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PROPERTIES OF THE PREISACH MODEL FOR HYSTERESIS

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APRIL 1988

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Abstract.
As shown by Krasnosel'skii, the classical Preisach model allows to construct a hysteresis operator W between spaces of real functions of time. This construction, via the definition of a measure μ in the so-called Preisach plane, is recalled. Characterizations in terms of μ are given for several mapping and continuity properties of W in various function spaces, for the invertibility of W and for the corresponding mapping and continuity properties of the inverse.

§ 1 Introduction

The mathematical investigation of hysteresis models is still largely open, except for the rather special case of plasticity, despite of their obvious importance in applications. However, a remarkable effort in this direction is to be acknowledged to the Soviet school, cf. the monograph [6]; and interest in this subject has been spreading also among western mathematicians in the last few years.
It was already in 1935 that the physicist Preisach proposed in [9] a mathematical model for representing ferromagnetic hysteresis; this then became popular among scientists and engineers [2,3,7,8,12], and was also used for modeling other hysteresis phenomena. In fact, this model has an appealing geometrical interpretation, which also allows to perform efficient numerical computations. Actually it can be regarded as the most satisfactory mathematical model of hysteresis currently available.
The key idea of the Preisach model, also known as the independent domain model, consists in representing a general hysteresis loop by "adding" several elementary rectangular loops. Following Krasnosel'skii and Pokrovskii, compare e.g. [5,6], this construction is formalized here by defining a *hysteresis operator* acting between spaces of time-dependent functions, which correspond to the input and the output of a system with hysteresis. The formulation is presented in section 2, where the basic definitions of *elementary hysteresis operator* (or "*delay*") w_ρ, of the *Preisach plane P* and of the *Preisach operator* W are introduced. The latter is obtained by integrating the elementary operators w_ρ with respect to a measure μ defined in P.
The main aim of the present paper is to investigate the relationship between the properties of the measure μ and those of the corresponding operator W. In an effort to make our presentation as complete and self-contained as possible, also some results of [5,6,11] are reported here (with or without proof), aside several original ones; this shall be pointed out at each occurrence.
In section 2 the construction of the operator W is recalled

and a characterization of the case in which it operates in $C^o([0,T])$, $0<T<\infty$, is given. In section 3 it is proved that under the only condition that W operates in $C^o([0,T])$, it is also continuous with respect to the uniform topology of $C^o([0,T])$. This result was already stated in [6] and independently stated and proved in [11]. Our present argument is different from that of [11]; it is based on the analytical representation and treatment of the curve B(t), which in the Preisach plane separates the two phases characterized by $w_\rho=-1$ and $w_\rho=1$ respectively. We also analyze in detail the properties of B(t), which have an interest in themselves and can also be exploited in the numerical approximation, cf. [10].

In section 4 the cases in which W is uniformly continuous or even Lipschitz continuous in $C^o([0,T])$ are characterized in terms of the corresponding measure μ; these results are already included in [6], the second one without proof. Furthermore, necessary and sufficient conditions on μ are given for W to operate and have weak continuity properties in the Hölder spaces $C^{o,\lambda}([0,T])$ with $0<\lambda\leq 1$, in the Sobolev spaces $W^{1,p}(0,T)$ with $1\leq p\leq\infty$, and in $C^o([0,T]) \cap BV(0,T)$, where BV denotes the space of functions with bounded variation.

In section 5 the case in which W is invertible is characterized in terms of μ; it also turns out that W is one-to-one if and only if it is onto. Finally, necessary and sufficient conditions on μ are given for W^{-1} to operate and to be weak star continuous in the spaces $C^{o,\lambda}([0,T])$, $W^{1,p}(0,T)$ and $C^o([0,T]) \cap BV(0,T)$, as well as to be Lipschitz continuous in $C^o([0,T])$.

In section 6 we extend W to functions which depend also on space, not only on time. If $\Omega \subset \mathbb{R}^N$ is a Euclidean domain and $u = u(y,t)$ is a Carathéodory function, we set

$$[\tilde{W}(u)](y,t) = [W(u(y,\cdot))](t)$$

and extend the mapping and continuity properties of W and its inverse to corresponding properties of $\tilde{W}$ and its inverse in spaces of vector valued functions such as

$C^o(\Omega;C^o([0,T]))$, $W^{\lambda,p}(\Omega;C^o([0,T]))$, $L^p(\Omega;C^{0,\lambda}([0,T]))$, $L^p(\Omega;W^{1,p}(0,T))$ and $L^p(\Omega;C^o([0,T]) \cap BV(0,T))$.

These results do not exhaust the properties of the Preisach operator. For instance a question, which is not considered here, is how W changes as μ changes in the space of measures on the Preisach plane. This question arises if one wants to identify the measure μ, which is of interest in applications. Some results concerning the identification problem can be found in [1,4], but several questions are still open.
The Preisach model is also well-behaved for coupling with various partial differential equations, cf. [11], and this looks as an interesting area of research.

§ 2 The Preisach hysteresis operator

An elementary hysteresis operator formalizes the input-output-behaviour of the rectangular loop as depicted in figure 2.1, with switching thresholds ρ_1 and ρ_2.

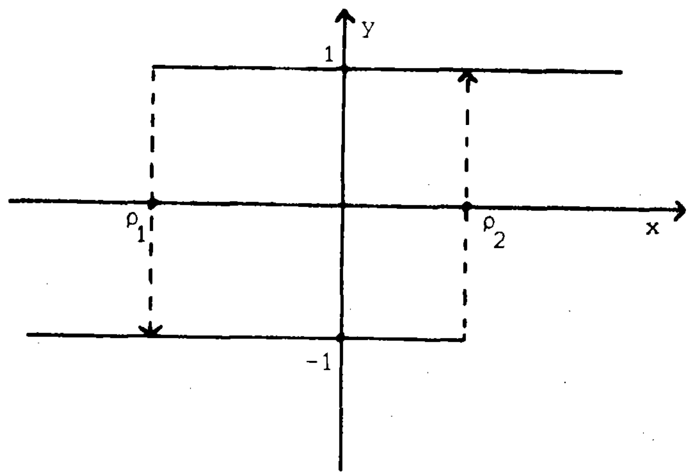

Figure 2.1

Definition 2.1

The Preisach (half-) plane P is defined by

$$P = \{(\rho_1, \rho_2) \in \mathbb{R}^2 : \rho_1 \leq \rho_2\}.$$

For any $\rho \in \mathrm{int}(P) = \{\rho \in P: \rho_1 < \rho_2\}$ the elementary hysteresis operator w_ρ maps an $x \in C\,[0,T])$ and an $\eta = -1$ or 1 to a function

$$y_\rho = w_\rho(x,\eta) : [0,T] \to \{-1,1\},$$

defined in the following way:

$$y_\rho(0) = \begin{cases} -1\ , & \text{if } x(0) \leq \rho_1 \\ \eta\ , & \text{if } \rho_1 < x(0) < \rho_2 \\ 1\ , & \text{if } x(0) \geq \rho_2 . \end{cases}$$

For $t \geq 0$, we set

$$A_t = \{\tau : 0 < \tau \leq t,\ x(\tau) = \rho_1 \text{ or } x(\tau) = \rho_2\}$$

and define

$$y_\rho(0) = \begin{cases} y_\rho(0) & , \text{ if } A_t = 0 \\ -1 & , \text{ if } A_t \neq 0 \text{ and } x(\max A_t) = \rho_1 \\ 1 & , \text{ if } A_t \neq 0 \text{ and } x(\max A_t) = \rho_2 . \end{cases}$$

Note that y_ρ is well defined and depends measurably upon t, ρ and η. □

We state the main properties of the elementary hysteresis operators w_ρ.

Proposition 2.2

Let $\rho \in \mathrm{int}(P)$. Then w_ρ has the following properties for all arguments in their respective domains:

(i) Causality: If $x_1 = x_2$ in $[0,t]$, then
$$[w_\rho(x_1,\eta)](t) = [w_\rho(x_2,\eta)](t).$$

(ii) Rate independence: If $s: [0,T] \to [0,T]$ is a monotone homeomorphism, then
$$[w_\rho(x,\eta)](t) = [w_\rho(x \circ s^{-1},\eta)](s(t)).$$

(iii) Transition or semigroup property: If $t_1 < t_2$, then
$$[w_\rho(x,\eta)](t_2) = [w_\rho(x(t_1+\cdot),[w_\rho(x,\eta)](t_1))](t_2-t_1).$$

(iv) Piecewise monotonicity: If x is either non-increasing or non-decreasing in some interval $I \subset [0,T]$, then so is $w_\rho(x,\eta)$ in the same interval I.

(v) Order preservation: If $x_1 \le x_2$ in $[0,t]$ and $\eta_1 \le \eta_2$, then
$$[w_\rho(x_1,\eta_1)](t) \le [w_\rho(x_2,\eta_2)](t).$$

(vi) BV-regularization: The function $w_\rho(x,\eta)$ is piecewise constant, and
$$\mathrm{Var}[w_\rho(x,\eta)] \le \frac{2T}{\omega(x,\rho_2-\rho_1)} + 2,$$
where for any $x \in C^0([0,T])$ and any $h>0$ we set
$$\omega(x,h) = \sup\{r \in \mathbb{R}: |t_1-t_2| < r \Rightarrow |x(t_1)-x(t_2)|<h\}.$$

(vii) Boundedness in BV: If $x \in C^0([0,T]) \cap BV(0,T)$, then
$$\mathrm{Var}[w_\rho(x,\eta)] \le \frac{2}{\rho_2-\rho_1}\mathrm{Var}[x] + 2$$

<u>**Proof:**</u> Properties (i) - (v) follow immediately from definition 2.1. The last two properties are consequences of the following facts: First, if the function $w_\rho(x,\eta)$ has a jump at $t\geq 0$, the next jump occurs only after x has gone from ρ_1 to ρ_2 or conversely; second, the number of such oscillations of x between ρ_1 and ρ_2 is finite, as x is uniformly continuous.

□

We remark that, for any reasonable choice of function spaces X and Y, the elementary hysteresis operator w_ρ: X → Y is discontinuous. Also, for ease of exposition we so far excluded the case $\rho_1 = \rho_2$, i.e. a switch without memory. We will return to this point at the end of this section.
We now introduce the Preisach hysteresis operator.

<u>**Definition 2.3**</u>
Let μ be a finite Borel measure on int(P), let S denote the set of all Borel measurable mappings on int(P) with values in $\langle -1,1\rangle$. Then the Preisach operator W maps any $x \in C^0([0,T])$ and any $\eta \in S$ to the function $W(x,\eta) : [0,T] \to \mathbb{R}$ defined by

$$[W(x,\eta)](t) = \int_{\mathrm{int}(P)} [w_\rho(x,\eta(\rho))](t)\, d\mu(\rho).$$

□

Note that the argument in the integral above is Borel measurable.
We can immediately extend the basic properties of w_ρ in proposition 2.2 to the Preisach operator.

<u>**Proposition 2.4**</u>
Let μ be a finite Borel measure on int(P). If we replace w_ρ by W in 2.2, then the properties of causality and rate independence and, if $\mu\geq 0$, also of piecewise monotonicity and order preservation remain true. The semigroup property has to be modified in the following way:

(iii') If $t_1 \leq t_2$ and $\eta \in S$, then
$[W(x,\eta)](t_2) = [W(x(t_1+\cdot),\eta_1](t_2-t_1)$,
where $\eta_1 \in S$ is given by $\eta_1(\rho) = [w_\rho(x,\eta(\rho))](t_1)$.

The BV-properties are, if $|\mu|$ denotes the variation of μ:

(vi') BV-regularization: For any $x \in C^0([0,T])$, if

$$L_1(x) = \int_{\mathrm{int}(P)} \frac{1}{\omega(x,\rho_2-\rho_1)}\, d|\mu|(\rho) < \infty,$$

then $W(x,\eta) \in BV(0,T)$ for all $\eta \in S$, and

$$\mathrm{Var}[W(x,\eta)] \leq 2L_1(x)\cdot T + 2|\mu|(\mathrm{int}\ P).$$

(vii') Boundedness in BV: If

$$L_2 = \int_{\mathrm{int}(P)} \frac{1}{\rho_2-\rho_1}\, d|\mu|(\rho) < \infty,$$

then $W(\cdot,\eta)$ maps $C^0([0,T]) \cap BV(0,T)$ into $BV(0,T)$ for all $\eta \in S$, and

$$\mathrm{Var}[W(x,\eta)] \leq 2L_1\mathrm{Var}[x] + 2|\mu|(\mathrm{int}\ P).$$

Proof: This is immediate from proposition 2.2 and definition 2.3.

□

The next theorem asserts that the output function $W(x,\eta)$ is continuous if and only if the measure μ of each horizontal and vertical line is zero.

Theorem 2.5

Let μ be a finite Borel measure on int(P). Then $W(\cdot,\eta)$ maps $C^0([0,T])$ into $C^0([0,T])$ for any $\eta \in S$ if and only if

$$|\mu|(L(r,i)) = 0$$

for any $r \in \mathbb{R}$ and $i \in \{1,2\}$, where

$$L(r,i) = \{\rho \in P: \rho_i = r\}.$$

Proof: We shall just prove the "if"-part, the converse being obvious. Let us fix any $\tilde{t} \in [0,T]$. For any $\rho \in P$ such that $\rho_1 \neq x(\tilde{t})$ and $\rho_2 \neq x(\tilde{t})$, there exists a $\delta>0$ such that $\rho_1 \neq x(t)$ and $\rho_2 \neq x(t)$ if $|t-\tilde{t}| \leq \delta$. Hence, by the assumption on μ,

$$\lim_{t\to\tilde{t}} [w_\rho(x,\eta)] = [w_\rho(x,\eta)](\tilde{t}) \qquad \mu\text{ - a.e. in } P,$$

therefore

$$\lim_{t \to \tilde{t}} [W(x,\eta)](t) = [W(x,\eta)](\tilde{t})$$

by Lebesgue's dominated convergence theorem, since $|w_\rho| \leq 1$ and μ is finite.

□

We now present an extremely useful geometric interpretation, which in fact dates back to the origin of the Preisach model and has accompanied it ever since.
Let us fix any $(x,\eta) \in C^0([0,T]) \times S$ and set

$$y_\rho(t) = [w_\rho(x,\eta(\rho))](t).$$

For any $t \in [0,T]$, define

$$A^+(t) = \{\rho \in \mathrm{int}(P): y_\rho(t) = 1\}$$
$$A^-(t) = \{\rho \in \mathrm{int}(P): y_\rho(t) = -1\}.$$

Now, from figure 2.1 and the definition of w_ρ it is apparent how the sets $A^+(t)$ and $A^-(t)$ change in time: As x increases in time, the boundary of $A^+(t)$ moves upwards; as x decreases in time, the boundary of $A^-(t)$ moves to the left, compare figure 2.2.

