Constructive Approximation and Numerical Methods in Geodetic Research Today – An Attempt of a Categorization Based on an Uncertainty Principle

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Abstract

This review article reports current activities and recent progress on constructive approximation and numerical analysis in physical geodesy. The paper focuses on two major topics of interest, namely trial systems for purposes of global and local approximation and methods for adequate geodetic application. A fundamental tool is an uncertainty principle, which gives appropriate bounds for the quantification of space and momentum localization of trial functions. The essential outcome is a better understanding of constructive approximation in terms of radial basis functions such as splines and wavelets.

Key Words: uncertainty principle, trial systems, approximation methods, geodetic applications.

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1 Introduction

Physical geodesy is much concerned with the space $\mathcal{L}^2(\Omega)$ of square–integrable functions on the unit sphere Ω. The quantity $\|F\|_{\mathcal{L}^2(\Omega)}$ is called the energy of the 'signal' $F \in \mathcal{L}^2(\Omega)$. 'Signals' $F \in \mathcal{L}^2(\Omega)$ possess Fourier transforms $F^\wedge(n,k)$ defined by

$$F^\wedge(n,k) = \int_\Omega F(\xi) Y_{n,k}(\xi)\, d\omega(\xi) \tag{1}$$

in terms of $\mathcal{L}^2(\Omega)$–orthonormal spherical harmonics $\{Y_{n,k}\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$. From Parseval's identity we have

$$\|F\|^2_{\mathcal{L}^2(\Omega)} = (F,F)_{\mathcal{L}^2(\Omega)} = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \left(F^\wedge(n,k)\right)^2 .$$

Usually one works more with the '*amplitude spectrum*'

$$\left\{F^\wedge(n,k)\right\}_{\substack{n=0,1,\ldots, \\ k=1,\ldots,2n+1}}$$

than with the 'original signal' $F \in \mathcal{L}^2(\Omega)$. The 'inverse Fourier transform'

$$F = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} F^\wedge(n,k) Y_{n,k} \tag{2}$$

allows the geodesist to think of the function F as a sum of 'wave functions' $Y_{n,k}$ of different frequencies. One can think of their measurements as operating on an 'input signal' F to produce an output signal $G = \Lambda F$, where Λ is an operator acting on $\mathcal{L}^2(\Omega)$. Fortunately, it is the case that large portions of interest can be well approximated by operators that are linear, rotation–invariant pseudodifferential operators (cf. Svensson, S.L. (1983)). If Λ is such an operator on $\mathcal{L}^2(\Omega)$, this means that

$$\Lambda Y_{n,k} = \Lambda^\wedge(n) Y_{n,k}, \qquad n = 0,1,\ldots; k = 1,\ldots,2n+1, \tag{3}$$

where the so–called symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ is a sequence of real values (independent of k). Here $\mathbb{N}_0$ denotes the set of all non-negative integers and $\mathbb{N}$ denotes the set of all positive integers. Thus we have the fundamental fact that the spherical harmonics are the eigenfunctions of the operator Λ. Different pseudo-differential operators Λ are characterized by their eigenvalues $\Lambda^\wedge(n)$. The 'amplitude spectrum' $\{G^\wedge(n,k)\}$ of the response of Λ is described in terms of the amplitude spectrum of functions (signals) by a simple multiplication by the 'transfer' $\Lambda^\wedge(n)$.

Physical devices do not transmit spherical harmonics of arbitrarily high frequency without severe attenuation. The 'transfer' $\Lambda^\wedge(n)$ usually tends to zero with increasing n. It follows from (3) that the amplitude spectra of the responses (observations) to functions (signals) of finite energy also are negligibly small beyond some finite frequency. Thus, both because of the frequency limiting nature of the used devices and because of the nature of the 'transmitted signals', the geoscientist is soon led to consider bandlimited functions. These are the functions $F \in \mathcal{L}^2(\Omega)$ whose 'amplitude spectra' vanish for all $n > N$ ($N \in \mathbb{N}$ fixed). In other words, each bandlimited function $F \in \mathcal{L}^2(\Omega)$ can be written as a finite Fourier transform

$$F = \sum_{n=0}^{N} \sum_{k=1}^{2n+1} F^\wedge(n,k) Y_{n,k}. \tag{4}$$

A function F of the form (4) is said to be *bandlimited with the band* N. In analogous manner, $F \in \mathcal{L}^2(\Omega)$ is said to be *locally supported (spacelimited) with spacewidth* ρ around an axis $\eta \in \Omega$, if for some $\rho \in (-1, 1)$ the function F vanishes on the set of all $\xi \in \Omega$ with $-1 \leq \xi \cdot \eta \leq \rho$. From (4) it readily follows that bandlimited functions are infinitely often differentiable everywhere. Moreover, it is clear that F is an analytic function. From the analyticity it follows immediately that a non–trivial bandlimited function cannot vanish on any (non–degenerate) subset of Ω. The only function that is both bandlimited and spacelimited is the trivial function. Now, in addition to bandlimited but non–spacelimited functions, numerical analysis would like to deal with spacelimited functions. But as we have seen, such a function (signal) of finite (space) support cannot be bandlimited, it must contain spherical harmonics of arbitrary large frequencies. Thus there is a dilemma of seeking functions that are somehow concentrated in both space and (angular) momentum domain. There is a way of mathematically expressing the impossibility of simultaneous confinement of a function to space and (angular) momentum, namely the *uncertainty principle.*

2 Trial Systems and Basis Property

The representation of a function $F \in \mathcal{L}^2(\Omega)$ (such as the gravitational potential on the earth's surface) in terms of countable (Hilbert) bases $\{B_n\}_{n=0,1,\ldots} \subset \mathcal{L}^2(\Omega)$ is one of the most interesting and important problems in physical geodesy.

