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STABILITY AND ROBUSTNESS PROPERTIES OF
UNIVERSAL ADAPTIVE CONTROLLERS FOR
FIRST ORDER LINEAR SYSTEMS

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0. Introduction

The question: "What is an adaptive controller?" is as old as the word "adaptive control" itself. In this paper we will adopt a pragmatic viewpoint which identifies adaptive controllers with nonlinear feedback controllers, designed for classes (families) of linear systems. In contrast to classical linear feedback controllers which are designed for individual systems, these nonlinear controllers are required to achieve a specific design objective (such as e.g. stability, tracking or decoupling) for a whole prescribed family of linear systems

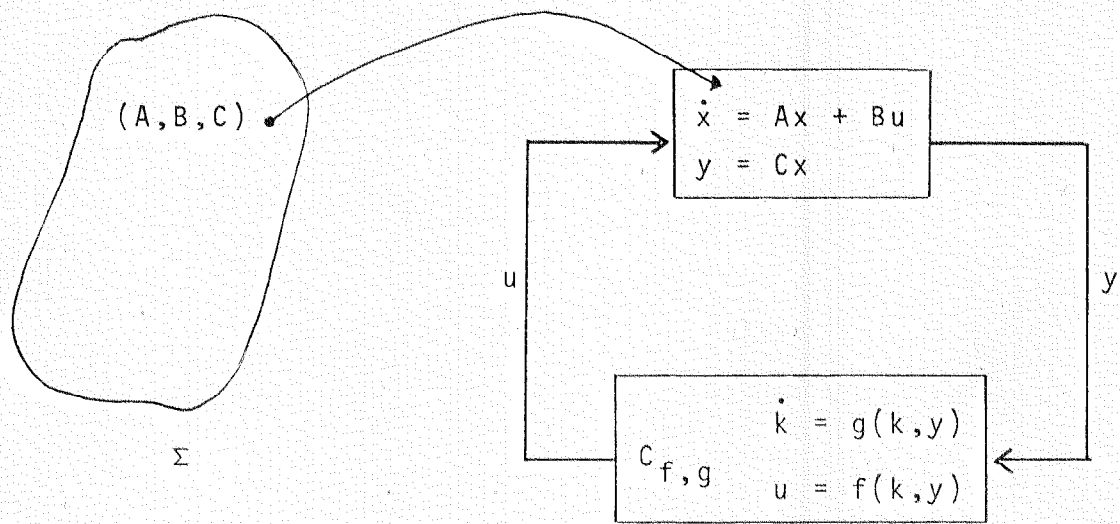


Fig. 1: *Universal adaptive controller*

In the above picture, Σ denotes a given class of linear systems (A, B, C) with a fixed number m, p of inputs resp. outputs. The functions

$$f: \mathbb{R}^{p+q} \rightarrow \mathbb{R}^m, \quad g: \mathbb{R}^{p+q} \rightarrow \mathbb{R}^q,$$

which determine the controller dynamic $C_{f,g}$, are required to belong to some function space F . Examples are the space of smooth functions $F = C^\infty(\mathbb{R}^{p+q}, \mathbb{R}^{m+q})$, or suitable other spaces of analytic functions, piecewise C^∞ functions, etc.

Concerning the control objective of adaptive stabilization, the controller $C_{f,g}$ is called a *universal adaptive stabilizer* (UAS_q) for Σ of dimension q , provided for any fixed system $(A, B, C) \in \Sigma$ and for all initial data $(x(0), k(0))$ the closed loop system

$$(0.1) \quad \begin{aligned} \dot{x} &= Ax + Bf(k, Cx) \\ \dot{k} &= g(k, Cx) \end{aligned}$$

satisfies

$$(0.2a) \quad \lim_{t \rightarrow \infty} x(t) = 0.$$

$$(0.2b) \quad \exists M > 0 \text{ such that } |k(t)| \leq M \text{ for all } t \in [0, \infty).$$

See Morse (1983) or Byrnes, Helmke, Morse (1986), where instead of (0.2b) the more restrictive assumption

$$"(0.2b') \quad \lim_{t \rightarrow \infty} k(t) = k_{\infty} \text{ exists}."$$

is used.

Since the control functions f, g are not allowed to depend upon the parameters $(A, B, C) \in \Sigma$, the existence problem of such universal controllers (f, g) is quite subtle. This is true even for the class Σ^1 of scalar first order controllable and observable systems

$$(0.3) \quad \begin{aligned} \dot{x} &= \alpha x + \beta u \\ y &= \gamma x \end{aligned}$$

where $\alpha \in \mathbb{R}$, $\beta \neq 0$, $\gamma \neq 0$.

Morse (1983) conjectured the non-existence of first order ($q=1$) universal stabilizers for the class Σ^1 of all systems (0.3). Later on, R. Nussbaum (1984) gave the following counter example to Morse's conjecture, proving the global adaptive stability (0.2a), (0.2b') of

$$(0.4a) \quad \dot{y} = (\alpha + \beta\gamma(k^2+1)\cos\frac{\pi k}{2} \cdot e^{k^2})y$$

$$(0.4b) \quad \dot{k} = \gamma(k^2+1)$$

for all $(\alpha, \beta, \gamma) \in \Sigma^1$.

Morse's and Nussbaum's pioneering work initiated considerable recent progress in designing universal adaptive stabilizers

for various classes of linear systems. We mention in particular Byrnes and Willems (1984), Heymann, Lewis and Meyer (1985), Mårtensson (1985a, 1985b, 1986), Ilchmann, Owens and Prätzel-Wolters (1986), Owens, Prätzel-Wolters and Ilchmann (1987), Morse and Mudgett (1985), Morse (1984a, 1984b, 1985).

This culminated in the recently found necessary and sufficient conditions for universal adaptive stabilization, see Byrnes, Helmke and Morse (1986) and Mårtensson (1986).

Despite of this encouraging progress in the adaptive stabilization problem, we are still far away from a complete understanding. In particular, the robustness properties of such universal stabilizers are only very poorly understood. For example, one would like to have answers to the following questions:

- (a) How does the closed-loop system (0.1) behave under - say - additive noise or structural perturbations in f, g ?
- (b) Does there exist something like a "structurally stable" adaptive controller in analogy to structural stable vector fields ?
- (c) Is "bursting", as e.g. documented in Anderson (1983), *necessary* for adaptive stabilization ?

In order to obtain a first understanding of question (a) we propose to study the following specific problem

- (P) Characterize the set $UAS_1 \subset C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ of all first order universal adaptive stabilizers (f, g) for Σ^1 !
What are the interior points of UAS_1 ?

While we will not fully solve this problem, we derive a simple *necessary condition* for a pair (f, g) to belong to UAS_1 , depending only on the parameter adaption function g . This condition is close to be sufficient in the sense that - up to a technical additional assumption - for each such g one can in fact construct an f with $(f, g) \in UAS_1$ (see Thm. 3.1).

Throughout this paper we will restrict ourselves to the simplest possible case: the adaptive stabilization of first-order systems (0.3) by one-dimensional controllers.

We proceed as follows:

In section 1 we study systems having a *proper first integral*. This concept is useful for the general stability analysis of first order scalar systems, controlled by a one-dimensional controller and is implicit in previous research (e.g. Willems-Byrnes (1984)). The necessary condition is derived in section 2. Section 3 is based on section 2 and deals with general classes of UAS's for Σ^1 . As special cases of our general analysis we re-obtain the stability results of Byrnes/Willems, Nussbaum and Heymann/Lewis/Meyer. Finally, section 4 deals with robustness properties of first order universal stabilizers. Different types of perturbations are studied and we report extensive simulation experiments. These results demonstrate considerable qualitative differences in the dynamical behaviour of various adaptive stabilizers. While previously proposed controllers all exhibit the bursting phenomena, we describe one explicit controller which does not seem to exhibit bursting.

1. First Integrals and Stability

For autonomous systems

$$(1.1) \quad \dot{x}(t) = f(x(t)), \quad f: \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad t \in \mathbb{R},$$

a *first integral* E is defined as a smooth, non constant, function $E: \mathbb{R}^n \rightarrow \mathbb{R}$ which satisfies

$$(1.2a) \quad \frac{d E(x(t))}{dt} = \sum_{j=1}^n \frac{\partial E}{\partial x_j} f_j(x(t)) \equiv 0$$

for every solution $x(t)$ of (1.1). Thus E is required to be constant along the trajectories of (1.1) or, equivalently, whose gradient $\nabla E = \left(\frac{\partial E}{\partial x_i} \right)_{i \in \underline{n}}$ is orthogonal to the vector field $f(x) = (f_1(x), \dots, f_n(x))$, for all $x \in \mathbb{R}^n$.

A *proper first integral* for (1.1) is a first integral $E \in C^1(\mathbb{R}^n, \mathbb{R})$ which satisfies the extra condition of properness:

$$(1.2b) \quad K \subset \mathbb{R} \text{ compact} \Rightarrow E^{-1}(K) \text{ compact.}$$

In any of the subsequent results one can replace properness by the weaker condition

$$(1.2c) \quad E^{-1}(a) \text{ is compact for all } a \in \mathbb{R}$$

without changing the validity of the statements.

As a first observation we note that the existence of a proper first integral implies the boundedness of the solutions of (1.1).

Lemma 1.1

If the system (1.1) has a first proper integral E then every solution $x(\cdot)$ of (1.1) is bounded and exists for all $t \in \mathbb{R}$.

Proof: Let $x(t)$ be any solution of (1.1), $a := E(x(0))$. By (1.2), $x(t) \in E^{-1}(a)$ for all t . The result follows from the compactness of $E^{-1}(a)$. □

In the sequel we consider only planar vector fields ($n=2$):

$$(1.3) \quad \begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \quad , \quad f, g \in C^1(\mathbb{R}^2, \mathbb{R}).$$

We assume throughout that (1.3) has as its equilibria set

$$(1.4) \quad E = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = g(x, y) = 0\}$$

a *closed non-compact submanifold* of \mathbb{R}^2 which is entirely contained in the y -axis:

$$(1.5) \quad E \subset \{(x, y) \in \mathbb{R}^2 \mid x = 0\}.$$

Proposition 1.2

Let E denote the equilibria set of a system (1.3) which has a proper first integral E . Assume that E is nowhere locally constant. Then either

(i) There is a continuum of periodic orbits of (1.3)

or

(ii) (1.3) has no periodic orbits and for all solutions

$$\begin{aligned} \underline{x}(\cdot) = (x(\cdot), y(\cdot)) \text{ of (1.3)} \\ \lim_{t \rightarrow \pm\infty} \text{dist}(\underline{x}(t), E) = 0. \end{aligned}$$

Furthermore, in case (ii), the ω -limit set (α -limit set) of every trajectory of (1.3) is a point $(x_\infty, y_\infty) \in E$.

Proof: Assume that (i) does not hold. By Lemma 1.1 the solutions $(x(\cdot), y(\cdot))$ of (1.3) are bounded. Let C denote the ω -limit set (resp. α -limit set) of $(x(\cdot), y(\cdot))$. C is non-empty, compact and connected. By Poincaré-Bendixson, [cf. M.W.Hirsch and S. Smale (1974)] there are only the following 4 possibilities for C :

- (a) C is a periodic orbit.
- (b) C is a continuum of equilibria.

- (c) C is a finite set of equilibria together with trajectories joining them.
- (d) C is a single equilibrium point, $C = \{(x_\infty, y_\infty)\}$,

Let us consider first case (a). Since (i) does not hold, we may assume w.l.o.g. that C is a (possibly one-sided) limit cycle of (1.3). Thus all solutions, starting in a (possibly one-sided) open neighborhood of C tend to C . But then E must be actually constant on that neighborhood. Contradiction. Similarly in each of the cases (b) and (c) there would exist an open neighborhood (possibly one-sided) of C such that the trajectories starting there would converge to C .

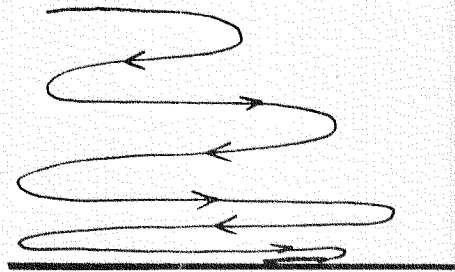


Figure 1: Flow near a continuum of equilibria

But this forces the integral E to take on a constant value on that neighborhood. Contradiction. Thus we are left with case (d), which completes the proof.

□

There are some simple a priori situations in which case (i) cannot occur. For example, let

$$(1.6) \quad g(x,y) \geq 0 \text{ (} g(x,y) \leq 0 \text{) for all } (x,y) \in \mathbb{R}^2.$$

Then $y(t)$ is monotonically increasing (resp. decreasing). But this implies that (1.3) has no periodic solutions at all.

A similar argument as in the proof of Prop. 1.2 shows:

Proposition 1.3

Consider the system (1.3) with equilibria set given by
 $E = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$ and assume

$$(1.7) \quad g(x, y) = 0 \Rightarrow x = 0.$$

Then the ω -limit set (resp. α -limit set) of each solution
 $(x(\cdot), y(\cdot))$ is either empty or an equilibrium point (x_∞, y_∞) ,
 $x_\infty = 0$.

Proof: The upper resp. lower half planes $\{(x, y) \mid x \geq 0\}$ resp.
 $\{(x, y) \mid x \leq 0\}$ are invariant under the flow of (1.3). By (1.7),
 $y(t)$ is monotonic on each half plane. Thus there does not
exist any periodic solution. Suppose the ω -limit set C of
 $(x(\cdot), y(\cdot))$ is non-empty. By monotonicity of $y(\cdot)$, the
cases (b) and (c) in the proof of Proposition 1.2 are excluded.
Thus $(x(t), y(t))$ must converge to an equilibrium point
 (x_∞, y_∞) with $x_\infty = 0$.

