Index Insurance for Farmers: Modeling, Demand and Combination with Further Products

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Abstract

In this thesis we focus on weather index insurance for agriculture risk. Even though such an index insurance is easily applicable and reduces information asymmetries, the demand for it is quite low. This is in particular due to the basis risk and the lack of knowledge about it's effectiveness. The basis risk is the difference between the index insurance payout and the actual loss of the insured. We evaluate the performance of weather index insurance in different contexts, because proper knowledge about index insurance will help to use it as a successful alternative for traditional crop insurance. In addition to that, we also propose and discuss methods to reduce the basis risk.

We also analyze the performance of an agriculture loan which is interlinked with a weather index insurance. We show that an index insurance with actuarial fair or subsidized premium helps to reduce the loan default probability. While we first consider an index insurance with a commonly used linear payout function for this analysis, we later design an index insurance payout function which maximizes the expected utility of the insured. Then we show that, an index insurance with that optimal payout function is more appropriate for bundling with an agriculture loan. The optimal payout function also helps to reduce the basis risk. In addition, we show that a lender who issues agriculture loans can be better off by purchasing a weather index insurance in some circumstances.

We investigate the market equilibrium for weather index insurance by assuming risk averse farmers and a risk averse insurer. When we consider two groups of farmers with different risks, we show that the low risk group subsidizes the high risk group when both should pay the same premium for the index insurance. Further, according to the analysis of an index insurance in an informal risk sharing environment, we observe that the demand of the index insurance can be increased by selling it to a group of farmers who informally share the risk based on the insurance payout, because it reduces the adverse effect of the basis risk. Besides of that we analyze the combination of an index insurance with a gap insurance. Such a combination can increase the demand and reduce the basis risk of the index insurance if we choose the correct levels of premium and of gap insurance cover. Moreover our work shows that index insurance can be a good alternative to proportional and excess loss reinsurance when it is issued at a low enough price.

Zusammenfassung

In dieser Dissertation betrachten wir Wetterindexversicherungen für landwirtschaftliche Risiken. Auch wenn eine solche Indexversicherung leicht anwendbar ist und Informationsasymmetrien reduziert, ist die Nachfrage nach ihr recht gering. Dies liegt insbesondere an dem Basisrisiko und dem mangelnden Wissen über dessen Effektivität. Das Basisrisiko ist die Differenz zwischen der Auszahlung der Indexversicherung und dem tatsächlichen Schaden des Versicherten. Wir evaluieren die Leistungen von Wetterindexversicherungen in verschiedenen Kontexten. Dieses Wissen über die Indexversicherung kann helfen, sie als erfolgreiche Alternative für die traditionelle Ernteversicherung zu nutzen. Darüber hinaus schlagen wir Methoden zur Reduzierung des Basisrisikos vor und diskutieren diese.

Wir analysieren auch die Performance eines Agrarkredits, der mit einer Wetterindexversicherung verknüpft ist. Wir zeigen, dass eine Indexversicherung mit versicherungsmathematisch fairer oder subventionierter Prämie dazu beiträgt, die Kreditausfallwahrscheinlichkeit zu reduzieren. Während wir zunächst eine Indexversicherung mit einer der üblicherweise verwendeten linearen Auszahlungsfunktionen für diese Analyse nehmen, entwickeln wir später eine Indexversicherungs-Auszahlungsfunktion, die den erwarteten Nutzen des Versicherten maximiert. Dann zeigen wir, dass eine Indexversicherung mit dieser optimalen Auszahlungsfunktion besser geeignet ist, um sie mit einem Agrarkredit zu bündeln. Außerdem hilft diese optimale Auszahlungsfunktion, das Basisrisiko zu reduzieren. Zusätzlich zeigen wir, dass in manchen Fällen eine Indexversicherung auch für den Kreditgeber lohnenswert sein kann.

Wir untersuchen auch das Marktgleichgewicht für eine Wetterindexversicherung unter der Annahme risikoaverser Landwirten und eines risikoaversen Versicherers. Wenn wir zwei Gruppen von Landwirten mit verschiedenem Risiko betrachten, erkennen wir, dass die Gruppe mit geringem Risiko die Gruppe mit hohem Risiko subventioniert, wenn beide die gleiche Prämie für die Indexversicherung zahlen sollen. Durch eine Analyse der informellen Index-basierten Risikoteilung in einer Gruppe von Farmern zeigen wir ferner, dass die Nachfrage nach Indexversicherung steigen kann, da diese unerwünschte Auswirkungen der Risikoteilung auf das Basisrisiko reduziert. Außerdem analysieren wir die Kombination einer Indexversicherung mit einer Gap-Versicherung. Eine solche Kombination kann die Nachfrage erhöhen und das Basisrisiko der Indexversicherung reduzieren, wenn wir die richtige Höhe der Prämie und des Gap-Versicherungsschutzes wählen. Außerdem zeigt unsere Arbeit, dass eine Indexversicherung eine gute Alternative zu einer proportionalen oder Exzedenten Rückversicherung sein kann, wenn sie zu einem ausreichend niedrigen Preis angeboten wird.

Contents

Chapter 1

Introduction

Agriculture insurance has a long history. As outlined in Reyes et al. (2017) in the late 19th century many European countries and the United States had crop insurance schemes. The agriculture sector requires insurance due to its uncertainty. Uncertainty of agriculture mainly depends on the nature of the weather. Nevertheless, the traditional crop insurance, which is based on actual losses may not be a good crop insurance. One of the major reasons for that are information asymmetries arising due to the complicated nature of agriculture risk.

Index based insurance is an alternative way to cover agriculture risk. Basically there are two types of index insurance and those are area yield index insurance and weather index insurance. The weather index insurance insures the harvest against natural catastrophes and weather events (e.g., floods, earthquakes and hurricanes). In this work we basically focus on the weather index insurance. As defined in World-Bank (2011), "the weather index insurance is a simplified form of insurance in which indemnity payments are based on the values obtained from an index that serves as a proxy for losses rather than upon the assessed losses of each individual policyholder". Further, it is independent of the information asymmetries, because it depends on some well known index which does not have an influence from the farmers.

In order to design a successful index insurance, it is very important to identify the relationship between the variable of interest and the underlying index. As explain in World-Bank (2011), in order for the underlying index to be a good signal for loss, it should be based on an objective measure such as rainfall, temperature, which is strongly correlated with the variable of interest (crop yield for weather based agriculture index insurance).

The weather index insurance became a good replacement for the traditional crop insurance due to it's favorable features that cannot be seen in traditional insurance. However, the basis risk is the most unfavorable feature of index insurance. As defined in World-Bank (2011) it is the difference between the payout as measured by the index and the actual loss of the farmer. There is no loss assessment done and the payout is exclusively based on the corresponding index. As a result of that, the payout can be either higher or lower than the actual loss. More details about the advantages and disadvantages of an index insurance are given in Table 1.1.

Table 1.1: Advantages and Disadvantages of Index Insurance

Advantages	Disadvantages
Reduces risk of adverse selection	Basis risk
Reduces moral hazard	Data availability
Field loss assessment is eliminated	Integrity of weather station
Reduces information requirements	Need for farmers' capacity building
Facilitation of reinsurance	
Transparency	
Facilitating access to financial services	

Source: World-Bank (2011)

Weather index insurance is still not a well established insurance scheme among farmers, specially due to the basis risk and the poor knowledge about its performance. This may be one reason that researchers try to explore index insurance further to identify its performance in different contexts. With the same motive, in this work we investigate selected interesting aspects of the index insurance. We believe that deeper analysis of the index insurance is important to realize its appropriateness and effectiveness as a crop insurance. Since index insurance is easy to handle, cheaper and has some other good features, if one can ensure that the index insurance performs well as a crop insurance it become a good alternative for crop insurance for both insured and insurer. Our main aim in this thesis is to identify the circumstance and conditions under which the index insurance performs as an effective insurance for agriculture risk. In order to achieve our goal, we consider many settings which are already analyzed by the researchers such as interlinked loan and index insurance. But also, we focus on some areas, which are not deeply discussed in the literature such as market equilibrium of weather index insurance.

Different aspects of the agriculture index insurance are vastly studied in the literature. The work done by Clarke (2016) is one of the most cited papers about weather index insurance. He builds up a mathematical model to study the demand of weather index insurance. We consider the same model setup with slight changes in several sections of our work to discuss different facts about index insurance. One of the emerging topics in this area is how index insurance affects an agriculture loan. Due to the uncertainty of the agriculture, it is difficult for farmers to obtain credit, such as bank loans. Because of that, the farmers need a collateral to increase their credit worthiness. Since an index insurance leads to an increase of the wealth of the farmer, an index insurance for agriculture risk can act as a collateral. Miranda and Gonzalez-Vega

(2011), Farrin and Miranda (2015) and Carter et al. (2011) build up mathematical models to study the performance of interlinked loan and index insurance. Miranda and Gonzalez-Vega (2011) numerically show that an index insurance with subsidized premiums makes a positive impact on the borrowers by reducing the loan default probability. In this work, we directly analyze the combination of index insurance and agriculture loan. In addition to that, we discuss this combination in other contexts such as market equilibrium. Another interesting aspect of index insurance is addressed in Zhang et al. (2018). They develop a numerical procedure to derive the optimal payout function of an index insurance. In addition to these areas there are lots of other works which analyze and discuss different aspects of index insurance (Breustedt et al. (2008), Carriquiry and Osgood (2012), Collier (2020), Dercon et al. (2014), Elabed et al. (2013), Golden et al. (2007), Doherty and Richter (2002)).

This thesis is organized as follows: In Chapter 2, we explain the basic mathematical concepts we use throughout our work. Then in Chapter 3, we discuss the performance of interlinked credit and index insurance similar to Miranda and Gonzalez-Vega (2011). We develop three mathematical models to analyze the interlink between credit and index insurance under three scenarios. In Chapter 4, we design optimal index insurance by following the procedure in Zhang et al. (2018) with some modifications. As an extension we build up an optimal index insurance for a lender who issues agriculture loans. Chapter 5 is about the market equilibrium with index insurance. In order to build up the equilibrium model for weather index insurance we consider the idea in Shen and Odening (2013), in which area yield index insurance is considered. Also, we develop an equilibrium model for a market with farmers in two risk groups and a single insurer. We follow Sass and Seifried (2014) in order to build up and analyze the market equilibrium for two groups. In Chapter 6, we study an index insurance in an informal risk sharing environment. We basically consider three different informal risk sharing methods given in Boucher and Delpierre (2014), Dercon et al. (2014) and Santos et al. (2021). We discuss how index insurance together with informal risk sharing affects the farmers and the demand for index insurance in an informal risk sharing environment. Chapter 7 is about another two aspects of index insurance. Since Clarke (2016) suggests that gap insurance is a possible option to increase the demand of an index insurance, first we examine how a gap insurance influences the demand for index insurance. Then we discuss the performance of the index insurance as a reinsurance. We consider the index insurance as an alternative for proportional reinsurance and excess loss reinsurance. Our work is related to Zeng (2005), but in contrast to that we consider an expected utility framework instead of a mean-variance framework. In the Appendix we state the proofs of some theorems, propositions and lemmas. In addition we provide additional information and figures.

Chapter 2

Preliminaries

In this chapter we discuss about the mathematical concepts, theorems and definitions which are used to develop our work.

2.1 The Indemnity Payment of the Index Insurance

We can define the indemnity payment of the index insurance as a function of the underlying index. Let x be the realized value of the underlying index and k be the threshold level or the strike of the index. Some of the possible forms of the indemnity function are given in Martin et al. (2001). We describe them as follows.

- $$\text{Indemnity} = \begin{cases} 0 & \text{if } x > k, \\ \lambda(k - x) & \text{if } x \leq k, \end{cases}$$

  where λ is some predetermined value per unit of index. It is called the tick of the insurance contract. In Wollenweber et al. (2003) and Conradt et al. (2015) λ is normalized to unity.

- $$\text{Indemnity} = \begin{cases} 0 & \text{if } x > k, \\ \frac{(k-x)}{k} \times \text{liability} & \text{if } x \leq k, \end{cases}$$

  where liability is a choice variable that establishes the maximum possible indemnity. Here the insured receives a fraction of liability depending on the realized value of the index.

- $$\text{Indemnity} = \begin{cases} 0 & \text{if } x > k_u, \\ \lambda(k_u - x) & \text{if } k_l < x \leq k_u, \\ \text{liability} & \text{if } x \leq k_l, \end{cases}$$

  where k_u and k_l are the upper and lower triggers of the contract and λ is the tick. When we consider that liability=$\lambda(k_u - k_l)$, then it is the maximum possible payout.

2.2 Utility Theory

Utility theory is one of the most frequently used concepts throughout our work. Therefore, here we state some important details on utility theory. It is a frequently used assumption that individuals have preferences over monetary values, such as level of wealth. That preference under certain conditions can be represented as expected utility by a von-Neumann-Morgenstern utility function

$$u : \mathbb{R} \to \mathbb{R}.$$

As given in Pablo and Cosimo (2017) we assume the following properties,

- u is twice continuously differentiable,
- u is strictly increasing, i.e. $u' > 0$.

Some commonly used utility functions are given below.

- Exponential utility $u(x) = \frac{1-e^{-\gamma x}}{\gamma}$, $x \in \mathbb{R}$, $\gamma > 0$
- Power utility $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $x \in (0, \infty)$, $\gamma > 0$ and $\gamma \neq 1$
- Logarithmic utility $u(x) = \log(x)$, $x \in (0, \infty)$.

Based on the expected utility the attitude towards the risk of individuals on their wealth can be categorized in to three types (Autor (2016)). Those are given below.

- $\mathbb{E}[u(X)] \leq u(\mathbb{E}[X]) \Rightarrow$ Risk Averse,
- $\mathbb{E}[u(X)] \geq u(\mathbb{E}[X]) \Rightarrow$ Risk Seeking,
- $\mathbb{E}[u(X)] = u(\mathbb{E}[X]) \Rightarrow$ Risk Neutral,

where $\mathbb{E}[u(X)]$ is the expected utility on wealth and $u(\mathbb{E}[X])$ is the utility of expected wealth. Jensen's inequality gives a major result about expectations and concave functions (Pablo and Cosimo (2017)).

Proposition 2.1. *(Jensen's Inequality)*

- *Suppose* $u : \mathbb{R} \to \mathbb{R}$ *is concave. Then* $\mathbb{E}[u(X)] \leq u(\mathbb{E}[X])$ *for every random variable* X *with* $\mathbb{E}[|X|] < \infty$.
- *Suppose* $u : \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies* $\mathbb{E}[u(X)] \leq u(\mathbb{E}[X])$ *for every random variable* X *with* $\mathbb{E}[|X|] < \infty$. *Then* u *is concave.*

From the above proposition it is easy to see that an individual is risk averse if and only if u is concave. The three utility functions given above are concave. Therefore, those utility functions represent the utility of risk averse individuals. As given in Levin (2006) the different ways to measure risk aversion are defined below.

Definition 2.2. (Certainty Equivalent) Let u be a given utility function. The certainty equivalent CE of is defined by

$$u(CE) = \mathbb{E}[u(X)].$$

Definition 2.3. For any twice differentiable utility function u, the **Arrow-Pratt coefficient of absolute risk aversion** is

$$A(x) = -\frac{u''(x)}{u'(x)}.$$

Definition 2.4. For any twice differentiable utility function u, the **Arrow-Pratt coefficient of relative risk aversion** is

$$R(x) = -\frac{xu''(x)}{u'(x)}.$$

2.3 Fixed-Point and Function Iteration

In our work we use a function iteration method to solve a Bellman equation to find the fixed point. Burden and Faires (2011) define a fixed-point as follows.

Definition 2.5. (Fixed-Point) x^* is a fixed-point of a function g if $g(x^*) = x^*$.

Burden and Faires (2011) state a theorem about the existence and uniqueness of the fixed-point which we repeat in the following.

Theorem 2.6. *(Existence and Uniqueness)*

(i) If g is continuous on $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has a fixed-point in $[a, b]$.

(ii) If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \text{ for all } x \in (a, b),$$

then there is exactly one fixed-point in $[a, b]$.

Proof.

(i)
- If $g(a) = a$, or $g(b) = b$, then g has a fixed point at the endpoint.
- Otherwise, $g(a) > a$ and $g(b) < b$.
- Define a new function $h(x) = g(x) - x$. Then $h(a) = g(a) - a > 0$ and $h(b) = g(b) - b < 0$. Also h is continuous.
- By the intermediate value theorem, there exists $x^* \in (a, b)$ for which $h(x^*) = 0$. Thus $g(x^*) = x^*$.