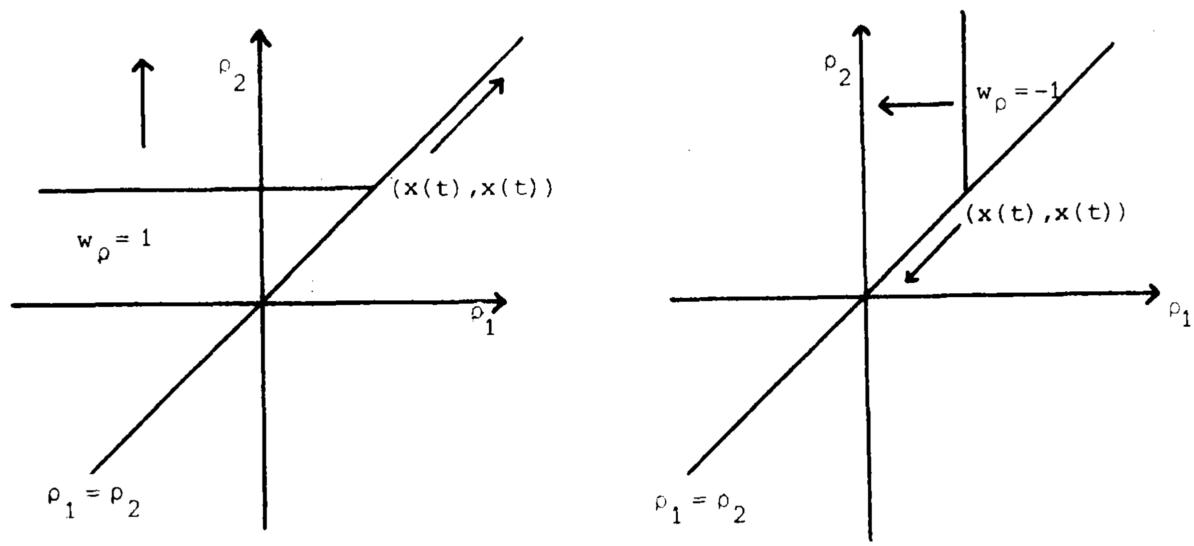

Figure 2.2

Moreover, for any $\rho, \rho' \in \mathrm{int}(P)$, if the following property holds at the initial time t=0, then it holds for all $t \geq 0$:

if $\rho \in A^+(t)$ and $\rho' \leq \rho$, then $\rho' \in A^+(t)$;
if $\rho \in A^-(t)$ and $\rho \leq \rho'$, then $\rho' \in A^-(t)$;
here $\rho = \rho'$ means $\rho_1 \leq \rho_1'$ and $\rho_2 \leq \rho_2'$.
Thus, the boundary

$$B(t) = \partial A^+(t) \cap \partial A^-(t)$$

is a maximal antimonotone graph, which intersects the main diagonal $\rho_1 = \rho_2$ at the point $(x(t), x(t))$, see figure 2.3. At any t, B(t) determines the function $\rho \to [w_\rho(x, \eta(\rho))](t)$, hence also $[W(x,\eta)](t)$. Thus, B(t) represents the state of the system at time t; it can also be viewed as the memory of the system at time t.

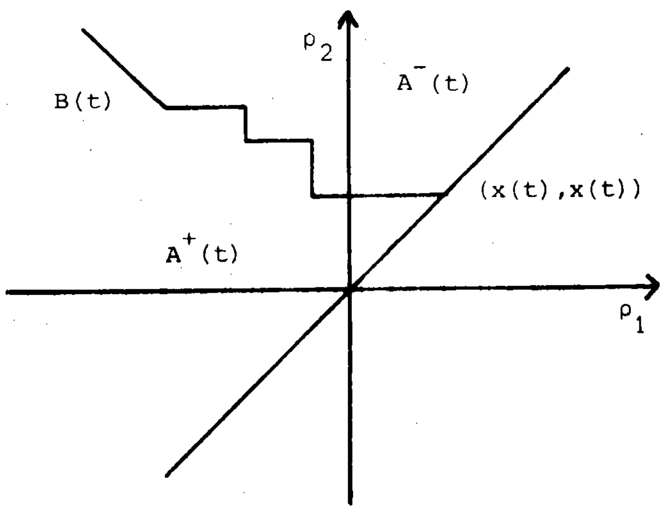

Figure 2.3

As x evolves in time, new arcs appearing in B(t) must be parallel to the axes ρ_1 and ρ_2 by the construction just given; any portion of B(t) with a different shape must have been present in the initial configuration B(0) determined by η and x(0). In the case of a virgin ferromagnetic material, namely of a system which never experienced magnetization,
$B(0) = \{\rho \in P: \rho_1 = -\rho_2\}$.
It is not difficult to imagine how the Preisach model generates continuous hysteresis loops in the input-output-plane. Let us consider a periodic input $x \in C^0([0,T])$ oscillating between two values $x_1 < x_2$; this corresponds to a periodic movement of B(t) within the triangle $\Delta(x_1, x_2)$ with vertices

(x_1,x_1), (x_1,x_2) and (x_2,x_2) in the Preisach plane. The output $W(x,\eta)$ therefore also is periodic with the same period, and the area of the loop bounded by the pair $(x(t),[W(x,\eta)](t))$, i.e. the resulting hysteresis loop, is equal to

$$\int_{\Delta(x_1,x_2)} (\rho_2-\rho_1)\, d\mu(\rho).$$

We want to close this section with a remark concerning the main diagonal $\rho_1=\rho_2$, the boundary of P. If we consider a measure μ concentrated along the main diagonal, being zero at every single point (which in this case is equivalent to the condition in theorem 2.5), then the geometric interpretation yields

$$[W(x,\eta)](t) = \mu(\{(r,r): r \le x(t)\} - \mu(\{(r,r): r \ge x(t)\}).$$

This means that the hysteresis operator W degenerates into a superposition operator. That is only natural since the main diagonal does not represent memory. On the other hand, an arbitrary finite Borel measure μ on P can be written as $\mu = \mu_o+\mu_\Delta$, where μ_o is concentrated on int(P) and μ_Δ on the main diagonal; the corresponding Preisach operator is the sum of a superposition operator and an operator as defined in 2.3. These considerations also prove:

<u>Corollary 2.6</u>

Let μ be a finite Borel measure on P. Then the assertions of theorem 2.5 hold.

□

§ 3 Continuity of the Preisach operator in $C^0([0,T])$

One aim of this section is to prove the following theorem.

Theorem 3.1
Let μ be a finite Borel measure on the Preisach plane P. If $|\mu|(L) = 0$ for every $L \subset P$ which is a horizontal or vertical straight line, then the Preisach operator W maps $C^0([0,T])$ into $C^0([0,T])$ and is continous with respect to the uniform topology.

□

From theorem 2.5 and corollary 2.6 we see that theorem 3.1 actually characterizes the strong continuity of $W: C^0([0,T]) \to C^0([0,T])$. We note as an immediate corollary of theorem 3.1 and corollary 2.6 a result which is well known for the special case of superposition operators as well as for certain integral operators in various function spaces.

Corollary 3.2
The Preisach operator W is strongly continuous on $C^0([0,T])$ if and only if it maps $C^0([0,T])$ into itself.

□

Theorem 3.1 has been stated already in [5,6] and proved in [11]. The considerations of [5,6,11] rest on the geometric interpretation described in section 2, whereas the boundary curve B(t) separating the "+"-region and the "-"-region in the Preisach plane represents the state of the system. One therefore is induced to study the evolution of B(t) in time.
We now propose - and this is the second aim of this section - to decompose the Preisach operator W into an input-state map F and a state-output map E, and actually to use this as an alternative definition of W (compare the discussion at the end of this section). It then turns out that F is Lipschitz continuous, whereas the properties of E obviously depend upon the measure μ; in particular, E may be smooth. This is obtained rather directly. The main goal of this section is to

analyze the operator F, i.e. the time evolution of the boundary curve B(t), in more detail in order to prove some of the characterizations given later as well as theorem 3.1. To give an analytical representation of the curve B(t), it seems convenient to change to the coordinates

$$u = \frac{\rho_2-\rho_1}{2} \quad , \qquad v = \frac{\rho_2+\rho_1}{2} \; ,$$

which are, in fact, the coordinates originally used by Preisach [9], see also [6]. A given boundary curve will be written as $v = \psi(u)$, where $\psi \in \Psi$ for some set Ψ; later we will write $v = \psi(t)u$ to include evolution in time, so that ψ: $[0,T] \rightarrow \Psi$.

<u>Definition 3.3</u>

Let

$$\Psi_o = \{\psi:\ \psi \in C^o([0,\infty)),\ \psi \text{ vanishes at } +\infty\}$$

$$\Psi_1 = \{\psi:\ \psi \in \Psi_o,\ \psi \text{ has bounded support and is Lipschitz continuous with Lipschitz constant } \leq 1\}$$

$$\hat{\Psi} = \{\psi:\ \psi \in \Psi_o, \text{ and there exists } \langle u_k\rangle_{k\in N} \text{ such that } u_k \geq 0,\ \lim u_k=0,\ u_{k+1}<u_k \text{ if } u_k>0,\ \psi|[u_1,\infty) = 0,\ \psi|[u_{k+1},u_k] \text{ is a straight line of slope either } +1 \text{ or } -1\}.$$

Let d_∞ and $d_{1,p}$ denote the distances corresponding to the norms

$$\|\psi\|_\infty = \max_{u\geq 0} |\psi(u)|$$

$$|\psi|_{1,p} = (\int_0^\infty |\psi\}(u)|^p)^{1/p} \quad , \qquad 1 \leq p < \infty.$$

For $\phi,\psi \in \Psi_o$, we say as usual that $\phi\leq\psi$ if $\phi(u) \leq \psi(u)$ for all $u\geq 0$. We define that $\phi<\psi$, if $\phi\leq\psi$ and $\phi(0) < \psi(0)$.

□

Obviously, $\hat{\Psi} \subset \Psi_1 \subset \Psi_o$. It will turn out that $\hat{\Psi}$ is the set of reachable internal states, if we fix $\psi_o=0$. We will use Ψ_1 since it is simpler to deal with. At last, Ψ_o is defined in order that $(\Psi_o, \|\cdot\|_\infty)$ is a separable Banach space which contains everything of interest.

We will obtain the time evolution F of the internal state by a completion argument, analogous to the definition of a hysteron of first kind in [6]. First, we give an analytical representation of the motion of the straight lines which form the boundary curve.

Definition 3.4
Define $G: \mathbb{R} \times \Psi_o \to \Psi_o$ by

$$G(x,\psi)(u) = \min\{x+u, \max\{x-u, \psi(u)\}\}.$$

This means that one projects graph(ψ) in the v-direction onto the cone

$$K_o(x) = \{(u,v): x-u \leq v \leq x+u\},$$

see figure 3.1.

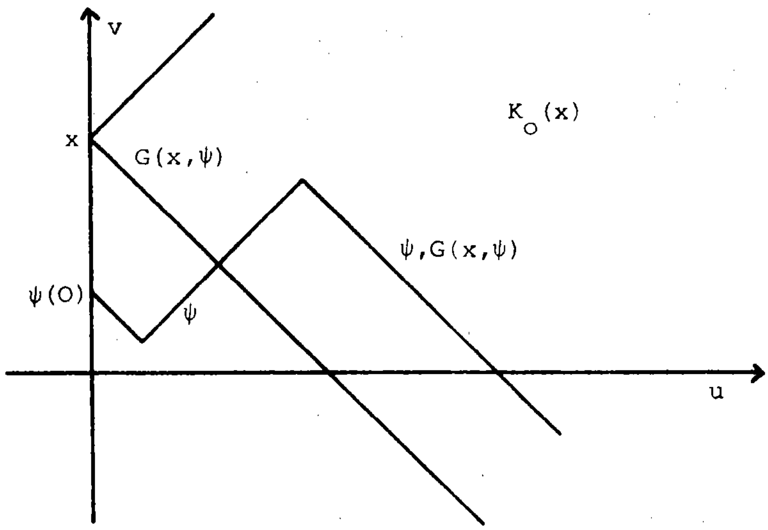

Figure 3.1

Comparing with the geometrical interpretation of the Preisach model in section 2, one easily is convinced that $G(x(t),\psi_o)$ gives the correct internal state at time t for an initial state ψ_o, if $x \in C^o([0,T])$ is monotone.
We state some properties of the mapping G.

Lemma 3.5

(i) We have $G: \mathbb{R} \times \Psi_1 \to \Psi_1$, $G: \mathbb{R} \times \hat{\Psi} \to \hat{\Psi}$.

(ii) $$G(x_2, G(x_1,\psi)) = G(x_2,\psi)$$
for all $x_1, x_2 \in \mathbb{R}$, $\psi \in \Psi_1$ with $\psi(0) \leq x_1 \leq x_2$ or

$\psi(0) \geq x_1 \geq x_2$.

(iii) If $x_1 \leq x_2$ and $\psi_1 \leq \psi_2$, then $G(x_1,\psi_1) \leq G(x_2,\psi_2)$; if $x_1 < x_2$ and $\psi_1 \leq \psi_2$, then $G(x_1,\psi_1) < G(x_2,\psi_2)$; this holds for all $x_1,x_2 \in \mathbb{R}$ and all $\psi_1,\psi_2 \in \Psi_o$.

(iv) $G(x,\psi)(0) = x$ for all $x \in \mathbb{R}$, $\psi \in \Psi_o$.

(v) $G(\psi(0),\psi) = \psi$ for all $\psi \in \Psi_1$.

(vi) $\|G(x_1,\psi_1)-G(x_2,\psi_2)\|_\infty \leq \max\{|x_1-x_2|,\|\psi_1-\psi_2\|_\infty\}$ for all $x_1,x_2 \in \mathbb{R}$ and all $\psi_1,\psi_2 \in \Psi_o$.

<u>Proof:</u> The justification of (i) - (v) is straightforward. To prove (vi) we observe that for all $u \geq 0$

$$|G(x_1,\psi_1)(u)-G(x_2,\psi_2)(u)| =$$
$$= |\min\{x_1+u,\max\{x_1-u,\psi_1(u)\}\} - \min\{x_2+u,\max\{x_2-u,\psi_2(u)\}\}|$$
$$\leq \max\{|x_1-x_2|,|\max\{x_1-u,\psi_1(u)\} - \max\{x_2-u,\psi_2(u)\}|\}$$
$$\leq \max\{|x_1-x_2|,\max\{|x_1-x_2|,|\psi_1(u)-\psi_2(u)|\}\}.$$

□

We describe the evolution of the internal state for a piecewise monotone continuous input.

<u>**Definition 3.6**</u>

Set

$$C^o_{pm}([0,T]) = \{x: x \in C^o([0,T]),\ x \text{ is piecewise monotone}\}.$$

For any $x \in C^o_{pm}([0,T])$ and $\psi_o \in \Psi_1$ we define

$$F(x,\psi_o): [0,T] \to \Psi_1$$

by

$$F(x,\psi_o)(0) = G(x(0),\psi_o)$$
$$F(x,\psi_o)(t) = G(x(t),F(x,\psi_o)(t_i)),\ \text{if } t_i<t\leq t_{i+1},$$

where $0 = t_o < \dots < t_N = T$ is a partition such that $x|[t_i,t_{i+1}]$ is monotone.