□

By combining the Propositions (1.2) and (1.3) we obtain:

Corollary 1.4

Consider the system (1.3) with equilibrium set
 $E = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$ and

$$(1.7) \quad g(x, y) = 0 \Rightarrow x = 0.$$

Assume (1.3) has an analytic function $E: \mathbb{R}^2 \rightarrow \mathbb{R}$ as a proper
first integral. Then for all solutions $(x(\cdot), y(\cdot))$ of (1.3)
one has:

$$(1.8) \quad \lim_{t \rightarrow \pm\infty} x(t) = 0.$$

$$(1.9) \quad \exists \lim_{t \rightarrow \pm\infty} y(t) = y_{\pm\infty} \in \mathbb{R}.$$

□

2. A necessary Condition for Universal Stabilizers

Let Σ^1 denote the class of all first order linear systems

$$(2.1a) \quad \dot{x} = \alpha x + \beta u$$

$$(2.1b) \quad y = \gamma x$$

with $\alpha, \beta, \gamma \in \mathbb{R}$, $\beta\gamma \neq 0$.

Definition 2.1

A *universal analytic adaptive stabilizer* (UAS) for Σ^1 is given by a pair of analytic functions

$$f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$$

such that for all initial conditions $x(0) \in \mathbb{R}$, $k(0) \in \mathbb{R}$ and all systems $(\alpha, \beta, \gamma) \in \Sigma^1$ the closed loop ordinary differential equation

$$(2.2) \quad \begin{array}{l} \dot{x} = \alpha x + \beta f(k, \gamma x) \\ \dot{k} = g(k, \gamma x) \end{array}$$

satisfies

$$(2.3a) \quad \lim_{t \rightarrow \infty} x(t) = 0$$

$$(2.3b) \quad \exists M \in \mathbb{R}^+ \text{ such that } |k(t)| < M \\ \text{for all } t \in [0, \infty).$$

If (2.3b) is replaced by the (apparently) stronger condition

$$(2.3b') \quad \exists \lim_{t \rightarrow \infty} k(t) = k_\infty < \infty$$

the pair (f, g) is called a *strict* universal analytic adaptive stabilizer.

□

As a first observation we note

Lemma 2.2

If (f, g) is universal for Σ^1 , then

$$(2.4) \quad g(k, y) = 0 \iff y = 0.$$

Proof: Suppose $g(k_0, y_0) = 0$ for some (k_0, y_0) with $y_0 \neq 0$.

Let $y = 1, \beta = -1, x(0) = y_0$ and $\alpha = \frac{f(k_0, y_0)}{y_0}$. For these choices, $(x(0), k_0)$ is an equilibrium point for (2.2) which does not satisfy (2.3a). Contradiction. □

Corollary 2.3

The function $g: R^2 \rightarrow R$ must have a fixed sign on $R \times R^+$ resp. $R \times R^-$. □

This corollary is an immediate consequence of the intermediate value theorem. In the following we show that for every universal (f, g) the component function $g_k: R \rightarrow R, k \rightarrow g(k, 0)$, cannot have finitely many zeroes.

Lemma 2.4

Let (f, g) be an analytic universal stabilizer for Σ^1 and assume $g_k \in R[k]$. Then necessarily $g_k \equiv 0$ and there exist an analytic function $\bar{g}: R^2 \rightarrow R$ and an integer $r \geq 1$ such that

$$(2.5) \quad g(k, y) = y^r \bar{g}(k, y) \quad \text{for all } (k, y) \in R^2$$

$$(2.6) \quad \text{Either } \bar{g}(k, y) \geq 0 \quad \text{or} \quad \bar{g}(k, y) \leq 0 \\ \text{for all } (k, y) \in R^2.$$

Proof: Clearly, by (2.3b), g_k must have a zero. Suppose g_k has only finitely many zeroes $-\infty < k_1 < \dots < k_N < \infty$. Let $D := \{(k, y) \mid k > k_N\}$. By Lemma 2.2 we have either

$$(2.7) \quad g(k, y) > 0 \quad \text{on } D$$

or

$$(2.8) \quad g(k,y) < 0 \quad \text{on } D$$

W.l.o.g. assume (2.7). Then the domain D is invariant under the flow. For each initial condition $(k_0, y_0) \in D$, $k(t)$ is monotonically increasing, hence converges by (2.3b) to some finite value $k_\infty > k_N$. Since $y(t) \xrightarrow{t \rightarrow \infty} 0$, $(0, k_\infty)$ must be an equilibrium point for (2.2). Contradiction.

Thus $g_k \equiv 0$. By the division theorem for analytic functions there exists $r \geq 1$ and an analytic function $\bar{g}: \mathbb{R}^2 \rightarrow \mathbb{R}$ with

$$g(k,y) = y^r \bar{g}(k,y) \quad \text{for all } (k,y) \in \mathbb{R}^2.$$

By choosing r maximally, $\bar{g}_k := \bar{g}(\cdot, 0)$ has only a discrete set of zeroes. The complement of this set of zeroes is a connected subset of \mathbb{R}^2 , hence $\bar{g}(k,y)$ cannot change sign on \mathbb{R}^2 .

□

Actually, this proof shows a little bit more:

Corollary 2.5

Let (f,g) be an analytic universal adaptive stabilizer for Σ^1 . Then for all $(\alpha, \beta, \gamma) \in \Sigma^1$, the closed loop system (2.2) has infinitely many equilibria points.

□

Proposition 2.6

Every UAS (f,g) for Σ^1 is a strict UAS.

Proof: Let E denote the equilibria set for (2.2). Consider first the case where $E = \{(k,y) | y = 0\}$. Then by Lemma 2.4 and Proposition 1.3 the solution $(k(t), y(t))$ converges to an equilibrium point $(0, k_\infty)$ and we are done. Now suppose that E is discrete. Because of Corollary 2.3 $k(\cdot)$ is monotonic on each half plane $\{(k,y) | y > 0\}$ resp. $\{(k,y) | y < 0\}$. Hence either the trajectory converges to some equilibrium point $(0, k_\infty)$ or winds around a finite set of equilibria $\{(0, k_1), \dots, (0, k_r)\}$, $k_1 < k_2 < \dots < k_r$.

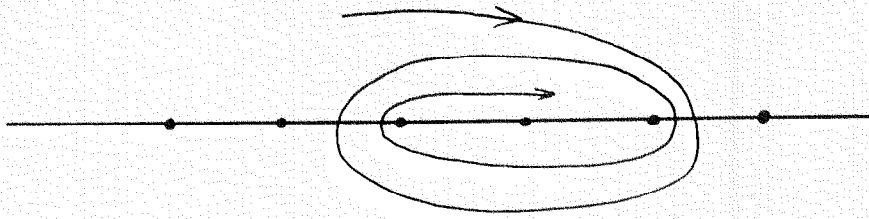


Fig. 2: An impossible situation

But this would imply that the whole enclosed interval $\{(0,k); k_1 \leq k \leq k_r\}$ consists of equilibria. Contradiction. □

Remark 2.7

Examples for admissible g 's which satisfy the conditions of Lemma 2.4 are

$$g(k,y) = y^r \cdot L, \quad L \neq 0, \quad r \in \mathbb{N}$$

$$g(k,y) = y^r \cdot e^k, \quad r \in \mathbb{N}$$

$$g(k,y) = y^r \cdot \left[\sum_{i=1}^p a_i y^{2i} + \sum_{i=1}^q b_i k^{2i} + L \right]$$

$$p, q \in \mathbb{N}, \quad a_i, b_i \in \mathbb{R}^+, \quad L > 0, \quad r \in \mathbb{N}$$

$$g(k,y) = y^r [L + \text{sink}], \quad L > 1$$

Excluded g 's are for example:

$$g(k,y) = y^2 + \lambda k, \quad \lambda \neq 0$$

$$g(k,y) = \pi_1(y) + \pi_2(k), \quad \pi_1 \in \mathbb{R}[y], \quad \pi_2 \in \mathbb{R}[k], \quad \pi_2 \neq 0$$

A function for which Lemma 2.4 cannot be applied is:

$$g(k,y) = y^2 + 1 + \text{sink}$$

Here $g(k,0) = 1 + \text{sink}$ has infinitely many zeroes and from the results given in this and the following section it is not clear, whether g can be extended to a UAS (f,g) for Σ^1 .

3. Classes of Universal Stabilizers

In this section we analyze certain families of universal adaptive controllers (2.2) which are shown to stabilize all first order systems:

$$(3.1)$$

$$\begin{array}{l} \dot{x} = \alpha x + \beta u \\ y = \gamma x \end{array}$$

$$, \quad (\alpha, \beta, \gamma) \in \Sigma^1 .$$

Previous universal adaptive stabilizers for Σ^1 , as those proposed by R. Nussbaum (1983), J.C. Willems and C.I. Byrnes (1984) and M. Heymann et.al. (1985), appear as special cases in these larger families of adaptive controllers.

Consider the system

$$(3.2a) \quad \dot{\xi} = f(\xi, \eta)\xi$$

$$(3.2b) \quad \dot{\eta} = \xi^r g(\xi, \eta)$$

where $r \geq 1$ and f, g are C^1 functions. We assume that g satisfies the necessary conditions for adaptive stabilization:

$$(3.3a) \quad g(\xi, \eta) = 0 \Rightarrow \xi = 0$$

$$(3.3b) \quad g(\xi, \eta) \geq 0 \quad \forall (\xi, \eta) \in \mathbb{R}^2$$

(or $g \leq 0$). We further assume that $\frac{f(\xi, \eta)}{g(\xi, \eta)}$ can be decomposed as

$$(3.4) \quad \frac{f(\xi, \eta)}{g(\xi, \eta)} = \tilde{f}(\eta) + h(\xi, \eta).$$

Here \tilde{f}, h are supposed to satisfy the growth conditions

$$(3.5)$$

There exist a compact set $K \subset \mathbb{R}^2$
and $M \in \mathbb{R}^+$ such that
 $|h(\xi, \eta)| \leq M \quad \forall (\xi, \eta) \in \mathbb{R}^2 \setminus K.$

$$(3.6a) \quad \sup_{\eta > 0} \frac{1}{\eta} \int_0^{\eta} \mathcal{F}(\sigma) d\sigma = +\infty$$

$$(3.6b) \quad \sup_{\eta < 0} \frac{1}{\eta} \int_0^{\eta} \mathcal{F}(\sigma) d\sigma = +\infty$$

$$(3.6c) \quad \inf_{\eta > 0} \frac{1}{\eta} \int_0^{\eta} \mathcal{F}(\sigma) d\sigma = -\infty$$

$$(3.6d) \quad \inf_{\eta < 0} \frac{1}{\eta} \int_0^{\eta} \mathcal{F}(\sigma) d\sigma = -\infty.$$

One of the main technical results in this paper can now be stated as follows:

Theorem 3.1

Suppose f, g satisfy (3.3) - (3.6). Then for all initial conditions $(\xi_0, \eta_0) \in \mathbb{R}^2$ the solutions of the system

$$\begin{aligned} \dot{\xi} &= f(\xi, \eta)\xi \\ \dot{\eta} &= \xi^r g(\xi, \eta), \quad r \in \mathbb{N} \end{aligned}$$

satisfy

$$(i) \quad \lim_{t \rightarrow \pm\infty} \xi(t) = 0$$

$$(ii) \quad \exists \lim_{t \rightarrow \pm\infty} \eta(t) = \eta_{\pm\infty} \in \mathbb{R}.$$

Proof: Consider

$$\begin{aligned} \dot{\xi} \xi^{r-1} &= f(\xi, \eta) \xi^r = \frac{f(\xi, \eta)}{g(\xi, \eta)} \dot{\eta} \\ &= (\mathcal{F}(\eta) + h(\xi, \eta)) \dot{\eta} \end{aligned}$$

resp.

$$(3.7) \quad \frac{1}{r} \xi^r(t) - \frac{1}{r} \xi^r(0) = \int_0^t \mathcal{F}(\eta) \dot{\eta} d\tau + \int_0^t h(\xi, \eta) \dot{\eta} d\tau.$$

By a change of variables this gives

$$(3.8) \quad \frac{1}{r} \xi^r(t) - \frac{1}{r} \xi^r(0) = \int_{\eta(0)}^{\eta(t)} \mathcal{F}(\sigma) d\sigma + \int_0^t h(\xi, \eta) \dot{\eta} d\tau.$$

We first show that $|\eta(\cdot)|$ is bounded. Note by (3.3) that $\eta(\cdot)$ is monotonic. Let $]t_-, t_+[$, $-\infty \leq t_- < 0 < t_+ \leq \infty$, denote the maximal time interval for which the solution $(\xi(\cdot), \eta(\cdot))$ exists. Suppose that $|\eta(t)| \rightarrow \infty$ for $t \rightarrow t_+$ (similarly for $t \rightarrow t_-$). By (3.5) there exists $0 < T < t_+$ such that

$$\frac{1}{r} \xi^r(t) - \frac{1}{r} \xi^r(0) \leq \int_0^T h(\xi, \eta) \dot{\eta} d\tau + \frac{\eta(t)}{\eta(0)} \int_0^{\eta(t)} \tilde{f}(\sigma) d\sigma + M|\eta(t) - \eta(T)|$$

and

$$\frac{1}{r} \xi^r(t) - \frac{1}{r} \xi^r(0) \geq \int_0^T h(\xi, \eta) \dot{\eta} d\tau + \frac{\eta(t)}{\eta(0)} \int_0^{\eta(t)} \tilde{f}(\sigma) d\sigma - M|\eta(t) - \eta(T)|.$$

Since $\int_0^T h(\xi, \eta) \dot{\eta} d\tau$ is finite these estimates yield:

$$(3.9a) \quad \frac{1}{r} \xi^r(t) - \frac{1}{r} \xi^r(0) \leq \text{const} + |\eta(t) - \eta(T)| \left[M + \frac{1}{|\eta(t) - \eta(T)|} \frac{\eta(t)}{\eta(0)} \int_0^{\eta(t)} \tilde{f}(\sigma) d\sigma \right]$$

resp.