(ii)
- $|g'(x)| \leq k < 1$. Suppose we have two fixed points x_1^* and x_2^*.
- By the mean value theorem (MVT), there is ξ between x_1^* and x_2^* with $g'(\xi) = \frac{g(x_1^*) - g(x_2^*)}{x_1^* - x_2^*}$.
- Thus $|x_1^* - x_2^*| = |g(x_1^*) - g(x_2^*)| = |g'(\xi)||x_1^* - x_2^*| \leq k|x_1^* - x_2^*| < |x_1^* - x_2^*|$, which is a contradiction.
- This shows the supposition is false. Hence, the fixed point is unique.

□

The function iteration is a simple technique which can be used to find the fixed point. As explained in Burden and Faires (2011) and Miranda and Fackler (2004), it starts with an initial guess x_0 and a sequence of values $\{x_n\}_{n=0}^{\infty}$ generated by iterations $x_n = g(x_{n-1})$. If the sequence converges to x^*, it is the fixed point of g, that is $g(x^*) = x^*$. Burden and Faires (2011) state the following theorem about the conditions which guarantee the convergence of function iteration,

Theorem 2.7. *(Fixed-Point Theorem) Let g be continuous on $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$. In addition, assume that $g'(x)$ on (a, b) and a positive constant $0 < k < 1$ exist with $|g'(x)| \leq k$ for all $x \in (a, b)$. Then, for any x_0 in $[a, b]$, the sequence defined by $x_n = g(x_{n-1})$ converges to the unique fixed point x^* in $[a, b]$.*

Proof.

$$\begin{aligned}|x_n - x^*| &= |g(x_{n-1}) - g(x^*)| \\ &= |g(\xi_n)||x_{n-1} - x^*| \quad \text{by MVT} \\ &\leq k|x_{n-1} - x^*|\end{aligned}$$

(since $k < 1$, the distance to fixed point is shrinking every iteration). Keep doing the above procedure:

$$|x_n - x^*| \leq k|x_{n-1} - x^*| \leq k^2|x_{n-2} - x^*| \leq \ldots \leq k^n|x_0 - x^*|,$$

and hence

$$\lim_{n\to+\infty} |x_n - x^*| \leq \lim_{n\to+\infty} k^n|x_0 - x^*| = 0.$$

□

Figure 2.1 graphically shows how to find the fixed point by function iteration.

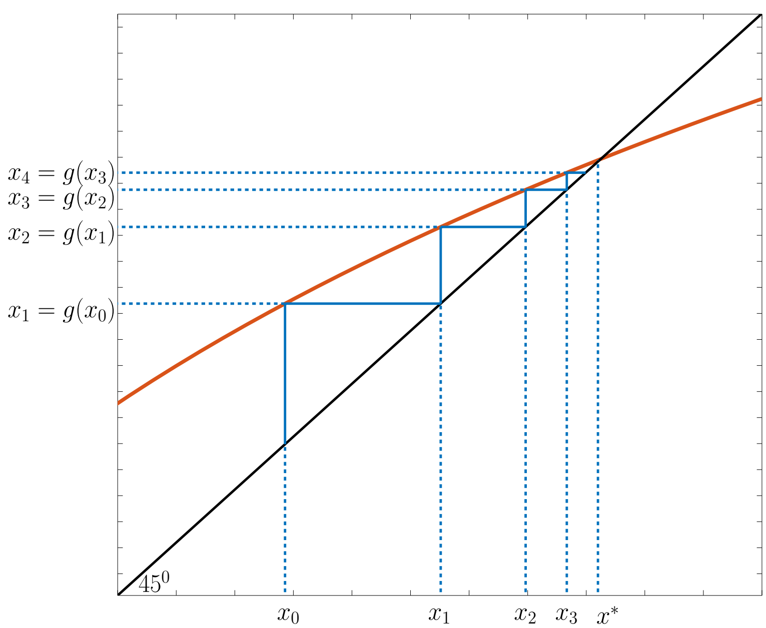

Figure 2.1: Function Iteration.

2.4 Optimality Conditions for Unconstrained Problems

In our work we solve several optimization problems to derive our results. Therefore, it is important to discuss optimization problems. Consider the following optimization problem.

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t} \quad & x \in X, \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ and X is an open set. f is differentiable on X if $f(x)$ is differentiable for all $x \in X$. The gradient vector is the vector of partial derivatives. It is given by

$$\nabla f(\bar{x}) = \left(\frac{\partial f(\bar{x})}{\partial x_1}, \frac{\partial f(\bar{x})}{\partial x_2}, \ldots, \frac{\partial f(\bar{x})}{\partial x_n} \right)$$

f is twice differentiable on X if $f(x)$ is twice differentiable for all $x \in X$. The Hessian is the matrix of second partial derivatives. The ij^{th} element of Hessian matrix H is given by

$$H(\bar{x})_{ij} = \frac{\partial^2 f(\bar{x})}{\partial x_i \partial x_j}.$$

A theorem which explains the sufficient condition for local optimality of the above optimization problem is given in Freund (2004). We repeat that theorem in the following.

Theorem 2.8. *Suppose that $f(x)$ is twice differentiable at $\bar{x}$. If $\Delta f(\bar{x}) = 0$ and $H(\bar{x})$ is positive definite, then $\bar{x}$ is a strict local minimum.*

Note: If $\Delta f(\bar{x}) = 0$ and $H(\bar{x})$ is negative definite, then $\bar{x}$ is a strict local maximum Freund (2004).

2.5 Correlation Coefficient

Let X and Y be two random variables with means μ_X and μ_Y, with standard deviations $\sigma_x > 0$ and $\sigma_y > 0$, respectively. The correlation coefficient of X and Y, denoted Corr(X, Y) or ρ_{XY} is defined as

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_x \sigma_y},$$

where $\mathrm{Cov}(X,Y)$ is the covariance of X and Y. It is defined as:

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= \mathbb{E}[XY] - \mu_X\mu_Y.\end{aligned}$$

The correlation coefficient of a sample of data is defined as follows

$$\rho_{XY} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2\sum_{i=1}^{n}(y_i-\bar{y})^2}},$$

where $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$, $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$ and n is the sample size.

Chapter 3

Interlinked Index Insurance and Multi Period Loan

Smallholder farmers always have financial issues for farming. As a solution to this matter farmers can take agriculture loans. But at the same time it creates some difficulties due to the loan repayment at the end of the period. Also, it may be difficult for farmers to take loan due to low creditworthiness. These difficulties can be managed successfully by purchasing an insurance, because then the insurance works as a collateral to the loan and it increases the creditworthiness of the farmers (Mahul and Stutley (2010), Tadesse et al. (2015), World-Bank (2011)). We know that the borrower of an agriculture loan or any other loan is obliged to repay the loan at the end of a given period. Generally, when the borrower is in a good wealth state at the end of the period he is willing to repay the loan, specially because of maintaining a good relationship with the lender for further loans. But sometimes the borrower may default on the loan due to lack of money (Miranda and Gonzalez-Vega (2011)). As explained in Miranda and Gonzalez-Vega (2011), when the borrower is unable to repay the loan at the end of the loan period, rather than going for costly legal actions against the borrower the bank can restructure the loan allowing to pay later when the farmer is in a good financial state. In this chapter we analyze this kind of multi period loan for farmers which is interlinked with weather index insurance. Here we basically consider three models. The first model or the model without savings is similar to the model in Miranda and Gonzalez-Vega (2011). The other two models, namely a model with savings and a model for the extreme weather events are two extensions of the first model. In addition to that we discuss the effect on the lender's wealth by issuing loans which are interlinked with index insurance.

3.1 The Model without Savings

As explained in Miranda and Gonzalez-Vega (2011) in this model the borrowers or the farmers face common systemic shocks and idiosyncratic shocks. A systemic shock has an effect on the amount of the yield by some weather event while an idiosyncratic shock is uncorrelated with the systemic shocks and uncorrelated across the borrowers. At the beginning of each period each borrower decides whether to repay the loan or to default on the loan. Due to the heterogeneity across the farmers at the end of each period some farmers repay the loan and some not.

Table 3.1: Description of the Notations

Notation	**Description**
l	Loan amount.
h	Cost of high technology. Where $h \geq l$ (Farrin and Miranda (2015)).
g_h, g_l	g_h for high technology and g_l for low technology. Typically $g_l = 1$ and $g_h > 1$ (Miranda and Gonzalez-Vega (2011)).
X	Systemic shock or weather index realization. Where $x_{min} \leq X \leq x_{max}$ with pdf $f(x)$.
Y	Idiosyncratic shock with pdf $g(y)$.
μ	Unit price of the harvest.
δ	Periodic interest rate.
ρ	Periodic interest rate on the loan.
$I(X)$	Indemnity payment of the index insurance.
α	Insurance cover.
m	Premium load/ Insurance pricing multiple.
$u(.)$	Utility function of the farmer. It is twice differentiable with $u' > 0$ and $u'' < 0$.

Now we consider an infinitely lived individual who borrows a fixed amount l at any period. Here we assume that the main purpose of borrowing is investing high technology for farming. This loan is interlinked with a mandatory index insurance. Therefore, whenever the farmer takes a loan he should purchase the index insurance which is bundled with the loan. The main notations we use in this chapter are given in Table 3.1.

Let $X \in [x_{min}, x_{max}]$ be the systemic shock with probability density function $f(x)$. Then the premium of the index insurance with the indemnity payment $I(X)$, insurance cover α and premium load m is given by

$$\pi = \frac{\alpha m}{1+\delta} \int_{x_{min}}^{x_{max}} I(x) f(x) dx, \tag{3.1}$$

where δ is the periodic interest rate. The individual can repay the loan and the interest as a single payment within $N \in \mathbb{Z}^+$ periods. He cannot take a new loan until he repays the outstanding loan. If the individual does not pay back the loan within N periods he will be blacklisted and he cannot take any loan in the future. In this model we assume that the individual has no access to savings facilities, therefore he consumes all of his disposable wealth. Also we assume that the farmer does not purchase any insurance other than the index insurance which is interlinked with the loan and that insurance is only valid for the first period of the loan. Figure 3.1 briefly explains the model.

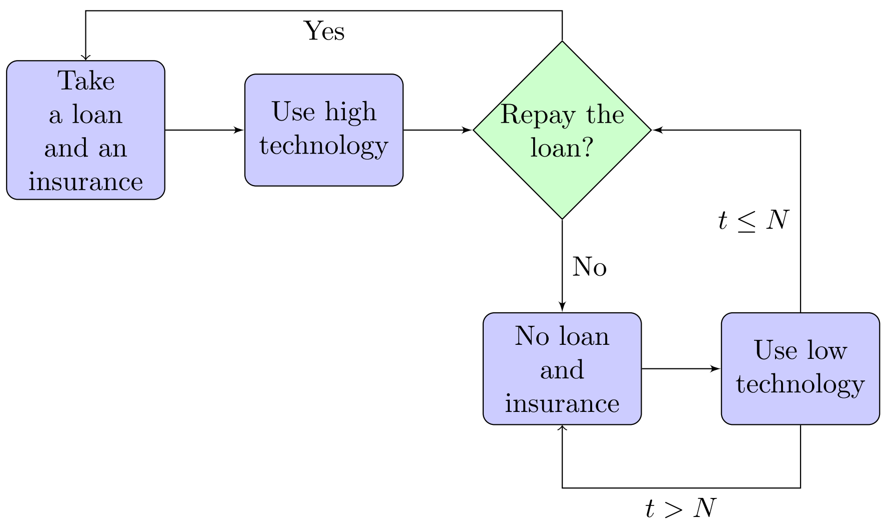

Figure 3.1: The Model without Savings.

At the end of the first period the borrower decides whether he will repay the outstanding loan or not. The borrower compares the expected present value of his current and future consumption in two cases. In the first case he repays the outstanding loan at the end of the period and in the second case he does not repay the outstanding loan at the end the period. If he does not repay the loan at the end of the first period he can again make a decision about the loan repayment at the end of the second period and he can continue like this up to N periods. We assume that whenever the farmer repays the loan within N periods he takes a new loan and an index insurance. By assuming a risk averse insured with utility function u, we state the insured's optimization problem by Bellman equations. For $t = 1, 2, \ldots, N$ and wealth $w \geq 0$, let

$z_t(w) = w + l - h - l(1+\rho)^t - \pi.$

$$v_t(w) = \begin{cases} \max\left\{u(z_t(w)) + \frac{1}{(1+\delta)}\mathbb{E}[v_1(W_1)], u(w) + \frac{1}{(1+\delta)}\mathbb{E}[v_{t+1}(W_{t+1})]\right\} & \text{if } z_t(w) \geq 0 \\ \\ u(w) + \frac{1}{(1+\delta)}\mathbb{E}[v_{t+1}(W_{t+1})] & \text{if } z_t(w) < 0, \end{cases} \tag{3.2}$$

subject to the boundary condition

$$v_{N+1}(w) = u(w) + \frac{1}{\delta}\mathbb{E}[u(W_{N+1})]. \tag{3.3}$$

Here

$$W_t = \begin{cases} \mu g_h XY + \alpha I(X) & t = 1 \\ \mu g_l XY & t = 2, 3, \ldots, N+1, \end{cases}$$

and ρ is the periodic interest rate on the loan, Y is the idiosyncratic shock of the farmer, μ is the unit price of the harvest and g_h and g_l represent high and low technology respectively. For $t = 2, 3, \ldots N$, v_t is the value function if the individual has a t period old outstanding loan. v_{N+1} is the value function if the individual has no loan and no insurance.

Remark 3.1. Note that we have stated directly the Bellman equation in (3.2) without specifying the underlying processes depending on time before. The index "t" rather denotes the time since taking the last loan. However, that the Bellman equation describes the optimal strategy (choosing whether to pay back the loan or not) requires that we have an underlying Markovian structure. This is true when assuming that the shocks and the yields are serially independent as we did. Then indeed $v_1(w)$ provides the optimal discounted expected utility of consumption when having capital w and taking a loan.

3.1.1 Performance of the loan

It is important to evaluate the performance of the loan which is interlinked with an index insurance. Miranda and Gonzalez-Vega (2011) use the loan default probability and the expected internal rate of return (IRR) of the bank to measure the performance of the interlinkage between the loan and the index insurance. We define the loan default probability of this model as the probability that the individual does not repay the loan within N periods. Let p_t be the probability that the individual repays the loan at the end of period t when he did not repay the loan previously. Let $Z_t = W_t + l - h - l(1+\rho)^t - \pi$. Then

$$p_t = P\Big(u(Z_t) + \frac{1}{(1+\delta)}\mathbb{E}[v_1(W_1)] > u(W_t) + \frac{1}{(1+\delta)}\mathbb{E}[v_{t+1}(W_{t+1})], Z_t \geq 0\Big). \tag{3.4}$$

Now the probability of loan default is given by

$$(1-p_1)(1-p_2)\dots(1-p_N). \tag{3.5}$$

Let i be the expected internal rate of return of the bank. We can derive it by setting the expected net present value (NPV) to zero, i.e. by stating

$$\mathbb{E}[NPV] = -l + \frac{l(1+\rho)}{(1+i)}p_1 + \frac{l(1+\rho)^2}{(1+i)^2}(1-p_1)p_2 + \dots + \frac{l(1+\rho)^N}{(1+i)^N}(1-p_1)\dots p_N = 0. \tag{3.6}$$

3.1.2 Solving the Optimization Problem

We need the values of $\mathbb{E}[v_t(W_t)]$ for $t = 1, 2, 3, \dots N+1$ in order to calculate the loan default probability and the expected IRR. Let $\mathbb{E}V_t = \mathbb{E}[v_t(W_t)]$. Then by Equation (3.2) we can compute $\mathbb{E}V_t$ as

$$\begin{aligned}\mathbb{E}V_t = \mathbb{E}\Big[&\max\Big\{u(Z_t) + \frac{1}{(1+\delta)}\mathbb{E}V_1, u(W_t) + \frac{1}{(1+\delta)}\mathbb{E}V_{t+1}\Big\}\mathbb{1}_{(Z_t \geq 0)} \\ &+ \Big(u(W_t) + \frac{1}{(1+\delta)}\mathbb{E}V_{t+1}\Big)\mathbb{1}_{(Z_t<0)}\Big],\end{aligned} \tag{3.7}$$

for $t = 1, 2, 3, \dots, N$. We can state the boundary value $\mathbb{E}V_{N+1}$ by taking the expectation of Equation (3.3) and it is given by

$$\mathbb{E}V_{N+1} = \frac{(1+\delta)}{\delta}\mathbb{E}[u(W_{N+1})]. \tag{3.8}$$

According to the form of $\mathbb{E}V_{N+1}$ it is clear that we can compute it without any knowledge about rest of $\mathbb{E}V_t$'s. Then starting from $t = N$, each $\mathbb{E}V_t$ can be written in terms of $\mathbb{E}V_1$. When $t = 1$ we can solve the problem by computing the fixed-point of, $\mathbb{E}V_1 = \mathcal{G}(\mathbb{E}V_1)$ of a function $\mathcal{G}$ from $\mathbb{R}$ to $\mathbb{R}$, since $\mathcal{G}$ is differentiable with respect to $\mathbb{E}V_1$ and $\mathcal{G}' = \frac{1}{(1+\delta)^n}$, for some $n \geq 1$, $\mathcal{G}' < 1$. As a result of that, by Theorem 2.7 there exists a unique $\mathbb{E}V_1$ which satisfies Equation (3.7). We can use the function iteration method to find the fixed-point, see Miranda and Fackler (2004). But in this model the function $\mathcal{G}$ is not a simple function of $\mathbb{E}V_1$. Therefore some further explanation about the model is required to solve the problem.

