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On a Recursive Approximation of Singularly Perturbed Parabolic Equations (Extended Version)

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Abstract

The asymptotic analysis of IBVPs for the singularly perturbed parabolic PDE $\partial_t u + \partial_x u = \varepsilon\,\partial_{xx} u$ in the limit $\varepsilon \to 0$ motivate investigations of certain recursively defined approximative series ("ping–pong expansions"). The recursion formulae rely on operators assigning to a boundary condition at the left or the right boundary a solution of the parabolic PDE. Sufficient conditions for uniform convergence of ping–pong expansions are derived and a detailed analysis for the model problem $\partial_t u + \partial_x u = \varepsilon\,\partial_{xx} u$ is given.

1 Introduction

The recursive approximation derived in this paper arises from investigations on a singularly perturbed two–phase Stefan problem: If one of the two phases is characterized by slow diffusion, then a boundary layer at the phase change will yield a modified Stefan condition for the unperturbed one–phase problem.
Using matched asymptotic expansions a zeroth order correction term has been derived in [SU]. This correction term is sufficiently accurate as long as the moving interface stays away from a fixed boundary. If the moving interface approaches this fixed boundary, the whole problem will become – due to interacting layers – quite complicated. Moreover, the derivation of higher order corrections can not be performed in a straightforward manner using standard matching techniques from asymptotic analysis.
To have a close insight to the singularly perturbed phase, it turned out to be necessary to develop a seemingly new (compare [Bob, GFLRT, RST]) asymptotic analysis for the model problem

$$\partial_t u_\varepsilon + \partial_x u_\varepsilon = \varepsilon\,\partial_{xx} u_\varepsilon, \quad u_\varepsilon(0,x) = 1-x, \quad u_\varepsilon(t,0) = 1, \quad u_\varepsilon(t,1) = 0. \quad (1.1)$$

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with $\varepsilon \ll 1$.
After re–scaling t and x one gets from (1.1) a half-space problem on $[0,\infty[$, which yields the approximation

$$v_\varepsilon(t,x) = 1 - \frac{x-t}{2}\,\mathrm{erfc}\left(\frac{t-x}{2\sqrt{\varepsilon\, t}}\right) - \frac{x+t}{2}\,\mathrm{e}^{x/\varepsilon}\,\mathrm{erfc}\left(\frac{t+x}{2\sqrt{\varepsilon\, t}}\right).$$

The function v_ε is, for small values of ε and away from the right boundary $x=1$, an excellent approximation for u_ε, such that essential properties of u_ε [Bob, GFLRT] may be deduced from an almost elementary discussion of v_ε, i.e.

$$\lim_{\varepsilon\to 0} v_\varepsilon(t,x) = \begin{cases} 1-(x-t) & , \quad x-t>0 \\ 1 & , \quad t-x<0 \end{cases}$$

The following three questions arise naturally:

1) Why is v_ε such a good approximation for u_ε ?

2) How may one derive a correction term to handle the boundary layer at $x=1$?

3) How can higher order terms be constructed ?

In the course of the discussion of the questions above it turned out that certain "one–sided" operators that assign to a given boundary condition at the left *or* the right boundary a solution of the PDE play the most important role. To clarify the importance of these operators a more general setting is appropriate.

We shall therefore be concerned with a general class of initial–boundary value problems of the form

$$\begin{cases} \partial_t u_\varepsilon - A_\varepsilon[u_\varepsilon] =: P_\varepsilon[u_\varepsilon] = f_\varepsilon, \\ u_\varepsilon(0,x) = u^I_\varepsilon(x), \quad u_\varepsilon(t,0) = \alpha_\varepsilon(t), \quad u_\varepsilon(t,1) = \beta_\varepsilon(t), \end{cases} \tag{1.2}$$

where $\varepsilon \ll 1$ is a "small" parameter.
The time variable t ranges in $J=]0,T[$ with $T\in]0,\infty[$, the spatial variable x in $\omega=]0,1[$ and f_ε belongs to $C_B(\omega_T)$, where $\omega_T = J\times\omega$. Moreover, we assume $u^I_\varepsilon \in C^2_B(\omega)$ and $\alpha_\varepsilon,\beta_\varepsilon \in C_B(J)$. The ε–dependent operator A_ε is defined on $C^2(\omega_T)$ by

$$A_\varepsilon[v] = a_\varepsilon\, v + b_\varepsilon\, \partial_x v + c_\varepsilon\, \partial_{xx} v,$$

where $a_\varepsilon, b_\varepsilon, c_\varepsilon \in C_B(\omega_T)$.

We shall make use of the concept of "C_2–solutions" of (1.2).

Definition 1. *u_ε is a* C_2–solution of (1.2) *iff*

1) $u_\varepsilon \in \mathsf{C}_2$, *where*

$$\mathsf{C}_2 := \{v\in C^2(\omega_T)\cap C_B(\omega_T) : v(t,0+) \textit{ and } v(t,1-) \textit{ exist for all } t\in J, \textit{ and } v(\cdot,0+),\ v(\cdot,1-) \textit{ belong to } C_B(J)\}.$$

2) For all $(t_0, x_0) \in \omega_T$*: partial*$_t\, u_\varepsilon(t_0, x_0) = A_\varepsilon[u_\varepsilon](t_0, x_0)$.

3) $u_\varepsilon(\cdot, 0+) = \alpha_\varepsilon$ *and* $u_\varepsilon(\cdot, 1-) = \beta_\varepsilon$.

4) $\lim_{t\to 0} \int_{\omega_T} |u_\varepsilon(t,z) - u_\varepsilon^I(z)|\, dz = 0.$

We deal with distinguished *recursively* defined series $(\sum_{\zeta=0}^{\nu} u_\varepsilon^\zeta)_{\nu\in\mathbb{N}_0}$ to approximate u_ε. The recursions rely on linear operators $\Gamma_\varepsilon^l, \Gamma_\varepsilon^r : \mathsf{C}_2 \to \mathsf{C}_2$, and sequences

$$(u_\varepsilon^0,\ u_\varepsilon^1,\ u_\varepsilon^2,\ \dots) = (I_\varepsilon + l_\varepsilon^0 + r_\varepsilon^0,\quad l_\varepsilon^1 + r_\varepsilon^1,\quad l_\varepsilon^2 + r_\varepsilon^2,\quad \dots)$$

such that

E1) $I_\varepsilon \in \mathsf{C}_2$ satisfies

$$P_\varepsilon\left[I_\varepsilon\right] = f_\varepsilon, \qquad I_\varepsilon(0,x) = u_\varepsilon^I(x).$$

E2) $l_\varepsilon^0, r_\varepsilon^0 \in \mathsf{C}_2$ satisfy

$$P_\varepsilon\left[l_\varepsilon^0\right] = 0, \qquad l_\varepsilon^0(0,x) = 0, \quad l_\varepsilon^0(t,0) = \alpha_\varepsilon(t) - I_\varepsilon(t,0+),$$

$$P_\varepsilon\left[r_\varepsilon^0\right] = 0, \qquad r_\varepsilon^0(0,x) = 0, \quad r_\varepsilon^0(t,1) = \beta_\varepsilon(t) - I_\varepsilon(t,1-) - l_\varepsilon^0(t,1-),$$

and for all $k \in \mathbb{N}$:

E3) $l_\varepsilon^k = \Gamma_\varepsilon^l\left[r_\varepsilon^{k-1}\right]$ satisfies

$$P_\varepsilon\left[l_\varepsilon^k\right] = 0, \qquad l_\varepsilon^k(0,x) = 0, \quad l_\varepsilon^k(t,0) = -r_\varepsilon^{k-1}(t,0+),$$

E4) $r_\varepsilon^k = \Gamma_\varepsilon^r\left[l_\varepsilon^k\right]$ satisfies

$$P_\varepsilon\left[r_\varepsilon^k\right] = 0, \qquad r_\varepsilon^k(0,x) = 0, \quad r_\varepsilon^k(t,1) = -l_\varepsilon^k(t,1-),$$

E5) $\|u_\varepsilon^{k-1}\|_\infty \le g_{k-1}(\varepsilon)$ and $\left\| u_\varepsilon - \sum_{\nu=0}^{k-1} u_\varepsilon^\nu \right\|_\infty \le g_k(\varepsilon)$,

where $\|.\|_\infty$ is the standard norm on $C_B(\omega_T)$ and $(g_\nu)_{\nu\in\mathbb{N}_0}$ is a sequence of order functions $g_\nu : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $k \in \mathbb{N}$ we have

$$\lim_{\varepsilon\to 0} g_k(\varepsilon) = 0, \qquad g_k = o(g_{k-1}) \quad \text{as } \varepsilon \to 0.$$

In this approach the operators $\Gamma_\varepsilon^{r,l}$ and the "initial" functions $I_\varepsilon, l_\varepsilon^0, r_\varepsilon^0$ play the most prominent roles.

a) According to E1) the function I_ε satisfies $P[I_\varepsilon] = f_\varepsilon$ and fulfills the initial conditions. It is *not* assumed that I_ε satisfies the boundary conditions.

b) Due to E1), E2) the function $u_\varepsilon^0 = I_\varepsilon + l_\varepsilon^0 + r_\varepsilon^0$ is a C_2–solution of

$$\begin{cases} P[u_\varepsilon^0] = f_\varepsilon, \quad u_\varepsilon^0(0,x) = u_\varepsilon^I(x), \\ u_\varepsilon^0(t,0) = \alpha_\varepsilon(t) + r_\varepsilon^0(t,0+), \quad u_\varepsilon^0(t,1) = \beta_\varepsilon(t), \end{cases}$$

i.e. u_ε^0 satisfies the parabolic PDE (1.2), the correct initial condition and the correct boundary condition at $x = 1$. By E5) we have

$$\lim_{\varepsilon \to 0} \left\| u_\varepsilon - u_\varepsilon^0 \right\|_\infty \leq \lim_{\varepsilon \to 0} g_1(\varepsilon) = 0.$$

Hence u_ε^0 is for "small" values of ε an approximation for u_ε. Furthermore, since

$$\|r_\varepsilon^0\|_T = \|u_\varepsilon(\cdot,0+) - u_\varepsilon^0(\cdot,0+)\|_T \leq \left\| u_\varepsilon - u_\varepsilon^0 \right\|_\infty = O(g_1(\varepsilon)) \quad \text{as } \varepsilon \to 0,$$

– where $\|.\|_T$ is the standard norm in $C_B(J)$ – we get $\lim_{\varepsilon \to 0} \|r_\varepsilon^0\|_T = 0$, such that u_ε^0 satisfies approximatively the boundary condition at $x = 0$.

c) The recursion formulae for l_ε^k, r_ε^k, $k \in \mathbb{N}$ imply

$$l_\varepsilon^k = \left(\Gamma_\varepsilon^l \circ \left(\Gamma_\varepsilon^r \circ \Gamma_\varepsilon^l\right)^{k-1}\right)[r_\varepsilon^0], \quad r_\varepsilon^k = \left(\Gamma_\varepsilon^r \circ \Gamma_\varepsilon^l\right)^k [r_\varepsilon^0], \tag{1.3}$$

which shows the distinctive importance of r_ε^0.

d) Putting for $\nu \in \mathbb{N}_0$

$$\Sigma_\varepsilon^\nu := \sum_{\zeta=0}^{\nu} u_\varepsilon^\zeta = u_\varepsilon^0 + \sum_{\zeta=1}^{\nu} \left(\Gamma_\varepsilon^l + \Gamma_\varepsilon^r \circ \Gamma_\varepsilon^l\right) \circ \left(\Gamma_\varepsilon^r \circ \Gamma_\varepsilon^l\right)^{\zeta-1} [r_\varepsilon^0], \tag{1.4}$$

we see as in b) that Σ_ε^ν is a C_2–solution of

$$\begin{cases} P[\Sigma_\varepsilon^\nu] = f_\varepsilon, \quad \Sigma_\varepsilon^\nu(0,x) = u_\varepsilon^I(x), \\ \Sigma_\varepsilon^\nu(t,0) = \alpha_\varepsilon(t) + r_\varepsilon^\nu(t,0+), \quad \Sigma_\varepsilon^\nu(t,1) = \beta_\varepsilon(t), \end{cases} \tag{1.5}$$

with $\|r_\varepsilon^\nu(.,0+)\|_T = O(g_{\nu+1})$ as $\varepsilon \to 0$. Hence we deduce from E5) and the properties assumed for the sequence $(g_\nu)_{\nu \in \mathbb{N}_0}$ that the functions

$$u_\varepsilon^0 = \Sigma_\varepsilon^0, \quad \Sigma_\varepsilon^1, \quad \Sigma_\varepsilon^2, \quad \Sigma_\varepsilon^3, \quad \ldots, \quad \Sigma_\varepsilon^k, \quad \ldots$$

are approximations of zeroth, first, second, third,...,k–th,... order for u_ε.