$$(3.9b) \quad \frac{1}{r} \xi^r(t) - \frac{1}{r} \xi^r(0) \geq \text{const} - |\eta(t) - \eta(T)| \left[M - \frac{1}{|\eta(t) - \eta(T)|} \frac{\eta(t)}{\eta(t)} \int_0^{\eta(t)} \tilde{f}(\sigma) d\sigma \right].$$

Because of (3.6) the right hand sides of these two inequalities admits for $|\eta(t)| \rightarrow \infty$ arbitrary large positive and negative values. But this is a contradiction since clearly $\xi^r(t)$ cannot change its sign. Thus $|\eta(t)|$ is bounded for all t . Hence by (3.9) $|\xi(t)|$ is also bounded. By Proposition 1.3 this implies (i) and (ii). \square

Consider the following adaptive control scheme:

$$(3.10) \quad \begin{cases} \dot{y} = (\alpha + \beta \gamma f(k) g(k, y)) y \\ \dot{k} = \gamma^r g(k, y) \end{cases},$$

where f and g are analytic functions (actually C^1 would be enough for our arguments) and $(\alpha, \beta, \gamma) \in \Sigma^1$. We make the following assumption on g :

- (3.11) There exist a compact subset $K \subset \mathbb{R}^2$ and $m \in \mathbb{R}^+$ such that
 $g(k, y) \geq m > 0 \quad \forall (k, y) \in \mathbb{R}^2 \setminus K.$

Theorem 3.2

Let g satisfy (3.3), (3.11) and let f satisfy the Nussbaum type condition (3.6). Then:

(3.10)
$$\begin{cases} \dot{y} = (\alpha + \beta \gamma f(k)g(k, y))y \\ \dot{k} = \gamma^\top g(k, y) \end{cases}$$

is a universal adaptive stabilizer for Σ^1 ; i.e. for all initial conditions $(y(0), k(0)) \in \mathbb{R}^2$:

- (i) $\lim_{t \rightarrow \pm\infty} y(t) = 0$
(ii) $\lim_{t \rightarrow \pm\infty} k(t) = k_{\pm\infty} \in \mathbb{R}$ exists.

Proof: The proof is an immediate consequence of Theorem 3.1, since the pair (f, g) with $f(k, y) := \alpha + \beta \gamma f(k)g(k, y)$ satisfies the assumptions (3.3) - (3.6). \square

Special universal adaptive stabilizers of the general form (3.10) have been previously described by R. Nussbaum (1983) and J.C. Willems and C.I. Byrnes (1984). We give a brief discussion:

Example 1 (Nussbaum Controller)

For $g(k, y) = 1 + k^2$ and f satisfying (3.6), Theorem 3.2 guarantees global adaptive stability universal for all $(\alpha, \beta, \gamma) \in \Sigma^1$, for the closed loop system:

$$(3.12) \quad \begin{cases} \dot{y} = (\alpha + \beta \gamma f(k)(1+k^2))y \\ \dot{k} = y^r(1+k^2). \end{cases}$$

Here $r \geq 1$ is arbitrary. For $r = 1$ this is precisely the universal stabilizer found by R. Nussbaum. Note that the term $1+k^2$ is unnecessary to achieve global stability. This is discussed in the next example.

Example 2 (Byrnes/Willems Controller)

C.I. Byrnes and J.C. Willems have shown that the following controller:

$$(3.13a) \quad \dot{k} = y^2$$

$$(3.13b) \quad u = s(k)ky,$$

where $s: \mathbb{R} \rightarrow \mathbb{R}$ is bounded on compact sets and satisfies

$$(3.14a) \quad \sup_{k \geq 1} \frac{1}{k} \int_0^k s(\sigma)\sigma d\sigma = +\infty$$

$$(3.14b) \quad \inf_{k \geq 1} \frac{1}{k} \int_0^k s(\sigma)\sigma d\sigma = -\infty,$$

is a UAS for systems of the form (3.1). The closed loop system

$$(3.15a) \quad \dot{y} = (\alpha + \beta \gamma s(k)k)y$$

$$(3.15b) \quad \dot{k} = y^2$$

is a special case of (3.10) with

$$f(k) = s(k)k, \quad g(k,y) \equiv 1, \quad r = 2.$$

Thus the stabilizing property of (3.15) is an immediate consequence of Theorem 3.2. Moreover Theorem 3.2 implies that -

for one and the same choice of $f(k)$ - one can replace the gain adaptation law (3.15) by any formula of the type

$$(3.16) \quad \dot{k} = y^r g(k,y), \quad r \geq 1,$$

as long as $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ can be lower bounded on a neighborhood of ∞ . In particular the linear gain adaptation law is always sufficient to guarantee stability.

Remark

In a recent paper A. Ilchmann, D.H. Owens and D. Prätzel-Wolters (1987) showed that a controller of the form:

$$(3.17a) \quad u(t) = f(k) \cdot \psi(t) y(t)$$

$$(3.17b) \quad \dot{k}(t) = y^2(t) \cdot \psi(t)$$

where $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is any piecewise continuous function bounded from 0:

$$\exists \epsilon \in \mathbb{R}: \quad \psi(t) \geq \epsilon > 0 \quad \forall t \geq 0$$

globally stabilizes every system $(\alpha, \beta, \gamma) \in \Sigma^1$ (resp. every scalar system (A, b, c) which is controllable, observable, minimum phase and of relative degree 1). In one sense this type of controller opens more possibilities for stabilization than the corresponding specialization ($r=2$) of the adaptive configuration described in (3.10). $\frac{\dot{k}(t)}{y^2(t)}$ is not necessarily a function of k and y . In particular by choice of time functions $\psi(t)$ the adaptation process can be manipulated externally independent of the values of the gain $k(t)$ and output $y(t)$. On the other hand $g(k,y)$ in (3.10) is not required to be bounded from 0 and not necessarily positive. An extension of the general adaptation law $\dot{k} = g(k,y)$ in the definition of an UAS (cf. Definition 2.1) to $\dot{k} = g(t,k,y)$ would perhaps provide a common framework for the alternatives (3.10) and (3.17).

□

From a practical point of view, the BW-controller (3.15) is not quite suitable due to the following properties (cf. M. Heymann et.al. (1985)):

- The limiting dynamic $\hat{\alpha}_\infty = \lim_{t \rightarrow \infty} [\alpha + \beta y_s(k(t)k(t))]$ is unpredictable and depends in an erratic way on the initial data.
- The gain $k(t)$ tends to become unbounded as the initial disturbances of the plant get large (cf. Fig. 3).
- Initially highly stable systems may become periodically destabilized and the trajectories may be forced during the adaptation process to perform undesired oscillations (cf. Fig. 4).

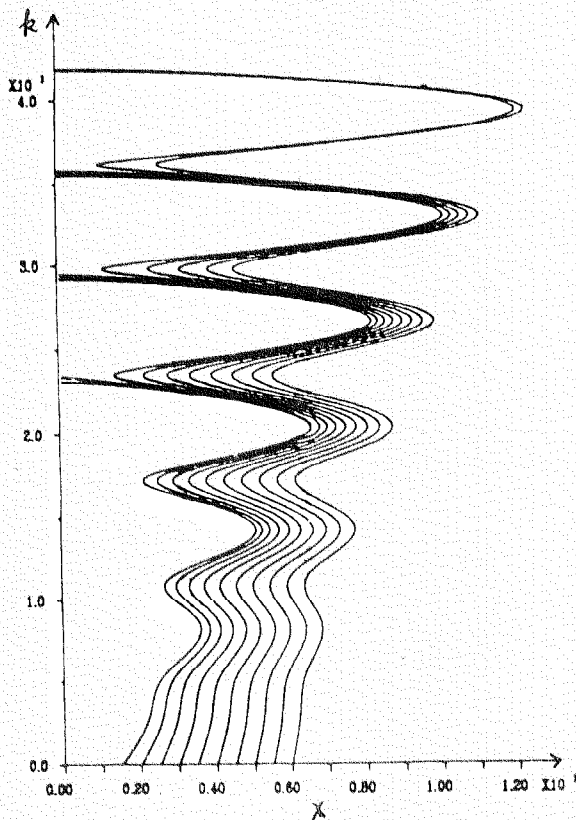


Figure 3:

Phase portrait BW-controller

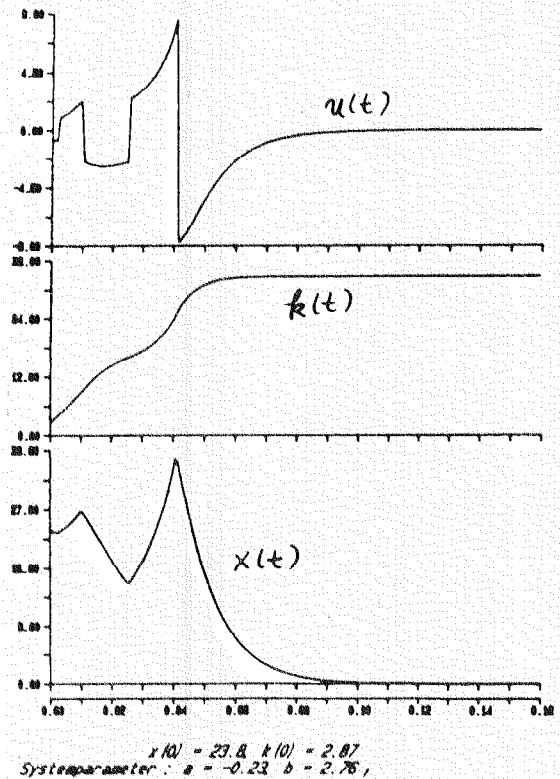


Figure 4:

Oscillations of a stable system

In view of these phenomena, M. Heymann, J.H. Lewis and G. Meyer (1985) proposed a new type of an adaptive controller which is based on a combination of a proportional and an integral gain adaptation:

$$(3.18a) \quad \dot{k} = \dot{y} + Ky^2, \quad K > 0$$

$$(3.18b) \quad u = (k^2 \cos k) \cdot y.$$

Written in closed loop form this yields the *Heymann-Lewis-Meyer (HLM) controller*:

$$(3.19a) \quad \dot{y} = (\alpha + \beta y k^2 \cos k)y$$

$$(3.19b) \quad \dot{k} = (K + \alpha + \beta y k^2 \cos k)y^2.$$

Observe, that the controller (3.18), although globally stabilizing, is not in the form of a u.a.s. in the sense of Definition (2.1). However, using the reparametrization

$$(3.20) \quad \xi = y, \quad \eta := k - \frac{1}{2}y^2$$

the system (3.18) takes on the following equivalent form

$$(3.19a') \quad \dot{\xi} = (\alpha + \beta y(\eta + \frac{1}{2}\xi^2)^2 \cos(\eta + \frac{1}{2}\xi^2))\xi$$

$$(3.19b') \quad \dot{\eta} = K\xi^2.$$

While this clearly has the form of an universal adaptive controller for Σ^1 , Theorem 3.1 does not apply to this. In the next result we describe a rather general stability result for such types of controllers.

Theorem 3.2

For any integer $r \geq 1$ and real numbers $\lambda > 0$, $K > 0$ consider the system

$$(3.21a) \quad \dot{x} = f(y + \lambda x^r)x$$

$$(3.21b) \quad \dot{y} = Kx^r$$

Suppose that $f: \mathbb{R} \rightarrow \mathbb{R}$ is analytic with the property:
For all $\lambda > 0$, $K > 0$ there exists countably many zeroes
 $(z_j)_{j \in \mathbb{Z}}$ of the equation $\lambda f(z) + K = 0$ with:

$$(3.22) \quad z_j < z_{j+1}, \quad j \in \mathbb{Z}.$$

$$(3.23) \quad z_j \rightarrow +\infty \quad \text{as } j \rightarrow \infty$$

$$(3.24) \quad z_j \rightarrow -\infty \quad \text{as } j \rightarrow -\infty.$$

Then for all initial conditions $(x_0, y_0) \in \mathbb{R}^2$ the solution
 $(x(\cdot), y(\cdot))$ of (3.21) satisfies:

$$(3.25a) \quad \lim_{t \rightarrow \infty} x(t) = 0$$

$$(3.25b) \quad \lim_{t \rightarrow \infty} y(t) = y_\infty \in \mathbb{R} \quad \text{exists.}$$

Proof: a) Set $\xi = x^r$, $\eta = y$. Then any solution $(x(\cdot), y(\cdot))$
of (3.21) gives a solution of

$$\dot{\xi} = r f(\eta + \lambda \xi) \xi$$

$$\dot{\eta} = K \xi.$$

Thus it suffices to prove Theorem 3.2 for $r = 1$.

b) Let $r = 1$. Using the new change of variables

$$\xi := x, \quad \eta := y + \lambda x,$$

we obtain the equivalent system

$$(3.26a) \quad \dot{\xi} = f(\eta) \xi$$

$$(3.26b) \quad \dot{\eta} = (K + \lambda f(\eta)) \xi.$$

(3.25) is equivalent to

$$(3.25a') \quad \lim_{t \rightarrow \infty} \xi(t) = 0$$

$$(3.25b') \quad \lim_{t \rightarrow \infty} \eta_{\infty} \in \mathbb{R} \text{ exists.}$$

Let $\Gamma(K, \lambda) = \bigcup_{j \in \mathbb{Z}} \mathbb{R} \times \{\eta_j\}$ denote the union of vertical lines in the (ξ, η) -plane defined by the zeroes η_j of $\lambda f(\eta) + K = 0$. From the form of (3.26) it is immediate that each such vertical line $\mathbb{R} \times \{\eta_j\}$ is invariant under the flow of (3.26), hence consists of trajectories. Since $\lambda f(\eta_j) + K = 0$ implies

$$f(\eta_j) = -\frac{K}{\lambda} < 0,$$

these trajectories must converge exponentially to 0.