For $w \geq 0$, $z_t(w) > 0$, and $t = 1, 2, 3, \ldots, N$, let

$$\mathcal{H}_t(w) = u(z_t(w)) + \frac{1}{(1+\delta)}\mathbb{E}V_1 - u(w) - \frac{1}{(1+\delta)}\mathbb{E}V_{t+1}. \tag{3.9}$$

At period t the individual repays the loan if $\mathcal{H}_t(w) > 0$ and defaults if $\mathcal{H}_t(w) < 0$. Due to the properties of u, $\mathcal{H}_t(w)$ is continuous and strictly increasing for $z_t > 0$ (that is $w > (1+\rho)^t l + \pi + h - l$). Thus, it is possible to find unique w_t^* such that when $w > w_t^*$ the individual repays the loan. It can be obtained as follows. $\mathcal{H}_t(w)$ is always negative in some cases (for an example when $\mathbb{E}V_{t+1} > \mathbb{E}V_1$), then w_t^* is equal to the maximum possible wealth level. Otherwise there exist a unique root $w_t^* > (1+\rho)^t l + \pi + h - l$. At wealth level w_t^* the individual is indifferent between repaying and defaulting the loan at the end of period t.

As given in the model for any given systemic shock $X = x$ and idiosyncratic shock $Y = y$ the level of wealth at time $t = 2, 3, \ldots, N$ is given by $w_t = \mu g_l xy$. Then for $t = 2, 3, \ldots, N$, at any given level $Y = y$, w_t is increasing on x. Therefore at any given level of $Y = y$ there exist a unique $x_{t,y}^*$, such that $w_t^* = s g_l x_{t,y}^* y$ and the farmer repays the loan at the end of period t if $x > x_{t,y}^*$.

When $t = 1$, deep understanding about w_1 as a function of x is required to identify the range of X in which the farmer repays the loan. Since most commonly used indemnity functions of index insurance are linear monotonically decreasing functions of x, now we assume that $I(x)$ is linearly monotonically decreasing in x. Also the insurer makes a payment only at small levels of X.

By considering an indemnity function which works as above, we can find the range of X in which the farmer repays the loan as follows. We can divide Y into two subsets Y_1 and Y_2 ($Y_1 \cup Y_2 = Y$) such that when $y \in Y_1$, w_1 first decreases and then increases as X increases and when $y \in Y_2$, w_1 increases as X increases. (More details about Y_1 and Y_2 are given in Appendix B). Then we can show that

- Also at a given $y \in Y_1$ there exist unique $x_y^{**}, x_y^* \in X$ and the farmer repays the loan at the end of the first period if $x < x_y^{**}$ or $x > x_y^*$.
- At given $y \in Y_2$ there exist unique $x_{1,y}^* \in X$ and the farmer repays the loan at the end of the first period if $x > x_{1,y}^*$.

More details about finding $x_{1,y}^*$, x_y^{**} and x_y^* are given in Appendix B. By considering $x_{t,y}^*$, x_y^* and x_y^{**}, now we state $\mathbb{E}V_t$ as follows.

$$\mathbb{E}V_1 = \mathop{\mathbb{E}}_{y \in Y_1}\Big[\mathop{\mathbb{E}}_{x}[u(\mu g_h XY + \alpha I(X) + l - h - (1+\rho)^t l - \pi) + \frac{1}{(1+\delta)}\mathbb{E}V_1]\mathbb{1}_{(X<x_Y^{**}, X>x_Y^*)}$$

$$
\begin{aligned}
&+ \mathop{\mathbb{E}}_{x}\Big[u(\mu g_h XY + \alpha I(X)) + \frac{1}{(1+\delta)}\mathbb{E}V_2\Big]\mathbb{1}_{(x_Y^{**} \le X \le x_Y^{*})}\Big] \\
&+ \mathop{\mathbb{E}}_{y \in Y_2}\Big[\mathop{\mathbb{E}}_{x}\Big[u(\mu g_h XY + \alpha I(X) + l - h - (1+\rho)^t l - \pi) + \frac{1}{(1+\delta)}\mathbb{E}V_1\Big]\mathbb{1}_{(X > x_{1,Y}^{*})} \\
&+ \mathop{\mathbb{E}}_{x}\Big[u(\mu g_h XY + \alpha I(X)) + \frac{1}{(1+\delta)}\mathbb{E}V_2\Big]\mathbb{1}_{(X \le x_{1,Y}^{*})}\Big]. \qquad (3.10)
\end{aligned}
$$

For $t = 2, 3, \ldots, N$

$$
\begin{aligned}
\mathbb{E}V_t = &\mathop{\mathbb{E}}_{y}\Big[\mathop{\mathbb{E}}_{x}\Big[u(\mu g_l XY + l - h - (1+\rho)^t l - \pi) + \frac{1}{(1+\delta)}\mathbb{E}V_1\Big]\mathbb{1}_{(X > x_{t,Y}^{*})} \\
&+ \mathop{\mathbb{E}}_{x}\Big[u(\mu g_l XY) + \frac{1}{(1+\delta)}\mathbb{E}V_{t+1}\Big]\mathbb{1}_{(X \le x_{t,Y}^{*})}\Big]. \qquad (3.11)
\end{aligned}
$$

Then we can state the probability of loan repayment at each period, which is given in Equation (3.4) as

$$
p_1 = \mathop{\mathbb{E}}_{y}[1 + F(x_Y^{**}) - F(x_Y^{*})]\mathbb{1}_{Y_1}(y) + \mathop{\mathbb{E}}_{y}[1 - F(x_{1,Y}^{*})]\mathbb{1}_{Y_2}(y), \qquad (3.12)
$$

and for $t = 2, 3, \ldots, N$

$$
p_t = \mathop{\mathbb{E}}_{y}[1 - F(x_{t,Y}^{*})]. \qquad (3.13)
$$

We build up the following numerical scheme to compute $\mathbb{E}V_1$, the loan default probability and the expected IRR based on the function iteration method and the facts which are discussed above.

Step 1: Choose a suitable indemnity function $I(X)$ and then calculate the premium $\pi = \frac{\alpha m}{1+\delta}\mathbb{E}[I(X)]$.

Step 2: Calculate $\mathbb{E}V_{N+1} = \mathbb{E}[u(W_{N+1})] + \frac{1}{\delta}\mathbb{E}[u(W_{N+1})]$.

Step 3: Choose an initial guess for $\mathbb{E}V_1$. It is denoted by $\mathbb{E}V_1^0$.

Step 4: Find $x_{N,y}^{*}$. Then use $\mathbb{E}V_{N+1}$ and $\mathbb{E}V_1^0$ to compute $\mathbb{E}V_N$ by Equation (3.11).
Find $x_{N-1,y}^{*}$. Then use $\mathbb{E}V_N$ and $\mathbb{E}V_1^0$ to compute $\mathbb{E}V_{N-1}$ by Equation (3.11).
⋮
Find $x_{1,y}^{*}$, x_y^{*} and x_y^{**}. Then use $\mathbb{E}V_2$ and $\mathbb{E}V_1^0$ to compute $\mathbb{E}V_1$ by Equation (3.10).

Step 5: Verify whether $| \mathbb{E}V_1 - \mathbb{E}V_1^0] | < \epsilon$, for a given tolerance ϵ.
If yes, go to step 6.
If no, set $\mathbb{E}V_1^0 = \mathbb{E}V_1$ and go back to step 4.

Step 6: For $t = 1, 2, \ldots, N$ calculate p_t by Equations (3.13) and (3.12).

Step 7: Calculate the loan default probability by Equation (3.5) and the expected IRR by Equation (3.6).

3.1.3 Numerical Results of the Model without Savings

We consider a rainfall based index insurance and assume that the rainfall index is gamma distributed, see Sharma and Singh (2010). Let $I(x) = \max\{\lambda(k - x), 0\}$, where λ is the tick value and k is the strike value of the index insurance. Similar to Conradt et al. (2015) we assume that the insurer considers that there is a loss when the income from yield is below 80% of the expected income from yield. Therefore let $k = 0.8 \cdot \mathbb{E}[X]$. Also λ should be chosen such that it is equal to the increment in the income from yield per one unit increase in the rainfall index. Since Y is a random variable we consider $\lambda = \mu g_h \mathbb{E}[Y]$. For simplicity of the model we assume that Y is a discrete variable with only two values, $y_h \geq 1$ and $0 < y_l < 1$. Under this setting when $Y = y_l$ the actual loss is greater than the indemnity payment and when $Y = y_h$ the actual loss is less than the indemnity payment. This leads to basis risk, but it is a common feature of index insurance. Where basis risk is one of the major disadvantages of the index insurance and it occurs when the insurance payout is different to the actual loss.

We consider that the farmer's utility is given by the power utility function, $u(w) = \frac{w^{(1-\gamma)}}{1-\gamma}$, where $w \geq 0$, $\gamma > 0$ and $\gamma \neq 1$. Let $N = 5, l = 25, h = 25$ and $k = 30$. Also we consider index insurances with three different premium loads. Those three cases are subsidized insurance ($m < 1$), actuarial fair insurance ($m = 1$) and unsubsidized insurance ($m > 1$). In addition to that we consider the case where the loan does not interlinked with an index insurance. We choose some fixed amount α for all the cases by assuming that there is only one index insurance contract available in the market. The loan default probability and the expected IRR are given in Figure 3.2 against the interest rate on the loan.

As shown in Figure 3.2 it is clear that the subsidized and the actuarial fair insurances always give lower loan default probabilities than the no insurance case. But unsubsidized insurance gives lower default probability than the no insurance case only for low interest rates. When the interest rate is below 15%, in the subsidized and actuarial fair cases the bank can reach quite low loan default probabilities. For considering the IRR it is clear that subsidized and actuarial fair cases always have high IRR compared to the no insurance case. But unsubsidized insurance gives higher IRR than no insurance case only for low interest rates. It is clear that high premiums and high interest rates lead to high loan default probabilities and low expected IRR. Even though the insurance helps to cover some losses, at high interest rates when the premium is high the possibility of repaying the loan is lower than in the no insurance case. This may be the reason for the above mentioned situation. However, around 15% interest rate the

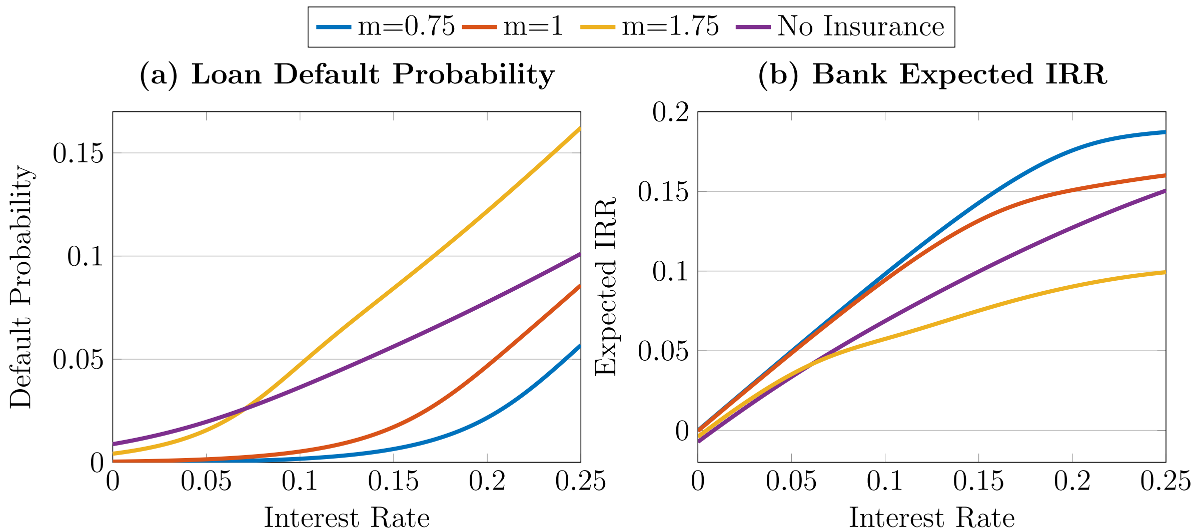

Figure 3.2: Loan Default probability and Expected IRR – No Savings.

lender can obtain significantly low loan default probability and considerably high IRR under the subsidized and actuarial fair insurances.

Also according to Figure 3.2 it is clear that at a given interest rate the loan default probability and expected IRR change with the premium load. Therefore it is important for the insurer to choose a proper level of premium load which leads to low level of default probability and high expected IRR. In order to get an idea about this, the loan default probability and the expected IRR for different premium loads are given in Figure 3.3. The loan default probability and the expected IRR of the no insurance case are included in the figure for comparison. We use 15% interest rate on the loan for the numerical computations. According to Figure 3.3 it is clear

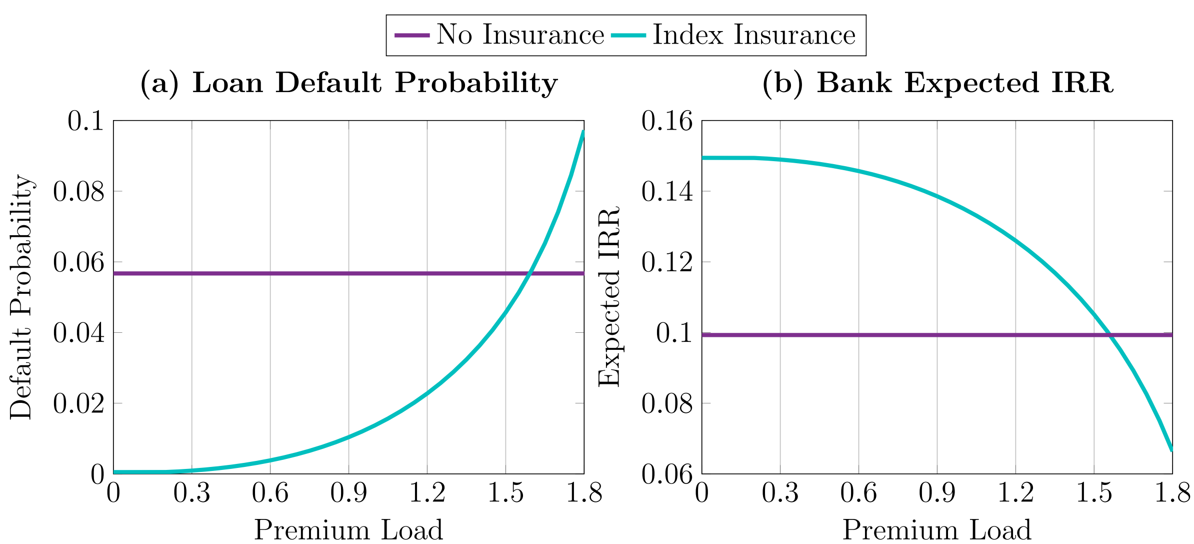

Figure 3.3: Loan Default probability and Expected IRR vs Premium Load.

that the loan default probability increases and expected IRR decreases as the premium load increases. Let us denote the premium load at the intersection point of the default probability (or expected IRR) with and without insurance as m^*. m^* is in between 1.5 and 1.6. If the insurer imposes a premium load which is above m^*, then the loan default probability is higher than in the no insurance case and the IRR is lower than in the no insurance case. Therefore that kind of index insurance is not a good choice to be better off by interlinkage. According to this numerical example we suggest that the lender should choose a premium load which is below m^* in order to get some benefit from the interlinkage of the loan with index insurance.

