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STRATIFICATION OF SOLIDS
A NEW PERSPECTIVE IN
THREE DIMENSIONAL COMPUTER AIDED DESIGN

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JUNI 1987

1. Introduction

1.1 We want to study solid objects in real three dimensional space aiming at two issues:

(i1) modelling solids subject to boolean set algebra, including wire models,

(i2) determining the behaviour of moving solids, e.g. when they collide and the resulting points of contact.

This research has been initiated by the FORD Motor Company, Cologne. It is motivated by the intention to provide for a model of an automatical car gear, which gives a high precision basis to the optimization of moving tolerances.

1.2 Basically we have to choose a suitable representation of the intended model. In CAD the choice will be somewhere between sets of formulae, describing it as a mathematical entity, and collections of coordinate values.

Commonly in CAD surface patches are used, thereby linking machine representation to coordinate values (see e.g. [E-S], [N-S]). This is well suited to graphical display and finite element analysis for instance. To our issues however this approach implies time and space consuming brute force searches involving great numbers of patches (cf. [Gr]). Still not always good approximations to intersection curves, as desired in (i1) result. Concerning the second issue these methods do not guarantee collision points to be found and little can be done regarding their exactness. So the requirements of an intention like the one above seem to be hard to match.

1.3 We want to introduce some of the geometrical ideas of differential topology and algebraic geometry to link machine representation to mathematical description.

Our treatment rests on a careful a priori analysis of the families of real (semi)algebraic sets in question[1] with respect to mappings which can be chosen quite arbitrarily, the underlying scheme being what we shall call a stratification by rank. For intersection curves for example one deduces how they can be built from/represented by "singular points" and parametrized arcs. These are given by few algebraic criteria and parametrizations which are closed form in almost all interesting cases and one variable problems else.

We expect to achieve a very economical machine representation and a quick production of high precision data meeting the requirements intended.

1.4 In this paper we primarily want to explain a conceptual idea which we believe is new. To this end we treat a not too complicated special case first, becoming more involved and general later. So we start by analyzing the intersection curve problem, basic to the first issue above, for two arbitrary cylindrical surfaces, deriving the representation of intersection curves.

Next we show how to solve the collision point problem of the second issue for two solid cylinders, one of them moved by an arbitrary translation or rotation. Not surprisingly the intersection curve problem comes in here.

In part three we try to give an idea of how to proceed in general. The computations of the first parts fit in here but we want to convince the reader that much more can be achieved.

The fourth part on programming indicates how to integrate things to get and manipulate moving wire models.

In the appendix we give a more rigorous treatment of stratification by rank. We show how the representation and

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1) This applies to families of solids defined by BÉZIER- or SPLINE-patches too.

the type of an intersection curve fit into the representation of the whole solid. We propose the notion of stratification as the proper mathematical framework for solid geometry.

1.5 We wish to mention that, along with the studies on two arbitrary cylinders, we have built a prototypical computer program to explore numerical realizability and behaviour of our concept.

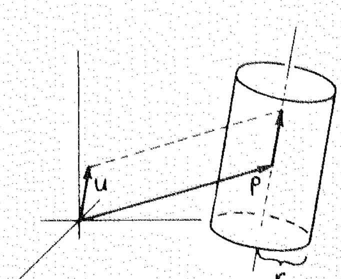
1.6 Acknowledgements: The author expresses his thanks to the FORD Motor Company, Cologne, for supporting this work, in particular to ... Hierlwimmer and ..... Schultheiß who initiated it and accompanied it with their suggestions.

The author is indebted to Professor Neunzert of the Laboratory of Technomathematics at the University of Kaiserslautern, whose sponsorship rendered this cooperation possible.

The members of the singularities/algebraic geometry/topology group at the University of Kaiserslautern were of great help in discussions on the mathematical background.

2. Two cylinders: a case study

Cylinders of course are ubiquitous in technical geometry, occurring as bars, tubes or drilling holes etc. We suppose a general infinite cylinder to be given by a point p on its axis, a directional vector u (of unit length eventually) and its radius r.



Taking an equation $f = 0$ for its surface[2], we can assume one of the sectors

$$f \geq 0 \quad \text{or} \quad f \leq 0$$

of $\mathbb{R}^3$ to be "empty" or "full", i.e. solid.

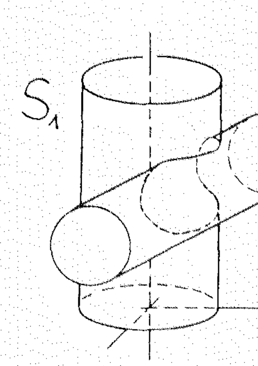
Similarly for a solid composed of two cylinders the solid region, the surface and the intersection curve of the two constituting cylindrical surfaces are determined by equations and inequalities.

2.1 Intersection curves of two cylindrical surfaces S_1, S_2.

2.1.1 By a suitable translation followed by a rotation we assume them to be in a standardized position where

S_1 is given by $p^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $u^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, any $r^{(1)}$,

S_2 is given by some arbitrary $p^{(2)}$, $u^{(2)}$, $r^{(2)}$.



So one pair of cylinders in standardized position corresponds to one point in a parametrizing set $\mathcal{H}$, subset of the total space of parameters $\mathbb{R}^8$ with coordinates

$$r^{(1)},\ r^{(2)},\ p_k^{(2)},\ u_k^{(2)} \qquad k = 1, 2, 3.$$

In fact this set $\mathcal{H}$ is the HILBERT-scheme of all pairs of cylinders in standard position in $\mathbb{R}^3$.