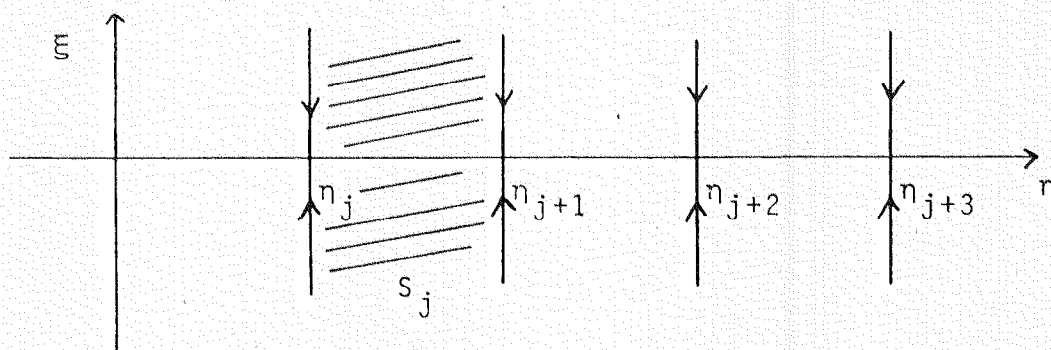


Fig. 5: *Invariant stripes*

Thus it remains to verify (3.25a'), (3.25b') in each (invariant) stripe $S_j = \mathbb{R} \times]\eta_j, \eta_{j+1}[$.

To this end, we distinguish between two cases.

Case I: $\lambda f(\eta) + K < 0$ for $\eta \in]\eta_j, \eta_{j+1}[$.

Then $f(\eta) < 0$ for all $\eta \in]\eta_j, \eta_{j+1}[$. It follows that $\xi(t)$ and $\eta(t)$ both decrease or increase monotonically and hence must converge to some $(\xi_{\infty}, \eta_{\infty}) \in \overline{S_j}$. Since $(\xi_{\infty}, \eta_{\infty})$ must be an equilibrium point, $\xi_{\infty} = 0$.

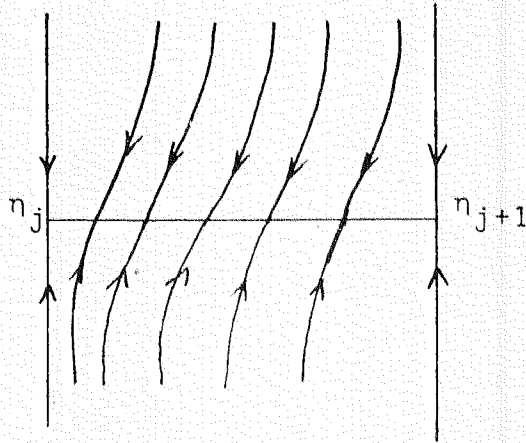


Fig. 6: Phase portrait, Case I

Case II: $\lambda f(\eta) + K > 0$ on $]\eta_j, \eta_{j+1}[$.

Then $f(\eta)$ has finitely many zeroes y_1, \dots, y_k in $]\eta_j, \eta_{j+1}[$ with:

$$y_0 := \eta_j < y_1 < \dots < y_k < \eta_{j+1} =: y_{k+1}$$

and $f(\eta)$ has constant sign on each interval $]y_\ell, y_{\ell+1}[$, $0 \leq \ell \leq k$. Furthermore by continuity of $f(\eta)$:

$$f(\eta) < 0 \text{ on }]y_0, y_1[\cup]y_k, y_{k+1}[$$

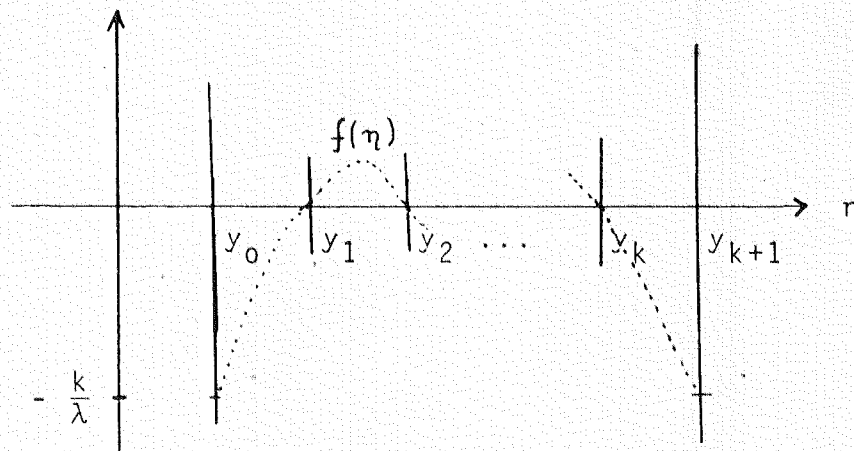


Fig. 7: $f(\eta)$ between 2 zeroes of $\lambda f(\eta) + K = 0$.

In the sector $R_- \times]y_0, y_1[$ (or in $R_+ \times]y_k, y_{k+1}[$) $\xi(t)$ resp. $\eta(t)$ are monotonically increasing resp. decreasing (or decreasing resp. increasing). Hence the trajectories starting $R_- \times]y_0, y_1[$ or in $R_+ \times]y_k, y_{k+1}[$ must remain there and converge to an

equilibrium point $(\xi_\infty, \eta_\infty)$, $\xi_\infty = 0$.

Consider now trajectories starting in $R \times [y_1, y_k]$. Because $\lambda f(\eta) + k$ is bounded from 0 on $[y_1, y_k] \subset S_j^0$ there exists a constant $C > 0$ such that

$$(3.27) \quad \left| \frac{d\xi}{d\eta} \right| = \frac{|f(\eta)|}{\lambda f(\eta) + k} \leq C \quad \text{for all } \eta \in [y_1, y_k]$$

holds for the solution of (3.26). Thus any solution of (3.26) starting in resp. entering the sector $R \times [y_1, y_k]$ either converges there to an equilibrium point $(0, \eta_\infty)$ or leaves this sector in a finite time, entering either $R_+ \times]y_k, y_{k+1}[$ or $R_- \times]y_0, y_1[$.

Similarly, consider any trajectory $(\xi(\cdot), \eta(\cdot))$ starting in $R_+ \times]y_0, y_1[$ resp. in $R_- \times]y_k, y_{k+1}[$. In the first case, $\xi(t)$ is monotonically decreasing and $\eta(t)$ increasing, while in the second case $\xi(t)$ is increasing and $\eta(t)$ decreasing. In any case, $(\xi(\cdot), \eta(\cdot))$ thus converges to some equilibrium point $(0, \eta_\infty)$. The result follows. \square

Consider now the HLM-type controller of the form:

$$(3.18a) \quad \dot{\xi} = (\alpha + \beta \gamma (\eta + \frac{1}{2} \xi^2)^2 \cos(\eta + \frac{1}{2} \xi^2)) \xi$$

$$(3.18b) \quad \dot{\eta} = \kappa \xi^2.$$

Here $\lambda = \frac{1}{2}$, $r = 2$ and

$$f(z) = \alpha + \beta \gamma z^2 \cos z$$

clearly satisfies the assumptions (i) - (iii) of Theorem 3.2 and we see that (3.18) is globally stable, thus rederiving Heymann's et.al. stability result. Furthermore, we obtain as an immediate Corollary of Theorem 3.2 the following extension of the HLM-controller:

Corollary 3.3 For all $\lambda \geq 0$, $K > 0$, $r \geq 1$

$$(3.28) \quad \begin{cases} \dot{\xi} = (\alpha + \beta\gamma(\eta + \lambda\xi^r)^2 \cos(\eta + \lambda\xi^r))\xi \\ \dot{\eta} = K\xi^r \end{cases}$$

is a u.a.s. for all $(\alpha, \beta, \gamma) \in \Sigma^1$. □

Let $r = 2$. For $\lambda = 1/2$ (3.28) gives the Heymann et.al. controller (3.18) while for $\lambda = 0$ we obtain a Byrnes-Willems type controller. This shows that the BW-controller can be viewed as a degenerate form of the controllers (3.21). This - perhaps - explains a little bit more why e.g. the HLM controller (3.19) shows better performance behaviour than the BW controller (3.15). Note further, that Theorem 3.2 shows global stability for a much wider class of controllers than those studied by Heymann et.al.

Obviously, the above classes of controllers do not at all exhaust the possible choices for designing universally adaptive stabilizers. We describe one further example of an UAS in order to demonstrate how some interesting types of differential equations arise in the context of adaptive stabilization.

Theorem 3.4

Consider the adaptive control scheme with $f, g \in C^1$ -functions:

$$(3.29a) \quad \dot{y} = (\alpha + \beta f(k) + \beta g(k)y)y$$

$$(3.29b) \quad \dot{k} = y$$

Let $G(k) := \int_0^k g(\sigma) d\sigma$ and $\Phi(k) := \int_0^k e^{-\beta G(\sigma)} (\alpha + \beta f(\sigma)) d\sigma$.

Assume that for all $\alpha, \beta, \beta \neq 0$:

$$(3.30a) \quad \sup_{k>0} \Phi(k) = +\infty$$

$$(3.30b) \quad \inf_{k>0} \Phi(k) = -\infty$$

$$(3.30c) \quad \sup_{k < 0} \Phi(k) = +\infty$$

$$(3.30d) \quad \inf_{k < 0} \Phi(k) = -\infty.$$

Then for all initial conditions (y_0, k_0) and all $\alpha \in \mathbb{R}$, $\beta \neq 0$

$$(i) \quad \lim_{t \rightarrow \pm\infty} y(t) = 0$$

$$(ii) \quad \lim_{t \rightarrow \pm\infty} k(t) = k_{\pm\infty} \in \mathbb{R} \text{ exists.}$$

Proof: The half planes $\{y|y > 0\}$ resp. $\{y|y < 0\}$ are invariant under the flow of (3.29). Since $k \neq 0$ on each half plane we can solve (3.29) by thinking of y as a function of k . We obtain the equivalent linear differential equation

$$(3.31) \quad \frac{dy}{dk} = y'(k) = \alpha + \beta f(k) + \beta g(k)y.$$

For initial conditions (k_0, y_0) integration of (3.31) gives:

$$\begin{aligned} y(k) &= e^{\beta[\sigma(k)-G(k_0)]} \left[y_0 + \int_{k_0}^k e^{-\beta[G(\sigma)-G(k_0)]} (\alpha + \beta f(\sigma)) d\sigma \right] \\ &= e^{\beta[G(k)-G(k_0)]} \left[y_0 + e^{\beta G(k_0)} (\Phi(k) - \Phi(k_0)) \right]. \end{aligned}$$

Let k_{\max} resp k_{\min} be the first values of $k, k > k_0$ resp. $k < k_0$ such that

$$\Phi(k) = \Phi(k_0) - y_0 e^{-\beta G(k_0)}.$$

By (3.30), k_{\max} and k_{\min} do exist.

Thus:

$$\lim_{k \rightarrow k_{\max}} y(k) = \lim_{k \rightarrow k_{\min}} y(k) = 0.$$

Let now $(y(\cdot), k(\cdot))$ denote the solution of (3.29) with $y(0) = y_0, k(0) = k_0$. By the above argument $(y(\cdot), k(\cdot))$

must exist for all $t \in \mathbb{R}$ and satisfy:

(i) $\lim_{t \rightarrow \pm\infty} y(t) = 0$

(ii) $\lim_{t \rightarrow \infty} k(t) = k_{\max}$, $\lim_{t \rightarrow -\infty} k(t) = k_{\min}$.

□

A special pair of functions f, g for which the assumption (3.30) of Theorem 3.4 is satisfied is e.g. given by:

$$g(k) \equiv 1, \quad f(k) = e^{k^2} \cos k.$$

Thus:

$$\begin{aligned} \dot{y} &= (\alpha + \beta e^{k^2} \cos k + \beta y)y \\ \dot{k} &= y \end{aligned}$$

is a UAS for Σ^1 .

More generally one might consider adaptive controllers of the form

$$(3.32) \quad \begin{aligned} \dot{y} &= \left(\alpha + \beta \sum_{j=0}^d f_j(k) y^j \right) y, \quad d \geq 0 \\ \dot{k} &= y. \end{aligned}$$

For $d = 2$ this leads to the *Riccati equation*

$$(3.33) \quad y' = \alpha + \beta(f_0(k) + f_1(k)y + f_2(k)y^2),$$

while for $d = 3$ we obtain *Abel's equation*

$$(3.34) \quad y' = \alpha + \beta(f_0(k) + f_1(k)y + f_2(k)y^2 + f_3(k)y^3)$$

for the reparametrized solutions $y(k)$ of (3.32).

4. Robustness Properties

It should be obvious from the previous results that the class of universal stabilizers for the set Σ^1 of first order systems forms a thin subset UAS of the class of all vector fields in the plane. In particular the stability requirements (2.3) are highly non generic and the resulting closed loop systems (2.2) are quite degenerate as vector fields in the plane. Thus one should expect that even small perturbations of the control systems (2.2) will in general lead to unpredictable changes in the dynamical behaviour. We will see that this is not quite true. Actually the proposed controllers share a certain amount of robustness under perturbations. However this does not mean that they all behave similarly under e.g. additive noise.

In this section we analyze the effect of different types of perturbations on the stabilizability properties of UAS's. Beside presenting some theoretical results we demonstrate the differences in the dynamical behavior (for example bursting phenomena) by means of simulation experiments.

