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda} < \infty\right\}.$$

$C^{k,\lambda}(\Omega)$ is called the space of k-times Lipschitz continuous differentiable functions for $\lambda = 1$ or of k-times Hölder continuous differentiable functions for $\lambda < 1$.

Theorem 4 (Trace Theorem). [8, p.307], [14], [21, p.130]

- *Let $p > 1$, $k \in \mathbb{N}, \Omega \in C^{k-1,1}$. There exists a unique continuous linear mapping $T_k : H_p^k(\Omega) \longrightarrow \prod_{l=0}^{k-1} H_p^{k-l-\frac{1}{p}}(\partial\Omega)$ such that*

$$T_k u = \left(u, \frac{\partial u}{\partial n}, \ldots, \frac{\partial^{k-1}}{\partial n^{k-1}} u\right) \qquad \forall u \in C^\infty(\bar{\Omega}).$$

n denotes the external unit normal vector, T_k is the trace operator. For $k=1$, $\Omega \in C^{0,1}$, we have $T_1 u = u|_{\partial\Omega}$ "in the trace sense".

- *For $p=2$,*

$$-\frac{1}{2}+k < s < k-1+\lambda, 0<\lambda<\frac{1}{2}, k \in \mathbb{N}, \Omega \in C^{k,\lambda},$$

T_k maps $H^s(\Omega) \longrightarrow \prod_{l=0}^{k-1} H^{s-l-\frac{1}{2}}(\partial\Omega)$ continuously.

COROLLARY. The estimates hold:

$$||T_k u||_{p,\partial\Omega} = \sum_{l=0}^{k-1} \left\| \frac{\partial^l u}{\partial n^l} \right\|_{{H_p}^{k-l-\frac{1}{p}}(\partial\Omega)} \leq C||u||_{H_p^k(\Omega)}$$

$$||T_k u||_{\partial\Omega} = \sum_{l=0}^{k-1} \left\| \frac{\partial^l u}{\partial n^l} \right\|_{H^{s-l-\frac{1}{2}}(\partial\Omega)} \leq C||u||_{H^s(\Omega)}.$$

The proof needs a lot of techniques such as partition of unity, local mappings onto $\mathbb{R}^N_+$, estimates in $\mathbb{R}^N_+$, inverse mappings [8], [21], [6].
The trace operator is invertible, that means there is an extension from the boundary values onto the whole domain.

Theorem 5 (Extension Theorem). [14], [8, p.338]

- *Let $p>1, k \in \mathbb{N}, \Omega \subset C^{k,1}$. There exists a continuous linear mapping*

$$F_k : \prod_{l=0}^{k-1} H_p^{k-l-\frac{1}{p}}(\partial\Omega) \longrightarrow H_p^k(\Omega),$$

such that for each $(u_0, u_1, \ldots, u_{k-1}) \in \prod_{l=0}^{k-1} {H_p}^{k-l-\frac{1}{p}}(\partial\Omega)$ with $F_k(u_0, u_1, \ldots, u_{k-1}) = v$ it follows that $\frac{\partial^l v}{\partial n^l} = u_l$ on $\partial\Omega$, $l = 0, 1, \ldots, k-1$.

- *For $k=1$, $p>1$ we need only that $\Omega \in C^{0,1}$.*

- *For $p=2$, $-\frac{1}{2}+k<s<k+\lambda-1, 0<\lambda<\frac{1}{2}, k \in \mathbb{N}, \Omega \in C^{k,\lambda}$ there exists a continuous linear mapping*

$$F_k : \prod_{l=0}^{k-1} H^{s-l-\frac{1}{2}}(\partial\Omega) \longrightarrow H^s(\Omega) \quad \textit{(see [21, p.133])}.$$

Lemma 6.

- *Let be $\Omega \in C^{0,1}$, then $\overset{\circ}{H}{}^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$ in the trace sense.*

- *Let be $\Omega \in C^{k,1}$, $k \in \mathbb{N}, p \geq 1$ then*

$$\begin{aligned} \overset{\circ}{H}{}^k_p(\Omega) &= \{u \in H^k_p(\Omega) : u|_{\partial\Omega} = \frac{\partial u}{\partial n}|_{\partial\Omega} = \ldots = \frac{\partial^{k-1} u}{\partial n^{k-1}} = 0\} \\ &= \{u \in H^k_p(\Omega) : D^\alpha u|_{\partial\Omega} = 0 \text{ for } 0 \leq |\alpha| \leq k-1\}. \end{aligned}$$

Proof. We have to prove that $\ker T_k = \overset{\circ}{H}{}^k_p(\Omega)$. The step $\overset{\circ}{H}{}^k_p(\Omega) \subset \ker T_k$ is trivial; in order the equality to prove one needs some technical considerations [21, p.135], [6, p.120]. □

1.1.6 The space $H^s(\Omega)$ with $s < 0$

We remind of the definition of a dual space X' of a Banach space X on the field of real (complex) numbers $\mathbb{R}$ or $\mathbb{C}$. The dual space X' is the space of all bounded linear mappings F from X into $\mathbb{R}$ (or $\mathbb{C}$), equipped with the operator norm

$$\|F\|_{X'} = \sup_{u \neq 0} \frac{|F(u)|}{\|u\|_X}.$$

Definition 8. *Let be $1 < p < \infty$, $s \geq 0$ a real number.*

$$H^{-s}_p(\Omega) = [\overset{\circ}{H}{}^s_q]'$$

where $1/p + 1/q = 1$ and $\|u\|_{H^{-s}_p(\Omega)} = \sup_{v \neq 0} \frac{|(u,v)_0|}{\|v\|_{H^s_q(\Omega)}}$.

The restriction of $F \in H^{-k}_p(\Omega)$ on $C^\infty_0(\Omega)$ defines a distribution (a bounded linear functional on $C^\infty_0(\Omega)$, where $C^\infty_0(\Omega)$ is equipped with a "convergence definition"). The following theorem characterizes the elements of $H^{-k}_p(\Omega)$, $k \in \mathbb{N}$.

Theorem 6. [8, p.294]
Let $p \in (1, \infty), k \in \mathbb{N}$. Then $F \in H^{-k,p}(\Omega)$ if and only if there exists a family $\{f_\alpha\}_{|\alpha| \leq k} \in L_p(\Omega)$ such that

$$F = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha f_\alpha, \tag{1.5}$$

where $D^\alpha f_\alpha$ denotes the distributional derivative. Moreover,

$$\|F\|_{H^{-l,p}(\Omega)} = \inf \sum_{|\alpha| \leq k} \|f_\alpha\|_{L_p(\Omega)},$$

where the infimum is taken over all families $\{f_\alpha\}_{|\alpha| \leq k}$ such that F can be expressed by formula (1.5).

For $p = 2$ the trace spaces are defined

$$H^{-s}(\partial\Omega) = [H^s(\partial\Omega)]',$$

where the duality can be understood by the Riesz-representation theorem as a scalar product in $L_2(\partial\Omega)$.

1.2 Operator equations and their variational formulations

The basic idea of functional analysis is to formulate differential and integral equations in terms of operator equations

$$Au = f, \qquad A : X \longrightarrow Y, \tag{1.6}$$

where X and Y are appropriate function spaces, $u \in X$ is the unknown. For example, A is the Laplacian, $X = \{u \in C^2(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$, $Y = C(\bar{\Omega})$. Then (1.6) describes the Dirichlet problem

$$\begin{aligned} \triangle u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

If X is a Banach Space, $Y = X'$, then the equation (1.6) means

$$\langle Au, v\rangle = \langle f, v\rangle \qquad \forall v \in X$$

or

$$a(u, v) := \langle Au, v\rangle = \langle f, v\rangle \qquad \forall v \in X. \tag{1.7}$$

The relations (1.7) are called variational (or weak) formulation of the problem (1.6) and read: Find an element $u \in X$ such that

$$a(u, v) = \langle f, v\rangle \qquad \forall v \in X.$$

Furthermore, we can introduce a functional

$$F(u) = \frac{1}{2}a(u, u) - \langle f, u\rangle$$

The extremal problem reads: Find an element $u_0 \in X$ such that

$$\min_{u \in X} F(u) = F(u_0). \tag{1.8}$$

Under certain assumptions u_0 is a solution of (1.7) and (1.6).
In the following we describe the connections between the problems (1.6) , (1.7) and (1.8) with regard to the solvability and uniqueness.

1.2.1 Example – The Dirichlet problem for the Laplacian

Proposition 1. Let be $\Omega \subset \mathbb{R}^N$ a bounded domain with a piecewise smooth boundary $\partial\Omega$, $f \in C(\bar{\Omega}), g \in C(\partial\Omega)$. If $u \in C^2(\bar{\Omega})$ solves one of the following three problems, then it solves the other ones too.

(1) $-\triangle u = f$ in Ω, $\quad u = g$ on $\partial\Omega$

(2) for all $v \in M = \{v \in C^1(\bar{\Omega}) : v|_{\partial\Omega} = 0\}$ and $u|_{\partial\Omega} = g$:

$$a(u, v) = \int_\Omega \sum_{i=1}^N \partial_i u \partial_i v dx = \langle f, v\rangle = \int_\Omega f(x)v(x)dx$$

(3) $\min_{\substack{u\in C^1(\bar\Omega)\\ u|_{\partial\Omega}=g}} F(u) = \int_\Omega(\frac{1}{2}\sum_{i=1}^N(\partial_i u)^2 - uf)dx = \frac{1}{2}a(u,u) - \langle f,u\rangle$

Proof. (3) $\to$ (2): Let be $u_0 \in C^2(\bar\Omega)$ a solution of the problem (3). We show that u_0 solves problem (2). Besides u_0 the following functions are admissible functions for finding the minimum of $F(u)$:

$$u_0 + tv, \qquad v \in M, t \in \mathbb{R}.$$

Let be

$$\begin{aligned}\varphi(t) = F(u_0+tv) &= \int_\Omega \Big[\frac{1}{2}\sum_{i=1}^N [D_i(u_0+tv)]^2 - (u_0+tv)f\Big]dx \\ &= \frac{1}{2}a(u_0+tv, u_0+tv) - \langle f, u_0+tv\rangle \\ &\overset{\text{symmetry}}{=} \frac{1}{2}\big[a(u_0,u_0) + 2ta(u_0,v) + t^2a(v,v)\big] \\ &\quad -\langle f,u_0\rangle - t\langle f,v\rangle \quad \forall t\in\mathbb{R}.\end{aligned}$$

Since $\min_{t\in\mathbb{R}}\varphi(t) = \varphi(0)$, the necessary condition for an extremal value reads:

$$\varphi'(t)|_{t=0} = a(u_0,v) + ta(v,v) - \langle f,v\rangle\Big|_{t=0} = 0.$$

Hence

$$a(u_0,v) = \langle f,v\rangle \quad \forall v\in M.$$

(2) $\longrightarrow$ (1): Applying integration by parts to (2) we get

$$\int_\Omega(-\triangle u_0 - f)v dx = 0 \qquad \forall v \in M$$

and especially for $v \in C_0^\infty(\Omega)$. Due to the variational Lemma 3 and $u_0 \in C^2(\bar\Omega)$ we have $-\triangle u_0 = f$; furthermore, it holds $u_0|_{\partial\Omega} = g$ in (2).
(1) $\longrightarrow$ (2): Multiplying (1) by elements $v \in M$ and integrating by parts we get (2).
(2) $\longrightarrow$ (3): Let be $a(u_0,v) = \langle f,v\rangle$ $\forall v\in M$. Then is $\varphi'(0) = a(u_0,v) - \langle f,v\rangle = 0$ $\forall v \in M$ and zero is a critical value for the real function

$$\varphi(t) = \frac{1}{2}a(u_0+tv, u_0+tv) - (f, u_0+tv).$$

Since $\varphi''(t)\big|_{t=0} = a(v,v) > 0$ for $v \neq 0$, it follows that $t=0$ yields the minimum. Hence $\varphi(0) = F(u_0) \le \varphi(t) = F(u_0+tv)$. The admissible set $\{u \in C^1(\bar\Omega), u\big|_{\partial\Omega} = g\} = m$ for searching the minimum coincides with the set $\{u_0+tv\}_{t\in R, v\in M} = \tilde m$. Indeed, it is evident that $\tilde m \subset m$. For $u \in m$ we have $u = u - u_0 + u_0 = v + u_0 \in \tilde m$, hence $m \subset \tilde m$.
□

REMARK. In order to show the equivalence of problems (2) and (3) we only need that $u_0 \in C^1(\bar\Omega)$. There are situations, where no solutions in $C^2(\bar\Omega)$ exist.

EXAMPLE: Let η be a cut-off function with support in a neighborhood of P (See Fig. 2).

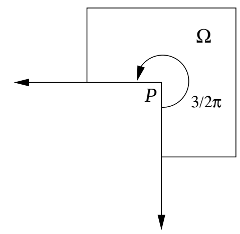


Figure 2

The function

$$u_0 = \eta r^{\frac{2}{3}} \sin \frac{2}{3}\varphi$$

is not contained in $C^1(\bar{\Omega})$, but it solves the Poisson equation with a smooth right hand side:

$$\triangle u_0 = \eta \left[\frac{2}{3}\left(\frac{2}{3} - 1\right) + \frac{2}{3} - \left(\frac{2}{3}\right)^2 \right] r^{\frac{2}{3}-2} \sin \frac{3}{2}\varphi + \text{smooth remainder},$$

where

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \varphi^2}\frac{1}{r^2}$$

is the Laplace operator.

1.2.2 Bilinear forms

In the proof of Proposition 1 we have used, that $a(\cdot,\cdot)$ is symmetric and $a(v,v) > 0$ for $v \in M, v \neq 0$, that means $a(\cdot,\cdot)$ is positive.
These properties are important for more general considerations.

Definition 9. *Let V be a Hilbert space. The mapping $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called a real bilinear form if*

$$a(u, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 a(u, v_1) + \alpha_2 a(u, v_2)$$
$$a(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1 a(u_1, v) + \alpha_2 a(u_2, v)$$

for all $u, v, u_i, v_i, \in V, \alpha_1, \alpha_2 \in \mathbb{R}$ (in the complex case we have sesquilinear forms and $a(u, \alpha_1 v_1 + \alpha_2 v_2) = \overline{\alpha_1} a(u, v_1) + \overline{\alpha_2} a(u, v_2)$) The bilinear form is called continuous (or bounded) if there is a $C > 0$ with

$$|a(u,v)| \leq C\|u\|_V \|v\|_V \quad \forall u, v \in V.$$

Lemma 7. *For every bounded bilinear form there exists an uniquely determined operator $A \in L(V, V')$ such that*

$$a(u,v) = \langle Au, v \rangle \quad \forall u, v \in V \tag{1.9}$$

and

$$\|A\|_{L(V,V')} \le C.$$

Conversly, every operator $A \in L(V, V')$ creates a bilinear form (1.9).

Proof.
a) $a(\cdot,\cdot)$ is given. We fix an element $u \in V$ and define the functional

$$F_u v := a(u,v) \quad \forall v \in V.$$

F_u is linear and

$$|F_u v| = |a(u,v)| \le C\|u\|_V\|v\|_V = C'\|v\|_V.$$

Consequently $F_u \in V'$ and $\|F_u\| \le C' = C\|u\|_V$. We define $Au := F_u \quad \forall u \in V$. The operator A is linear and bounded:

$$\|Au\|_{V'} = \|F_u\|_{V'} \le C\|u\|_V$$

and $\|A\| \le C$.
b) $A \in L(V,V')$ is given. We define $a(u,v) := \langle Au, v\rangle$, $a(\cdot,\cdot)$ is linear. Since

$$|a(u,v)| \le \|Au\|_{V'}\|v\|_V \le \|A\|_{L(V,V')}\|u\|_V\|v\|_V,$$

$a(\cdot,\cdot)$ is bounded. Let us remark that

$$\begin{aligned}\|A\|_{L(V,V')} &= \sup_{\|u\|_V=1} \|Au\|_{V'}\\ &= \sup_{\|v\|_V=1}\{\sup_{\|u\|_V=1} |\langle Au, v\rangle|\}\\ &= \sup_{\|v\|=1,\|u\|=1} |a(u,v)|.\end{aligned}$$

The adjoint bilinear form is defined

$$a^*(u,v) = a(v,u) = \langle A'u, v\rangle.$$

A' denotes the adjoint (dual) operator $\|A\| = \|A'\|$. The bilinear form $a(\cdot,\cdot)$ is symmetric, if $a(u,v) = a(v,u)$ $\forall u,v \in V$. The operator equation $Au = f$ is satisfied iff $\langle Au, v\rangle = \langle f, v\rangle$ $\forall v \in V$. □

Lemma 8. (Lax-Milgram) [14, 6] *Let be $A \in L(V,V')$ and $a(\cdot,\cdot)$ the corresponding bilinear form. Then the following statements are equivalent:*

(1) $A^{-1} \in L(V',V)$ exists.

(2) There exist real numbers $\varepsilon, \varepsilon' > 0$ with

$$\inf\{\sup\{|a(u,v)|, v \in V, \|v\|_V = 1\} : u \in V, \|u\|_V = 1\} = \varepsilon > 0 \qquad (1.10)$$

$$\inf\{\sup\{|a(u,v)|, u \in V, \|u\|_V = 1\} : v \in V, \|v\|_V = 1\} = \varepsilon' > 0 \qquad (1.11)$$

(3) The inequality (1.10) and the inequality

$$\sup\{|a(u,v)| : u \in V, \|u\|_v = 1\} > 0 \tag{1.12}$$

are valid for $v \in V, v \neq 0$.

REMARK. The condition

$$\inf\{\sup\{|a(u,v)|, v \in V, \|v\|_V = 1\} u \in V, \|u\|_V = 1\} \geq \varepsilon > 0 \tag{1.13}$$

together with (1.12) are called Ladyshenskaja-Babushka-Brezzi (or LBB-condition). Furthermore, $A^{-1} \in L(V', V)$ means: $\|A^{-1}f\| = \|A^{-1}Au\| = \|u\| \leq \|A^{-1}\| \, \|f\|$.

Proof. (1) $\to$ (2): For $A^{-1}u' = u$ it holds

$$\begin{aligned}
\inf_{\substack{u\in V\\ u\neq 0}} \sup_{\substack{v\in V\\ v\neq 0}} \frac{|a(u,v)|}{\|u\|_V\|v\|_V} &= \inf_u \sup_v \frac{|\langle Au, v\rangle|}{\|u\|\|v\|} = \inf_{u'\in V'} \sup_{v\in V} \frac{|\langle AA^{-1}u', v\rangle|}{\|A^{-1}u'\|_V\|v\|_V} \\
&= \inf_{u'\in V'} \sup_{v\in V} \frac{|\langle u', v\rangle|}{\|A^{-1}u'\|_V\|v\|_V} \\
&= \inf_{u'\in V'} \frac{1}{\|A^{-1}u'\|_V}\|u'\|_{V'} \\
&= \frac{1}{\sup_{u'\in V'} \frac{\|A^{-1}u'\|_V}{\|u'\|_{V'}}} = \frac{1}{\|A^{-1}\|_{L(V',V)}} = \varepsilon.
\end{aligned}$$

(1.12) can be realized analogically and we get even $\varepsilon = \varepsilon'$ since $A'^{-1} = (A^{-1})'$ and $V'' = V$. For the other statements compare [6, p.128]. □

Definition 10. *A bilinear form is V-elliptic, if it is bounded on $V \times V$ and if there is a positive constant C with*

$$a(u,u) \geq C\|u\|^2 \quad \forall u \in V. \tag{1.14}$$

Lemma 9. *The V-ellipticity implies (1.10) and (1.11) with $\varepsilon = \varepsilon'$.*

Proof.

$$\inf_{\|u\|=1} [\sup_{\|v\|=1} |a(u,v)|] \geq \inf_{\|u\|=1} |a(u,u)| \geq \inf_{\|u\|=1} C\|u\|^2 = C$$

and

$$\inf_v [\sup_u a(u,v)] \geq \inf_v |a(v,v)| \geq C.$$

Hence $A^{-1} \in L(V, V')$ exists and $\varepsilon = \varepsilon'$. □

1.2.3 Extremal problems

Theorem 7 (Main theorem on quadratic variational problems). *Suppose that V_0 is a closed linear subspace of the real Hilbert space V, $u_0 \in V$ fixed.*

1. *$a : V \times V \to \mathbb{R}$ is a symmetric, bounded, V_0-elliptic bilinear form*
2. *$f : V \to \mathbb{R}$ is a linear continuous functional on V. Then the following statements hold true:*
   - *The extremal problem*

$$\frac{1}{2}a(u,u) - f(u) = \min! \quad u \in \{u_0 + V_0\} \tag{1.15}$$

   has a unique solution for a fixed element $u_0 \in V$.

   - *The problem (1.15) is equivalent to the variational equation: Find a solution $u \in \{u_0 + V_0\}$ with*

$$a(u,v) = f(v) \quad \forall v \in V_0. \tag{1.16}$$

Proof.
Step 1 – Equivalence of (1.15) and (1.16). As in the proof of Proposition 1 we denote $F(u) = \frac{1}{2}a(u,u) - f(u)$ and we introduce $\varphi(t) := F(u + tv)$ for fixed $u \in V, v \in V_0$ and the variable $t \in \mathbb{R}$. Due to the symmetry and linearity we obtain

$$\varphi(t) = \frac{1}{2}t^2 a(u,v) + t[a(u,v) - f(v)] + \frac{1}{2}a(u,u) - f(u).$$

Since $a(v,v) > 0$ for all $v \in V_0, v \neq 0$, the problem (1.15) has a solution $u \in \{u_0 + V_0\}$ if and only if the real quadratic function $\varphi = \varphi(t)$ has a minimum at the point t=0 for each fixed $v \in V_0$, i.e. $\varphi'(0) = 0 \Leftrightarrow a(u,v) = (f,v)$ for all $v \in V_0$.
Step 2 – Uniqueness. Let u_1 and u_2 be solutions of (1.15) or (1.16). Then

$$\begin{aligned} C\|u_1 - u_2\|^2 &\leq a(u_1 - u_2, u_1 - u_2) = a(u_1, v) - a(u_2, v) \\ &= f(v) - f(v) = 0 \text{ for } v = u_1 - u_2 \in V_0. \end{aligned}$$

Step 3 – Existence. Instead of (1.15) we consider the problem: Find a solution $w \in V_0$ with $a(w,v) = \langle f,v\rangle - a(u_0,v) = \langle \tilde{f},v\rangle \quad \forall v \in V_0$. $\tilde{f} \in V_0'$ and the Lemma of Lax-Milgram yields that $w \in V_0$ exists. Now we set $u = w + u_0$. □

Let us summarize the results: *Let V be a Hilbert space,*

- *$V_0 = V$. The operator equation*

$$Au = f, \quad A : V \to V', A \in L(V,V') \tag{1.17}$$

- *generates a bilinear form*

$$a(u,v) = \langle Au, v\rangle = \langle f,v\rangle \quad \forall v \in V \tag{1.18}$$

- *and a functional*

$$F(u) = \frac{1}{2}a(u,v) - \langle f, u\rangle. \tag{1.19}$$

(1.17) has a uniquely determined solution $u \in V \Leftrightarrow$ (1.18) has a uniquely determined solution $u \in V \Leftrightarrow$ the conditions of the Lax-Milgram Lemma are valid. (1.19) has a uniquely determined minimizer in V, if the bilinear form $a(\cdot,\cdot)$ is symmetric and V-elliptic.