e) Trivial choices for $I_\varepsilon, l_\varepsilon^0, r_\varepsilon^0$ and $\Gamma_\varepsilon^{r,l}$ would be $I_\varepsilon = u_\varepsilon$, $l_\varepsilon^0 = r_\varepsilon^0 = 0$ and $\Gamma_\varepsilon^{r,l} = 0$, the zero–operator. This choice is however not of interest here, because we are in particular concerned with "complicated" functions u_ε, for which approximations shall be constructed.

f) According to (1.3)–(1.5) the boundary conditions at $x = 0$ and $x = 1$ enter the recursion formula in different manners. Indeed a much more important role is played by r_ε^0 and the boundary condition at $x = 1$, which is satisfied by each Σ_ε^ν, than by l_ε^0 and the boundary condition at $x = 0$, which will usually not be satisfied by any Σ_ε^ν. It is however not difficult to interchange the role of the boundary conditions.

g) Loosely speaking the construction of the sequence $(u_\varepsilon^0, u_\varepsilon^1, u_\varepsilon^2, \dots)$ has something in common with "ping–pong": consider for $\nu \in \mathbb{N}_0$ the function r_ε^ν. Then one constructs $l_\varepsilon^{\nu+1}$ by solving a "left–hand boundary value problem", whose purpose is the elimination of $r_\varepsilon^\nu(.,0+)$ at the *left* boundary ("ping"). After this intermediate step one finds $r_\varepsilon^{\nu+1}$ by doing the very same thing, but now with $l_\varepsilon^{\nu+1}$ at the right boundary: one solves a "right–hand boundary value problem", whose purpose is the elimination of $l_\varepsilon^{\nu+1}(.,1-)$ at the *right* boundary ("pong"). Therefore, one can motivate to call the approximation defined by E1)–E5) a "ping–pong expansion".

The paper is organized as follows: in Section 2 we show how recursively defined series E1)–E5) arise at hand of the model problem

$$\partial_t u_\varepsilon = -\partial_x u_\varepsilon + \varepsilon\, \partial_{xx} u_\varepsilon, \qquad u_\varepsilon(0,x) = 1 - x, \qquad u_\varepsilon(t,0) = 1, \quad u_\varepsilon(t,1) = 0. \tag{1.6}$$

The investigations are settled on solutions z_ε of "half–space problems", i.e. z_ε satisfy IBVPs associated with $\partial_t z_\varepsilon = \partial_x z_\varepsilon + \varepsilon\, \partial_{xx} z_\varepsilon$ on intervals $]0,\infty[$ and $]-\infty,1[$, respectively. In Section 3 ping–pong series of the type E1)–E5) are investigated from an abstract point of view. The main result is the derivation of sufficient conditions for convergence, where the proof relies on a geometric series argument. The investigations of Section 2 and the theoretical result of Section 3 are combined in Section 4 to investigate properties of ping–pong expansions for

$$\begin{cases} \partial_t u_\varepsilon = -\partial_x u_\varepsilon + \varepsilon\, \partial_{xx} u_\varepsilon, & u_\varepsilon(0,x) = u_\varepsilon^I(x), \\ u_\varepsilon(t,0) = \alpha_\varepsilon(t), & u_\varepsilon(t,1) = \beta_\varepsilon(t) \end{cases}$$

2 An Introductionary Example

In this section we are concerned with the model problem

$$\partial_t u_\varepsilon = -\partial_x u_\varepsilon + \varepsilon\, \partial_{xx} u_\varepsilon, \qquad u_\varepsilon(0,x) = 1 - x, \qquad u_\varepsilon(t,0) = 1, \quad u_\varepsilon(t,1) = 0. \tag{2.1}$$

for "small" values of ε. In this case an asymptotic analysis of (2.1) has been performed in [Bob] and - in a more general setting - in [GFLRT]. Four aspects dominate the behaviour of u_ε as $\varepsilon \to 0$ (see Figure 1):

1) u_ε converges as $\varepsilon \to 0$ in a "rather good" (i.e. without oscillations) sense to the function

$$u_0 : \omega_T \to \mathbb{R}$$
$$(t,x) \mapsto \left\{ \begin{array}{rcl} 1-(x-t) & , & t \leq x \\ 1 & , & x < t \end{array} \right\},$$

i.e. u_0 solves the transport equation

$$\partial_t u_0 + \partial_x u_0 = 0, \qquad u_0(0,x) = 1-x, \qquad u_0(t,0) = 1.$$

2) The limiting function u_0 is less regular than any u_ε, $\varepsilon > 0$. The loss of regularity is due to the competition between initial values and boundary values (at $x=0$) along the characteristics $\{(t,x) \in \omega_T : x = t\}$.
3) The boundary condition at $x=1$ is lost in the limit $\varepsilon \to 0$. This amounts to the appearance of boundary layers for u_ε, $\varepsilon > 0$, at $x=1$.
4) Away from the "critical" regions $\{(t,x) \in \omega_T : x = t\}$ and $x=1$ we have *uniform* convergence of u_ε to u_0 as $\varepsilon \to 0$.

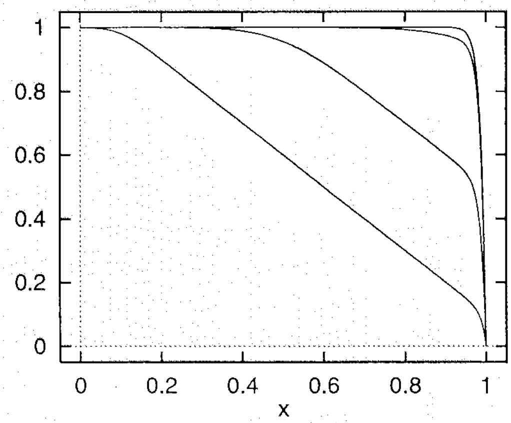

Figure 1: Numerical solution $u_\varepsilon(t,x)$ for $\varepsilon = 0.01$ at different times $t = 0.1$, 0.5, 1.0 and 2.0 (from left ro right).

One may try to look for these properties of u_ε for $\varepsilon \approx 0$ in the "explicit" solution of (2.1) which is – due to the constant coefficients – available from the separation method as follows: introducing

$$w_\varepsilon := \exp\left(-\frac{x}{2\varepsilon}\right) u_\varepsilon - (1-x),$$

w_ε satisfies

$$\left\{ \begin{array}{l} \partial_t w_\varepsilon = -\dfrac{1-x}{4\varepsilon} - \dfrac{1}{4\varepsilon} w_\varepsilon + \varepsilon\, \partial_{xx} w_\varepsilon, \\ w_\varepsilon(0,x) = \left(\exp\left(-\dfrac{x}{2\varepsilon}\right) - 1\right)(1-x), \quad w_\varepsilon(t,0) = w_\varepsilon(t,1) = 0 \end{array} \right.$$

For $l \in \mathbb{N}$ let

$$\mu_\varepsilon^l := \frac{1}{4\varepsilon^2} + l^2\pi^2, \quad \lambda_\varepsilon^l := \varepsilon\, \mu_l,$$

and

$$\kappa_\varepsilon^l := \frac{1}{2}\int_0^1 \exp\left(-\frac{x}{2\varepsilon}\right)(1-x)\sin(l\,\pi\,x)\,dx - \frac{l\,\pi}{2\,\mu_\varepsilon^l}$$
$$= -\frac{l\,\pi}{2\,\lambda_\varepsilon^l\,\mu_\varepsilon^l}\left(1-\exp\left(-\frac{1}{2\varepsilon}\right)(-1)^l\right),$$
$$C_\varepsilon^l := \frac{1}{2\,l\,\pi\,(1+4\,\varepsilon^2\,l^2\,\pi^2)}.$$

Then, in sense of $L^2(]0,1[)$,

$$w_\varepsilon = \sum_{l=1}^{\infty}\left[\kappa_\varepsilon^l\,\exp\left(-\lambda_\varepsilon^l\,t\right) - C_\varepsilon^l\right]\,\sin(l\,\pi\,x),$$

such that – again $L^2(]0,1[)$–sense –

$$u_\varepsilon = \exp\left(\frac{x}{2\varepsilon}\right)\left(1-x+\sum_{l=1}^{\infty}\left[\kappa_\varepsilon^l\,\exp\left(-\lambda_\varepsilon^l\,t\right) - C_\varepsilon^l\right]\,\sin(l\,\pi\,x)\right).$$

Obviously it is rather difficult to verify 1)-4) at hand of this series expansion.

The best way to proceed with an asymptotic analysis of (2.1) is to try to find an outer expansion of u_ε. Speaking phenomenologically for small values of ε the boundary value at $x=1$ is "far away" from the behaviour of u_ε on $J\times]0,\theta]$. Taking this idea literally, one may think to shift the boundary condition from $x=1$ to $x\to\infty$ – and thus to ignore it. Hence, we are led to consider the half–space problem

$$\partial_t v_\varepsilon = -\partial_x v_\varepsilon + \varepsilon\,\partial_{xx} v_\varepsilon, \qquad v_\varepsilon(0,x) = 1-x, \qquad v_\varepsilon(t,0)=1, \tag{2.2}$$

with $(t,x)\in J\times\mathbb{R}^+$ and $v_\varepsilon(x\to\infty)=0$. It may be reasonable to think that v_ε will be a good approximation of u_ε, at least for $(t,x)\in J\times]0,\theta]$.
Equation (2.2) is explicitly solvable by means of the Sinus–Fourier–Transformation (see Section 4, compare [Hir]), which yields

$$v_\varepsilon(t,x) = 1 - \frac{x-t}{2}\,\mathrm{erfc}\left(\frac{t-x}{2\sqrt{\varepsilon\,t}}\right) - \frac{x+t}{2}\,\mathrm{e}^{x/\varepsilon}\,\mathrm{erfc}\left(\frac{t+x}{2\sqrt{\varepsilon\,t}}\right).$$

We set $I_\varepsilon := v_\varepsilon\big|_{\omega_T}$ and, since I_ε already satisfies the correct boundary values at $x=0$, $l_\varepsilon^0 := 0$. According to Figure 2, I_ε is an excellent approximation of u_ε for small values of ε, naturally away from $x=1$.
What to do with the boundary condition at $x=1$? As I_ε exhibits no boundary layer at $x=1$, one has to find a correction term.

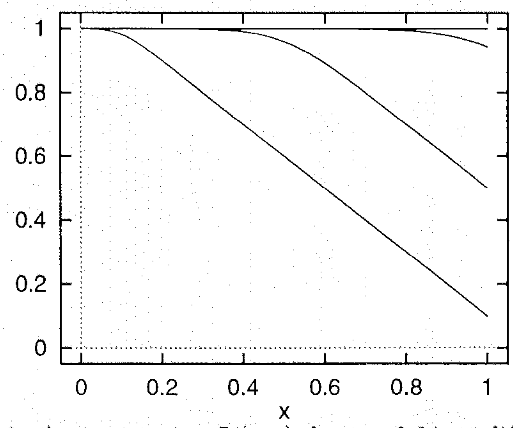

Figure 2: Approximation $I_\varepsilon(t,x)$ for $\varepsilon = 0.01$ at different times $t = 0.1$, 0.5, 1.0 and 2.0 (from left ro right).

The canonical way to proceed would be to re–scale the equations close to $x = 1$ and to apply matching procedures afterwards. This strategy however, though promising, leads us astray from recursively defined approximating series for u_ε and a different strategy is needed.

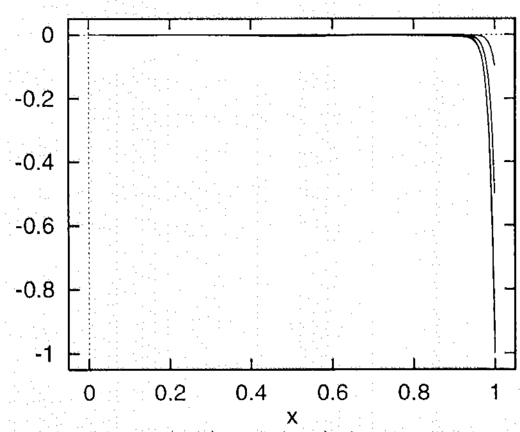

Figure 3: Difference $u_\varepsilon(t,x) - I_\varepsilon(t,x)$ for $\varepsilon = 0.01$ at different times $t = 0.1$, 0.5, 1.0 and 2.0 (from right to left).