2) On one hand each real algebraic set can be described by one equation (the real zero set of $f_1, \ldots, f_p$ is the same as that of $g := f_1^2 + \ldots + f_p^2$). On the other hand an algebraic subset of $\mathbb{R}^3$ is purely 2-dimensional if and only if every polynomial function which vanishes on it is a polynomial multiple of one certain polynomial function defining it (see [P] 2.3.13, 2.3.14).

2.1.2 A general point $a \in S_i$ on cylinder surface S_i has to meet the equation

$$\| a-p^{(i)} - \langle a-p^{(i)}, u^{(i)}\rangle u^{(i)} \|^2 = (r^{(i)})^2 \quad i = 1, 2.$$

To parametrize the intersection curve

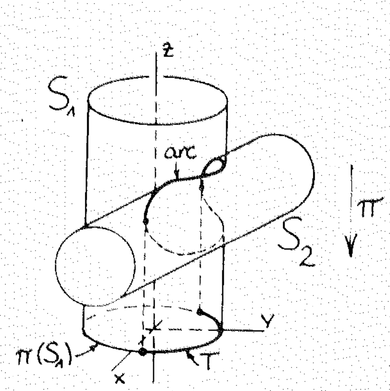
$$C = S_1 \cap S_2$$

means to cover C by arcs, the points on each given by a one-to-one mapping

$$t \longmapsto a(t)$$

of the points t of an interval or a suitable curve T onto it (everything depending, of course, on our point in $\mathcal{H}$).

2.1.3 We observe that the orthogonal projection π of x,y,z-space onto the x,y-plane maps S_1 to a circle. We let T be the subset of this circle given by projection of a corresponding **maximal arc of C.**



In this setting a point $a \in C$ on C is given by the conditions

(1) it has to lie on cylinder S_2,
and (2) it has to project to a point of $\pi(S_1)$,

or equivalently

$$(1) \quad \| a-p^{(2)} - \langle a-p^{(2)}, u^{(2)}\rangle u^{(2)} \|^2 = (r^{(2)})^2$$

$$\text{and (2)} \quad a = \begin{pmatrix} \alpha \\ \beta \\ z \end{pmatrix} \text{ with } \alpha^2 + \beta^2 = (r^{(1)})^2.$$

To parametrize an arc of C we have to determine the subset T of $\pi(S_1)$ and z as a function

$$T \ni (\alpha,\beta) \longmapsto z(\alpha,\beta)$$

which is one-to-one onto the arc.

2.1.4 Clearly (1) is quadratic in z and can be turned to the form

$$(1) \qquad Az^2 + Bz + C = 0.$$

Keeping the side condition (2), this can be used to determine z on one hand. On the other hand (1) and (2) decide whether t exists at all, i.e. where to look for the parametrizing subset $T \subset \pi(S_1)$. z of course exists if and only if along with (2)

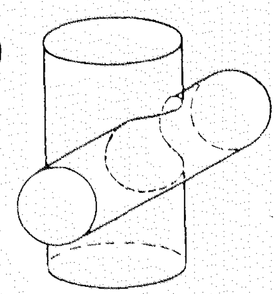
$$(1a) \qquad A \neq 0 \quad \text{and} \quad B^2 - AC \geq 0$$

$$\text{or } (1b) \qquad A = 0 = C \quad (B = 0 \text{ follows here})$$

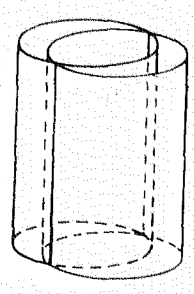
can be satisfied.

Under (1a) there are two values for z (counted with multiplicities) and for an arcwise parametrization we have to select one. If (1b) holds the two cylinders are parallel. There exist infinitely many solutions to z and the arcs, in which S_1 and S_2 intersect, are one or two straight lines, or S_1 and S_2 coincide. Clearly one obtains exactly <u>two general parametrization formulae</u> in closed form.

(1a)



(1b)



2.1.5 Having dismissed of the variable z in (1a), (1b) we can push the question of existence one step further. We switch to the <u>boundary</u> or <u>exceptional conditions</u> on $(\alpha,\beta) \in T \subset C\ \pi(S_1)$ by taking equality in (1a) instead of "$\geq$":

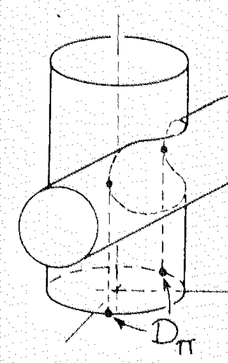
$$(1a)' \quad A \neq 0 \text{ and } B^2 - AC = 0$$

$$\text{or } (1b)' \quad A = 0 = C$$

keeping (2). Together they determine the subset

$$D_\pi := \{ (\alpha,\beta) \in \pi(S_1) \mid B^2 - AC = 0 \}$$

of $\pi(S_1)$ which we call the discriminant of $\pi_{|C}$.



It is the image of the singular set of $\pi_{|C}$ which bounds the arcs to be parametrized or coincides with them if they map to a point.

2.1.6 Both (1a)' and (1b)' turn out to be equivalent to sets of quadratic equations (again!) in (α,β).

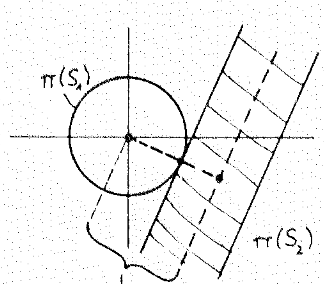
As above with z, these can be used to determine (α,β) on the boundary of T on one hand and on the other to decide on the existence of these points and thus of T itself.