1.2.4 The energetic space

The properties "symmetry" and "V_0-ellipticity" can be defined directly for the operator $A : D(A) \subset V \to V$.

Definition 11. .

- *A is symmetric, if $(Au, v)_V = (u, Av)_V$ $\forall u, v \in D(A)$. $(\cdot,\cdot)_V$ denotes the scalar product in V.*
- *A is positive, if*

$$\begin{aligned}(Au, u) &\geq 0 \quad \forall u \in D(A)\\ (Au, u) &= 0 \Leftrightarrow u = 0.\end{aligned}$$

- *A is positive definite, if*

$$(Au, u) \geq C\|u\|^2 \quad \forall u \in D(A).$$

(Au, u) is the energy of the element u with respect to A.

EXAMPLE. $A = -\triangle, \quad V = L_2(\Omega), \quad D(A) = C_0^\infty(\Omega)$.

$$\begin{aligned}(Au, v) &= \int_\Omega -\triangle uv\,dx = \int_\Omega \operatorname{grad} u \operatorname{grad} v\,dx = (u, Av).\\ (Au, u) &= \int_\Omega |\operatorname{grad} u|^2 dx \geq 0.\\ (Au, u) &= 0 \Leftrightarrow u = 0 \text{ in } D(A)\\ (Au, u) &\geq C\|u\|_{L_2}^2 \text{ (Friedrichs-Poincaré-inequality.)}\end{aligned}$$

Lemma 10. *Let $A : D(A) \subset V \to V$ be a linear symmetric, positive operator. The mapping $[\cdot,\cdot] : D(A) \times D(A) \to \mathbb{R}$, $[u, v] := (Au, v)$ is a scalar product in $D(A)$, the so called energetic scalar product.*

We introduce the energetic norm in $D(A)$

$$\|u\|^2 = [u, u]. \tag{1.20}$$

Definition 12. *The energetic space V_A is the completion of $D(A)$ with respect to the norm (1.20)*

$$V_A = \overline{D(A)}^{[u,u]^{1/2}}.$$

Lemma 11. *$V_A \subset V$ provided A is linear, symmetric and positive definite.*

Proof.

$$\|u\|_V^2 \le \frac{1}{c}(Au,u) = \frac{1}{c}[u,u].$$

□

Lemma 12. *Let $A : D(A) \subset V \to V$ be a linear, symmetric and positive definite operator and $f \in V$. Then $\inf_{u\in D(A)} F(u) = \inf_u \frac{1}{2}(Au,u) - (f,u) = F(u_0)$, $u_0 \in V_A$.*

Proof. For $u \in V_A$ and $f \in V$ it holds

$$|(f,u)_V| \le \|f\|_V \|u\|_V \le \|f\|_V \frac{1}{\sqrt{c}} \|u\|_{V_A}.$$

Therefore f generates a linear continuous functional f_A on V_A:

$$f_A(u) = \langle f_A, u\rangle = (f,u)_V \qquad \forall u \in V_A.$$

The Riesz's representation theorem yields that there is an element $u_0 \in V_A$ with

$$f_A(u) = [u,u_0]_{V_A} = (f,u)_V.$$

This element realizes the minimum of $F(u)$.

$$\begin{aligned} F(u) &= \frac{1}{2}(Au,u)_V - (u,f)_V = \frac{1}{2}\|u\|_{V_A}^2 - (f,u)_V \\ &= \frac{1}{2}\|u\|_{V_A}^2 - [u,u_0]_{V_A} = \frac{1}{2}[u,u]_{V_A} - [u,u_0]_{V_A} \\ &= \frac{1}{2}\{[u,u-u_0] - [u,u_0] + [u_0,u_0] - [u_0,u_0]\} \\ &= \frac{1}{2}\{[u-u_0,u-u_0] - [u_0,u_0]\} \end{aligned}$$

$$\inf_{u\in D(A)} F(u) = -\frac{1}{2}[u_0,u_0] =: F(u_0)$$

□

Example Let be $A = -\triangle$, $V = L_2(\Omega)$, $D(A) = C_0^\infty(\Omega)$ and $[u,u] = \int_\Omega |\operatorname{grad} u|^2 dx$. Then $V_A = \overset{\circ}{H}{}^1(\Omega)$.

1.3 Elliptic boundary value problems

As an example we have considered the boundary value problem

$$
\begin{aligned}
-\triangle u &= f \text{ in } \Omega, \\
u &= 0 \text{ on } \partial\Omega,
\end{aligned}
\tag{1.21}
$$

$f \in H^{-1}(\Omega) = [\overset{\circ}{H}{}^1(\Omega)]'$. The variational formulation reads: Find an element $u \in \overset{\circ}{H}{}^1(\Omega) = V$ with

$$
a(u,v) = \int_\Omega \operatorname{grad} u \operatorname{grad} v \, dx = \langle f, v\rangle \qquad \forall v \in V. \tag{1.22}
$$

The extremal formulation is: Find an element $u \in V$ which realizes the minimum

$$
\min_{u\in V} F(u) = \min_u \left[\frac{1}{2}\int_\Omega |\operatorname{grad} u|^2 dx - \langle f, u\rangle \right]. \tag{1.23}
$$

Let us show, that $a(u,v)$ is bounded:

$$
\begin{aligned}
|a(u,v)| &= \left| \sum_{i=1}^N \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx \right| \le \sum_i \left| \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx \right| \\
&\le \sum_i \left(\int_\Omega |\frac{\partial u}{\partial x_i}|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |\frac{\partial v}{\partial x_i}|^2 dx \right)^{\frac{1}{2}} \le c\|u\|_V \|v\|_V .
\end{aligned}
$$

From the Lax-Milgram-Lemma 8 and the main Theorem 7 about quadratic variational problems it follows: There is an uniquely determined solution from $u \in \overset{\circ}{H}{}^1(\Omega) = V$ of (1.21), (1.22) and (1.23). The inhomogeneous boundary value problem

$$
\begin{aligned}
-\triangle u &= f \text{ in } \Omega \\
u &= g \text{ on } \partial\Omega,
\end{aligned}
\tag{1.24}
$$

$f \in H^{-1}(\Omega)$, $g \in H^{\frac{1}{2}}(\Omega)$, can be written as problem with a homogeneous boundary datum. To this end we consider an extension $\tilde{g} \in H^1(\Omega)$ of g and we set $w = u - \tilde{g}$. Then we have

$$
\begin{aligned}
a(w,v) &= \int_\Omega \operatorname{grad} w \operatorname{grad} v \, dx = \langle f, v\rangle - \int_\Omega \operatorname{grad} \tilde{g} \operatorname{grad} v \, dx \\
&= \langle \tilde{f}, v\rangle \qquad \forall v \in H^1(\Omega)
\end{aligned}
\tag{1.25}
$$

and $\tilde{f} \in V'$. From the Lax-Milgram Lemma 8 it follows that there is a solution $w \in \overset{\circ}{H}{}^1(\Omega)$. Setting $u = w + \tilde{g}$ we have an uniquely determined solution $u \in H^1(\Omega)$ (or $w \in \overset{\circ}{H}{}^1(\Omega)$) of (1.21), (1.25) and (1.26), where (1.26) reads: Find

$$
\min_{u\in\{\tilde{g}+V_0\}} F(u). \tag{1.26}
$$

(1.21) and (1.24) are examples for elliptic boundary value problems. We shall study more general boundary value problems in what follows.

1.3.1 Boundary value problems and Green's formulae

Let be

$$A(x,D)=\sum_{\substack{|\alpha|\le m\\|\beta|\le m}}(-1)^{|\alpha|}D^{\alpha}(a_{\alpha\beta}(x)D^{\beta}(\cdot)) \tag{1.27}$$

a linear partial differential operator of the order $2m$ with variable coefficients in a divergence form.
EXAMPLE

$$A(x,D)=-\triangle=-\operatorname{div}(\operatorname{grad}(\cdot))=-\sum_{\substack{|\alpha|=1\\|\beta|=1}}D^{\alpha}(\delta_{\alpha\beta}D^{\beta}(\cdot)),$$

where $\delta_{\alpha\beta}$ denotes the Kronecker symbol. Furthermore, we consider a normal system of boundary operators $\{b_j(x,D)\}_{j=1,\dots,m}$

$$b_j(x,D)=\sum_{|\beta_j|\le m_j}b_{\beta_j}(x)D^{\beta_j} \tag{1.28}$$

and $m_j\le 2m-1$.

Definition 13. *The system $\{b_j(x,D)\}_{j=1,\dots,m}$ is normal on $\partial\Omega$, if*

1. $\operatorname{ord} b_j=m_j\neq m_i=\operatorname{ord} b_i$ *for* $j\neq i$.
2. $b_{j,0}(x,\xi)=\sum_{|\beta_j|=m_j}b_{\beta_j}(x)\xi^{\beta_j}\neq 0$ *for* $0\neq\xi\in\mathbb{R}^N$, *where* $\xi=(\xi_1,\dots,\xi_N)$ *is a normal vector in* $x\in\partial\Omega$, $\xi^{\beta_j}=\xi_1^{\beta_{j1}}\xi_2^{\beta_{j2}}\cdots\xi_N^{\beta_{jN}}$.
3. *If additionally* $m_j=j-1, j=1,\dots,m$, *then* $\{b_j(x,D)\}_j$ *is called a Dirichlet-system of the order* m.

The operators $A(x,D)$ and $r|_{\partial\Omega}b_j(x,D)$ define a boundary value problem in a bounded domain $\Omega\in\mathbb{R}^N$ with a sufficiently smooth boundary $\partial\Omega$

$$\begin{aligned}A(x,D)u&=f\text{ in }\Omega\\ b_j(x,D)u\big|_{\partial\Omega}&=g_j\text{ on }\partial\Omega,\quad j=1,\dots,m.\end{aligned} \tag{1.29}$$

Let be $V_0\subset H^m(\Omega)$. Under which conditions for $A((x,D),\{b_j(x,D)\}_{j=1,\dots,m}$ can we guarantee that a "weak" solution $u\in H^m(\Omega)$ exists?

- *1st step*: We create an appropriate bilinear form on $V_0\times V_0$ with the help of a Green's formula.
- *2nd step*: We try to check the conditions of the Lax-Milgram Lemma 8.

Green's Formulae. We remind of the Green's formulae for the Laplacian:
First Green's formula:

$$\int_{\Omega}\operatorname{grad}u\operatorname{grad}v\,dx=-\int_{\Omega}\triangle uv\,dx+\int_{\partial\Omega}v\frac{\partial u}{\partial n}d\sigma.$$

Second Green's formula:

$$\int_\Omega \triangle uv\,dx - \int_\Omega u\triangle v\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v(x) d\sigma - \int_{\partial\Omega} u \frac{\partial v}{\partial n} d\sigma.$$

We introduce the bilinear form for the operator (1.27)

$$a(u,v) = \int_\Omega \sum_{\substack{|\alpha|\le m\\|\beta|\le m}} a_{\alpha\beta}(x) D^\beta u D^\alpha v dx$$

and demand the validity of the first Green's formula: There is a system of boundary operators $c_j(x,D)$ with $\operatorname{ord} c_j = 2m-1-m_j$ and

$$a(u,v) = \int_\Omega Auv\,dx - \sum_{j=1}^m \int_{\partial\Omega} c_j u b_j v\,d\sigma \quad \forall u,v \in H^{2m}(\Omega), \tag{1.30}$$

where the boundary operators $b_j(x,D)$ are given in (1.29).

Lemma 13. *If A(x,D) is elliptic in $\bar\Omega$, $\{b_j\}_j$ is a Dirichlet system, then exists a normal system of boundary operators, such that (1.30) is valid.*

The definition of the ellipticity of $A(x,D)$ will be given later. Here we consider some examples:

(i.) *The Neumann problem for the Laplacian*:

$$\begin{aligned} -\triangle u &= f \text{ in } \Omega \\ b_j u = \frac{\partial u}{\partial n}\Big|_{\partial\Omega} &= g \text{ on } \partial\Omega, j=1. \end{aligned}$$

We have

$$-\int \triangle uv\,dx + \int_{\partial\Omega} v\frac{\partial u}{\partial n} d\sigma = \int_\Omega fv\,dx - \int_{\partial\Omega} c_j v b_j u\,d\sigma = a(u,v)$$

with $c_j v = -v|_{\partial\Omega}$, $b_j u = \frac{\partial u}{\partial n}|_{\partial\Omega}$ and $j=1$.

(ii.) *The Dirichlet problem for the Lamé operator* (linear elasticity in homogeneous isotropic materials). The problem is: find a sufficient smooth solution $\vec u$ of

$$\begin{aligned} -\big[\mu\triangle\vec u + (\lambda+\mu)\operatorname{grad}\operatorname{div}\vec u\big] = -L(\vec u) &= \vec f \text{ in } \Omega \\ \vec u|_{\partial\Omega} &= \vec g \text{ on } \partial\Omega. \end{aligned} \tag{1.31}$$

$\vec u = (u_1,u_2,u_3)^\top$ is the displacement field, $\vec f$ denotes the density of volume forces, $\vec g$ the density of surface forces. Introducing the stress and strain tensors $\sigma = (\sigma_{ij})_{ij}$, $\varepsilon = (\varepsilon_{ij})_{ij}$ with

$$\begin{aligned} \varepsilon_{ij} &= \tfrac12(\partial_i u_j + \partial_j u_i), \\ \sigma_{ij} &= \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})\partial_{ij} + 2\mu\varepsilon_{ij} \end{aligned}$$

then we can write (1.31) in divergence form

$$-\operatorname{div}(\sigma) = -\begin{pmatrix} \partial_1\sigma_{11} + \partial_2\sigma_{12} + \partial_3\sigma_{13} \\ \partial_1\sigma_{21} + \partial_2\sigma_{22} + \partial_3\sigma_{23} \\ \partial_1\sigma_{31} + \partial_2\sigma_{32} + \partial_3\sigma_{33} \end{pmatrix} = \vec{f}. \tag{1.32}$$

Multiplying (1.32) with a function $\vec{v} \in [C^2(\bar{\Omega})]^3$ and integrating by parts we get:

$$\begin{aligned}
-\int_\Omega \operatorname{div}(\sigma)\vec{v}dx &= -\int_\Omega \Big\{[\partial_1\sigma_{11} + \partial_2\sigma_{12} + \partial_3\sigma_{13}]v_1 + [\partial_1\sigma_{21} + \\
&\qquad \partial_2\sigma_{22} + \partial_3\sigma_{23}]v_2 + [\partial_1\sigma_{31} + \partial_2\sigma_{32} + \partial_3\sigma_{33}]v_3\Big\}dx \\
&= \int_\Omega \Big\{(\sigma_{11}, \sigma_{12}, \sigma_{13})\nabla v_1 + (\sigma_{21}, \sigma_{22}, \sigma_{23})\nabla v_2 \\
&\qquad +(\sigma_{31}, \sigma_{32}, \sigma_{33})\nabla v_3\Big\}dx - \int_{\partial\Omega} \sigma n \cdot \vec{v}\, d\sigma \\
&= \int_\Omega \sigma(\vec{u}) : \nabla\vec{v}\, dx - \int_{\partial\Omega} \sigma(\vec{u})n \cdot \vec{v}\, d\sigma = \int_\Omega \vec{f} \cdot \vec{v}\, dx \\
&= \int_\Omega \sigma(\vec{u}) : \varepsilon(\vec{v})dx - \int_{\partial\Omega} \sigma(\vec{u})n \cdot \vec{v}\, d\sigma.
\end{aligned}$$

Here we have used the notation $A : B = \operatorname{tr} A^\top B$,

$$\nabla\vec{v} = \begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 & \partial_3 v_1 \\ \partial_1 v_2 & \partial_2 v_2 & \partial_3 v_2 \\ \partial_1 v_3 & \partial_2 v_3 & \partial_3 v_3 \end{pmatrix}.$$

It follows that a Green's formula holds

$$\boxed{a(\vec{u}, \vec{v}) = \int_\Omega \sigma(\vec{u}) : \varepsilon(\vec{v})dx = \langle \vec{f}, \vec{v}\rangle + \int_{\partial\Omega} \sigma(\vec{u})n \cdot \vec{v}\, d\sigma}$$

or

$$\begin{aligned}
a(\vec{u}, \vec{v}) &= \int_\Omega \sigma(\vec{u}) : \varepsilon(\vec{v})\, dx - \int_{\partial\Omega} c_j(\vec{u}) \cdot b_j\vec{v}\, d\sigma \\
&= \langle \vec{f}, \vec{v}\rangle.
\end{aligned}$$

We remark, that we can write $\sigma(\vec{u}) = \lambda \operatorname{tr} \varepsilon(\vec{u})I + 2\mu\varepsilon(\vec{u})$ and

$$\begin{aligned}
a(\vec{u}, \vec{v}) &= \int_\Omega \{\lambda \operatorname{tr} \varepsilon(\vec{u}) \operatorname{tr} \varepsilon(\vec{v}) + 2\mu\varepsilon(u) : \varepsilon(v)\}\, dx - \int_{\partial\Omega} \sigma(\vec{u})n \cdot \vec{v}\, d\sigma \\
&= \langle \vec{f}, \vec{v}\rangle.
\end{aligned} \tag{1.33}$$

(iii.) *Boundary value problems for the biharmonic equation.* The biharmonic operator

$$\triangle^2 = \triangle\triangle = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4}$$

is an operator used in the Kirchhoff plate theory and the problem

$$\triangle^2 u = f \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0 \text{ and } \frac{\partial u}{\partial v} = 0 \text{ on } \partial\Omega \tag{1.34}$$

determines the bending u of a thin plate (membran) under forces which are orthogonal to the plate and and which is clamped on the boundary. There are different possibilities to derive bilinear forms $a(u, v)$. The first idea is to multiply the equation (3.14) with a smooth function v and to integrate by parts twice. This leads to (compare [15, p. 265-269])

$$\begin{aligned} a_1(u,v) &= \int_\Omega \left(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} + 2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2} \right) dx \\ &= (\triangle^2 u, v) - \int_{\partial\Omega} \Bigg[v \Bigg\{ \frac{\partial}{\partial n}(\triangle u) + \frac{\partial}{\partial s}\Bigg[-\frac{\partial^2 u}{\partial x_1^2} n_1 n_2 \\ &\quad + \frac{\partial^2 u}{\partial x_1 \partial x_2}(n_1^2 - n_2^2) + \frac{\partial^2}{\partial x_2^2} n_1 n_2 \Bigg]\Bigg\} + \frac{\partial v}{\partial n}\frac{\partial^2 u}{\partial n^2} \Bigg] ds. \end{aligned}$$

Another possibility is to use the second Green's formula for the Laplacian

$$\int_\Omega (\triangle u v - u \triangle v) dx = \int_{\partial\Omega} \left[\frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \right] ds.$$

Insert for u the expression $\triangle u$. It follows that

$$a_2(u,v) = \int_\Omega \triangle u \triangle v dx = \int_\Omega \triangle^2 u v dx - \int_{\partial\Omega} \left[\frac{\partial \triangle u}{\partial n} v - \frac{\partial v}{\partial n} \triangle u \right] ds.$$

A third possibility, which realizes the influence of a material parameter $\sigma = \frac{\lambda}{2(\lambda+\mu)}$ (σ is the Poisson ratio) is to use the identity:

$$\begin{aligned} \triangle^2 u &= \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 u}{\partial x_1^2} + \sigma \frac{\partial^2 u}{\partial x_2^2} \right) + 2\left(\frac{\partial^2}{\partial x_1 \partial x_2}\left[(1-\sigma)\frac{\partial^2 u}{\partial x_1 \partial x_2} \right] \right) \\ &\quad + \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 u}{\partial x_2^2} + \sigma \frac{\partial^2 u}{\partial x_1^2} \right). \end{aligned}$$

Multiplying with v and integrating by parts we get:

$$\begin{aligned} a_3(u,v) &= \int_\Omega \frac{\partial^2 u}{\partial x_1^2}\left(\frac{\partial^2 v}{\partial x_1^2} + \sigma \frac{\partial^2 v}{\partial x_2^2} \right) + 2(1-\sigma)\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} \\ &\quad + \frac{\partial^2 u}{\partial x_2^2}\left(\frac{\partial^2 v}{\partial x_2^2} + \sigma \frac{\partial^2 v}{\partial x_1^2} \right) dx \\ &= (\triangle^2 u, v) + \int_{\partial\Omega} \left[v N u + \frac{\partial v}{\partial n} M u \right] ds, \end{aligned}$$

where

$$\begin{aligned} Nu &= -\frac{\partial}{\partial n}(\triangle u) + (1-\sigma)\frac{\partial}{\partial s}\Bigg[\frac{\partial^2 u}{\partial x_1^2} n_1 n_2 \\ &\quad - \frac{\partial^2 u}{\partial x_1 \partial x_2}(n_1^2 - n_2^2) - \frac{\partial^2 u}{\partial x_2^2} n_1 n_2 \Bigg], \\ Mu &= \sigma \triangle u + (1-\sigma)\frac{\partial^2 u}{\partial n^2}. \end{aligned}$$

For $\sigma = 0$ we get $a_1(u,v) = a_3(u,v)$. For the Dirichlet data $b_1 v = v|_{\partial\Omega}$, $b_2 v = \frac{\partial v}{\partial n}|_{\partial\Omega}$ we have

$$
\begin{aligned}
a_1(u,v) &= (\triangle^2 u, v) - \int_{\partial\Omega} b_1 v c_1 u ds + \int_{\partial\Omega} b_2 v c_2 u ds \\
a_2(u,v) &= (\triangle^2 u, v) - \int_{\partial\Omega} b_1 v \tilde{c_1} u ds + \int_{\partial\Omega} b_2 v \tilde{c_2} u ds \\
a_3(u,v) &= (\triangle^2 u, v) - \int_{\partial\Omega} b_1 v (-Nu) ds + \int_{\partial\Omega} b_2 v (Mu) ds,
\end{aligned}
$$

where $\operatorname{ord} c_1 = \operatorname{ord} \tilde{c_1} = 2m - 1 - m_1 = 3$, $\operatorname{ord} c_2 = 2 = \operatorname{ord} \tilde{c_2}$.