Let us check the difference $s_\varepsilon^{-1} := u_\varepsilon - I_\varepsilon$ for small values of ε, indicated in Figure 3: naturally, s_ε^{-1} has a boundary layer at $x = 1$. The equation satisfied by s_ε^{-1} is

$$\begin{cases} \partial_t s_\varepsilon^{-1} = -\partial_x s_\varepsilon^{-1} + \varepsilon\, \partial_{xx} s_\varepsilon^{-1}, & s_\varepsilon^{-1}(0,x) = 0, \\ s_\varepsilon^{-1}(t,0) = 0, \quad s_\varepsilon^{-1}(t,1) = -I_\varepsilon(t,1-) \end{cases} \tag{2.3}$$

and we deduce from Figure 3 a very important property of (2.3): away from $x = 1$ the operator $A_\varepsilon[v] := -v' + \varepsilon\, v''$ has the tendency to make $|s_\varepsilon^{-1}|$ rather small. From this point of view it is not important to prescribe the boundary value 0 at $x = 0$, i.e. we may replace (2.3) by the half–space problem (see [VaRo])

$$\partial_t W_\varepsilon = -\partial_x W_\varepsilon + \varepsilon\, \partial_{xx} W_\varepsilon, \quad W_\varepsilon(0,x) = 0, \quad W_\varepsilon(t,1) = -I_\varepsilon(t,1-), \tag{2.4}$$

with $W_\varepsilon \in J\times]-\infty,1[$. For W_ε a convolution-type representation is available in the form

$$W_\varepsilon(t,x) = -\frac{1-x}{\sqrt{4\,\pi\,\varepsilon}} \int_0^t s^{-3/2} \exp\left(-\frac{(1-x+s)^2}{4\,\varepsilon\,s}\right) I_\varepsilon(t-s,1-)\,ds. \qquad (2.5)$$

Since W_ε solves a "right" boundary value problem, it is appropriate to introduce the notation

$$r_\varepsilon^0 := W_\varepsilon\Big|_{\omega_T}.$$

We put $u_\varepsilon^0 := I_\varepsilon + r_\varepsilon^0$ $(=I_\varepsilon + l_\varepsilon^0 + r_\varepsilon^0)$ and observe that the function $\Sigma_\varepsilon^0 := u_\varepsilon^0$ satisfies the problem

$$\begin{cases} \partial_t\,\Sigma_\varepsilon^0 = -\partial_x \Sigma_\varepsilon^0 + \varepsilon\,\partial_{xx}\Sigma_\varepsilon^0, & \Sigma_\varepsilon^0(0,x) = 1-x, \\ \Sigma_\varepsilon^0(t,0) = 1 + r_\varepsilon^0(t,0+), & \Sigma_\varepsilon^0(t,1) = 0, \end{cases} \qquad (2.6)$$

i.e. the correct equation, the correct initial condition, the correct boundary condition at $x = 1$ and – since $|r_\varepsilon^0(t,0+)|$ is rather small (see Figure 4) – "almost" the boundary condition at $x = 0$.

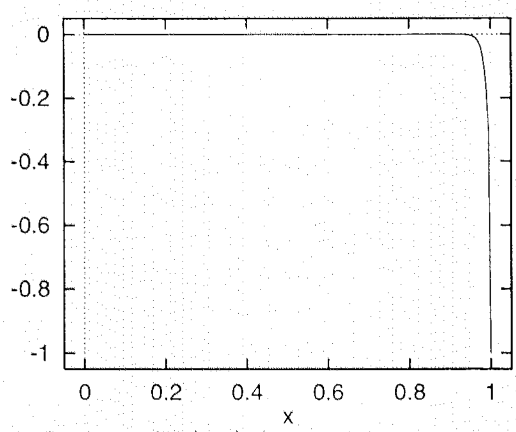

Figure 4: Function $W_\varepsilon(t,x)$ for $\varepsilon = 0.01$ at time $t = 2.0$.

What is the reason for the excellent approximation properties of u_ε^0 ?
Let us consider the difference $s_\varepsilon^0 := u_\varepsilon - u_\varepsilon^0$, which satisfies

$$\begin{cases} \partial_t\, s_\varepsilon^0 = -\partial_x s_\varepsilon^0 + \varepsilon\,\partial_{xx} s_\varepsilon^0, & s_\varepsilon^0(0,x) = 0, \\ s_\varepsilon^0(t,0) = -r_\varepsilon^0(t,0+), & s_\varepsilon^0(t,1) = 0 \end{cases}$$

From (2.5) we have

$$r_\varepsilon^0(t,0+) = \frac{1}{\sqrt{4\,\pi\,\varepsilon}} \int_0^t s^{-3/2} \exp\left(-\frac{(1+s)^2}{4\,\varepsilon\,s}\right) I_\varepsilon(t-s,1-)\,ds,$$

where we note that the term

$$I_\varepsilon(\sigma,1-) = 1 - \frac{1-\sigma}{2}\,\mathrm{erfc}\left(\frac{\sigma-1}{2\sqrt{\varepsilon\,\sigma}}\right) - \frac{1+\sigma}{2}\,\mathrm{e}^{1/\varepsilon}\,\mathrm{erfc}\left(\frac{\sigma+1}{2\sqrt{\varepsilon\,\sigma}}\right)$$

is uniformly bounded, i.e. there is $K \in]0, \infty[$ such that

$$\forall \varepsilon \in \mathbb{R}^+, \forall \sigma \in J: \qquad |I_\varepsilon(\sigma, 1-)| \leq K.$$

Hence we get for all $\varepsilon \in \mathbb{R}^+$ the estimate

$$\|r_\varepsilon^0(., 0+)\|_T \leq 2\,K \operatorname{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right),$$

where $\|.\|_T$ is the supremum norm on $C_B(]0, T[)$. Using the maximum principle [LSU] we can conclude that

$$\|u_\varepsilon - u_\varepsilon^0\|_\infty \leq 2\,K \operatorname{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right),$$

and thus

$$\|u_\varepsilon - u_\varepsilon^0\|_\infty = O\left(\operatorname{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right) \qquad \text{as } \varepsilon \to 0.$$

Now let us try to derive higher correction terms and let us consider (2.6): the two solutions u_ε and u_ε^0 differ from each other according to the boundary condition at $x = 0$, such that one may think to substract from u_ε^0 a function satisfying (2.1) with vanishing initial values and with boundary condition $u_\varepsilon^0(t, 0+) - 1 = r_\varepsilon^0(t, 0+)$ at $x = 0$.
In principle one may wish to fulfill the boundary condition at $x = 1$ as well – but then we would run into the same troubles as for u_ε: no "easy–to–handle" representation of this function would be available.
On the other hand, our experiences with half–space solutions are so far excellent and it seems appropriate to consider (the restriction to ω_T of) a half–space solution as a correction for u_ε^0. To make this idea more precise let us introduce for $a \in C_B(J)$ and $(t, y) \in J \times \mathbb{R}^+$ the function

$$U_\varepsilon[a](t, y) := \frac{y}{\sqrt{4\,\varepsilon\,\pi}} \int_0^t s^{-3/2} \exp\left(-\frac{(y-s)^2}{4\,\varepsilon\,s}\right) a(t-s)\,ds. \tag{2.7}$$

Then $U_\varepsilon = U_\varepsilon[a]$ satisfies

$$\partial_t U_\varepsilon = -\partial_x U_\varepsilon + \varepsilon\,\partial_{xx} U_\varepsilon, \qquad U_\varepsilon(0, y) = 0, \quad U_\varepsilon(t, 0) = a(t),$$

such that it is nearby to set

$$u_\varepsilon^{1/2} := l_\varepsilon^1 := U_\varepsilon[-u_\varepsilon^0(., 0+)]\Big|_{\omega_T}.$$

But, $\Sigma_\varepsilon^{1/2} := u_\varepsilon^0 + u_\varepsilon^{1/2}$ is not necessarily a better approximation than u_ε^0: to verify this let us consider the IBVP satisfied by $\Sigma_\varepsilon^{1/2}$:

$$\begin{cases} \partial_t \Sigma_\varepsilon^{1/2} = -\partial_x \Sigma_\varepsilon^{1/2} + \varepsilon\,\partial_{xx} \Sigma_\varepsilon^{1/2}, & \Sigma_\varepsilon^{1/2}(0, x) = 1 - x, \\ \Sigma_\varepsilon^{1/2}(t, 0) = 1, \quad \Sigma_\varepsilon^{1/2}(t, 1) = l_\varepsilon^1(t, 1-), \end{cases}$$

such that the difference $s_\varepsilon^{1/2} := u_\varepsilon - \Sigma_\varepsilon^{1/2}$ satisfies

$$\begin{cases} \partial_t s_\varepsilon^{1/2} = -\partial_x s_\varepsilon^{1/2} + \varepsilon\, \partial_{xx} s_\varepsilon^{1/2}, & s_\varepsilon^{1/2}(0,x) = 0, \\ s_\varepsilon^{1/2}(t,0) = 0, \quad s_\varepsilon^{1/2}(t,1) = -l_\varepsilon^1(t,1-). \end{cases}$$

From the maximum principle we get

$$\|u_\varepsilon - (u_\varepsilon^0 + u_\varepsilon^{1/2})\|_\infty \le \|l_\varepsilon^1(\cdot,1-)\|_T,$$

where $\|.\|_T$ is the supremum norm on $C_B(]0,T[)$. Using

$$l_\varepsilon^1(t,1-) = -\frac{1}{\sqrt{4\,\varepsilon\,\pi}} \int_0^t s^{-3/2} \exp\left(-\frac{(1-s)^2}{4\,\varepsilon\, s}\right) r_\varepsilon^0(t-s,0+)\, ds,$$

for $t \in J$, we get the estimate

$$\|l_\varepsilon(.,1-)\|_T = O(\|r_\varepsilon^0(.,0+)\|_T) = O\left(\operatorname{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right) \qquad \text{as } \varepsilon \to 0,$$

and therefore

$$\|u_\varepsilon - (u_\varepsilon^0 + u_\varepsilon^{1/2})\|_\infty = O\left(\operatorname{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right) \qquad \text{as } \varepsilon \to 0,$$

i.e. it may happen that $\|u_\varepsilon - (u_\varepsilon^0 + u_\varepsilon^{1/2})\|_\infty$ is of the same order of magnitude as $\|u_\varepsilon - u_\varepsilon^0\|_\infty$.
This intermediate result is not entirely surprising: the norm $\|u_\varepsilon - u_\varepsilon^0\|_\infty$ is determined by the difference of u_ε and u_ε^0 at the left boundary. This order of magnitude is – due to the specific structure of the solution operator $U[\cdot]$ – also the order of magnitude of l_ε^1 at the *right* boundary. (As $\varepsilon \to 0$ the PDE $\partial_t u_\varepsilon = A_\varepsilon[u_\varepsilon]$ becomes more and more a transport equation "transporting" left boundary values into ω). Hence the difference between u_ε and $u_\varepsilon^{1/2} = u_\varepsilon^0 + l_\varepsilon^1$ is at least of the order of magnitude of $l_\varepsilon^1(.,1-)$, i.e. the order of magnitude of $u_\varepsilon - u_\varepsilon^0$.
These considerations show that there is no chance to get higher order corrections just by adding half–space terms, which compensate the wrong boundary condition at $x = 0$. One has to take into account corrections at the right boundary as well.
Similar to $U[\cdot]$, we introduce for $b \in C_B(J)$ and $(t,y) \in J \times]-\infty, 1[$ the function

$$W_\varepsilon(t,y) = \frac{1-y}{\sqrt{4\,\pi\,\varepsilon}} \int_0^t s^{-3/2} \exp\left(-\frac{(1-y+s)^2}{4\,\varepsilon\, s}\right) b(t-s)\, ds. \tag{2.8}$$

Then $W_\varepsilon = W_\varepsilon[b]$ satisfies

$$\partial_t W_\varepsilon = -\partial_x W_\varepsilon + \varepsilon\, \partial_{xx} W_\varepsilon, \qquad W_\varepsilon(0,y) = 0, \quad W_\varepsilon(t,1) = b(t),$$

such that it is now nearby to define

$$r_\varepsilon^1 := W_\varepsilon[-l_\varepsilon^1(\cdot,1-)]\Big|_{\omega_T}.$$

If we introduce

$$u_\varepsilon^1 := l_\varepsilon^1 + r_\varepsilon^1,$$

the difference $s_\varepsilon^1 := u_\varepsilon - (u_\varepsilon^0 + u_\varepsilon^1)$ satisfies the problem

$$\begin{cases} \partial_t s_\varepsilon^1 = -\partial_x s_\varepsilon^1 + \varepsilon\, \partial_{xx} s_\varepsilon^1, & s_\varepsilon^1(0,x) = 0, \\ s_\varepsilon^1(t,0) = r_\varepsilon^1(t,0+), & s_\varepsilon^1(t,1) = 0, \end{cases}$$

and using the maximum principle we have

$$\|u_\varepsilon - (u_\varepsilon^0 + u_\varepsilon^1)\|_\infty = O\left(\|r_\varepsilon^1(\cdot,0+)\|_T\right) \quad \text{as } \varepsilon \to 0.$$

On the other hand it follows as before that

$$\|r_\varepsilon^1(.,0+)\|_T \leq 2\, \|l_\varepsilon^1(.,1-)\|_T \operatorname{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right),$$

and therefore

$$\|r_\varepsilon^1(.,0+)\|_T = O\left(\left[\operatorname{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right]^2\right) \quad \text{as } \varepsilon \to 0,$$

Hence, we get

$$\|u_\varepsilon - (u_\varepsilon^0 + u_\varepsilon^1)\|_\infty = O\left(\left[\operatorname{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right]^2\right) \quad \text{as } \varepsilon \to 0,$$

i.e. u_ε^1 is a higher order correction term.