Explicitly written out in mathematical language the problem is to find, to every value $\varepsilon > 0$ and every element $F \in \mathcal{L}^2(\Omega)$, a linear combination $F^{(N)} = \sum_{n=0}^{N} a_n B_n$ such that $\left\| F - F^{(N)} \right\|_{\mathcal{L}^2(\Omega)} \leq \varepsilon$. Written down in functional analytic notation the *basis property* of $\{B_n\}_{n=0,1,\ldots}$ in $\mathcal{L}^2(\Omega)$ equivalently means that the space $\mathcal{L}^2(\Omega)$ is the completion of the set $span_{n=0,1,\ldots}(B_n)$ of all finite linear combinations of functions B_n (with respect to the $\| \cdot \|_{\mathcal{L}^2(\Omega)}$–topology):

$$\mathcal{L}^2(\Omega) = \overline{\underset{n=0,1,\ldots}{span}\ (B_n)}^{\| \cdot \|_{\mathcal{L}^2(\Omega)}}. \tag{5}$$

In the literature a collection of georelevant basis systems $\{B_n\}_{n=0,1,\ldots}$ satisfying (5) are known from which we list the most interesting ones (for notational purposes and more detailed information on the function systems the reader is referred to Freeden, W. et al. (1998a)).

Spherical Harmonics. A particular role in physical geodesy is played by the system

$$\{Y_{n,k}\}_{n=0,1,\ldots;\ k=1,\ldots,2n+1}$$

of spherical harmonics. Spherical harmonics are usually defined as the restrictions of homogeneous harmonic polynomials to the sphere. The polynomial structure has tremendous advantages. Spherical harmonics of different degrees are orthogonal. The space $Harm_n$ of spherical harmonics of degree n is finite–dimensional: $dim(Harm_n) = 2n + 1$. The basis property of $\{Y_{n,k}\}_{n=0,1,\ldots;\ k=1,\ldots,2n+1}$ is equivalently characterized by the completion of the direct sum $\bigoplus_{n=0}^{\infty} Harm_n$, i.e.:

$$\mathcal{L}^2(\Omega) = \overline{\bigoplus_{n=0}^{\infty} Harm_n}^{\| \cdot \|_{\mathcal{L}^2(\Omega)}}. \tag{6}$$

As a matter of fact, spherical harmonic expansions (i.e. multipole expansions) are the classical approaches to earth's gravitational potential modelling. Particularly important spherical harmonic models are OSU91A (cf. Rapp, R.H. et al. (1991), Rapp, R.H. (1997a,b)), GRIM4 (cf. Schwintzer, P. et al. (1997)), GRIM4-S4 (cf. Schwintzer, P. (1997)), and EGM96 (cf. Lemoine, F.G. et al. (1996)). Some working groups are continuously improving these models on global and/or regional scale (for example, Groten, E. (1996), Groten, E. et al. (1998), König, R. et al. (1996), Rapp, R.H. (1997), Wenzel, G. (1998)). Many efforts are made to overcome the difficulties arising in the numerical application of spherical harmonics, for example, Nyquist rate, aliasing, etc. (cf. e.g. Jekeli, C. (1996), Schuh, W.-D. (1996), Strakhov, V.N. et al. (1998)). For a historical perspective on global spherical harmonic analysis the reader is referred to Sneeuw, N. (1994).

Bandlimited Radial Basis Functions. Kernel functions of the representation

$$K_j(\xi,\eta) = \sum_{n=2^j}^{2^{j+1}-1} A_n^{-2} \sum_{k=1}^{2n+1} Y_{n,k}(\xi) Y_{n,k}(\eta), \qquad (\xi,\eta) \in \Omega^2,$$

where $A_n \neq 0$ for all relevant n, may be used to establish the basis property in $\mathcal{L}^2(\Omega)$ as follows: Choose a fundamental system X_{N_j} of $N_j = \sum_{n=2^j}^{2^{j+1}-1}(2n+1)$ points $\eta_1, \ldots, \eta_{N_j}$ on Ω. Then the space $Harm_{2^j,\ldots,2^{j+1}-1}$ of all spherical harmonics of degree n with $2^j \leq n \leq 2^{j+1}-1$ can be written in the form $Harm_{2^j,\ldots,2^{j+1}-1} = span_{i=1,\ldots,N_j} K_j(\eta_i,\cdot)$. Each bandlimited function of class $Harm_{2^j,\ldots,2^{j+1}-1}$ can be represented exactly in terms of the functions $K_j(\eta_1,\cdot),\ldots,K_j(\eta_{N_j},\cdot)$. In other words, the advantage is when using bandlimited functions we do not need the function at all positions. It suffices to know a finite set for each scale $j \in \mathbb{N}_0$. In conclusion,

$$\mathcal{L}^2(\Omega) = \overline{\bigoplus_{j=0}^{\infty} \underset{i=1,\ldots,N_j}{span} K_j(\eta_i,\cdot)}^{\|\cdot\|_{\mathcal{L}^2(\Omega)}}. \tag{7}$$

It should be noted that the addition theorem of the theory of spherical harmonics allows the following reformulation of the kernel $K_j(\cdot,\cdot)$ in terms of Legendre polynomials P_n:

$$K_j(\xi,\eta) = \sum_{n=2^j}^{2^{j+1}-1} A_n^{-2} \frac{2n+1}{4\pi} P_n(\xi\cdot\eta), \quad (\xi,\eta) \in \Omega^2. \tag{8}$$

Thus the kernel (8) constitutes a radial basis function, i.e. a (one–dimensional) function depending only on the spherical distance of the unit vectors ξ and η. As example of a bandlimited kernel function we mention the Shannon kernels $K_j(\cdot,\cdot)$ with $A_n = 1$ for $n = 2^j,\ldots,2^{j+1}-1$, $j = 0, 1, \ldots$

In physical geodesy bandlimited kernel functions are usually obtained by truncation of non-bandlimited ones. The significance of bandlimited kernels will increase in future wavelet research, since finite-dimensional scale and detail spaces may be generated exactly by bandlimited radial basis kernels (cf. Freeden, W., Schreiner, M. (1998)). Moreover, fast evaluation can be organized in form of pyramid schemata (cf. Schreiner, M. (1997b), Freeden, W. et al. (1998c)).