3.2 The Model with Savings

Now we consider that the individual has access to the savings facilities and he saves some amount of money for the future. Then at the end of any given period in addition to the income from farming the farmer may have some more money from the savings that he has done at the beginning of the period. At the end of each period after the farmer makes the decision about the loan repayment he makes a decision about savings, that is the amount of savings. We assume that the individual saves some fraction $0 < \theta < 1$ of his remaining wealth.

We state the insured's optimization problem by Bellman equations. For $t = 1, 2, \ldots, N$ and wealth $w \geq 0$, let $z_t(w) = w + l - h - (1+\rho)^t l - \pi$ as before and

$$v_t^s(w) = \begin{cases} \max\{u(z_t(w) - s_t) + \frac{1}{(1+\delta)}\mathbb{E}[v_1^s(W_1 + s_t(1+\delta))], \\ u(w - \bar{s}_t) + \frac{1}{(1+\delta)}\mathbb{E}[v_{t+1}^s(W_{t+1} + \bar{s}_t(1+\delta))]\} & \text{if } z_t(w) \geq 0 \\ \\ u(w - \bar{s}_t) + \frac{1}{(1+\delta)}\mathbb{E}[v_t^s(W_{t+1} + \bar{s}_t(1+\delta))] & \text{if } z_t(w) < 0, \end{cases} \tag{3.14}$$

subject to the boundary condition

$$v_{N+1}^s(w) = u(w - \bar{s}_{N+1}) + \frac{1}{(1+\delta)}\mathbb{E}[v_{N+1}^s(W_{N+1} + \bar{s}_{N+1}(1+\delta))], \tag{3.15}$$

where

$$s_t = \theta z_t(w),$$

$$\bar{s}_t = \theta w$$

and as before

$$W_t = \begin{cases} \mu g_h XY + \alpha I(X) & t = 1 \\ \mu g_l XY & t = 2, 3, \ldots, N+1. \end{cases}$$

In this model for $t = 2, 3, \ldots, N$, v_t is the value function if the individual has a t period outstanding loan and can do savings. v^s_{N+1} is the value function if the individual has no loan, no insurance but can do savings.

As we discussed above the model without savings can be solved by function iteration method, because we can write $\mathbb{E}[v_1(W_1)]$ as a fixed point of a function. In this model the value function does not only depend on the time but also on the amount of savings. The amount of savings changes from period to period. Therefore we cannot express $\mathbb{E}[v^s_1(W_1 + s_1(1+\delta))]$ as a fixed point of a function. Due to this complicated nature of this model, we cannot apply function iteration method. Therefore, we consider a simplified version of the model with savings. In this simplified model a fixed level of savings (s_0) is considered. At the end of the period if the remaining wealth after savings is at least equal to the minimum level of consumption c, then the individual continues with savings and otherwise continues with no savings.

We state the optimization problem of the insured's with savings by Bellman equation. For $t = 1, 2, \ldots, N$ and wealth $w \geq 0$, let $z_t(w) = w + l - h - (1+\rho)^t l - \pi$.

$$v^{s_0}_t(w) = \begin{cases} \max\{u(z_t(w) - s_t) + \frac{1}{(1+\delta)}(\mathbb{E}[v^{s_0}_1(W_1 + s_0(1+\delta))]\mathbb{1}_{\{s_t = s_0\}} + \mathbb{E}[v^{s_0}_1(W_1)]\mathbb{1}_{\{s_t=0\}}), \\ u(w - \bar{s}_t) + \frac{1}{(1+\delta)}(\mathbb{E}[v^{s_0}_{t+1}(W_{t+1} + s_0(1+\delta))]\mathbb{1}_{\{\bar{s}_t = s_0\}} + \mathbb{E}[v^{s_0}_{t+1}(W_{t+1})]\mathbb{1}_{\{\bar{s}_t=0\}})\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } z_t(w) \geq 0 \\ u(w - \bar{s}_t) + \frac{1}{(1+\delta)}(\mathbb{E}[v^{s_0}_{t+1}(W_{t+1} + s_0(1+\delta))]\mathbb{1}_{\{\bar{s}_t = s_0\}} + \mathbb{E}[v^{s_0}_{t+1}(W_{t+1})]\mathbb{1}_{\{\bar{s}_t=0\}}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } z_t(w) < 0, \end{cases} \tag{3.16}$$

subject to the boundary condition

$$\begin{aligned} v^{s_0}_{N+1}(w) =& u(w - \bar{s}_{N+1}) + \frac{1}{(1+\delta)}(\mathbb{E}[v^{s_0}_{N+1}(W_{N+1} + s_0(1+\delta))]\mathbb{1}_{\{\bar{s}_{N+1} = s_0\}} \\ &+ \mathbb{E}[v^{s_0}_{N+1}(W_{N+1})]\mathbb{1}_{\{\bar{s}_{N+1}=0\}}). \end{aligned} \tag{3.17}$$

Where

$$s_t = \begin{cases} s_0 & z_t(w) \geq s_0 + c \\ 0 & z_t(w) < s_0 + c, \end{cases}$$

$$\bar{s}_t = \begin{cases} s_0 & w \geq s_0 + c \\ 0 & w < s_0 + c, \end{cases}$$

and as before

$$W_t = \begin{cases} \mu g_h XY + \alpha I(X) & t = 1 \\ \mu g_l XY & t = 2, 3, \ldots, N+1. \end{cases}$$

We can compute $\mathbb{E}[v_{N+1}^{s_0}(W_{N+1} + s_0(1+\delta))]$ and $\mathbb{E}[v_{N+1}^{s_0}(W_{N+1})]$ by direct calculations. At a given level of $\mathbb{E}[v_1^{s_0}(W_1)]$ we can compute $\mathbb{E}[v_1^{s_0}(W_1 + s_0(1+\delta))]$ by the fixed-point iteration method. Similarly at a given level of $\mathbb{E}[v_1^{s_0}(W_1 + s_0(1+\delta))]$, $\mathbb{E}[v_1^{s_0}(W_1)]$ can be computed by the fixed-point iteration method. Then by considering an initial guess for $\mathbb{E}[v_1^{s_0}(W_1)]$ and $\mathbb{E}[v_1^{s_0}(W_1 + s_0(1+\delta))]$, the problem can be solved iteratively. But at a given value of y finding the range of X in which the farmer repays the loan is more complicated than in the no saving model. But it is still possible by considering similar arguments as in the no savings model.

3.2.1 Numerical Results of the Model with Savings

We compute the loan default probability and expected IRR of the simplified model with savings and the obtained results are given in Figure 3.4. In the figure the dotted curves give the loan default probability and expected IRR of no savings model.

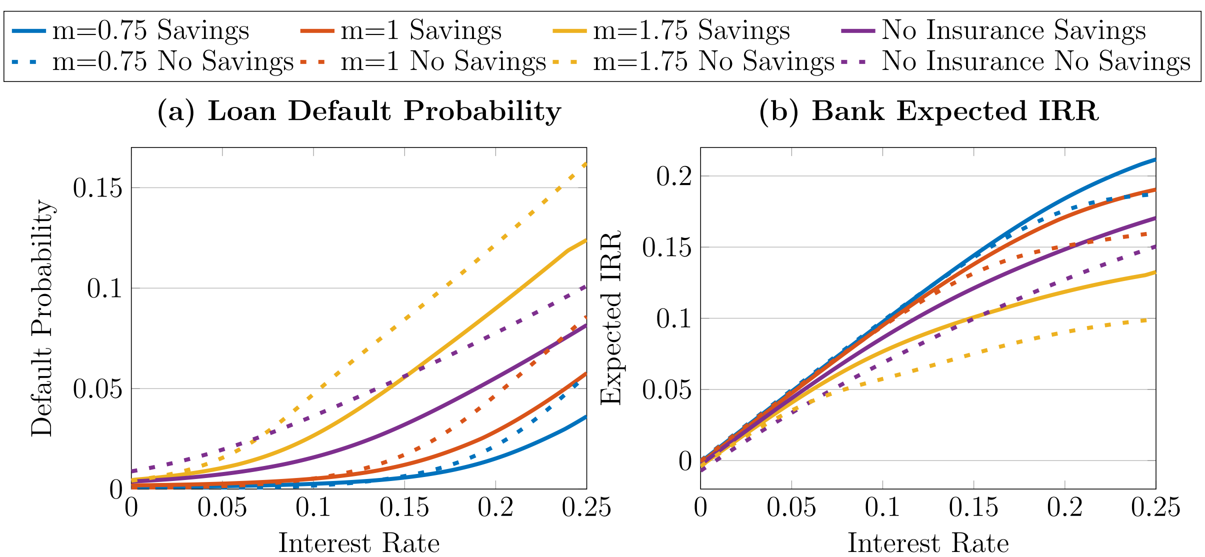

Figure 3.4: Loan Default Probability and Expected IRR – with Savings.

According to Figure 3.4 it is clear that the loan default probability of the model with savings always below the corresponding loan default probability of the no savings model. Similarly the expected IRR of the model with savings is always above the corresponding IRR of the no savings model. This numerical example shows that savings of the farmer makes a good impact on the lender in terms of the loan default probability and the expected IRR. We can suggest that one of the possible reason behind this effect is the increment of the farmer's wealth due to the savings in bad periods and thus it may help to repay the loan.

3.3 The Effect of Extreme Weather Events

The above two models are applicable for the crops in which the harvest increases with the corresponding weather index value. Also those models are compatible for the regions with no or very low probability of high values of the corresponding weather event. Then there is no considerable effect to the harvest from the excessive size of the weather events. But there are some crops which need only moderate level of some weather events and when there is an excessive amount of those weather events the crop will be destroyed and it will lead to low yield. For example, excessive rainfall leads to low maize yield in the United States, see Li et al. (2019).

In this section we build up a model for a crop which is destroyed due to very low levels and also to extreme high levels of some weather event. Here we assume that the yield of the crop first increases as the value of the underlying weather index increases up to a certain level. Then the yield keeps unchanged until the value of the weather index increases up to another level. Finally the yield decreases as the value of the weather index increases up to it's maximum level. As a result of that the income of the farmer first increases then keeps unchanged and then decreases as the value of the weather index increases. Because of this the farmer will get less income not only for low values of the weather index but also extreme high values of the weather index. Then the insurance company makes a payment in both cases where the level of the weather index is extremely low and extremely high.

We modify the above model with no savings to fit it to the situation discussed above. Let the yield increase up to the level of weather index x_1 and then keep unchanged until x_2. Also let x_{min} and x_{max} be the minimum value and the maximum value of the weather index respectively. For the simplicity of the model we assume that the yield at the minimum and maximum levels of the index are the same. The farmers receive a payment from the insurer when the value of the weather index is below k_1 ($x_{min} < k_1 < x_1$) and above k_2 ($x_2 < k_2 < x_{max}$). The yield at k_1 and k_2 should be equal. The behavior of the yield at a specific level of Y and with respect to the value of the weather index is given in Figure 3.5.

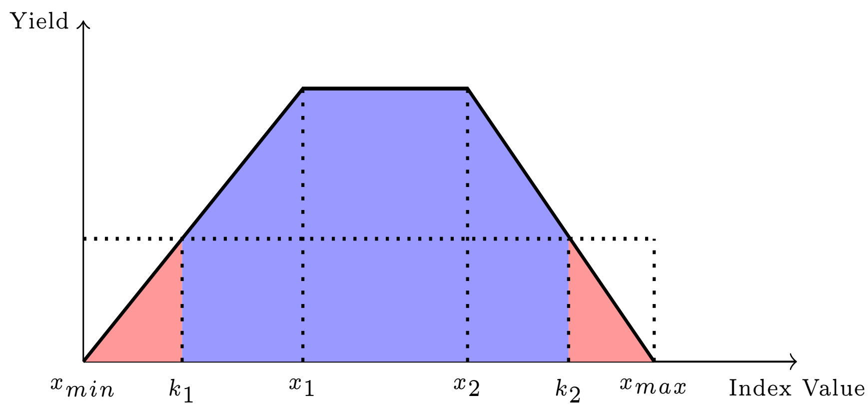

Figure 3.5: Yield vs Weather Index Value.

We can formulate the model similar to the model without savings by Equations (3.2) and (3.3). We restructure the form of W_t and $I(X)$ as follows.

$$W_t = \begin{cases} \mu g_h \tilde{X} Y + \alpha I(X) & t = 1 \\ \mu g_l \tilde{X} Y & t = 2, 3, \ldots, N+1 \end{cases}$$

where $\tilde{X}$ is a function of X given by

$$\tilde{X} = \begin{cases} X & X \leq x_1 \\ x_1 & x_1 < X \leq x_2 \\ aX + b & X > x_2 \end{cases}$$

for $a < 0$ and $b > 0$, where

$$a = \frac{x_1 - x_{min}}{x_2 - x_{max}}$$

and

$$b = \left(\frac{x_1 - x_{min}}{x_{max} - x_2}\right) x_{max} + x_{min}.$$

Further,

$$I(X) = \begin{cases} \max\{\lambda_1(k_1 - X), 0\} & X \leq x_1 \\ \max\{\lambda_2(X - k_2), 0\} & X > x_2 \end{cases},$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$. With similar argument as in Section 3.1.2 for $t = 2, 3, \ldots, N$ for any given $Y = y$ there exist unique $x^*_{t,y} \leq x_1$ such that the farmer repays the loan at the end of period t if $x > x^*_{t,y}$. Then we can find $x^{**}_{t,y} \geq x_2$ such that wealth at $x^{**}_{t,y}$ is equal to the wealth at $x^*_{t,y}$. At a given $Y = y$ if $x^*_{t,y} < x_1$ (then $x^{**}_{t,y} > x_2$) then the farmer repays the loan at the end of period t if $x^*_{t,y} < x < x^{**}_{t,y}$. If $x^*_{t,y} = x_1$ (that is $x^{**}_{t,y} = x_2$) then the farmer defaults on the loan at time t for any given level of X. When $t = 1$ by similar arguments as in Section 3.1.2 and by the arguments above we can find the intervals for X in which the farmer repays the loan. Based on these facts we compute the loan default probability and the expected IRR. The numerical results are given in Figure 3.6.

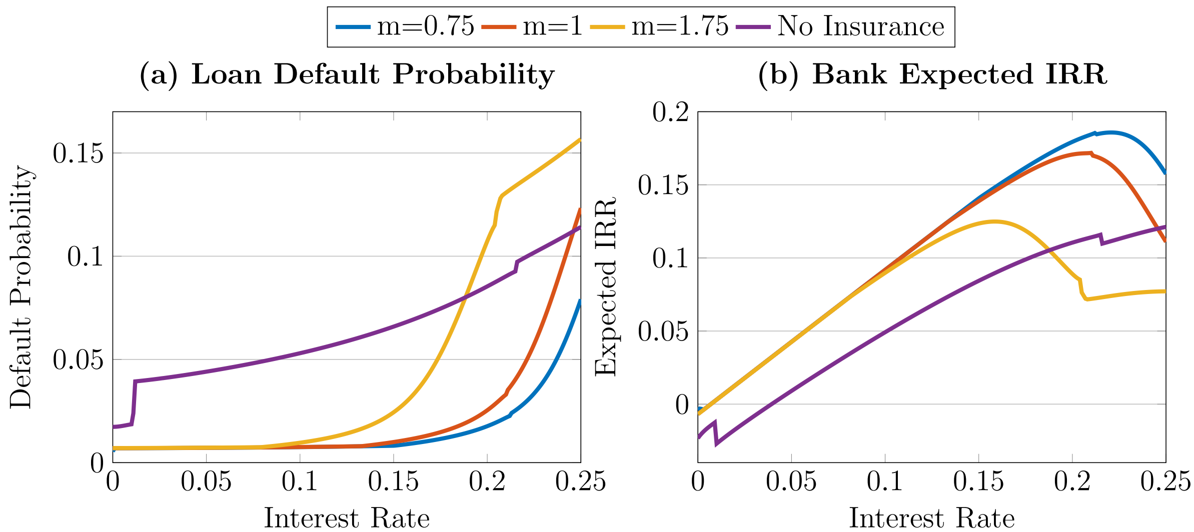

Figure 3.6: Loan Default probability and Expected IRR – Extreme Weather Events.