Let us repeat the single steps for the construction of u_ε^1 out of r_ε^0:

1) set $l_\varepsilon^1 := \Gamma_\varepsilon^l\left[r_\varepsilon^0\right] := -U_\varepsilon[r_\varepsilon^0(\cdot,0+)]\Big|_{\omega_T}$.

2) set $r_\varepsilon^1 := \Gamma_\varepsilon^r\left[l_\varepsilon^1\right] := -W_\varepsilon[l_\varepsilon^1(\cdot,1-)]\Big|_{\omega_T}$.

3) set $u_\varepsilon^1 := l_\varepsilon^1 + r_\varepsilon^1$.

It is straightforward to deduce from 1)–3) a recursive definition of $l_\varepsilon^k, r_\varepsilon^k$ and u_ε^k and it can be left to the reader to verify that the sequence

$$(u_\varepsilon^0,\ u_\varepsilon^1,\ u_\varepsilon^2,\ \ldots) = (I_\varepsilon + l_\varepsilon^0 + r_\varepsilon^0,\quad l_\varepsilon^1 + r_\varepsilon^1,\quad l_\varepsilon^2 + r_\varepsilon^2,\quad \ldots)$$

actually satisfies E1)–E4). The verification of property E5) however – the crucial point to accept $\left(\sum_{\zeta=0}^{\nu} u_\varepsilon^\zeta\right)_{\nu\in\mathbb{N}_0}$ as asymptotic expansion of u_ε – is a bit technical. The argumentation is therefore outlined in a more general setting in the next section.

3 A Convergence Result for Ping–Pong Expansions

The investigations in this section will be settled on the following assumptions on the singularly perturbed parabolic equation (1.2)

A1) $\varepsilon \in]0, \varepsilon_0[$ with $\varepsilon_0 \in]0, \infty[$.

A2) $a_\varepsilon, b_\varepsilon, c_\varepsilon \in C_B(\omega_T)$ and $f_\varepsilon \in C_B(\omega_T)$.

A3) $u_\varepsilon^I \in C_B^2(\omega)$.

A4) $\alpha_\varepsilon, \beta_\varepsilon \in C_B(J)$.

A5) Problem (1.2) has exactly one C_2–solution u_ε.

Concerning the beginning condition I_ε of E1) we assume

A6) The function $I_\varepsilon \in \mathsf{C}_2$ satisfies $P_\varepsilon[I_\varepsilon](t,x) = f_\varepsilon(t,x)$ for all $(t,x) \in \omega_T$ and

$$\lim_{t\to 0} \int_\omega |I_\varepsilon(t,x) - u_\varepsilon^I(x)|\, dx = 0.$$

Following E2), l_ε^0 and r_ε^0 shall satisfy an IBVP as (1.2); but with vanishing initial data and prescribed boundary conditions only at $x = 0$ *or* $x = 1$, respectively. Due to this observation the functions l_ε^0, r_ε^0 are quite similiar to l_ε^k, r_ε^k with $k \in \mathbb{N}$. In order to keep things simple we shall assume that l_ε^ν, r_ε^ν, $\nu \in \mathbb{N}_0$ are "generated" by the same operators L_ε and R_ε, i.e. we assume

A7) $L_\varepsilon : C_B(J) \to \mathsf{C}_2$ is linear, $\|\cdot\|_T$-$\|\cdot\|_\infty$–bounded,

$$\|L_\varepsilon\|_{T,\infty} := \sup\{\|L_\varepsilon[a]\|_\infty : a \in C_B(J), \|a\|_T \le 1\} < \infty,$$

and for all $a \in C_B(J)$,

A7a) $P_\varepsilon\left[L_\varepsilon[a]\right](t,x) = 0$, for all $(t,x) \in \omega_T$,

A7b) $(L_\varepsilon[a])(\cdot, 0+) = a$.

A7c) $\lim_{t\to 0} \int_\omega |(L_\varepsilon[a])(t,z)|\, dz = 0$.

A8) $R_\varepsilon : C_B(J) \to \mathsf{C}_2$ is linear, $\|\cdot\|_T$–$\|\cdot\|_\infty$–bounded,

$$\|R_\varepsilon\|_{T,\infty} := \sup\{\|R_\varepsilon[b]\|_\infty : b \in C_B(J), \|b\|_T \le 1\} < \infty,$$

and for all $b \in C_B(J)$,

A8a) $P_\varepsilon\left[R_\varepsilon[b]\right](t,x) = 0$, for all $(t,x) \in \omega_T$,

A8b) $(R_\varepsilon[b])(\cdot, 1-) = b$.

A8c) $\lim_{t\to 0} \int_\omega |(R_\varepsilon[b])(t,z)|\, dz = 0$.

In accordance with E3), E4) we set for $v \in \mathsf{C}_2$

$$\Gamma^l_\varepsilon[v] := L_\varepsilon[-v(\cdot,0+)], \quad \Gamma^r_\varepsilon[v] := R_\varepsilon[-v(\cdot,1-)]. \tag{3.1}$$

It is obvious that the operators defined by (3.1) are linear, $\|.\|_\infty$-$\|.\|_\infty$-bounded operators with

$$\|\Gamma^l_\varepsilon\|_{\infty,\infty} := \sup\left\{\|\Gamma^l_\varepsilon[v]\|_\infty : v \in \mathsf{C}_2, \|v\|_\infty \le 1\right\} \le \|L_\varepsilon\|_{T,\infty} \ ,$$

$$\|\Gamma^r_\varepsilon\|_{\infty,\infty} \le \|R_\varepsilon\|_{T,\infty}.$$

It is straightforward to prove

Proposition 1. *Assume A1)–A8) and let* Γ^l_ε, Γ^r_ε *be given by* (3.1). *Moreover, let*

$$l^0_\varepsilon := L_\varepsilon\left[\alpha_\varepsilon - I_\varepsilon(\cdot,0+)\right], \quad r^0_\varepsilon := R_\varepsilon\left[\beta_\varepsilon - I_\varepsilon(\cdot,1-) - l^0_\varepsilon(\cdot,1-)\right], \tag{3.2}$$

and for $k \in \mathbb{N}$

$$l^k_\varepsilon := \Gamma^l_\varepsilon\left[r^{k-1}_\varepsilon\right], \quad r^k_\varepsilon := \Gamma^r_\varepsilon\left[l^k_\varepsilon\right].$$

Then E1)–E4) hold.

Now we are in the position to formulate the main theoretical result (whose proof can be found in Appendix 1):

Theorem 2. *Assume A1)–A8),* Γ^l_ε, Γ^r_ε *given by* (3.1) *and for* $\nu \in \mathbb{N}_0$ *let* $l^\nu_\varepsilon, r^\nu_\varepsilon$ *be as in Proposition 1. Assume furthermore*

B1) There is $K \in]0,\infty[$ *such that for all* $\varepsilon \in]0,\varepsilon_0[$,

$$(1 + \|\Gamma^r_\varepsilon\|_{\infty,\infty}) \times (\|R_\varepsilon\|_{T,\infty}\|\beta_\varepsilon\|_T + \|I_\varepsilon\|_\infty + \|L_\varepsilon\|_{T,\infty}\|\alpha_\varepsilon - I_\varepsilon(\cdot,0+)\|_T) \le K,$$

B2) There is for each $\varepsilon \in]0,\varepsilon_0[$ *a number* $\Theta(\varepsilon) \in]0,1[$ *such that*

$$\lim_{\varepsilon\to 0}\Theta(\varepsilon) = 0,$$

and for all $k \in \mathbb{N}$

$$\left\|\left[\Gamma^l_\varepsilon \circ \Gamma^r_\varepsilon\right]^k\right\|_{\infty,\infty} \le [\Theta(\varepsilon)]^k,$$

B3) There is $K_1 \in]0,\infty[$ *such that for all* $\varepsilon \in]0,\varepsilon_0[$ *and all* $a \in C_B(J)$*: if* $v_\varepsilon = v_\varepsilon[a]$ *is a* C_2*–solution of*

$$P_\varepsilon[v_\varepsilon] = 0, \qquad v_\varepsilon(0,x) = 0, \quad v_\varepsilon(t,0) = a, \quad v_\varepsilon(t,1) = 0,$$

then

$$\int_{\omega_T} |v_\varepsilon(\tau,z)|\, d\tau dz \le K_1\, \|a\|_T.$$

Then we have for all $k \in \mathbb{N}$

$$\|u_\varepsilon^k\|_\infty \le K\,[\Theta(\varepsilon)]^k, \tag{3.3}$$

$$\left\| u_\varepsilon - \sum_{\nu=0}^{k-1} u_\varepsilon^\nu \right\|_\infty \le \frac{K}{1-\Theta(\varepsilon)}\,[\Theta(\varepsilon)]^k, \tag{3.4}$$

i.e. E5) is satisfied with

$$g_0(\varepsilon) := \max\{1, \|u_\varepsilon^0\|_\infty\}, \qquad g_k(\varepsilon) := \frac{K}{1-\Theta(\varepsilon)}\,[\Theta(\varepsilon)]^k, k \in \mathbb{N}.$$

Some remarks will clarify the theorem above:

1) In Theorem 2 no strong maximum principle is assumed. This is in accordance with [GFLRT] whose asymptotic analysis is settled only on a *weak* maximum principle. In Theorem 2 assumptions A1)–A8) allow for a replacement of the maximum principle by the weaker assumption B3).

2) According to (3.4) we have *for each* $\varepsilon \in]0, \varepsilon_0[$

$$\lim_{k\to\infty} \left\| u_\varepsilon - \sum_{\nu=0}^{k} u_\varepsilon^\nu \right\|_\infty = 0,$$

i.e. the serie $(\Sigma_\varepsilon^\nu)_{\nu\in\mathbb{N}_0}$ converges uniformly to u_ε.
Actually, this improves E5), where for fixed $\varepsilon \in]0, \varepsilon_0[$ we only get the estimate

$$\limsup_{k\to\infty} \left\| u_\varepsilon - \sum_{\nu=0}^{k} u_\varepsilon^\nu \right\|_\infty \le \limsup_{k\to\infty} g_k(\varepsilon),$$

with a right–hand side perhaps larger than 0.