Once more we have to discuss the vanishing of coefficients and discriminant of the respective equation. The union of their vanishing sets (keeping (2)) is a subset D of $\mathcal{H}$ called discriminant of the intersection curve problem.

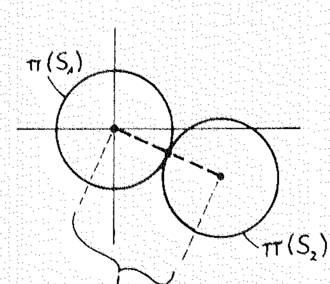
D seems to be the image of the singular set of the projection $\mathbb{R}^3 \times \mathbb{R}^8 \longrightarrow \mathbb{R}^8$ restricted to the family of all standardized cylinder pairs.

2.1.7 In our case D is interpreted quite simply as characterizing non-generic or exceptional values of the distance d between the projections of the axes of our cylinders.

(1a)'



(1b)'



2.1.8 We are left with the variables

$$r^{(1)},\ r^{(2)},\ p_k^{(2)},\ u_k^{(2)} \qquad k = 1,\ 2,\ 3$$

whose value characterizes the intersection curve problem.

We define the type of an intersection curve by the number of its parametrized arcs, the ranks of $\pi_{|C}$ within them, and the ranks of $\pi_{|C}$ in their endpoints.

As $r^{(1)}, r^{(2)}, p^{(2)}, u^{(2)}$ vary, the corresponding type will not change unless D is crossed or, varying within D, unless certain subsets of D are crossed (viz. $A = 0 = C$ within $B^2 - AC = 0$ in our case). So $\mathcal{K}$ and D are stratified by the type conditions into certain subsets which can be seen to be semialgebraic (for this notion see the appendix).

2.1.9 We are ready to put things together now and define the representation of an intersection curve to be given by the following data:

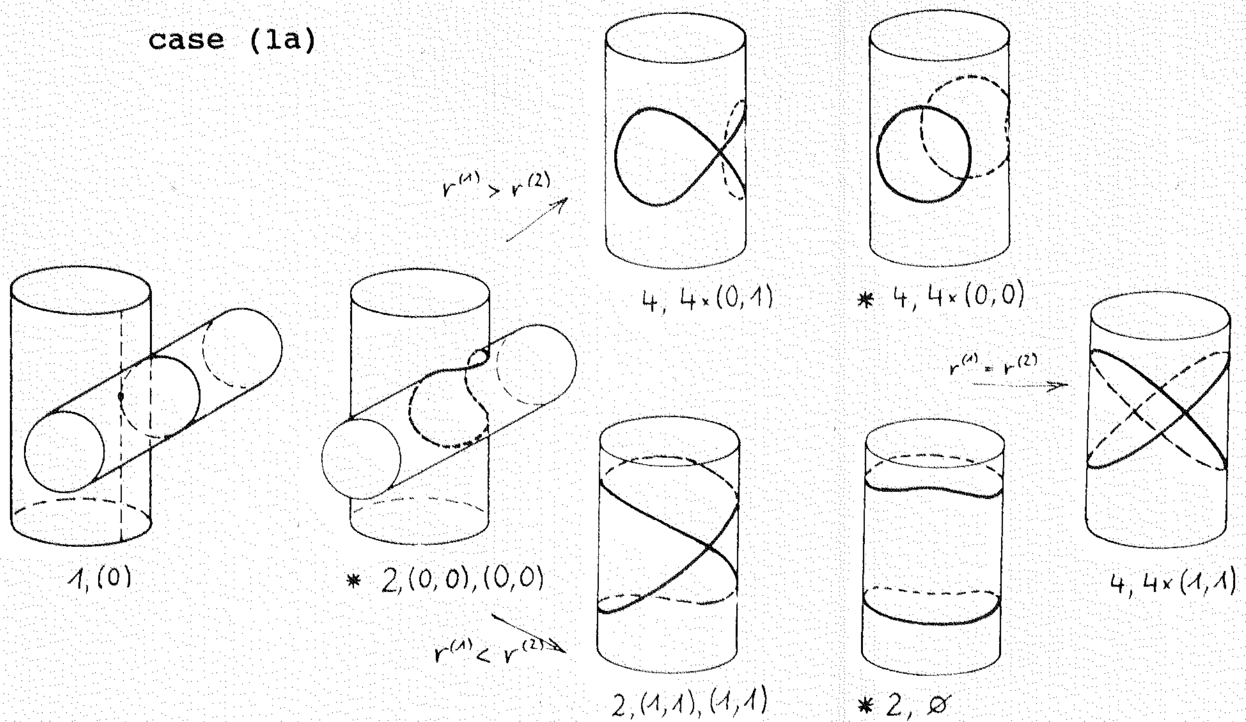
(r1) the data of the two intersecting surfaces S_1, S_2

(r2) the endpoints of the parametrizing subsets of $\pi(S_1)$.

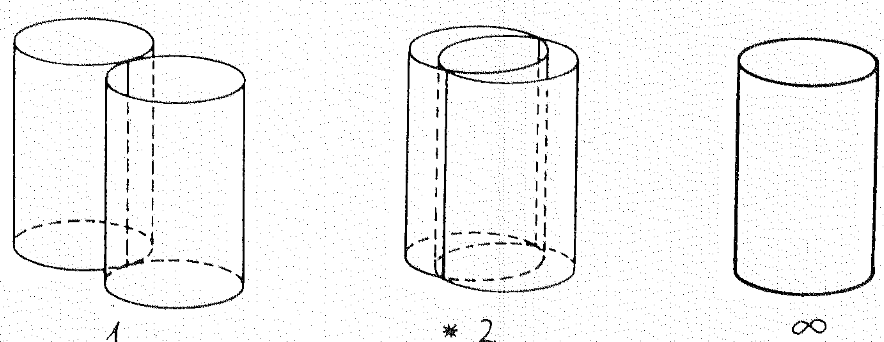
This representation shows how to build the intersection curve from its parametrized arcs using the two general cylinder/cylinder parametrization formulae of 2.1.4.