Non–bandlimited Radial Basis Functions. Non–bandlimited kernel functions of the form

$$K(\xi,\eta) = \sum_{n=0}^{\infty} A_n^{-2} \frac{2n+1}{4\pi} P_n(\xi\cdot\eta), \quad (\xi,\eta) \in \Omega^2, \tag{9}$$

Figure 1: Legendre Polynomial $P_3(\cos(\vartheta))$ and Legendre coefficients

Figure 2: Shannon Wavelet $K_3(\cos(\vartheta))$ and Legendre coefficients

satisfying the 'summability condition' $\sum_{n=0}^{\infty} A_n^{-2}((2n+1)/4\pi) < \infty$ have been extensively used in physical geodesy. Examples are the *Green kernel of the Beltrami operator* Δ^* ($A_n^{-1} = (n(n+1))^{-2}, n = 1, 2, \ldots$), the *(Abel–) Poisson kernel* ($A_n^{-1} = h^{n/2}$, $0 < h < 1$, $n = 0, 1, \ldots$), the *Stokes kernel* ($A_n^{-1} = h^{n/2}\frac{1}{n-1}$, $0 < h < 1$, $n = 0, 2, 3, \ldots$), etc. The basis property of non–bandlimited kernel functions with $A_n \neq 0$ for all n can be guaranteed as follows: Assume that $\{\eta_0, \eta_1, \ldots\}$ is a countable dense set of points on the sphere Ω. Then it follows that

$$\mathcal{L}^2(\Omega) = \overline{\underset{n=0,1,\ldots}{span} \; K(\eta_n, \cdot)}^{\|\cdot\|_{\mathcal{L}^2(\Omega)}}. \quad (10)$$

Non–bandlimited kernel functions as presented above are radial basis functions depending only on the inner product of the unit vectors ξ and η. They are of basic significance in spline and/or wavelet theory of functions on the sphere (cf. Freeden, W. et al. (1998a)). As examples we illustrate the Abel–Poisson kernel (cf. Figure 3) and the Abel–Poisson wavelet (cf. Figure 4).
Non-bandlimited kernels (such as the Abel-Poisson, Gauß-Weierstraß, Green, rational, Stokes, Tikhonov, and locally (cap) supported kernels) have been extensively applied in physical geodesy, mainly for purposes of local approximation. The palette of different approximation techniques is large (for example, Albertella, A., Sacerdote, F. (1995), Bian, S., Menz, J. (1998), Brand, R. et al. (1996), Cui, J. et al. (1992), De Santis, A., Torta, J.M. (1997), Grafarend, E., Engels, J. (1992), Martinez, Z., Grafarend, E.W. (1997), Schreiner, M. (1994, 1997a), Thalhammer, M. (1995)).

It should be noted that the statement (5) can be extended to the $C(\Omega)$–norm for all continuous

Figure 3: Abel–Poisson kernel $Q_r(\cos(\vartheta))$ and Legendre coefficients, $r = 0.3, 0.5, 0.7, 0.9$

Figure 4: Abel–Poisson Wavelet $\Psi_j(\cos(\vartheta)), j = 0, 1, 2$ and Legendre coefficients

trial functions of the above type

$$C(\Omega) = \overline{\underset{n=0,1,\ldots}{span}\ (B_n)}^{\|\cdot\|_{C(\Omega)}}, \tag{11}$$

i.e. any continuous function on Ω, therefore, admits a uniform approximation on Ω in terms of the basis system $\{B_n\}_{n=0,1,\ldots}$. Moreover, the basis property (5) can be formulated in a variety of different topologies (e.g. Sobolev topology (cf. Freeden, W. et al. (1998a)).

For some geodetic applications, such as the determination of the earth's density distribution, one has to deal with functions whose domain is not only a sphere $B = \left\{x \in \mathbb{R}^3 \;\middle|\; |x| = \beta\right\}$ but a whole ball $\overline{B_{int}} = \left\{x \in \mathbb{R}^3 \middle| |x| \leq \beta\right\}$. Note that in this case the harmonic functions are not dense in $\mathcal{L}^2\left(\overline{B_{int}}\right)$, i.e. the approximation of a square-integrable function by a harmonic function is no more guaranteed. Moreover, an infinite-dimensional space of so–called anharmonic functions, which are functions that are (in the sense of $\mathcal{L}^2\left(\overline{B_{int}}\right)$) orthogonal to all harmonic functions, has to be taken into account, too. Hence, a basis $\{B_n\}_{n=0,1\ldots} \subset \mathcal{L}^2\left(\overline{B_{int}}\right)$ satisfying

$$\mathcal{L}^2\left(\overline{B_{int}}\right) = \overline{\underset{n=0,1,\ldots}{span}\ (B_n)}^{\mathcal{L}^2\left(\overline{B_{int}}\right)} \tag{12}$$

can, for example, be obtained by taking the inner harmonics $\left\{H^{int}_{m,k}\right\}_{\substack{m=0,1,\ldots \\ k=1,\ldots,2m+1}}$ and an appropriate anharmonic basis $\{\mathcal{A}_i\}_{i=0,1,\ldots}$, such that

$$\{B_n\}_{n=0,1\ldots} = \left\{H^{int}_{m,k}\right\}_{\substack{m=0,1,\ldots \\ k=1,\ldots,2m+1}} \cup \{\mathcal{A}_i\}_{i=0,1,\ldots}. \tag{13}$$

For more details the reader is referred to Ballani, L. et al. (1993), Ballani, L., Stromeyer, D. (1990), Michel, V. (1998, 1999).

Seen from a superficial point of approximation theory our foregoing treatment of basis systems would suggest that, if we are looking for an approximation of a function on Ω from discrete data, all possible choices of basis systems would be applicable for geodetic purposes. Actually it is necessary, in the case where several choices are possible, to choose the trial systems in close adaption to the specific properties of the data. The essential reason is the uncertainty principle, which offers a clarification of the mathematical characteristics of the aforementioned basis systems $\{B_n\}_{n=0,1,\ldots}$.