□

The definition of $F(x,\psi_o)$ does not depend on the choice of the partition $\{t_i\}$, by properties (ii) and (iv) of G in lemma 3.5. One immediately obtains the Lipschitz continuity of F:

Proposition 3.7
The mapping F of definition 3.6 can be uniquely extended to a Lipschitz continuous mapping

$$F\colon C^o([0,T]) \times \Psi_1 \longrightarrow C^o([0,T];\ (\Psi_o, \|\cdot\|_\infty))$$

with

$$\|F(x_1,\psi_{o1})-F(x_2,\psi_{o2})\| \leq \max\{\|x_1-x_2\|_\infty, \|\psi_{o1}-\psi_{o2}\|_\infty\}$$

for all $x_1,x_2 \in C^o([0,T])$ and all $\psi_{o1},\psi_{o2} \in \Psi_1$, where we take for $\psi = F(x,\psi_o) \in C^o([0,T];\Psi_o)$ the standard norm

$$\|\psi\| = \sup_{t\in[0,T]} \|\psi(t)\|_\infty .$$

Proof: Let $\psi_o \in \Psi_1$, $x \in C^o_{pm}([0,T])$ with partition $\{t_i\}$, then $F(x,\psi_o)$ is continuous in (t_i,t_{i+1}) and at t=0 by lemma 3.5, (vi). Furthermore,

$$\lim_{t\downarrow t_i} F(x,\psi_o)(t) = G(x(t_i),F(x,\psi_o)(t_i)) = F(x,\psi_o)(t_i)$$

by lemma 3.5, (vi) and (v), since $x(t_i) = (F(x,\psi_o)(t_i))(0)$ by (iv) applied to $F(x,\psi_o)(t_i) = G(x(t_i),F(x,\psi_o)(t_{i-1}))$. Therefore, F maps $C^o_{pm} \times \Psi_1$ into $C^o([0,T];\Psi_o)$. For the Lipschitz continuity, consider $x_1,x_2 \in C^o_{pm}([0,T])$ and $\psi_{o1},\psi_{o2} \in \Psi_1$. Again, by lemma 3.5 (vi)

$$\|F(x_1,\psi_{o1})(0)-F(x_2,\psi_{o2})(0)\|_\infty \leq \max\{|x_1(0)-x_2(0)|, \|\psi_{o1}-\psi_{o2}\|_\infty\},$$

and for all $t \in (t_i,t_{i+1}]$

$$\|F(x_1,\psi_{o1})(t)-F(x_2,\psi_{o2})(t)\|_\infty =$$
$$= \|G(x_1(t),F(x_1,\psi_{o1})(t_i))-G(x_2(t),F(x_2,\psi_{o2})(t_i))\|_\infty$$
$$\leq \max\{|x_1(t)-x_2(t)|, \|F(x_2,\psi_{o1})(t_i)-F(x_2,\psi_{o2})(t_i)\|_\infty\},$$

so the asserted inequality follows by induction on i.
Since $C^o_{pm}([0,T])$ is dense in $C^o([0,T])$, the proposition holds. □

Remark 3.8
(i) Later on in lemma 3.19 we will see that for $\psi = F(x,\psi_o)$ we also have $\psi(t) \in \Psi_1$, if $\psi_o \in \Psi_1$, as well as $\psi(t) \in \hat{\Psi}$

(ii) F cannot be continuous w.r.t. x in L^p-norm, $p<\infty$, because needle-like variations of the input with arbitrarily small L^p-norm can change the state permanently by a fixed magnitude.

(iii) From the definitions of G and F it is clear that Lipschitz continuity will be the optimal regularity of F.

□

The semigroup property of G extends to F.

Lemma 3.9

Let $x \in C^o([0,T])$, $\psi_o \in \Psi_1$, set $\psi = F(x,\psi_o)$. Then

$$\psi(t)(0) = x(t) \qquad \text{for all } t \geq 0$$

$$\psi(t+s) = F(x(\cdot+t),\psi(t))(s) \qquad \text{for all } t,s \geq 0$$

(with slight abuse of notation).

Proof: For piecewise monotone x, it follows from lemma 3.5. Passing to the limit, we obtain it also for arbitrary x, since F is continuous by proposition 3.7.

□

The cone K_o from definition 3.4 and figure 3.1 preserves its meaning for arbitrary continuous inputs:

Lemma 3.10

Let $x \in C^o([0,T])$, $\psi_o \in \Psi_1$, set $\psi = F(x,\psi_o)$. Then for all t we have:

(i) $\psi(t): [0,\infty) \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant not larger than 1.

(ii) $$\text{graph}(\psi(t)) \subset K_o(x(t)) \quad , \quad \text{i.e.}$$
$$x(t)-u \leq \psi(t)u \leq x(t)+u \qquad \text{for all } u \geq 0.$$

(iii) "If $\psi(t)\bar{u}$ lies on the boundary of $K_o(x(t))$, then $\psi(t)$ is identical with that boundary up to $\bar{u}$":

$$x(t) = \psi(t)\bar{u}+\bar{u} \implies x(t) = \psi(t)u+u \qquad \text{for all } u \in [0,\bar{u}]$$
$$x(t) = \psi(t)\bar{u}-\bar{u} \implies x(t) = \psi(t)u-u \qquad \text{for all } u \in [0,\bar{u}].$$

Proof: Let $x_n \in C^o_{pm}([0,T])$ with $x_n \to x$ uniformly. Then (i) holds for $\psi_n = F(x_n,\psi_o)$ by lemma 3.5, (i), and also for ψ, because $\psi_n(t) \to \psi(t)$ uniformly on $[0,\infty]$ by proposition 3.7. Since $\psi(t)(0) = x(t)$ by lemma 3.9, the rest is now a direct consequence of (i). □

The order preserving properties of G extends to F.

<u>Lemma 3.11</u>
Let $x \in C^o([0,T])$, $\psi_o \in \Psi_1$, set $\psi = F(x,\psi_o)$. Let x be monotone on some interval $I \subset [0,T]$. If $t,s \in I$, then $x(t) \leq x(s)$ implies $\psi(t) \leq \psi(s)$, and $x(t) < x(s)$ implies $\psi(t) < \psi(s)$.

<u>Proof:</u> This is immediate from 3.5, (iii).

□

<u>Lemma 3.12</u>
Let $x_1,x_2 \in C^o([0,T])$ and $\psi_{o1},\psi_{o2} \in \Psi_1$ with $x_1 \leq x_2$ and $\psi_{o1} \leq \psi_{o2}$. Set $\psi_1 = F(x_1,\psi_{o1})$ and $\psi_2 = F(x_2,\psi_{o2})$. Then $\psi_1(t) \leq \psi_2(t)$ for all $t \in [0,T]$, and if $x_1(t) < x_2(t)$ for some t, then $\psi_1(t) < \psi_2(t)$.

<u>Proof:</u> If x_1 and x_2 are piecewise monotone, it follows from lemma 3.5, (iii) and (iv). If not, take piecewise linear interpolates x_{1n},x_{2n} such that $x_{in} \to x_i$ uniformly and $x_{1n} \leq x_{2n}$, and note that F is continuous by proposition 3.7 and that $\psi_i(t)(0) = x_i(t)$ by lemma 3.9.

□

We investigate the input-state map F in more detail, in order to get properties which will be used in section 5. (A reader who is mainly interested in the definition of the Preisach operator W may jump immediately to theorem 3.24.)

<u>Definition 3.13</u>
Given $x \in C^o([0,T])$, associate with it the quantities

$$m(t) = \min_{t\leq s\leq T} x(s) \quad , \quad M(t) = \max_{t\leq s\leq T} x(s) \quad ,$$

$$d(t) = \frac{1}{2}\,(M(t)-m(t))$$

as well as closed regions $\Delta(t)$, $K(t)$, $S_+(t)$ and $S_-(t)$ whose definition is immediate from figure 3.2. □

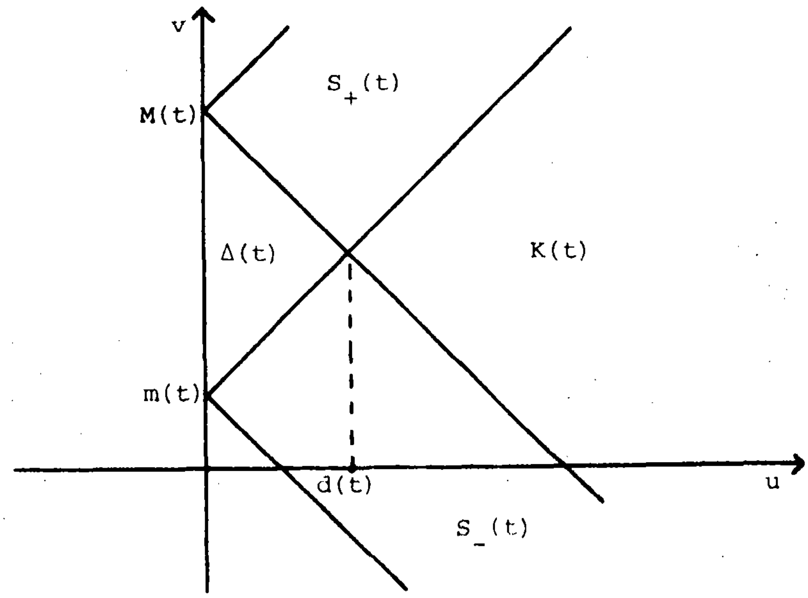

Figure 3.2

Lemma 3.14

The functions m,M,d: $[0,T] \to \mathbb{R}$ in definition 3.13 are continuous, $m(T) = M(T) = x(T)$, $d(T) = 0$, m is nondecreasing, M and d are nonincreasing. If $t \geq s$, then $K(t) \supset K(s)$ and $\Delta(t) \subset \Delta(s)$.

Proof: Immediate from the continuity of x.

□

As it is intuitively obvious, the cone K(t) has the following property: The portion of graph $\psi(t)$, which lies within K(t), will remain unchanged for the rest of the time.

Lemma 3.15

Let $x \in C^o([0,T])$, $\psi_o \in \Psi_1$, set $\psi = F(x,\psi_o)$. For $t \in [0,T]$ and

$$K(t) = \{(u,v): u \geq d(t),\ M(t)-u \leq v \leq m(t)+u\}$$

we have:

(i) If $(u,\psi(t)u) \in K(t)$, then $\psi(s)u = \psi(t)u$ for all $s \geq t$.

(ii) If $(u,\psi(s)u) \in \mathrm{int}(K(t))$ for some $s \geq t$, then

$\psi(\tau)u = \psi(t)u$ for all $\tau \geq t$.

Proof: For $s>t$, define a comparison input $\bar{x}$ by $\bar{x}=x$ in $[0,t]$, $\bar{x} = M(t)$ in $[s,T]$, $\bar{x}\geq x$ and $\bar{x}$ monotone in $[t,s]$. If $\bar{\psi} = F(\bar{x},\psi_p)$, we have by lemma 3.12 that $\bar{\psi}\geq\psi$ and in particular

$$\psi(s)u \leq \bar{\psi}(s)u = \max\{M(t)-u,\ \psi(t)u\}.$$

The assumptions of (i) and (ii) both imply that the max is attained at $\psi(t)u$ and therefore $\psi(s)u \leq \psi(t)u$. Bounding x from below, the reverse inequality is obtained. This proves (i) directly and reduces (ii) to (i).

□

The next lemma states that the variation of $\psi(t)$ in time takes place within the set $\Delta(t) \cup S_+(t) \cup S_-(t)$ from figure 3.2 and sweeps at least the triangle $\Delta(t)$.

Lemma 3.16

Let $x \in C^0([0,T])$, $\psi_0 \in \Psi_1$, set $\psi = F(x,\psi_0)$. Then we have for all $t \in [0,T]$:

(i) For all $s,\bar{s} \geq t$, the region between graph($\psi(s)$) and graph($\psi(\bar{s})$) is contained either in $\Delta(t) \cup S_+(t)$ or in $\Delta(t) \cup S_-(t)$.

(ii) If $M(t) > m(t)$, then there exists $s_m, s_M \geq t$ such that $\psi(s_m) < \psi(s_M)$ and $\psi(s_m)(0) = m(t)$, $\psi(s_M)(0) = M(t)$.

Proof: Let $s,\bar{s} \geq t$. Since by lemma 3.15, graph($\psi(s)$) and graph($\bar{\psi}(s)$) coincide in int($K(t)$), there is a common entry point $(u,v) \in \partial K(t)$ with

$$u = \inf\{w\colon (w,\psi(s)w) \in \mathrm{int}(K(t))\}$$
$$= \inf\{w\colon (w,\bar{\psi}(s)w) \in \mathrm{int}(K(t))\}.$$

Because of the Lipschitz bound 1 for $\psi(s)$ and $\psi(\bar{s})$ and because of lemma 3.10, (i) is proved. To prove (ii), choose $s_m, s_M \geq t$ with $x(s_m) = m(t)$, $x(s_M) = M(t)$. If, say, $s_M \leq s_m$, define a comparison input $\bar{x}$ by $\bar{x}=x$ on $[0,s_M]$ and $\bar{x} = M(t)$ on $[s_M,T]$. Then $\bar{\psi} = F(\bar{x},\psi_0) \geq \psi$ and in particular $\psi(s_m) \leq \bar{\psi}(s_m) = \psi(s_M)$.

□

We now describe formally, how the corners of graph($\psi(T)$) are formed by the movement of the straight lines. For any given point $(u,\psi(T)u) \in$ graph ψ we want to denote by $l(u)$ the last time this point is touched by a straight line.

Definition 3.17

Let $x \in C^0([0,T])$, $\psi_0 \in \Psi_1$, $\psi = F(x,\psi_0)$. Let $K(t)$ be as in 3.13 and 3.15. We set

$$u_* = \inf\{u:\ u \geq 0,\ (u,\psi_0(u)) \in \operatorname{int} K(0)\},$$

and define $l: [0,\infty) \to [0,T]$ by $l(u) = 0$ if $u \geq u_*$,

$$l(u) = \sup\{s:\ \psi(T)u = x(s)+u \quad \text{or} \quad \psi(T)u = x(s)-u\}$$

if $u \leq u_*$. □

Lemma 3.18

Let $x \in C^0([0,T])$, $\psi_0 \in \Psi_1$, $\psi = F(x,\psi_0)$. Then for any $\bar{u} \geq 0$ we have

$$\psi(t)u = \psi(T)u$$

for all $u \geq \bar{u}$ and $t \geq l(\bar{u})$. In particular, $\psi(t)u = \psi_0(u)$ for all $u \geq u_*$ and all $t \geq 0$. Moreover, $\psi(l(\bar{u}))$ restricted to $[0,\bar{u}]$ is a straight line of slope +1 or −1 for all $\bar{u} \geq 0$.

Proof: On (u_*,∞), all $\psi(t)$ coincide with ψ_0 by lemma 3.15 (ii) and definition of u_*. Let now $\bar{u} \leq u_*$. Since $\psi: [0,T] \to (\Psi_0, \|\cdot\|_\infty)$ is continuous, it is enough to prove the first assertion for $T \geq t > l(\bar{u})$. In this case, we have

$$\psi(T)\bar{u}-\bar{u} < m(t) \leq x(T) \leq M(t) < \psi(T)\bar{u}+\bar{u},$$

because $x(T) = \psi(T)(0)$, $\psi(T)$ has Lipschitz constant 1 and the definition of $l(\bar{u})$. Therefore, $(u,\psi(T)u) \in \operatorname{int}(K(t))$ for all $u \geq \bar{u}$ and we obtain $\psi(t)u = \psi(T)u$ from lemma 3.15, (ii).
The second assertion follows directly from lemma 3.10, (iii).

□

It now turns out that the evolution defined by F leaves invariant the internal state sets ψ_1 and $\bar{\Psi}$.

Proposition 3.19

Let $\psi = F(x,\psi_0)$, where $\psi_0 \in \Psi_1$ and $x \in C^0([0,T])$. Then $\psi(t) \in \Psi_1$ for all $t \geq 0$. If $\psi_0 \in \hat{\Psi}$, then also $\psi(t) \in \hat{\Psi}$ for all $t \geq 0$.