2.1.10 Finishing this section we give pictures of the distinct types of cylinder/cylinder intersection curves on the next page.

case (1a)



case (1b)



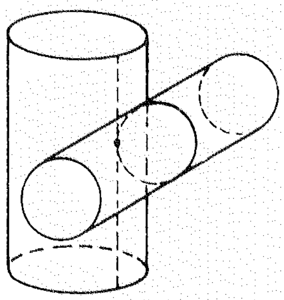
The curves are marked by #arcs, (type of left endpoint, type of right endpoint). The non-intersecting case, to be marked 0, has been left out.

This <u>classification</u> is <u>complete</u> and each class corresponds to a connected component of $\mathcal{H} \setminus D$ (these classes are marked with an *) or to one "stratum" of D in the HILBERT-scheme $\mathcal{H}$.

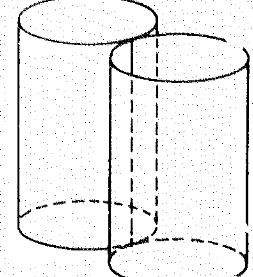
2.2 <u>Contact points</u> of two moving solid cylinders.

According to the preceding discussion two solid cylindrical bars can touch in essentially two ways:

(1 a)



(1 b)



2.2.1 We assume cylinder S_1 to rest in place and allow cylinder S_2 to move in two ways, namely

(m1) S_2 is translated along a directional vector v (of unit length if suitable)

or (m2) S_2 is rotated with respect to some axis, given by a point q and a directional vector v (of unit length if suitable)

In fact, we can assume S_1 in the same position as before (2.1.1). So now we have a standard situation, varying according to the translational resp. rotational parameters.

2.2.2 We just look whether the varying standard situation parameters hit the discriminant D of (2.1.6)[3] by substitution:

(m1) $p^{(2)}$ is replaced by $p^{(2)} + \lambda v$

(m2) $p^{(2)}$ is replaced by $\theta_v (p^{(2)}-q) + q$,
$u^{(2)}$ is replaced by $\theta_v u^{(2)}$

where θ_v is the orthogonal matrix of a general rotation of 3-space with axis vector v, parametrized by (σ,τ) on a unit circle.

2.2.3 The discriminant equation simply gives a quadratic equation in λ in case of (m1). From (m2) at first look one obtains fourth-degree equations in (σ,τ). By taking the contour lines of $\pi(S_2)$ instead of its axis however (see 2.1.7) they can be split into second degree equations. Anyway all equations can be solved in closed form again.

As to programming we have an alternative here which occurs quite frequently in the present context. Either we solve numerically for σ, τ or λ here, move the cylinders to the respective position and compute the contact points as degenerate intersection curves (left endpoint of arc = right endpoint).

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3) We remark, that it is not reasonable to look for further touching conditions.

Or we solve algebraically for σ, τ or λ, substitute the expressions obtained in

$$\theta_V(p^{(2)}-q) + q, \quad \theta_V(u^{(2)}) \quad \text{or} \quad p^{(2)} + \lambda v$$

respectively and derive formal, closed form solutions for the contact points.

Mathematically of course both amounts to the same.

3. Towards stratification

We try to give an idea of stratification by rank. A more rigorous treatment is postponed to the appendix.

3.1 As indicated in section 2 already, starting with one equation

$$f = 0$$

(f a polynomial with real, coefficients) for an algebraic surface, we further get the regions

$$f \geq 0, \quad f \leq 0$$

of space which can be considered the empty or full regions of a solid as desired.

Note, that $\pm f$ or f^2 give the same surface but different regions and remember that the respective gradient always points to the "≥ 0" side.

3.2 To get more complicated solids we can of course form intersections or unions of one-equation-solids (hypersurfaces, see [P] 2.3). A finite solid cylinder for example is the intersection of an infinite cylinder with two half spaces. This leads to the general definition of a solid as a semialgebraic set. Of course one will try to restrict to some suitable set of primitives.

For the intentions of 1.1

plane, cylinder, cone, sphere, torus

have been proposed. Mathematically these are distinguished clearly by their low genus (0,1,0,0,1 respectively).

However also solids defined by BÉZIER- or SPLINE-polynomial patches fit into the definition of solid we use (the point is in how we use them).

3.3 There are several particular subsets of a solid: its interior, the smooth and the singular points of its surface (i.e. singular points like the tip of a cone or those lying on several of the constituting surfaces).

The singular points of the surface form the wire model of the solid. Intersection curves belonging to them maybe meet or have singular points themselves. Of course the representation of intersection curves has to take their intersections and singular points into account.

This partitioning is the first, absolute stage of stratification.

3.4 We introduce a mapping π to $\mathbb{R}^2$ in order to get a description of a solid with respect to its image under π. In fact π can be chosen quite generally.

The second, relative stratification stage subdivides the subsets obtained so far into smaller smooth strata on which the restriction of π has constant rank.

We get one set of strata parametrized by their image, the restriction of π to them being 1-1. They are pieces of the surface of the solid or are the parametrized arcs of section 2.

The other strata constitute the interior of the solid, or border the parametrized strata or are cartesian products essentially.