3) A close screening of the proof shows that the assumption "$\lim_{\varepsilon\to 0}\Theta[\varepsilon] = 0$" is not essential to get the estimates (3.3) and (3.4):

Theorem 3. *Assume A1)–A8), Γ_ε^l, Γ_ε^r given by (3.1) and for $\nu \in \mathbb{N}_0$ let $l_\varepsilon^\nu, r_\varepsilon^\nu$ be as in Proposition 1. Assume B1) and B3) of Theorem 2 and furthermore*

B) For each $\varepsilon \in]0, \varepsilon_0[$ there is a number $\Theta(\varepsilon) \in]0, 1[$ such that for all $k \in \mathbb{N}$*

$$\left\| \left[\Gamma_\varepsilon^l \circ \Gamma_\varepsilon^r\right]^k \right\|_{\infty,\infty} \le [\Theta(\varepsilon)]^k.$$

Then we have for all $k \in \mathbb{N}$

$$\|u_\varepsilon^k\|_\infty \le K\,[\Theta(\varepsilon)]^k, \tag{3.5}$$

$$\left\| u_\varepsilon - \sum_{\nu=0}^{k-1} u_\varepsilon^\nu \right\|_\infty \le \frac{K}{1-\Theta(\varepsilon)}\,[\Theta(\varepsilon)]^k. \tag{3.6}$$

Theorem 3 ensures *uniform convergence* of the serie $(\sum_{\zeta=0}^{\nu} u_\varepsilon^\zeta)_{\nu\in\mathbb{N}_0}$ to u_ε as long as the estimate of B*) holds, i.e. $(\sum_{\zeta=0}^{\nu} u_\varepsilon^\zeta)_{\nu\in\mathbb{N}_0}$ provides an approximation for u_ε *as long as* ε *is fixed.* Here, three aspects are of interest:

3a) If $\Theta :]0,\varepsilon_0[\to]0,1[$ is *increasing*, then we will get from (3.6) for each $\eta \in]0,\infty[$ a number $k(\eta) \in \mathbb{N}$ – independent of $\varepsilon \in]0,\varepsilon_0[$! – such that

$$\forall \varepsilon \in]0,\varepsilon_0[: \quad \left\| u_\varepsilon - \sum_{\nu=0}^{k(\eta)} u_\varepsilon^\nu \right\|_\infty \leq \eta,$$

i.e. we can choose a fixed "order" of the expansion to achieve a prescribed accuracy of the approximation, independently of $\varepsilon \in]0,\varepsilon_0[$.

3b) If Θ is *not increasing*, then the norm $\left\| u_\varepsilon - \sum_{\nu=0}^{k} u_\varepsilon^\nu \right\|_\infty$ will possibly grow for fixed k. In this case an increasingly number (as $\varepsilon \to 0$) of terms in the expansion of u_ε may be necessary to achieve a certain accuracy of the approximation.

3c) The additional assumption "$\lim_{\varepsilon\to 0}\Theta(\varepsilon) = 0$" ensures that u_ε^0 is for all *sufficiently small* ε an acceptable approximation for u_ε.

4 An Application of Ping–Pong Expansions

In this section we deal with singularly perturbed IBVPs of the form

$$\begin{cases} P_\varepsilon[u_\varepsilon] := \partial_t u_\varepsilon + \partial_x u_\varepsilon - \varepsilon\, \partial_{xx} u_\varepsilon = 0, \quad u_\varepsilon(0,x) = u_\varepsilon^I(x), \\ u_\varepsilon(t,0) = \alpha_\varepsilon(t), \quad u_\varepsilon(t,1) = \beta_\varepsilon(t), \end{cases} \tag{4.1}$$

where we assume that ε, u_ε^I, $\alpha_\varepsilon, \beta_\varepsilon$ satisfy A1), A3) and A4). It can be left to the reader to verify that A5) holds under these assumptions.
We will make use of the fact that $u_\varepsilon^I \in C_B^2(\omega)$ implies that

$$u_\varepsilon^I(0+),\ u_\varepsilon^I(1-),\ (u_\varepsilon^I)'(0+),\ (u_\varepsilon^I)'(1-) \text{ exist.}$$

In order to avoid technical inconveniences we additionally assume that

D1) $(u_\varepsilon^I)''(1-)$ exists.

D2) There is $K_4 \in]0,\infty[$ such that for all $\varepsilon \in]0,\varepsilon_0[$:

$$\|u_\varepsilon^I\|_\infty, \|\alpha_\varepsilon\|_T, \|\beta_\varepsilon\|_T \leq K_4.$$

According to $u_\varepsilon^I \in C_B^2(\omega)$ and due to D1) there is $G_\varepsilon \in C^2(\mathbb{R}^+)$ with $u_\varepsilon^I = G_\varepsilon\big|_\omega$. Hence, we can assume without loss of generality

D3) For all $\varepsilon \in]0,\varepsilon_0[$, $u_\varepsilon^I = G_\varepsilon\big|_\omega$ with $G_\varepsilon \in C^2(\mathbb{R}^+)$ and there are $K_2, K_3 \in]0,\infty[$ such that for all $\varepsilon \in]0,\varepsilon_0[$ and all $y \in \mathbb{R}^+$:

$$\left|\left(G_\varepsilon' - \varepsilon\, G_\varepsilon''\right)(y)\right| \leq K_2\, \left(1 + y^{K_3}\right).$$

Our aim is to construct I_ε, R_ε, L_ε such that the assumptions of Theorem 2 are satisfied. As in Section 2 the argumentation is settled on half-space problems:

Remark 1. *The proofs of the statements given in this section can be found in Appendix 2.*

a) The initial function I_ε.

We set $I_\varepsilon := \Phi_\varepsilon\big|_{\omega_T}$, where $\Phi_\varepsilon \in C^2(J \times \mathbb{R}^+)$ satisfies

$$P_\varepsilon[\Phi_\varepsilon] = 0, \qquad \Phi_\varepsilon(0,y) = G_\varepsilon(y), \quad \Phi_\varepsilon(t,0) = u_\varepsilon^I(0+). \tag{4.2}$$

We like to derive a convolution–type representation for Φ_ε. This can be achieved by introducing for $(t,y) \in J \times \mathbb{R}^+$ the function

$$\phi_\varepsilon(t,y) := \exp\left(-\frac{y}{2\,\varepsilon}\right)\,(\Phi_\varepsilon(t,y) - G_\varepsilon(y)).$$

Then ϕ_ε satisfies

$$\partial_t \phi_\varepsilon + \frac{1}{4\,\varepsilon}\,\phi_\varepsilon = \varepsilon\,\partial_{xx}\phi_\varepsilon + p_\varepsilon, \qquad \phi_\varepsilon(0,y) = 0, \quad \phi_\varepsilon(t,0) = 0, \tag{4.3}$$

where

$$p_\varepsilon(y) := \exp\left(-\frac{y}{2\,\varepsilon}\right)\,\left(\varepsilon\,G_\varepsilon''(y) - G_\varepsilon'(y)\right). \tag{4.4}$$

We introduce (formally) the Sine–Fourier transformations of ϕ_ε and p_ε by

$$\mathsf{S}[\phi_\varepsilon](t,\xi) := V_\varepsilon(t,\xi) := \int_0^\infty \phi_\varepsilon(t,y)\,\sin(\xi\,y)\,dy,$$
$$\mathsf{S}[p_\varepsilon](\xi) := \int_0^\infty p_\varepsilon(y)\,\sin(\xi\,y)\,dy,$$

where $\xi \in \mathbb{R}^+$, such that (4.3) formally becomes

$$\partial_t V_\varepsilon + \left(\frac{1}{4\,\varepsilon} + \varepsilon\,\xi^2\right)\,V_\varepsilon = \mathsf{S}[p_\varepsilon], \qquad V_\varepsilon(0,\xi) = 0. \tag{4.5}$$

The solution of (4.5) is given by

$$V_\varepsilon(t,\xi) = \frac{\mathsf{S}[p_\varepsilon](\xi)}{\frac{1}{4\,\varepsilon} + \varepsilon\,\xi^2}\,\left(1 - \exp\left(-\frac{t}{4\,\varepsilon} - \varepsilon\,t\,\xi^2\right)\right),$$

such that we get applying again the Sine-Fourier transformation [Obe]

$$\phi_\varepsilon(t,y) = \frac{2}{\pi\,\varepsilon}\,\mathsf{S}_{\xi\to y}\left[\mathsf{S}[p_\varepsilon]\,\frac{1}{\frac{1}{4\,\varepsilon^2} + \xi^2}\right](t,y)$$
$$- \frac{2}{\pi\,\varepsilon}\,\mathsf{S}_{\xi\to y}\left[\mathsf{S}[p_\varepsilon]\,\frac{\exp\left(-\xi^2\,\varepsilon\,t\right)}{\frac{1}{4\,\varepsilon^2} + \xi^2}\right](t,y) \tag{4.6}$$

for all $(t,y) \in J \times \mathbb{R}^+$, where naturally

$$\mathsf{S}_{\xi\to y}[h(t,\xi)](t,y) = \int_0^\infty h(t,\xi)\,\sin(y\,\xi)\,d\xi.$$

The arguments of the $\mathsf{S}_{\xi\to y}$-operators of (4.6) are *products* of functions. It can be expected that $\mathsf{S}_{\xi\to y}$ transforms these products into *convolutions* of transformed functions. On behalf of well–known properties of the Fourier transformation (see, e.g. [Rud]) it is not very difficult to verify

Proposition 4. *Let $F, G \in L^2(\mathbb{R})$. Assume that*

F is odd and G is even.

Then (in sense of $L^2(\mathbb{R})$),

$$\mathsf{S}_{\xi\to y}\left[\mathsf{S}[F]\,g\right](y) = \frac{1}{2}\left(F \star \mathsf{C}^{even}[G]\right)(y) = \frac{1}{2}\int_{-\infty}^{\infty} F(y-\xi)\,\mathsf{C}^{even}[G](\xi)\,d\xi,$$

where $\mathsf{C}^{even}[G]$ the even extension (to $\mathbb{R}$) of the Cosine–Fourier transform $\mathsf{C}[G]$ of G:

$$\forall \xi \in \mathbb{R}^+ : \qquad \mathsf{C}[G](\xi) = \int_0^\infty G(y)\,\cos(\xi\,y)\,dy.$$

From (4.6) we get with the aid of some well–known Cosine-Fourier transformations [Obe] for all $(t,y) \in J \times \mathbb{R}^+$

$$\phi_\varepsilon(t,y) = [p_\varepsilon^{odd} \star k_\varepsilon](t,y), \tag{4.7}$$

where p_ε^{odd} is the odd extension of p_ε to $\mathbb{R}$ and for $(t,z) \in J \times \mathbb{R}$,

$$\begin{aligned} k_\varepsilon(t,z) &= \exp\left(-\frac{|z|}{2\,\varepsilon}\right) \\ &+ \frac{1}{2}\left(\exp\left(-\frac{z}{2\,\varepsilon}\right)\,\mathrm{erfc}\left(\frac{t-z}{2\sqrt{\varepsilon\,t}}\right) + \exp\left(\frac{z}{2\,\varepsilon}\right)\,\mathrm{erfc}\left(\frac{t+z}{2\sqrt{\varepsilon\,t}}\right)\right). \end{aligned} \tag{4.8}$$

Equation (4.7) involves the odd extension of p_ε to $\mathbb{R}$. It is however sometimes desireable to have a representation of ϕ_ε just in terms of p_ε. To get such a formula for ϕ_ε we can make use of a proposition, which is again straightforward to prove:

Proposition 5. *Let $F, G \in L^1(\mathbb{R})$. Assume that*

F is odd and G is even.

Then for all $y \in \mathbb{R}^+$,

$$\begin{aligned} (F \star G)(y) &= \int_{-\infty}^{\infty} F(y-z)\,G(z)\,dz \\ &\int_0^y F(y-z)\,G(z)\,dz + \int_0^\infty F(y+z)\,G(z)\,dz - \int_y^\infty F(z-y)\,G(z)\,dz. \end{aligned}$$

Now we are in the position to formulate our main result concerning I_ε:

Lemma 6. *Assume A1), A3) and D1)–D3). For* $(t,x) \in \omega_T$ *let*

$$I_\varepsilon(t,x) := \varepsilon \left(\left(u_\varepsilon^I\right)'(x) - \left(u_\varepsilon^I\right)'(0) \right) + u_\varepsilon^I(0+)$$
$$+ \int_0^\infty \exp\left(-\frac{y}{\varepsilon}\right) \left[H_\varepsilon(x+y) - H_\varepsilon(y)\right] dy$$
$$+ \frac{1}{2} \int_0^x H_\varepsilon(x-y) \left[\operatorname{erfc}\left(\frac{t-y}{2\sqrt{\varepsilon t}}\right) + \exp\left(\frac{y}{2\varepsilon}\right) \operatorname{erfc}\left(\frac{t+y}{2\sqrt{\varepsilon t}}\right)\right] dy$$
$$+ \frac{1}{2} \int_0^\infty H_\varepsilon(x+y) \left[\exp\left(-\frac{y}{\varepsilon}\right) \operatorname{erfc}\left(\frac{t-y}{2\sqrt{\varepsilon t}}\right) + \operatorname{erfc}\left(\frac{t+y}{2\sqrt{\varepsilon t}}\right)\right] dy$$
$$- \frac{1}{2} \int_0^\infty H_\varepsilon(y) \exp\left(-\frac{y}{\varepsilon}\right) \operatorname{erfc}\left(\frac{t-x-y}{2\sqrt{\varepsilon t}}\right) dy$$
$$- \frac{1}{2} \int_0^\infty H_\varepsilon(y) \exp\left(\frac{x}{\varepsilon}\right) \operatorname{erfc}\left(\frac{t+x+y}{2\sqrt{\varepsilon t}}\right) dy,$$

where for $y \in \mathbb{R}^+$

$$H_\varepsilon(y) := \varepsilon\, G_\varepsilon''(y) - G_\varepsilon'(y).$$

Then

1) I_ε *satisfies A6) (with* $f_\varepsilon = 0$*).*

2) $I_\varepsilon(.,0+) = u_\varepsilon^I(0+)$.