3 Uncertainty Principle

Localization in Space. Suppose that F is of class $\mathcal{L}^2(\Omega)$. Assume, without loss of generality, that

$$\|F\|_{\mathcal{L}^2(\Omega)} = \left(\int_\Omega (F(\eta))^2 \, d\omega(\eta) \right)^{1/2} = 1. \tag{14}$$

We associate to F the normal (radial) field $O_\eta F = \eta F(\eta)$, $\eta \in \Omega$, i.e. $O : F \mapsto O_\eta F, \eta \in \Omega$, maps $\mathcal{L}^2(\Omega)$ into the associated set of normal fields on Ω. The 'centre of gravity of the spherical window' is defined by the *expectation value in the space domain*

$$g_F^O = \int_\Omega (O_\eta F(\eta)) \, F(\eta) \, d\omega(\eta) = \int_\Omega \eta (F(\eta))^2 \, d\omega(\eta) \in \mathbb{R}^3, \tag{15}$$

thereby interpreting $(F(\eta))^2 \, d\omega(\eta)$ as surface mass distribution over the sphere Ω embedded in Euclidean space $\mathbb{R}^3$. It is clear that g_F^O lies in the space $\overline{\Omega^{int}}$ of Ω: $\left|g_F^O\right| \leq 1$. The *variance in the space domain* is understood in canonical sense as the variance of the operator O

$$\begin{aligned} \sigma_F^O &= \int_\Omega \left| \left(O_\eta - g_F^O\right) F(\eta) \right|^2 \, d\omega(\eta) \\ &= \int_\Omega \left| \eta - g_F^O \right|^2 (F(\eta))^2 \, d\omega(\eta) \in \mathbb{R}. \end{aligned} \tag{16}$$

Observing the identity $\left|\eta - g_F^O\right|^2 = 1 + (g_F^O)^2 - 2\eta \cdot g_F^O$, $\eta \in \Omega$, it follows immediately that $\sigma_F^O = 1 - (g_F^O)^2$. Obviously, $0 \leq \sigma_F^O \leq 1$.

Figure 5 gives a geometric interpretation of g_F^O and σ_F^O. We associate to g_F^O, $g_F^O \neq 0$, and its normalization $\eta_F^O = g_F^0 / \left|g_F^O\right|$ the spherical cap $C = \left\{\eta \in \Omega \,\middle|\, 1 - \eta \cdot \eta_F^O \leq 1 - \left|g_F^O\right| \right\}$. Then the boundary ∂C is a circle with radius $(\sigma_F^O)^{1/2}$.

Localization in Momentum. Next the *expectation value in the 'momentum domain'* (more accurately, the space of Fourier transforms which is understood analogously to the momentum space in physics) is introduced to be the *expectation value of the negative Beltrami operator* $-\Delta^*$ on Ω. Then, for $F \in \mathcal{H}_{2l}(\Omega)$, $l \in \mathbb{N}$, i.e. for all $F \in \mathcal{L}^2(\Omega)$ such that there exists a function $G \in \mathcal{L}^2(\Omega)$

Figure 5: Localization in a spherical cap

with $G^\wedge(n,k) = (-n(n+1))^l F^\wedge(n,k)$ for all n,k we have

$$g_F^{-\Delta^*} = \int\limits_\Omega \left(-\Delta_\eta^* F(\eta)\right) F(\eta) d\omega(\eta) \in \mathbb{R}.$$

Correspondingly, the *variance in the 'momentum domain'* is given by

$$\sigma_F^{-\Delta^*} = \int\limits_\Omega \left(\left(-\Delta_\eta^* - g_F^{-\Delta^*}\right) F(\eta)\right)^2 d\omega(\eta) \in \mathbb{R}.$$

The square roots, $\sqrt{\sigma^O}$ and $\sqrt{\sigma^{-\Delta^*}}$, are called the *uncertainties* in O and $-\Delta^*$, respectively.

	operator	**expectation value**
space	O	$g_F^O = \int\limits_\Omega \left(O_\eta F(\eta)\right) F(\eta) d\omega(\eta)$
momentum	$-\Delta^*$	$g_F^{-\Delta^*} = \int\limits_\Omega \left(-\Delta_\eta^* F(\eta)\right) F(\eta) d\omega(\eta)$

	operator	**variance**
space	O	$\sigma_F^O = \int\limits_\Omega \left(\left(O_\eta - g_F^O\right) F(\eta)\right)^2 d\omega(\eta)$
momentum	$-\Delta^*$	$\sigma_F^{-\Delta^*} = \int\limits_\Omega \left\lvert\left(-\Delta_\eta^* - g_F^{-\Delta^*}\right) F(\eta)\right\rvert^2 d\omega(\eta)$

'Space/momentum' Localization.

The *uncertainty relation* measures the trade-off between 'space localization' and 'momentum localization' ('spread in momentum'). It states that *sharp localization in space and 'momentum'*

are mutually exclusive.
For the uncertainties we get from (Freeden, W., Windheuser, U. (1997)) the following theorem.

Theorem 3.1 *Let F be of class $\mathcal{H}_4(\Omega)$ such that $\|F\|_{\mathcal{L}^2(\Omega)} = 1$. Then*

$$\sigma_F^O \sigma_F^{-\Delta^*} \geq |g_F^O| \frac{g_F^{(-\Delta^*)^2} - \left(g_F^{-\Delta^*}\right)^2}{g_F^{-\Delta^*}}, \tag{17}$$

provided that $g_F^{-\Delta^*} \neq 0$. If the right hand side of (17) is non–vanishing, then

$$\Delta_F^O \Delta_F^{-\Delta^*} \geq 1, \tag{18}$$

where

$$\Delta_F^{-\Delta^*} = \left(\frac{\sigma_F^{-\Delta^*}}{\frac{g_F^{(-\Delta^*)^2} - \left(g_F^{-\Delta^*}\right)^2}{g_F^{-\Delta^*}}} \right)^{1/2} = \left(g_F^{-\Delta^*}\right)^{1/2}$$

and

$$\Delta_F^O = \left(\frac{\sigma_F^O}{|g_F^O|} \right)^{\frac{1}{2}}.$$

Finally we discuss some examples which are of particular interest for us:

Localization of Spherical Harmonics. We know that $\int\limits_\Omega (Y_{n,k}(\xi))^2 \, d\omega(\xi) = 1$. Now it is easy to see that

$$g_{Y_{n,k}}^O = 0, \qquad \sigma_{Y_{n,k}}^O = 1.$$

Moreover, we find

$$g_{Y_{n,k}}^{-\Delta^*} = n(n+1), \qquad \sigma_{Y_{n,k}}^{-\Delta^*} = 0.$$

In other words, *spherical harmonics show an ideal momentum localization, but no space localization.*
Therefore, the use of spherical harmonic expansions of higher and higher degrees for the determination of the gravitational field becomes more and more difficult. In particular, local changes of a function under consideration affect the whole amplitude spectrum, which is an unfortunate feature for constructive approximation.