The stratification by the rank of π we have obtained thus tells how to build the solid from parametrized and exceptional subsets.

The mapping of a solid onto a screen of course too is a mapping in this sense and the related stratification for instance gives the contour lines of the image.

3.5 We make use of course of the equations of the image of parametized areas/arcs and their boundaries (see 2.1.5), i.e. the images of strata. Now they are semialgebraic and there are general methods to compute effectively the image of semialgebraic sets (equations and inequalities) under all the mappings we have in mind (see section 5).

One can use classical elimination algorithms - modern treatments in the context of CAD have been given by [Go], [G-S-A], [M-T] for instance. Only recently these algorithmus have been supplemented by the new standard base approach due to HIRONAKA and BUCHBERGER (see e.g. [L-J]).

4. Programming considerations

4.1 We give a sketch only. A computer program solving the "wire model" and the "colliding solids" problems could be structured as follows:

(p1) c o n t r o l section: receives requirements from outside and calls appropriate actions; location of the wire model representation.

(p2) i n t e r s e c t i o n c u r v e section: routines computing the representation of specific surface pair intersections.

(p3) i n t e r s e c t i o n p o i n t section: computer intersection points of pairs of intersection curves (resp. of triples of surfaces)

(p4) g e n e r a l mathematical subroutines.

(p5) C A D - b a ck g r o u n d defining requirements and receiving data produced

4.2.1 The representation of intersection curves as computed in (p2) by (2.1.9) will consist of

(r2) the endpoints of the parametrizing subsets of $\pi(S1)$

essentially, the intersecting surfaces being input.

4.3. The representation of a wire model resides in (p1). It consists of a graph with the data of the involved solids/surfaces as nodes, the data of their intersections on the solid to be modelled as edges, and some additional information on other singular points, contour lines etc. as desired.

If a new mathematical solid (a primitive) is added or subtracted from the model, the graph is modified as follows:

(1) The surface data are stored as a <u>new node</u>, solid/empty regions indicated according to whether to add or to subtract the new solid.

(2) Intersection curves with the existing surfaces are computed.

(3) Intersection points of the new curves from (2) among themselves and with the old ones are computed.

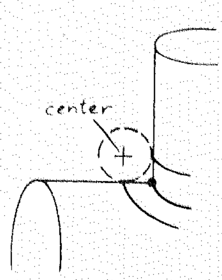
At this stage the new curves of (2) are subdivided into their natural arcs and further into sub-arcs by the intersection points of (3).

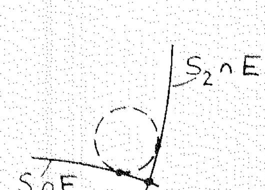
(4) If an endpoint of a sub-arc lies inside the "empty" region of one of the constituting sufaces it is omitted; otherwise it is kept as a <u>new edge</u> of the graph representing the wire model

4.4 <u>Additional features</u>

4.4.1 Sometimes one wishes to have only a certain number of points on an intersection curve, with a <u>predetermined spacing</u> or distribution. This can be achieved using *grad* z (as we did in our prototypical program) or approximations to arc length.

4.4.2 Another important feature are <u>offset curves</u> to intersection curves $C = S_1 \cap S_2$ as the paths of a cutter touching both S_1 and S_2 (cf. [F-P] p.268 ff. where the offset curve is the path of the center of the cutter) or as the boundary of a welding seam.





In "each" point of C we consider the plane E perpendicular to C. Then a disk of prescribed small radius in E is computed, which touches both $S_1 \cap E$ and $S_2 \cap E$ on a prescribed side of S_1 and S_2 respectively. The touching points are displayed. This approximates the path of a ball rolling along C while touching both S_1 and S_2.

Appendix: Stratification by rank

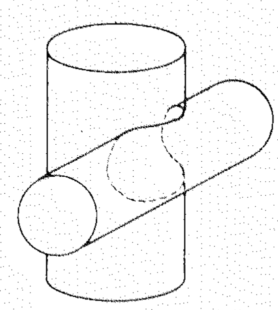
A1. Our computations of section 2 remind of the concept of stratification of sets and mappings which was introduced by WHITNEY and THOM. We believe that this is, general as it may look, the economic and effective mathematical framework for the geometry of real solids, which moreover provides for a proper understanding.

For reference see [T: 1956, 1969], [Ma], [G-G].

As an attempt let's describe a stratification which allows to do a computation like ours for arbitrary solids (semi-algebraic sets) with respect to a large class of mappings.

Its range of validity is greater than that of the so-called THOM-BOARDMANN-stratification, with which it coincides for smooth solids and "generic" mappings (see [G-G]). Much finer stratifications have been employed in singularity theory (see the references).

For illustration we take, say, a solid S, composed of two solid cylindrical bars S_1, S_2 in standard position (as in 2.), intersecting in a proper curve as depicted below:



A.2 Definition A (family of) solid(s) is a semialgebraic subset of $(\mathbb{R}^n \times)$ $\mathbb{R}^3$, i.e. one that is obtained by a finite combination of union, intersection and taking complements of subsets of $(\mathbb{R}^n \times)$ $\mathbb{R}^3$ given by conditions of the form

$f \geq 0$, f a polynomial with real coefficients.