(iv.) *The Stokes system.* The steady state Stokes problem reads: Find a velocity field

$$\vec{u} = (u_1, u_2, u_3)$$

and a pressure field p such that

$$
\left.
\begin{aligned}
-\triangle \vec{u} + \nabla p &= \vec{f} \\
\operatorname{div} \vec{u} &= 0
\end{aligned}
\right\} \quad \text{in } \Omega,
$$

$$\vec{u} = 0 \qquad \text{on } \partial\Omega \text{ (Dirichlet Datum).} \tag{1.35}$$

The pressure field is not uniquely defined (add a constant function), $p = \tilde{p} + const.$ We get bilinear forms, multiplying (1.35) with $\vec{v}$ and integrating on Ω

$$
\begin{aligned}
a(\vec{u}, \vec{v}) + b(p, \vec{v}) &= \int_\Omega \operatorname{grad} \vec{u} \operatorname{grad} \vec{v}\, dx - \int_\Omega p \operatorname{div} \vec{v}\, dx + \\
&\quad + \int_{\partial\Omega} \frac{\partial \vec{u}}{\partial n} \vec{v}\, ds + \int_{\partial\Omega} p \vec{v} n\, ds.
\end{aligned}
$$

1.3.2 V-Ellipticity and V-Coerciveness

We remind of our examples

$$\triangle \ : \ a(u,v) = \int_\Omega \operatorname{grad} u \operatorname{grad} v\, dx = \langle f, v\rangle + \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds \tag{1.36}$$

$$\text{Lamé} \ : \ a(\vec{u}, \vec{v}) = \int_\Omega \sigma(\vec{u}) : \varepsilon(\vec{v}) dx = \langle \vec{f}, \vec{v}\rangle + \int_{\partial\Omega} \sigma(\vec{u}) n \vec{v} ds \tag{1.37}$$

$$
\begin{aligned}
\triangle^2 \ : \ a_3(u,v) &= \int_\Omega \left\{ \frac{\partial^2 u}{\partial x_1^2} \left(\frac{\partial^2 v}{\partial x_1^2} + \sigma \frac{\partial^2 v}{\partial x_2^2} \right) + \frac{\partial^2 u}{\partial x_2^2} \left(\frac{\partial^2 v}{\partial x_2^2} + \sigma \frac{\partial^2 v}{\partial x_1^2} \right) + \right. \\
&\qquad \left. + 2(1-\sigma) \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} \right\} dx \\
&= \langle f, v\rangle - \int_{\partial\Omega} \left[vNu + \frac{\partial v}{\partial n} Mu \right] ds.
\end{aligned} \tag{1.38}
$$

In general we have:

$$\begin{aligned} a(u,v) &= \int_\Omega \sum_{\substack{|\alpha|\le m \\ |\beta|\le m}} a_{\alpha\beta} D^\beta u D^\alpha v dx \\ &= \int_\Omega f v dx + \sum_{i=1}^m \int_{\partial\Omega} -c_j u b_j v d\sigma. \end{aligned} \tag{1.39}$$

Now, we introduce a space $V \subset H^m$, which depends on the system of boundary conditions $\{b_j\}$:

$$V = \{v \in H^m : b_j v\big|_{\partial\Omega} = 0, m_j \le m-1\}.$$

The boundary conditions with order $m_j \le m-1$ are called essential boundary conditions while the boundary condition with order $m_j > m-1$ are called natural boundary conditions. If

$$b_j v = \frac{\partial^j v}{\partial n^j}, \quad j = 0, \dots, m-1,$$

then

$$V = \mathring{H}^m(\Omega).$$

The variational formulation reads: *Find $u \in V$, such that $a(u,v) = \langle f, v\rangle$ for all $v \in V$.*

Lemma 14. *The bilinear forms (1.36),(1.37),(1.38) are $\mathring{H}^m(\Omega)$-elliptic. That means*

$$\begin{aligned} a(u,v) &\le c_1 \|u\|_V \|v\|_V \\ a(u,u) &\ge c_2 \|u\|_V^2 \end{aligned}$$

for all $u, v \in V = \mathring{H}^m(\Omega)$.

Proof. For (1.36) both conditions have been already shown. We consider problem (1.37). Since

$$\begin{aligned} a(\vec{u},\vec{v}) &= \int_\Omega \sigma(\vec{u}) : \varepsilon(\vec{v}) dx = \int_\Omega \sum_{i,j} \sigma_{ij}(\vec{u}) \varepsilon_{ij}(\vec{v}) dx \\ &\le \sum_{i,j} \|\sigma_{ij}(\vec{u})\|_{L_2(\Omega)} \|\varepsilon_{ij}\vec{v})\|_{L_2(\Omega)} \le C \|\vec{u}\|_V \|\vec{v}\|_V, \end{aligned}$$

the bilinear form is bounded. We now show the second inequality for $\lambda > 0, \mu > 0$. We have

$$\begin{aligned} a(\vec{u},\vec{u}) &= \int_\Omega \lambda [tr\varepsilon(\vec{u})]^2 + 2\mu[\varepsilon(\vec{u}) : \varepsilon(\vec{u})] dx \\ &= \int_\Omega \left\{ \lambda \left[\varepsilon_{11}(\vec{u}) + \varepsilon_{22}(\vec{u}) + \varepsilon_{33}[\vec{u}]\right]^2 + 2\mu \sum_{i,j} [\varepsilon_{ij}(\vec{u})]^2 \right\} dx \end{aligned}$$

$$\begin{aligned}
&\geq 2\mu \int_\Omega \sum_{i,j} [\varepsilon_{ij}(\vec{u})]^2 dx \\
&\geq C \int_\Omega \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j}\right)^2 dx \\
&\geq C_1 \|\vec{u}\|_V^2,
\end{aligned}$$

due to Friedrichs' inequality. The last inequality says that the norm in V is equivalent to the seminorm

$$|u| = \int_\Omega \sum_{i,j} [(\varepsilon_{ij}(\vec{u}))^2 dx]^{\frac{1}{2}}$$

and is called **Korn's inequality**.
Finally we pass to the problem (1.38), assuming $\sigma = \frac{\lambda}{2(\mu+\lambda)} \in [0,1)$. It is easy to see that

$$|a_3(u,v)| \leq c\|u\|_{H^2(\Omega)}\|v\|_{H^2(\Omega)}$$

Furthermore,

$$a_3(u,u) = \int_\Omega \left[\left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + 2\sigma \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 + 2(1-\sigma)\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2\right] dx.$$

Since for any numbers a, b, σ

$$a^2 + 2\sigma ab + b^2 - (1-\sigma)(a^2+b^2) = \sigma(a+b)^2$$

and for $\sigma \geq 0$

$$a^2 + 2\sigma ab + b^2 \geq (1-\sigma)(a^2+b^2)$$

we get from the Friedrichs' inequality

$$\begin{aligned}
a_3(u,u) &\geq \int_\Omega (1-\sigma)\left[\left(\frac{\partial^2 u}{\partial x_1}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)\right] dx \\
&\geq C\|u\|_{H^2(\Omega)}^2
\end{aligned}$$

for all $u \in \overset{\circ}{H}{}^2(\Omega)$. □

It cannot be expected that the general bilinear form (1.39) is $\overset{\circ}{H}{}^m$-elliptic. We need the "ellipiticity" of the operator

$$A(x, D_x) = \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} (-1)^{|\beta|} D^\beta a_{\alpha\beta}(x) D^\alpha,$$

which is defined for the principal part A_0 of A

$$A_0(x, D_x) = (-1)^m \sum_{|\alpha|,|\beta|=m} D^\beta a_{\alpha\beta}(x) D^\alpha.$$

Definition 14. *A is uniformly elliptic in* Ω*, if there is a positive constant* C *with*

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta} \geq C|\xi|^{2m} \qquad \forall x \in \Omega, \quad \xi \in \mathbb{R}^n.$$

Lemma 15. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$*. Assume* $a_{\alpha\beta}$ *are constant for* $|\alpha| = |\beta| = m$*,* $a_{\alpha\beta} = 0$ *for* $0 < |\alpha| + |\beta| \leq 2m-1$*,* $a_{00}(x) \geq 0$*. If A is uniformly elliptic, then* $a(\cdot,\cdot)$ *is* $\mathring{H}^m(\Omega)$*-elliptic.*

Proof.
We use the Fourier transform

$$F(u) = \hat{u}(\xi) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-i(\xi,x)} u(x)dx. \tag{1.40}$$

The properties hold:

$$\begin{aligned} F(D^\alpha u) &= i^{|\alpha|}\xi^\alpha \hat{u}(\xi) \\ \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi &= \int_{\mathbb{R}^n} u(x)v(x)dx \qquad \text{(Parseval identity [21, p.40])} \end{aligned}$$

The boundedness of $a(\cdot,\cdot)$ is obvious. Let us show the positive definiteness.

$$a(u,u) = \sum_{|\alpha|,|\beta|=m} \int_\Omega a_{\alpha\beta} D^\alpha u(x) D^\beta u(x)dx + \int_\Omega a_{00}(x)u^2(x)dx$$

Extending the elements $u \in \mathring{H}^m(\Omega)$ by zero onto $\mathbb{R}^n$ and using the Fouriertransform we get:

$$\begin{aligned} a(u,u) &= \sum_{|\alpha|,|\beta|=m} a_{\alpha\beta} \int_{\mathbb{R}^n} \left[(i\xi)^\alpha \hat{u}(\xi)\overline{(i\xi)^\beta \hat{u}(\xi)}d\xi + \int_\Omega a_{00}(x)u^2(x)dx \right. \\ &= \int_{\mathbb{R}^n} \sum_{|\alpha|,|\beta|=m} a_{\alpha\beta}\xi^{\alpha+\beta}|\hat{u}(\xi)|^2 d\xi + \int_\Omega a_{00}(x)u^2dx \\ &\geq C\int_{\mathbb{R}^n} |\xi|^{2m}|\hat{u}(\xi)|^2 d\xi \\ &\geq C\int_{\mathbb{R}^n} \sum_{|\alpha|=m} |\xi^\alpha|^2|\hat{u}(\xi)|^2 d\xi = C\int_{\mathbb{R}^n} \sum_{|\alpha|=m} i^{|\alpha|}\xi^\alpha\hat{u}\overline{i^{|\alpha|}\xi^\alpha\hat{u}}\, d\xi \\ &\geq C\int_{\mathbb{R}^n} \sum_{|\alpha|=m} D^\alpha u D^\alpha u\, dx \geq \tilde{c}\|u\|_V^2. \end{aligned}$$

□

Besides the Dirichlet problem we can study so called "Neumann" problems (natural boundary conditions)

$$\begin{aligned} \triangle &: \frac{\partial u}{\partial n} = g \\ \text{Lamé} &: \sigma(\vec{u})n = \vec{g} \\ \triangle^2 &: N[u] = g_1,\ M[u] = g_2. \end{aligned}$$

In general $c_j(u) = g_j$, $V = H^m(\Omega)$. This leads to the following variational formulation:

- $\triangle$: Find an element $u \in H^1(\Omega)$ with

$$a(u,v) = \int_\Omega \operatorname{grad} u \operatorname{grad} v \, dx = \langle F, v\rangle := \langle f, v\rangle + \int_{\partial\Omega} gv \, d\sigma$$

  for all $v \in H^1(\Omega)$.

- Lamé: Find an element $\vec{u} \in [H^1(\Omega)]^3$ with

$$a(\vec{u},\vec{v}) = \int_\Omega \sigma(\vec{u}) : \varepsilon(\vec{v}) dx = \langle \vec{F}, \vec{v}\rangle := \langle \vec{f}, \vec{v}\rangle + \int_{\partial\Omega} \vec{g}\vec{v} d\sigma.$$

  for all $\vec{v} \in [H^1(\Omega)]^3$.

- $\triangle^2$: Find an element $u \in H^2(\Omega)$ with

$$a_3(u,v) = \int_\Omega \ldots = \langle F, v\rangle := \langle f, v\rangle - \int_{\partial\Omega} \big[v g_1 + \frac{\partial v}{\partial n} g_2\big] d\sigma$$

  for all $v \in H^2(\Omega)$ and

- in general: Find an element $u \in H^m(\Omega)$ with

$$\begin{aligned} a(u,v) &= \int_\Omega \sum_{\substack{|\alpha|\le m \\ |\beta|\le m}} a_{\alpha\beta} D^\beta u D^\alpha v dx = \langle F, v\rangle \\ &:= \int f v dx + \sum_{j=1}^m \int_{\partial\Omega} -c_j u b_j v d\sigma \quad \forall v \in H^m(\Omega). \end{aligned}$$

These bilinear forms cannot be positive definite. The solutions are not uniquely defined and they are not solvable for every right hand sides f and g; e.g. choosing $v = 1$ or $\vec{v} = \binom{0}{1}$, we get the solvability conditions

$$\begin{aligned} \triangle: &\quad 0 = \int_\Omega f \, dx + \int_{\partial\Omega} g \, d\sigma \\ \text{Lamé}: &\quad 0 = \int_\Omega f_1 \, dx + \int_{\partial\Omega} g_1 \, d\sigma \\ \triangle^2: &\quad 0 = \int_\Omega f \, dx + \int_{\partial\Omega} g_1 \, d\sigma. \end{aligned}$$

Nevertheless, the variational formulations are meaningful under taking into account solvability and uniqueness conditions. The corresponding bilinear forms are V-coercive.

Definition 15. *Let be $V \subset H^m(\Omega)$. The bilinear form $a(\cdot,\cdot) : V \times V \to V'$ is V-coercive if there exist positive constants c_1 and c_2 that a Gårding inequality holds*

$$a(u,u) \ge c_1 \|u\|_m^2 - c_2 \|u\|_0^2 \qquad \forall u \in V. \tag{1.41}$$

EXAMPLE. $\triangle u = f$ in Ω, $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$.

$$\begin{aligned} a(u,v) &= \int_\Omega \operatorname{grad} u \operatorname{grad} v \, dx = \langle f, v\rangle \\ a(u,u) &= \int_\Omega \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx = \|u\|^2_{H^1(\Omega)} - \|u\|^2_{L_2(\Omega)}. \end{aligned}$$

The importance of the V-coerciveness is based on the fact that a **Fredholm alternative** is valid:
There is a uniquely determined solution $u \in V$ for every $f \in V'$ such that

$$a(u,v) = \langle f, v\rangle \quad \forall v \in V$$

or
$\dim \ker A = \dim \ker A^* > 0$, *where A is defined by $\langle Au, v\rangle = a(u,v)$.*
There exists a solution (not uniquely defined) of $a(u,v) = \langle f, v\rangle$, iff $f \perp \ker A^*$, that means $\langle f, e^*\rangle = 0 \quad \forall e^* \in ker\, A^*$.

Proof. [21, p.266]. □

S. Agmon [2] has formulated, under which conditions the boundary value problem $(A_1, b_1, \dots, b_m)$ is V-coercive, where V is generated by the homogeneous essential boundary conditions [21, p. 286].
The more general class of elliptic boundary value problems is characterized by the condition: The operator

$$\begin{aligned} &\mathcal{A} = (A, b_1, \dots, b_m) : \\ &H^{2m}(\Omega) \to L_2(\Omega) \times \prod_{j=1}^m H^{2m-m_j-\frac{1}{2}}(\partial\Omega) \end{aligned}$$

belonging to a boundary value problem is a Fredholm operator. That means:
$\ker \mathcal{A} < \infty, \operatorname{coker} \mathcal{A} < \infty$, $\mathcal{A}$ is linear and continuous.

1.3.3 Ellipticity

Scalar differential operators

Let Ω be a domain in $\mathbb{R}^n$, $A(x,D) = \sum_{|\alpha| \le M} a_\alpha(x) D^\alpha$ a linear differential operator of order M with smooth (in general complex valued) coefficients; $\alpha = (\alpha_1, \dots \alpha_n)$ is a multiindex, $D^\alpha = \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, $|\alpha| = \sum_{i=1}^n \alpha_i$, $\alpha_i \in \mathbb{N} \cup \{0\}$.

$$A(x,D) : H^{k+M}(\Omega) \to H^k(\Omega), \qquad k \in \mathbb{N} \cup \{0\}$$

Definition 16. *$A_0(x,D) = \sum_{|\alpha|=M} a_\alpha(x) D^\alpha$ is the principal part of $A(x,D)$,*

$$A_0(x,\xi) = \sum_{|\alpha|=M} a_\alpha(x)\xi^\alpha, \quad \xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$$

denotes the polynomial expression (symbol) belonging to $A_0(x,D)$.

Definition 17. *A(x,D) is elliptic in a point $x \in \bar{\Omega}$, if*

$$A_0(x,\xi) \neq 0 \qquad \forall \xi \neq 0 \in \mathbb{R}^n.$$

$A(x,D)$ is elliptic in $\bar{\Omega}$, if $A(x,D)$ is elliptic in all points x of $\bar{\Omega}$.

The ellipticity is determined by the principal part.
EXAMPLES.

(i.) We consider the ordinary linear differential operator

$$A(x,D)u = \sum_{j=0}^{M} a_j(x) u^{(j)}(x).$$

The principal part is

$$A_0(x,D) = a_M(x) u^{(M)}(x)$$

with the polynomial expression

$$A_0(x,\xi) = a_M(x)\xi^M.$$

$A(x,D)$ is elliptic in $[a,b]$ if $A_M(x) \neq 0 \quad \forall x \in [a,b]$.

(ii.) Let $A(x,D)$ be a linear differential operator of second order:

$$A(x,D)u = \sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j} + a_0(x)u.$$

The polynomial

$$A_0(x,\xi) = \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k$$

is a quadratic form. If the coefficients $a_{jk}(\cdot)$ are real valued and $A_0(x,\xi)$ positive (or negative) definite, then $A(x,D)$ is elliptic in $\bar{\Omega}$.

Systems of differential operators

Let be given a matrix with operator elements:

$$\begin{aligned} A(x,D) &= (A_{ij}(x,D))_{i,j=1,\dots,n}, \\ A_{ij}(x,D) &= \sum_{|\alpha|\leq k_{ij}} c_\alpha^{ij}(x) D^\alpha. \end{aligned}$$

The differential equation system reads:

$$A(x, D)\vec{u} = \vec{f}.$$

The orders of the elements can be equal (Lamé system) or unequal (Stokes system). In order to create the principal parts in a uniform way we introduce numbers $m_1, m_2, \ldots, m_n, m'_1, \ldots, m'_n$ such that

$$k_{ij} \leq m_i + m'_j, \quad 1 \leq i, j \leq n,$$

and

$$A(x, D) : H^{k-m_1} \times H^{k-m_2} \times \ldots \times H^{k-m_n} \longrightarrow H^{k-m'_1} \times H^{k-m'_2} x \ldots \times H^{k-m'_n}.$$

Note, that $m_1, m_2, \ldots, m_n$ and $m'_1, \ldots, m'_n$ are not uniquely defined (choose $m_i - l, m'_i + l$).

Definition 18. *The principal part of $A(x, D)$ is*

$$A_0(x, D) = \left(A^0_{ij}(x, D)\right)_{ij} = \left(\sum_{|\alpha| = m_i + m'_j} c^{ij}_\alpha(x) D^\alpha \right)_{ij}.$$

If $k_{ij} < m_i + m'_j$, put $c^{ij}_\alpha = 0$.

Examples.

- Lamé system: $k_{ij} = 2$ for $1 \leq i, j \leq n$. Set $m_i = 2, m'_j = 0$, or $m_i = 1, m'_j = 1$ for $1 \leq i, j \leq n$, then $A_0(x, D) = A(x, D)$.
- Stokes system for (u_1, u_2, p)

$$A(x, D) = A(D) = \begin{pmatrix} -\triangle & 0 & \frac{\partial}{\partial x_1} \\ 0 & -\triangle & \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & 0 \end{pmatrix}$$

  Here is $k_{ii} = 2$ for $i \leq 2$, $k_{ij} = k_{ji} = 1$ for $i \leq 2, j = 3$, $k_{ij} = 0$ else. Set $m_i = m'_i = 1$ for $i = 1, 2$; $m_3 = m'_3 = 0$. Then

$$A_0(x, D) = A(x, D).$$

Douglis-scheme.

	m_1	m_2	m_3
m'_1	$m'_1 + m_1$	$m'_1 + m_2$	$m'_1 + m_3$
m'_2	...	...	...
m'_3	...	...	...

Definition 19. *(Douglis-Nirenberg-ellipticity) The matrix-operator $A(x, D)$ is elliptic in a point $x \in \bar{\Omega}$, if*

$$\det(A^0_{ij}(x, \xi)) \neq 0 \quad \forall \xi \neq 0 \in \mathbb{R}^n.$$

$A(x,D)$ is uniformly elliptic in Ω, if there is a constant $c>0$ with

$$|\det A^0_{ij}(x,\xi)| \geq c|\xi|^{2m} \qquad \forall x\in\Omega, \xi\in\mathbb{R}^n,$$

where $2m=\sum_{i=1}^n (m_i+m_i')$.

EXAMPLE. *Stokes system*

$$\left|\det\begin{pmatrix} -|\xi|^2 & 0 & \xi_1 \\ 0 & -|\xi|^2 & \xi_2 \\ -\xi_1 & -\xi_2 & 0 \end{pmatrix}\right| = \left| -\xi_1^2|\xi|^2 - \xi_2^2|\xi|^2\right| = |\xi|^4$$

Boundary differential operators

We consider on $\partial\Omega$ the boundary operators

$$b_j(x,D)u = \sum_{|\beta_j|\leq m_j} b_{\beta_j}(x)D^{\beta_j}u\Big|_{\partial\Omega},$$

b_{β_j} are smooth coefficients.
To ensure the Fredholm property for the whole boundary value problem we need a connection between the differential operator in the domain and the boundary differential operators. This connection is expressed through a Lopatinskij-Shapiro condition (or covering condition) formulated locally at the boundary points.
We explain this condition in short. Let be $\partial\Omega$ sufficiently smooth and $x_0\in\partial\Omega$. We choose x_0 as origin and introduce the coordinates $x=(x',x_n)$; $x'=(x_1,...,x_{n-1})$ is from the tangential plane, x_n lies in the direction of the interior normal vector.

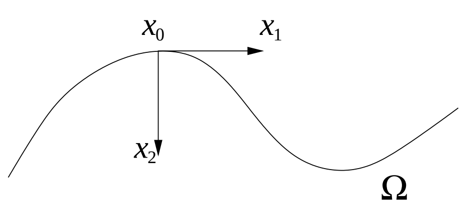


Figure 3

We consider the principal parts of $A(x,D)$ and $b_j(x,D)$ with coefficients frozen in x_0, $A_0(x_0,D)$, $b_j(x_0,D)$, $j=1,...,m$. Writing the derivatives $\frac{\partial}{\partial x_j}$ as $\frac{1}{i}\frac{\partial}{\partial x_j}$, $i^2=-1$, $j=1,\ldots,n$, and applying the Fourier transform

$$\mathcal{F}_{n-1}[f]=\hat{f}(\xi)=\int_{\mathbb{R}^{n-1}} e^{-i(x,\xi)}f(x)dx,$$

we get the Fourier transformed operators,

$$\mathcal{F}_{n-1}A_0(x_0,D) = A_0\left(x_0,\xi',\frac{1}{i}\frac{\partial}{\partial x_n}\right) = A_0\left(x_0,\xi',\frac{1}{i}\frac{d}{dt}\right)$$
$$\mathcal{F}_{n-1}b_{j,0}(x_0,D) = b_{j,0}\left(x_0,\xi',\frac{1}{i}\frac{\partial}{\partial x_n}\right) = b_{j,0}\left(x_0,\xi',\frac{1}{i}\frac{d}{dt}\right).$$

Let be

$$P(z) = A_0\left(x_0, (\xi', z)\right) \tag{1.42}$$

the characteristic polynomial in z belonging to the ordinary differential equation

$$A_0\left(x_0, \xi', \frac{1}{i}\frac{d}{dt}\right) v(t) = 0. \tag{1.43}$$

With M^+ we denote the set of solutions of (1.43) generated by the roots of (1.42) with positive imaginary part.