3) *There is* $K_5 = K_5(K_2, K_3, K_4, \varepsilon_0) \in]0,\infty[$ *such that for all* $\varepsilon \in]0, \varepsilon_0[$:

$$\|I_\varepsilon\|_\infty \le K_5.$$

Remark 2.

1) *In applications one may wish to use* (4.6) *to compute* ϕ_ε *and pass to* $I_\varepsilon(t,x) = \exp\left(\frac{x}{2\varepsilon}\right) \phi_\varepsilon(t,x) + G_\varepsilon(y)$ *afterwards. This strategy is of advantage whenever the Sine-Fourier transforms of* (4.6) *can explicitly be calculated (which is the case, e.g., for* $G_\varepsilon(x) = u_\varepsilon^I(x) = 1 - x$ *of the introductionary example in Section 2).*

2) *The estimate on* $\|I_\varepsilon\|_\infty$ does not depend on T.

b) The operator L_ε.

The purpose of L_ε is to map left-boundary data a into C_2 such that $P_\varepsilon[L_\varepsilon[a]] = 0$ and $L_\varepsilon[a](0,\cdot) = 0$. The operator L_ε is constructed as in Section 2 as restriction of a half-space operator U_ε. The construction of U_ε can be performed as in **a)**, such that one gets for $\varepsilon \in]0, \varepsilon_0[$, $a \in C_B(J)$ and $(t,y) \in J \times \mathbb{R}^+$,

$$U_\varepsilon[a](t,y) := \frac{y}{\sqrt{4\,\varepsilon\,\pi}} \int_0^t s^{-3/2} \exp\left(-\frac{(y-s)^2}{4\,\varepsilon\,s}\right) a(t-s)\, ds. \tag{4.9}$$

Then we have

Lemma 7. *For* $\varepsilon \in]0, \varepsilon_0[$ *and* $a \in C_B(J)$ *let*

$$L_\varepsilon[a] := U_\varepsilon[a]\big|_{\omega_T}.$$

Then for all $\varepsilon \in]0, \varepsilon_0[$*:*

1) $L_\varepsilon : C_B(J) \to \mathsf{C}_2,\ a \mapsto L_\varepsilon[a]$, *is linear.*

2) For all $a \in C_B(J)$ *and all* $(t,x) \in \omega_T$*:*

$$\begin{aligned} &L_\varepsilon[a](t,x) \\ &\quad = \frac{1}{\sqrt{\pi}} \int_{\frac{x-t}{\sqrt{4\,t\,\varepsilon}}}^{\infty} \mathrm{e}^{-s^2}\, a\left(t - \frac{x^2}{\varepsilon\, s^2 + x + 2\sqrt{\varepsilon}\, s\, \sqrt{\varepsilon\, s^2 + x}}\right) \\ &\qquad \times \left(1 + \frac{\sqrt{\varepsilon}\, s}{\sqrt{\varepsilon\, s^2 + x}}\right) ds. \end{aligned} \tag{4.10}$$

3) L_ε *satisfies A7).*

4) $\|L_\varepsilon\|_{T,\infty} \le 2$.

c) The operator R_ε.

The purpose of R_ε is to map right–boundary data b into C_2 such that $P_\varepsilon[R_\varepsilon[b]] = 0$ and $R_\varepsilon[b](0,\cdot) = 0$. The operator R_ε is constructed as in Section 2 as restriction of a half-space operator W_ε. The construction of W_ε can be performed as in **a)** such that one gets for $\varepsilon \in]0, \varepsilon_0[$, $b \in C_B(J)$ and $(t,y) \in J \times]-\infty, 1[$,

$$W_\varepsilon[b](t,y) = \frac{1-y}{\sqrt{4\,\pi\,\varepsilon}} \int_0^t s^{-3/2} \exp\left(-\frac{(1-y+s)^2}{4\,\varepsilon\, s}\right) b(t-s)\, ds. \tag{4.11}$$

Then one gets

Lemma 8. *For* $\varepsilon \in]0, \varepsilon_0[$ *and* $b \in C_B(J)$ *let*

$$R_\varepsilon[b] := W_\varepsilon[b]\big|_{\omega_T}.$$

Then for all $\varepsilon \in]0, \varepsilon_0[$*:*

1. $R_\varepsilon : C_B(J) \to \mathsf{C}_2,\ b \mapsto R_\varepsilon[b]$, *is linear.*

2. For all $b \in C_B(J)$ *and all* $x \in \overline{\omega} = [0,1]$*:*

$$\|R_\varepsilon[b](.,x)\|_T \le 2\, \|b\|_T \operatorname{erfc}\left(\sqrt{\frac{1-x}{4\,\varepsilon}}\right). \tag{4.12}$$

3. R_ε *satisfies A.8.*

4. $\|R_\varepsilon\|_{T,\infty} \le 1$.

d) Ping–pong asymptotics for (4.1).

We wish to define a ping–pong serie as in Section 3 for (4.1). This is done in several steps. We always assume that A1), A3), A4) and D1)–D3) hold. The operators $\Gamma_\varepsilon^{r,l}$ are introduced as in (3.1), i.e. $\Gamma_\varepsilon^{r,l} : \mathsf{C}_2 \to \mathsf{C}_2$ with

$$\Gamma_\varepsilon^l[v] := L_\varepsilon[-v(\cdot, 0+)], \quad \Gamma_\varepsilon^r[v] := R_\varepsilon[-v(\cdot, 1-)]. \tag{4.13}$$

The ping–pong asymptotics will work because of the following essential Lemma:

Lemma 9. *For all* $\varepsilon \in \mathbb{R}^+$*:*

$$\|\Gamma_\varepsilon^l \circ \Gamma_\varepsilon^r\|_{\infty,\infty} \le 4\,\mathrm{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right).$$

Now we can prove

Theorem 10. *For* $\varepsilon \in]0, \varepsilon_0[$ *let* I_ε *as in Lemma 6,* L_ε *as in Lemma 7 and* R_ε *as in Lemma 8. For* $\varepsilon \in]0, \varepsilon_0[$ *and* $\nu \in \mathbb{N}_0$ *let* l_ε^ν *and* r_ε^ν *be given by* (3.2) *with* Γ_ε^l*,* Γ_ε^r *as in* (4.13).
Furthermore, let $\varepsilon_\star \in]0, \infty[$ *such that*

$$4\,\mathrm{erfc}\left(\frac{1}{2\sqrt{\varepsilon_\star}}\right) = 1, \quad i.e. \quad \varepsilon_\star = 0.3778422150\ldots,$$

Then, for all $\varepsilon \in]0, \varepsilon_*[$ *the sequence*

$$(u_\varepsilon^0,\ u_\varepsilon^1,\ u_\varepsilon^2,\ \ldots) = (I_\varepsilon + l_\varepsilon^0 + r_\varepsilon^0,\quad l_\varepsilon^1 + r_\varepsilon^1,\quad l_\varepsilon^2 + r_\varepsilon^2,\quad \ldots)$$

satisfies E1)–E4),

$$\left\|u_\varepsilon^0\right\|_\infty \le 3\,(K_4 + K_5),$$

where K_4 *is as in D2),* K_5 *as in Lemma 6, and for all* $k \in \mathbb{N}$ *we have*

$$\left\|u_\varepsilon^k\right\|_\infty \le 6\,(K_4 + K_5)\,\left(4\,\mathrm{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right)^k,$$

$$\left\|u_\varepsilon - \sum_{\nu=0}^{k-1} u_\varepsilon^\nu\right\|_\infty \le \frac{6\,(K_4 + K_5)}{1 - 4\,\mathrm{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right)}\,\left(4\,\mathrm{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right)\right)^k,$$

Moreover, the ping–pong serie $\left(\sum_{\zeta=0}^{\nu} u_\varepsilon^\zeta\right)_{\nu \in \mathbb{N}_0}$ *converges uniformly to* u_ε *as* $\nu \to \infty$*:*

$$\lim_{\nu\to\infty} \left\|u_\varepsilon - \sum_{\zeta=0}^{\nu} u_\varepsilon^\zeta\right\|_\infty = 0.$$

Remark 3. *The ping–pong serie for* u_ε *converges for all* $\varepsilon \in]0, \varepsilon_*[$ *with* ε_* *as given above* – independently *of the choices of* u_ε^I*,* α_ε*,* β_ε *(as long as A3), A4), D1)–D3) are satisfied). The norms of these functions determine to some extent the rate of convergence of the ping–pong serie, but not whether the ping–pong series converge at all.*

e) Discussion

In accordance with the discussion of Section 2 one might expect that $I_\varepsilon + l_\varepsilon^0$ is away from $x = 1$ for all sufficiently small ε an excellent approximation for u_ε. This is indeed the case, because it readily follows from Lemma 8 for all $\theta \in]0,1[$ that

$$\forall t \in J, \forall x \in]0,\theta[: \quad \left|r_\varepsilon^0\right|(t,x) \le 6\,(K_4 + K_5)\,\mathrm{erfc}\left(\frac{\sqrt{1-\theta}}{2\sqrt{\varepsilon}}\right),$$

such that due to Theorem 10

$$\sup_{(t,x)\in J\times]0,\theta[} \left|u_\varepsilon - (I_\varepsilon + l_\varepsilon^0)\right|(t,x) \le 6\,(K_4+K_5)\,\frac{2-C}{1-C}\,C$$

$$C = \mathrm{erfc}\left(\frac{\sqrt{1-\theta}}{2\sqrt{\varepsilon}}\right)$$

i.e. $I_\varepsilon + l_\varepsilon^0 \to u_\varepsilon$ uniformly on $J\times]0,\theta[$ as $\varepsilon \to 0$.

Now let us discuss the behaviour of $I_\varepsilon + l_\varepsilon^0$ as $\varepsilon \to 0$. Here the initial and boundary data will play a prominent role. In order to keep things simple let us assume that $u_\varepsilon^I = u^I$ and $\alpha_\varepsilon = \alpha \in C([0,T])$ are ε–independent. Then it is easy to deduce from lemma 6 and lemma 7 with the aid of Lebesgue's dominated convergence theorem

$$\forall (t,x) \in \omega_T: \quad \text{If } x - t \neq 0, \text{ then } \quad \lim_{\varepsilon\to 0}(I_\varepsilon + l_\varepsilon^0)(t,x) = u_0(t,x),$$

where

$$\begin{aligned} u_0 \ : \ \omega_T &\to \mathbb{R} \\ (t,x) &\mapsto \begin{cases} u^I(x-t) &, \ x-t>0 \\ \dfrac{\alpha(0)}{2} + u^I(0) &, \ x=t \\ \alpha(t-x) + u^I(0) &, \ x-t<0 \end{cases} \end{aligned}$$

is a weak solution of the transport equation

$$\partial_t u_0 + \partial_x u_0 = 0, \qquad u_0(0,x) = u^I(x), \quad u_0(t,0) = \alpha(t).$$

A close screening of the estimates actually gives a more detailled result:

$$I_\varepsilon + l_\varepsilon^0 \to u_0 \text{ uniformly on each compact } \mathbf{K} \subset\subset \{(t,x) \in \omega_T : x - t \neq 0\}\,.$$

Furthermore, since $I_\varepsilon + l_\varepsilon^0$ is uniformly (i.e. independent of $\varepsilon \in]0,\varepsilon_\star[$) bounded on ω_T, we deduce from the pointwise convergence almost everywhere

$$\forall p \in [1,\infty[: \quad \lim_{\varepsilon\to 0}\int_{\omega_T} \left|u_0(s,z) - \left(I_\varepsilon + l_\varepsilon^0\right)(s,z)\right|^p \,ds\,dz = 0.$$

Uniform convergence on ω_T of $I_\varepsilon + l_\varepsilon^0$ to u_0 is usually not available because the limiting function u_0 is continuous iff the additional assumption $\alpha(0) = 0$ holds.