In Figure 3.6 (a) and (b) there are two jumps in the curves of no insurance case and the shape of the curves for unsubsidized insurance suddenly changes at 20.8% interest rate. We explain the reason behind these jumps and shape changes in Appendix B. As shown in Figure 3.6 similar to the other models the loan default probability increases as interest rate increases and also as the pricing multiple increases. The loan default probability of subsidized and actuarial fair cases are below 0.01 when the interest rate is below 15%. For the unsubsidized case, the loan default probability is below 0.01 when the interest rate is below 10%. In contrast to the previous models the loan default probability of the no insurance case is greater than the loan default probability of the unsubsidized case for most of the interest rates. Therefore the loan default probability can be reduced even by interlinking index insurance with $m = 1.75$ when interest rate is below 18.8%. But the loan default probability of the no insurance case is less than for the actuarial fair case for high interest rates. We see the corresponding features in the figure of expected IRR. The expected IRR of the subsidized and the actuarial fair cases have their the maxima around 20% interest rate. For the unsubsidized insurance the maximum is around 15% interest rate.

3.4 The Wealth of the Lender

Here we discuss how index insurance effects the wealth of the lender. The income of the lender from the loan repayment might change when the borrower purchases an index insurance compared to the case with no insurance. We compute the wealth of the lender at the end of a given period by a similar method as in Miranda and Gonzalez-Vega (2011). We assume that the number of borrowers is large enough to apply the law of large numbers across the idiosyncratic shocks. Now the ratio of the borrowers who repay the loan of age $t = 1, 2, ..., N$ at a given x is given by

$$q_1(x) = P(\mu g_h xY + \alpha I(x) > w_1^*) = P\left(Y > \frac{w_1^* - \alpha I(x)}{\mu g_h x}\right), \tag{3.18}$$

and for $t = 2, 3, ..., N$

$$q_t(x) = P(\mu g_l xY > w_t^*) = P\left(Y > \frac{w_t^*}{\mu g_l x}\right). \tag{3.19}$$

where w_t^* is the wealth level in which the farmer is indifferent between repaying the loan and defaulting on the loan of age $t = 1, 2, ..., N$. Now let M_i be an $N \times 1$ vector whose tth element is the number of borrowers who hold a loan of age t at the end of period $i = 1, 2, \ldots$ (before making the decision of loan repayment). Let n_i be the number of new borrowers at time i (beginning of period $i + 1$). Then

$$M_{i+1} = Q(x_i)M_i + n_i v, \tag{3.20}$$

where

$$Q(x) = \begin{bmatrix} q_1(x) & q_2(x) & \ldots & q_{N-1}(x) & q_N(x) \\ 1 - q_1(x) & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 - q_{N-1}(x) & 0 \end{bmatrix}$$

and v is a $N \times 1$ vector whose first element is 1 and all the other elements are 0. We assume that the lender invests all his wealth in loans starting from initial wealth E_0. That is initially the lender issues E_0/l number of loans. Therefore the wealth of the lender at time i is the total amount repaid by the borrowers at time i and it is given by

$$E_i = \sum_{t=1}^{N} = M_{it} q_t(x_i) l(1 + \rho)^t. \tag{3.21}$$

The borrowers who repay the loan take a new loan for the next period. Then the remaining wealth of the lender is used to issue loans for new borrowers. Now the number of new borrowers at time t is given by

$$n_i = \sum_{t=1}^{N} M_{it} q_t(x_i)((1+\rho)^t - 1). \tag{3.22}$$

We compute the wealth of the lender after 10 periods by assuming four cases. Those are that the borrowers purchase index insurance with on of the three levels of premium loads 0.75, 1 or 1.75 or that the borrowers do not purchase any insurance. We randomly choose the weather index value of each period from the corresponding distribution of X. Then we compute the ratio of the borrowers who repay the loan based on those weather index values. We assume that initially 1000 loans are issued. We generate 10000 random samples of 10 periods weather index values and then compute the wealth of the lender after 10 periods under all the samples. Then we compute the average wealth. This average wealth of the lender after 10 periods under all four cases for different interest rates on the loan from 0 to 25% is given in Figure 3.7.

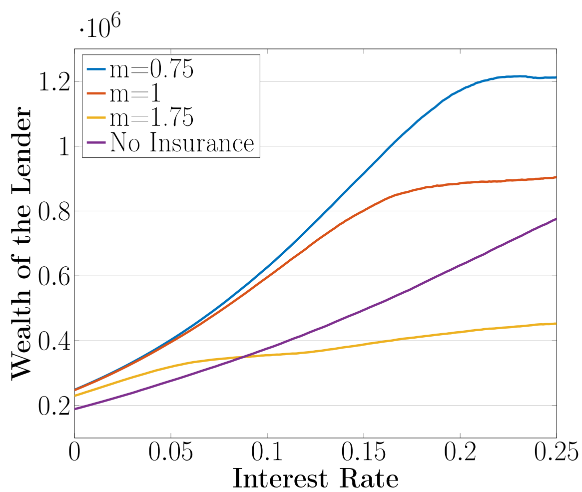

Figure 3.7: Wealth of the Lender.

As shown in Figure 3.7 the wealth of the lender is always higher when the loan is interlinked with subsidized or actuarial fair insurance compared to the no insurance case. But the wealth of the lender when the loan is interlinked with unsubsidized insurance is higher than the no insurance case only for small interest rates. In the case with subsidized insurance the wealth of the lender first dramatically increases and then slightly decreases as the interest rate increases. Also in the case with actuarial fair insurance the wealth of the lender first increases nearly as fast and then slowly increases as the interest rate increases. As discussed earlier, the loan default probability increases as interest rate increases. Then for the high interest rates the lender receives less money from the borrowers. This may be the reason for the above mentioned pattern of the wealth (flattering of the curve for high interest rates) of the lender.

3.5 Summary

The main findings of this chapter are:

- The loan default probability can be reduced and expected IRR can be increased by combining index insurance with an agriculture loan.
- Not only the subsidized or actuarial fair insurance, but also the unsubsidized insurance with proper level of premium load (not very high) is a good choice to combine with agriculture loans.
- The lender's wealth increases when the agriculture loans are interlinked with index insurance.

Chapter 4

Optimal Index Insurance Design

The most commonly used indemnity functions of weather index insurance are piecewise linear functions of the corresponding weather event (Conradt et al. (2015), Martin et al. (2001), Stoppa and Hess (2003)). But there is a possibility that the payment based on the chosen indemnity function does not properly cover the actual loss. This leads to high basis risk. Therefore it is important to choose the correct form of the indemnity function to design a proper index insurance. Zhang et al. (2018) develop a method to find the optimal indemnity function for an index insurance. They have shown that the designed optimal index insurance performs better than index insurance with a linear indemnity function by reducing the basis risk. In this chapter we use the procedure in Zhang et al. (2018) to derive an optimal index insurance with some modifications and extensions.

The model in Zhang et al. (2018) consists of the initial wealth, insurance premium, insurance payout and loss of farming. In the numerical results they compute the loss by deducting the maximum detrended yield from the detrended yield of the considered 20 years. This leads to a loss (no profit) in each year. Since in practice both loss and profit are possible, instead of loss of farming we use the income from farming. We derive the optimal indemnity function for this slightly changed model by using the method in Zhang et al. (2018). This does not make a significant difference to the model and the analytical expression of the optimal indemnity function in Zhang et al. (2018). We consider a special case by assuming the income from farming has the same form as the model in Chapter 3. Also we compute the optimal indemnity by assuming the farmer's preference on wealth is given by power utility function, which is not addressed in Zhang et al. (2018). In addition to that we discuss the impact on the repayment of a loan which is interlinked with the optimal index insurance. Motivated by Collier (2020) here we discuss the optimal index insurance for the lender as an extension. An index insurance against El Nino for the lender is discussed in Collier (2020).

4.1 The Optimization Problem

Most of the parameters are similar to the parameters of the model in Chapter 3.1. Let X denote the value of the corresponding weather index with the support $[x_{min}, x_{max}]$ and the probability density function $f(x)$. Then $I(X)$ be the indemnity function with $0 \leq I(X) \leq M$, where M is the maximum payout of the index insurance. Let Z denote the income from the yield. We consider the following two cases based on the nature of Z.

- General case : X and Z are correlated but the relationship is unknown.
- Special case: $Z = \mu g_h XY$.

For the general case we assume that Z is a continuous variable with the support $[z_{min}, z_{max}]$ and probability density function h. We also assume that X and Z are correlated. Let p be the joint density function of X and Z. In the special case we use the form of income from farming in Chapter 3.1 by assuming the farmer uses high technology for farming. Then the income $Z = \mu g_h XY$, where μ is the unit price of the yield, g_h is the high technology parameter and Y is the idiosyncratic shock with probability density function g. It is assumed that X and Y are independent. Depending on the nature of Y this special case also has two cases. Those two cases are,

- Case 1: Y has only two states, bad shock (y_l) and good shock (y_h) with $P(Y = y_l) = p$ and $P(Y = y_h) = 1 - p$. Where $0 < p < 1$,
- Case 2: Y is a continuous random variable with support $[y_{min}, y_{max}]$ and probability density function g.

Below we state the optimization problems for all those cases. We assume that the premium of this insurance is the expectation of the indemnity payment, discounted and multiplied by the premium load. So it is given by

$$\pi = \frac{m}{1+\delta} \int_{x_{min}}^{x_{max}} I(x) f(x) dx. \tag{4.1}$$

where $m \geq 1$ is the premium load and δ is the periodic interest rate. Our main focus in this work is to design an index insurance which maximizes the expected utility of the wealth of the insured. An insured with utility function u is assumed, where u is a strictly concave and non decreasing strictly concave function ($u'(.) \geq 0, u''(.) < 0$) and $u''(.)$ is a continuous function.

First the optimization problem in the general case is given by

$$\sup_I \; E[u((w_0 - (1-\theta)\pi)(1+\delta) + I(X) + Z)]$$

$$s.t \quad \pi = \frac{m}{1+\delta}\int_{x_{min}}^{x_{max}} I(x)f(x)dx. \tag{4.2}$$

where w_0 is the initial wealth of the farmer. $0 \le \theta \le 1$ is the subsidy rate on the premium which is provided by the government or another party. When the premium of the insurance is π, the insured should only pay the amount $(1-\theta)\pi$ and the rest of the premium, $\theta\pi$ is paid by the subsidy provider.

Now the optimization problem of the special case is given by

$$\sup_I \; E[u((w_0 - (1-\theta)\pi)(1+\delta) + I(X) + \mu g_h XY)]$$

$$s.t \quad \pi = \frac{m}{1+\delta}\int_{x_{min}}^{x_{max}} I(x)f(x)dx. \tag{4.3}$$

4.2 Uniqueness and Existence of the Optimal Solution

We discuss the uniqueness and existence of the optimal solution of problem (4.2) similar to Zhang et al. (2018). A proposition regarding the uniqueness of the optimal solution is given in Zhang et al. (2018) (Prop.1), which we repeat in the following. It is based on the strict concavity of the utility function.

Proposition 4.1. *The optimal solution to problem (4.2) is unique up to equality almost everywhere if it exists.*

We introduce the Lagrange multiplier λ to problem (4.2) in order to discuss the existence of the optimal solution and solve problem (4.2).

$$\begin{aligned} K(I,\lambda) &= E[u(w + I(X) + Z)] + \lambda\Big(\pi - \frac{m}{1+\delta}\int_{x_{min}}^{x_{max}} I(x)f(x)dx\Big) \\ &= E[u(w + I(X) + Z)] + \lambda\int_{x_{min}}^{x_{max}} \Big(\pi - \frac{m}{1+\delta}I(x)\Big)f(x)dx \\ &= \int_{x_{min}}^{x_{max}}\int_{z_{min}}^{z_{max}} u(w + I(x) + z)p(x,z)dzdx + \int_{x_{min}}^{x_{max}} \lambda\Big(\pi - \frac{m}{1+\delta}I(x)\Big)f(x)dx \\ &= \int_{x_{min}}^{x_{max}} \Big[\int_{z_{min}}^{z_{max}} u(w+I(x)+z)p(z|x)dz + \lambda\Big(\pi - \frac{m}{1+\delta}I(x)\Big)\Big]f(x)dx \end{aligned} \tag{4.4}$$

where $w = (w_0 - (1-\theta)\pi)(1+\delta)$ and $p(z|x) = p(x,z)/f(x)$. In the following we slightly restructure a lemma from Zhang et al. (2018) (Lemma 1) in order to discuss the optimal solution to problem (4.2) by the maximizer of $K(I,\lambda)$.

Lemma 4.2. *Let I_λ denote the maximizer of $K(I,\lambda)$ defined by Equation (4.4) for every $\lambda \in \mathbb{R}$. If there exists λ^* such that $\mathbb{E}[I_{\lambda^*}] = \pi(1+\delta)/m$, then $I^* = I_{\lambda^*}$ solves problem (4.2).*

Based on the idea of Lemma 4.2, first we investigate the maximizer of $K(I,\lambda)$ with respect to I for a given λ. According to Equation (4.4) it is sufficient to consider the pointwise maximization in $x \in [x_{min}, x_{max}]$ of

$$H(I(x),x,\lambda) = \int_{z_{min}}^{z_{max}} u(w+I(x)+z)p(z|x)dz + \lambda\Big(\pi - \frac{m}{1+\delta}I(x)\Big). \tag{4.5}$$

The derivative of $H(I(x),x,\lambda)$ with respect to $I(x)$ is given by

$$\dot{H}(I(x),x,\lambda) = G(I(x),x) - \lambda\frac{m}{1+\delta}, \tag{4.6}$$

where

$$\begin{aligned} G(i,x) &= \int_{z_{min}}^{z_{max}} u'(w+i+z)p(z|x)dz \\ &= \mathbb{E}[u'(w+i+Z)|X=x]. \end{aligned} \tag{4.7}$$

Since u is strictly concave, u' is strictly decreasing. Therefore $G(i,x)$ is strictly decreasing in i. Then $G(i,x)$ attains its maximum value at $i=0$ and its minimum value at $i=M$. Based on this and similar to Zhang et al. (2018) we define the following three sets.

$$S_1^\lambda = \Big\{x \in [x_{min}, x_{max}] |\ G(0,x) < \lambda\frac{m}{1+\delta}\Big\}. \tag{4.8}$$

$$S_2^\lambda = \Big\{x \in [x_{min}, x_{max}] |\ G(M,x) > \lambda\frac{m}{1+\delta}\Big\}. \tag{4.9}$$

$$S_3^\lambda = \Big\{x \in [x_{min}, x_{max}] |\ G(M,x) \le \lambda\frac{m}{1+\delta} \le G(0,x)\Big\}. \tag{4.10}$$

Since $G(i,x)$ is strictly decreasing in i for a given x, there is no common x which satisfies the conditions in both S_1^λ and S_2^λ. Then $S_1^\lambda \cap S_2^\lambda = \emptyset$. Thus S_1^λ, S_2^λ and S_3^λ are three disjoint subsets of $[x_{min}, x_{max}]$ and $S_1^\lambda \cup S_2^\lambda \cup S_3^\lambda = [x_{min}, x_{max}]$. So they form a partition. The nature of $\dot{H}(I(x),x,\lambda)$ in the above three sets may help to show the existence of an optimal solution. We observe the following results regarding $H(I(x),x,\lambda)$.

- For any $x \in S_1^\lambda$, $\dot{H}(I(x),x,\lambda) < 0$. Therefore $H(I(x),x,\lambda)$ is decreasing in $I(x) \in [0,M]$.

- For any $x \in S_2^\lambda$, $\dot{H}(I(x), x, \lambda) > 0$. Therefore $H(I(x), x, \lambda)$ is increasing in $I(x) \in [0, M]$.
- For any $x \in S_3^\lambda$, $\dot{H}(0, x, \lambda) > 0$ and $\dot{H}(M, x, \lambda) < 0$. Therefore $H(I(x), x, \lambda)$ is first increasing in $I(x) \in [0, M]$ and then decreasing.

According to the above observations we can easily understand that

$$I_\lambda(x) = \underset{I(x)\in[0,M]}{\operatorname{argmax}}\ H(I(x), x, \lambda) = \begin{cases} 0 & \text{for } x \in S_1^\lambda \\ M & \text{for } x \in S_2^\lambda \\ \hat{I}_\lambda(x) & \text{for } x \in S_3^\lambda, \end{cases} \tag{4.11}$$

where $\dot{H}(\hat{I}_\lambda(x), x, \lambda) = 0$, that is,

$$G(\hat{I}_\lambda(x), x) = \lambda \frac{m}{1+\delta}. \tag{4.12}$$

4.2.1 Uniqueness and Existence of the Optimal Solution for Special Cases

The form of Z does not matter for the uniqueness of the optimal solution discussed in Proposition 4.1. Therefore, Proposition 4.1 not only discusses the uniqueness of the optimal solution of general case but also the special case. Nevertheless, we require some modifications in order to discuss the existence of the optimal solution and solve the optimization problem for special case.