Definition 20. (Lopatinskij-Shapiro condition) *The initial problem*

$$\begin{aligned} A_0\left(x_0, \xi', \frac{1}{i}\frac{d}{dt}\right) v(t) &= 0 \qquad \text{for} \quad t > 0 \\ b_{j,0}\left(x_0, \xi', \frac{1}{i}\frac{d}{dt}\right) v(t)\bigg|_{t=0} &= h_j, \qquad j = 1, \dots, m, \end{aligned} \tag{1.44}$$

is for every $h = (h_1, ..., h_m) \in \mathbb{R}^m$ (or $\mathbb{C}^m$) and $0 \neq \xi' \in \mathbb{R}^{n-1}$ uniquely solvable in M^+.

Definition 21. *The boundary value problem:*

$$\begin{aligned} A(x, D)u &= f \quad \text{in} \quad \Omega \\ b_j(x, D)u &= g_j \quad \text{on} \quad \partial\Omega, \qquad j = 1, \dots, m, \end{aligned}$$

is called elliptic, if the operator $A(x, D)$ of the order $2m = M$ is elliptic in $\bar{\Omega}$ and $(A, b_1, ..., b_m) = \mathcal{A}$ satisfies the Lopatinskij-Shapiro conditions for all $x \in \partial\Omega$.

Theorem 8 (Main theorem). [21, p.189]
Let the coefficients $a_\alpha(\cdot)$ and $b_{\beta j}(\cdot)$ as well the boundary $\partial\Omega$ be sufficiently smooth. The following statements are equivalent:

(1) The boundary value problem is elliptic.

(2) The operator

$$\mathcal{A} = (A, b_1, ..., b_m) : H^{2m+l}(\Omega) \to H^l(\Omega) \times \prod_{j=1}^{m} H^{2m+l-mj-\frac{1}{2}}(\partial\Omega)$$

is a Fredholm operator for $l = 0, 1, 2, \dots$

(3) The apriori estimate holds for all $u \in H^{2m+l}(\Omega)$: There is a positive constant c that

$$\|u\|_{2m+l} \leq c\left\{ \|Au\|_l + \sum_{j=1}^{m} \|b_j u\|_{2m+l-mj-\frac{1}{2}} + \|u\|_{2m+l-1} \right\}.$$

EXAMPLES We consider the Laplace operator

$$A(x,D) = \triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \sum_{j=1}^{n} - \left(\frac{1}{i}\frac{\partial}{\partial x_j}\right)\left(\frac{1}{i}\frac{\partial}{\partial x_j}\right),$$

$\operatorname{ord}\triangle = 2$, $m = 1$. One boundary condition on $\partial\Omega$ is allowed with order $b_1 \leq 1$. b_1 has the general form

$$b_1(x,D) = b_0(x) + \sum_{i=1}^{n-1} b_j(x)\frac{\partial}{\partial x_j} + b_n(x)\frac{\partial}{\partial x_n},$$

assuming $x = (x', x_n)$ are already the local coordinates. We have:

$$\begin{aligned} A_0(x,\xi',\frac{1}{i}dt) &= -|\xi'|^2 - \left(\frac{1}{i}\frac{d}{dt}\right)^2 = -|\xi'|^2 + \frac{d^2}{dt^2}, \quad \xi' \in \mathbb{R}^{n-1}, \\ P(z) &= A_0(x,\xi',z) = -|\xi'|^2 - z^2. \end{aligned}$$

The zeros of $P(z)$ are $z_{1,2} = \pm i|\xi'|$, $z_1 = i|\xi'|$ is situated in the upper half plane. Therefore:

$$M^+ = \{c_1 e^{iz_1 t} = c_1 e^{-|\xi'|t}, \quad c_1 \in \mathbb{C}\}.$$

First case: Let be $\operatorname{ord} b_1 = 0$. Then

$$b_1\left(x,\xi',\frac{1}{i}\frac{d}{dt}\right)v(0) = b_0(x_0)c_1 e^{-|\xi'|0} = b_0(x_0)c_1 = h_1.$$

If $b_0(x_0) \neq 0$ then an uniquely determined solution c_1 exists for every h_1.
Second case: Let be $\operatorname{ord} b_1 = 1$,

$$b_{1,0}(x_0,D) = \sum_{i=1}^{n} b_j(x_0) i\left(\frac{1}{i}\frac{\partial}{\partial x_j}\right).$$

Then

$$\begin{aligned} b_{1,0}\left(x,\xi',\frac{1}{i}\frac{d}{dt}\right)v(0) &= \left[\sum_{j=1}^{n-1} b_j(x_0) i\xi_j + b_n(x_0)\frac{d}{dt}\right] v(t)\bigg|_{t=0} \\ &= \left[\sum_{j=1}^{n-1} b_j(x_0)\xi_j i c_1 + b_n(x_0)(-c_1|\xi'|)\right] \\ &= \left\{ i\sum_{j=1}^{n-1} b_j(x_0)\xi_j - b_n(x_0)|\xi'| \right\} c_1 = h_1. \end{aligned}$$

This equation is uniquely solvable if the factor does not vanish for $|\xi'| \neq 0$. For the Neuman problem we have $b_0 = b_1 = \ldots = b_{n-1} = 0$, $b_n = 1$. The condition is satisfied.

Assume that under the m boundary operators there are p essential boundary operators $0 \leq p \leq m$. Setting $\tilde{a}_{\alpha\beta} = \frac{a_{\alpha\beta}+\overline{a_{\beta\alpha}}}{2}$ we consider analogously to (1.44) the boundary value problem

$$\tilde{A}_0\left(x_0, \xi', \frac{1}{i}\frac{d}{dt}\right)v(t) = 0 \text{ for } t > 0 \tag{1.45}$$

$$b_{j,0}\left(x_0, \xi', \frac{1}{i}\frac{d}{dt}\right)v(t)\Big|_{t=0} = 0 \text{ for } j = 1, \ldots, p. \tag{1.46}$$

Introducing M^+ analogously as before and writing $D_t = \frac{1}{i}\frac{d}{dt}$ we get the
Condition of Agmon. *Let be $v(t) \neq 0 \in M^+$ an arbitrary solution of the initial problem (1.45), (1.46). It holds*

$$Re \int_0^\infty \sum_{\substack{|\alpha'|+k=m \\ |\beta'|+l=m}} a_{(\alpha',k)(\beta',l)} \xi'^{\alpha'} \xi'^{\beta'} D_t^k v(t) \overline{D_t^l v(t)} dt > 0 \text{ for } 0 \neq \xi' \in \mathbb{R}^{n-1} \tag{1.47}$$

If $p = 0$ then all solutions of (1.45) should satisfy (1.47). If $p = m$ and the Lopatinskij-Shapiro condition holds, then $v(t) = 0$ and (1.47) is meaningless (trivially satisfied).
EXAMPLE.
$\triangle u = f$ in Ω, $m = 1$, $n = 2$, $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$, $a_{\alpha\beta} = \delta_{\alpha\beta}$, $(\alpha', k) = (0,1) = (\beta', l)$ or $(\alpha', k) = (1,0) = (\beta', l)$. It follows that

$$\int_0^\infty \left[\xi_1\xi_1 v(t)\bar{v}(t) + D_t v(t) D_t \overline{v(t)}\right] dt > 0.$$

Theorem 9 (Theorem of Agmon). *For the boundary value problem*

$$\sum_{\alpha,\beta}(-1)^\alpha D^\beta a_{\alpha\beta}(x) D^\alpha u(x) = f(x) \text{ in } \Omega$$

$$b_j(x, D)u(x)\big|_{\partial\Omega} = 0, \qquad j = 1, \ldots, m,$$

we consider the bilinear form

$$a(u, v) = \sum_{\alpha,\beta} \int_\Omega a_{\alpha\beta}(x) D^\alpha u(x) D^\beta v(x) dx$$

on $V \times V$, where

$$V = \{u \in H^m(\Omega),\ b_j(x, D)u\big|_{\partial\Omega} = 0 \text{ for } m_j \leq m - 1\}.$$

If the coefficients of A and b are sufficiently smooth, the domain is smooth enough and the condition of Agmon (1.47) is satisfied, then $a(\cdot, \cdot)$ is V-coercive.

REMARK. The Fredholm-property follows from the V-coerciveness, and therefore we get the ellipticity of the boundary value problem.

Chapter 2

The Numerics

This Chapter is devoted to the numerical computation of the solution of variational problems. We use Finite-Element-Methods (FEM) and Boundary-Element-Methods (BEM). Both methods belong to the class of Galerkin-Methods.

2.1 Galerkin-Methods

In 1909 W. Ritz [16] has published a method, nowadays called Ritz-method, for numerical solving of the extremal problem (1.8), while B. Galerkin [4] has developed a numerical method for solving variational problems. The basic idea of both methods is: Solve the problems not in Sobolev spaces of infinite dimension, but solve it in finite dimensional subspaces. This leads to linear equation systems. Let us start with the Ritz-Method.

2.1.1 Ritz Method

Let be V a Hilbertspace and V_0 a closed linear subspace of V. Assume the bilinear form $a(\cdot,\cdot)$ is symmetric, bounded and V_0-elliptic. Then Theorem 7 yields that the extremal problem

$$F(u) = \frac{1}{2}a(u,u) - f(u) = \min!$$

for $u \in \{u_0 + V_0\}$, u_0 is a fixed element of V, has a uniquely defined solution. For simplicity we assume $u_0 = 0$, $V_0 = V$. Instead of V we consider a finite dimensional subspace V_N of dimension N and with a basis $\{e_1, \dots, e_N\}$. Every element $w \in V_N$ has a uniquely determined representation

$$w = \sum_{i=1}^{N} w_i e_i.$$

Let be $\vec{w} = (w_1, \dots, w_N)^T$ and $P\vec{w} = \sum w_i e_i = w$. P is an isomorphism between $\mathbb{R}^N$ and V_N. The problem in V_N reads: Find coefficients w_i, such that

$$\begin{aligned} F(w) &= H(w_1,\dots,w_N) \\ &= \frac{1}{2}a\Big(\sum_i w_ie_i,\sum_j w_je_j\Big)-(f,\sum_i w_ie_i)=min! \end{aligned} \tag{2.1}$$

Lemma 16. *Assume the bilinear form $a(\cdot,\cdot):V\times V\to\mathbb{R}$ is V-elliptic and symmetric. Then problem (2.1) has a uniquely defined solution.*

Proof. The nessecary condition for an extremal value reads:

$$\frac{\partial H}{\partial w_i}=0 \text{ for } i=1,\dots,N. \tag{2.2}$$

Since

$$\begin{aligned} H(w_1,\dots,w_N) &= \frac{1}{2}w_1\sum_{j=1}^N w_ja(e_1,e_j)-w_1(f,e_1) \\ &+ \frac{1}{2}w_2\sum_{j=1}^N w_ja(e_2,e_j)-w_2(f,e_2) \\ &\vdots \\ &+ \frac{1}{2}w_N\sum_{j=1}^N w_ja(e_N,e_j)-w_N(f,e_N), \end{aligned}$$

the equations (2.2) mean:

$$\begin{aligned} \frac{\partial H}{\partial w_1} &= w_1a(e_1,e_1)+\frac{1}{2}w_2a(e_1,e_2)+\dots+\frac{1}{2}w_Na(e_1,e_N)-(f,e_1) \\ &\quad +\frac{1}{2}w_2a(e_2,e_1)+\dots+\frac{1}{2}w_Na(e_N,e_1)=0 \\ &\vdots \\ \frac{\partial H}{\partial w_N} &= w_Na(e_N,e_N)+\frac{1}{2}w_1a(e_N,e_1)+\dots+\frac{1}{2}w_{N-1}a(e_N,e_{N-1})-(f,e_N) \\ &\quad +\frac{1}{2}w_1a(e_1,e_N)+\dots+\frac{1}{2}w_{N-1}a(e_{N-1},e_N)=0. \end{aligned}$$

These relations lead to the linear equation system

$$\begin{pmatrix} a(e_1,e_1) & a(e_1,e_2) & \dots & a(e_1,e_N) \\ \vdots & & & \\ a(e_N,e_1) & \dots & \dots & a(e_N,e_N) \end{pmatrix}\begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}=\begin{pmatrix} (f,e_1) \\ \vdots \\ (f,e_N) \end{pmatrix}. \tag{2.3}$$

The resulting matrix M is symmetric and positive definite; indeed

$$\begin{aligned} (M\vec{w},\vec{w}) &= \sum_i w_i\left(\sum_j a(e_i,e_j)w_j\right)=\sum_i w_ia(e_i,w) \\ &= a(P\vec{w},P\vec{w})\geq c\|P\vec{w}\|^2>0 \text{ for } w\neq 0. \end{aligned}$$

Therefore there exists a uniquely defined solution of (2.3).
If the Hesse-matrix

$$\begin{pmatrix} \frac{\partial^2 H}{\partial w_1^2} & \frac{\partial^2 H}{\partial w_2 \partial w_1} & \cdots & \frac{\partial^2 H}{\partial w_N \partial w_1} \\ \vdots & & & \vdots \\ \frac{\partial^2 H}{\partial w_1 \partial w_N} & \cdots & \cdots & \frac{\partial^2 H}{\partial w_N \partial w_N} \end{pmatrix}$$

is positive definite, then the solution of (2.1) is a minimum. The Hesse matrix coincides with the matrix M and we have proved the Lemma 16. □

The Ritz method is a special case of the Galerkin method. We will study the problem of convergence in a more general framework.

2.1.2 Galerkin solutions

We study now the variational formulation: Find an element $u \in V$, such that

$$a(u, v) = \langle f, v \rangle \qquad \forall v \in V. \tag{2.4}$$

Assume that $a(\cdot,\cdot) : V \times V \to V'$ is linear and bounded. Let $V_N \subset V$ be a N-dimensional subspace of V.

Definition 22. *$u_N \in V_N$ is a Galerkin solution of (2.4) if*

$$a(u_N, v) = \langle f, v \rangle \qquad \forall v \in V_N. \tag{2.5}$$

An equivalent formulation of the Galerkin solution is: $u_N \in V_N$ is Galerkin solution if

$$a(u_N, e_i) = \langle f, e_i \rangle \qquad \text{for } i = 1, \dots, N. \tag{2.6}$$

We now define a linear equation system, whose solution generates the Galerkin solution.

Lemma 17. *Let be $M = (a(e_i, e_j))_{i,j}$ the matrix and $\vec{f} = \langle f, e_i \rangle_{i=1,\dots,N}$ the vector defined by the basis elements of V_N. The problems*

$$\begin{aligned} M\vec{w} &= \vec{f} \\ \textit{and} \quad a(P\vec{w}, v) &= \langle f, v \rangle \qquad \forall v \in V_N \end{aligned} \tag{2.7}$$

are equivalent and there exist solutions of both problems for every $f \in V'$.

Proof.

(1) Assume $u_N \in V_N$ is Galerkin solution. Insert $u_N = \sum_{i=1}^N w_i e_i$ into the equation (2.6). Using the bilinearity of the form $a(\cdot,\cdot)$ we get (2.7).

(2) Let $\vec{w}$ be solution of (2.7). The scalar multiplication of (2.7) by an arbitrary vector $\vec{v}$ leads to

$$M\vec{w} \cdot \vec{v} = \vec{f}\vec{v} = \sum_j \left(\sum_i a(e_i, e_j) w_i \right) v_j = a(P\vec{w}, P\vec{v}) = \langle f, v \rangle.$$

□

COROLLARY. There is a Galerkin solution for every $f \in V'$ iff the matrix M from (2.7) is regular ($\det M \neq 0$). If $a(\cdot,\cdot)$ is symmetric and V-elliptic, then the matrix M is symmetric and positive definite (see above).

Lemma 18. *Assume the LBB condition*

$$\inf\{\sup\{|a(u,v)| : v \in V_N, \|v\|_V = 1\} u \in V_N, \|u\|_V = 1\} = \varepsilon_N > 0.$$

Then there is a Galerkin solution u_N for every $f \in V'$ and

$$\|u_N\| \leq \frac{1}{\varepsilon_N}\|f\|_{V_N'} \leq \frac{1}{\varepsilon_N}\|f\|_{V'}. \tag{2.8}$$

Proof. Since $V_N \subset V$ it follows $V' \subset V_N'$. Therefore we can apply the Lax-Milgram theorem to V_N instead to V. The conditions (1.10) and (1.11) coincide for finite dimensional spaces. (Note that the unit sphere is a compact set and the inf and sup will be realised on it.) Furthermore, $A^{-1} \in L(V_N', V_N)$ means

$$\begin{aligned}\|A^{-1}f\| &= \|A^{-1}Au_N\| = \|u_N\| \leq \|A^{-1}\|\,\|f\|_{V_N'} = \\ &= \frac{1}{\varepsilon_N}\|f\|_{V_N'} \leq \frac{1}{\varepsilon_N}\|f\|_{V'}.\end{aligned}$$

□

REMARK. If the bilinear form is V-coercive and a solution exists for a right hand side f, then the linear equation system (4.7) is not always solvable in general.

EXAMPLE: Let be

$$\begin{aligned}a(u,v) &= \int_0^1 (u'v' - 10uv)dx = \int_0^1 fvdx. \\ V &= \overset{0}{H^1}(0,1).\end{aligned}$$

The corresponding classical problem reads:

$$\begin{aligned}+u'' + 10u &= g \text{ in } (0,1) \\ u(0) &= u(1) = 0.\end{aligned}$$

It is

$$\begin{aligned}&a(u,u) = \int_0^1 u'^2 dx - 10\int_0^1 u^2 dx \geq C_3\|u\|_V^2 - 10\|u\|_{L_2}^2 \text{ (V-coerciveness)} \\ &V_N = V_1 = \operatorname{span}\{x(1-x)\} \subset V,\ e_1 = x(1-x) = x - x^2. \\ &a(e_1,e_1) = a\left(x(1-x), x(1-x)\right) = \int_0^1 (1-2x)^2 - 10x^2(1-x)^2\,dx = 0\end{aligned}$$

and the matrix M vanishes. The inf-sup condition is violated since

$$\inf_{\substack{u\in V_1\\ \|u\|=1}} \sup_{\substack{v\in V_1\\ \|v\|=1}} a(u,v) = a\left(\frac{e_1}{\|e_1\|}, \frac{e_1}{\|e_1\|}\right) = 0.$$

REMARK. M is called stiffness matrix, especially in continuum mechanics.

We estimate the error between the weak solution $u \in V$ and the Galerkin solution $u_N \subset V_N$.

Lemma 19. (Céa, 64) *Let $a(\cdot,\cdot)$ be a bilinear form on $V \times V$ with*

$$\begin{aligned} |a(u,v)| &\leq c_1\|u\|_V\|v\|_V && (2.9)\\ a(u,u) &\geq c_2\|u\|_V^2, && (2.10)\end{aligned}$$

and $f \in V'$. For the solution $u \in V$ to a(u,v)=(f,v) and for the Galerkin solution u_N it holds the error estimate

$$\|u-u_N\| \leq \frac{c_1}{c_2}\|u-v_N\| \quad \forall v_N \in V_N. \tag{2.11}$$

Proof. We have

$$\begin{aligned} a(u,v) &= (f,v) \quad \forall v \in V_N\\ a(u_N,v) &= (f,v) \quad \forall v \in V_N.\\ \text{Hence} \quad a(u-u_N,v) &= 0 \quad \forall v \in V_N, \end{aligned} \tag{2.12}$$

that means $u - u_N \perp v$ with respect to the energetic scalar product. Thanks to (2.10) we have

$$\begin{aligned} \|u-u_N\|_V^2 &\leq \frac{1}{c_2}|a(u-u_N,u-u_N)| \overset{(2.12)}{=} \frac{1}{c_2}a(u-u_N,u-u_N)|\\ &\overset{(2.10)}{\leq} \frac{c_1}{c_2}\|u-u_N\|_V\|u-v_N\|_V^2. \end{aligned}$$

Dividing by $\|u-u_N\| \neq 0$ we get the assertion. □

Lemma 19 can be weakened.

Lemma 20. *Let $a(\cdot,\cdot)$ be a bilinear form on $V \times V$ with*

- $a(u,v) \leq c_1\|u\|_V\|v\|_V \qquad \forall u,v \in V.$
- $V_N \subset V$, dim $V_N = N < \infty$.
- $\inf_{\substack{u\in V_N\\ \|u\|=1}} \{\sup_{\substack{v\in V_N\\ \|v\|=1}} |a(u,v)|\} = \varepsilon_N > 0.$

Let $u \in V$ be a solution to $a(u,v) = (f,v)$ for $f \in V'$ and let be $u_N \in V_N$ the Galerkin solution. Then

$$\begin{aligned} \|u - u_N\| &\leq \left(1 + \frac{c_1}{\varepsilon_N}\right) \inf_{v_N \in V_N} \|u - v_N\|_V \\ &= \left(1 + \frac{c_1}{\varepsilon_N}\right) \mathrm{dist}(u, V_N). \end{aligned} \tag{2.13}$$

Proof. As in the Lemma before we have

$$a(u_N - u, v) = 0 \quad \forall v \in V_N.$$

For any v and $w \in V_N$ with $\|v\| = 1$ we get

$$a(u_N - w, v) = a\left([u_N - u] + [u - w], v\right) = a(u - w, v)$$

and

$$|a(u_N - w, v)| \leq c_1 \|u - w\|_V.$$

From the third assumption it follows

$$\|u\|_V \leq \frac{1}{\varepsilon_N} \sup\{|a(u,v)|,\ v \in V_N,\ \|v\| = 1\} \quad \forall\, u \in V_N,$$

hence

$$\begin{aligned} \|u_N - w\| &\leq \frac{1}{\varepsilon_N} \sup\left\{|a(u_N - w, v| : v \in V_N,\ \|v\| = 1\right\} \\ &\leq \frac{c_1}{\varepsilon_N} \|u - w\|_V. \end{aligned}$$

Applying the inequality

$$\|u - u_N\|_V \leq \|u - w\|_V + \|u_N - w\|_V \leq \left(1 + \frac{c_1}{\varepsilon_N}\right) \|u - w\|_V$$

and having in mind that w is arbitrary we get the assertion. □

Let us underline that we have not assumed that an uniquely defined weak solution exists.

The formulae (2.11) and (2.13) lead to error estimates. The error converges to zero if $\mathrm{dist}(u, V_N)$ tends to zero for $N \to \infty$. Therefore it is necessary to introduce a sequence of finite-dimensional subspaces which converges to V. Let us denote $V_i = V_{N_i}$ $(i \in \mathbb{N})$ and assume

$$\lim_{i \to \infty} \mathrm{dist}(u, V_i) = 0 \quad \forall\, u \in V. \tag{2.14}$$

If $V_1 \subset V_2 \subset \ldots \subset V_i \ldots \subset V$, $\cup_{i=1}^{\infty} V_i$ is dense in V, then the condition (2.14) holds [6].

Theorem 10. *Let be $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ a bounded bilinear form. Assume that (2.14) is satisfied and the third condition of Lemma 20 is "uniformly" valid with constants*

$\varepsilon_{N_i} \geq \tilde{\varepsilon} > 0$.
Then exists a uniquely defined solution $u \in V$ *of* $a(u,v) = \langle f, v\rangle$ *for every* $f \in V'$*, and the Galerkin solution* $u_{N_i} = u_i$ *converges to* u*:*

$$\|u - u_i\|_V \to 0 \text{ for } i \to \infty. \tag{2.15}$$

Proof.

(a) Assume that a solution u exists. It follows from (2.13) that

$$\|u - u_i\| \leq \left(1 + \frac{c_1}{\varepsilon_{N_i}}\right) \operatorname{dist}(u, V_i).$$

From (2.14) it follows that u is uniquely defined.