(I) Nonlinear structural perturbations

We consider perturbations of the system (3.1) of the form:

$$(4.1) \quad \dot{y} = \alpha y + \beta \gamma u + h(t, k, y) \cdot y$$

where the C^1 function $h: \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfies

$$(4.2) \quad \exists L \in \mathbb{R}^+ \text{ such that } |h(t, k, y)| \leq L \\ \text{for all } (t, k, y) \in \mathbb{R}^3.$$

For example, $h(t, k, y)$ can be given as a time varying perturbation caused by feedback loops or by nonlinear effects in the plant sensor or actuator. We note the following result:

Theorem 4.1

Let f satisfy (3.6) and suppose $h: \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfies (4.2). Then

$$(4.3a)$$

$$\dot{y} = (\alpha + \beta y f(k) + h(t, k, y))y$$

$$(4.3b)$$

$$\dot{k} = y^r, \quad r \geq 1$$

is a UAS, i.e. for all $(\alpha, \beta, \gamma) \in \mathbb{R}^1$ and all initial conditions $(k(0), y(0)) \in \mathbb{R}^2$:

$$(i)$$

$$\lim_{t \rightarrow \infty} y(t) = 0$$

$$(ii)$$

$$\lim_{t \rightarrow \pm\infty} k(t) = k_{\infty} \in \mathbb{R} \text{ exists.}$$

Proof: As in the proof of Theorem 3.1 one shows that $|y(t)|$ and $|k(t)|$ are bounded for all $t \in \mathbb{R}$. $k(t)$ is monotonic and thus converges for $t \rightarrow \pm\infty$: $\lim_{t \rightarrow \pm\infty} k(t) = k_{\pm\infty}$. From (4.3b) we have: $y \in L^r(\mathbb{R})$ and (4.3a) implies that $\dot{y} \in L^r(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$. Thus $y(t) \rightarrow 0$ for $t \rightarrow \pm\infty$ (cf. Appendix). \square

Theorem 4.1 has the following immediate consequence:

Corollary 4.2

Assume $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -function which is uniformly bounded on \mathbb{R} . Then for every f satisfying (3.6) and any nonzero β, γ the closed loop system:

$$(4.4)$$

$$\begin{cases} \dot{y} = (\alpha(t) + \beta y f(k))y \\ \dot{k} = y^r, \quad r \geq 1 \end{cases}$$

is globally stable, i.e. satisfies

$$(i)$$

$$y(t) \rightarrow 0 \text{ for } t \rightarrow \pm\infty$$

$$(ii)$$

$$\lim_{k \rightarrow \pm\infty} k(t) = k_{\pm\infty} \in \mathbb{R} \text{ exists.}$$

\square

Thus any Nussbaum type function f satisfying (3.6) also stabilizes adaptively linear time varying systems

$$(4.5) \quad \begin{cases} \dot{x} = \alpha(t)x + \beta u \\ y = \gamma x \end{cases}$$

for arbitrary bounded $\alpha(\cdot)$. While this is rather obvious from

Theorem 3.1 if $\alpha(\cdot)$ is supposed to be 'slowly varying' we see that this is also true in general, at least as long as $\alpha(\cdot)$ remains bounded.

We demonstrate Corollary 4.2 by some simulation experiments.

The following figures illustrate the behaviour of the discontinuous BW-controller:

$$(*) \quad \dot{k} = y^2, \quad u = s(k)ky$$

$$s(k) = \begin{cases} 1 & \text{if } n^2 \leq k < (n+1)^2, \quad n = 0, 2, \dots \\ -1 & \text{if } n^2 \leq k \leq (n+1)^2, \quad n = 1, 3, \dots \end{cases}$$

applied to the time varying-linear plant:

$$\dot{y} = (1 + \sigma \cos \omega t)y - 0.5u$$

The initial values for all simulations are $(x_0, k_0) = (2, 2)$.

The pictures show that for fixed σ the variation of the frequency ω has nearly no influence on the solutions. Furthermore, very rapid oscillations with high amplitudes increase the oscillations in the trajectories. However they do not force the system to generate large output values $|y(t)|$.

We emphasize one apparent paradoxon. As the simulations (\cdot) show, the adaptive control scheme $(*)$ behaves particularly nice *the larger the perturbation amplitude is.*

Simulations (A)

$$\Sigma : \quad \dot{x} = (1 + \sigma \cos \omega t)x - 0.5u, \quad y = 0.5x$$
$$\dot{k} = y^2, \quad u = s(k)ky$$

$$s(k) = \begin{cases} 1 & \text{if } n^2 \leq k < (n+1)^2, \quad n = 0, 2, \dots \\ -1 & \text{if } n^2 \leq k \leq (n+1)^2, \quad n = 1, 3, \dots \end{cases}$$