<u>**Proof:**</u> Lemma 3.18 implies $\psi(t) \in \Psi_1$. Now let $\psi_o \in \Psi$. Since obviously $\psi(0) \in \hat{\Psi}$, it is enough to show $\psi(T) \in \hat{\Psi}$. We want to construct a monotone decreasing sequence $\langle u_k \rangle$ such that $\psi(T) | [u_{k+1}, u_k]$ has slope +1 or -1. Since we already know from lemma 3.18 that $\psi(T)$ has the required form on $[u_*, \infty]$ with u_* from definition 3.17, we may set $u_1 = u_*$. We define $t_1 = l(u_1)$ and with d from definition 3.13

$$u_k = d(t_{k-1}) \quad , \quad t_k = l(u_k) \quad , \quad k \geq 2.$$

If we assume for definiteness $\psi(t_1)u_1 = \psi(T)u_1 = x(t_1) - u_1$ then we can visualize the sequences as in figure 3.3.

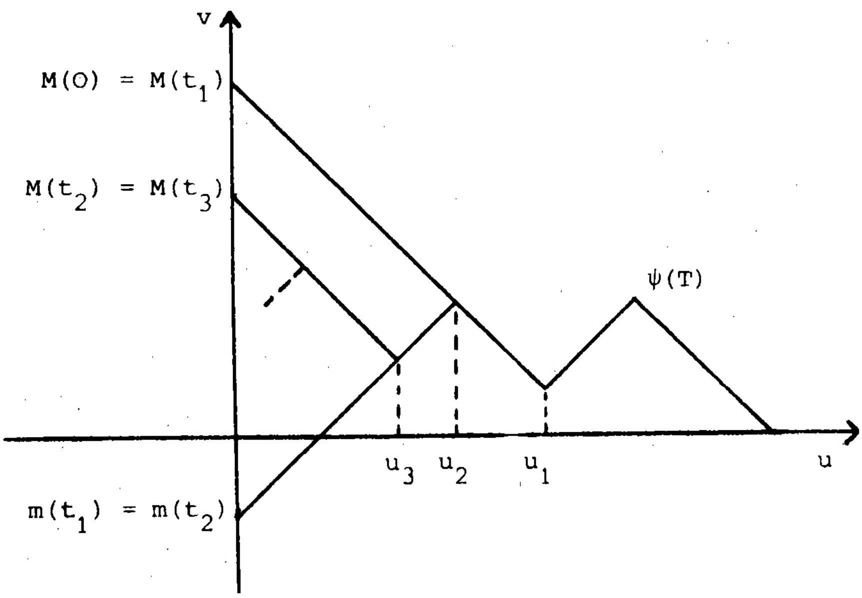

Figure 3.3

Since $(u_1, \psi(t_1)u_1) \in \partial K(0)$, we have $x(t_1) = M(t_1) = M(0)$ and $\psi(t_1)u = x(t_1) - u$ for $u \in [0, u_1]$ from lemma 3.18, also $m(t_1) + u_1 \geq \psi(t_1)u_1$. If $u_1 = 0$, we are done. If $u_1 < 0$, then by definition of $t_1 = l(u_1)$ we have $m(t_1) + u_1 > \psi(t_1)u_1$ and therefore $u_2 = d(t_1) < u_1$. Furthermore, since $(u, \psi(t_1)u) \in K(t_1)$ for $u \in [u_2, u_1]$, we have $\psi(T) | [u_2, u_1] = \psi(t_1) | [u_2, u_1]$ by lemma 3.15, (i), which has slope -1. Again if $u_2 > 0$ we have, by definition of $t_2 = l(u_2)$, that $\psi(T)u_2 = x(t_2) + u_2 = \psi(t_2)u_2$ as well as $x(t_2) = m(t_1) = m(t_2)$. Continuing in this manner, by induction we obtain a sequence $\langle u_k \rangle$, with the slope $\psi(T) | [u_{k+1}, u_k]$ as required, which either has $u_k = 0$ for some $k \in N$ or is strictly decreasing. In the latter case, $\langle t_k \rangle$ is increasing to some limit, and since by construction

$$u_{k+1} = d(t_k) = \frac{1}{2} |x(t_{k+1})-x(t_k)|$$

the continuity of x implies that $\langle u_k \rangle$ converges to zero.

□

We remark that theorem 38.2 in [6] gives a description similar to lemma 3.19.
We can also prescribe precisely the support of $\psi(t)$, which obviously cannot decrease in time.

<u>Lemma 3.20</u>
For any $\phi \in \Psi_1$ we denote by $u_M(\phi)$ the maximal point in the support of ϕ. Let $\psi = F(x,\psi_o)$ with $\psi_o \in \Psi_1$ and $x \in C^o([0,T])$. Then

$$u_M(\psi(t)) = \max\langle u_M(\psi_o), \max_{0 \leq s \leq t} |x(t)|\rangle .$$

<u>Proof:</u> Since obviously

$$u_M(G(\xi,\psi_o)) = \max\langle u_M(\psi_o), |\xi|\rangle$$

the assertion holds for t=0, and it is enough to prove it for t=T. The construction in the proof of proposition 3.19 shows that $u_M(\psi(T)) \geq u_M(\psi_o)$, so the semigroup property of F implies that $u_M(\psi(t))$ does not decrease in time. If x is piecewise monotone, the assertion follows from the definition of F. In general, let $x_n \to x$ with $x_n \in C_{pm}$, then for $\psi_n = F(x_n,\psi_o)$ we have $u_M(\psi_n(T)) \to \max\langle u_M(\psi_o), \|x\|_\infty\rangle$ and $\psi_n(T) \to \psi(T)$ in Ψ_1, so $u_M(\psi(T)) \leq \max\langle u_M(\psi_o), \|x\|_\infty\rangle$. Since always $|x(t)| \leq u_M(\psi(t))$ and the latter does not decrease, $\|x\|_\infty \leq u_M(\psi(T))$. This ends the proof. □

If we have two inputs x_1 and x_2, then it is immediately clear that $\psi_1(t) \neq \psi_2(t)$ if $x_1(t) \neq x_2(t)$, where $\psi_i = F(x_i,\psi_o)$. However, for the characterization of injectivity of W the following stronger property of F is needed.

<u>Proposition 3.21</u>
Let $x_1, x_2 \in C[0,T]$, $\psi_o \in \Psi_1$, $\psi_i = F(x_i,\psi_o)$. If $\psi_1(T) \neq \psi_2(T)$ then there exists a $t \in [0,T]$ such that

$$\psi_1(t) < \psi_2(t) \quad \text{or} \quad \psi_2(t) < \psi_1(t).$$

Proof: The idea of the proof is to look at the corner, where $\psi_1(T)$ and $\psi_2(T)$ branch (as seen from the right), and to investigate its formation. By M_i, l_i ... we denote the functions and sets from definitions 3.13 and 3.17 corresponding to x_i. Set

$$u_o = \sup\{u: u \geq 0,\ \psi_1(T)u \neq \psi_2(T)u\}.$$

Then $0 < u_o < \infty$ by lemma 3.19. Set $t = \max\{l_1(u_o), l_2(u_o)\}$, assume that $t = l_1(u_o)$. By lemma 3.18 we have

$$\psi_1(s)u = \psi_2(s)u = \psi(T)u \quad \forall\ u \geq u_o \quad \forall\ s \geq t,$$

and $\psi_1(t)|[0,u_o]$ is a straight line of slope +1 or -1, assume it to be +1. Then $\psi_1(t) \leq \psi_2(t)$, since $\psi_2(t) \in \Psi_1$. If $x_1(t) < x_2(t)$, we are done; let us assume $x_1(t) = x_2(t)$ and derive a contradiction. Set $M_o = \max\{M_1(t), M_2(t)\}$ and $d_o = \max\{d_1(t), d_2(t)\}$, see figure 3.4. We have $M_o < x_1(t)+2u_o$ by definition of l_1, l_2 and u_o, and therefore also $d_o < u_o$.

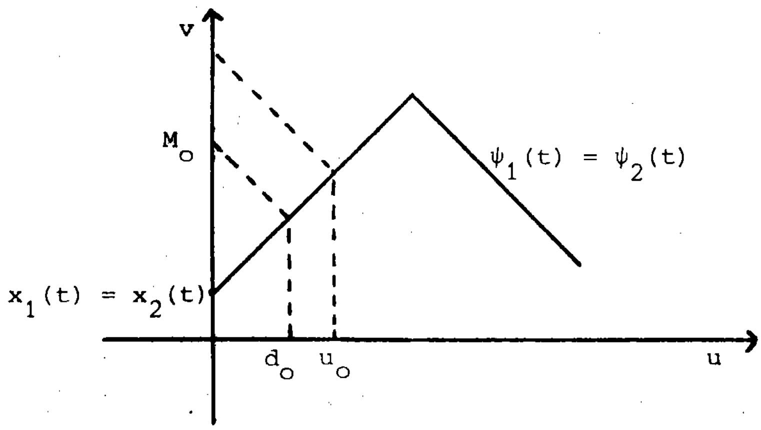

Figure 3.4

Now for $u \in [d_o,u_o]$ we have $(u,\psi_i(t)u) \in K_i(t)$, and from lemma 3.15 we conclude

$$\psi_1(T)u = \psi_1(t)u = \psi_2(t)u = \psi_2(T)u$$

for all $u \in [d_o,u_o]$, which contradicts the definition of u_o.

□

The special structure of the set $\hat{\Psi}$ of attainable internal states renders the distance $d_{1,p}$ almost equivalent to the uniform distance, yielding a certain amount of continuity of F in the $W^{1,p}$-norm.

Lemma 3.22

We have $\phi_n \to \phi$ in $(\hat{\Psi},d_{1,p})$ for $1 \le p < \infty$ if and only if $\phi_n \to \phi$ in $(\hat{\Psi},d_\infty)$ and $u_M(\phi_n) \to u_M(\phi)$, where $u_M(\phi)$ again denotes the largest point of the support of ϕ.

Proof: Let $\phi_n \to \phi$ in $(\hat{\Psi},d_{1,p})$. Between $u_M(\phi_n)$ and $u(\phi_n)$, $|\phi_n'-\phi'| = 1$, therefore

$$|u_M(\phi_n)-u_M(\phi)| \le \|\phi_n'-\phi'\|_p^p,$$

and obviously $\phi_n \to \phi$ in $\|\cdot\|_\infty$. For the converse we note that $\phi_n' \to 0$ in $L^p([u_M(\phi),\infty))$. In $L^p([0,u_M(\phi)])$ the sequence $\langle\phi_n'\rangle$ is bounded, therefore $\phi_n' \to \phi'$ weakly. Since $|\phi_n'| \le 1 = |\phi'|$ a.e., the convergence is strong.

□

Lemma 3.23

Let $x_n \in C^0([0,T])$, $\psi_{on} \in \hat{\Psi}$. If $t_n \to t$ in $[0,T]$, $x_n \to x$ in $C^0([0,T])$ and $\psi_{on} \to \psi_o$ in $(\hat{\Psi},d_{1,p})$, $1 \le p < \infty$, then $F(x_n,\psi_{on})(t_n) \to F(x,\psi_o)(t)$ in $(\hat{\Psi},d_{1,p})$.

Proof: Set $\phi_n = F(x_n,\psi_{on})(t_n)$, $\phi = F(x,\psi_o)(t)$. Lemma 3.22 implies $\psi_{on} \to \psi_o$ uniformly and $u_M(\psi_{on}) \to u_M(\psi_o)$. From proposition 3.7 we have $\phi_n \to \phi$ uniformly, and from lemma 3.20 we obtain $u_M(\phi_n) \to u_M(\phi)$. Again by lemma 3.22 we finally conclude $\phi_n \to \phi$ in $(\hat{\Psi},d_{1,p})$. □

We may summarize the continuity properties of the mapping F.

Theorem 3.24

The mapping F from definition 3.6 defines a unique Lipschitz-continuous (with Lipschitz constant 1) mapping

$$F: C^0([0,T]) \times \hat{\Psi} \to C^0([0,T];(\hat{\Psi},d_\infty)).$$

Also, the mapping $(t,x,\psi_o) \longmapsto F(x,\psi_o)(t)$ is continuous from $[0,T] \times C^0([0,T]) \times (\hat{\Psi},d_{1,p})$ to $(\hat{\Psi},d_{1,p})$ for all $1 \le p < \infty$.

Moreover, $\hat{\Psi}$ is the set of internal states which can be reached from $\psi_o=0$, i.e.

$$\hat{\Psi} = \{F(x,0)(t) \ : \ t \in [0,T], \ x \in C^0([0,T])\}.$$

<u>Proof:</u> For a given $\Phi \in \hat{\Psi}$ one can easily define an input x whose local maxima and minima produce the corners of Φ.
All other assertions have already been stated and proved in propositions 3.7, 3.19 and lemma 3.22.

□

This completes the first part of this section.
We now define the state-output map E and discuss some of its properties. As $E \circ F$ must correspond to the hysteresis operator W defined in 2.3, it is clear that E should measure the sets of switches, which are in the same position, i.e. the sets above and below an internal state curve $\psi \in \Psi_0$:

$$E(\psi,\mu) = c_+ \int_0^{\infty} \int_{-\infty}^{\psi(u)} d\mu(v,u) + c_- \int_0^{\infty} \int_{\psi(u)}^{\infty} d\mu(v,u),$$

where c_+ and c_- are the values taken on by an individual switch (in section 2 we assumed $c_+=1$, $c_-=-1$). This description makes apparent the analytical properties of E; formally it leads to difficulties if one wants to consider measures which are not absolutely continuous or which are only locally finite. To treat an infinite but locally finite measure (e.g. the Lebesgue measure without restrictions on support), one simply fixes a reference state ψ_μ and a reference output value $y_\mu = E(\psi_\mu, \mu)$. The other problem is harder; if the measure of a boundary curve graph$(\psi) \subset \mathbb{R}_+ \times \mathbb{R}$ is not zero, then one has to know the state of the switches on the boundary curve. This means that one has to decide how a switch with, say, an upper threshold ρ_2 behaves subject to an input with local maximum exactly equal to ρ_2. Does it switch or not? The situation is inherently ambiguous, the decision made implicitly in definition 2.1 is arbitrary. However, as already stated in section 2, if the measure of some set $\{\rho_2 = \text{const}\}$ or $\{\rho_1 = \text{const}\}$ is not zero, then the output y=y(t) is in general a discontinuous function. Therefore, in the continuous case this ambiguity does not matter. We state the relevant assumption, already given in theorems 2.5 and 3.1, formally in (u,v)-coordinates.

Definition 3.25

Let

$$M_\infty = \{\mu: \mu \text{ locally finite Borel measure on } \mathbb{R}_+ \times \mathbb{R}\}$$
$$M = \{\mu \in M_\infty: |\mu|(\mathbb{R}_+ \times \mathbb{R}) < \infty\}$$
$$M_+ = \{\mu \in M: \mu \geq 0\}.$$

We say that $\mu \in M_\infty$ has property (P_1), if $|\mu|(L) = 0$ for all straight lines L with slope 1 or -1.

□

Property (P_1) means that the measure of the graph of all possible boundary curves $\phi \in \hat{\Psi}$ is zero on the portion where $|\phi'| = 1$. We now define E in a convenient way, ignoring the exact history of the boundary itself.