(b) The image W of the operator $A : V \to V'$, $\langle Au, V\rangle := a(u,v)$ is closed. $W := \{Av,\, v \in V\} \subset V'$. For every $f \in W$ exists a solution $u \in V$ with $Au = f$ and there is a sequence $u_i \in V_i$ which converges to u (step (a)). Since

$$\|u_i\|_V \overset{(2.8)}{\leq} \frac{1}{\varepsilon_{N_i}} \|f\|_{V'} \leq \frac{1}{\tilde{\varepsilon}} \|f\|_{V'}$$

we have

$$\|u\|_V = \lim_{i\to\infty} \|u_i\|_V \leq \frac{1}{\tilde{\varepsilon}} \|f\|_{V'}.$$

Let be (f_n) a sequence of elements from W with

$$f_n \to f^* \text{ in } V' \text{ and } Au_n = f_n.$$

It follows

$$\|f_n - f_m\|_{V'} \to 0 \text{ and } \|u_n - u_m\| \leq \frac{1}{\tilde{\varepsilon}} \|f_n - f_m\|_{V'} \to 0.$$

Therefore $u^* = \lim_{n\to\infty} u_n \in V$ exists. Since $A \in L(V, V')$ is a continuous operator we have

$$f^* = \lim f_n = \lim Au_n = Au^* \text{ and } f^* \in W.$$

(c) A is a surjective mapping. Assume $W \neq V'$. Then exists an element $f \in W^{\perp}$ with $\|f\|_{V'} = 1$. Using the Riesz-isomorphism $J_V : V \to V'$ we define $v := J_V^{-1} f \in V$. It holds

$$a(u,v) = \langle Au, v\rangle = (Au, f)_{V'} = 0 \quad \forall\, u \in V,$$

in particular, for a Galerkin solution u_i with $a(u_i, v) = \langle f, v\rangle \quad \forall v \in V$ we have $a(u_i, v) = 0$. Splitting $v = v_i + r_i$, $v_i \in V_i$ then

$$\begin{aligned}
0 = a(u_i, v) &= a(u_i, v_i) + a(u_i, r_i) = \langle f, v_i\rangle + a(u_i, r_i) \\
&= \langle f, v\rangle - \langle f, r_i\rangle + a(u_i, r_i) \\
&= (f, f)_{V'} - \langle f, r_i\rangle + a(u_i, r_i) \\
&= 1 - \langle f, r_i\rangle + a(u_i, r_i).
\end{aligned}$$

It follows that

$$1 = |\langle f, r_i \rangle - a(u_i, r_i)| \leq c\,[\|f\|_{V'} + \|u_i\|_V]\,\|r_i\|_V. \tag{2.16}$$

Furthermore, it follows from the estimate (2.8) that the solution u_i are uniformly bounded,

$$\|u_i\| \leq \frac{1}{\tilde{\varepsilon}}\|f\|_{V'}$$

and $\|r_i\|_V \to 0$ for $i \to \infty$. This is a contradiction to the assumption that $W \neq V'$.

□

Remark The uniform stability condition (LBB-condition) is for V-elliptic bilinear forms satisfied. The verification is not easy in general.

2.2 Finite Element Methods (FEM)

The matrix M is dense in general. In particular for large N it is a disadvantage, since we have N^2 integrations in order to compute the elements of M, and N^3 operations for solving the resulting equation system. Therefore it is desirable to choose the spaces V_N and the basis $\{e_i\}$ such that the resulting matrix

$$M = (a(e_i, e_j))$$

is sparse. The best variant is, that $a(e_i, e_j) = \delta_{ij}$ (orthonormality with respect to the energetic scalar product) but, the orthonormality procedure is not efficient enough. The fundamental idea is now: Given is the bilinear form

$$a(u, v) = \int_\Omega \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u(x)\, D^\beta v(x)\, dx.$$

Choose a basis $(e_1, \dots, e_N)$ in V_N of functions with small support. (Such basis functions e_i are called "finite elements"). If supp $e_i \cap$ supp $e_j = m_{ij}$ is a set of the measure zero, then $a(e_i, e_j) = 0$; especially, if

$$\begin{aligned} int(\operatorname{supp} e_i) \cap int(\operatorname{supp} e_j) &= \emptyset, \\ \text{then} \quad a(e_i, e_j) &= 0. \end{aligned}$$

From the practical point of view the domain Ω is splitted into subdomains (triangles, polyhedrals) also called "finite elements", which yield the support of the basis functions. Furthermore, the basis function should be easy analytically differentiable and integrable, such as continuous, piecewise polynomials.

2.2.1 Linear Elements (Splines)

We start with $n = 1$, $\Omega = (a, b)$.

(1) *The boundary value problem*
Classical formulation: Find an element $u \in C^2(a, b)$ with

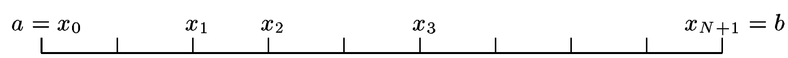
$$\begin{aligned} -\quad & u''(x) = f(x) \text{ for } a < x < b \\ & u(a) = u(b) = 0. \end{aligned}$$

Weak formulation: Find an element $u \in \overset{\circ}{H}{}^1(a, b)$ such that

$$a(u, v) = \int_a^b u'(x)v'(x)\,dx = \int_a^b f(x)v(x)\,dx \quad \forall v \in \overset{\circ}{H}{}^1(a, b).$$

(2) *Partition of $\bar{\Omega}$*
Let be choosen the nodes $\{x_i\} : a = x_0 < x_1 < \ldots < x_{N+1} = b$ which yield the intervalls $I_i = (x_{i-1}, x_i) \subset (a, b)$

$$\cup_{i=1}^{N+1} \bar{I}_i = \bar{\Omega} = [a, b].$$



(3) *Construction of $V_N \subset V$*

$$V_N = \{u \in C[a, b] : \; u|_{I_i} = a_i x + b_i, \; 1 \le i \le N + 1, \; u(a) = u(b) = 0\}$$

Figure 4 describes graphs of elements of V_N.

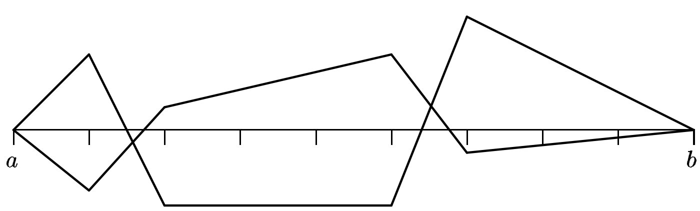


Figure 4

$u \in C[a, b]$ means, that $u(x_i + 0) = u(x_i - 0)$ for $i = 1, \ldots, N$.

Lemma 21. *Every element from V_N is through the values $u(x_i)$ in the node points x_i, $i = 1, \ldots, N$, uniquely defined,* $\dim V_N = N$ *and the basis elements can be choosen as:*

$$e_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \textit{for} \quad x_{i-1} < x \le x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \textit{for} \quad x_i < x < x_{i+1} \\ 0 & \textit{else}, \end{cases}$$

where supp $e_i = [x_{i-1}, x_{i+1}]$.

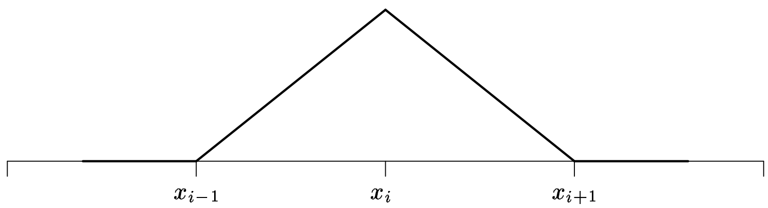


Figure 5. Graph of e_i

Proof.

(a) Let be $u(x_i)$ given

$$u(x)\Big|_{I_i} = a_i x + b_i$$

$$\left.\begin{array}{r} u(x_{i-1}) = a_i x_{i-1} + b_i \\ u(x_i) = a_i\, x_i + b_i \end{array}\right\} \Rightarrow \begin{array}{r} a_i = \frac{u(x_{i-1}) - u(x_i)}{x_{i-1} - x_i} \\ b_i = \frac{u(x_i)x_{i-1} - u(x_{i-1})x_i}{x_{i-1} - x_i} \end{array}$$

(b) $\{e_i\}_{i=1,\dots,N}$ are linearly independent: if

$$\sum_{i=1}^{N} \alpha_i e_i(x) = 0 \quad \forall x \in (a,b),$$

then

$$\sum_{i=1}^{N} \alpha_i e_i(x_j) = 0 = \sum_{i=1}^{N} \alpha_i \delta_{ij} = \alpha_j = 0 \text{ for } j = 1, \dots, N.$$

□

Furthermore,

$$u_N(x) = \sum_{i=1}^{N} u_i e_i(x) \textit{ and } u_N(x_j) = u_j.$$

(4) *Setting up the equation system.*
Since $a(u,v) = \int_\Omega u'v'dx$ we get $a(e_i, e_j) = 0$ if $|i-j| \geq 2$, i.e. M is a three-diagonal matrix.

$$\begin{aligned} a(e_i, e_{i-1}) &= \int_{x_{i-1}}^{x_i} \frac{1}{x_i - x_{i-1}} \left(\frac{-1}{x_i - x_{i-1}} \right) dx = -\frac{1}{x_i - x_{i-1}}, \\ a(e_i, e_i) &= \int_{x_{i-1}}^{x_i} \left(\frac{1}{x_i - x_{i-1}} \right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{-1}{x_{i+1} - x_i} \right)^2 dx \\ &= \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}, \\ a(e_i, e_{i+1}) &= -\frac{1}{x_{i+1} - x_i}. \end{aligned}$$

$$\begin{pmatrix} \frac{1}{x_1-x_0}+\frac{1}{x_2-x_1} & -\frac{1}{x_2-x_1} & 0 & 0 & \dots \\ -\frac{1}{x_2-x_1} & \frac{1}{x_2-x_1}+\frac{1}{x_3-x_2} & -\frac{1}{x_3-x_2} & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ u_N \end{pmatrix} =$$

$$\begin{pmatrix} \vdots \\ \int_{x_{i-1}}^{x_i} f(x)\frac{x-x_{i-1}}{x_i-x_{i-1}}dy + \int_{x_i}^{x_{i+1}} f\frac{x_{i+1}-x}{x_{i+1}-x_i}dx \\ \vdots \end{pmatrix}.$$

For the calculation of the right hand sides use numerical integration formulae e.g. $(f, e_i) \approx f(x_i)(x_{i+1} - x_{i-1})$ (rectangular formula).
Special case: equidistant meshing(uniform mesh)

$$\begin{aligned} x_i &= a + ih, \quad h = \frac{b-a}{N-1} \\ a(e_i, e_i) &= \frac{2}{h} \\ a(e_i, e_{i\pm1}) &= -\frac{1}{h}. \end{aligned}$$

Then

$$M = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \dots \\ -1 & 2 & -1 & 0 \\ 0 & \ddots & \ddots & \ddots \\ \dots & 0 & -1 & 2 \end{pmatrix}.$$

(5) *Solving of the equation system.*
Direct solvers, iterative methods.

The case $n = 2$

Let $\Omega \subset \mathbb{R}^2$ be a polygon.

(1) *The boundary value problem.*
Classical formulation: Find an element $u \in C^2(\bar{\Omega})$ with

$$\begin{aligned} -\triangle u &= f(x) \text{ for } x \in \Omega \\ u|_{\partial\Omega} &= 0. \end{aligned}$$

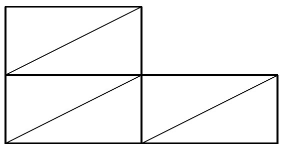
Weak formulation: Find an element $u \in \overset{0}{H^1}(\Omega) = V$ such that

$$a(u, v) = \int_\Omega \operatorname{grad} u \operatorname{grad} v \, dx = \int_\Omega f(x)v(x)dx \quad \forall v \in V.$$

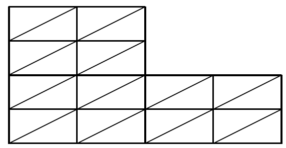
(2) *Partition of* $\bar{\Omega}$
We devide $\bar{\Omega}$ into a number of subregions, e.g. of triangles T_i or quadrilaterals. We consider here a triangulation.

We use the notation: t–number of triangles, N–number of interior nodes, and M–number of all nodes. Note that a node is a corner point of a triangle.

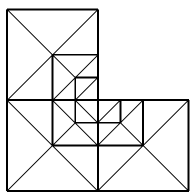
Examples.



$t = 6,\ \ N = 0,\ M = 8$



$t = 24,\ \ N = 5,\ \ M = 21$

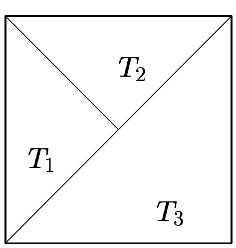


$t = 30, \quad N = 10, \quad M = 22$

Definition 23. (admissible triangulation)
$\tau = \{T_1, \dots, T_t\}$ is an admissible triangulation of Ω if

- *T_i $(i \leq i \leq t)$ are open triangles.*
- *The finite elements T_i are disjunct i.e $T_i \cap T_j = \emptyset$ for $i \neq j$.*
- *$\cup_{i=1,..,t} \bar{T}_i = \bar{\Omega}$*
- *for $i \neq j$ is $\bar{T}_i \cap \bar{T}_j = \begin{cases} \emptyset \\ \textit{common side} \\ \textit{common corner} \end{cases}$*

Remark. The last condition says that the follwing triangulation is not allowed:



$\bar{T}_1 \cap \bar{T}_3 \notin \{\emptyset,$ common side, common corner$\}$

(3) *Construction of V_N*

$$V_N = \{u \in C(\bar{\Omega}) : \ u|_{\partial\Omega} = 0, \ \ u|_{T_i} = a_{i1} + a_{i2}x + a_{i3}y\}$$

Lemma 22. *$V_N \subset \overset{\circ}{H}{}^1(\Omega)$. Every function $u \in V_N$ is uniquely defined by the values $u(x_i, y_i)$ in the interior nodes (x_i, y_i), $1 \leq i \leq N$.*

Proof. The weak derivative of u exists in $L_2(\Omega)$, furthermore, $u|_{\partial\Omega} = 0$ and therefore Lemma 6 implies that $V_N \subset \overset{\circ}{H}{}^1(\Omega)$. The three coefficients $a_{ij},\ j = 1, 2, 3$ are uniquely determined, inserting the node points in the equations

$$u_{ik} = u(x_{ik}, y_{ik}) = a_{i1} + a_{i2}x_{ik} + a_{i3}y_{ik}, \quad k = 1, 2, 3.$$

Then

$$\begin{pmatrix} 1 & x_{i1} & y_{i1} \\ 1 & x_{i2} & y_{i2} \\ 1 & x_{i3} & y_{i3} \end{pmatrix} \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{13} \end{pmatrix} = \begin{pmatrix} u_{i1} \\ u_{i2} \\ u_{i3} \end{pmatrix},$$

$$\begin{aligned} \det \begin{pmatrix} 1 & x_{i1} & y_{i1} \\ 1 & x_{i2} & y_{i2} \\ 1 & x_{i3} & y_{i3} \end{pmatrix} &= x_{i3}(y_{i1} - y_{i2}) + x_{i2}(y_{i3} - y_{i1}) + x_{i1}(y_{i2} - y_{i3}) \\ &= -\begin{vmatrix} x_{i3} - x_{i1} & y_{i3} - y_{i1} \\ x_{i2} - x_{i1} & y_{i2} - y_{i1} \end{vmatrix} \\ &= -2|T_i| \neq 0, \end{aligned} \tag{2.17}$$

if the vectors in Fig. 6 are linearly independent.

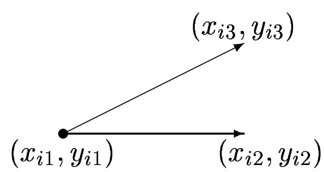


Figure 6 □

This lemma allows us to define basis functions through the relation

$$e_i(x_k, y_k) = \delta_{ik}.$$

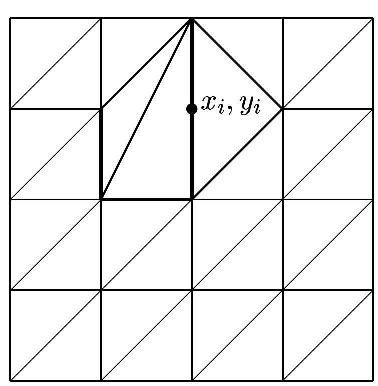


Figure 7

Let be T a triangle with the corner points (x_i, y_i), (x', y') and (x'', y''), then

$$e_i(x,y)\big|_T = \frac{(x-x')(y''-y') - (y-y')(x''-x')}{(x_i-x')(y''-y') - (y_i-y')(x''-x')}$$

and $\operatorname{supp} e_i = \cup_{i_k}\{\bar{T}_{ik} \subset \tau : \ (x_i, y_i) \text{ is corner point}\}$.

It holds that

$$int \operatorname{supp} e_i \cap int \operatorname{supp} e_j = \emptyset$$

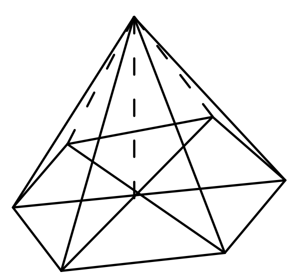
iff the nodes (x_i, y_i) and (x_j, y_j) are not directly connected by an edge.
Exercise. Show that the functions e_i are linearly independent.

(4) *Generation of the Equation System*
We have

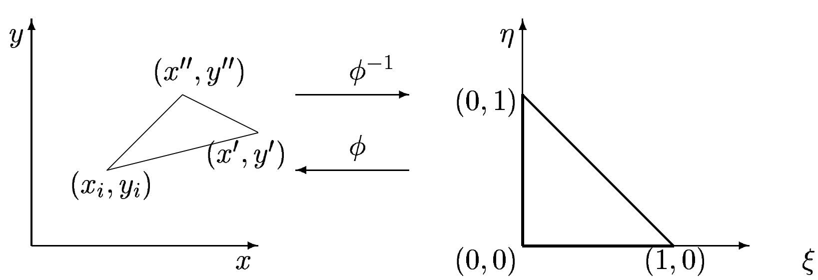
$$\begin{aligned} a(e_i, e_j) &= \int_\Omega \operatorname{grad} e_i \ \operatorname{grad} e_j \, dx \\ &= \sum_k \int_{T_k} \operatorname{grad} e_i \ \operatorname{grad} e_j \, dx, \end{aligned}$$

where we have to take the sum for following numbers k:

- If $i = j$, (x_i, y_i) is corner point of T_k (see Fig. 8)
- If $i \neq j$, (x_i, y_i) and (x_j, y_j) are corner points of T_k.



The integration over different T_k will be simplified if we take a mapping onto a unit triangle, which is called "master" (or "reference") element.



$$\begin{aligned} \phi^{-1}: \quad (x_i, y_i) &\longrightarrow (0,0) \\ (x', y') &\longrightarrow (1,0) \\ (x'', y'') &\longrightarrow (0,1) \end{aligned}$$

$$\phi \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \underbrace{\begin{pmatrix} x' - x_i & x'' - x_i \\ y' - x_i & y'' - y_i \end{pmatrix}}_{m} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since $\det m \neq 0$ we get

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = m^{-1} \begin{pmatrix} x - x_i \\ y - y_i \end{pmatrix} = \frac{1}{\det m} \begin{pmatrix} y'' - y_i & x_i - x'' \\ y_i - y' & x' - x_i \end{pmatrix} \begin{pmatrix} x - x_i \\ y - y_i \end{pmatrix},$$

with $\det m = 2|T_k|$, $|T_k| = \text{area } T_k$. Furthermore,

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = \det \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \det m$$

and

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{1}{2|T_k|}\left[\frac{\partial}{\partial \xi}(y'' - y_i) + \frac{\partial}{\partial \eta}(y_i - y')\right] \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial \eta} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{2|T_k|}\left[\frac{\partial}{\partial \xi}(x_i - x'') + \frac{\partial}{\partial \eta}(x' - x_i)\right] \end{aligned}$$

1^{st} case

$i = j$, (x_i, y_i) is corner point of T_k. The basis function $e_i(x,y)|_{T_k}$ is defined by the equations

$$\begin{aligned} e_i(x_i, y_i) &= 1 \\ e_i(x', y') &= 0 \\ e_i(x'', y'') &= 0. \end{aligned}$$

Defining $\hat{e}_i(\xi,\eta) := e_i(x,y)$ we have analogously

$$\begin{aligned} \hat{e}_i(0,0) &= 1 \\ \hat{e}_i(1,0) &= 0 \\ \hat{e}_i(0,1) &= 0 \end{aligned}$$

what implies $\hat{e}_i(\xi,\eta) = 1 - \xi - \eta$ (shape function). This leads to

$$\int_{T_k} (\text{grad } e_i)^2 \, dxdy = \int_{T_k} \left(\frac{\partial e_i}{\partial x}\right)^2 + \left(\frac{\partial e_i}{\partial y}\right)^2 dxdy$$

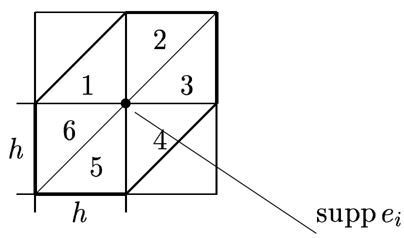
$$= \int_E \left[\frac{1}{2|T_k|}\left\{\frac{\partial(1-\xi-\eta)}{\partial \xi}(y''-y_i) + \frac{\partial(1-\xi-\eta)}{\partial \eta}(y_i-y')\right\}\right]^2$$

$$+ \left[\frac{1}{2|T_k|}\left\{\frac{\partial(1-\xi-\eta)}{\partial \xi}(x_i-x'') + \frac{\partial(1-\xi-\eta)}{\partial \eta}(x'-x_i)\right\}\right]^2 \cdot 2|T_k| d\xi d\eta$$

$$= \frac{1}{2|T_k|}\int_E \left[(-y''+y_i)+(y'-y_i)\right]^2 + \left[(x''-x_i)+(x_i-x')\right]^2 d\xi d\eta$$

$$= \frac{1}{2|T_k|}\left[(y'-y'')^2 + (x''-x')^2\right]\frac{1}{2}.$$

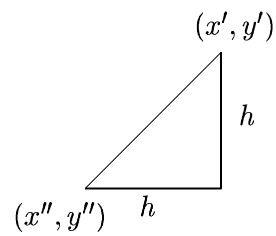
Special case: quadratic-mesh-triangulation



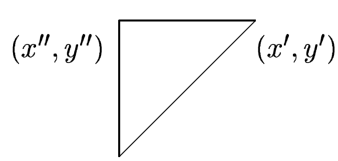
$$\int_\Omega (\text{grad}\, e_i)^2 \; dxdy = \int_{6 \text{ triangles}} (grad\, e_i)^2 \, dxdy$$

Triangles T_1, T_4:

$$x'' - x' = h$$
$$y' - y'' = h$$



Triangles T_2, T_3, T_5, T_6:



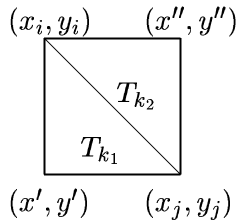
$$(y' - y'')^2 + (x'' - x')^2 = h^2$$

$|T_k| = \frac{h^2}{2}$. This leads to

$$\int_\Omega (\text{grad}\, e_i)^2 dxdy = \frac{1}{h^2}(8h^2) \cdot \frac{1}{2} = 4.$$

<u>2^{nd} case</u>

$i \neq j$



$$\hat{e_i}\Big|_{T_{k_1}} = 1 - \xi - \eta, \qquad \hat{e_i}\Big|_{T_{k_2}} = 1 - \xi - \eta$$
$$\hat{e_j}\Big|_{T_{k_1}} = \eta \qquad\qquad \hat{e_j}\Big|_{T_{k_2}} = \xi$$

since

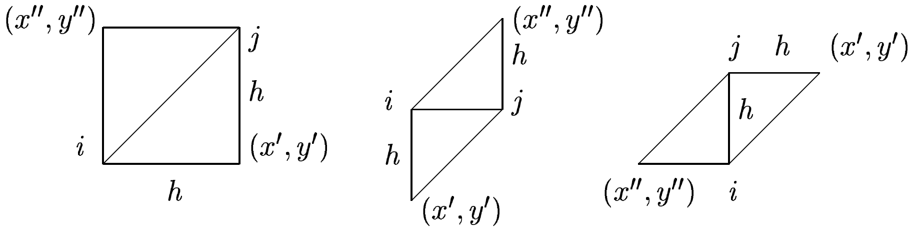
$$\begin{array}{lll} \hat{e_j}(1,0) &=& 0 \qquad \hat{e_j}(1,0) = 1 \\ \hat{e_j}(0,0) &=& 0 \qquad \hat{e_j}(0,0) = 0 \\ \hat{e_j}(0,1) &=& 1 \qquad \hat{e_j}(0,1) = 0. \end{array}$$

Hence

$$\begin{aligned} & \int_{T_{k_1}} \operatorname{grad} e_i \operatorname{grad} e_j \, dxdy \\ = \ & \frac{1}{2|T_{k_1}|} \int_E \left[\frac{\partial}{\partial \xi}(1-\xi-\eta)(y_j - y_i) \frac{\partial}{\partial \eta}(1-\xi-\eta)(y_i - y') \right] \left[\frac{\partial}{\partial \eta} \eta (y_i - y') \right] \\ & + \left[\frac{\partial}{\partial \xi}(1-\xi-\eta)(x_i - x_j) + \frac{\partial}{\partial \eta}(x' - x_j) \right] \left[\frac{\partial}{\partial \eta} \eta(-x_i + x') \right] d\xi d\eta \\ = \ & \frac{1}{4|T_{k_1}|} \left\{ \left[y_i - y_j + y' - y_i \right] (y_i - y') + (x_j - x_i + x_i - x')(-x_i + x') \right\} \\ = \ & \frac{1}{4|T_{k_1}|} \left\{ (y' - y_j)(y_i - y') + (x_j - x')(-x_i + x') \right\} \end{aligned}$$

$$\int_{T_{k_2}} \ldots = \frac{1}{4|T_{k_2}|} \left\{ (y_j - y'')(y'' - y_i) + (x'' - x_j)(x_i - x'') \right\}.$$

There are 3 cases for a quadratic-mesh-triangulation:

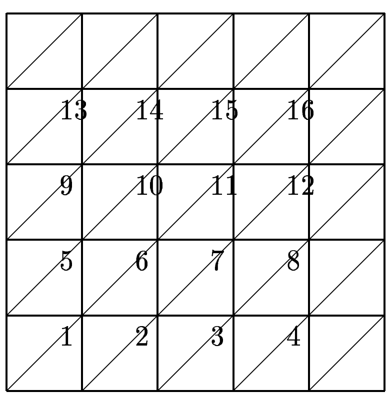


$y_j - y'' = 0$	$x_j - x'' = 0$	$y'' - y_i = 0$
$x'' - x_i = 0$	$x_i - x' = 0$	$y_j - y' = 0$
$y' - y_i = 0$	$\int_{T_{k_1} \cup T_{k_2}} = \frac{1}{2h^2}(-h^2)2 = -1$	$\int_{T_{k_1} \cup T_{k_2}} = \frac{1}{2h^2}(-h^2)2 = -1$
$x' - x_j = 0$		
$\int_{T_{k_1} \cup T_{k_2}} = 0$		

We have for $i \neq j$

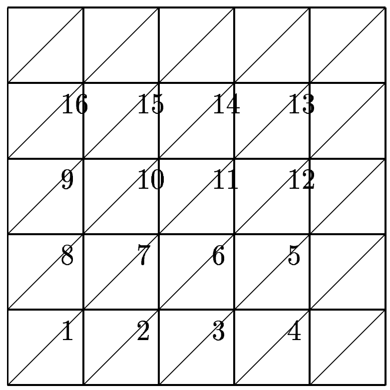
$$a(e_i, e_j) = \begin{cases} -1 & \text{if } \begin{pmatrix} x_i - x_j \\ y_i - y_j \end{pmatrix} = \begin{pmatrix} 0 \\ \pm h \end{pmatrix} \text{ or } \begin{pmatrix} \pm h \\ 0 \end{pmatrix}. \\ 0 & \text{else} \end{cases}$$

Example



$$\begin{pmatrix} 4 & -1 & 0 & 0 & -1 & 0 & 0 & \dots & 0 \\ -1 & 4 & -1 & 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & -1 & \dots & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\ 0 & & & & & \dots & & & \\ 0 & & & & & \dots & & & \\ \dots & & & & & \dots & & & \\ 0 & & & & & \dots & & & \end{pmatrix} \quad \text{9-diagonal band matrix}$$

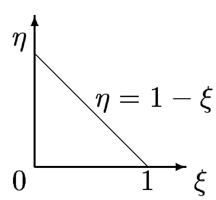
Problem: Minimize the width of the band by optimal numbering of the nodes. There are heuristical algorithms [18].



band width = 15

Computation of the right hand sides:

$$\begin{aligned}(f, e_j) &= \int_{\operatorname{supp} e_j} f(x,y)e_j(x,y)\,dx\,dy \\ &= \sum_{k=1}^{6} 2|T_k| \int_E \hat{f}(\xi,\eta)(1-\xi-\eta)\,d\xi\,d\eta \\ &= \sum_{k=1}^{6} 2|T_k| \int_0^1 \int_0^{1-\xi} \hat{f}(\xi,\eta)(1-\xi-\eta)\,d\eta\,d\xi\end{aligned}$$



EXAMPLE: $\hat{f}(\xi,\eta) = 1,\;\; 2|T_k| = h^2$, then $(1, e_j) = 6h^2 \cdot \frac{1}{6} = h^2$.

(5) *Solving of the equation system*
Direct solvers as Gauss-elimination, $L^T DL$-splitting and Cholesky-algorithm are robust, but the memory and the number of arithmetical operations is increasing if the mesh-parameter h tends to 0.
Iterative solvers need not so much place for memory, the number of arithmetical operations is optimal, but if the condition-number is high, then the number of iterations is increasing (preconditioning).

The case $n = 3$

We remark that the twodimensional considerations can be transmitted to the 3-dimensional case; Ω is a polyhedral domain which is divided into tetrahedral elements. V_N

consists of continuous functions which are linear on the subregions T_i.

$$u(x,y,z)|_{T_i} = a_{i1} + a_{i2}x + a_{i3}y + a_{i4}z$$

The basis elements are defined by the relation $e_i(x_j) = \delta_{ij}$, where $x_j = (x_{j1}, x_{j3}, x_{j3})$ are the corner points of the tetrahedrons.

2.2.2 Error estimates for linear elements

In section 2.1.2 we have defined error estimates of the form

$$\|u - u_N\|_V \leq C \inf_{v_N \in V_N} \|u - v_N\|.$$

Choosing V_N as piecewise linear elements with respect to a triangulation of $\bar{\Omega} \subset \mathbb{R}^2$ with a mesh parameter h the error estimate can be improved, namely,

$$\inf_{v_N \in V_N} \|u - v_N\|_{H^k(\Omega)} \leq C(\alpha_0) h^{2-k} \|u\|_{H^2(\Omega)}, \quad k = 0, 1. \tag{2.18}$$

Here is $\alpha_0 > 0$ the minimal interior angle of the triangles. The estimate (2.18) holds under certain assumptions and its proof demands some technical considerations.
Let $\Omega \subset \mathbb{R}^2$ be a polygon, and $\tau = \{T_i\}_i$ an admissible triangulation, $V_N \subset H^1(\Omega)$ the space of continuous piecewise linear elements, N is the number of interior node points if $V_N \subset \overset{\circ}{H}{}^1(\Omega)$ (Dirichletproblem) or the number of all nodes if $V = H^1(\Omega)$ (Neumann problem).
Furthermore, we assume, that the weak solution u is smooth enough, $u \in H^2(\Omega)$. This condition holds if the right hand side f belongs to $L_2(\Omega)$ and the polygon Ω is convex. For a polygon with reentrant corners we have only $u \in H^{1+\frac{\pi}{w_0}-\varepsilon}(\Omega)$, where $w_0 > \pi$ is the largest interior angle. If Ω has a smooth boundary, then $u \in H^2(\Omega)$.

Lemma 23. *Let be $E = \{(\xi, \eta), \quad \xi, \eta \geq 0, \quad \xi + \eta \leq 1\}$ the unit triangle. For any $u \in H^2(E)$ the following estimate holds:*

$$\|u\|^2_{H^2(E)} \leq C\left[|u(0,0)|^2 + |u(0,1)|^2 + |u(1,0)|^2 + \sum_{|\alpha|=2} \|D^\alpha u\|^2_{L_2(E)}\right]$$

Proof.

(1) We show that the bilinear form

$$a(u,v) = u(0,0)v(0,0) + u(0,1)v(0,1) + u(1,0)v(1,0) + \sum_{|\alpha|=2} (D^\alpha u, D^\alpha v)_{L_2(E)}$$

defined on $H^2(E) \times H^2(E)$, is bounded and $H^2(E)$-coercive.

- Boundedness.

$$\begin{aligned} |a(u,v)| &\leq |u(0,0)||v(0,0)| + |u(0,1)||v(0,1)| \\ &\quad + |u(1,0)||v(1,0)| + \|u\|_{H^2(E)}\|v\|_{H^2(E)}. \end{aligned}$$

Since $H^2(E) \subset C(\bar{E})$ and $\|u\|_{C(\bar{E})} = \max_{x\in\bar{E}} |u(x) \leq C_1\|u\|_{H^2(\Omega)}$ we get

$$|u(x_i, y_i)| \leq \max_{(x,y)\in\bar{E}} |u(x,y)| \leq C_1\|u\|_{H^2(\Omega)}$$

and finally

$$\begin{aligned} |a(u,v)| &\leq 3C_1^2\|u\|_{H^2(E)}\|v\|_{H^2(E)} + \|u\|_{H^2(E)}\|v\|_{H^2(E)} \\ &\leq (3C_1^2+1)[\|u\|_{H^2(E)}\|v\|_{H^2(E)}]. \end{aligned}$$

- Coerciveness. We apply the Lemma of Ehrling: If $X \subset Y \subset Z$, $X \underset{comp}{\subset} Y$ then $\forall\varepsilon > 0$ exists $C(\varepsilon)$ with

$$\|x\|_Y \leq \varepsilon\|x\|_X + C(\varepsilon)\|X\|_Z \ \ \forall x \in X.$$

In our case is $H^2(E) \subset H^1(E) \subset L_2(E)$, taking $\varepsilon = \frac{1}{2}$ we get

$$\begin{aligned} \left(\|u\|_{H^1(E)}\right)^2 &\leq \left[\frac{1}{2}\|u\|_{H^2(E)} + C\left(\frac{1}{2}\right)\|u\|_{L_2(E)}\right]^2 \\ &\overset{(a+b)^2\leq 2a^2+2b^2}{\leq} \frac{1}{2}\|u\|^2_{H^2(E)} + 2C^2\left(\frac{1}{2}\right)\|u\|^2_{L_2(E)} \end{aligned}$$

for all $u \in H^2(E)$,

$$\begin{aligned} a(u,u) &= u^2(0,0) + u^2(1,0) + u^2(0,1) + \sum_{|\alpha|=2}\|D^\alpha u\|^2_{L_2(E)} \\ &\geq \sum_{|\alpha|=2}\|D^\alpha u\|^2_{L_2(E)} = \|u\|^2_{H^2(E)} - \|u\|^2_{H^1(E)} \\ &\geq \frac{1}{2}\|u\|^2_{H^2(E)} - 2C^2\left(\frac{1}{2}\right)\|u\|^2_{L_2(E)}. \end{aligned}$$

(2) $a(u,v)$ is $H^2(E)$-elliptic.
The $H^2(E)$-coerciveness yields a Fredholm property: For the operator A defined by

$$\langle Au, v\rangle = a(u,v) \ \ A : H^2(\Omega) \to H^2(\Omega)'$$

holds: either A^{-1} exists or $\lambda = 0$ is an eigenvalue of A. Assume $\lambda = 0$ is an eigenvalue of A. Then exists a nontrivial element $e \in H^2(E)$ with $Ae = 0$ and $\langle Ae, e\rangle := a(e,e) = 0$. Hence

$$\sum_{|\alpha|=2}\|D^\alpha e\|^2 = 0$$

and $e(x,y)=\alpha+\beta x+\gamma y$. Since $e(0,0)=e(0,1)=e(1,0)=0$ we get $e(x,y)=0$ and $\lambda=0$ is not an eigenvalue of A. From the existence of A^{-1} it follows the $H^2(E)$-ellipticity. [6, p. 168]

□

Lemma 24. *Let be* $E_h = hE = \{(\hat\xi,\hat\eta) : \hat\xi,\hat\eta \ge 0,\ \ \hat\xi+\hat\eta \le h\}$, $u\in H^2(E_h)$, $|\beta|\le 2$. *Then*

$$\begin{aligned}\|D^\beta u\|^2_{L^2(E_h)} \;\le\; & C\Big\{h^{2-2|\beta|}[u^2(0,0)+u^2(0,h)+u^2(h,0)] \\ & + h^{4-2|\beta|}\sum_{|\alpha|=2}\|D^\alpha u\|^2_{L_2(E_h)}\Big\}.\end{aligned}$$

Proof. We have $\hat\xi = h\xi$, $\hat\eta = h\eta$, where $(\xi,\eta)\in E$
$\frac{\partial}{\partial\hat\xi} = \frac{\partial}{\partial\xi}\frac{\partial\xi}{\partial\hat\xi} + \frac{\partial}{\partial\eta}\frac{\partial\eta}{\partial\hat\eta} = \frac{1}{h}\frac{\partial}{\partial\xi}$, $\frac{\partial}{\partial\hat\eta} = \frac{1}{h}\frac{\partial}{\partial\eta}$, $D^\beta_{(\hat\xi,\hat\eta)} = \frac{1}{h^{|\beta|}}D^\beta_{(\xi,\eta)}$. Therefore for $u(\hat\xi,\hat\eta)=v(\xi,\eta)$

$$\begin{aligned}\|D^\beta u\|^2_{L_2(E_h)} &= \int_{E_h}|D^\beta_{(\hat\xi,\hat\eta)}u(\hat\xi,\hat\eta)|^2 d\hat\xi d\hat\eta = \int_E |h^{-|\beta|}D^\beta_{(\xi,\eta)}v(\xi,\eta)|^2 h^2 d\xi d\eta \\ &= h^{2-2|\beta|}\|D^\beta v\|^2_{L_2(E)} \le h^{2-2|\beta|}\|v\|^2_{H^2(E)} \\ &\overset{\text{Lemma 23}}{\le} h^{2-2|\beta|}\Big[v^2(0,0)+v^2(1,0)+v^2(0,1)+\sum_{|\alpha|=2}\|D^\alpha v\|^2_{L_2(E)}\Big] \\ &= h^{2-2|\beta|}\Big[u^2(0,0)+u^2(h,0)+u^2(0,h)+h^2\sum_{|\alpha|=2}\|D^\alpha u\|^2_{L_2(E_h)}\Big]\end{aligned}$$

since $\int_E |D^\alpha v|^2 d\xi d\eta = \int_{E_h} h^4 |D^\alpha u|\frac{1}{h^2}d\hat\xi d\hat\eta$ for $|\alpha|=2$. □

Lemma 25. *Let* $T\in\tau$ *be any triangle with sides whose length is less then* h_{max}, *and interior angles* $\ge \alpha_0 > 0$, $|\beta|\le 2$. *For every* $u\in H^2(T)$ *it holds:*

$$\|D^\beta u\|^2_{L_2(T)} \le C(\alpha_0)\Big[h_{max}^{2-2|\beta|}\sum_{\substack{(x_i,y_i)\\ \text{corner of } T}}|u(x_i,y_i)|^2 + h_{max}^{4-2|\beta|}\sum_{|\alpha|=2}\|D^2 u\|^2_{L_2(T)}\Big],$$

where $|\beta|\le 2$.

Proof. We consider the mapping $\underset{(x,y)}{T} \to \underset{(\xi,\eta)}{E} \to \underset{(\hat\xi,\hat\eta)}{E_h}$, where h is the minimal side-length of the triangle T. Then

$$\int_T \dots dxdy = 2|T|\int_E \dots d\xi d\eta = \frac{2|T|}{h^2}\int_{E_h}\dots d\hat\xi d\hat\eta.$$

Since $\frac{h^2}{2|T|} \le \frac{1}{\sin\alpha_0}$ we get after some calculations the assertion. □

Theorem 11. *Let* V_N *be the space of continuous piecewise linear functions with respect to an admissible triangulation of the polygon* Ω. *Let be* $\alpha_0 > 0$ *the smallest* *interior angle of the triangles and h the largest length of the sides of the triangles. Then*

$$\inf_{v\in V_N} |u-v|_{H^k(\Omega)} \le C(\alpha_0)h^{2-k}\|u\|_{H^2(\Omega)}$$

for $k=0,1$ and all $u \in H^2(\Omega)\cap V$.

Proof. Let be $u \in H^2(\Omega)\cap V$. We define

$$v_N(x,y) = \sum_{i=1}^{N} u(x_i,y_i)e_i(x,y).$$

The function $w(x,y) = u(x,y) - v_N(x,y)$ vanishes in the nodes and

$$D^\alpha w|_{T_k} = D^\alpha u|_{T_k} \text{ for } |\alpha| = 2 \text{ for all } T_k \in \tau.$$

From Lemma 25 it follows that for $|\beta| \le 2$

$$\begin{aligned} \|D^\beta w\|^2_{L_2(\Omega)} &= \sum_{T_k\in\tau} \|D^\beta w\|^2_{L_2(T_k)} \le C(\alpha_0)h^{4-2|\beta|} \sum_{T_k\in\tau}\sum_{|\alpha|=2} \|D^\alpha u\|^2_{L_2(T_k)} \\ &= C(\alpha_0)h^{4-2|\beta|}\sum_{|\alpha|=2}\|D^\alpha u\|^2_{L_2(\Omega)} \le C(\alpha_0)h^{4-2|\beta|}\|u\|^2_{H^2(\Omega)}. \end{aligned}$$

(1) $k=0$: We have

$$\inf_{V\in V_N} \|u-v\|^2_{L_2(\Omega)} \le \|u-v_N\|^2_{L_2(\Omega)} = \|w\|^2_{L_2(\Omega)} \le C(\alpha_0)h^4\|u\|^2_{H^2(\Omega)}.$$

(2) $k=1$: Then

$$\inf_{V\in V_N} |u-v|^2_{H^1(\Omega)} \le |w|^2_{H^1(\Omega)} \le C(\alpha_0)h^2\|u\|_{H^2(\Omega)}.$$

For $h<1$

$$\inf_{v\in V_N} \|u-v\|^2_{H^1(\Omega)} \le \|w\|^2_{H^1(\Omega)} \le C(\alpha_0)h^2\|u\|_{H^2(\Omega)}.$$

□

REMARK. The choice of the mesh parameter is crucial. Therefore we write V_h instead of $V_N = V_{N(h)}$ in what follows.

Definition 24. *A sequence of triangulations τ_i with $h_i \to 0$ is quasiuniform if*

$$\alpha_{0i} \ge \alpha_0 > 0.$$

Summarizing the results we get:

Theorem 12 (H^1-convergence) . *Assume that a sequence of quasiuniform triangulations is given. Let be $u \in H^2(\Omega)\cap V$ be a weak solution and u_h a sequence of Galerkin solutions with $\|u-u_h\| \le C\inf_{v_h\in V_h}\|u-u_h\|$. Then*

$$\|u-u_h\|_{H^1(\Omega)} \le C\, h\|u\|_{H^2(\Omega)} = O(h)$$

(asymptotical error estimate). $O(h)$ is the optimal convergence rate.

Remarks to error estimates in other spaces.

(1) $L_2(\Omega)$

(a) Assume that the adjoint problem

$$a^*(u,v) := a(v,u) = \langle f, v\rangle$$

has a solution $u \in H^2(\Omega) \cap V$ for every $f \in L_2(\Omega) \subset V'$ with $|u|_2 \leq C\|f\|_{L_2(\Omega)}$.

(b) The bilinear form $a(\cdot,\cdot)$ is bounded on $V \times V$ and

$$\inf\{\sup\{|a(u,v)| : v \in V_h,\ \|v\|_V = 1\} u \in V_h,\ \|u\|_V = 1\} = \varepsilon_h \geq \tilde{\varepsilon} > 0.$$

(c) $\inf_{v\in V_h} |u-v|_1 \leq C_0 h|u|_2 \quad \forall u \in H^2(\Omega) \cap V$.

(d) Let $u \in V$ be a weak solution and $u_h \in V_h \subset V$ the Galerkin solution.

Then

$$\begin{aligned} \|u-u_h\|_{L_2(\Omega)} &\leq C_1 h|u|_1. \\ \|u-u_h\|_{L_2(\Omega)} &\leq C_2 h^2|u|_2. \end{aligned}$$

The proof is based on the so called Nitsche trick [6, p.172],[9, p.102].