5 Conclusion

In the previous sections we derived a recursive approximation for singularly perturbed parabolic equations of the form

$$\partial_t u_\varepsilon = a_\varepsilon u_\varepsilon + b_\varepsilon \partial_x u_\varepsilon + c_\varepsilon \partial_{xx} u_\varepsilon, \quad u_\varepsilon(0,x) = u_\varepsilon^I(x)$$

for $x \in]0,1[$ with time–dependent boundary conditions at $x = 0$ and $x = 1$, repectively. The approximation is derived from sucessive solutions of related half–space problems, where the intermediate boundary conditions at $x = 0$ and $x = 1$ are alternately shifted to infinity. This motivates to call the recursive approximation a "ping–pong" expansion.
We gave a detailed convergence analysis of the new asymptotic method and applied the method to a certain model problem. As mentioned in the introduction, our present investigations originated from an asymptotic analysis of a singularly perturbed two–phase Stefan problem and the application of ping–pong expansions to this problem is currently under investigation.

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A Appendix 1

Proof (of Theorem 2): We observe that

$$r_\varepsilon^0 = \Gamma_\varepsilon^r[\rho_\varepsilon], \qquad \rho_\varepsilon := -R_\varepsilon[\beta_\varepsilon] + I_\varepsilon + l_\varepsilon^0.$$

Hence by (1.3) we have for all $k \in \mathbb{N}$,

$$l_\varepsilon^k = \left(\Gamma_\varepsilon^l \circ \Gamma_\varepsilon^r\right)^k [\rho_\varepsilon], \qquad r_\varepsilon^k = \Gamma_\varepsilon^r \circ \left[\Gamma_\varepsilon^l \circ \Gamma_\varepsilon^r\right]^k [\rho_\varepsilon],$$

such that we get

$$u_\varepsilon^k = l_\varepsilon^k + r_\varepsilon^k = (\mathsf{id} + \Gamma_\varepsilon^r) \circ \left[\Gamma_\varepsilon^l \circ \Gamma_\varepsilon^r\right]^k [\rho_\varepsilon].$$

It follows that

$$\|u_\varepsilon^k\|_\infty \leq \|\mathsf{id} + \Gamma_\varepsilon^r\|_{\infty,\infty} \left\| \left[\Gamma_\varepsilon^l \circ \Gamma_\varepsilon^r\right]^k \right\|_{\infty,\infty} \|\rho_\varepsilon\|_\infty,$$

and due to B2)

$$\|u_\varepsilon^k\|_\infty \leq \left(1 + \|\Gamma_\varepsilon^r\|_{\infty,\infty}\right) [\Theta(\varepsilon)]^k \, \|\rho_\varepsilon\|_\infty. \tag{A.1}$$

Now we like to estimate $\|\rho_\varepsilon\|_\infty$: from

$$\rho_\varepsilon = -R_\varepsilon[\beta_\varepsilon] + I_\varepsilon + l_\varepsilon^0 = -R_\varepsilon[\beta_\varepsilon] + I_\varepsilon + L_\varepsilon[\alpha_\varepsilon - I_\varepsilon(\cdot, 0+)],$$

we get

$$\|\rho_\varepsilon\|_\infty \leq \|R_\varepsilon\|_{T,\infty} \|\beta_\varepsilon\|_T + \|I_\varepsilon\|_\infty + \|L_\varepsilon\|_{T,\infty} \|\alpha_\varepsilon - I_\varepsilon(\cdot, 0+)\|_T,$$

and due to B1) and (A.1) one has

$$\|u_\varepsilon^k\|_\infty \leq K \, [\Theta(\varepsilon)]^k,$$

which gives estimate (3.3).
Moreover, according to B2), we have $\Theta[\varepsilon] < 1$. Hence the serie $(\Sigma_\varepsilon^\nu)_{\nu \in \mathbb{N}_0}$ – with $\Sigma_\varepsilon^\nu = u_\varepsilon^0 + \ldots + u_\varepsilon^\nu$, $\nu \in \mathbb{N}_0$ – converges in $C_B(\omega_T)$ to – let's say – w_ε. From (3.3) we have for all $k \in \mathbb{N}$ the estimate

$$\left\| w_\varepsilon - \Sigma_\varepsilon^{k-1} \right\|_\infty = \left\| w_\varepsilon - \sum_{\nu=0}^{k-1} u_\varepsilon^\nu \right\|_\infty \leq \sum_{\nu=k}^{\infty} \|u_\varepsilon^\nu\|_\infty \leq \frac{K}{1 - \Theta(\varepsilon)} [\Theta(\varepsilon)]^k.$$

Hence, in order to prove (3.4) it suffices to show that $u_\varepsilon = w_\varepsilon$.
Let us introduce the function $s_\varepsilon^k := u_\varepsilon - \Sigma_\varepsilon^k$, which is a C_2–solution of

$$P_\varepsilon[s_\varepsilon^k] = 0, \qquad s_\varepsilon^k(0, x) = 0, \quad s_\varepsilon^k(t, 0) = -r_\varepsilon^k(t, 0+), \quad s_\varepsilon^k(t, 1) = 0.$$

Then, due to B3), we have the estimate

$$\int_{\omega_T} |s_\varepsilon^k(\tau, z)| \, d\tau dz \leq K_1 \, \|r_\varepsilon^k\|_T. \tag{A.2}$$

On the other hand

$$r_\varepsilon^k = \Gamma_\varepsilon^r \circ \left[\Gamma_\varepsilon^l \circ \Gamma_\varepsilon^r\right]^k [\rho_\varepsilon],$$

for all $k \in \mathbb{N}$ and therefore, due to B1) and previous estimates, we get

$$\|r_\varepsilon^k\|_T \le K\,[\Theta(\varepsilon)]^k,$$

such that we deduce from (A.2)

$$\int_{\omega_T} |s_\varepsilon^k(\tau, z)|\; d\tau dz \le K\; K_1\; [\Theta(\varepsilon)]^k \to 0 \quad \text{as } k \to \infty,$$

which immediately gives $w_\varepsilon(t,x) = u_\varepsilon(t,x)$ for almost all $(t,x) \in \omega_T$. Since w_ε and u_ε are continuous, it follows that $w_\varepsilon = u_\varepsilon$. □

B Appendix 2

Proof (of Lemma 6): It is easy to deduce from (4.7) that ϕ_ε satisfies

$$\partial_t \phi_\varepsilon(t_0, y_0) + \frac{1}{4\,\varepsilon}\,\phi_\varepsilon(t_0, y_0) = \varepsilon\,\partial_{xx}\phi_\varepsilon(t_0, y_0) + p_\varepsilon(t_0, y_0),$$

for all $(t_0, y_0) \in J \times \mathbb{R}^+$ with

$$\phi(t, 0+) = 0, \quad \text{and} \quad \lim_{t \to 0} \int_\omega |\phi_\varepsilon(t, z)|\; dz = 0.$$

From these properties of ϕ_ε and due to $I_\varepsilon = \Phi_\varepsilon\big|_{\omega_T}$ with

$$\Phi_\varepsilon(t, y) = \exp\left(-\frac{y}{2\,\varepsilon}\right)\,\phi_\varepsilon(t, y) + G_\varepsilon(y)$$

for $(t, y) \in J \times \mathbb{R}^+$ one readily obtains 1) and 2)
The function I_ε is of the form

$$I_\varepsilon(t,x) = E_\varepsilon^1(x) + E_\varepsilon^2(x) + E_\varepsilon^3(t,x) + E_\varepsilon^4(t,x) - E_\varepsilon^5(t,x) - E_\varepsilon^6(t,x),$$

where $(t,x) \in \omega_T$ and we estimate term-by-term: first,

$$E_\varepsilon^1(x) = u_\varepsilon^I(x) + \int_0^x H_\varepsilon(\xi)\; d\xi.$$

According to D3), D4)

$$|E_\varepsilon^1(x)| \le K_5^1 := K_4 + K_2\left(1 + \frac{1}{1 + K_3}\right),$$

where K_5^1 only depends on K_2, K_3, K_4. Concerning $E_\varepsilon^2(x)$ we have

$$|E_\varepsilon^2(x)| \le 2\,K_2 \int_0^1 \exp\left(-\frac{y}{\varepsilon_0}\right)\,(1 + y^{K_3})\; dy \le 4\,K_2 =: K_5^2,$$

where K_5^2 only depends on K_2. We proceed with an observation whose proof can be left to the reader:

$$\exists \kappa \in \mathbb{R}^+ : \quad \forall \gamma, \delta \in \mathbb{R}^+ : \quad \mathrm{e}^{\delta} \operatorname{erfc}\left(\frac{1}{2\gamma} + \gamma\,\delta\right) \leq \kappa.$$

Hence

$$|E_\varepsilon^3(t,x)| \leq \frac{K_2}{2} \int_0^1 \left(1 + y^{K_3}\right) [2 + \kappa]\; dy \leq (2+\kappa)\, K_2 =: K_5^3,$$

where K_5^3 only depends on K_2. The first term of $E_4(t,x)$ can be estimated as

$$\begin{aligned}
\frac{1}{2}\left| \int_0^\infty H_\varepsilon(x+y) \exp\left(-\frac{y}{\varepsilon}\right) \operatorname{erfc}\left(\frac{t-y}{2\sqrt{\varepsilon\, t}}\right)\, dy \right| \\
\leq K_2 \int_0^\infty \left(1 + (x+y)^{K_3}\right) \exp\left(-\frac{y}{\varepsilon_0}\right)\, dy \\
\leq K_2 \int_0^\infty \left(1 + (1+y)^{K_3}\right) \exp\left(-\frac{y}{\varepsilon_0}\right)\, dy =: K_5^4,
\end{aligned}$$

where K_5^4 only depends on K_2, K_3, ε_0. In order to estimate the second term of $E_4(t,x)$ we need an auxiliary result whose proof again can be left to the reader:

$$\forall \gamma \in \mathbb{R}^+ : \quad \mathrm{e}^{\gamma^2} \operatorname{erfc}(\gamma) \leq \min\left\{1, \frac{1}{\sqrt{\pi}\,\gamma}\right\}. \tag{B.1}$$

We observe

$$\begin{aligned}
\frac{1}{2}\left| \int_0^\infty H_\varepsilon(x+y) \operatorname{erfc}\left(\frac{t+y}{2\sqrt{\varepsilon\, t}}\right)\, dy \right| \\
\leq \frac{K_2}{2} \int_0^\infty \left(1 + (1+y)^{K_3}\right) \operatorname{erfc}\left(\frac{t+y}{2\sqrt{\varepsilon_0\, t}}\right)\, dy.
\end{aligned}$$

and proceed by a case-distinction. If $4\,\varepsilon_0^2 < \pi\, t$, then

$$\forall y \in \mathbb{R}^+ : \quad \frac{t+y}{2\sqrt{\varepsilon_0\, t}} > \frac{1}{\sqrt{\pi}},$$

and due to (B.1),

$$\operatorname{erfc}\left(\frac{t+y}{2\sqrt{\varepsilon_0\, t}}\right) \leq \frac{1}{\sqrt{\pi}} \frac{2\sqrt{\varepsilon_0\, t}}{t+y} \exp\left(-\left(\frac{t+y}{2\sqrt{\varepsilon_0\, t}}\right)^2\right) \leq \exp\left(-\frac{y}{2\,\varepsilon_0}\right),$$

such that we get in this case

$$\begin{aligned}
\frac{1}{2}\left| \int_0^\infty H_\varepsilon(x+y) \operatorname{erfc}\left(\frac{t+y}{2\sqrt{\varepsilon\, t}}\right)\, dy \right| \\
\leq \frac{K_2}{2} \int_0^\infty \left(1 + (1+y)^{K_3}\right) \exp\left(-\frac{y}{2\,\varepsilon_0}\right)\, dy =: K_5^5,
\end{aligned}$$

where we note that K_5^5 only depends on K_2, K_3, ε_0. If $4\,\varepsilon_0^2 \geq \pi\, t$, then

$$\frac{1}{2\,\sqrt{\varepsilon_0\, t}} \geq \frac{\sqrt{\pi}}{4\,\varepsilon_0^{3/2}},$$

and therefore due to (B.1)

$$\operatorname{erfc}\left(\frac{t+y}{2\,\sqrt{\varepsilon_0\, t}}\right) \leq \operatorname{erfc}\left(\frac{y}{2\,\sqrt{\varepsilon_0\, t}}\right) \leq \operatorname{erfc}\left(\frac{y\,\sqrt{\pi}}{4\,\varepsilon_0^{3/2}}\right) \leq \exp\left(-\frac{y^2\,\pi}{16\,\varepsilon_0^3}\right),$$

such that in this case

$$\frac{1}{2}\left|\int_0^\infty H_\varepsilon(x+y)\operatorname{erfc}\left(\frac{t+y}{2\,\sqrt{\varepsilon\, t}}\right)\,dy\right| \leq \frac{K_2}{2}\int_0^\infty \left(1+(1+y)^{K_3}\right)\,\exp\left(-\frac{y^2\,\pi}{16\,\varepsilon_0^3}\right)\,dy =: K_5^6,$$

where we note that K_5^6 only depends on $K_2, K_3.\varepsilon_0$. Summarizing the discussion we have in any case

$$\frac{1}{2}\left|\int_0^\infty H_\varepsilon(x+y)\operatorname{erfc}\left(\frac{t+y}{2\,\sqrt{\varepsilon\, t}}\right)\,dy\right| \leq K_5^5 + K_5^6,$$

and thus

$$|E_\varepsilon^4(t,x)| \leq K_5^4 + K_5^5 + K_5^6,$$

where K_5^4, K_5^5 and K_5^6 only depend on K_2, K_3, ε_0. E_ε^5 can easily be estimated as

$$|E_\varepsilon^5(t,x)| \leq \frac{1}{2}\int_0^\infty |H_\varepsilon(y)|\,\exp\left(-\frac{y}{\varepsilon}\right)\left|\operatorname{erfc}\left(\frac{t-x-y}{2\,\sqrt{\varepsilon\, t}}\right)\right|\,dy \leq K_2\int_0^\infty \left(1+y^{K_3}\right)\,\exp\left(-\frac{y}{\varepsilon_0}\right)\,dy =: K_5^7,$$

where K_5^7 only depends on K_2, K_3, ε_0. It remains to look at

$$E_\varepsilon^6(t,x) = \frac{1}{2}\int_0^\infty H_\varepsilon(y)\,\exp\left(\frac{x}{\varepsilon}\right)\,\operatorname{erfc}\left(\frac{t+x+y}{2\,\sqrt{\varepsilon\, t}}\right)\,dy.$$