Localization of the Abel–Poisson Kernel. Consider the function $Q_r : [-1,+1] \to \mathbb{R}$, $r < 1$, given by

$$Q_r(t) = \frac{1}{4\pi} \frac{1 - r^2}{(1 + r^2 - 2rt)^{3/2}} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} r^n P_n(t).$$

An easy calculation gives us

$$\|Q_r\|_{\mathcal{L}^2[-1,+1]} = (Q_{r^2}(1))^{1/2} = \left(\frac{1+r^2}{4\pi} \right)^{1/2} \frac{1}{1-r^2}.$$

Furthermore, for $\tilde{Q}_r(t) = \|Q_r\|^{-1}_{\mathcal{L}^2[-1,+1]}\ Q_r(t),\ t \in [-1,+1]$, we obtain after an elementary calculation

$$g^{O}_{\tilde{Q}_r} = \frac{2r}{1+r^2}, \quad \sigma^{O}_{\tilde{Q}r} = \left(\frac{1-r^2}{1+r^2}\right)^2,$$

$$g^{-\Delta^*}_{\tilde{Q}_r} = \frac{6r^2}{(1-r^2)^2}, \quad \sigma^{-\Delta^*}_{\tilde{Q}_r} = \frac{12r^2(r^4+5r^2+1)}{(1-r^2)^4}$$

and

$$\Delta^{O}_{\tilde{Q}_r} = \frac{1-r^2}{2r}, \quad \Delta^{-\Delta^*}_{\tilde{Q}_r} = \frac{\sqrt{6}r}{1-r^2}.$$

Thus we finally obtain

$$\Delta^{O}_{\tilde{Q}_r}\,\Delta^{-\Delta^*}_{\tilde{Q}_r} = \frac{\sqrt{6}}{2} = \sqrt{\frac{3}{2}} > 1.$$

Note that in this case the value $\Delta^{O}_{\tilde{Q}_r}\,\Delta^{-\Delta^*}_{\tilde{Q}_r}$ is independent of r. Letting r formally tend to 1 we

space domain momentum domain

Figure 6: Abel–Poisson kernel Uncertainty Principle: $\Delta^{O}_{\tilde{Q}_h}$ (left), $\Delta^{-\Delta^*}_{\tilde{Q}_h}$ (right) and the constant value $\Delta^{O}_{\tilde{Q}_h} \cdot \Delta^{-\Delta^*}_{\tilde{Q}_h}$

are able to interpret the localization properties of the *Dirac kernel* on Ω:

$$\delta(t) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} P_n(t), \qquad t = \xi \cdot \eta, \qquad \xi, \eta \in \Omega.$$

As a matter of fact, letting r tend to 1 shows us that the variances in the space domain take the constant value 0. On the other hand, the variances in the momentum domain converge to ∞.

Hence, the *Dirac kernel shows ideal space localization, but no momentum localization.* Thus, all intermediate cases of 'space–momentum localization' occur when discussing the Abel–Poisson kernel. It should be pointed out that the Abel–Poisson kernel does not satisfy a minimum uncertainty state.

The minimum uncertainty state is provided by the bell–shaped (Gaussian) probability density function.

Localization of the Gaussian Function. Consider the function

$$G_\lambda(t) = e^{-(\lambda/2)(1-t)}, \qquad t \in [-1,+1], \quad \lambda > 0,$$

An elementary calculation shows us that

$$\tilde{G}_\lambda(t) = \gamma(\lambda) e^{-(\lambda/2)(1-t)}, \quad \gamma(\lambda) = \left(1/\sqrt{4\pi}\right)\left(\frac{1}{2\lambda}\left(1 - e^{-2\lambda}\right)\right)^{-1/2},$$

satisfies $\left\|\tilde{G}_\lambda\right\|_{\mathcal{L}^2[-1,+1]} = 1$. Furthermore, it is not difficult to deduce (cf. Freeden, W. et al. (1997c)) that $\Delta_K^O \Delta_K^{-\Delta^*} \to 1$ as $\lambda \to \infty$. This shows us that the best value of the uncertainty principle (Theorem 3.1) is 1.

Figure 7: Gaussian Function Uncertainty Principle: $\Delta^O_{\tilde{G}_\lambda}$ (left), $\Delta^{-\Delta^*}_{\tilde{G}_\lambda}$ (right) and $\Delta^O_{\tilde{G}_\lambda} \cdot \Delta^{-\Delta^*}_{\tilde{G}_\lambda}$

Summarizing our results we are led to the following conclusions: The uncertainty principle represents a trade-off between two 'spreads', one for the position and the other for the momentum. The main statement is that *sharp localization in space and in momentum are mutually exclusive.* The reason for the validity of the uncertainty relation (Theorem 3.1) is that the operators O and $-\Delta^*$ do not commute. Thus O and $-\Delta^*$ cannot be sharply defined simultaneously. Extremal members in the space/momentum relation are the polynomials (i.e. spherical harmonics) and the Dirac function(al)s. An asymptotically optimal kernel is the Gaussian function.

For the considerations of three-dimensional functions $G \in \mathcal{L}^2\left(\mathbb{R}^3\right)$ we assume that a separation ansatz $F(r\eta) = G(r)H(\eta)$, $r \in \mathbb{R}_0^+$, $\eta \in \Omega$, is possible. Furthermore, we deal with functions in $\mathcal{H}_{1;0}(\mathbb{R})$, i.e. square-integrable functions G that are differentiable almost everywhere, such that G' is square-integrable and $G(0) = 0$. In this context we introduce the operator Q with $Q_r G(r) = rG(r)$ for the space localization and the operator P with $P_r G(r) = G'(r)$ for the momentum localization. Note that P is anti-symmetric: $P^* = -P$.
For the investigation of the localization property of the spherical part we can use the results derived above. Concerning the radial part G the expectation values

$$g_G^Q = \int_0^\infty r(G(r))^2\,dr, \tag{19}$$

$$g_G^P = \int_0^\infty G(r)\frac{d}{dr}G(r)\,dr \tag{20}$$

and the variances

$$\sigma_G^Q = \int_0^\infty \left(\left(r - g_G^Q\right) G(r)\right)^2 dr \tag{21}$$

$$\sigma_G^P = \int_0^\infty \left(\left(\frac{d}{dr} - g_G^P\right) G(r)\right)^2 dr \tag{22}$$

occur. Also in this situation an uncertainty principle can be derived, which is similar to Heisenberg's uncertainty principle in quantum mechanics.