$x_0 = 2, \quad k_0 = 2$ Simulation time: 10s
step size: 10000

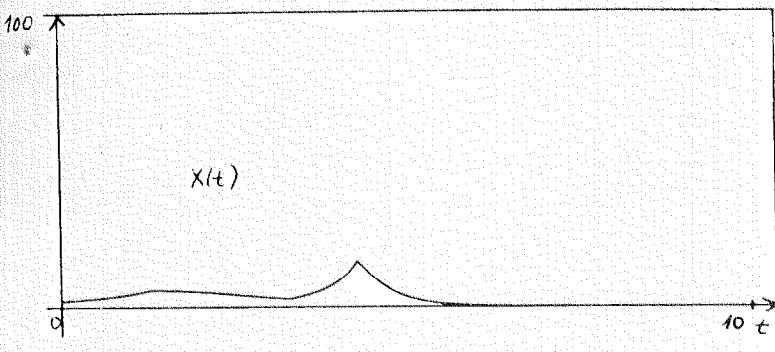


Fig. 8a: $\sigma = \omega = 0$

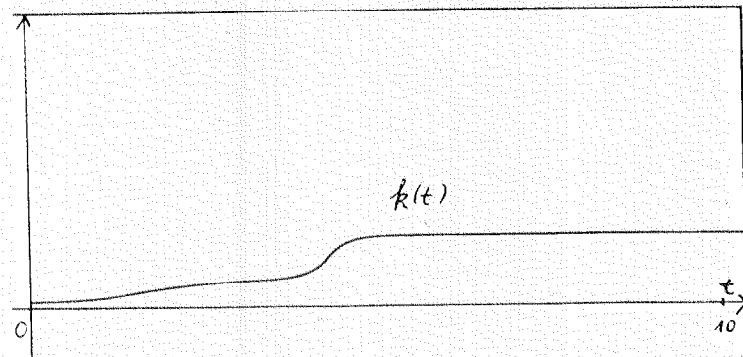


Fig. 8b: $\sigma = \omega = 0$

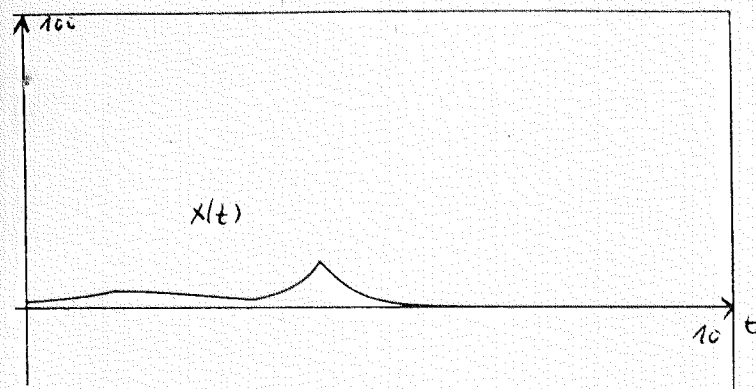


Fig. 8c: $\sigma = 1, \omega = 1000$

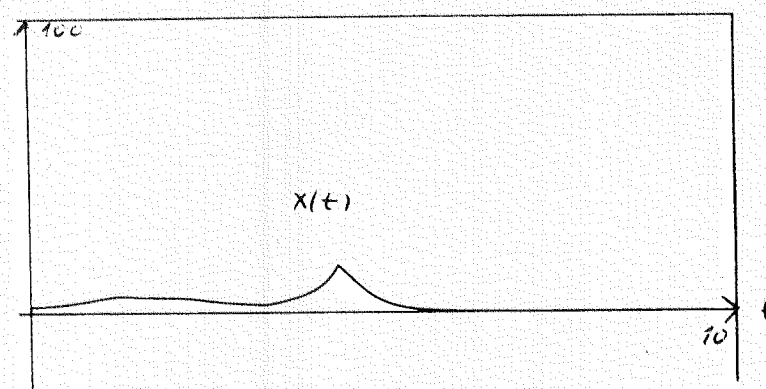


Fig. 8d: $\sigma = 1, \omega = 10$

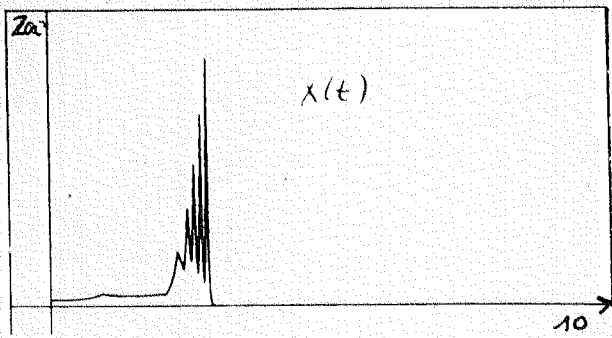


Fig. 8e: $\sigma = 3, \omega = 1$

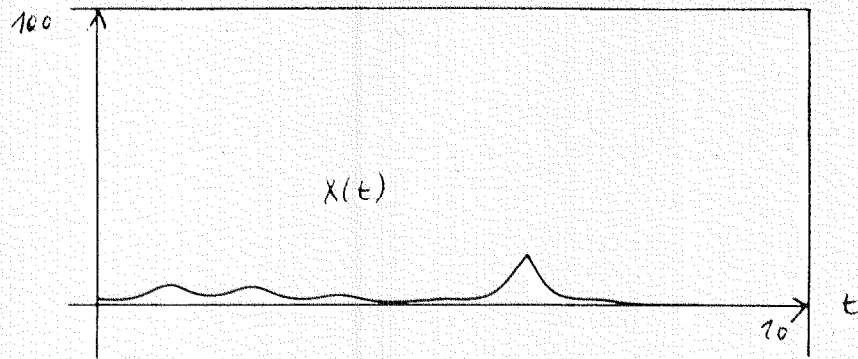


Fig. 8f: $\sigma = 8, \omega = 10$

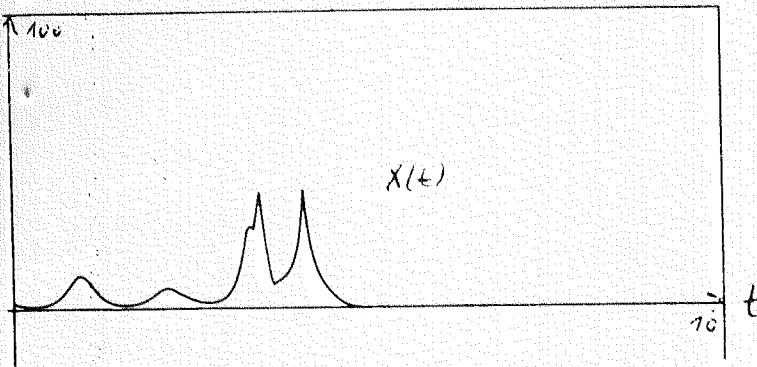


Fig. 8g: $\sigma = 20, \omega = 10$

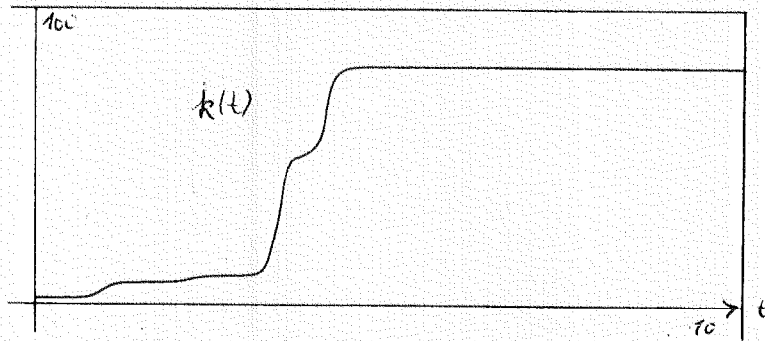


Fig. 8h: $\sigma = 20, \omega = 10$

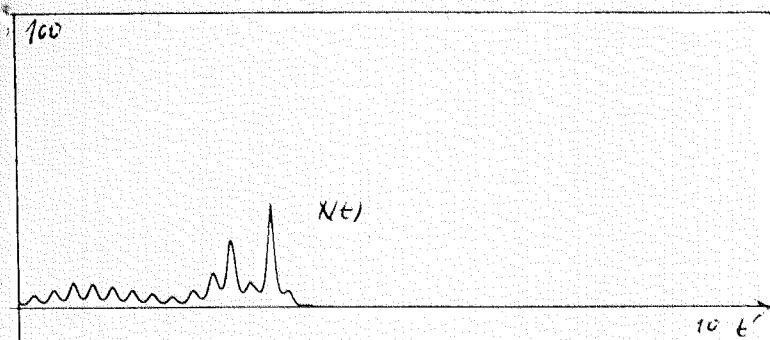


Fig. 8i: $\sigma = 30, \omega = 30$

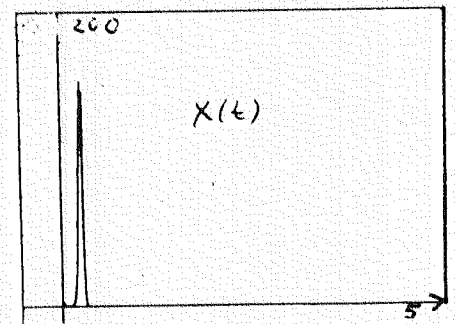


Fig. 8j: $\sigma = 60, \omega = 10$

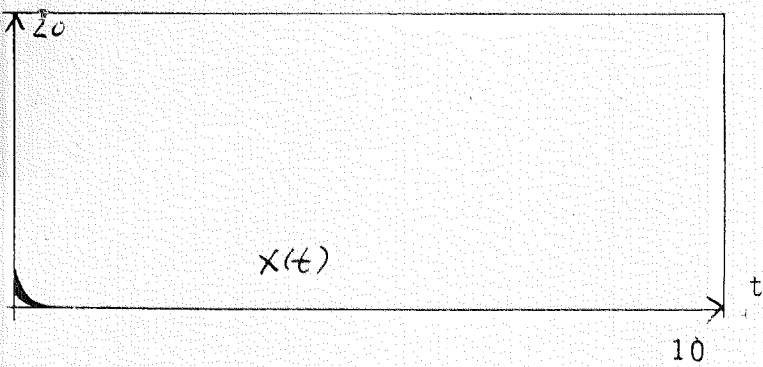


Fig. 8k: $\sigma = 500, \omega = 500$

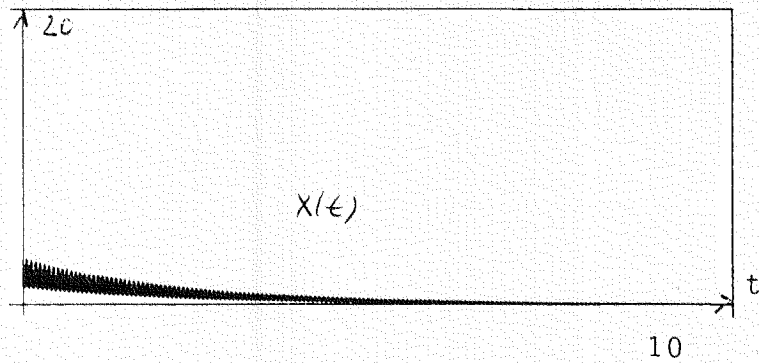


Fig. 8l: $\sigma = 200, \omega = 200$

A further simulation experiment shows the behaviour of the time varying linear plant:

$$\dot{y}(t) = 4.9y(t) - 0.5u(t) + \sigma \cos(\omega \cdot y(t)) \cdot y(t)$$

controlled by the BW-controller. Here the time varying perturbations:

$$h(t, y, k) = \sigma \cos(\omega \cdot y(t))$$

depend explicitly on the current values of the output $y(t)$. Initial values for the following simulation pictures are $(x_0, k_0) = (2.7, 2.2)$. Again these figures show that a change in the parameters σ and ω (in a reasonable range) has no drastic effect on the behavior of the closed loop system.

Simulations (B)

$$\Sigma : \quad \dot{x}(t) = 4.9 x(t) - 0.5u(t) + \sigma \cos (\omega \cdot x(t)) \cdot x(t) \\ = [4.9 + \sigma \cos[\omega \cdot x(t)]]x(t) - 0.5u$$

$$\dot{k} = y^2, \quad y = 0.5x$$

$$u = s(k)ky, \quad x_0 = 2.7, \quad k_0 = 2.2$$

Simulation time: $T = 5s$
Step size: $h = 5000$

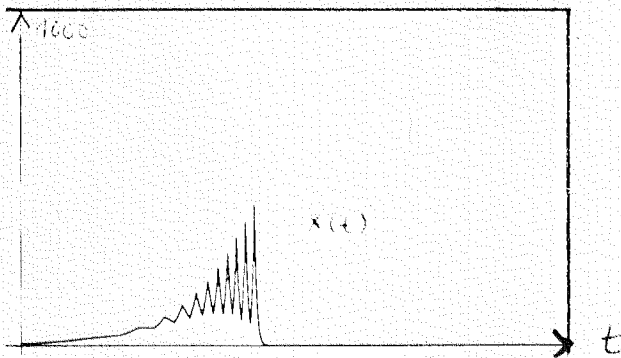


Fig. 9a: $\sigma = \omega = 1$

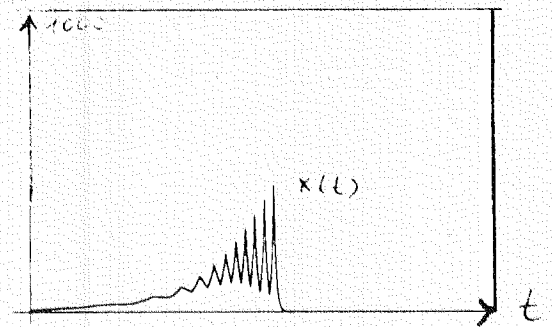


Fig. 9b: $\sigma = 5, \omega = 1$

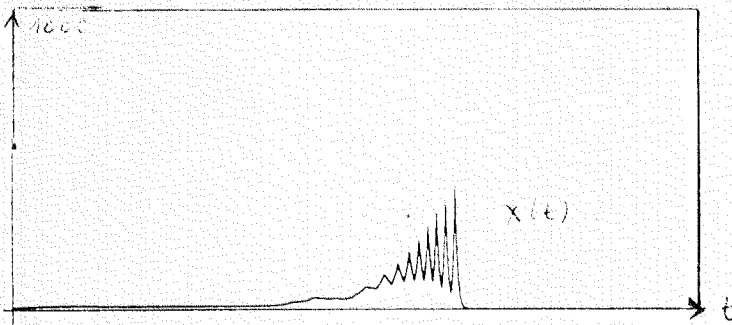


Fig. 9c: $\sigma = 15, \omega = 1$

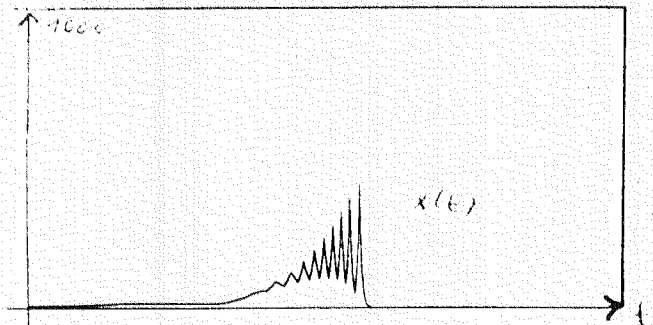


Fig. 9d: $\sigma = 10, \omega = 500$

(II) Additive input perturbations

As shown in the previous section, any gain adaptation law $\dot{k} = y^r$, $r \in \mathbb{N}$, can be combined with a Nussbaum type feedback law $u = f(k)y$ to generate a UAS for Σ^1 . Furthermore these adaptive stabilizers are tolerant with respect to nonlinear structural perturbations, independent of the choice of r . However if additive input perturbations $e(t)$ are considered:

$$(4.6a) \quad \dot{y} = (\alpha + \beta \gamma f(k))y + e(t)$$

$$(4.6b) \quad \dot{k} = y^r$$

the stability behavior of the closed loop system (4.6) essentially differs if r is odd or even. If r is even then arbitrary small sinusoidal perturbations $e(t)$ (or other noisy signals) generate bursting phenomena for the trajectories of (4.6). Since the integrator (4.6b) is continuously excited by $e(t)$, $k(t)$ will wind up and highly negative poles can become positive when the switching functions $f(k)$ changes its sign.

However for the *linear* adaptation law $\dot{k} = y$, *these bursting phenomena do not occur*. Even large disturbances like $e(t) = 1000 \cos 10t$ only marginally influence the closed loop trajectories $y(t)$. We have the following result:

Theorem 4.3

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfy (3.6) and $e: \mathbb{R} \rightarrow \mathbb{R}$ be piecewise continuous with $|\int_0^t e(\tau) d\tau| < \infty$ for all t . Then any solution of

$$(4.7a) \quad \dot{y} = (\alpha + \beta f(k))y + e(t)$$

$$(4.7b) \quad \dot{k} = y$$

satisfies:

$$(i) \quad |k(t)| \leq M \text{ is uniformly bounded on } \mathbb{R}.$$

$$(ii) \quad |\int_0^t y(\tau) d\tau| \text{ is uniformly bounded on } \mathbb{R}.$$

Proof: Set $F(k) := \int_{k_0}^k (\alpha + \beta f(\sigma)) d\sigma$. Integrating (4.7a) gives

$$\dot{k} = F(k) + \int_0^t e(\tau) d\tau + \text{const.}$$

Since $\int_0^t e(\tau) d\tau$ is bounded this yields the differential inequality:

$$(4.8) \quad F(k) - C \leq \dot{k} \leq F(k) + C$$

By (3.6), the functions $F(k) + C$ and $F(k) - C$ have infinitely many zeroes which tend to $+\infty$ (resp. $-\infty$) as k goes to $+\infty$ (resp. $-\infty$). Hence every solution of:

$$(4.9) \quad \dot{k} = F(k) + C \quad \text{resp.} \quad \dot{k} = F(k) - C$$

is bounded between two such successive zeroes of $F(k) = -C$ resp. $F(k) = +C$. Since every solution $k(t)$ of (4.7) is upper bounded (lower bounded) by a solution of $\dot{k} = F(k) + C$ (resp. $\dot{k} = F(k) - C$) the result follows. □

Simulations (c)

Linear adaptation $\dot{k} = y$ versus square adaptation $\dot{k} = y^2$

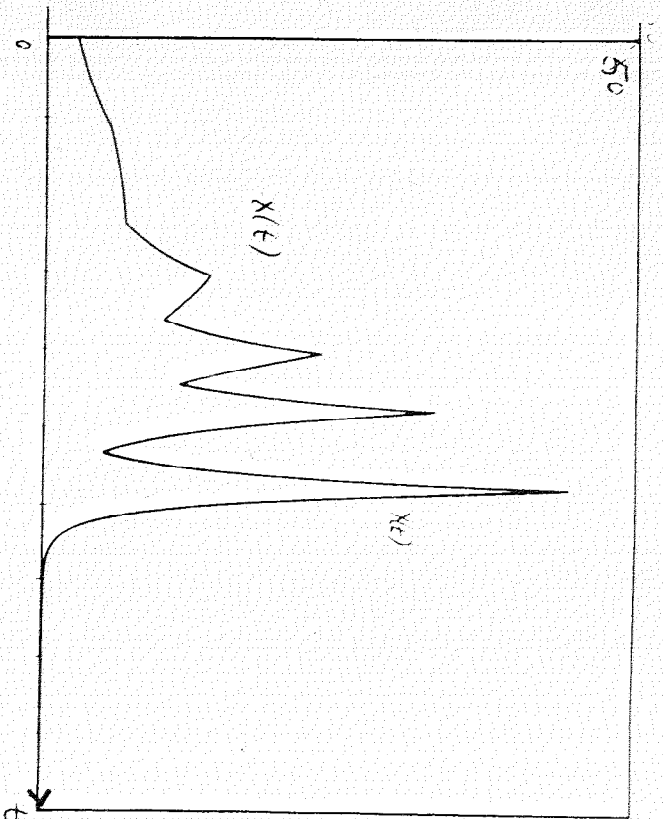
$$\Sigma : \quad \dot{x} = 4.9x - 0.5u + \sigma \cos \omega t$$

$$\dot{k} = x^2$$

resp. $\dot{k} = x$

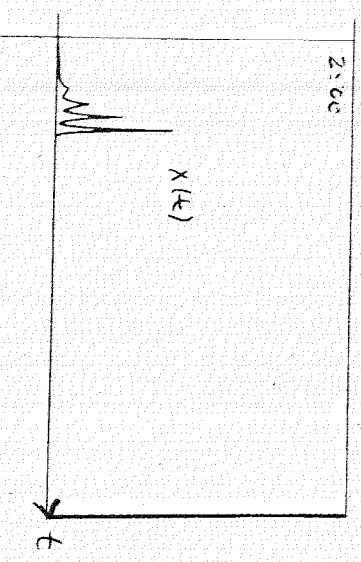
$$u = s(k)kx, \quad (x_0, k_0) = (2.7, 2.2)$$