Now we introduce the Lagrange multiplier to problem (4.3). Then we modify $K(I, \lambda)$ in Equation (4.4) for Case 1 and Case 2, which we state in the following as $K_1(I, \lambda)$ and $K_2(I, \lambda)$ respectively.

$$K_1(I, \lambda) = \int_{x_{min}}^{x_{max}} \Big[u(w+I(x)+\mu g_h x y_l)p + u(w+I(x)+\mu g_h x y_h)(1-p) + \lambda\Big(\pi - \frac{m}{1+\delta} I(x)\Big)\Big] f(x) dx, \tag{4.13}$$

$$K_2(I, \lambda) = \int_{x_{min}}^{x_{max}} \Big[\int_{y_{min}}^{y_{max}} u(w+I(x)+\mu g_h x y) g(y) dy + \lambda\Big(\pi - \frac{m}{1+\delta} I(x)\Big)\Big] f(x) dx. \tag{4.14}$$

By considering pointwise maximization, we modify $G(i, x)$ in Equation (4.7) for Case 1 and Case 2 as $G_1(i, x)$ and $G_2(i, x)$. Those are given by

$$G_1(i, x) = u'(w+i+\mu g_h x y_l)p + u'(w+i+\mu g_h x y_h)(1-p), \tag{4.15}$$

$$G_2(i, x) = \int_{y_{min}}^{y_{max}} u'(w+i+\mu g_h x y) g(y) dy. \tag{4.16}$$

Now we can define the modified versions of the three sets S_1^λ, S_2^λ and S_3^λ for Case 1 and Case 2 based on $G_1(i,x)$ and $G_2(i,x)$. We can easily understand that, $I_\lambda(x)$ of Case 1 and 2 have the same form as $I_\lambda(x)$ in Equation (4.11). But instead of Equation (4.12) $I_\lambda(x)$ of Case 1 and 2 satisfy $G_1(\hat{I}_\lambda(x),x) = \lambda\frac{m}{1+\delta}$ and $G_2(\hat{I}_\lambda(x),x) = \lambda\frac{m}{1+\delta}$ respectively. The uniqueness and existence of the optimal solution is discussed in details in Zhang et al. (2018).

4.3 The ODE Method

It is important to derive a closed form expression for the optimal solution. In order to do that we need the form of $S_1^{\lambda^*}$, $S_2^{\lambda^*}$, $S_3^{\lambda^*}$ and $\hat{I}_{\lambda^*}(x)$, where λ^* is given in Lemma 4.2. $\hat{I}_{\lambda^*}(x)$ is only defined on the set $S_3^{\lambda^*}$ and solved by Equation (4.12). Zhang et al. (2018) develop a method to derive the optimal solution. Since it is based on an ordinary differential equation, they named it as ODE method. That method is also based on the fact that the analytical form of $\hat{I}_{\lambda^*}(x)$ can be extended to the whole interval $[x_{min}, x_{max}]$. Lemma 2 in Zhang et al. (2018) is about the extension of $\hat{I}_\lambda(x)$ from S_3^λ to the whole interval $[x_{min}, x_{max}]$ is true for this model and which we rewrite in the following.

Lemma 4.3. *Let λ be a constant such that $S_3^\lambda \neq \emptyset$ and assume that a solution $\hat{I}_\lambda(x)$ from Equation (4.12) exists on $[x_{min}, x_{max}]$. Then the optimal solution to maximize $K(I,\lambda)$ is*

$$I_\lambda(x) = [(\hat{I}_\lambda(x)) \vee 0] \wedge M. \tag{4.17}$$

According to the above lemma we do not require to determine the set S_3^λ to derive $\hat{I}_\lambda(x)$. Because when we obtain the analytical form of $\hat{I}_\lambda(x)$ by solving Equation (4.12), then we can derive the optimal solution $I_\lambda(x)$ by Equation (4.17). As suggested in Zhang et al. (2018) we can transfer problem (4.2) into an ODE problem by using Lemma 4.3. In the ODE method first we derive the analytical form of $\hat{I}_{\lambda^*}(x)$ in $S_3^{\lambda^*}$ and then we extend it to the whole interval $[x_{min}, x_{max}]$ as explained in Lemma 4.3, where λ^* satisfies $\mathbb{E}[I_{\lambda^*}(X)] = \pi(1+\delta)/m$. Therefore non-emptiness of $S_3^{\lambda^*}$ is important. We slightly change Proposition 3 in Zhang et al. (2018) regarding the non-emptiness of $S_3^{\lambda^*}$ and provide it in the following.

Proposition 4.4. *$S_3^{\lambda^*}$ is a non empty subset of $[x_{min}, x_{max}]$, where λ^* is any constant such that $\mathbb{E}[I_{\lambda^*}(X)] = \pi(1+\delta)/m$.*

The following theorem explains how to solve the optimization problem (4.2) by the ODE approach.

Theorem 4.5. *Suppose that the derivative* $\frac{\partial}{\partial x}p(z|x)$ *exists and is continuous on* $[x_{min}, x_{max}] \times [z_{min}, z_{max}]$*, and a function* $\hat{L} : [x_{min}, x_{max}] \mapsto \mathbb{R}$ *which solves the following ODE problem:*

$$\frac{dL}{dx} = F(x, L)$$

$$\pi = \frac{m}{1+\delta}\mathbb{E}[(L(x) \vee 0) \wedge M], \tag{4.18}$$

where the function $F : [x_{min}, x_{max}] \times \mathbb{R} \mapsto \mathbb{R}$ *is defined by*

$$F(x, L) = -\frac{\int_{z_{min}}^{z_{max}} u'((w_0 - (1-\theta)\pi)(1+\delta) + L + z)\frac{\partial}{\partial x}p(z|x)dz}{\int_{z_{min}}^{z_{max}} u''((w_0 - (1-\theta)\pi)(1+\delta) + L + z)p(z|x)dz}. \tag{4.19}$$

Then $L^* = ((\hat{L}(x))\vee) \wedge M$ *is the solution to the optimization problem (4.2).*

4.3.1 The ODE Method for $Z = \mu g_h XY$

One can expect that when $Z = \mu g_h XY$, one can use a classical linear indemnity function. Then the optimal indemnity function can be derived by estimating the optimal values of the parameters of the considered linear indemnity function which maximize the expected utility of the insured. But even if the relationship between Z and X is known, it is still convenient to use the ODE approach given in Zhang et al. (2018) without predetermining the form of the indemnity function.

Lemma 4.3 and Proposition 4.4 are valid for Case 1 and Case 2 under the special case. Therefore we can use the ODE method to solve problem (4.3) for Case 1 and Case 2. We state the ODE approach to solve the optimization problem (4.3) under both cases in the following two theorems.

Theorem 4.6. *Assuming Case 1, there exist a function* $\hat{L} : [x_{min}, x_{max}] \mapsto \mathbb{R}$ *which solves the following ODE problem:*

$$\frac{dL}{dx} = F(x, L)$$

$$\pi = \frac{m}{1+\delta}\mathbb{E}[(L(x) \vee 0) \wedge M], \tag{4.20}$$

where the function $F : [x_{min}, x_{max}] \times \mathbb{R} \mapsto \mathbb{R}$ *is defined by*

$$F(x, L) = -\frac{\mu g_h(u''(w_h)(1-p)y_h + u''(w_l)py_l)}{u''(w_h)(1-p) + u''(w_l)p}, \tag{4.21}$$

and $w_h = (w_0 - (1-\theta)\pi)(1+\delta) + L + \mu g_h x y_h$ *and* $w_l = (w_0 - (1-\theta)\pi)(1+\delta) + L + \mu g_h x y_l$*. Then* $L^* = ((\hat{L}(x))\vee) \wedge M$ *is the solution to the optimization problem (4.3).*

Theorem 4.7. *Assuming Case 2, there exist a function* $\hat{L} : [x_{min}, x_{max}] \mapsto \mathbb{R}$ *which solves the following ODE problem:*

$$\frac{dL}{dx} = F(x, L)$$
$$\pi = \frac{m}{1+\delta}\mathbb{E}[(L(x) \vee 0) \wedge M], \tag{4.22}$$

where the function $F : [x_{min}, x_{max}] \times \mathbb{R} \mapsto \mathbb{R}$ *is defined by*

$$F(x, L) = -\frac{\mu g_h \int_{y_{min}}^{y_{max}} u''((w_0 - (1-\theta)\pi)(1+\delta) + L + \mu g_h x y) y g(y) dy}{\int_{y_{min}}^{y_{max}} u''((w_0 - (1-\theta)\pi)(1+\delta) + L + \mu g_h x y) g(y) dy}. \tag{4.23}$$

Then $L^* = ((\hat{L}(x))\vee) \wedge M$ *is the solution to the optimization problem (4.3).*

4.4 The Optimal Indemnity under Different Utility Functions

As discussed above the optimal indemnity depends on the utility function of the insured. In this section we discuss the nature of the optimal indemnity for some commonly used utility functions. We consider all the above mentioned cases in order to derive the optimal index insurance under different utility functions.

4.4.1 Exponential Utility

We suppose the insured's preference on wealth is given by an exponential utility function, i.e $u(w) = \frac{1-e^{-\gamma w}}{\gamma}$, where $\gamma > 0$. We derive the optimal index insurance for different cases by Theorem 4.5, 4.6 and 4.7. We state them in the following propositions.

Proposition 4.8. *In the general case suppose that the derivative* $\frac{\partial}{\partial x}p(z|x)$ *exists and continuous on* $[x_{min}, x_{max}] \times [z_{min}, z_{max}]$. *If the insured's utility function is* $u(w) = \frac{1-e^{-\gamma w}}{\gamma}$, *where* $\gamma > 0$, *then the optimal index insurance is given by*

$$I^*(x) = \left[\left(\frac{1}{\gamma} ln \mathbb{E}[e^{-\gamma Z}|X = x] + \eta^*\right) \vee 0\right] \wedge M, \tag{4.24}$$

where η^* *is determined by*

$$\mathbb{E}[I^*(X)] = \mathbb{E}\left[\left[\left(\frac{1}{\gamma} ln \mathbb{E}[e^{-\gamma Z}|X] + \eta^*\right) \vee 0\right] \wedge M\right] = \frac{\pi(1+\delta)}{m}.$$

Proposition 4.9. *In the special case, if the insured's utility function* $u(w) = \frac{1-e^{-\gamma w}}{\gamma}$*, where* $\gamma > 0$*, then the optimal index insurance is given by*

$$I^*(x) = \Big[\Big(\frac{1}{\gamma}ln\mathbb{E}[e^{-\gamma\mu g_h xY}] + \eta^*\Big) \vee 0\Big] \wedge M, \tag{4.25}$$

where η^* *is determined by*

$$\mathbb{E}[I^*(X)] = \mathbb{E}\Bigg[\Big[\Big(\frac{1}{\gamma}ln\mathbb{E}[e^{-\gamma\mu g_h XY}] + \eta^*\Big) \vee 0\Big] \wedge M\Bigg] = \frac{\pi(1+\delta)}{m}.$$

In the two cases under the special case, $\mathbb{E}[e^{-\gamma\mu g_h xY}]$ can be computed according to the corresponding case.

4.4.2 Power Utility

Now we suppose the farmer's preference on wealth is given by a power utility function, i.e $u(w) = \frac{w^{(1-\gamma)}}{1-\gamma}$, where $w \geq 0$, $\gamma > 0$ and $\gamma \neq 1$. Due to the nature of the power utility function, it is difficult to simplify $F(x, L)$ to derive a closed form solution of the optimal index insurance. But it is possible to state the ODE problem which leads to the optimal solution. This means that we can derive the function $F(x, L)$ which is given in Theorem 4.5, 4.6 and 4.7. Then we can compute the optimal solution numerically. We state the form of $F(x, L)$ in all cases for power utility function in the following three propositions.

Proposition 4.10. *In the general case suppose that the derivative* $\frac{\partial}{\partial x}p(z|x)$ *exists and continuous on* $[x_{min}, x_{max}] \times [z_{min}, z_{max}]$*. If the insured's utility function is* $u(w) = \frac{w^{(1-\gamma)}}{1-\gamma}$*, where* $w \geq 0$*,* $\gamma > 0$ *and* $\gamma \neq 1$*, then* $F(x, L)$ *in Theorem 4.5 is given by*

$$F(x, L) = \frac{\int_{z_{min}}^{z_{max}}((w_0 - (1-\theta)\pi)(1+\delta) + L + z)^{-\gamma}\frac{\partial}{\partial x}p(z|x)dz}{\gamma\int_{z_{min}}^{z_{max}}((w_0 - (1-\theta)\pi)(1+\delta) + L + z)^{-\gamma-1}p(z|x)dz}. \tag{4.26}$$

Proposition 4.11. *In the special case with discrete Y (Case 1), if the insured's utility function is* $u(w) = \frac{w^{(1-\gamma)}}{1-\gamma}$*, where* $w \geq 0$*,* $\gamma > 0$ *and* $\gamma \neq 1$*, then* $F(x, L)$ *in Theorem 4.6 is given by*

$$F(x, L) = -\frac{\mu g_h(w_h^{-\gamma-1}(1-p)y_h + w_l^{-\gamma-1}py_l)}{w_h^{-\gamma-1}(1-p) + w_l^{-\gamma-1}p}, \tag{4.27}$$

where $w_h = (w_0 - (1-\theta)\pi)(1+\delta) + L + \mu g_h x y_h$ *and* $w_l = (w_0 - (1-\theta)\pi)(1+\delta) + L + \mu g_h x y_l$*.*

Proposition 4.12. *In the special case with continuous Y (Case 2), if the insured's has utility function* $u(w) = \frac{w^{(1-\gamma)}}{1-\gamma}$*, where* $w \geq 0$*,* $\gamma > 0$ *and* $\gamma \neq 1$*, then* $F(x, L)$ *in Theorem 4.7 is given by*

$$F(x, L) = -\frac{\mu g_h \int_{y_{min}}^{y_{max}} ((w_0 - (1-\theta)\pi)(1+\delta) + L + \mu g_h xy)^{-\gamma-1} y g(y) dy}{\int_{y_{min}}^{y_{max}} ((w_0 - (1-\theta)\pi)(1+\delta) + L + \mu g_h xy)^{-\gamma-1} g(y)) dy}. \tag{4.28}$$

Remark 4.13. Due to the unavailability of closed form solution to ODE problems for some utility functions Zhang et al. (2018) develop a numerical scheme. It is based on 4th order Runge-Kutta (RK4) method. We state that numerical scheme in B.

4.5 Numerical Computation of Optimal Indemnity

We consider the special case with $Z = \mu g_h XY$ for all numerical computations. We assume that X is gamma distributed and Y is also gamma distributed in Case 2. Let $\theta = 0$ and $m = 1$ for all the computations unless otherwise specified. We compute the optimal indemnity for both Case 1 and Case 2 under the exponential utility and power utility. Let $M = 30$. The obtained results are given in Figure 4.1 for the exponential utility and in Figure 4.2 for the power utility. We choose different levels of premiums in order to show the different structures of the optimal indemnity. Therefore for Case 1 we consider $\pi = 2, 3, 4, 5$ and for Case 2 we consider $\pi = 3, 5, 7, 9, 11$.

4.5.1 Exponential Utility

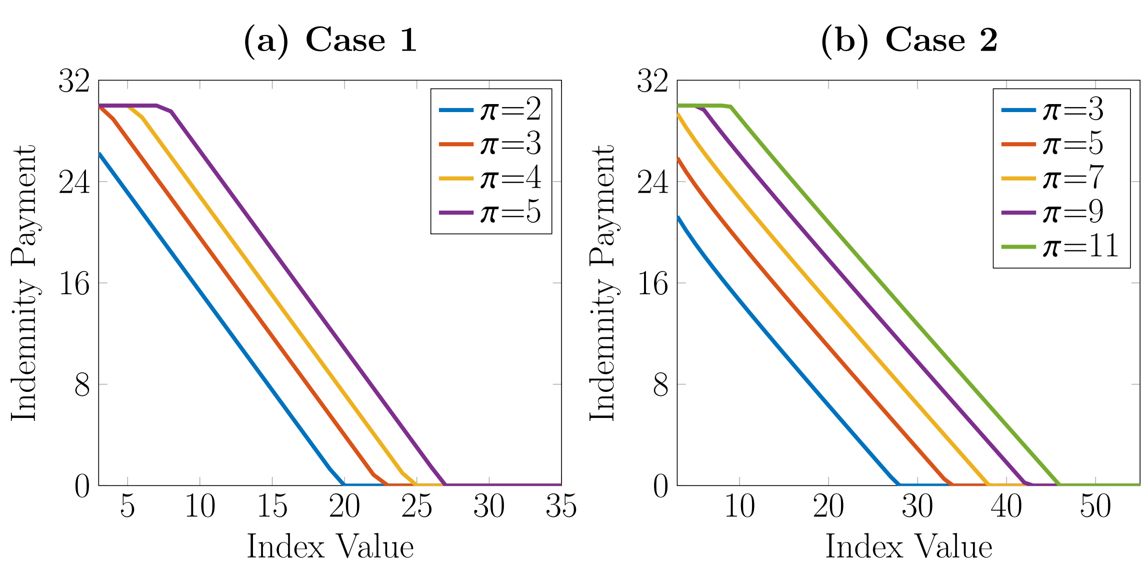

Figure 4.1: Optimal Indemnity under Exponential Utility.