Definition 3.26

Let $c_+, c_- \in \mathbb{R}$ with $c_+ \neq c_-$ be the two values attained by each individual switch. For $\psi \in \Psi_0$ and $\mu \in M$ we set

$$E(\psi,\mu) = c_+ \cdot \mu(\{(u,v): u \geq 0,\ v < \psi(u)\} + c_- \cdot \mu(\{(u,v): u \geq 0,\ \psi(u) \leq v\}.$$

□

Clearly, if μ is absolutely continuous with density $e \in L^1(\mathbb{R}_+ \times \mathbb{R})$, we can write definition 3.26 in the form

$$E(\psi,\mu) = c_+ \int_0^\infty \int_{-\infty}^{\psi(u)} e(u,v)\, dvdu + c_- \int_0^\infty \int_{\psi(u)}^\infty e(u,v)\, dvdu.$$

For $\mu \in M_\infty \setminus M$ we fix $\psi_\mu \in \hat{\Psi}$, $y_\mu \in \mathbb{R}$, set $E(\psi_\mu,\mu) = y_\mu$ and define $E(\psi,\mu)$ for $\psi \in \Psi_1$ by the signed difference

$$E(\psi,\mu) = y_\mu + (c_+ - c_-)[\mu(A_+(\psi,\psi_\mu)) - \mu(A_-(\psi,\psi_\mu))],$$

where

$$A_+(\psi,\psi_M) = \{(u,v) : u \geq 0,\ \psi_\mu(u) \leq v < \psi(u)\}$$
$$A_-(\psi,\psi_\mu) = \{(u,v) : u \geq 0,\ \psi(u) \leq v < \psi_\mu(u)\}.$$

For $\mu \in M$ this is equivalent to definition 3.26 if we define y_μ to be $E(\psi_\mu,\mu)$ from that definition. In the following, we mainly ignore the case $\mu \in M_\infty \setminus M$, since it only complicates the exposition.

The following order preserving property of E is obvious.

Lemma 3.27

In the situation of definition 3.26 let $\mu \in M_+$ and $\Psi_1, \Psi_2 \in \Psi_0$ with $\Psi_1 \leq \Psi_2$. Then

$$E(\Psi_1,\mu) \leq E(\Psi_2,\mu) \qquad \text{if} \quad c_- \leq c_+$$
$$E(\Psi_1,\mu) \geq E(\Psi_2,\mu) \qquad \text{if} \quad c_- \geq c_+$$

Also $\quad E(\Psi,\mu) \geq 0 \quad \text{for all } \Psi \in \Psi_0 \quad \text{if} \quad c_- \geq 0,\ c_+ \geq 0.$

Proof: Omitted. □

It turns out that continuity of E in the $W^{1,p}$-norm is characterized by property (P_1).

Lemma 3.28

Let $\mu \in M_\infty$, E as in definition 3.26. Then μ has property (P_1) if and only if

$$E(\cdot,\mu) : (\hat{\Psi}, d_{1,p}) \to \mathbb{R}$$

is continuous.

Proof: The "if"-part is obvious. For the other part, assume (P_1) holds. Let $\Psi_n \to \Psi$ in $(\hat{\Psi}, d_{1,p})$. We have

$$|E(\Psi_n,\mu) - E(\Psi,\mu)| \leq |c_+ - c_-| \cdot |\mu|(R_n)$$

where R_n denotes the region between graph(Ψ) and graph(Ψ_n). From lemma 3.22 we conclude that R_n is contained in the ε-neighbourhood of graph($\Psi|[0,u_M(\Psi)]$) with $\varepsilon = \|\Psi-\Psi_n\|_\infty + |u_M(\Psi)-u_M(\Psi_n)|$, which converges to zero for $n \to \infty$. Since (P_1) holds, also $|\mu|(R_n) \to 0$. □

We now compose E and F to define the Preisach operator.

Definition 3.29

For $x \in C^0([0,T])$, $\mu \in M_\infty$ and $\Psi_0 \in \Psi_1$ we define the Preisach operator W by

$$[W(x,\mu,\Psi_0)](t) = E(F(x,\Psi_0)(t),\mu).$$

If we keep Ψ_0 or μ fixed, we also write $W(x,\mu)(t)$, $W(x,\Psi_0)(t)$ or even $(Wx)(t)$. □

It is easy to see that the operator W defined in this way also has the properties of causality, rate independence and transition from proposition 2.2, if they are formulated in the obvious way. We state the order preservation and piecewise monotonicity property explicitly.

<u>Proposition 3.30</u>

Let $\mu \in M_\infty$, $\mu \geq 0$, let $c_+ \geq c_-$, where c_+ and c_- are the switching values at the right resp. left threshold.

(i) If $x_1 \leq x_2$ and $\psi_{o1} \leq \psi_{o2}$, then we have

$$W(x_1,\psi_{o1})(t) \leq W(x_2,\psi_{o2})(t) \quad \text{for all } t \in [0,T],$$

where $x_1, x_2 \in C^o([0,T])$ and $\psi_{o1}, \psi_{o2} \in \Psi_1$.

Let moreover $x \in C^o([0,T])$, $\psi_o \in \Psi_1$ and $I \subset [0,T]$ an interval.

(ii) If x is monotone on I (i.e., either nondecreasing or nonincreasing on I), then so is Wx, and $x(s) \leq x(t)$ implies $(Wx)(s) \leq (Wx)(t)$ for $t,s \in I$.

(iii) If Wx is strictly monotone on I, then so is x, and $(Wx)(s) < (Wx)(t)$ implies $x(s) < x(t)$ for $t,s \in I$.

<u>Proof:</u> Assertions (i) and (ii) immediately follow from the corresponding properties of F and E in lemmas 3.11, 3.12 and 3.27. To prove (iii), set $\psi = F(x,\psi_o)$ and consider the case that Wx is increasing. We show that for any $s<t$, the function x has a unique maximum on $[s,t]$ at $r=t$. Indeed, if there is a maximum at $r<t$, then $\psi(t) \leq \psi(r)$, by comparison with the input $\bar{x}$ which is equal to x on $[0,r]$ and constant on $[r,t]$, and therefore $(Wx)(t) \leq (Wx)(r)$, contradicting the assumption that Wx is increasing.

□

In order to prove theorem 3.1, we restate it in the terminology of this section.

<u>Theorem 3.31</u>

Let $\mu \in M_\infty$ and $\psi_o \in \Psi_1$. If μ has property (P_1), then definition 3.29 yields a continuous operator

$$W: C^o([0,T]) \to C^o([0,T]).$$

<u>**Proof:**</u> Let $x \in C^o([0,T])$, $\psi = F(x,\psi_o)$. By Theorem 3.24, the map ψ: $[0,T] \to (\hat{\Psi},d_{1,p})$ is continuous, therefore $Wx \in C^o([0,T])$ by lemma 3.28. Suppose now $x_n \to x$ in $C^o([0,T])$. Since $\mu = \mu_+ - \mu_-$ with $\mu_+ \geq 0$, $\mu_- \geq 0$, and E is linear with respect to μ, we may assume $\mu \geq 0$. We set

$$x_n^+ = x + \|x-x_n\|_\infty .$$

Then $x_n^+ \to x$ uniformly, and $F(x_n,\psi_o)(t) \to F(x,\psi_o)(t)$ in $(\hat{\Psi},d_{1,p})$ for all t by theorem 3.24, and again $(Wx_n^+)(t) \to (Wx)(t)$ pointwise in t by lemma 3.28. Since W is order preserving by proposition 3.30 (i), this convergence is monotone; we also have Wx_n^+, $Wx \in C^o([0,T])$, and Dini's theorem implies $Wx_n^+ \to Wx$ uniformly. The same holds for $x_n^- = x - \|x-x_n\|_\infty$, and $Wx_n^- \leq Wx_n \leq Wx_n^+$ then implies $Wx_n \to Wx$ uniformly.

□

We again comment on the relationship between the two different definitions of the Preisach operator in this and in the previous section. Definition 2.3 is slightly more general than definition 3.29, because it admits more general initial states, but usually this is no real advantage. Disregarding this aspect, both definitions are equivalent for piecewise monotone inputs if the measure μ has property (P_1), and the separate continuity proofs then show that they yield the same operator W: $C^o([0,T]) \to C^o([0,T])$. If μ does not satisfy (P_1), then the two definitions are not equivalent, and neither one defines an operator from $C^o([0,T])$ into itself.

§ 4 Further continuity properties

In this section we discuss various continuity properties of the Preisach operator W.

It was already proved in [6], theorem 38.3, that the following property of the measure μ characterizes the uniform continuity of W in $C^0([0,T])$.

Definition 4.1

Let $\mu \in M$. We say that μ has property (P_2), if

$$|\mu|(\text{graph } \psi) = 0 \qquad \text{for all } \psi \in \Psi_1.$$

□

One can express the modulus of continuity of $E(\cdot,\mu)\colon (\Psi_1, d_\infty) \to \mathbb{R}$ by means of the following quantity.

Definition 4.2

For any $\mu \in M$ and $\varepsilon > 0$ we define

$$\alpha(\varepsilon,\mu) = \sup_{\psi \in \Psi_1} |\mu|(N(\psi,\varepsilon)),$$

where

$$N(\psi,\varepsilon) = \{(u,v) : u \geq 0,\ \psi(u)-\varepsilon < v \leq \psi(u)+\varepsilon\}$$

□

Lemma 4.3

Let $\mu \in M_+$. Then we have for all $\varepsilon > 0$

$$|c_+ - c_-| \cdot \alpha(\varepsilon,\mu) = \sup\{|E(\phi,\mu)-E(\psi,\mu)| : \phi,\psi \in \Psi_1 \text{ and } \|\phi-\psi\|_\infty \leq 2\varepsilon\}.$$

Proof: The proof follows directly from the definition of E.

□

Theorem 4.4

Let $\mu \in M$. Then the following statements are equivalent:

(i) μ has property (P_2).

(ii) $E(\cdot,\mu) : (\Psi_1, d_\infty) \to \mathbb{R}$ is uniformly continuous.

(iii) $W\colon C^0([0,T]) \to C^0([0,T])$ is uniformly continuous.

<u>Proof:</u> The part "(i) $\Leftrightarrow$ (iii)" was already proved in [6], p. 254 f.; we shall restate that argument in our terminology. We may assume $\mu \geq 0$. To prove "(ii) => (iii)" we note that for $x_1, x_2 \in C^o([0,T])$ and $\psi_i = F(x_i, \psi_o)$ we have for all t

$$|(Wx_1)(t) - (Wx_2)(t)| = |E(\psi_1(t), \mu) - E(\psi_2(t), \mu)|,$$
$$\|\psi_1(t) - \psi_2(t)\|_\infty \leq \|x_1 - x_2\|_\infty$$

by theorem 3.24.

Next, we prove "(i) => (ii)". If E is not uniformly continuous, then by lemma 4.3 and the definition of $\alpha(\varepsilon, \mu)$ there exists an $\eta > 0$ and sequences $\psi_n \in \Psi_1$, $\varepsilon_n \downarrow 0$ with

$$\mu(N(\psi_n, \varepsilon_n)) \geq 2\eta > 0.$$

Choose $M > 0$ with $\mu([M,\infty) \times \mathbb{R}) \leq \eta$, set $K = [0,M] \times \mathbb{R}$. For a subsequence and some $\psi \in \Psi_1$, $\psi_n \to \psi$ uniformly on $[0,M]$. Then

$$K \cap N(\psi_n, \varepsilon_n) \subset K \cap N(\psi, \varepsilon_n + \|\psi - \psi_n\|_\infty),$$
$$\mu(K \cap N(\psi_n, \varepsilon_n)) \geq \eta,$$

so

$$\mu(K \cap \mathrm{graph}(\psi)) = \lim_{n\to\infty} \mu(K \cap N(\psi, \varepsilon_n + \|\psi - \psi_n\|_\infty)) \geq \eta,$$

and (i) does not hold. The implication "(iii) => (i)" is again proved indirectly. Let $\psi \in \Psi_1$ which $\mu(\mathrm{graph}(\psi)) > 0$. For some $M > 0$, with $K = [0,M] \times \mathbb{R}$ also $\mu(K \cap \mathrm{graph}(\psi)) > 0$. Since $\hat{\Psi}$ is dense in (Ψ_1, d_∞), for every $\varepsilon > 0$ one can find $\phi_\varepsilon \in \hat{\Psi}$ with $\phi_\varepsilon(u) < \psi(u) < \phi_\varepsilon(u) + \varepsilon$ for all $u \geq 0$. Moreover this can be done in a way such that on $[0,M]$

$$\phi_\varepsilon = F(x_\varepsilon, \psi_o)(t_\varepsilon), \quad \phi_\varepsilon + \varepsilon = F(x_\varepsilon + \varepsilon, \psi_o)(t_\varepsilon)$$

for some $x_\varepsilon \in C^o([0,T])$ and $t_\varepsilon \in [0,T]$.
This implies

$$|(W(x_\varepsilon + \varepsilon))(t) - (Wx_\varepsilon)(t)| \geq \mu(K \cap \mathrm{graph}(\psi)) > 0,$$

therefore W is not uniformly continuous.

□

In a similar manner one characterizes the Lipschitz continuity of W in $C^o([0,T])$.

<u>Theorem 4.5</u>

Let $\mu \in \mathcal{M}$. Then the following assertions are equivalent:

(i) There exists an L with

$$\alpha(\varepsilon,\mu) \leq L\varepsilon$$

for all $\varepsilon \geq 0$, where α is given in definition 4.2.

(ii) $E(\cdot,\mu) : (\Psi_1, d_\infty) \to \mathbb{R}$ is Lipschitz continuous.

(iii) $W: C^o([0,T]) \to C^o([0,T])$ is Lipschitz continuous.

__Proof:__ The equivalence of (i) and (ii) is clear from lemma 4.3. The implication (ii) => (iii) again follows directly from theorem 3.24. For the reverse, if $\mu \geq 0$ and

$$|E(\phi_1,\mu)-E(\phi_2,\mu)| \geq L\|\phi_1-\phi_2\|_\infty$$

for some $\phi_1,\phi_2 \in \Psi_1$, one constructs $\phi_1^\varepsilon \in \hat{\Psi}$, $x_1^\varepsilon \in C^o([0,T])$, $t_\varepsilon \geq 0$ with the properties

$$\min\{\phi_1,\phi_2\}-\varepsilon \leq \phi_1^\varepsilon \leq \min\{\phi_1,\phi_2\},$$

$$\phi_1^\varepsilon = F(x_1^\varepsilon,\psi_o)(t_\varepsilon),$$

$$\phi_2^\varepsilon = F(x_2^\varepsilon,\psi_o)(t_\varepsilon) \text{ on a large enough compact set,}$$

where $\phi_2^\varepsilon = \phi_1^\varepsilon + 2\varepsilon + \|\phi_1-\phi_2\|_\infty$, $x_2^\varepsilon = x_1^\varepsilon + 2\varepsilon + \|\phi_1-\phi_2\|_\infty$.

Then one easily checks that

$$|(Wx_1^\varepsilon)(t_\varepsilon)-(Wx_2^\varepsilon)(t_\varepsilon)| \geq L\|x_1^\varepsilon-x_2^\varepsilon\|,$$

if ε is small enough. □

The equivalence of (i) and (iii) in theorem 4.5 was already stated in [6], section 38.6, without proof.
We formulate the most important special case of theorem 4.5 separately.