(2) $L_\infty(\Omega), \quad C(\bar{\Omega})$
Assume (a)–(d) as above. Then

$$\|u-u_h\|_{L_\infty} = \max_{x\in\bar{\Omega}} |u(x)-u_h(x)| \leq ch^k|u|_{k+1(\Omega)}, \quad k=0,1.$$

Since $H^1(\Omega) \not\subset C(\bar{\Omega})$ we cannot estimate the norm in $C(\bar{\Omega})$ through the norm in $H^1(\Omega)$. Therefore we have to use other tools such that as inverse inequalities

$$\|v_h\|_{L_\infty(\Omega)} \leq Ch^{-1}\|v_h\|_{L_2(\Omega)} \text{ for } v_h \in V_h$$

and the Bramble-Hilbert Lemma:

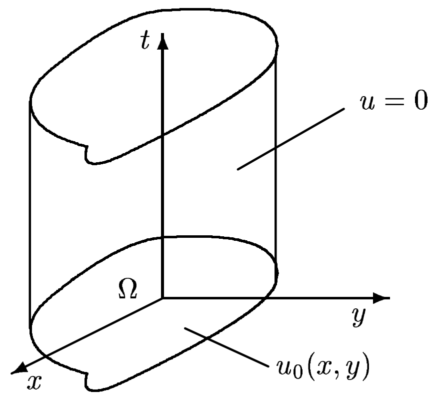
Lemma 26. *Let f be a bounded linear functional on $H^{k+1}(\Omega)$ and $\langle f,u\rangle = 0 \quad \forall u \in P_k(\bar{\Omega})$={polynomials of degree at most k}. Then there exists a constant c such that*

$$|\langle f,u\rangle| \leq c|u|_{k+1} \qquad \forall u \in H^{k+1}(\Omega).$$

2.2.3 Initial-Boundary value problems

We consider an initial boundary value problem for the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} - \triangle u &= f(x,y,t) \text{ for } (x,y,t) \in \Omega \times (0,\infty) \\ u(x,y,t)\Big|_{\partial\Omega} &= 0 \text{ for } t > 0 \\ u(x,y,0) &= u_0(x,y) \text{ for } (x,y) \in \Omega. \end{aligned} \tag{2.19}$$



Assume $f(x,y,t) \in L_2(\Omega)$ for all $t > 0$ and $u_0(x,y) \in L_2(\Omega)$. One possibility is to derive a semidiscrete problem applying a finite element scheme with respect to the space variables. We consider the space $V = \overset{\circ}{H}{}^1(\Omega)$. Multiplying the heat equation with an element $v \in V$ and integrating by parts on Ω we get for every fixed t:

$$\begin{aligned} &\int_\Omega \frac{\partial u(x,y,t)}{\partial t} v(x,y)dxdy + \int_\Omega \operatorname{grad} u(x,y,t) \operatorname{grad} v(x,y)\, dxdy \\ = &\int_\Omega f(x,y,t)v(x,y)dxdy \end{aligned}$$

shortly written as

$$\left(\frac{\partial u(t,\cdot)}{\partial t}, v\right)_{L_2(\Omega)} + a\left(u(t,\cdot), v\right) = \left(f(t,\cdot), v\right)_{L_2(\Omega)}.$$

Let be $H^1\left((0,T); V, L_2\right)$ the space of all functions $u : (0,T) \to V$ with

$$\|u\|^2_{L_2(0,T)} = \int_0^T \|u(t)\|^2_V dt < \infty$$

and with the property that

$$\frac{\partial u}{\partial t} : (0,T) \to V' \text{ and } \left\|\frac{\partial u}{\partial t}\right\|_{L_2(0,T)} = \int_0^T \left\|\frac{\partial u}{\partial t}\right\|^2_{V'} dt < \infty$$

where V' is the dual space of V. The norm in $H^1((0,T);V,L_2)$ is defined as

$$\|u\|_{H^1((0,T);V,L_2)} := \|u\|_{L_2(0,T)} + \left\|\frac{\partial u}{\partial t}\right\|_{L_2(0,T)}.$$

The **weak formulation** of problem (2.19) reads:
Find an element $u \in H^1((0,T);V,L_2)$ with

$$\begin{aligned} u(0,\cdot) &= u_0(\cdot) \in L_2(\Omega) \\ \textit{or} \quad (u(0,\cdot),v)_{L_2(\Omega)} &= (u_0,v) \quad \forall v \in L_2(\Omega) \end{aligned} \tag{2.20}$$

such that

$$\frac{d}{dt}(u(t),v)_{L_2(\Omega)} + a(u(t),v) = \langle f(t),v\rangle_{L_2(\Omega)} \quad \forall v \in V,$$

where $f \in L_2((0,T),V')$, that means $\int_0^T \|f\|^2_{V'} dt < \infty$.

Let be $V_h \subset V$ an appropriate finite-element-space. For every $t \in T$ we define the **Galerkin solution** $u_h(t)$ as follows: Find an element $u_h(t) \in V_h$ with

$$\begin{aligned} \frac{d}{dt}(u_h(t),v_h)_{L_2(\Omega)} + a(u_h(t),v_h)_\Omega &= \langle f(t),v_h\rangle_\Omega \ \ \forall v_h \in V_h \\ (u_h|_{t=0}.v_h)_{L_2(\Omega)} &= \langle u_0,v_h\rangle_{L_2(\Omega)}. \end{aligned} \tag{2.21}$$

We choose a basis $\{e_i^{(h)}(x,y)\}_{i=1,\dots,N(h)}$ in V_h (piecewise linear functions with respect to a triangulation) and we represent

$$u_h(x,y,t) = \sum_{i=1}^{N(h)} a_i(t)e_i^{(h)}(x,y) = u_h \tag{2.22}$$

with $a_i(t) = u_h(x_i,y_i,t)$ provided $e_i^{(h)}(x_j,y_j) = \delta_{ij}$; (x_i,y_i) are the relevant nodes. Inserting (2.22) into (2.21) we get for $v_h = e_j^{(h)}$

$$\begin{aligned} \frac{d}{dt}\left(\sum_{i=1}^{N(h)} a_i(t)e_i^{(h)}(x,y), e_j^{(h)}(x,y)\right) &+ \sum_{i=1}^{N(h)} a(a_i(t)e_i^{(h)}(x,y), e_j^{(h)}(x,y)) \\ &= \langle f(t), e_j^{(h)}\rangle_\Omega \\ \left(\sum_{i=1}^{N(h)} a_i(0)e_i^{(h)}(x,y), e_j^{(h)}(x,y)\right) &= (u_0, e_j^{(h)}) \end{aligned}$$

and finally

$$\begin{pmatrix} (e_1^{(h)},e_1^{(h)}) & \dots & (e_N^{(h)},e_1^{(h)}) \\ \vdots & & \vdots \\ (e_1^{(h)},e_N^{(h)}) & \dots & (e_N^{(h)},e_N^{(h)}) \end{pmatrix} \begin{pmatrix} \frac{d}{dt}a_1(t) \\ \vdots \\ \frac{d}{dt}a_N(t) \end{pmatrix} +$$

$$\begin{pmatrix} a(e_1^{(h)},e_1^{(h)}) & \dots & a(e_N^{(h)},e_1^{(h)}) \\ \vdots & & \vdots \\ a(e_1^{(h)},e_N^{(h)}) & \dots & a(e_N^{(h)},e_N^{(h)}) \end{pmatrix} \begin{pmatrix} a_1(t) \\ \vdots \\ a_N(t) \end{pmatrix} = \begin{pmatrix} \langle f,e_1^{(h)}\rangle \\ \vdots \\ \langle f,e_N^{(h)}\rangle \end{pmatrix}$$

$$\begin{pmatrix} (e_1^{(h)}, e_1^{(h)}) & \dots & (e_N^{(h)}, e_1^{(h)}) \\ \vdots & & \vdots \\ (e_1^{(h)}, e_N^{(h)}) & \dots & (e_N^{(h)}, e_N^{(h)}) \end{pmatrix} \begin{pmatrix} a_1(0) \\ \vdots \\ a_N(0) \end{pmatrix} = \begin{pmatrix} \langle u_0, e_1^{(h)} \rangle \\ \vdots \\ \langle u_0, e_N^{(h)} \rangle \end{pmatrix}.$$

Shortly

$$\begin{aligned} E\frac{d}{dt}\vec{a}(t) + M\vec{a}(t) &= \vec{F}(t) \qquad (2.23) \\ E\vec{a}(0) &= \vec{u}_0 \,, \end{aligned}$$

where $F_i(t) = \langle f, e_i^{(h)} \rangle, \quad u_{0i} = \langle u_0, e_i^{(h)} \rangle$. The problem (2.23) is an initial problem for a first order system of ordinary differential equations.
The symmetric matrix E is regular, since the basis element $e_i^{(h)}$ are linearly independent. E is in general not a diagonal matrix.
EXAMPLE. We consider the interval $(0,1) = \Omega$ and $t \in (0,T)$ and a uniform partition of $(0,1)$. Then

$$M = \frac{1}{h} \begin{pmatrix} 2 & -1 & 0 & \vdots & \vdots & 0 \\ -1 & 2 & -1 & 0 & \vdots & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & \vdots & \vdots & \vdots & -1 & 2 \end{pmatrix} = \frac{1}{h} M'$$

and

$$E = \frac{h}{6} \begin{pmatrix} 4 & 1 & 0 & \ddots & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & 1 & 4 \end{pmatrix} = hE'.$$

We calculate E:

$$\begin{aligned} (e_i^{(h)}, e_i^{(h)}) &= \int_{x_{i-1}}^{x_i} \frac{(x - x_{i-1})^2}{h^2} dx + \int_{x_{i-1}}^{x_i} \frac{(x_{i+1} - x)^2}{h^2} dx \\ &= \frac{1}{h^2} \left(\int_0^h y^2 dy + \int_{-h}^0 y^2 dy \right) = \frac{1}{h^2} \left(\frac{h^3}{3} + \frac{h^3}{3} \right) = \frac{2}{3} h \\ (e_i^{(h)}, e_{i-1}^{(h)}) &= \frac{1}{h^2} \int_{x_{i-1}}^{x_i} (x - x_{i-1})(x_i - x) dx = \frac{1}{h^2} \int_0^h y(h-y) dy = \frac{h}{6} \\ (e_i^{(h)}, e_{i+1}^{(h)}) &= \frac{1}{h^2} \int_{x_i}^{x_{i+1}} (x_{i+1} - x)(x - x_i) dx = \frac{1}{h^2} \int_0^h y(h-y) dy = \frac{h}{6}. \end{aligned}$$

Furthermore, if $f(x,y,t) = 1$ then

$$\begin{aligned} (f, e_i^{(h)}) &= \frac{1}{h} \left(\int_{x_{i-1}}^{x_i} (x - x_{i-1}) dx + \int_{x_i}^{x_{i+1}} (x_{i+1} - x) dx \right) \\ &= \frac{1}{h} \left(\int_0^h y dy + \int_0^h y dy \right) = h \end{aligned}$$

Therefore the problem (2.23) reads:

$$\begin{aligned} hE'\frac{d}{dt}\vec{a}(t) + \frac{1}{h}M'\vec{a}(t) &= \vec{h} \\ hE'\vec{a}(0) &= \vec{u}_0 \end{aligned}$$

and finally

$$\begin{aligned} E'\frac{d}{dt}\vec{a}(t) + \frac{1}{h^2}M'\vec{a}(t) &= \vec{1} \\ E'\vec{a}(0) &= \frac{1}{h}\vec{u}_0 \end{aligned} \tag{2.24}$$

There are error estimates for the semidiscrete solution (Assuming that the solution of problem (2.23) is exactly known), namely [5, p. 311]

$$\begin{aligned} \|u(t) - u_h(t)\|_{L_2(\Omega)} \leq\; & e^{-c_2 t}\|u_0 - u_h^0\|_{L_2(\Omega)} \\ & ch^r\left\{\|u\|_{H^r} + \int_0^t e^{-c_2(t-s)}\|\dot{u}(t)\|_{H^r}ds\right\}, \end{aligned}$$

where c_2 is the constant in the estimate

$$a(u,u)_\Omega \geq c_2\|u\|^2 \quad \forall u \in V.$$

Further,

$$\|u - u_h\|_\infty \leq ch^2|\ln h|\,\|u\|_{H^2_\infty} + ch^2(\ln\ h)^{\frac{1}{2}}\left(\int_0^t \|u\|_2^2 d\tau\right)^{\frac{1}{2}}$$

provided the solution u is smooth enough.
Now, we consider the time-discretisation of problem (2.23). We remark, that the matrix E can be replaced approximately through a diagonal matrix D by lumping: e.g.

$$d_{jj} = \sum_k e_{jk}\ \text{(row-sum)}$$

Multiplying the system (2.23) with D^{-1} we get the standard problem

$$\begin{aligned} \frac{d}{dt}\vec{a}(t) + B\vec{a}(t) &= \vec{g}(t) \\ \vec{a}(0) &= \vec{w_0}. \end{aligned} \tag{2.25}$$

Note that the above example shows, that $B = D^{-1}\frac{1}{h^2}M'$. Indeed, the problem (2.25) depends on the mesh parameter h in general. Therefore the classical stability for the Euler or Runge-Kutta algorithms can be disturbed and a careful analysis is necessary. [5, p. 315]
One possibility is to use a time-discretisation already for the problem (2.21), writing the time derivative as difference quotient and solving the problem on some discrete time levels.

Let be the intervall $[0,T]$ divided in n subintervalls (t_{i-1},t_i) with $t_i = i\frac{T}{n} = i\tau$, $i = 0,\dots,n$. Let be

$$u(t_i) \approx U^i \approx u_h(t_i).$$

The problem (2.21) reads: start from $U^0(x,y) = u_0(x,y)$. Find $U^{k+1}(x,y)$, $k = 0,1,2,\dots$ such that

$$\begin{aligned}\left(\tfrac{U^{k+1}-U_k}{\tau}, v_h\right) + a\left(\sigma U^{k+1} + (1-\sigma)U^k, v_h\right)& \\ = \langle \sigma f^{k+1} + (1-\sigma) f^k, v_h\rangle& \qquad (2.26)\end{aligned}$$

with

$$\begin{aligned}\tau &= t_{k+1} - t_k,\\ f^k(x,y) &= f(x,y,t_k) \text{ and } 0 \le \sigma \le 1.\end{aligned}$$

$\sigma = 0$ corresponds to an explicit, $\sigma = 1$ to an implicit Euler algorithm, $\sigma = \frac{1}{2}$ implies a Crank-Nicolson-scheme. For every k the problem (2.26) is an elliptic problem and leads to an equation system for the vektor $\vec{a}^{l+1} = \vec{a}(t_{k+1})$ with the matrix $E + \tau\sigma M$. The controlling of the error with respect to t yields an adaptive generated time grid. We remark, that hyperbolic or parabolic problems can be handled similar, if the space-term is V-elliptic.

2.2.4 Nonlinear boundary value problems

We have considered linear operators $A : V \to V'$. If

$$\langle Au, v\rangle \le C_1\|u\|_V\|v\|_V \quad \forall u,v \in V \qquad (2.27)$$
$$\langle Au, u\rangle \ge C_2\|u\|_V^2 \quad \forall u \in V, \qquad (2.28)$$

then A^{-1} exists and is continuous. A Galerkin scheme works and the Lemma of Ceá yields

$$\|u - u_h\|_V \le \frac{C_1}{C_2}\inf_{v_h\in V_h}\|u - v_h\|_V.$$

We consider now an nonlinear operator $A : V \to V'$, where V is a Banach space, V' its dual space. We generalize the condition (2.28), defining

(a) A is **monotone** $\Leftrightarrow$ $\langle Au - Av, u - v\rangle \ge 0 \quad \forall u,v \in V$,

(b) A is **strictly monotone** $\Leftrightarrow$ $\langle Au - Av, u - v\rangle > 0 \quad \forall u,v \in V$, $u \ne v$

(c) A is **uniformly monotone** $\Leftrightarrow$ $\langle Au - Av, u - v\rangle \ge b(\|u - v\|)\|u - v\| \quad \forall u,v \in V$ where $b : [0,\infty) \to \mathbb{R}$ is a strongly monotone increasing continuous function with $b(0) = 0, \quad b(t) \to \infty$ for $t \to \infty$

(d) A is **strongly monotone** $\Leftrightarrow$ $\exists c > 0$ with $\langle Au - Av, u - v\rangle \ge c\|u - v\|^2$ $\forall u,v \in V$,

(e) A is **coercive** $\Leftrightarrow$ $\frac{\langle Au,u\rangle}{\|u\|_V} \to \infty$ for $\|u\| \to \infty$.

It holds:

$$(d) \Rightarrow (c) \begin{array}{l} \Rightarrow (b) \quad \Rightarrow (a) \\ \Rightarrow (e) \end{array}$$

Theorem 13. *Let be V a reflexive Banach space with countable basis, $A : V \to V'$ and*

(i) A is monotone,

(ii) A is coercive,

(iii) A is continuous.

Then exists for every $f \in V'$ an element $u \in V$ with

$$Au = f,$$

that means $\langle Au, v\rangle = \langle f, v\rangle \quad \forall v \in V$.

EXAMPLE We consider the boundary value problem in $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned} \operatorname{div}(D(x)\operatorname{grad} u(x)) + F(x, u(x)) &= 0 \text{ in } \Omega \\ u|_{\partial\Omega} &= 0. \end{aligned}$$

$D(x)$ is a matrix of continuous functions with

$$K_1|z|^2 \geq z^T D(x) z \geq K_2 |z|^2 \quad \forall x \in \Omega,\ z \in \mathbb{R}^2.$$

$F : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a function with

$$\begin{aligned} |F(x,s) - F(x,t)| &\leq L|s-t| \quad \forall s,t \in \mathbb{R},\ x \in \Omega \\ (F(x,s) - F(x,t))(s-t) &\geq 0. \end{aligned}$$

We take $V = \overset{\circ}{H}{}^1(\Omega)$ and have

$$\begin{aligned} \langle Au, v\rangle &= \int_\Omega \Big[\operatorname{div}\left(D(x)\operatorname{grad} u(x)\right) v(x) + F(x,u(x))v(x)\Big] dx \\ &= \int_\Omega \Big[(\nabla u)^T D^T \nabla v + F(x,u(x))v(x)\Big] dx. \end{aligned}$$

The operator A is strongly monotone

$$\begin{aligned} \langle Au - Av, u - v\rangle &= \int_\Omega \Big[(\nabla(u-v))^T D^T \nabla(u-v) \\ &\qquad + \big[F(x,u) - F(x,v)\big](u-v)\Big] dx \\ &\geq K_2 \int_\Omega |\nabla(u-v)|^2 dx \overset{\text{Friedr. ineq.}}{\geq} K_2 C \|u-v\|_V^2. \end{aligned}$$

The operator A is Lipschitz continuous:

$$\begin{aligned} \langle Au - Ay, v\rangle &= \left|\int_\Omega \Big\{(\nabla u - \nabla y)^T D^T \nabla v + \big[F(x,u) - F(x,y)\big]v\Big\}dx\right| \\ &\leq L\int_\Omega |u-y||v|dx \leq K\|u-y\|_V\|v\|_V \end{aligned}$$

Theorem 14. *Let be $A : V \to V'$ strongly monotone and Lipschitz continuous, $f \in V'$. Assume that $V_h \subset V$ and dim $V_h < \infty$. Then exists a uniquely determined $u_h \in V_h$ with*

$$\begin{aligned} \langle Au_h, v_h\rangle &= \langle f, v_h\rangle \textit{ for all } v_h \in V_h. \\ \textit{and } \|u-u_h\|_V &\leq C \inf_{v_h\in V_h} \|u - v_h\|_V. \end{aligned}$$

Proof. The Galerkin solution $u_h = \sum c_i e_i^{(h)}(x,y)$ is solution of a nonlinear equation system. Since

$$\langle Au_h - Au, v_h\rangle = 0 \quad \forall v_h \in V_h$$

it follows

$$\begin{aligned} \|u_h - u\|_V^2 &\leq \frac{1}{C_2}\langle Au_h - Au, u_h - u\rangle = \frac{1}{C_2}\langle Au_h - Au, v_h - u\rangle \\ &\leq \frac{C_1}{C_2}\|u_h - u\|_V\|v_h - u\| \end{aligned}$$

and finally

$$\|u_h - u\|_V \leq \frac{C_1}{C_2}\|v_h - u\| \quad \forall v_h \in V_h.$$

□

2.3 Boundary element methods

Besides the Finite Element Methods there are Boundary Element Methods developed for numerical solving of elliptic boundary value problems. The original problem is formulated as equivalent boundary integral equations and these equations are solved by Galerkin schemes. Instead of finite elements, defined on the whole body, there will be created finite boundary elements.

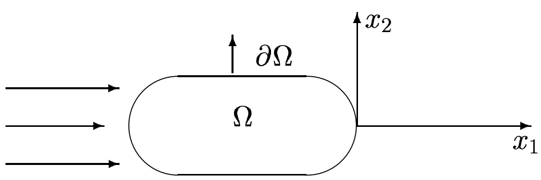
The advantages are: The reduction of the dimension, exterior boundary value problems will be restricted on the boundary of a finite domain. Disadvantages: Derivation of an appropriate boundary integral equation, computation of the Galerkin matrix (singular integrals) and solving of the resulting dense equation systems.

2.3.1 Derivation of the boundary integral equations

The knowledge of the "fundamental solutions" and the representation of solutions of boundary value problems as potentials with respect to the fundamental solutions are essential for deriving of integral equations. Therefore there are considered mostly boundary value problems for the Laplacian, the Helmholtz equation, the biharmonic equation, the Stokes system and the linear elasticity system, for which the fundamental solutions are well known. For the sake of simplicity we demonstrate the derivation of the boundary integral equations for few examples.

The exterior Neumann problem for the Laplacian

[17, p.223] Let us consider an exterior stationary divergence-free irrotational two-dimensional flow in the domain Ω^c, exterior to a given obstacle Ω with boundary $\partial\Omega$.



Let be $\vec{v}(\vec{x})$ the velocity field. It can be expressed by a potential u:

$$\vec{v}(\vec{x}) = v_\infty \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\lambda}{2|\pi|} \frac{1}{|\vec{x}|^2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} + \nabla u(\vec{x}) \text{ in } \Omega^c,$$

where v_∞ is the velocity at infinity, λ is a given circulation, $\nabla = (\partial x_1, \partial x_2)^T$ denotes the nabla-operator. The potential u is solution of the exterior Neumann problem

$$\begin{aligned} \triangle u &= 0 \text{ in } \Omega^c \\ \partial_{\vec{n}} u|_{\partial\Omega} &= \vec{n}(\vec{x}) \cdot \nabla u(\vec{x})|_{\partial\Omega} = g(\vec{x})|_{\partial\Omega}, \end{aligned}$$

where

$$g(\vec{x}) = -v_\infty n_1(\vec{x}) - \frac{\lambda}{2\pi} \frac{1}{|\vec{x}|^2}(x_1 n_2(\vec{x}) - x_2 n_1(\vec{x}))$$

(that means $\vec{v}(x)|_{\partial\Omega} = 0$) and u satisfies the decay condition at infinity

$$u(\vec{x}) = 0(|\vec{x}|^{-1}) \text{ as } |\vec{x}| \to \infty.)$$

Assume that $\partial\Omega$ is a sufficiently smooth Jordan curve.

Definition of the fundamental solution
Let $\delta_{\vec{y}}(\vec{x})$ be the Dirac distribution concentrated at $\vec{y} \in \mathbb{R}^2$. The solution $F = F(\vec{x}, \vec{y})$ of the Laplacian is called fundamental solution if

$$-\triangle_{\vec{x}} F(\vec{x}, \vec{y}) = \delta_{\vec{y}}(\vec{x}).$$

F is not uniquely defined. We choose the translation-invariant solution

$$F(\vec{x}, \vec{y}) = -\frac{1}{2\pi} \ln |\vec{x} - \vec{y}|.$$

We cite a classical result in the potential theory.