Theorem 3.2 *Let $G \in \mathcal{L}^2\left(\mathbb{R}_0^+\right)$ be differentiable, where*

$$(G,G)_{\mathcal{L}^2(\mathbb{R}_0^+)} = \int_0^\infty (G(r))^2\,dr = 1.$$

Then

$$\sigma_G^Q \sigma_G^P \geq \frac{1}{4}, \tag{23}$$

i.e.

$$\Delta_G^Q \Delta_G^P \geq \frac{1}{2}, \tag{24}$$

where

$$\Delta_G^Q = \left(\sigma_G^Q\right)^{\frac{1}{2}} \text{ and } \Delta_G^P = \left(\sigma_G^P\right)^{\frac{1}{2}}. \tag{25}$$

Proof: Using the $\mathcal{L}^2\left(\mathbb{R}_0^+\right)$-scalar product $(.,.)_{\mathcal{L}^2(\mathbb{R}_0^+)}$ we obtain:

$$((PQ - QP)G, G)_{\mathcal{L}^2(\mathbb{R}_0^+)}$$

$$
\begin{aligned}
&= \left(\left(P - g_G^P\right) \left(Q - g_G^Q\right) G - \left(Q - g_G^Q\right) \left(P - g_G^P\right) G, G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} \\
&= \left(\left(P - g_G^P\right) \left(Q - g_G^Q\right) G, G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} - \left(\left(Q - g_G^Q\right) \left(P - g_G^P\right) G, G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} \\
&= \left(\left(Q - g_G^Q\right) G, \left(P^* - g_G^P\right) G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} - \left(\left(P - g_G^P\right) G, \left(Q - g_G^Q\right) G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} \\
&= \left(\left(Q - g_G^Q\right) G, \left(P^* - g_G^{P^*} + g_G^{P^*} - g_G^P\right) G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} - \left(\left(P - g_G^P\right) G, \left(Q - g_G^Q\right) G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} \\
&= \left(g_G^{P^*} - g_G^P\right) \left(\left(Q - g_G^Q\right) G, G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} + \left(\left(Q - g_G^Q\right) G, \left(P^* - g_G^{P^*}\right) G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} \\
&\quad - \left(\left(P - g_G^P\right) G, \left(Q - g_G^Q\right) G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} \\
&= \left(\left(Q - g_G^Q\right) G, \left(P^* - g_G^{P^*}\right) G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} - \left(\left(P - g_G^P\right) G, \left(Q - g_G^Q\right) G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} \\
&= -2 \left(\left(P - g_G^P\right) G, \left(Q - g_G^Q\right) G \right)_{\mathcal{L}^2(\mathbb{R}_0^+)} .
\end{aligned}
$$

The Cauchy-Schwarz inequality implies

$$
((PQ - QP)G, G)^2_{\mathcal{L}^2(\mathbb{R}_0^+)} \leq 4 \sigma_G^P \sigma_G^Q .
$$

An easy calculation yields

$$
\begin{aligned}
((PQ - QP)G, G)_{\mathcal{L}^2(\mathbb{R}_0^+)} &= \int_0^\infty \left(\frac{d}{dr}(rG(r)) - r \frac{d}{dr} G(r) \right) G(r)\, dr \\
&= \int_0^\infty \left(G(r) + r \frac{d}{dr} G(r) - r \frac{d}{dr} G(r) \right) G(r)\, dr \\
&= 1.
\end{aligned}
$$

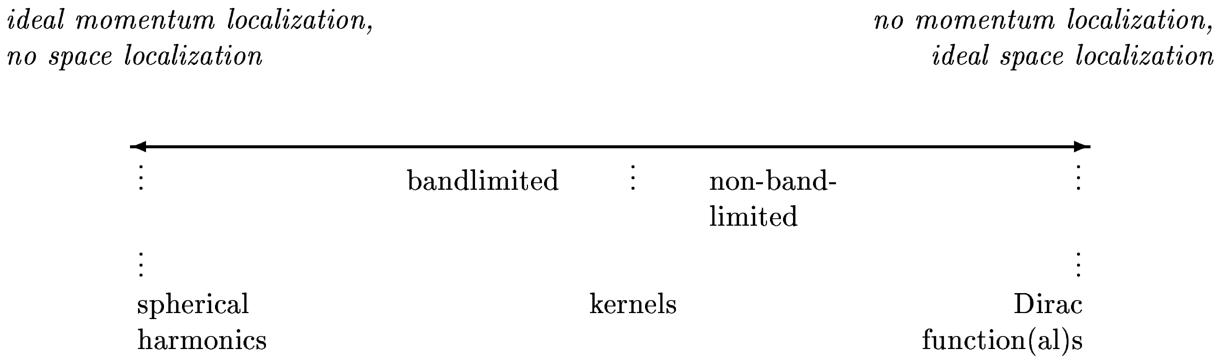
Hence, the uncertainty principle is valid.

The estimate (Theorem 3.1) allows us to give a quantitative classification in form of a canonically defined hierarchy of the space/momentum localization properties of kernel functions of the form

$$
K(x,y) = \sum_{n=0}^{\infty} a_n B_n(x) B_n(y), \tag{26}
$$

where x and y are in the relevant domain. In view of the amount of space/momentum localization it is also important to distinguish bandlimited kernels (i.e. $a_n = 0$ for all $n \geq N$) and non–bandlimited ones. Non–bandlimited kernels show a much stronger space localization than their comparable bandlimited counterparts. Actually, if $a_n \approx a_{n+1} \approx 1$ for many successive integers n, then the support of (26) in space domain is small.