__Corollary 4.6__
Let $\mu \in M$ have a bounded density f with support in $[0,M] \times \mathbb{R}$.
Then

$$\|W(x_1,\mu)-W(x_2,\mu)\|_\infty \leq |c_+-c_-|\cdot M\cdot\|f\|_\infty\cdot\|x_1-x_2\|_\infty$$

for all $x_1,x_2 \in C^o([0,T])$.

__Proof:__ We use lemma 4.3 and definition 4.2 directly, noting that $|\mu|(N(\psi,c)) \leq 2M\|f\|_\infty$. □

We remark that one could obtain corollary 4.6 immediately from proposition 3.7 and the definition of E without further analysis. This is one of the attractive features of the decomposition of section 3.

We will now consider boundedness and continuity properties of W in spaces other than $C^o([0,T])$. We first formulate an auxiliary lemma, which will enable us to conclude weak star continuity of W from boundedness of W.

<u>Lemma 4.7</u>

Let X, S_1, S_2 be metric spaces such that $S_1 \subset X$ and $S_2 \subset X$ with continuous injections. Let $f: X \to X$ be continuous and such that it maps relatively compact subsets of S_1 into relatively compact subsets of S_2, the compactness being here with respect to the topologies of S_1 and S_2. Then $f: S_1 \to S_2$ is continuous with respect to the topologies of S_1 and S_2.

<u>Proof:</u> Fix any $x \in S_1$ and any sequence $\{x_n \in S_1\}$ such that $x_n \to x$ in S_1; then the set $\{f(x_n)\}$ is relatively compact in S_2, therefore $f(x_{n'}) \to w$ in S_2 for some $w \in S_2$ and some subsequence. By continuity of f in X we also have $f(x_{n'}) \to f(x)$ in X; thus $w=f(x)$. As the limit does not depend on the subsequence, the whole sequence $f(x_n)$ converges to $f(x)$ in S_2.
$\square$

In section 5 we will establish the continuity of the inverse of W. Since the lemma needed there is similar to the one just proved, we present it at once.

<u>Lemma 4.8</u>

Let X, Y be metric spaces, let $f: S \to Y$ be continuous and $\tilde{Y} \subset f(X)$ be dense in Y. Also assume that for any relatively compact set $K \subset \tilde{Y}$, the set $f^{-1}(K)$ is relatively compact. Then $f(X) = Y$, and if moreover f is injective, then $f^{-1}: Y \to X$ is continuous.

<u>Proof:</u> Fix any $y \in Y$ and let the sequence $\{y_n \in \tilde{Y}\}$ converge to y. Then for any choice of $x_n \in f^{-1}(y_n)$, the sequence $\{x_n\}$ is relatively compact, hence $x_{n'} \to x$ for some subsequence $\{x_{n'}\}$ and some $x \in X$. Since f is continuous, we have $f(x) = y$. So f is surjective. Let f now be injective. Fix any $y \in Y$ and any

sequence $\langle y_n \in Y \rangle$ with $y_n \to y$. By the same argument as above we conclude that $f^{-1}(y_n) \to x = f^{-1}(y)$, since x does not depend on the subsequence. □

In order to obtain bounds on Wx in various norms, we want to estimate the oscillation of Wx. To do this, we have to relate the variation of the internal state, already described in lemma 3.16, to the measure μ.

Lemma 4.9

Let $\mu \in M_+$ and $c_+, c_- \in \mathbb{R}$ with $c_+ \neq c_-$. We define $k: \mathbb{R}_+ \to \mathbb{R}$ by

$$k(\delta) = |c_+ - c_-| \cdot \sup\langle \mu(R_i(\lambda, \lambda+\delta)) : \lambda \in \mathbb{R},\ i=1 \text{ or } 2 \rangle,$$

where $R_i(\lambda_1, \lambda_2)$ is the vertical resp. horizontal strip $\langle \lambda_1 \leq \rho_i \leq \lambda_2 \rangle$ in the Preisach plane in (ρ_1, ρ_2) coordinates; thus in (u,v) coordinates we have

$$R_1(\lambda_1, \lambda_2) = \langle (u,v) : u \geq 0,\ \lambda_1 + u \leq v \leq \lambda_2 + u \rangle$$
$$R_2(\lambda_1, \lambda_2) = \langle (u,v) : u \geq 0,\ \lambda_1 - u \leq v \leq \lambda_2 - u \rangle.$$

Here c_+ and c_- are again the values attained by each individual switch. □

Definition 4.10

For any $x \in C^0([0,T])$ and any $[t_1, t_2] \subset [0,T]$, we define the oscillation of x in $[t_1, t_2]$ by

$$\operatorname*{osc}_{[t_1,t_2]} x = \max_{t \in [t_1,t_2]} x(t) - \min_{t \in [x_1,x_2]} x(t) .$$

□

The crucial property of W is given in the next lemma.

Lemma 4.11

Let $\mu \in M_+$ have property (P_1) as defined in 3.25. Then we have, with k as defined in 4.9,

$$\operatorname*{osc}_{[t_1,t_2]} Wx \leq k(\operatorname*{osc}_{[t_1,t_2]} x)$$

for all $x \in C^0([0,T])$ and all $[t_1, t_2] \subset [0,T]$.

Proof: If we apply lemma 3.16 (i) with [0,T] replaced by $[t_1, t_2]$, the assertion follows directly from the definition of k and W, recalling from figure 3.2 that

$R_1(m(t),M(t)) = \Delta(t) \vee S_+(t)$ and $R_2(m(t),M(t)) = \Delta(t) \vee S_-(t)$.

□

For any $\nu \in (0,1]$, let us denote as usual by $C^{0,\nu}([0,T])$ the space of Hölder continuous functions with seminorm

$$|x|_\nu := \sup_{t_1,t_2 \in [0,T]} \frac{|x(t_1)-x(t_2)|}{|t_1-t_2|^\nu} .$$

We then can characterize a boundedness property for W in Hölder spaces.

Theorem 4.12

Let $\mu \in M_+$, assume that (P_1) holds. Let $0 < \nu_1,\nu_2 \leq 1$ and $C>0$. Then the following statements are equivalent:

(i) We have

$$k(\xi) \leq C\xi^{\nu_2/\nu_1} \quad \text{for all } \xi>0.$$

(ii) We have

$$|Wx|_{\nu_2} \leq C|x|_{\nu_1}^{\nu_2/\nu_1} \quad \text{for all } x \in C^{0,\nu_1}([0,T]).$$

In particular, W then maps $C^{0,\nu_1}([0,T])$ into $C^{0,\nu_2}([0,T])$, and is sequentially weakly star continuous.

Proof: Let us first prove that (ii) implies that W is sequentially weakly star continuous. This is a consequence of lemma 4.7. We set $X = C^0([0,T])$ and take for S_i bounded subsets of $C^{0,\nu_i}([0,T])$ such that $W: S_1 \to S_2$ by (ii). We have only to note that $W: X \to X$ is continuous by theorem 3.1, that $C^{0,\nu_i}([0,T])$ has a separable predual and therefore, with respect to the weak star topology, the S_i are precompact and metrizable.

We now prove that (i) implies (ii). Let $x \in C^{0,\nu_1}([0,T])$. Let us fix any $t_1,t_2 \in [0,T]$ with $t_1 < t_2$. By lemma 4.11 and (i) we obtain

$$|(Wx)(t_2)-(Wx)(t_1)| \leq \operatorname*{osc}_{[t_1,t_2]} Wx \leq$$

$$\leq k(\operatorname*{osc}_{[t_1,t_2]} x) \leq C(\operatorname*{osc}_{[t_1,t_2]} x)^{\nu_2/\nu_1} \leq$$

$$\leq C\Big[|x|_{\nu_1}(t_2-t_1)^{\nu_1}\Big]^{\nu_2/\nu_1} = C|x|_{\nu_1}^{\nu_2/\nu_1}(t_2-t_1)^{\nu_2},$$

therefore (ii) holds.
Let us now prove that (ii) implies (i). First, assume that μ has bounded support. Take any $\varrho>0$ and $\lambda \in \mathbb{R}$. To obtain a bound on $\mu(R_2(\lambda,\lambda+\varrho))$, we construct a suitable $x \in C^{0,\nu_1}([0,T])$. Now fix a t_1 with $0 < t_1 < T$ and set $x(t_1) = \lambda$, $x(0) = x_o$, where x_o is such that the support of μ lies above graph$(\psi(0))$. For arbitrary t_2 with $t_1 < t_2 < T$ consider the corresponding piecewise linear x defined by $x(0) = x_o$, $x(t_1) = \lambda$, $x(t_2) = \lambda+\varrho$ and $x(T) = x(t_2)$. Fix t_2 such that

$$|x|_{\nu_1} \leq (1+\varepsilon)\varrho(t_2-t_1)^{-\nu_1},$$

this is possible for any $\varepsilon>0$ since

$$|x|_{\nu_1;[t_1,t_2]} = \varrho(t_2-t_1)^{-\nu_1},$$

$$|x|_{\nu_1;[0,T]} \leq |x|_{\nu_1;[0,t_1]} + |x|_{\nu_1;[t_1,t_2]}.$$

By construction of x we have

$$|(Wx)(t_2)-(Wx)(t_1)| = |c_+-c_-|\mu(R_2(\lambda,\lambda+\varrho)).$$

and with assumption (ii) we may estimate

$$|(Wx)(t_2)-(Wx)(t_1)| \leq |Wx|_{\nu_2}\cdot(t_2-t_1)^{\nu_2}$$

$$\leq C|x|_{\nu_1}^{\nu_2/\nu_1}\cdot(t_2-t_1)^{\nu_2}$$

$$\leq C\Big[(1+\varepsilon)\varrho(t_2-t_1)^{-\nu_1}\Big]^{\nu_2/\nu_1}(t_2-t_1)^{\nu_2}$$

$$= C(1+\varepsilon)^{\nu_2/\nu_1}\cdot\varrho^{\nu_2/\nu_1},$$

which gives the desired bound for $\mu(R_2(\lambda,\lambda+\varrho))$ and therefore proves (i). If μ is finite, but has unbounded support, a suitable choice of x_o entails

$$|(Wx)(t_2)-(Wx)(t_1)| = \eta|c_+-c_-|\mu(R_2(\lambda,\lambda+\xi))$$

for some η with $\frac{1}{2} \leq \eta \leq 1$, so the argument above remains valid. □

A similar theorem holds, if we consider the Sobolev spaces $W^{1,p}(0,T)$.

Theorem 4.13

Let $\mu \in M_+$, assume that (P_1) holds, let $C \geq 0$. Then the following statements are equivalent:

(i) We have

$$k(\xi) \leq C\xi \qquad \text{for all } \xi \geq 0.$$

(ii) We have

$$\left|\frac{d}{dt}(Wx)(t)\right| \leq C\left|\frac{d}{dt}x(t)\right| \quad \text{a.e. in } (0,T)$$

for all $x \in W^{1,1}(0,T)$.

In particular, W then maps $W^{1,p}(0,T)$ into itself for all $1 \leq p \leq \infty$; moreover, for $p>1$, W is sequentially weakly star continuous in $W^{1,p}(0,T)$.

Proof: The last statement again is a consequence of lemma 4.7, applied as in the proof of theorem 4.12. Now, starting from (i), (ii) is easily established for any $x \in W^{1,1}(0,T) \cap C^o_{pm}([0,T])$; as this space is dense in $W^{1,1}(0,T)$, it holds in general. For the converse we obtain a bound on $\mu(R_2(\lambda,\lambda+\xi))$ constructing x as in the proof of theorem 4.12; this time, t_2 may be arbitrary, and the corresponding estimate is

$$|(Wx)(t_2)-(Wx)(t_1)| \leq \int_{t_1}^{t_2} \left|\frac{d}{dt}Wx\right| ds$$

$$\leq C\int_{t_1}^{t_2} \left|\frac{d}{dt}x\right| ds = C|x(t_2)-x(t_1)| = C\xi.$$

□

Now let us consider inputs of bounded variation. We remind the reader that by definition

$$x_n \to x \quad \text{weakly star in } BV(0,T)$$

if and only if

$$\int_0^T \Phi \, dx_n \longrightarrow \int_0^T \Phi \, dx$$

for all $\Psi \in C^0([0,T])$, these integrals being in the sense of Lebesgue-Stieltjes.

Theorem 4.14

Let $\mu \in M$, assume that (P_1) holds. If

$$L = \int_P \frac{1}{\rho_2 - \rho_1} \, d|\mu|(\rho) < \infty,$$

then W maps $C^0([0,T]) \cap BV(0,T)$ into itself and

$$\mathrm{Var}(Wx) \leq 2L \, \mathrm{Var}(x) + 2|\mu|(P)$$

for all $x \in C^0([0,T]) \cap BV(0,T)$. Moreover, W is continuous in the sense that for any sequence $\langle x_n \rangle$ in this space, if $x_n \to x$ strongly in C^0 and weakly star in BV, then also $Wx_n \to Wx$ strongly in C^0 and weakly star in BV.

Proof: Since $L < \infty$ implies that μ is concentrated on int(P), the first part is already given in proposition 2.4 (vii'). For the rest, the strong convergence is given in theorem 3.1 and the weak star convergence follows from lemma 4.7 in the same manner as in the proof of theorem 4.12, since BV(0,T) has the separable predual $C^0([0,T])$.

□

We remark that the convergence of the integral in theorem 4.14 is only a sufficient, but not a necessary condition for the inequality stated there to hold. To check this, consider a measure concentrated on the main diagonal $\rho_1 = \rho_2$ with bounded (one-dimensional) density; in this case the hysteresis operator degenerates into a superposition operator, but the integral obviously diverges.

§ 5 Inverse Preisach operator

Here we shall determine the conditions, under which there exists the inverse W^{-1} of the Preisach operator W, and study its properties. In contrast to the second part of section 4, here geometrical objects to be considered are triangles of the form

$$\{\rho \in P:\ \lambda_1 \leq \rho_1 \leq \rho_2 \leq \lambda_2\}$$

which correspond to sets of switches whose thresholds both lie between fixed bounds λ_1 and λ_2. We write down a formal definition in (u,v)-coordinates.

Definition 5.1

For any $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \leq \lambda_2$, we define

$$\Delta(\lambda_1,\lambda_2) = \{(u,v):\ u \geq 0,\ \lambda_1 + u \leq v \leq \lambda_2 - u\}.$$

Note that, in the terminology of figure 3.2 and definition 3.13, we have $\Delta(m(t),M(t)) \equiv \Delta(t)$.

If $\lambda_1 \geq \lambda_2$, we define $\Delta(\lambda_1,\lambda_2) = \Delta(\lambda_1,\lambda_2)$.

□

For convenience, we describe at once the setting in which we will formulate the results of this section.

Assumption 5.2

Let us fix $\psi_0 \in \hat{\Psi}$ and $c_+ = 1$, $c_- = 0$. Moreover, fix $a, b \in \mathbb{R}$ with $a < b$ and let $\mu \in M_+$ be a measure with support in $\Delta(a,b)$, satisfying property (P_1) in definition 3.25. We then define

$$X = \{x:\ x \in C^0([0,T]),\ a \leq x(t) \leq b \quad \text{for all } t\}$$

$$Y = \{y:\ y \in C^0([0,T]),\ 0 \leq y(t) \leq \mu(\Delta(a,b)) \quad \text{for all } t\}.$$

□

It is obvious from the definition of W that W maps X into Y. In the study of the inverse of W and its properties, a key role will be played by the following condition, which expresses the strict positivity of the measure μ in the neighbourhood of the main diagonal $\{\rho \in P:\ \rho_1 = \rho_2\}$ resp. the v-axis $\{(u,v) \in \mathbb{R}^2:\ u=0\}$.