In Figure 4.1 (a) all the optimal indemnity functions monotonically decrease and go to zero at index values between 20 and 30 and then stay at zero for rest of the index values. In Figure 4.1 (b) the optimal indemnity functions go to zero at index values between 25 and 45. For Case 1 index values are given up to 35 and for Case 2 up to 55. Because the rest of the index values up to x_{max} do not give any particular information since the indemnity payment is zero for those index values. The optimal indemnities decrease linearly (according to the appearance). For high premium levels the optimal indemnity is equal to the maximum level $M = 30$ for very small index values and then strictly decreases. Therefore the insured receives the maximum indemnity payment when the weather index indicates a very small value and then the insured receives indemnity payment less than M.

4.5.2 Power Utility

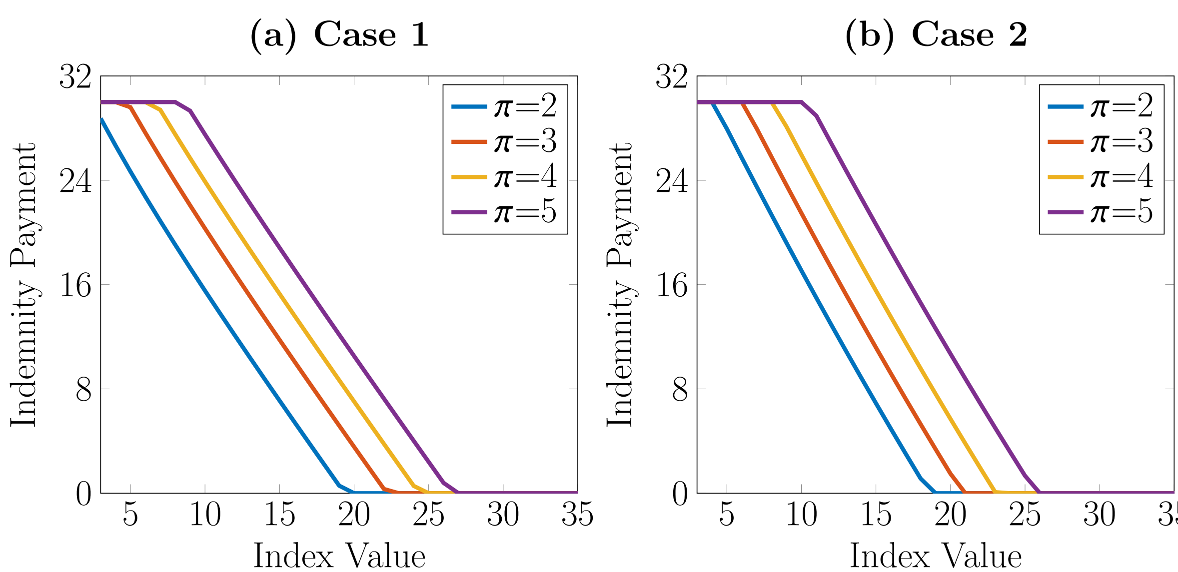

Figure 4.2: Optimal Indemnity under Power Utility.

According to Figure 4.2 (a) it is clear that the optimal indemnity functions for the insured with power utility function is mostly similar to the the optimal indemnity functions for the insured with exponential utility function under Case 1. But under Case 2 for $\pi = 2, 3, 4, 5$ the optimal indemnities are equal to the maximum level $M = 30$ for small index values and then strictly decrease and then become zero. This feature can be seen under exponential utility only for high premium levels ($\pi > 7$). Further, for Case 2 the optimal indemnity under power utility becomes zero at a smaller index value compared to the corresponding optimal indemnity under exponential utility. This implies an index insurance which pays less within a wider range of index values is optimal for a farmer with exponential utility function while an index insurance which pays more within a smaller range of the index values is optimal for a farmer with power utility function.

4.6 Effectiveness of the Optimal Index Insurance

In this section we discuss the effectiveness of the optimal index insurance in terms of the basis risk. It is a well known fact that basis risk is one of the major problematic features of an index insurance. Therefore we can measure the effectiveness of the optimal index insurance by computing the basis risk of the insurance. The basis risk arises when there is a difference between the actual loss and the indemnity payment. Here we need to obtain the loss of farming in order to compute the basis risk. We consider that there is a loss when the income of farming is below 80% of the expected income of farming, see Conradt et al. (2015). That is, when the actual income is $\tilde{z}$, the Loss is given by,

$$\text{Loss} = \max\{0.8 \cdot \mathbb{E}[Z] - \tilde{z}, 0\}. \tag{4.29}$$

Now, similar to Zhang et al. (2018) we calculate the basis risk as the standard deviation of the residual risk, that is the standard deviation of [Loss $- I^*(X)$]. As per the definition of the loss in Equation (4.29) when the actual income is above 80% of the expected income the loss is set to zero instead of considering it as a negative loss (or profit). Because in order to compute the basis risk we only require the amounts which are underpaid and overpaid. At a given x if there is a profit ($\tilde{z} \geq 0.8 \cdot \mathbb{E}[Z]$) but the insured receives an indemnity payment then the overpaid amount is $I^*(x)$ regardless of the amount of profit. Because of that the overpaid or underpaid amount at x can be simply given by [Loss $- I^*(x)$].

We consider the expectation of the income as $\mathbb{E}[\mu g_h XY]$ by assuming Case 2. We simulate a data set to represent the actual income by considering a discrete set of values for X in $[x_{min}, x_{max}]$ and generating a random value y for each x from the corresponding distribution of Y. Then we calculate the basis risk as explained above for the different levels of premium in the interval [0, 10] and the different levels of M in the interval [10, 55]. We consider the power utility function for the numerical computation. The obtained results are given in Figure 4.3.

According to Figure 4.3 when the premium is equal to zero, that is, there is no insurance, the basis risk is around 3.86 for all the levels of M. In that case, since there is no index insurance, the basis risk simply equal to the standard deviation of the Loss. The basis risk of all the other cases are below that level. This already shows that there is a positive impact from this index insurance in terms of the basis risk. The basis risk monotonically decreases as the maximum indemnity level increases. Also when $M \geq 20$, the basis risk decreases as the

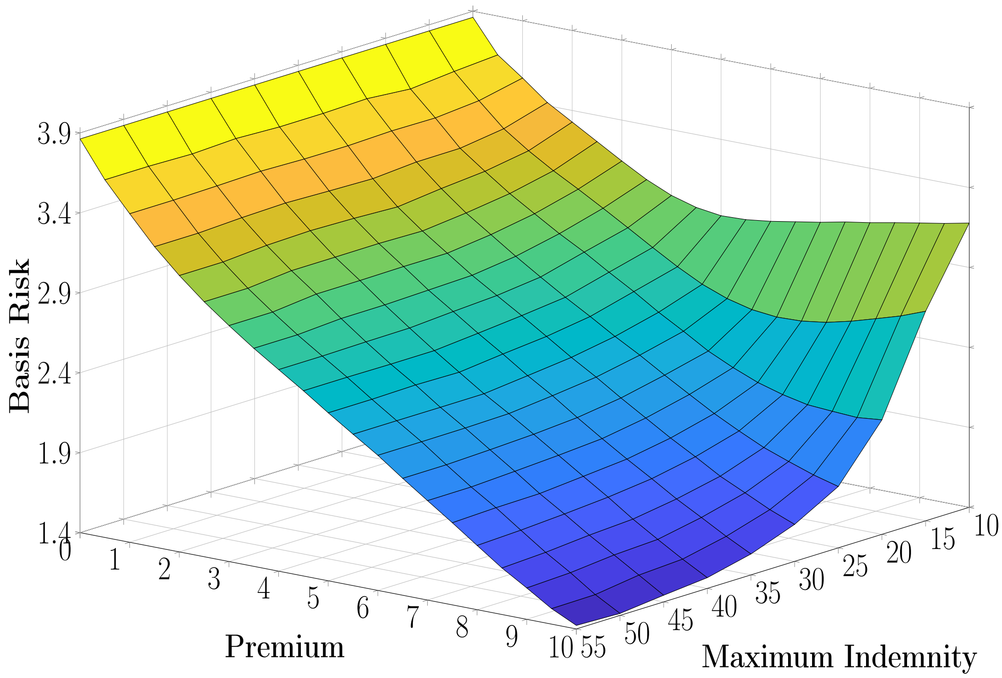

Figure 4.3: Basis Risk for Different Levels of Premium and Maximum Indemnity.

premium increases. For small levels of M (e.g M=10,15) the basis risk first decreases and then increases as the premium increases. In the cases where M is small and the premium is large, the index insurance pays more than enough for small losses as a result of the high premium, but it still does not properly cover the large losses since M is small. Therefore the basis risk is high. Not only for small M but also for large M when the premium is very small, the basis risk is high, because in those cases the insurance does not properly cover even a small loss due to the very small premium. When M is small, a moderate level of premium works better, because it does not leads to too much underpaid or overpaid small losses. The insurance with high levels of M and high premium levels are the most effective in terms of reducing the basis risk. In those cases the M is high enough to cover the large losses in other hand the high premiums allows to cover the losses properly. Then the effect of overpaid and underpaid are very low.

As mentioned in Zhang et al. (2018) this kind of figure helps governments to determine the premium subsidy amount according to M. For an example let us assume that the government is willing to provide an amount of 3 as premium subsidy. The farmer has only amount 4 to pay the premium. With the premium subsidy now the farmer can purchase an index insurance with $\pi = 7$. Now let $M = 30$. In this case the premium subsidy helps the farmer to take an insurance with less basis risk compared to the insurance he can purchase without subsidy. Because the insurance with $M = 30$ and $\pi = 7$ has lower basis risk compared to the insurance with $M = 30$ and $\pi = 4$. Now let $M = 10$. In this case the basis risk of the index insurance with $\pi = 7$ is greater than the index insurance with $\pi = 4$. In this case the premium subsidy of 3 does not help the farmer go for a better insurance. But the premium subsidy of amount 1 let farmer to go for a index insurance with less basis risk. In this manner the results in Figure 4.3 help to choose an appropriate subsidy level for a given M.

4.7 Comparison between Optimal and Linear Indemnities

We make a comparison between the optimal indemnity function which is calculated by the ODE method and a linear indemnity function. We consider the linear indemnity function $I(x) = \max\{\lambda(k - x), 0\}$ with $k = 30$ and $\lambda = 1$. The actuarial fair premium of the index insurance with this linear indemnity is approximately 4.5 and the maximum indemnity is 27 since $x_{min} = 3$. Therefore we compare it with the optimal indemnity function which is derived by ODE approach with $\pi = 4.5$ and $M = 27$. We compute the optimal indemnity for both Case 1 and Case 2. The obtained results are given in 4.4.

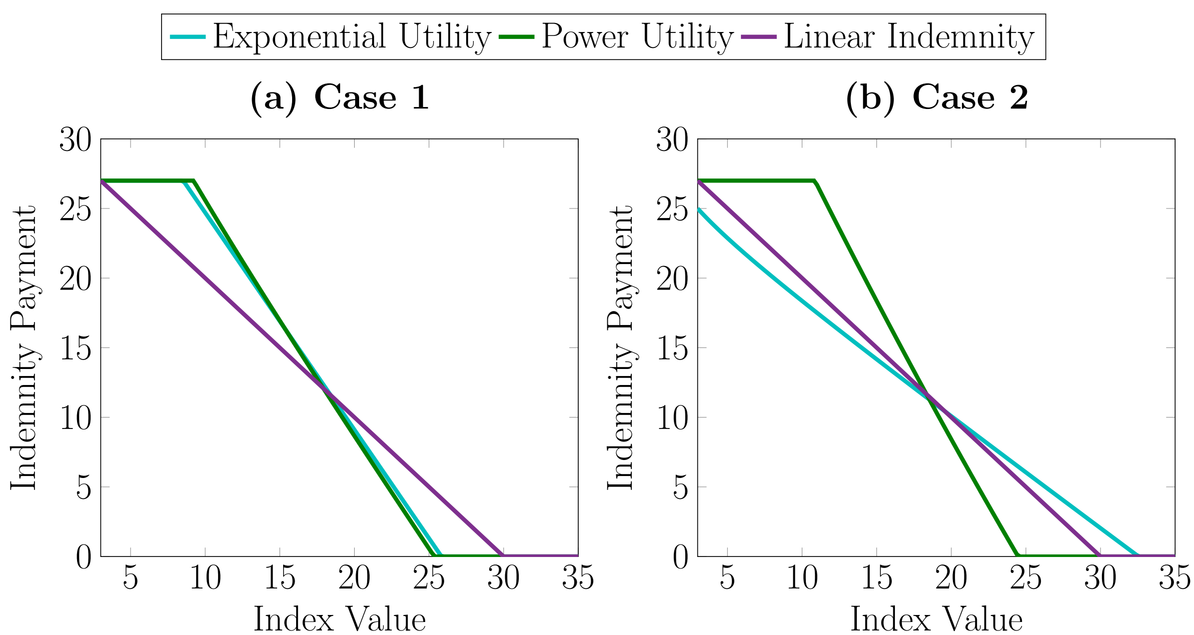

Figure 4.4: Comparison between Optimal Indemnity and Linear Indemnity for $M = 27$.

Figure 4.4 shows the difference between the optimal indemnity and the linear indemnity. Under Case 1 the optimal indemnities of both exponential and power utilities are mostly similar. But the linear indemnity is different to them up to some extent. Under Case 2 the linear indemnity and the optimal indemnity of exponential utility are mostly similar, but the optimal indemnity of power utility is somewhat different to them. According to Figure 4.4 (a) under power utility or exponential utility in order to maximize the expected utility of the insured's wealth more payment is required for low index values ($x < 20$) and less payment is required for high index values ($x > 20$) compared to the linear indemnity. In Figure 4.4 (b) similar comparison can be seen between the optimal indemnity under power utility and linear indemnity. But those facts are totally different when comparing the optimal indemnity of exponential utility and linear indemnity under Case 2. Because those two indemnity functions are mostly similar.

The results in Figure 4.3 shows that high premium and high M lead to low basis risk. Therefore we make a comparison between the optimal and the linear indemnities for high premium and high M. Now let $\lambda = 2$ and keep $k = 30$. Then the premium of the insurance with this linear indemnity is 9. We compare it with optimal indemnity with $\pi = 9$ and $M = 54$. The comparison is given in Figure 4.5.

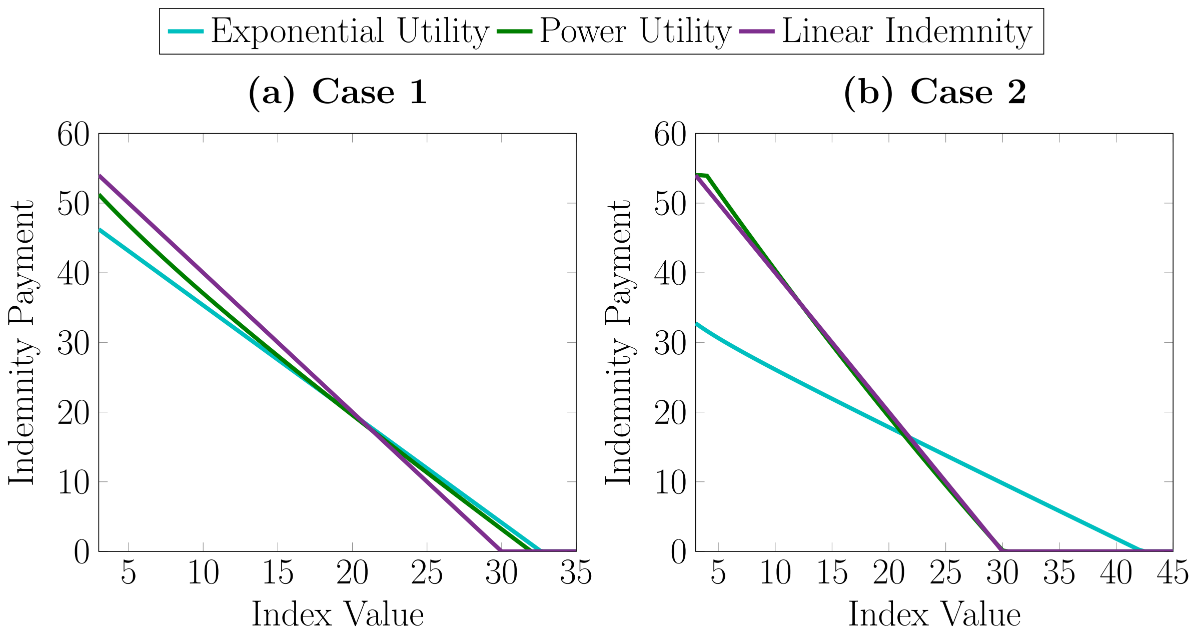

Figure 4.5: Comparison between Optimal Indemnity and Linear Indemnity for $M = 54$.