The figure below gives a rough qualitative illustration of the consequences of the uncertainty principle in constructive approximation:



To be more specific, on the left end of this scheme we have the spherical harmonics with their ideal momentum localization. However, they have no space localization, as they are restrictions of polynomials. The present standard way in physical geodesy of increasing the accuracy is to increase the maximum degree of the spherical harmonics expansions under consideration.
On the right end of the scheme there is the Dirac functional which maps a function to its value at a certain point. Hence those functionals have ideal space localization but no momentum localization. Consequently, they are used in a finite pointset approximation (see, for example, Cui, J. (1995) and the references therein).
Radial basis functions exist as bandlimited and non-bandlimited functions. Every bandlimited radial basis function refers to a finite number of frequencies. This reduction of the momentum localization allows a finite variance of the space in the uncertainty principle, i.e. this method has both a momentum localization and a space localization. If we move from bandlimited to non-bandlimited radial basis functions the momentum localization decreases and the space localization increases in accordance to the uncertainty principle. In consequence, if the accuracy has to be increased in radial basis approximation, a denser point grid is required in the region under investigation.
Note that the classical radial basis functions have been constructed for Euclidean spaces like the plane. Recently radial basis functions have been studied for the sphere (cf. Freeden, W. et al. (1998a) and the references therein). Canonical generalizations to georelevant manifolds (like ellipsoid (cf. e.g. Thong, N.C., Grafarend, E.W. (1989)), spheroid, and actual earth's surface) are under current development. For more details about constructive approximation on closed (regular) surfaces the reader is referred to Freeden, W. (1987), Freeden, W., Schneider, F. (1998a,b). The varieties of the intensity of the localization on the sphere can be illustrated by considering the kernel function

$$K(\xi,\eta) = \sum_{n=0}^{\infty} A_n^{-2} \frac{2n+1}{4\pi} P_n(\xi\cdot\eta),\ (\xi,\eta)\in\Omega^2. \tag{27}$$

By choosing $A_n = \delta_{nm}$ we obtain a spherical harmonic in $Harm_m(\Omega)$, i.e. we arrive at the left end of our scheme. On the other hand, if we define $A_n = 1$ for all $n \in \mathbb{N}$, we obtain the kernel which is the Dirac functional in the Hilbert space $\mathcal{L}^2(\Omega)$. Bandlimited kernels have the property $A_n = 0$ for all $n \geq N$, N fixed. For radial basis functions we usually define A_n in a way such that $\lim_{n\to\infty} A_n^{-2} = 0$. The slower this convergence is, the lower is the momentum localization and the higher is the space localization.

Seen from numerical point of view, the scheme motivates that spherical harmonic based methods of constructive approximation are efficiently suitable for recovering globally and homogeneously the long wavelengths of the earth's gravitational potential. Spherical harmonics diagonalize invariant pseudodifferential operators. Geodetic observables are connected by use of rotation–invariant pseudodifferential operators within a simply structured Meissl scheme (cf. Rummel, R., Van Gelderen, M. (1995)). But spherical harmonics as non–space localizing polynomial structures certainly need an adaptive uniformly dense coverage of data all over the sphere. Local changes are not treatable locally; they affect all constituting elements of a function (e.g. the whole table of orthogonal coefficients). The crucial point besides numerical calamities is that equidistributed material of sufficiently small data width is simply not available on global basis and will not be for the forseeable future. In the opinion of the authors the future use of spherical harmonics is therefore limited in physical geodesy.

What we really need for the future satellite scenario are more and more space localizing basis systems in order to model medium–to–short–wavelengths features of the earth's gravitational potential. In this respect it should be mentioned that satellite–to–satellite tracking (SST) is understandable as a transition from spherical harmonic (multipole) modelling being responsible for the 'low frequency contribution' to spline and/or wavelet modelling being appropriate for 'higher frequency parts'. Therefore, seen in the interdependence of increasing space localization and decreasing frequence localization (cf. Freeden, W. (1998), Freeden, W. et al. (1999)), satellite–to–satellite tracking may be mathematically considered the interface of global expansion by ideally frequency localizing, but non–space localizing polynomials (spherical harmonics) and multiscale expansion (multiresolution analysis) by frequency as well as space localizing radial basis functions (e.g. splines and/or wavelets).

But in view of the amount of space/frequency localization it is also worth distinguishing bandlimited kernel functions from non–bandlimited ones. As a matter of fact, non–bandlimited radial basis functions show a much stronger space localization than their bandlimited counterparts. Thus we are led to the conclusion that for the 'medium–wavelength contributions' bandlimited kernel functions or moderately space localizing non–bandlimited kernel functions should be used for approximation. The canonical interface from 'medium–to–short–wavelengths features' is satellite gravity gradiometry (SGG) that should be handled by a multiscale analysis of more and more space localizing non–bandlimited radial basis function. In the opinion of the authors we are thus led to the following scheme.

Current Status	Future Concepts	
Potential differences, geoid heights, satellite altimetry etc.	satellite – to satellite tracking	satellite gravity gradiometry
Fourier (orthogonal) expansion by spherical harmonics	spline/wavelet expansion by bandlimited kernels	wavelet expansion by non–bandlimited kernels

4 Approximation Methods

It is only relatively recently that radial basis function techniques such as splines and wavelets play a fundamental role in modern numerical analysis on the sphere. Starting point of spherical splines are the early eighties (Freeden, W. (1981, 1987, 1990), Wahba, G. (1981, 1987)).
Spline functions are canonical generalizations of 'spherical polynomials', i.e. spherical harmonics, having desirable characteristics as interpolating, smoothing, and best approximating functions (cf. Freeden, W. (1981, 1990), Freeden, W. et al. (1996a,b, 1998a), Wahba, G. (1990)). By spline interpolation we mean a variational problem of minimizing an 'energy'-norm of a suitable Sobolev space. According to the choice of the norm, bandlimited as well as non-bandlimited splines can be distinguished. Spherical splines have been successfully applied to linear inverse problems of satellite geodesy (cf. Schneider, F. (1996)). Spherical tensor spline approximation in satellite gravity gradiometry (SGG) is due to Schreiner, M. (1994). It is also remarkable that the spherical interpolating processes by splines open new perspectives for solving boundary value problems of elliptic equations even for georelevant non-spherical boundaries (cf. Freeden, W. (1987)). Moreover, a variant formulated for the interior of the (spherical) earth enables modelling of the anharmonic part of the earth's density. Figure 8 shows the result of an anharmonic spline interpolation of PREM, which is a radially symmetric model for the earth's density distribution. More precisely, the harmonic part of PREM is constant, and the approximation to PREM has been obtained by adding an anharmonic spline, which was calculated according to given interpolation points, to the constant harmonic density (see the PhD thesis due to Michel, V. (1999)). The horizontal axes have the range -earthradius to +earthradius (the enumeration is based on the used point grid), and the vertical axes refer to the density.
The construction of spherical wavelets has seen an enormous increase of activities over the last five years. Methods to introduce wavelets by convolution kernels or related constructions are due to Freeden, W., Windheuser, U. (1995, 1996, 1997), Göttelmann, J. (1998), and Holschneider, M. (1996).
Spherical wavelets are building blocks that enable fast decorrelation of data on the sphere. Thus three features are incorporated in this way of thinking about georelevant wavelets, namely basis property, decorrelation, and fast computation. First of all, wavelets are building blocks for general data sets derived from functions. By virtue of the basis property each element of a general class of functions (e.g. the earth's gravitational potential seen as as member of a set of potentials within a Sobolev space framework) can be expressed in stable way as a linear combination of dilated and shifted copies of a 'mother function'. The role of the wavelet transform as a mapping from the class of functions into an associated two-parameter family of space and scale dependent functions is properly characterized by least squares properties. Secondly, wavelets have the power to decorrelate. In other words, the representation of data in terms of wavelets is somehow 'more compact' than the original representation. We search for an accurate approximation by only using a small fraction of the original information of a function. Typically the decorrelation is achieved by building wavelets which have a compact support (localization in space), which are smooth (decay towards high frequencies), and which have vanishing moments (decay towards low frequencies). Different types of wavelets can be found from certain constructions of space/momentum localization. In this respect the uncertainty principle tells us that sharp localization in 'space and momentum' is mutually exclusive. Nevertheless, it turns out that decay towards long and short wavelengths (i.e. in information theoretic jargon, bandpass filtering) can be assured without any difficulty. Moreover, vanishing moments of wavelets (see Freeden, W., Windheuser, U. (1997), Freeden, W., Schreiner, M. (1998)) enable us to combine (polynomial)