Definition 5.3

Let assumption 5.2 hold. We say that μ satisfies the triangle property (T), if

$$\mu(\Delta(\lambda_1,\lambda_2)) > 0 \quad \text{for all} \quad a \leq \lambda_1 < \lambda_2 \leq b.$$

We define the function $\chi\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$\chi(\rho) = \min\{\mu(\Delta(\lambda,\lambda+\rho)) : a \leq \lambda \leq b-\rho\}.$$

We remark that if (T) holds, then $\chi(\rho) > 0$ for any $\rho > 0$, since by property (P_1) the function defined by $\lambda \longmapsto \mu(\Delta(\lambda,\lambda+\rho))$ is continuous. □

Remark 5.4

The triangle property is satisfied, for example, in the following two cases:

(i) μ possesses a density, with respect to the two-dimensional Lebesgue measure, which is strictly positive a.e. in some right neighbourhood of $\{(u,v) \in \mathbb{R}^2 : u=0\}$.

(ii) μ is concentrated on the line $\{(u,v): u=0\}$ and has an a.e. strictly positive density with respect to the one-dimensional Lebesgue measure along this line. In this case, W is a superposition operator of the form $(Wx)(t) = f(x(t))$ for some absolutely continous and increasing function f. □

The triangle property is linked to piecewise monotonicity properties of the operator W.

Proposition 5.5

Let assumption 5.2 hold. Then the following statements are equivalent:

(i) μ satisfies the triangle property (T).

(ii) We have

$$E(\Phi_1,\mu) < E(\Phi_2\mu)$$

for all $\Phi_1,\Phi_2 \in \Psi_1$ with $\Phi_1 < \Phi_2$, which means as before that $\Phi_1 \leq \Phi_2$ and $\Phi_1(0) < \Phi_2(0)$.

(iii) If $x \in X$ is strictly monotone on some interval $I \subset [0,T]$, then so is Wx on I.

(iv) If $x \in X$ and Wx is monotone on some interval $I \subset [0,T]$, then so is x on I.

Proof: We prove the implications (i) => (ii) => (iii) => (i) and (ii) => (iv) => (i). To prove (ii) from (i), note that

$$E(\Phi_2,\mu)-E(\Phi_1,\mu) \geq \mu(\Delta(\Phi_1(0),\Phi_2(0)))$$

for all $\Phi_1,\Phi_2 \in \Psi_1$ with $\Phi_1 \leq \Phi_2$. By lemma 3.11 we obtain (iii) directly from (ii). To prove (i) from (iii), consider an arbitrary triangle $\Delta(\lambda_1,\lambda_2)$, define $x \in X$ as the linear interpolate corresponding to $x(0) = \lambda_2$, $x(T/2) = \lambda_1$ and $x(T) = \lambda_2$, and observe that

$$\mu(\Delta(\lambda_1,\lambda_2)) = (Wx)(T)-(Wx)(\tfrac{T}{2}).$$

Next, we obtain (iv) from (ii). Let Wx be nondecreasing on I. Assume that there exist $s,t \in I$ with $s<t$ and $x(t) < x(s)$. We then may also assume that $x(r) \leq x(s)$ for all $r \in [s,t]$. With $\psi = F(x,\psi_0)$, we then have $\psi(t) < \psi(s)$ by comparison and lemma 3.12, therefore $(Wx)(t) < (Wx)(s)$ by assumption (ii), which is a contradiction. Finally, we prove that (iv) does not hold if the triangle property is not satisfied. Take $\lambda_1<\lambda_2$ with $\mu(\Delta(\lambda_1,\lambda_2)) = 0$ and define $x \in X$ to be the linear interpolate corresponding to $x(0) = 0$, $x(T/2) = \lambda_2$, $x(T) = \lambda_1$. Then Wx is nondecreasing, but x is not monotone.

□

We remark that the reverse implications in statements (iii) and (iv) above always hold, as was proved in lemma 3.30, so the triangle condition is characterized by the statement, that $x \in X$ is (strictly) monotone on a subinternal if and only if Wx is (strictly) monotone on that interval.
The first result of this section states that the triangle property is equivalent to the injectivity of W.

Theorem 5.6
Let assumption 5.2 hold. Then $W: X \to Y$ is one-to-one if and only if μ satisfies the triangle property.

Proof: First, assume that (T) holds. Let $x_1,x_2 \in X$ with $x_1 \neq x_2$. Then $x_1(\hat{t}) \neq x_2(\hat{t})$ for some $\hat{t} \in [0,T]$, and also

$\psi_1(\hat{t}) \neq \psi_2(\hat{t})$ with $\psi_i = F(x_i,\psi_o)$. Now proposition 3.21 implies that for some $t \in [0,\hat{t}]$ we have either $\psi_1(t) < \psi_2(t)$ or $\psi_2(t) < \psi_1(t)$. But then, proposition 5.5 (ii) applied to $\phi_i = \psi_i(t)$ shows that $(Wx_1)(t) \neq (Wx_2)(t)$, so $Wx_1 \neq Wx_2$ as was to be proved.
For the converse, assume that $\mu(\Delta(\lambda_1,\lambda_2)) = 0$ for some $\lambda_1<\lambda_2$. Fix $t_1 \in (0,T)$ and define $x_\lambda \in X$ as the linear interpolate corresponding to $x_\lambda(0) = \lambda_2$, $x_\lambda(t_1) = \lambda_1$ and $x_\lambda(T) = \lambda$. Now it is obvious that Wx_λ yields the same function for all $\lambda \in [\lambda_1,\lambda_2]$, since Wx_λ is constant on $[t_1,T]$ for all such λ. Therefore W is not injective. □

The next lemma is in some sense dual to lemma 4.11.

<u>Lemma 5.7</u>
Let assumption 5.2 hold. Then we have, with $\varkappa$ as defined in 5.3,

$$\varkappa(\underset{[t_1,t_2]}{\operatorname{osc}} x) \leq \underset{[t_1,t_2]}{\operatorname{osc}} Wx ,$$

for all $x \in X$ and all $[t_1,t_2] \subset [0,T]$.

<u>Proof:</u> For any $x \in X$ and any $[t_1,t_2]$, we set $\psi = F(x,\psi_o)$ and apply lemma 3.16 (ii) with $[t,T]$ replaced by $[t_1,t_2]$. This yields $s_m,s_M \in [t_1,t_2]$ with

$$\underset{[t_1,t_2]}{\operatorname{osc}} x = x(s_M) - x(s_m)$$

and $\psi(s_m) < \psi(s_M)$, therefore we also have

$$\begin{aligned}\mu(\Delta(x(s_m),x(s_M))) &\leq E(\psi(s_m),\mu) - E(\psi(s_m),\mu)\\ &= (Wx)(s_M) - (Wx)(s_m).\end{aligned}$$

The assertion now follows from the definition of $\varkappa$.

□

The lemma just given allows us to characterize the range of W and to derive continuity properties of the inverse.

<u>Theorem 5.8</u>
Let assumption 5.2 hold. Then $W(X) = Y$ if and only if μ

satisfies the triangle property (T). In this case, moreover, $W^{-1}: Y \to X$ is continuous with respect to the uniform topology.

<u>Proof:</u> First, let us assume that (T) holds. We want to apply lemma 4.8 to $f \equiv W$. To do this, we note that, for every compact set $K \subset Y$, the set $W^{-1}(K)$ is compact in X by Ascoli's theorem and lemma 5.7. Moreover, we assert that

$$Y \cap C^{o}_{pm}([0,T]) \subset f(X).$$

This is true, because for every piecewise monotone $y \in Y$ we may construct explicitly an $x \in X$ with $y = Wx$, if we first choose $x(0)$ such that $y(0) = (Wx)(0)$ and then proceed from one monotonicity interval to the next. Therefore, $W(X)$ is dense in Y, and because of theorem 5.6, W is injective. Lemma 4.8 now yields that $W(X) = Y$ and that W^{-1} is continuous. For the converse, let us assume that $W(X) = Y$. We want to show that (T) holds. We first recall that, if $y = Wx$ is strictly monotone on some interval $I \subset [0,T]$, then so is x by proposition 3.30 (iii). For the interpolate y of $y(0) = 0$, $y(T) = \mu(\Delta(a,b))$, take x with $y = Wx$. Since x and y are increasing, $\mu(R_2(\lambda_1,\lambda_2)) \geq 0$ for all horizontal strips $R_2(\lambda_1,\lambda_2)$ with $a \leq \lambda_1 < \lambda_2 \leq b$. Now take any $\lambda_2 \geq a$ and $t_1 \in (0,T)$, and choose y as the interpolate of $y(0) = 0$, $y(t_1) = \mu(\Delta(a,\lambda_2))$, $y(T) = 0$. Also take $x \in X$ with $y = Wx$. By the above, $x(t_1) = \lambda_2$, and x is increasing on $[0,t_1]$ as well as decreasing on $[t_1,T]$. Therefore we have for all $\lambda_1 < \lambda_2$ that

$$\mu(\Delta(\lambda_1,\lambda_2)) = y(t_1) - y(t) \geq 0,$$

if we choose $t \geq t_1$ such that $x(t) = \lambda_1$.

□

<u>Corollary 5.9</u>

Let assumption 5.2 hold. Then $W: X \to Y$ is bijective if and only if μ satisfies the triangle property (T). If so, then $W^{-1}: Y \to X$ is continuous with respect to the uniform topology.

<u>Proof:</u> This is a direct consequence of theorems 5.6 and 5.8.

□

For the continuity of W^{-1} in other norms, we obtain results analogous to those in section 4. Again, we start with Hölder seminorms.

Theorem 5.10

Let assumption 5.2 hold, let $0 < \nu_1, \nu_2 \leq 1$, $C>0$. The following statements are equivalent:

(i) We have
$$\chi(\rho) \geq C\rho^{\nu_1/\nu_2} \quad \text{for all } \rho>0.$$

(ii) The inverse of $W: X \to Y$ exists, and we have
$$|W^{-1}y|_{\nu_2} \leq (C^{-1}|y|_{\nu_1})^{\nu_2/\nu_1}$$
for all $y \in Y \cap C^{0,\nu_1}([0,T])$.

In particular, W^{-1} then maps $Y \cap C^{0,\nu_1}([0,T])$ into $X \cap C^{0,\nu_2}([0,T])$ and is sequentially weakly star continuous with respect to the Hölder norms.

Proof: Since the proof is analogous to the proof of theorem 4.12, we will state it more briefly. To prove (ii) from (i), note that (i) implies (T) and therefore W is invertible by corollary 5.9. Moreover, for any $t_1 < t_2$ and any $y \in Y \cap C^{0,\nu_1}([0,T])$ we have, using lemma 5.7 and writing $x = W^{-1}y$,
$$\begin{aligned} C|x(t_2)-x(t_1)|^{\nu_1/\nu_2} &\leq \chi(|x(t_2)-x(t_1)|) \leq \\ &\leq \chi(\operatorname*{osc}_{[t_1,t_2]} x) \leq \operatorname*{osc}_{[t_1,t_2]} y \\ &\leq |y|_{\nu_1}(t_2-t_1)^{\nu_1}, \end{aligned}$$
which implies (ii). For the converse, construct $y \in Y$ and $x = W^{-1}y$ such that
$$x(0) = a,\ x(t_1) = \lambda+\rho,\ x(t_2) = x(T) = \lambda,$$
$$|y(t_2)-y(t_1)| = \mu(\Delta(\lambda,\lambda+\rho)),$$
$$|y|_{\nu_1} \leq (1+\varepsilon)\frac{|y(t_2)-y(t_1)|}{(t_2-t_1)^{\nu_1}}.$$

This gives us

$$\rho \leq |W^{-1}y|_{\nu_2} (t_2-t_1)^{\nu_2}$$

and, using (ii),

$$C\rho^{\nu_1/\nu_2} \leq (1+c)\mu(\Delta(\lambda,\lambda+\rho)),$$

which implies (i). Since W^{-1}: $Y \to X$ is continuous by corollary 5.9, the same argument as in the proof of theorem 4.12 yields that W^{-1} is sequentially weakly star continuous with respect to the Hölder norms.

□

We also obtain the corresponding theorem for the Sobolev spaces $W^{1,p}(0,T)$.

<u>Theorem 5.11</u>

Let assumption 5.2 hold, let $C>0$. Then the following statements are equivalent:

(i) We have

$$\chi(\rho) \geq C\rho \qquad \text{for all } \rho>0.$$

(ii) The inverse W^{-1} : $Y \to X$ exists, and we have

$$\left|\frac{d}{dt} (W^{-1}y)(t)\right| \leq C^{-1}\left|\frac{d}{dt} y(t)\right| \quad \text{a.e. in } (0,T)$$

for all $y \in W^{1,1}(0,T) \cap Y$.

In particular, W^{-1} then maps $W^{1,p}(0,T) \cap Y$ into $W^{1,p}(0,T) \cap X$ for all $1 \leq p \leq \infty$; moreover, for $p>1$, W^{-1} is sequentially weakly star continuous with respect to the norm of $W^{1,p}$.

<u>Proof:</u> Since property (i) implies (T), by corollary 5.9 it also implies the existence and continuity of W^{-1}: $Y \to X$. The proof is now completely analogous to that of theorem 4.13; for the converse one constructs x and y with $y = Wx$, $|x(t_2)-x(t_1)| = \rho$ and

$$|y(t_2)-y(t_1)| = \mu(\Delta(\lambda,\lambda+\rho)).$$

□

Let us now consider the BV properties.

Theorem 5.12

Let assumption 5.2 hold, assume that for some $C>0$

$$\chi(\rho) \geq C\rho \qquad \text{for all } \rho>0.$$

Then W^{-1} maps $Y \cap BV(0,T)$ into $X \cap BV(0,T)$, and

$$\mathrm{Var}(W^{-1}y) \leq C^{-1}\cdot \mathrm{Var}(y)$$

for all $y \in Y \cap BV(0,T)$. Moreover, W^{-1} is continuous in the sense that for any sequence $\langle y_n\rangle$ in $C^0([0,T]) \cap BV(0,T)$, if $y_n \to y$ strongly in C^0 and weakly star in BV, then also $W^{-1}y_n \to W^{-1}y$ strongly in C^0 and weakly star in BV.

Proof: Theorem 5.11 implies that

$$\mathrm{Var}(W^{-1}y) \leq C^{-1}\cdot \mathrm{Var}(y)$$

for all $y \in Y \cap W^{1,1}(0,T)$. The inequality remains true for $y \in Y \cap BV(0,T)$, because $W^{1,1}(0,T)$ is dense in $BV(0,T)$ with respect to the topology induced by the distance

$$d(y,z) = \|y-z\|_{L^1(0,T)} + |\mathrm{Var}(y)-\mathrm{Var}(z)|,$$

compare [13]. The second assertion again is a consequence of lemma 4.7 and corollary 5.9. □

We end this section with a characterization of Lipschitz continuity of W^{-1} in the norm of $C^0([0,T])$. We first state the inequality which is essential for this result.