Examples:

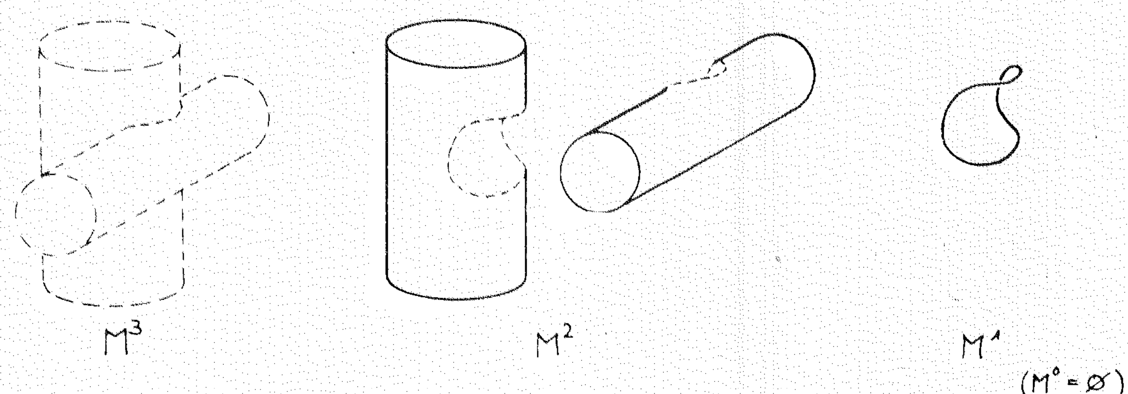
(i) the solid $S = S_1 \cup S_2$ above,

(ii) any solid defined locally by BÉZIER- or SPLINE-patches

(iii) the family of all pairs of cylinders in standard position

A.3 There is a decomposition of every solid into its interior $\mathring{S}$ and its boundary $\partial S = S - \mathring{S}$. If it is not empty, $\mathring{S}$ is a smooth three-dimensional, submanifold[4] M^3 of $\mathbb{R}^3$, ∂S is a semialgebraic set again, which decomposes into the semi-algebraic, at most 1-dimensional singular subset $(\partial S)_{sing}$ and its complement. $\partial S - (\partial S)_{sing}$ (if not empty) is a smooth, 2-dimensional submanifold M^2 of $\mathbb{R}^3$.

$(\partial S)_{sing}$ finally decomposes into a set of points and a smooth, 1-dimensional submanifold M^1 of $\mathbb{R}^3$. The remaining set of points forms a smooth, 0-dimensional submanifold M^0 of $\mathbb{R}^3$. The manifolds M^i may have several components, may be compact or not, even empty.



This is WHITNEY's rank-stratification of S and we call it

A.4 The absolute stratification step. Split S into a finite union

$$S = M^3 \cup M^2 \cup M^1 \cup M^0$$

of disjoint smooth submanifolds M^i (i=0,1,2,3) of $\mathbb{R}^3$, with

4) By (smooth) manifold we mean analytic manifold without boundary. Any (semi)algebraic subset of $\mathbb{R}^1$ without singular points also is an analytic submanifold of $\mathbb{R}^1$ (see [P] 2.4).

$$\bigcup_{i=0}^{k} M^i, \quad k = 0,1,2,3, \quad \dim M^i = i$$

semialgebraic. The connected components of the manifolds M^i are called the <u>absolute strata</u> of S.

A.5 We introduce now the projection

$$\pi: \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \pi(x,y,z) = (x,y).$$

It maps every semialgebraic subset of $\mathbb{R}^3$ to a semialgebraic set in $\mathbb{R}^2$. The image can be <u>effectively computed</u> by the methods of 2.3.4.

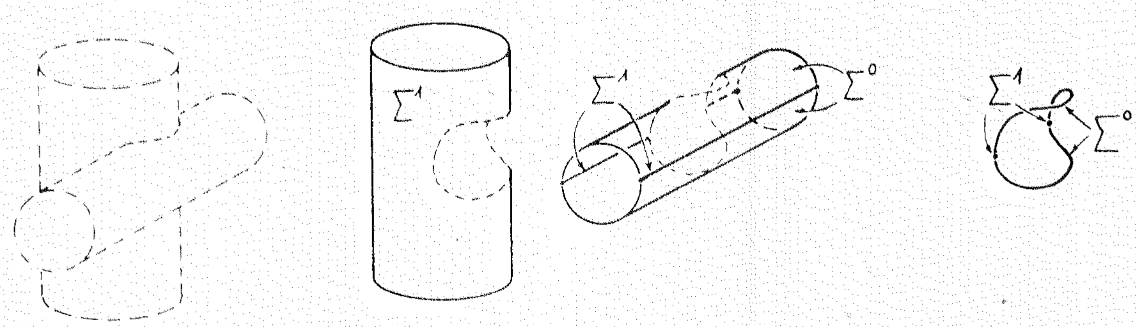
A.6 <u>The relative stratification step</u>: Split a smooth, m-dimensional manifold M (to be thought of as one of the M^i above) with respect to the restriction $\pi_{|M}$ of the projection π into

$$M = \bigcup_{i_0=0}^{3} \Sigma^{i_0}(\pi)$$

such that the rank of $\pi_{|M}$ will take on the value

$$\min(m, 2) - i_0$$

equally in each point of $\Sigma^{i_0}(\pi)$. Note that all the sets $\Sigma^{i_0}(\pi)$ are semialgebraic.



$M^3 = \mathring{S} = \Sigma^0(\pi_{|M^3})$ $\qquad$ M^2 $\qquad$ M^1

<u>Remarks</u>:

(i) $\Sigma^1(\pi_{|M^1})$ is the singular set, its image under π is the discriminant of 2.1.5.

(ii) If π is the projection onto the user's plane of observation (screen, paper), then

$$\bigcup_{i \geq 0} \Sigma^{i}(\pi_{|M^2})$$

maps to the visible contour of $\pi(M^2)$.