Figure 8: anharmonic spline interpolation of radially symmetric inner structures of the earth: PREM (top left), reconstruction (top right) and comparison of the profiles

outer harmonic expansions (responsible for the long-wavelength part of a function) with wavelet multiscale expansions (responsible for the medium–to–short–wavelengths contributions).

Thirdly, the main question of recovering a function on the sphere, e.g. the earth's gravitational potential is how to decompose the function into wavelet coefficients, and how to reconstruct efficiently the potential from the coefficients. There is a 'tree algorithm' or 'pyramid algorithm' that makes these steps simple and fast. In this respect it is desirable to switch between the original representation of the data and its wavelet representation in a time proportional to the size of the data. In fact, the fast decorrelation power of wavelets is the key to applications such as data compression, fast data transmission, noise cancellation, signal recovering, etc.

Multiresolution analysis of the gravity field based on classical Euclidean (wavelet) theory or related multilevel techniques has been presented by Arabelos, D., Tscherning, C.C. (1995), Belikov, M., Groten, E. (1995), Li, L.T. et al. (1998), Schwarz, K.P., Zuofa Li (1997), and Zuofa Li (1996). Multilevel methods for regional adaptive gravity field modelling have been proposed by Kusche, J. et al. (1998), Kusche, J. (1998), and Rudolph, S. (1998). First test computations for gravity field recovery using spherical wavelets are due to Bäcker, M. (1995) and Windheuser, U. (1996). Wavelet investigations dealing with inverse problems of satellite geodesy are due to Schneider, F. (1997). An integrated concept of physical geodesy by means of bandlimited harmonic wavelets has been proposed by Freeden, W., Schneider, F. (1998). Application in

scalar problems of airborne gravimetry are due to Bayer, M. (1996), Freeden, W., Schneider, F. (1998), and Schneider, F. (1997). Vectorial and tensorial wavelets on the sphere for use in satellite–to–satellite tracking and satellite gradiometry can be found in Bayer, M. et al. (1998), Freeden, W. et al. (1997a, 1998a, 1999), and Freeden, W., Schneider, F. (1998).
Three-dimensional wavelets for the inner space and the outer space of the sphere are constructed in Michel, V. (1998, 1999) as a method for the gravimetry problem to determine the earth's density from potential values. Figures 9 and 10 show a multiscale reconstruction of the harmonic part of the earth's density anomaly plotted on the earth's surface from EGM96 data.

5 Fast Evaluation

Our results have shown that future activities in gravitational field determination should concentrate on combined models, where expansions in terms of spherical harmonics have to be combined with more and more space localizing trial functions, for example, radial basis functions such as splines and wavelets. Even for local approximation the philosophy of the authors developed from the uncertainty principle is the following three step procedure: First an outer harmonic approach should be used to model the global trends, i.e. the low-wavelengths part. In a second step bandlimited wavelets showing moderate space localizing phenomena may be taken for the medium frequency band of the earth's gravitational potential. Finally, the third step consists of non-bandlimited wavelet approximation to analyze the fine structure, i.e. short-wavelengths phenomena for local areas within a global concept. An example of this approach is given in Freeden, W. et al. (1998).
For the use of combined approximation methods fast algorithms are required. Basic steps to future work are fast Fourier procedures, fast wavelet schemata, and fast summation techniques (such as panel clustering). Helpful are the following research notes: Arabelos, D., Tscherning, C.C. (1998), Bláha, T. et al. (1996), Dahmen, W. (1997), Freeden, W. et al. (1998b,c), Glockner, O. (1997), Jiang, Z. et al. (1997), Lanser, M. (1997), Lehmann, R. (1997), Lin, Q.W. et al. (1997), Risbo, T. (1996), Sideris, M.G. (1995), Sneeuw, N. (1996), and Schreiner, M. (1997), Sweldens, W. (1997), etc.

A challenge for future work is the problem of combining efficiently and economically data of different types and data coming from different heights (cf. Arabelos, D., Tscherning, C.C. (1998), Freeden, W. et al. (1999)). In particular, the vectorial and tensorial nature of satellite data (for example, satellite–to–satellite tracking, satellite gradiometry) requires adequate approximation procedures. Future numerical methods should be able to handle such problems automatically.

6 Final Remark

An attempt to categorize recent developments in the field of trial functions and approximation methods based on an uncertainty principle has, of course, an element of arbitrariness, and we apologize to any author who might consider his/her paper(s) misplaced or forgotten. Nevertheless, we believe that any such errors are compensated for by the constructive structure thus given to the paper.

Figure 9: bandlimited reconstruction of density anomalies on the earth's surface at scales 4 (top left) to 10 (bottom)

Figure 10: non-bandlimited reconstruction of density anomalies on the earth's surface at scales 4 (top left) to 10 (bottom)

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