Greens representation theorem
Let $u \in C^1(\Omega^c \cup \partial\Omega) \cap C^2(\Omega^c)$ be a solution of $\triangle u = 0$ in Ω^c with $u(\vec{x}) = O(|\vec{x}|^{-1})$ for $|\vec{x}| \to \infty$. Then u admits the representation:

$$\begin{aligned} u(\vec{x}) = & - \frac{1}{2\pi} \int_{\vec{y} \in \partial\Omega} u(\vec{y})(\partial_{\vec{n}_y} \ln(\vec{x} - \vec{y})) ds_{\vec{y}} \\ & + \frac{1}{2\pi} \int_{\vec{y} \in \partial\Omega} \ln |\vec{x} - \vec{y}| (\partial_{\vec{n}_y} u(\vec{y})) ds_{\vec{y}} y \quad \forall \vec{x} \in \Omega^c. \end{aligned} \tag{2.29}$$

If $n = -n_y$ denotes the exterior normal at boundary points on Ω, then

$$\mathcal{V}[\partial_n u] = -\int_{\partial\Omega} F(\bar{x}, \bar{y}) \partial_n u(\vec{y}) ds_{\vec{y}}$$

is the single layer potential and

$$U[u] = -\int_{\partial\Omega} \frac{\partial}{\partial n} F(x, \vec{y}) u(\vec{y}) ds_{\vec{y}}$$

denotes the double layer potential. Shortly (2.29) reads

$$u(\vec{x}) = -U[u] + \mathcal{V}[\partial_n u].$$

COROLLARY. The solution u is known if the Cauchy data $(u, \partial_n u)^T$ are known on $\partial\Omega$.

For our example $\partial_n u = g$ is known and $u|_{\partial\Omega}$ is to determine. For its determination we can obtain boundary integral equations from (2.29) sending $\vec{x} \to \vec{x_0} \in \partial\Omega$. The boundary traces of the right hand sides are known in terms of the so called "jump relations".

Lemma 27. [3] *The operators*

$$\begin{aligned} \gamma|_{\partial\Omega} \mathcal{V} &: H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega) \\ \gamma|_{\partial\Omega} U &: H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega) \\ \gamma|_{\partial\Omega} \frac{\partial}{\partial n} \mathcal{V} &: H^{-\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \\ \gamma|_{\partial\Omega} \frac{\partial}{\partial n} U &: H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \end{aligned}$$

are continuous and the boundary values are defined for sufficiently smooth functions in the sense of the Cauchy principal value almost everywhere:

$$\begin{aligned}
\lim_{\substack{x\in\Omega^c \\ x\to x_0\in\partial\Omega}} \mathcal{V}[\psi]x &= -\int_{\partial\Omega} F(x_0,y)\psi(y)ds_y =: V[\psi](x_0)\,, \\
\lim_{\substack{x\in\Omega^c \\ x\to x_0\in\partial\Omega}} U[\phi]x &= \frac{1}{2}\phi(x_0) + \int_{\partial\Omega} \frac{\partial}{\partial n_y} F(x_0,y)\phi(y)ds_y \\
&=: \frac{1}{2}\phi(x_0) - K[\phi](x_0)\,, \\
\lim_{\substack{x\in\Omega^c \\ x\to x_0\in\partial\Omega}} \frac{\partial}{\partial n_y}\mathcal{V}[\psi]x &= -\frac{1}{2}\psi(x_0) - K^*[\psi](x_0) \\
&= -\frac{1}{2}\psi(x_0) + \int_{\partial\Omega} \frac{\partial}{\partial n_y} F(x,y)\Big|_{x=x_o} \psi(y)ds_y\,, \\
\lim_{\substack{x\in\Omega^c \\ x\to x_0\in\partial\Omega}} \frac{\partial}{\partial n} U[\phi]x &= \frac{\partial}{\partial n_x}\int_{\partial\Omega} \frac{\partial}{\partial n_y} F(x,y)\Big|_{x=x_o} \phi(y)ds_y = D[\phi](x_0).
\end{aligned}$$

COROLLARY. The following relations between the Cauchy data hold on the boundary $\partial\Omega$:

$$\begin{aligned}
u(x) &= \frac{1}{2}u(x) - K[u](x) + V\left(\frac{\partial u}{\partial n}\right)(x) \\
\frac{\partial u(x)}{\partial n} &= D[u](x) + \frac{1}{2}\frac{\partial u}{\partial n}(x) + K^*\left(\frac{\partial u}{\partial n}\right)(x) \text{ for } x \in \partial\Omega.
\end{aligned}$$

With the help of the so called Calderon projector C the relations can be written:

$$\begin{pmatrix} u \\ \frac{\partial}{\partial n}u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K^* \end{pmatrix}\begin{pmatrix} u \\ \frac{\partial}{\partial n}u \end{pmatrix} = C\begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix}. \tag{2.30}$$

From the relation $C^2 = C$ we get the relations:

$$\begin{aligned}
VD &= \frac{1}{4}I - K^2 \\
DV &= \frac{1}{4}I - (K^*)^2 \\
K^*D &= DK \\
KV &= VK^*.
\end{aligned}$$

The unknown Cauchy datum $u|_{\partial\Omega}$ is solution of the boundary integral equations

$$\left[\frac{I}{2} + K\right](u(x)) = V[g](x) \tag{2.31}$$

$$\text{or} \quad Du = \left(\frac{1}{2}I - K^*\right)[g](x), \quad x \in \partial\Omega. \tag{2.32}$$

Let us discuss the solvability and uniqueness of the equation(2.32). The linear hypersingular integral operator D maps $V = H^{\frac{1}{2}}(\partial\Omega)$ into $V' = H^{-\frac{1}{2}}(\partial\Omega)$. We introduce a bilinear form

$$\langle Du, v\rangle = d(u,v) \text{ on } H^{\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega).$$

Since D is a continuous operator (Lemma 27) the bilinear form $d(\cdot,\cdot)$ is bounded.

Lemma 28. *The operator D is positive semidefinite; that means there is a constant C such that*

$$\langle Du, u\rangle \geq C\|u\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \qquad \forall u \in H^{\frac{1}{2}}(\partial\Omega)/\ker D$$

The kernel of D consists of the constant functions.

Proof. [10] A. W. Maue has shown in 1949 that the hypersingular boundary operator D and the single layer potential operator V are connected on $\partial\Omega$:

$$D = -\frac{d}{ds}V\frac{d}{ds}.$$

Here denotes s the arclength.
For the single layer potential is known [7] (compare the Lemma 29 hereafter) that there exists a constant c with

$$\langle Vg, g\rangle \geq c\|g\|^2_{H^{-\frac{1}{2}}(\partial\Omega)}, \qquad \forall g \in H^{-\frac{1}{2}}(\partial\Omega)$$

provided diameter $\Omega < 1$. Therefore, we get by partial integration for any $u \in H^{\frac{1}{2}}(\partial\Omega)/\{\text{const}\}$

$$\begin{aligned}\langle Du, u\rangle_{\partial\Omega} &= -\langle \frac{d}{ds}V\left(\frac{du}{ds}\right), u\rangle_{\partial\Omega} = \langle V\left(\frac{du}{ds}\right), \frac{du}{ds}\rangle_{\partial\Omega} \\ &\geq c\|\frac{du}{ds}\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \geq \tilde{c}\|u\|^2_{H^{\frac{1}{2}}(\partial\Omega)}.\end{aligned}$$

In the last estimate we have used that the differential operator $\frac{d}{ds} : H^{\frac{1}{2}}(\partial\Omega)/\{\text{const}\} \to H^{-\frac{1}{2}}(\partial\Omega)$ is a continuous bijective mapping. Let us remark that the condition, diameter < 1, can be removed, considering an appropriate scaling. □

Lemma 29. [7, p.453] *Let be $\partial\Omega$ smooth enough (e.g. C^2* [7] *, Lipschitz continuous* [3]*). The single layer operator*

$$V : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

is continuous and, if additionally diameter$(\Omega) < 1$, then there exists a positive constant c such that

$$\langle V\varphi, \varphi\rangle_{\partial\Omega} \geq c\|\varphi\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} \qquad \forall \varphi \in H^{-\frac{1}{2}}(\partial\Omega)$$

Proof. We consider an element $e \in H^{-\frac{1}{2}}(\partial\Omega)$, such that $\int_{\partial\Omega} e(s)ds = 1$ and $Ve = E = \text{const}$. Any $\varphi \in H^{-\frac{1}{2}}(\partial\Omega)$ can be written

$$\varphi = \varphi_0 + ek, \qquad k = \text{const}$$

where $\int_{\partial\Omega} \varphi_0 ds = 0$, $k = \text{const}$. Since V is selfadjoint and $\langle \varphi_0, kE\rangle = \langle ek, V\varphi_0\rangle = 0$, we get

$$\langle V\varphi, \varphi\rangle_{\partial\Omega} = \langle \varphi, V\varphi\rangle_{\partial\Omega} = \langle \varphi_0, V\varphi_0\rangle_{\partial\Omega} + Ek^2.$$

The single layer potential

$$\mathcal{V}\varphi_0(y) = -\frac{1}{2\pi}\int_{\partial\Omega} \ln|x(s) - y|\varphi_0(s)ds, \quad y \in \mathbb{R}^2$$

vanishes at infinity, hence one can apply the Green's formula in $\bar{\Omega}$ and $\mathbb{R}^2 \setminus \Omega$:

$$\int_{\Omega} \operatorname{grad}\mathcal{V}\varphi_0 \operatorname{grad}\mathcal{V}\varphi_0 dx = \int_{\partial\Omega} V\varphi_0 \frac{\partial V\varphi_0}{\partial n} ds \tag{2.33}$$

$$\int_{\mathbb{R}^2\setminus\Omega} \operatorname{grad}\mathcal{V}\varphi_0 \operatorname{grad}\mathcal{V}\varphi_0 dx = -\int_{\partial\Omega} V\varphi_0 \frac{\partial V\varphi_0}{\partial n} ds \tag{2.34}$$

Here denotes n the exterior normal vector on $\partial\Omega$.
The jump relation yields (i is used from interior, e from exterior)

$$\varphi_0 = \left(\left[\frac{\partial\mathcal{V}\varphi_0}{\partial n}\right]_i - \left[\frac{\partial\mathcal{V}\varphi_0}{\partial n}\right]_e\right) \qquad \text{on } \partial\Omega.$$

Adding (2.33) and (2.34) we get

$$\int_{\mathbb{R}^2} |\operatorname{grad}\mathcal{V}\varphi_0|^2 dx = \int_{\partial\Omega} V\varphi_0\varphi_0 ds$$

and

$$\int_{\partial\Omega} V\varphi\varphi ds = \int_{\partial\Omega} V\varphi_0\varphi_0 ds + Ek^2 \geq \int_{\Omega} |\operatorname{grad}\mathcal{V}\varphi_0|^2 dx + Ek^2.$$

It is

$$\operatorname{grad}\mathcal{V}\varphi = \operatorname{grad}\mathcal{V}\varphi_0 + k\operatorname{grad}\mathcal{V}e = \operatorname{grad}\mathcal{V}\varphi_0$$

and $\langle e, V\varphi\rangle_{\partial\Omega} = Ek$. Hence

$$\begin{aligned}
\langle V\varphi, \varphi\rangle_{\partial\Omega} = \int_{\partial\Omega} V\varphi\varphi ds &\geq \int_{\Omega} |\operatorname{grad}\mathcal{V}\varphi|^2 dx + \frac{1}{E}\left(\int_{\partial\Omega} eV\varphi ds\right)^2 \\
&\overset{\text{Th.2, estimates}}{\geq} c\|\mathcal{V}\varphi\|^2_{H^1(\Omega)} \overset{\text{Th.4}}{\geq} \tilde{c}\|V\varphi\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\geq \tilde{\tilde{c}}\|\varphi\|^2_{H^{-\frac{1}{2}}(\partial\Omega)}.
\end{aligned}$$

It is not easy to show the last estimate:

$$\|V\varphi\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \geq c\|\varphi\|^2_{H^{-\frac{1}{2}}(\partial\Omega)}.$$

One can prove that $V : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is a continuous bijective mapping and then Banach's theorem implies the continuity of the inverse mapping. The condition diameter$(\Omega) < 1$ is used, in order to excluse that V has eigensolutions. □

Some problems of elastostatics

In section (1.3.2) Lemma 14, we have already discussed the Dirichlet problem for the Lamé operator as an example for elastostatic problems. Elastostatic problems have been treated rather early by means of boundary integral equations and boundary element methods.
The displacement field $\vec{u}(\vec{x})$ of an ideal elastic homogeneous and isotropic material is governed by the Lamé equations

$$\mu\Delta\vec{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\vec{u} = \vec{0} \text{ in } \Omega \text{ (or } \Omega^c).$$

Usually, the displacement fields have to satisfy on one part $\Gamma_1 \subset \partial\Omega$ Dirichlet conditions and on the remainding part $\partial\Omega \setminus \bar{\Gamma_1} = \Gamma_2$ Neumann conditions:

$$\begin{aligned} \vec{u} &= \vec{g} \text{ on } \Gamma_1 \\ \sigma[\vec{u}]n &= \vec{t_0} \text{ on } \Gamma_2. \end{aligned}$$

Here is σ the stress tensor with the components

$$\sigma_{ij} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\delta_{ij} + 2\mu\varepsilon_{ij}$$

and $\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ are the components of the strain tensor; n denotes the exterior unit normal vector on $\partial\Omega$. As we have shown already, there are Green's formulas (in the linear elasticity called Betti's formulas).

First Betti's formula

$$\begin{aligned} a(\vec{u},\vec{v})_\Omega &= \sum_{ij}\int_\Omega \sigma_{ij}(\vec{u})\varepsilon_{ij}(v)dx \\ &= \sum_{ij}\left(\int_{\partial\Omega}\sigma_{ij}(\vec{u})n_j v_i ds_x - \int_\Omega \sigma_{ij,j}(\vec{u})v_i dx\right) = (L(\vec{u},\vec{v})_{L_2(\Omega)} \end{aligned}$$

Analogously to the definition of the fundamental solution for the Laplacian we define: the matrix $E(x,y)$ is a fundamental solution of L iff

$$\begin{aligned} -L_x E(x,y) &= \delta_x(y)I, \text{ or} \\ \langle -L_x E(x,y), \vec{u}(x)\rangle &= \langle \delta_x(y)I, \vec{u}(x)\rangle = \vec{u}(y). \end{aligned}$$

Let us remind that

$$L = \begin{pmatrix} \mu\Delta + (\lambda+\mu)\partial_1^2 & (\lambda+\mu)\partial_1\partial_2 & (\lambda+\mu)\partial_1\partial_3 \\ (\lambda+\mu)\partial_2\partial_1 & \mu\Delta + (\lambda+\mu)\partial_2^2 & (\lambda+\mu)\partial_2\partial_3 \\ (\lambda+\mu)\partial_3\partial_1 & (\lambda+\mu)\partial_2\partial_3 & \mu\Delta + (\lambda+\mu)\partial_3^2 \end{pmatrix}.$$

We choose as fundamental solution the "Kelvin solution"

$$E_{ij}(x,y) = \frac{(\lambda+\mu)}{4(n-1)\pi\mu(\lambda+2\mu)}\left[\frac{\lambda+3\mu}{\lambda+\mu}\delta_{ij}F(x,y) + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^n}\right],$$

where

$$F(x,y) = \begin{cases} -\ln|x-y| & for \;\; n=2 \\ \frac{1}{|x-y|} & for \;\; n=3 \end{cases}$$

is the fundamental solution of the Laplacian, except a factor.

The Green's representation theorem (Somigliana identity) reads for the components of a solution $\vec{u}$ from $L\vec{u} = \vec{0}$:

$$u_i(y) = \int_{\partial\Omega} t_j(x)E_{ij}(x,y)ds_x - \int_{\partial\Omega} T_{ij}(x,y)u_j(x)ds_x, \quad y \in \Omega, \tag{2.35}$$

where $t_j(x) = \sum_k \sigma_{jk}[\vec{u}](x)n_k(x)$ are the components of the traction vector (boundary stress vector) and $(T_{ij}(x,y))_{ij}$ is the tensor of the boundary stress of the fundamental matrix,

$$T_{ij}(x,y) = \sum_k \sigma_{jk}\left[\begin{pmatrix} E_{1i}(x,y) \\ \vdots \\ E_{ni}(x,y) \end{pmatrix}\right] n_k(x),$$

where σ_{jk} acts on the variable x.
We write (2.33) shortly with the help of single layer and double layer potentials

$$\vec{u}(y) = \mathcal{V}[\vec{t}](y) - U[\vec{u}](y).$$

There holds a lemma about the properties of $\mathcal{V}$ and U and the jump relations analogously to Lemma 27. Passing to the boundary in (2.33) we get the boundary integral equations after some rearrangements.

$$V(\vec{t})(y) = \left(\frac{1}{2}I + K\right)\vec{u}(y), \quad y \in \partial\Omega, \tag{2.36}$$

where

$$\begin{aligned} (V\vec{t})_i(y) &= \int_{\partial\Omega} E_{ij}(x,y)t_j(x)ds_x \\ (K\vec{u})_i(y) &= \int_{\partial\Omega} T_{ij}(x,y)u_j(x)ds_x \end{aligned}$$

and

$$D\vec{u}(y) = \left(\frac{1}{2}I - K^*\right)\vec{t}(y) \text{ for } y \in \partial\Omega. \tag{2.37}$$

The boundary integral equation can be used to determine the missing Cauchy data $\vec{u}$ or $\vec{t}$. If we know them, then the Somigliana identity yields the solution in the whole domain.
The equation (2.34) yields the Neumann datum $\vec{t}$ for a given displacement field $\vec{u}$ on $\partial\Omega$ and equation (2.35) yields the Dirichlet datum $\vec{u}$, if Neumann conditions are prescribed.

EXAMPLE. We consider the Dirichlet problem

$$\begin{aligned} L\vec{u} &= 0 \text{ in } \Omega \\ \vec{u} &= \vec{g} \text{ on } \partial\Omega. \end{aligned}$$

We use the boundary equation system (2.35) for the calculation of $\vec{t}$. The operator V maps $H^{-\frac{1}{2}}(\partial\Omega)$ into $H^{\frac{1}{2}}(\partial\Omega)$ continuously. The bilinear form $\langle V\vec{t},\vec{t}\rangle_{\partial\Omega} = c(\vec{t},\vec{t})$ is defined on

$$H^{-\frac{1}{2}}(\partial\Omega) \;\times\; H^{\frac{1}{2}}(\partial\Omega)$$

It turns out from Lemma 29 [7] that V is a positive definite operator, that means there is a constant c such that

$$\langle V\vec{t},\vec{t}\rangle_{\partial\Omega} \;\geq\; c\|\vec{t}\|^2_{H^{-\frac{1}{2}}(\partial\Omega)}.$$

The Steklov Poincaré operator

We introduce a boundary operator (pseudodifferential operator) which maps the Dirichlet datum on the Neumann datum. From (2.34) it follows

$$\left(\frac{1}{2}I + K\right)\vec{u} = V\vec{t}.$$

If V^{-1} exists (for *diam* $\Omega < 1$ it is valid), then

$$S = V^{-1}\left(\frac{1}{2}I + K\right) \quad : \quad H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

denotes the Steklov-Poincaré operator,

$$S\vec{u} = \vec{t}.$$

Using the second relation of (2.30) we have

$$\begin{aligned} D\vec{u} + \left(\frac{1}{2}I + K^*\right)\vec{t} &= D\vec{u} + \left(\frac{1}{2}I + K^*\right)V^{-1}\left(\frac{1}{2}I + K\right)\vec{u} \\ &= \vec{t}. \end{aligned}$$

$$S = \left(\frac{1}{2}I + K^*\right)V^{-1}\left(\frac{1}{2}I + K\right) + D$$

is another symmetric representation of the Steklov Poincaré operator. The inverse Steklov Poincaré operator is not defined in general, since $\ker S = \{const\}$ [19].

2.3.2 Boundary element Galerkin methods

We take a partition of the boundary curve $\Gamma = \partial\Omega$ ($n = 2$) with the help of a global parametric representation. If $\Gamma = \partial\Omega$ is a surface then we choose a triangulation (curvilinear triangles).
With respect to the partition of the boundary we introduce piecewise constant functions in order to approximate the traction, while picewise linear elements are used for the approximation of the boundary displacement fields.

$$\begin{aligned} \vec{t_h}(x) &= \sum_{k=1}^{N} t_k \vec{\Phi_k^0(x)} \in V_h^N. \\ \vec{u_h}(x) &= \sum_{k=1}^{M} u_k \vec{\Phi_k^1(x)} \in V_h^D. \end{aligned}$$

An essential problem is the computation of the elements (singular integrals) of the resulting Galerkin matrix. Here one has to use partly analytical integrations and integration by parts (especially for the hypersingular operator D). Now, we formulate an **error estimate**. We denote $\partial\Omega = \Gamma$. Let be

$$A : H^s(\Gamma) \to H^{s-2\alpha}(\Gamma).$$

e.g.

$$\begin{aligned} A &= V, \quad s = -\frac{1}{2},\ \alpha = -\frac{1}{2}, \quad V: \quad H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \\ A &= D, \quad s = \frac{1}{2},\ \alpha = \frac{1}{2}, \quad D: \quad H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma). \end{aligned}$$

We consider the operator equation

$$A\vec{u}(x) = \vec{f(x)}, \quad x \in \Gamma,$$

and assume

$$|\langle A\vec{u}, \vec{v}\rangle| \leq c_1 \|\vec{u}\|_{H^s(\Gamma)} \|\vec{v}\|_{H^s(\Gamma)} \quad \forall \vec{u}, \vec{v} \in H^s(\Gamma) \tag{2.38}$$

$$|\langle A\vec{u}, \vec{u}\rangle| \geq c_2 \|\vec{u}\|^2_{H^s(\Gamma)}. \tag{2.39}$$

Theorem 15. *Let be the conditions (2.36) and (2.37) satisfied and $u_h \in H^s(\Gamma)$ a Galerkin solution, that means*

$$\langle A\vec{u}_h, \vec{v}_h\rangle = \langle \vec{f}, \vec{v_h}\rangle \ \ \forall \vec{v_h} \in S_h^\nu(\Gamma).$$

For $s < \nu + \frac{1}{2}$, if $n = 2$; $s \leq \nu$, if $n = 3$ and $2s - \nu - 1 \leq \tau \leq s \leq \beta \leq \nu + 1$ there is an optimal error estimate

$$\|\vec{u} - \vec{u_h}\|_{H^\tau(\Gamma)} \leq Ch^{\beta-\tau} \|\vec{u}\|_{H^\beta(\Gamma)}.$$

EXAMPLES.

(a) $\nu = 0$, piecewise constant elements:

$$\begin{aligned} s &= -\frac{1}{2}, \quad 2s - \nu - 1 = -2 \le \tau \le -\frac{1}{2} \le \beta \le 1, \\ \tau &= s = -\frac{1}{2} \\ \beta &= \frac{1}{2} \\ \Rightarrow & \quad \|\vec{t} - \vec{t_h}\|_{H^{-\frac{1}{2}}(\Gamma)} \le Ch^1 \|\vec{t}\|_{H^{\frac{1}{2}}(\Gamma)} \end{aligned}$$

provided $\vec{t} \in H^{\frac{1}{2}}(\Gamma)$.

(b) $\nu = 1$, piecewise linear elements:

$$\begin{aligned} s &= \frac{1}{2}, \quad 2s - 2 \le \tau \le s \le \beta \le 2, \\ \tau &= s = \frac{1}{2}, \\ \beta &= \frac{3}{2} \\ \Rightarrow & \quad \|\vec{u} - \vec{u_h}\|_{H^{\frac{1}{2}}(\Gamma)} \le Ch^1 \|\vec{u}\|_{H^{\frac{3}{2}}(\Gamma)}. \end{aligned}$$

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