A.7 For each connected component M' of an M^i there is one (smallest) i_0 such that

$$M' \cap \Sigma^{i_0}(\pi)$$

is an open subset of M' and thus a manifold. Numbering all these we get the connected manifolds

$$\Sigma^{i_0}_{j_0}(\pi), \qquad j_0 \in J_0$$

of dimension at most 3.

The remaining sets $\Sigma^{i_0}(\pi)$ are stratified absolutely into the manifolds

$$\Sigma^{i_0}_{j_0}(\pi), \qquad j_0 \in J_0'.$$

of dimension at most 2.

Relative stratification of these yields

$$\Sigma^{i_0,i_1}_{j_0}(\pi) \quad := \quad \Sigma^{i_1}_{j_0}(\pi_{|\Sigma^{i_0}}), \; j_0 \in J_0'.$$

The intersection of some of these with the connected components of the $\Sigma^{i_0}_{j_0}$ (π) are open again in the latter; these intersections give us manifolds

$$\Sigma^{i_0,i_1}_{j_0,j_1}(\pi), \quad j_0 \in J_0', \; j_1 \in J_1$$

of dimension ≤ 2.

The remaining $\Sigma^{i_0,i_1}_{j_0}$ are stratified absolutely again into the manifolds

$$\Sigma^{i_0,i_1}_{j_0,j_1}(\pi), \quad j_0 \in J_0', \; j_1 \in J_1'$$

of dimension ≤ 1.

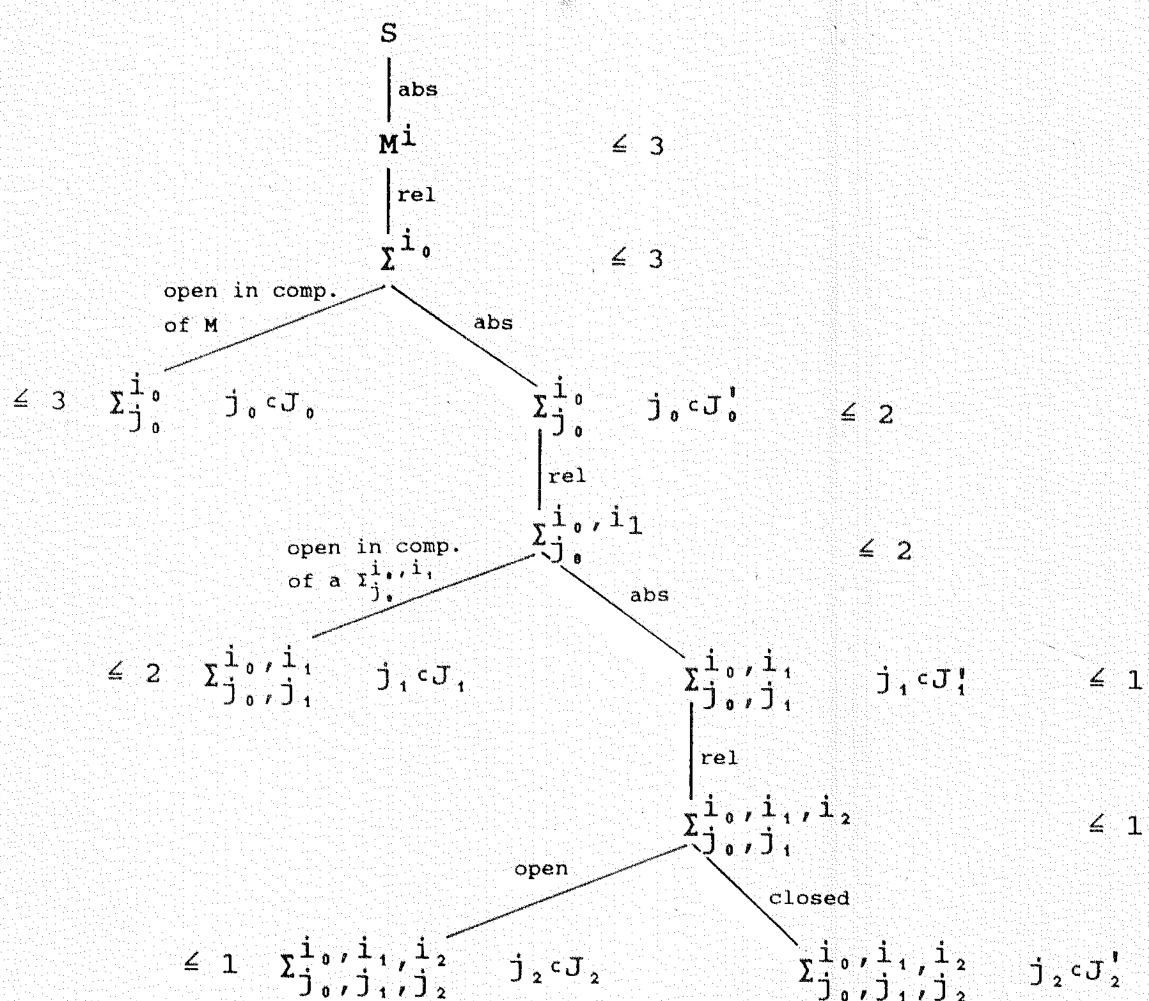
One last relative stratification gives the sets

$$\Sigma^{i_0,i_1,i_2}_{j_0,j_1,j_2}(\pi) \quad := \quad \Sigma(\pi|\Sigma^{i_0,i_1}_{j_0,j_1})$$

which are semialgebraic.

They are "≤1"-dimensional smooth manifolds in case they are open in the respective $\Sigma^{i_0,i_1}_{j_0,j_1}$ and points else.