Simulation time: $T = 10s$, Step size: $h = 10000^{-1}$



$$\dot{k} = x^2$$

Fig. 10a: $\sigma = \omega = 0$



$$\dot{k} = x$$

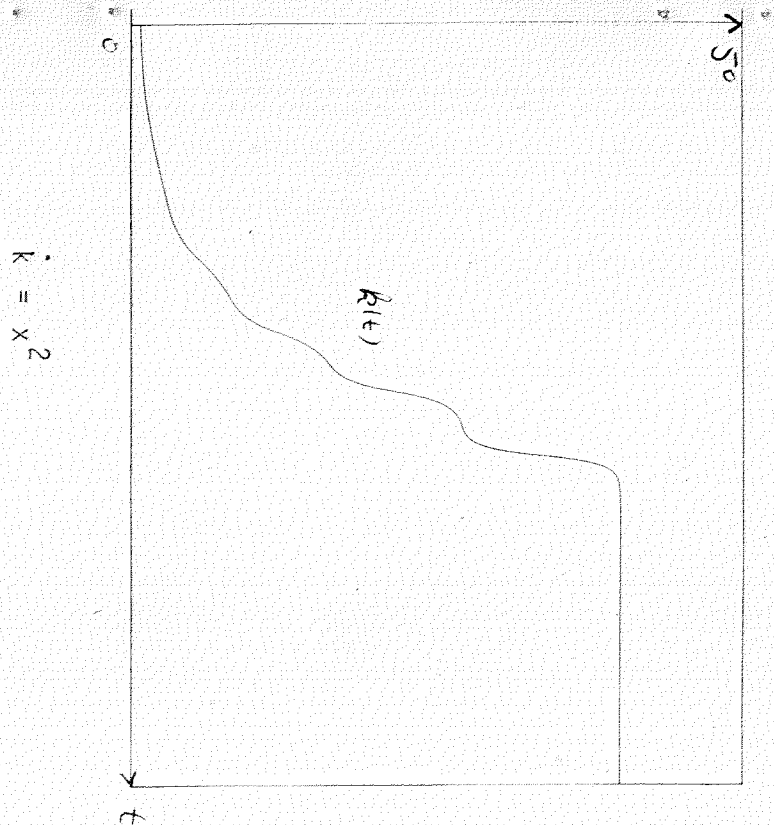


Fig. 10b: $\sigma = \omega = 0$

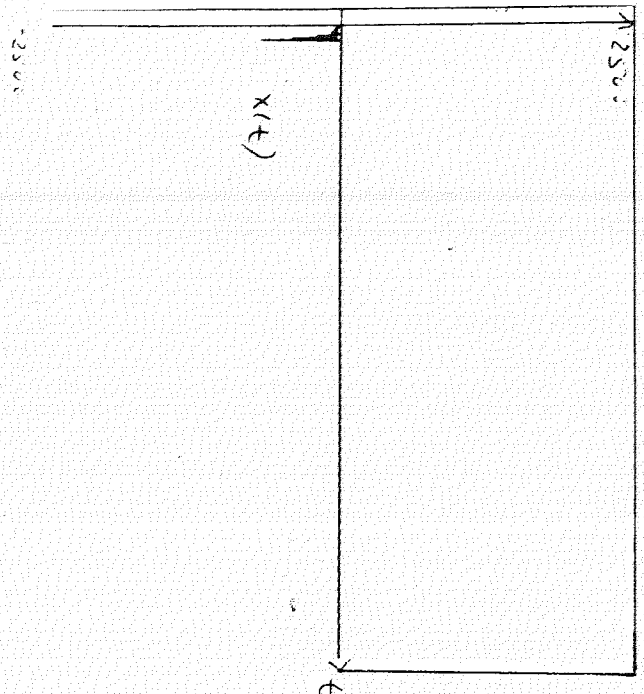
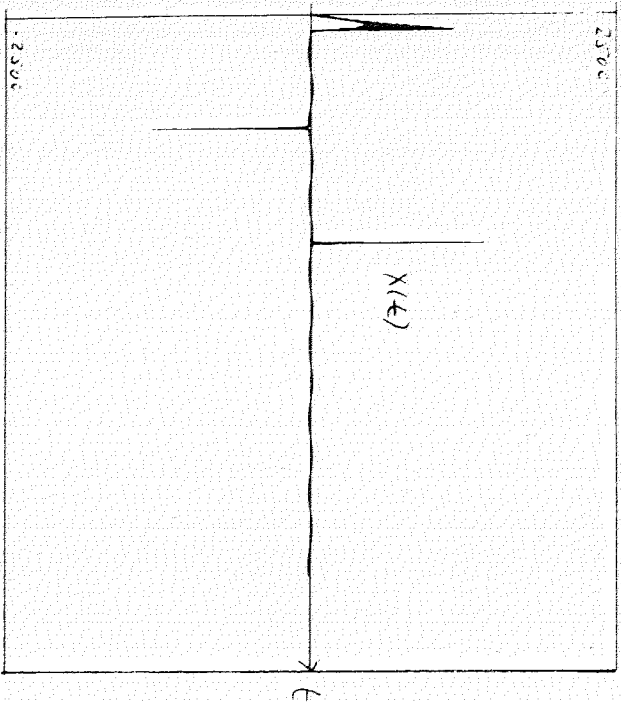
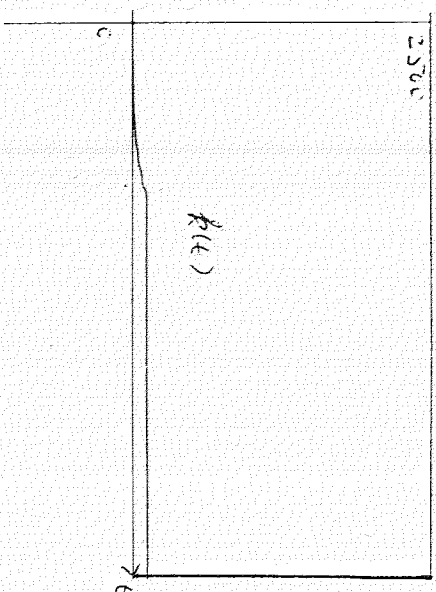
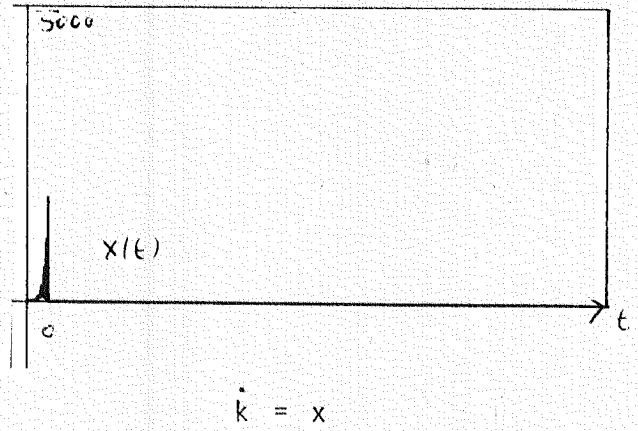
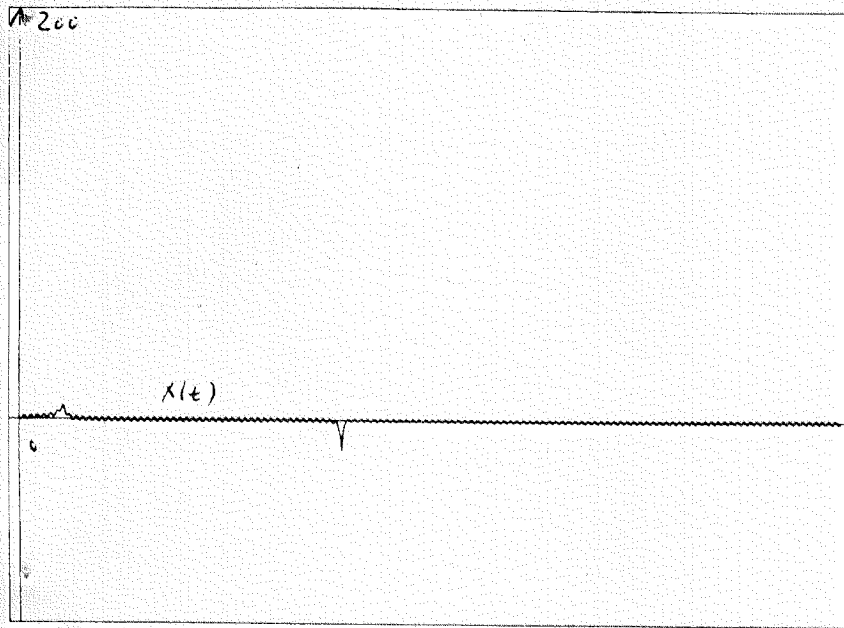
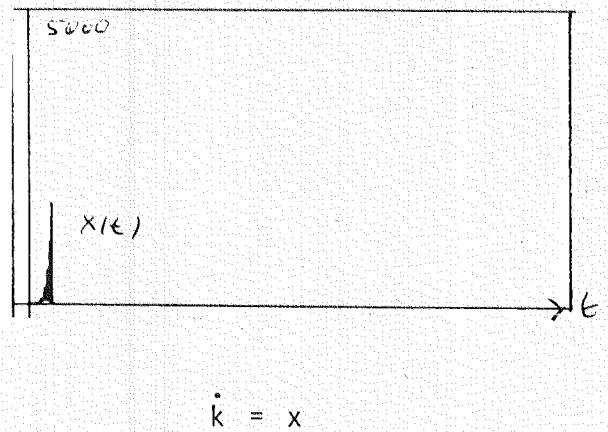
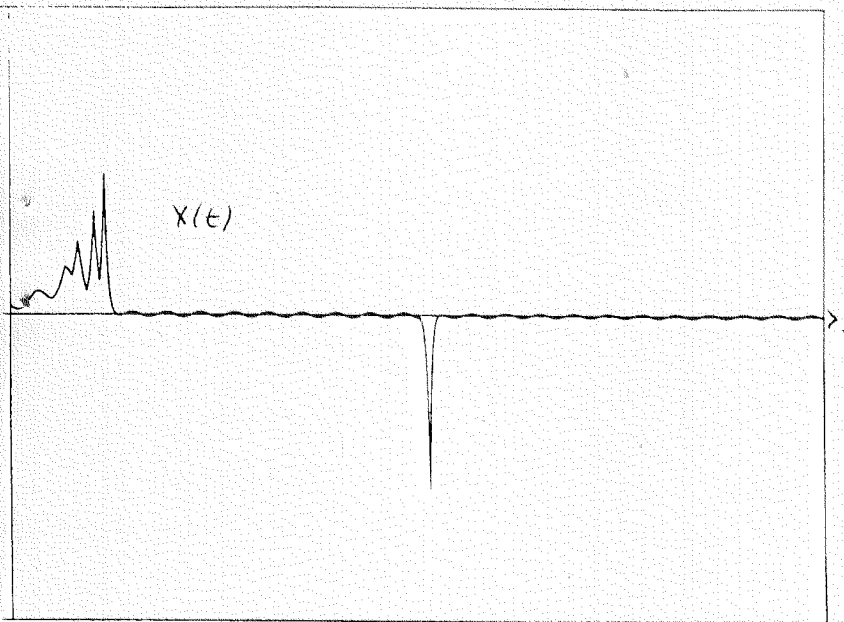


Fig. 10c: $\sigma = 1000, \omega = 10$



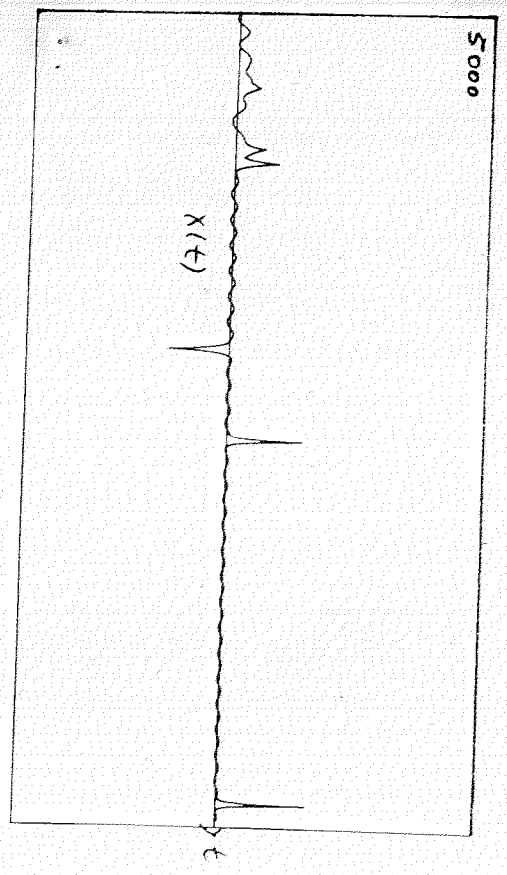
$$\dot{k} = x^2$$

Fig. 10d: $\sigma = 1000, \omega = 100$

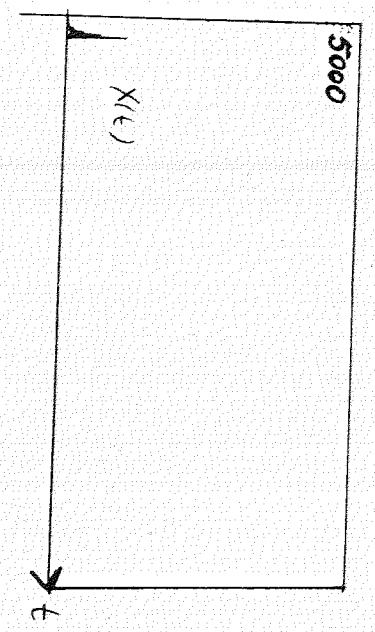


$$\dot{k} = x^2$$

Fig. 10.e: $\sigma = 100, \omega = 30$



$$\dot{k} = x^2$$



$$\dot{k} = x$$

Fig. 10f: $\sigma = 400, \omega = 40$

The next theorem shows that L_p -disturbances $e(t)$ ($\int_0^\infty |e(t)|^p dt < \infty$) are regulated to 0 by the 'adapted' adaptation law $\dot{k} = y^p$, if p is even.

Theorem 4.4

Let $p = 2q$ even and $f(\cdot)$ satisfy (3.6). Let $e: \mathbb{R} \rightarrow \mathbb{R}$ be continuous with $\int_0^\infty |e(t)|^p dt < \infty$. Then every solution of

$$(4.10a) \quad \dot{y} = (\alpha + \beta f(k))y + e(t)$$

$$(4.10b) \quad \dot{k} = y^p$$

satisfies:

$$(i) \quad \lim_{t \rightarrow \infty} y(t) = 0$$

$$(ii) \quad \lim_{t \rightarrow \infty} k(t) \in \mathbb{R} \text{ exists.}$$

Proof: Let $F(k) = \int_{k_0}^k (\alpha + \beta f(\sigma)) d\sigma$. Then by integrating $y^{p-1} \dot{y}$ we obtain:

$$\frac{1}{p} y^p = F(k) + \int_0^t e(\tau) y(\tau)^{p-1} d\tau + \text{const.}$$

By the Hölder inequality:

$$\left| \int_0^t e(\tau) y(\tau)^{p-1} d\tau \right| \leq \left[\int_0^t e(\tau)^p d\tau \right]^{\frac{1}{p}} \cdot \left[\int_0^t y(\tau)^p d\tau \right]^{\frac{p-1}{p}}$$

where $p = 2q$.