From the estimate

$$\forall \gamma,\delta,u \in \mathbb{R}^+: \quad \exp(u)\,\operatorname{erfc}(\gamma+\delta\, u) \leq \exp\left(-\frac{\gamma\,\delta-\frac{1}{4}}{\delta^2}\right).$$

we have

$$\exp\left(\frac{x}{\varepsilon}\right)\,\operatorname{erfc}\left(\frac{t+x+y}{2\,\sqrt{\varepsilon\, t}}\right) \leq \exp\left(-\frac{y}{\varepsilon}\right) \leq \exp\left(-\frac{y}{\varepsilon_0}\right),$$

such that

$$|E^6_\varepsilon(t,x)| \le \frac{K_2}{2}\int_0^\infty (1+y^{K_3})\,\exp\left(-\frac{y}{\varepsilon_0}\right)\,dy =: K_5^8,$$

where the constant K_5^8 only depends on K_2, K_3, ε_0. □

Proof (of Lemma 7): The verification of 1)–3) can be left to the reader. Concerning 4) we have for all $a \in C_B(J)$ with $\|a\|_T \le 1$ and for all $(t,x) \in \omega_T$ the estimate

$$\begin{aligned}
&|L_\varepsilon[a](t,x)| \\
&= \left|\frac{1}{\sqrt{\pi}}\int_{\frac{x-t}{2\sqrt{t\varepsilon}}}^\infty a\left(t - \frac{x^2}{\varepsilon s^2 + x + \sqrt{\varepsilon}\,s\,\sqrt{4\varepsilon s^2+4x}}\right)\,\mathrm{e}^{-s^2}\,\left(1+\frac{\sqrt{\varepsilon}\,s}{\sqrt{\varepsilon s^2+x}}\right)\,ds\right| \\
&\le \frac{1}{\sqrt{\pi}}\int_{\frac{x-t}{2\sqrt{t\varepsilon}}}^\infty \left|a\left(t - \frac{x^2}{\varepsilon s^2 + x + \sqrt{\varepsilon}\,s\,\sqrt{4\varepsilon s^2+4x}}\right)\right|\,\mathrm{e}^{-s^2}\,\left(1+\frac{\sqrt{\varepsilon}\,s}{\sqrt{\varepsilon s^2+x}}\right)\,ds \\
&\le \frac{1}{\sqrt{\pi}}\int_{\frac{x-t}{2\sqrt{t\varepsilon}}}^\infty \mathrm{e}^{-s^2}\,\left(1+\frac{\sqrt{\varepsilon}\,s}{\sqrt{\varepsilon s^2+x}}\right)\,ds \le \frac{2}{\sqrt{\pi}}\int_{-\infty}^\infty \mathrm{e}^{-s^2}\,ds = 2.
\end{aligned}$$

□

Proof (of Lemma 8): The verification of 1), 3) can be left to the reader. Concerning). we have for all $b \in C_B(J)$, $\|b\|_T \le 1$ and all $(t,x) \in \omega_T$ the estimates

$$\begin{aligned}
&|R_\varepsilon[b](t,x)| \\
&= \left|\frac{1-x}{2\sqrt{\pi\varepsilon}}\int_0^t b(t-s)\,\exp\left(-\frac{(1-x+s)^2}{4\,\varepsilon\,s}\right)\,s^{-3/2}\,ds\right| \\
&\le \frac{1-x}{\sqrt{4\,\pi\,\varepsilon}}\int_0^t |b(t-s)|\,\exp\left(-\frac{(1-x+s)^2}{4\,\varepsilon\,s}\right)\,s^{-3/2}\,ds \\
&\le \frac{1-x}{\sqrt{4\,\pi\,\varepsilon}}\int_0^t \exp\left(-\frac{(1-x+s)^2}{4\,\varepsilon\,s}\right)\,s^{-3/2}\,ds \\
&= \frac{1-x}{\sqrt{\pi\,\varepsilon}}\int_{\frac{1}{\sqrt{t}}}^\infty \exp\left(-\frac{((1-x)\,\sigma+\frac{1}{\sigma})^2}{4\,\varepsilon}\right)\,d\sigma \\
&\le \frac{1-x}{\sqrt{\pi\,\varepsilon}}\int_{\frac{1}{\sqrt{t}}}^\infty \exp\left(-\frac{(1-x)^2\,\sigma^2}{4\,\varepsilon}\right)\,d\sigma = \frac{2}{\sqrt{\pi}}\int_{\frac{1-x}{\sqrt{4\,t\,\varepsilon}}}^\infty \mathrm{e}^{-s^2}\,ds \\
&= \mathrm{erfc}\left(\frac{1-x}{\sqrt{4\,t\,\varepsilon}}\right) \le 1.
\end{aligned}$$

It remains to prove 2): let $b \in C_B(J)$ and $(t,x) \in \omega_T$. For the sake of brevity we put $\theta := 1 - x$. Then $\theta \in]0,1[$ and we have

$$
\begin{aligned}
&|R_\varepsilon[b](t,x)| \\
&\qquad = \left| \frac{1-x}{\sqrt{4\pi\varepsilon}} \int_0^t b(t-s) \exp\left(-\frac{(1-x+s)^2}{4\varepsilon s}\right) s^{-3/2}\, ds \right| \\
&\le \frac{\theta\,\|b\|_T}{\sqrt{4\pi\varepsilon}} \int_0^t \exp\left(-\frac{(\theta+s)^2}{4\varepsilon s}\right) s^{-3/2}\, ds = \frac{\theta\,\|b\|_T}{\sqrt{\pi\varepsilon}} \int_{\frac{1}{\sqrt{t}}}^\infty \exp\left(-\frac{(\theta\sigma+\frac{1}{\sigma})^2}{4\varepsilon}\right) d\sigma \\
&\qquad \le \frac{\theta\,\|b\|_T}{\sqrt{\pi\varepsilon}} \int_0^\infty \exp\left(-\frac{(\theta\sigma+\frac{1}{\sigma})^2}{4\varepsilon}\right) d\sigma \\
&= \frac{\theta\,\|b\|_T}{\sqrt{\pi\varepsilon}} \left(\int_0^{\frac{1}{\sqrt{\theta}}} \exp\left(-\frac{(\theta\sigma+\frac{1}{\sigma})^2}{4\varepsilon}\right) d\sigma + \int_{\frac{1}{\sqrt{\theta}}}^\infty \exp\left(-\frac{(\theta\sigma+\frac{1}{\sigma})^2}{4\varepsilon}\right) d\sigma \right) \\
&= \frac{\theta\,\|b\|_T}{\sqrt{\pi\varepsilon}} \left(\int_{\frac{1}{\sqrt{\theta}}}^\infty \exp\left(-\frac{(\theta\sigma+\frac{1}{\sigma})^2}{4\varepsilon}\right) \frac{1}{\theta\sigma^2}\, d\sigma + \int_{\frac{1}{\sqrt{\theta}}}^\infty \exp\left(-\frac{(\theta\sigma+\frac{1}{\sigma})^2}{4\varepsilon}\right) d\sigma \right) \\
&\qquad = \frac{\theta\,\|b\|_T}{\sqrt{\pi\varepsilon}} \int_{\frac{1}{\sqrt{\theta}}}^\infty \exp\left(-\frac{(\theta\sigma+\frac{1}{\sigma})^2}{4\varepsilon}\right) \left(\frac{1}{\theta\sigma^2}+1\right) d\sigma \\
&\qquad \le \frac{2\,\theta\,\|b\|_T}{\sqrt{\pi\varepsilon}} \int_{\frac{1}{\sqrt{4\varepsilon\theta}}}^\infty \exp\left(-\frac{(\theta\sigma+\frac{1}{\sigma})^2}{4\varepsilon}\right) d\sigma \\
&\le \frac{2\,\theta\,\|b\|_T}{\sqrt{\pi\varepsilon}} \int_{\frac{1}{\sqrt{\theta}}}^\infty \exp\left(-\frac{\theta^2\sigma^2}{4\varepsilon}\right) d\sigma = \frac{2\,\theta\,\|b\|_T}{\sqrt{\pi\varepsilon}} \frac{\sqrt{4\varepsilon}}{\theta} \int_{\sqrt{\frac{\theta}{4\varepsilon}}}^\infty e^{-\sigma^2}\, d\sigma \\
&\qquad = 2\,\|b\|_T \operatorname{erfc}\left(\sqrt{\frac{\theta}{4\varepsilon}}\right) = 2\,\|b\|_T \operatorname{erfc}\left(\sqrt{\frac{1-x}{4\varepsilon}}\right).
\end{aligned}
$$

the estimate 2) in case when $x = 0$ or $x = 1$ follows from this result by a continuity argument. □

Proof (of Lemma 9): Let $v \in \mathsf{C}_2$ with $\|v\|_\infty \le 1$. Then we have due to Lemma 7 and 8

$$
\begin{aligned}
\|\left(\Gamma_\varepsilon^l \circ \Gamma_\varepsilon^r\right)[v]\|_\infty &= \|\Gamma_\varepsilon^l[R_\varepsilon[-v(\cdot,1-)]]\|_\infty = \|L_\varepsilon[(R_\varepsilon[v(xi\cdot\cdot,1-)])(\cdot,0+)]\|_\infty \\
&\le \|L_\varepsilon\|_{T,\infty}\, \|R_\varepsilon[v(\cdot,1-)](\cdot,0+)\|_T \le 2\left(2\,\|v(\cdot,1-)\|_T \operatorname{erfc}\left(\sqrt{\frac{1-0}{4\varepsilon}}\right)\right) \\
&\le 4 \operatorname{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right).
\end{aligned}
$$

□

Proof (of Theorem 10): The result will follow from theorem 2. We therefore have to check its assumptions. By assumption we have A1), A3), A4) and A2) is trivially satisfied. Assumption A5) follows the standard theory of parabolic

PDEs, see e.g. [LSU]. A6) follows from Lemma 6, A7) from Lemma 7 and A8) from Lemma 8. Due to D2), Lemma 6, 7, 8 we have $\|I_\varepsilon\|_\infty \leq K_5$ and

$$(\|R_\varepsilon\|_{T,\infty} \|\beta_\varepsilon\|_T + \|I_\varepsilon\|_\infty + \|L_\varepsilon\|_{T,\infty} \|\alpha_\varepsilon - I_\varepsilon(.,0+)\|_T) \\ \leq (1 \cdot K_4 + +K_5 + 2\,(K_4 + K_5)) = 3\,(K_4 + K_5),$$

and $\|\Gamma_\varepsilon^r\|_{\infty,\infty} \leq 1$. Hence B1) of Theorem 2 holds with, e.g., $K = 6\,(K_4 + K_5)$. B2) follows from Lemma 9 with $\Theta(\varepsilon) = 4\,\mathrm{erfc}\left(\frac{1}{2\sqrt{\varepsilon}}\right)$. B3) follows from the maximum principle. □