A.8 The following tree-diagram many visualize the stratification prosess; its leaves are the strata; dimensions are indicated in the form "≤ dim"



The set of these strata as embedded in the solid S is called the <u>stratification</u> <u>of</u> S <u>by</u> (the) <u>rank</u> (of π).

Remarks:

(i) For generic maps the absolute stratification steps are trivial. The lower indices are superfluous then and the $\Sigma^{i_0,\ldots,i_n}$ are the universal singular sets of BOARDMAN-type (or THOM-BOARDMAN-strata of type) $i_0,\ldots,i_n$ (see [G-G]).

(ii) The indices i_k correspond to the type of 2.1.8.

(iii) The closure of each stratum is a union of strata. This is the most basic condition one usually asks on how strata should fit in with each other.

(iv) Every Σ^0 of some restriction of π to an absolute stratum is the complement of its singular set $\bigcup_{i>0}\Sigma^i$. The image of a singular set generally is called a discriminant. This notion occured already in several places of this work and is identical also with the "contour" of remark A6 (ii).

(v) For a family of solids over some parameter space $\mathcal{H}$ on one hand one has the family of all the stratifications of its members. On the other hand one has the stratification of the family with respect to the projection to $\mathcal{H}$. We have made first use of their relation in 2.1.6 - 2.1.8, 2.2.

A.9 S is the union of the disjoint strata. We summarize their properties.

(s1) Each stratum is a connected, semialgebraic set.

(s2) Each stratum is a smooth manifold (and thus a smooth submanifold of $\mathbb{R}^3$).

(s3) If π is restricted to any stratum, its rank is the same in each point of it.

(s4) The closure of each stratum is a union of strata.

A.10 As mentioned already the image of each stratum under π can be effectively computed by (s1).

(s2) and (s3) guarantee that we know exactly about the behaviour of π. Let m be the dimension of a stratum M and r be the rank of the restriction $\pi_{|M}$.

Then

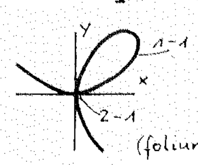
(a) if $r = m$: $\pi_{|M}$ is locally 1-1 onto it's image, i.e. its image parametrizes M locally.

(b) if $r < m$: $\pi_{|M}$ maps M to a lower dimensional set M' and M is an open subset $M \subset M' \times \mathbb{R}$.

These facts allow to construct the stratified solid from the images of the strata (note the incidence relations contained in (s4)).

We have carried out the details in section 2 for two cylinders.

Remark: In (a) one does not necessarily have a globally 1-1 mapping as the smooth space curve parametrized by



(folium cartesii)

$$x=\frac{3t}{1+t^3}, \quad y=\frac{3t^2}{1+t^3}, \quad z=t$$

shows. The projection to the x,y,-plane is 2-1 over $(x,y) = 0$, 1-1 else.

A.11 Satisfying as this really is, one frequently may wish to use different mappings. Of course one can switch to their graph and again have a projection. In the stratification process we did not make any use of specific properties of a projection except its differentiability. However the graph idea shows, which mappings in principle are computationally well suited to our approach: mappings with semialgebraic graph.

In fact the graph idea shows how to make use of a further way to relate domain and image, the use of which way become apparent in example (ii) below. For reference see [Mu].

A.12 Definition: An algebraic correspondence from a solid S to a subset of $\mathbb{R}^2$ is the restriction of a relation

$$Z \subset \mathbb{R}^3 \times \mathbb{R}^2$$

to $S \times \mathbb{R}^2$, Z given by algebraic equations. Similarly a correspondence $\mathcal{Z}$ of a family of solids is given by an algebraic relation

$$Z \subset (\mathbb{R}^n \times \mathbb{R}^3) \times (\mathbb{R}^n \times \mathbb{R}^2)$$

restricting to the identity on each

$$(\mathbb{R}^n \times \{a\}) \times (\mathbb{R}^n \times \{b\}), \quad b \in Z[a].$$

We call a correspondence Z <u>good</u> if

dim Z = 3, (respectively = n + 3 for a family)

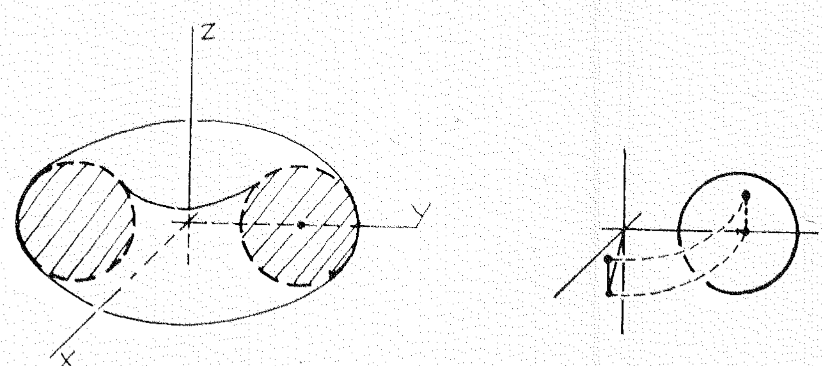
and if there are no fundamental points on the (family of) solid(s).

<u>Examples</u>:

(i) the projection $\pi(x,y,z) = (x,y)$ and more generally all polynomial maps, rational maps, maps involving roots.

(ii) A solid torus in x,y,z-space, symmetric with respect to the z-axis is mapped onto two disks in the y',z'-plane by the good correspondence

$$Z: \quad z' = z, \ (y')^2 = x^2 + y^2 \quad \text{in} \quad \mathbb{R}^3_{x,y,z} \times \mathbb{R}^2_{y,z}.$$



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