Thus for $E_\infty := \left(\int_0^\infty e(\tau)^p d\tau \right)^{\frac{1}{p}}$ we have:

$$\left| \int_0^t e(\tau) y(\tau)^{p-1} d\tau \right| \leq E_\infty \cdot \left[\int_0^t y(\tau)^p d\tau \right]^{\frac{p-1}{p}}$$

Now suppose that $|k(t)|$ is not bounded. By (4.10b) then $k(t)$ must monotonically increase to $+\infty$. Since $0 < \frac{p-1}{p} < 1$:

$$(k(t) - k(0))^{\frac{p-1}{p}} \leq (k(t) - k(0))$$

for $t \geq T$ large enough. Thus for $t \geq T$ we obtain:

$$\frac{1}{p} y^p(t) \leq F(k(t)) + E_\infty \cdot k(t) + \text{const.}$$

contradicting (3.6). Thus $k(t)$ is bounded and hence converges to some finite value $k_\infty \in \mathbb{R}$ as $t \rightarrow +\infty$. By (4.10b) this implies $y \in L^p[0, \infty)$ and by (4.10a) also $\dot{y} \in L^p[0, \infty)$. Thus $y(t) \rightarrow 0$ as $t \rightarrow \infty$ (cf. Appendix). □

Corollary 4.5

Let $f(\cdot)$ satisfy (3.6) and let $e: \mathbb{R} \rightarrow \mathbb{R}$ continuous, uniformly bounded and $\int_0^\infty |e(t)|^p dt < \infty$ for some $p \geq 1$. Then for every even q , $q \geq p$ the solutions of

$$\begin{aligned} \dot{y} &= (\alpha + \beta f(k))y + e(t) \\ \dot{k} &= y^q \end{aligned}$$

satisfy:

- (i) $\lim_{t \rightarrow \infty} y(t) = 0$
 - (ii) $\lim_{t \rightarrow \infty} k(t) \in \mathbb{R}$ exists.
-

This corollary is an immediate consequence of Theorem 4.4 and the fact that:

$$y \in L^\infty[0, \infty) \cap L^p[0, \infty) \Rightarrow y \in L^q[0, \infty) \quad \forall \quad q \geq p.$$

(III) HLM-controller with additive input noise

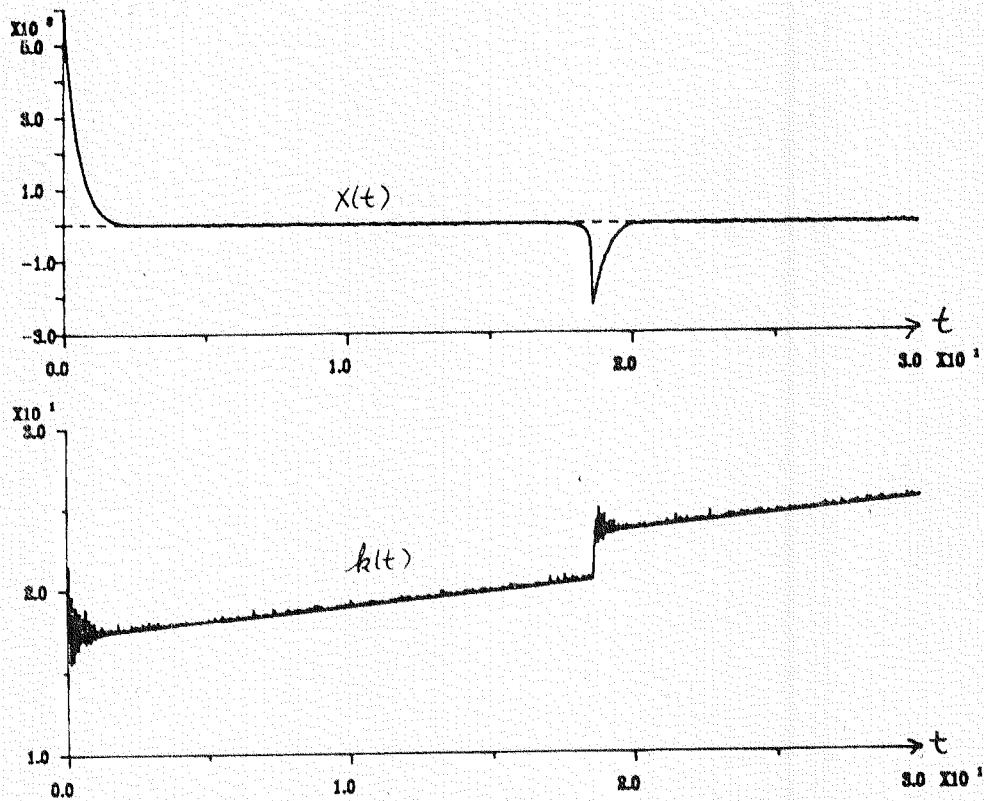
Finally we like to demonstrate that also the HLM-controller shows bursting phenomena if additive perturbations are applied to the controlled plants (cf. the Diploma thesis of J. Cordes and H.W. Philippson (1986)). The following simulation experiments show that changing the adaptation law from $\dot{k} = \frac{1}{2} \dot{x}x + Kx^2$ to the 'odd' adaptation law

$$\dot{k} = \dot{x}x^2 + Kx^3, \quad K > 0$$

does not prevent the system from 'bursting' if for example the systems are perturbed by additive gaussian noise $r(t)$:

$$\dot{x} = \alpha x + \beta u + r(t).$$

Here $r(t)$ is of mean value 0, standard deviation 0.3, and takes values in the intervall $[-1,+1]$. The simulations show an oscillating behavior of $x(t)$. The frequency of this oscillations depends on the initial value $x(0)$.



$$\dot{x} = x + 0.05 u$$

Fig.

*HLM-controller with
 x^2 -adaptation and input noise*

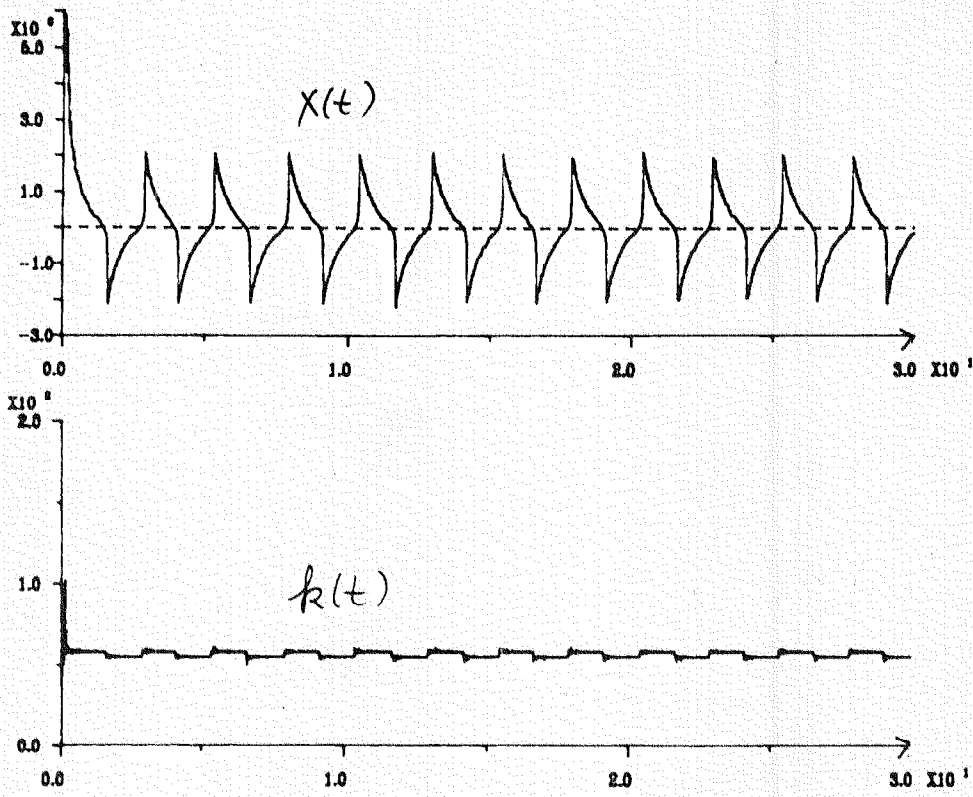


Fig. $\dot{x} = -x + 0.005 u, x(0) = 5$
HLM-controller with x^3 -adaptation and input noise

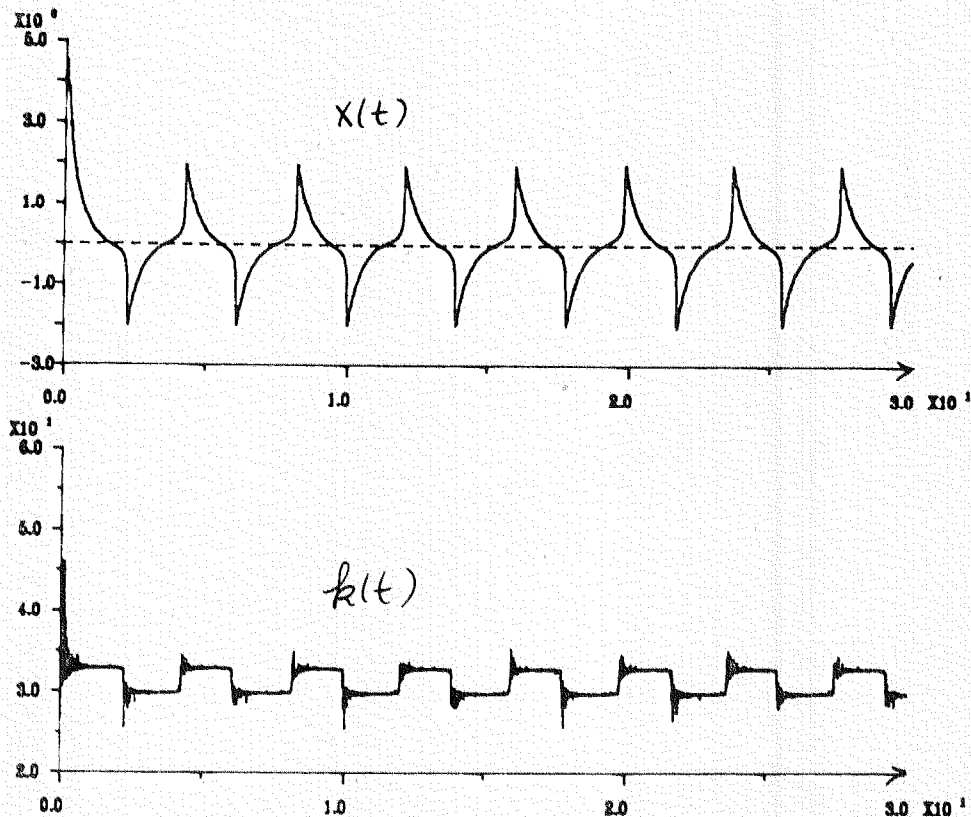


Fig. $\dot{x} = -x + 0.05 u, x(0) = 4$
HLM-controller with x^3 -adaptation and input noise

Appendix

The following result is frequently used in the adaptive control literature and also in this paper. However we could not find a proof in the references and hence we give one in this appendix.

Lemma

Let $p \in \mathbb{N}$. Then:

$$y(\cdot) \in L_{2p}[0, \infty) \text{ and } \dot{y}(\cdot) \in L_{2p}[0, \infty) \text{ implies } \lim_{t \rightarrow \infty} y(t) = 0.$$

Proof: We first prove the result for $p = 2$ $y(\cdot) \in L_2[0, \infty)$ implies that there exists a sequence $(t_k)_{k \in \mathbb{N}}$, $t_k \rightarrow \infty$ such that $\lim_{k \rightarrow \infty} y(t_k) = 0$. Integrating \dot{y} and applying Hölder's inequality we obtain for $t \geq t_k$:

$$\begin{aligned} \int_{t_k}^t \dot{y} dt &= \frac{1}{2} y(t)^2 - \frac{1}{2} y(t_k)^2 \\ &\leq \left(\int_{t_k}^t y^2 dt \right)^{\frac{1}{2}} \left(\int_{t_k}^t \dot{y}^2 dt \right)^{\frac{1}{2}} \\ &\leq \left(\int_{t_k}^{\infty} y^2 dt \right)^{\frac{1}{2}} \cdot \left(\int_{t_k}^{\infty} \dot{y}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

However $y(t_k) \xrightarrow{k \rightarrow \infty} 0$ and the right hand side of the last inequality tends to 0 if $k \rightarrow \infty$, thus $\lim_{t \rightarrow \infty} y(t) = 0$. In the general case $p \in \mathbb{N}$ it suffices to show

$$\dot{y}^p = p y^{p-1} \dot{y} \in L_2[0, \infty) \text{ because } y(\cdot) \in L_{2p}[0, \infty) \text{ implies}$$

$y^p(\cdot) \in L_2[0, \infty)$ and then $y(t) \rightarrow 0$ by the first part of this proof. Consider the pair $(p, \frac{2p}{2p-2})$, $p > 1$. Then $\frac{1}{p} + \frac{2p-2}{2p} = 1$, $\dot{y}^2 \in L_p[0, \infty)$ and $y^{2(p-1)} \in L_{\frac{2p}{2p-2}}[0, \infty)$. But then by Hölder's inequality $\dot{y}^2 y^{2(p-1)} \in L_1[0, \infty)$ and so $\dot{y} y^{p-1} \in L_2[0, \infty)$ \square

Finally if additionally $y(\cdot)$ and $\dot{y}(\cdot)$ are globally bounded on $[0, \infty)$ then the lemma is true for arbitrary L_p -spaces, because $y(\cdot), \dot{y}(\cdot) \in L_p[0, \infty) \cap L_\infty[0, \infty)$ implies $y(\cdot), \dot{y}(\cdot) \in L_q[0, \infty)$ for every $q, q \geq p$.

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