Lemma 5.13

Let assumption 5.2 hold, let $x_1,x_2 \in C^0([0,T])$. Then we have

$$\mu(\Delta(x_1(t),x_2(t)) \leq 2\cdot \sup_{0\leq s\leq t} |(Wx_1)(s)-(Wx_2)(s)|$$

for all $t \in [0,T]$.

Proof: The proof will be given after the proof of theorem 5.14. □

Theorem 5.14

Let assumption 5.2 hold. Then the following statements are equivalent:

(i) There exists a $C>0$ such that
$$\varkappa(\xi) \geq C\xi \qquad \text{for all } \xi>0.$$

(ii) The inverse $W^{-1}: Y \to X$ exists and is Lipschitz continuous with respect to the uniform norm.

<u>Proof:</u> If (i) holds, then also (T) holds, therefore W is invertible by corollary 5.9. Moreover, for all $t \in [0,T]$ and all $x_1, x_2 \in C^o([0,T])$ we have

$$\begin{aligned} C|x_1(t)-x_2(t)| &\leq \varkappa(|x_1(t)-x_2(t)| \\ &\leq \mu(\Delta(x_1(t),x_2(t))) \\ &\leq 2\cdot \sup_{0\leq s\leq t} |Wx_1)(s)-(Wx_2)(s)|, \end{aligned}$$

so W^{-1} is Lipschitz continuous with Lipschitz constant $2C^{-1}$. For the converse, take any triangle $\Delta(\lambda,\lambda+\xi)$. It is easy to construct $x_1, x_2 \in C^o([0,T])$ with $x_1(T) = \lambda$, $x_2(T) = \lambda+\xi$, $\|x_1-x_2\|_\infty = \xi$ and
$$(Wx_2)(T)-(Wx_1)(T) = \|Wx_2-Wx_1\|_\infty = \mu(\Delta(\lambda,\lambda+\xi)).$$
From this, assertion (i) follows immediately.

□

<u>Proof of lemma 5.13:</u> The proof is based upon a close inspection of the internal states $\psi_1(t)$ and $\psi_2(t)$, where $\psi_i = F(x_i,\psi_o)$ for $i=1,2$. Fix any $t \in [0,T]$. For $x_1(t) = x_2(t)$ the assertion is trivial, so let us assume that $x_1(t) < x_2(t)$. We can interpret the difference $(Wx_2)(t)-(Wx_1)(t)$ as the signed area between graph($\psi_2(t)$) and graph($\psi_1(t)$). Since $\Delta(x_1(t),x_2(t))$ is contained in that area, if $\psi_1(t) \leq \psi_2(t)$, then
$$\mu(\Delta(x_1(t),x_2(t))) \leq (Wx_2)(t)-(Wx_1)(t),$$
and we are done. The situation is more difficult if $\psi_1(t)u > \psi_2(t)u$ for some $u>0$, and the estimate above obviously is no longer valid. We introduce the following notation (compare figure 5.1):
$$u_* = \inf\{u\colon u\geq 0,\ \psi_1(t)u > \psi_2(t)u\},$$
$$u^* = \inf\{u\colon u>u_*,\ \psi_1(t)u = \psi_2(t)u\},$$

$$\Delta = \Delta(x_1(t), x_2(t)),$$

$$D_1 = \{(u,v): 0 \leq u \leq u_*, \ \psi_1(t)u \leq v \leq \psi_2(t)u\} - \Delta,$$

$$D_2 = \{(u,v): u_* \leq u \leq u^*, \ \psi_2(t)u \leq v \leq \psi_1(t)u\}.$$

Because $x_1(t) \neq x_2(t)$ and because $\text{supp}(\psi_i(t))$ is bounded,

$$0 < u_* < u^* < \infty.$$

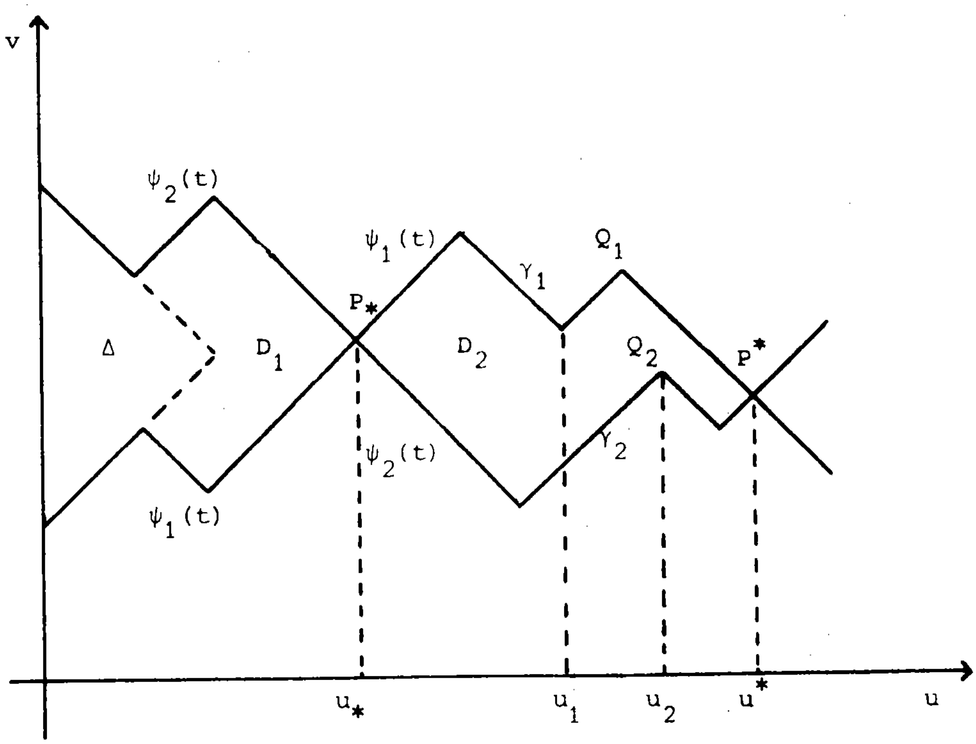

Figure 5.1

Now obviously

$$(Wx_1)(t)-(Wx_2(t) = \mu(\Delta)+\mu(D_1)-\mu(D_2)+m_\infty,$$

where m_∞ represents the contribution from the region where $u \geq u^*$. The idea of the proof is to find an $s \in [0,t]$ with

$$(*) \qquad (Wx_1)(s)-(Wx_2)(s) \geq \mu(D_2)-m_\infty.$$

This would be enough to prove the lemma, since then

$$\begin{aligned}\mu(\Delta) &= [(Wx_2)(t)-(Wx_1)(t)] - \mu(D_1) + \mu(D_2) - m_\infty \\ &\leq [(Wx_2)(t)-(Wx_1)(t)] + [(Wx_1(s)-(Wx_2)(s)].\end{aligned}$$

Let us prove (*). We consider the region D_2 and assume that

$u^* \leq \min\{u_M(\psi_1(t)), u_M(\psi_2(t))\}$, where u_M is defined in lemma 3.20, so that ∂D_2 consists of lines with slope 1 or -1 only, as in figure 5.1. (The proof is analogous if also horizontal portions appear.) We set

$$P_* = (u_*, \psi_1(t)u_*) = (u_*, \psi_2(t)u_*)$$

$$P^* = (u^*, \psi_1(t)u^*) = (u^*, \psi_2(t)u^*),$$

and observe that both curves graph($\psi_i(t)$), restricted to $[u_*, u^*]$, consist of at least two arcs of different slope, as is implied by the definition of u_* and u^*.
Let γ_i be the second arc of graph$(\psi_i(t))|[u_*, u^*]$, counted from left to right, and let $Q_i = (u_i, \psi_i(t)u_i)$ be the right endpoint of γ_i. (It may happen that $Q_i = P^*$ for one or both i.) We now set

$$s = \max\{l_1(u_1), l_2(u_2)\}.$$

where $l_i(u_i)$ denotes the time where the arc γ_i is formed on graph($\psi_i(t)$), i.e. the last time Q_i is touched by a straight line, according to definition 3.17. We claim that s satisfies (*). For this, assume without loss of generality that $s = l_1(u_1)$. From lemma 3.18 and the construction in the proof of proposition 3.19 it is easy to see that

$$\psi_1(s) = \max\{\psi_1(t), \phi_1\}$$

$$\psi_2(s) \leq \max\{\psi_2(t), \phi_*\} \quad \text{on } [0, u_2]$$

$$\psi_2(s) = \psi_2(t) \quad \text{on } [u_2, \infty),$$

where ϕ_1 and ϕ_* are the straight lines with slope -1 through Q_1 and P_*, respectively. Since the measure μ is nonnegative, this implies (*), and the proof is complete.

□

There is a more explicit version of the condition for μ in the previous theorems. (This observation is due to Stefan Luckhaus.)

Lemma 5.15

Let assumption 5.2 hold, let $C \geq 0$. Then the following statements are equivalent:

(i) We have

$$\chi(\xi) \geq C\xi \qquad \text{for all } \xi \geq 0.$$

(ii) The measure μ, restricted to the main diagonal, satisfies

$$\mu \geq C\lambda,$$

where λ is the one-dimensional Lebesgue measure.

__Proof:__ To prove (ii) from (i), consider any interval $I = [a,b]$ on the main diagonal and approximate it by a sequence of triangles

$$D_n = \bigcup_{i=1}^{n} \Delta(c_{i-1},c_i) \, , \qquad c_i = a + \frac{i}{n}\,(b-a).$$

Then $\mu(D_n) \longrightarrow \mu(I)$ and $\mu(D_n) \geq C(b-a)$. The converse is obvious. □

One therefore realizes that the continuity properties of W^{-1} characterized by condition 5.15 (i) are linked to the superposition part of the Preisach operator.

§ 6 Properties of W and W^{-1} in spaces of vector valued functions

Here we shall be concerned with the properties of the hysteresis operator W and of its inverse W^{-1} (if it exists) in the case that the input function depends not only on time, but also on another variable, typically representing space. So we introduce a Euclidean domain $\Omega \subset \mathbb{R}^N$, whose generic point will be denoted by y in order to avoid confusion with the previous use of the letter x; however, conforming to standard notation in the theory of PDE's, we will write u for the input function. In a general way, given any "memory operator"

$$H: C^o([0,T]) \longrightarrow C^o([0,T]),$$

for any function $u: \Omega \times [0,T] \longrightarrow \mathbb{R}$ we set

$$[\tilde{H}(u)](y,t) = [H(u(y,\cdot))](t)$$

for all $(y,t) \in \Omega \times [0,T]$, for the moment without specifying any regularity assumption. In this approach, the space variable y only plays the role of a parameter. The non-local dependence involves just the time variable; there is memory but no space interaction. Of course, this formulation may not be completely satisfactory for some application.
The results we shall present in a moment are intended to be applied for either $H = W$ or $H = W^{-1}$, whence the initial conditions have been prescribed. Note that also the initial conditions may depend on y, and hence must satisfy natural regularity conditions, such as measurability for example.
We set

$$Q = \Omega \times (0,T)$$

and introduce the set of Carathéodory functions

$$M(\Omega;C^o([0,T])) = \{u: Q \to \mathbb{R} \mid y \longmapsto u(y,t) \text{ is measurable for all } t,\ t \longmapsto u(y,t) \text{ is continuous in } [0,T] \text{ for almost all } y\}.$$

We consider the following function spaces, all Banach spaces endowed with natural norms:

$$W^{1,p}(0,T;L^p(\Omega)) = L^p(\Omega;W^{1,p}(0,T)) = \{u: u \in L^p(Q), \frac{\partial u}{\partial t} \in L^p(Q)\}. \quad 1 \le p < \infty,$$

$$L^p(\Omega;C^o([0,T])) = \left\{u:\ u \in M(\Omega;C^o([0,T])), \int_\Omega \|u(y,\cdot)\|_\infty^p\, dy < \infty\right\},$$

$$W^{1,p}(\Omega;C^o([0,T])) = \{u:\ u, \frac{\partial u}{\partial t} \in L^p(\Omega;C^o([0,T]))\}, \quad 1 \le p \le \infty,$$

$$W^{\lambda,p}(\Omega;C^o([0,T])) = \{u:\ u \in L^p(\Omega;C^o([0,T])),$$

$$\iint_{\Omega\times\Omega} \|u(y_1,\cdot)-u(y_2,\cdot)\|_\infty^p \cdot \|y_1-y_2\|^{-(N+\lambda p)} dy_1 dy_2 < \infty\},$$

$$1 \le p < \infty,\ 0 < \lambda \le 1,$$

$$W^{\lambda,\infty}(\Omega;C^o([0,T])) = \{u:\ u \in L^\infty(\Omega;C^o([0,T])),$$

$$\operatorname*{ess\,sup}_{y_1,\, y_2 \in \Omega} (\|u(y_1,\cdot) - u(y_2,\cdot)\|_\infty \cdot \|y_1-y_2\|^{-\lambda}) < \infty\},$$

$$0 < \lambda \le 1.$$

We recall that $W^{\lambda,\infty}(\Omega;C^o([0,T])$ coincides with the Hölder space $C^{o,\lambda}(\bar{\Omega};C^o([0,T]))$.

<u>Lemma 6.1</u>
Let $H:\ C^o([0,T]) \longrightarrow C^o([0,T])$ be continuous. Then

$$[\tilde{H}(u)](y,t) = [H(u(y,\cdot))](t)$$

defines an operator

$$\tilde{H}:\ M(\Omega;C^o([0,T])) \longrightarrow M(\Omega;C^o([0,T])).$$

<u>Proof:</u> It is sufficient to note that $y \longmapsto u(y,\cdot)$ defines a measurable mapping from Ω to $C^o([0,T])$ for any $u \in M(\Omega;C^o([0,T]))$. □

<u>Proposition 6.2</u>
Let X be a Banach space which can be embedded continuously in $C^o([0,T])$, let $H:\ C^o([0,T]) \longrightarrow C^o([0,T])$ be continuous. If H operates and is bounded in X, then $\tilde{H}$ operates and is bounded in $L^p(\Omega;X)$ for all $1 \le p \le \infty$.

<u>Proof:</u> The proof is straightforward and therefore omitted.

□

Due to the results of sections 3, 4 and 5, proposition 6.2 can be applied for $H = W$ or $H = W^{-1}$, and either

$$X = C^{o,\lambda}([0,T]) \;, \quad 0 < \lambda \leq 1 \;, \quad \text{or}$$
$$X = W^{1,p}(0,T) \;, \quad 1 \leq p \leq \infty \;, \quad \text{or}$$
$$X = C^{o}([0,T]) \cap BV(0,T).$$

Proposition 6.3
Let $H: C^o([0,T]) \longrightarrow C^o([0,T])$ be continuous. Then $\tilde{H}$ operates in $C^o(\Omega;C^o([0,T]))$.

Proof: Omitted. □

Proposition 6.4
Let $H: C^o([0,T]) \longrightarrow C^o([0,T])$ be Lipschitz continuous. Then the following assertions hold:

(i) $\tilde{H}$ operates and is Lipschitz continuous in $L^p(\Omega;C^o([0,T]))$ for $1 \leq p \leq \infty$.

(ii) $\tilde{H}$ operates and is bounded in $W^{\lambda,p}(\Omega;C^o([0,T]))$ for $0 < \lambda \leq 1$, $1 \leq p \leq \infty$.

Proof: Omitted. □

Acknowledgements

This paper was partially written during visits of the first author at CNR, Pavia, and at Università degli Studi di Trento, and of the second author at Universität Augsburg.

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