As shown in Figure 4.5 under Case 1 all the cases are mostly similar. It implies that a linear indemnity with high maximum payment (it leads to high premium) is a good choice of indemnity function when the farmers have either exponential or power utility. But under Case 2 the linear indemnity with high maximum payment is a good choice when the farmer has power utility. But it is not a good choice for the farmers with exponential utility, because the optimal indemnity under exponential utility is considerably different to the linear indemnity.

Now it is clear that there are situations in which the optimal indemnity is different to the considered linear indemnity. The linear indemnity does not differ based on the utility function of the insured. But the results in Figure 4.4 and 4.5 show that even the optimal indemnity function has a certain linear form, it can differ according to the utility function of the insured.

We can do further comparison to find which indemnity is more effective. Again we use the basis risk for the investigation. We compute the basis risk of the optimal and linear indemnities as explained earlier for different levels of premium and two levels of M. We consider the same linear indemnity function, $\max\{\lambda(k-x), 0\}$ and the optimal indemnity function under power utility. We choose the values of λ and k such that the premium and the maximum indemnity is equal to the given levels. Figure 4.6 shows the obtained results.

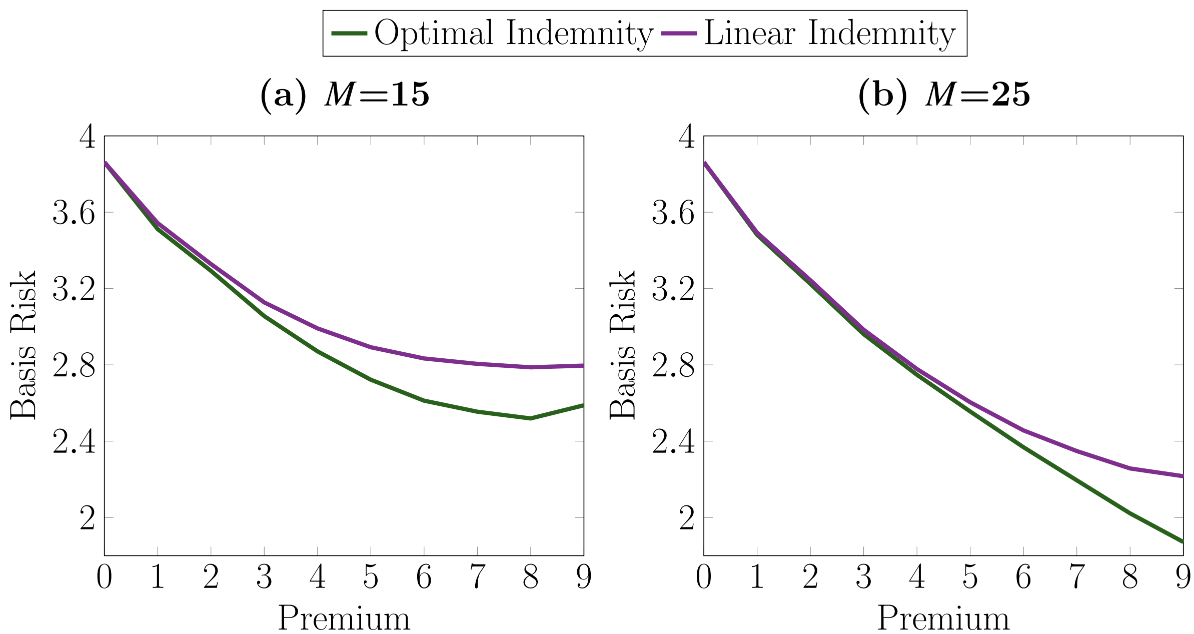

Figure 4.6: Comparison between Optimal Indemnity and Linear Indemnity.

According to Figure 4.6 the basis risk of the linear indemnity is greater than the basis risk of the optimal indemnity for both $M = 15$ and $M = 25$ for all the given premium levels except 0. In both cases the difference between the basis risks are bigger for the high premiums compared to the low premiums. However there is no considerably big difference between the the basis risks of two indemnity functions. When $M = 15$ the basis risk of both optimal and linear indemnities first decrease and then start to increase as premium increases. But when $M = 25$ both indemnities decrease as premium increases. Therefore small basis risk can be achieved for both indemnities with $M = 25$ compared to $M = 15$. Since there is no significant different between the optimal indemnity under power utility and linear indemnity as per the previous results, we cannot expect a big difference between the basis risk under two indemnity functions. Nevertheless we see, that the optimal indemnity clearly reduces the basis risk.

4.8 Effectiveness of the Optimal Index Insurance for Loan Repayment

Now we consider that the farmer takes a loan to use high technology for farming. We assume that the index insurance is interlinked with the agriculture loan of the farmer. In Chapter 3 we already discussed the loan repayment behavior of a farmer when the loan is interlinked with an index insurance. Since a linear indemnity function is considered there, now it is interesting to discuss how the optimal indemnity of interlinked index insurance affects the loan repayment behavior of the farmer. In this section we compare the loan repayment behavior of the farmer under the index insurance with the optimal indemnity function and a given linear indemnity function. We analyze the loan repayment behavior in terms of the loan default probability. Here we assume that the farmer defaults on the loan when his terminal wealth is not large enough to pay the total outstanding loan. Therefore the loan default probability is given by

$$\begin{aligned}\text{Loan Default Probabiity} &= P(\text{Terminal Wealth} < \text{Outstanding Loan})\\ &= P((w_0 - (1-\theta)\pi)(1+\delta) + I(X) + \mu g_h XY < l(1+\rho)),\end{aligned} \tag{4.30}$$

where l is the loan amount and ρ is the interest rate on the loan. For simplicity of the numerical computations we consider Case 1 (discrete Y). We assume that the farmer's preference is given by a power utility function. We also assume that the index insurance with actuarial fair price of 5 ($\mathbb{E}[I(X)]/(1+\delta) = 5$) and the maximum payout of 20 ($M = 20$). We consider the linear indemnity function $\max\{\lambda(k-x), 0\}$ and we choose λ and k such that actuarial fair price and the maximum payout are equal to the above given values. Also we consider three different types of index insurance based on the nature of the premium and those are listed below.

- Type A : Insurance with actuarial fair price ($m = 1$) and subsidy rate of 25% (θ=0.25).
- Type B: Insurance with actuarial fair price ($m = 1$) and no subsidy (θ=0).
- Type C: Insurance with premium load of 1.5 ($m = 1.5$) and no subsidy (θ=0).

The premium of Type A, B and C are 5, 5 and 7.5 respectively. But the amounts that the farmer should pay to the insurer are 3.75, 5 and 7.5 respectively. The comparison of the loan default probability is given if Figure 4.7.

According to Figure 4.7 the default probability of the loan which is interlinked with Type A insurance is 0 for the interest rates below 10% under both optimal and linear indemnities. Also the default probability of the loan which is interlinked with Type B insurance is 0 for the interest rates below 5% under both indemnities. But loan default probability with Type C insurance is

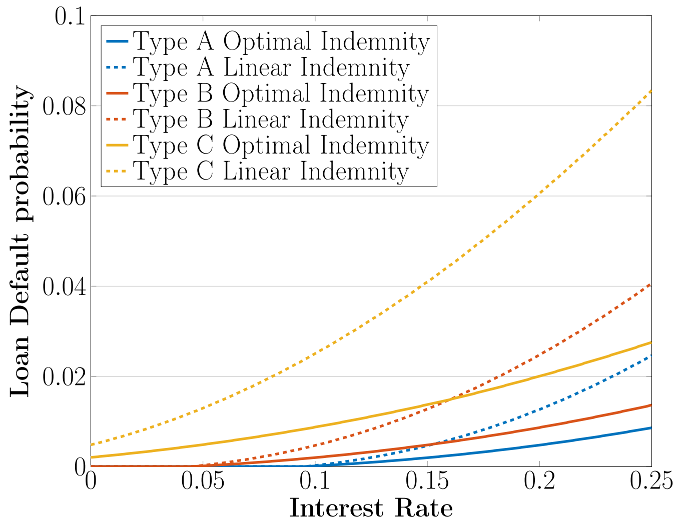

Figure 4.7: Comparison of Loan Default Probability.

always positive. The loan default probability under all types of optimal indemnities are below the loan default probability under the corresponding linear indemnity (except the cases with 0 default probability). The loan default probability is high when the portion of the premium which should be paid by the farmer is high. The premiums of both Type A and Type B insurances are the same but the loan default probability under Type A insurance is less than Type B insurance, because the amount paid by the farmer as his portion of the premium to the insurance company is smaller for type A compared to Type B. As the interest rate on the loan increases the gap between loan default probabilities under the optimal indemnity and the linear indemnity increases. Therefore when the interest rate is high the optimal indemnity makes a better influence on the loan default probability compared to the situation with a low interest rate.

4.9 Lender Level Index Insurance

In Section 4.8 we discuss about an index insurance which is interlinked with an agriculture loan. Motivated by the work in Collier (2020), in this section we focus on a lender who issue agriculture loans and we discuss a weather based index insurance for the lender. We assume that the income of the farmers varies based on a certain weather event. Obviously the loan repayment behavior of the farmers depends on their income. Since the lender's income is the amount of loan repayment, the underlying weather event indirectly affects the lender. Therefore it is reasonable to discuss weather index based insurance design for the lender.

We consider that at the beginning of the period the lender issues N agriculture loans to the farmers each with amount l. The periodic interest rate on the loan is ρ. The loan repayment rate at the end of the period is ϵ. Since the income of the farmers depends on a certain weather event, ϵ also depends on that weather event. Similar to the previous model, X denotes the value of the underlying weather index, then $\epsilon : X \to [0,1]$. Then the income of the lender at the end of the period is $Nl(1+\rho)\epsilon$. We assume that the lender purchases an index insurance which is based on the underlying weather event to cover the risk of defaulting loans. $I_l(X)$ denotes the indemnity function of the lender's index insurance. Further $0 \leq I_l(X) \leq M_l$, where M_l is the maximum payout of the index insurance. The lender pays the premium of the index insurance π_l at the beginning of the period and it is given by

$$\pi_l = \frac{m_l}{1+\delta}\int_{x_{min}}^{x_{max}} I_l(x)f(x)dx,$$

where m_l is the premium load of the index insurance and δ is the periodic interest rate. By assuming that the lender is risk averse and expected utility maximizer with initial wealth w_0 we focus on solving the following optimization problem.

$$\sup_{I_l} \; E[u_l((w_0-\pi_l)(1+\delta)+Nl(1+\rho)\epsilon+I_l(X))]$$

$$s.t \quad \pi_l = \frac{m_l}{1+\delta}\int_{x_{min}}^{x_{max}} I_l(x)f(x)dx, \tag{4.31}$$

where u_l is strictly concave and non decreasing function with $u_l'(.) \geq 0$, $u_l''(.) < 0$ and $u_l''(.)$ is a continuous function. In order find the optimal index insurance for the lender we solve problem (4.31) in a similar way as we derived the optimal index insurance for the farmers. Remember that, that procedure is adopted from Zhang et al. (2018).

Now we introduce the Lagrange multiplier λ in order to solve the above optimization problem.

$$\begin{aligned}
&K(I_l,\lambda)\\
&=E[u_l((w_0-\pi_l)(1+\delta)+Nl(1+\rho)\epsilon+I_l(X))]+\lambda\Big(\pi_l-\frac{m_l}{1+\delta}\int_{x_{min}}^{x_{max}} I_l(x)f(x)dx\Big)\\
&=E[u_l((w_0-\pi_l)(1+\delta)+Nl(1+\rho)\epsilon+I_l(X))]+\lambda\int_{x_{min}}^{x_{max}}\Big(\pi_l-\frac{m_l}{1+\delta}I_l(x)\Big)f(x)dx\\
&=\int_{x_{min}}^{x_{max}} u_l((w_0-\pi_l)(1+\delta)+Nl(1+\rho)\epsilon+I_l(x))f(x)dx+\int_{x_{min}}^{x_{max}}\lambda(\pi_l-\frac{m_l}{1+\delta}I_l(x))f(x)dx\\
&=\int_{x_{min}}^{x_{max}}\Big(u_l((w_0-\pi_l)(1+\delta)+Nl(1+\rho)\epsilon+I_l(x))+\lambda(\pi_l-\frac{m_l}{1+\delta}I_l(x))\Big)f(x)dx.
\end{aligned} \tag{4.32}$$

According to Equation (4.32) it is sufficient to consider the pointwise maximization of the integrand.

$$H_l(I_l(x), x, \lambda) = u_l((w_0 - \pi_l)(1+\delta) + Nl(1+\rho)\epsilon + I_l(x)) + \lambda(\pi_l - \frac{m_l}{1+\delta} I_l(x)). \tag{4.33}$$

The derivative of $H_l(I_l(x), x, \lambda)$ with respect to $I_l(x)$ is given by

$$\dot{H}_l(I_l(x), x, \lambda) = G_l(I_l(x), x) - \lambda \frac{m_l}{1+\delta}, \tag{4.34}$$

where

$$G_l(i, x) = u_l'((w_0 - \pi_l)(1+\delta) + Nl(1+\rho)\epsilon + i). \tag{4.35}$$

It is clear that $G_l(i, x)$ is strictly decreasing in i for a fixed x due to the strict concavity of u_l'. Similar to the previous model we can define three disjoint sets based on $G_l(0, x)$ and $G_l(M_l, x)$. Then we can discuss the nature of optimal indemnity in those three sets similar to the previous model. Proposition 4.1 and Lemma 4.2 which discuss the uniqueness and the existence of the optimal solution of Problem (4.2) are applicable for this model. Also Lemma 4.3 regarding extension of the optimal solution to the interval $[x_{min}, x_{max}]$ and Proposition 4.4 regarding non emptiness of $S_3^{\lambda^*}$ are true for this model. Therefore we can solve problem (4.31) by the ODE approach as given in the following theorem.

Theorem 4.14. *Let the function* $\hat{I} : [x_{min}, x_{max}] \mapsto \mathbb{R}$ *solves the following ODE problem:*

$$\frac{dI}{dx} = -Nl(1+\rho)\frac{d\epsilon}{dx} \tag{4.36}$$

$$\pi_l = \frac{m_l}{1+\delta}\mathbb{E}[(\hat{I}(x) \vee 0) \wedge M_l].$$

Then $I^* = (\hat{I}(x) \vee 0) \wedge M_l$ *is the optimal solution to problem (4.31).*

The solution to the ODE problem in Equation (4.36) is given by

$$\hat{I}(x) = -Nl(1+\rho)\epsilon + c, \tag{4.37}$$

where c is a constant which satisfies $\mathbb{E}[I^*(X)] = \frac{1+\delta}{m_l}\pi_l$.

4.9.1 Rate of Loan Repayment

According to the optimal solution it is clear that the loan repayment rate is one of the important ingredients to derive the optimal solution. Here we assume that the farmer repays the outstanding loan if his income is greater than the outstanding loan. By assuming that the number of borrowers is sufficiently large to apply the law of large numbers, the loan repayment rate is defined as the probability that the farmer's income is greater than the outstanding loan. More specifically, here we consider that the income of a farmer is similar to the special case of farmer's income in the previous model, that is $Z = \mu g_h XY$. Then by applying law of large numbers across the idiosyncratic shocks Y the rate of loan repayment at a fixed value x is given by

$$\epsilon(x) = P(\mu g_h xy > l(1+\rho)) = P\Big(y > \frac{l(1+\rho)}{\mu g_h x}\Big) = 1 - G\Big(\frac{l(1+\rho)}{\mu g_h x}\Big),$$

where G is the cumulative distribution of Y.

4.9.2 Numerical Computation of Optimal Indemnity for the Lender

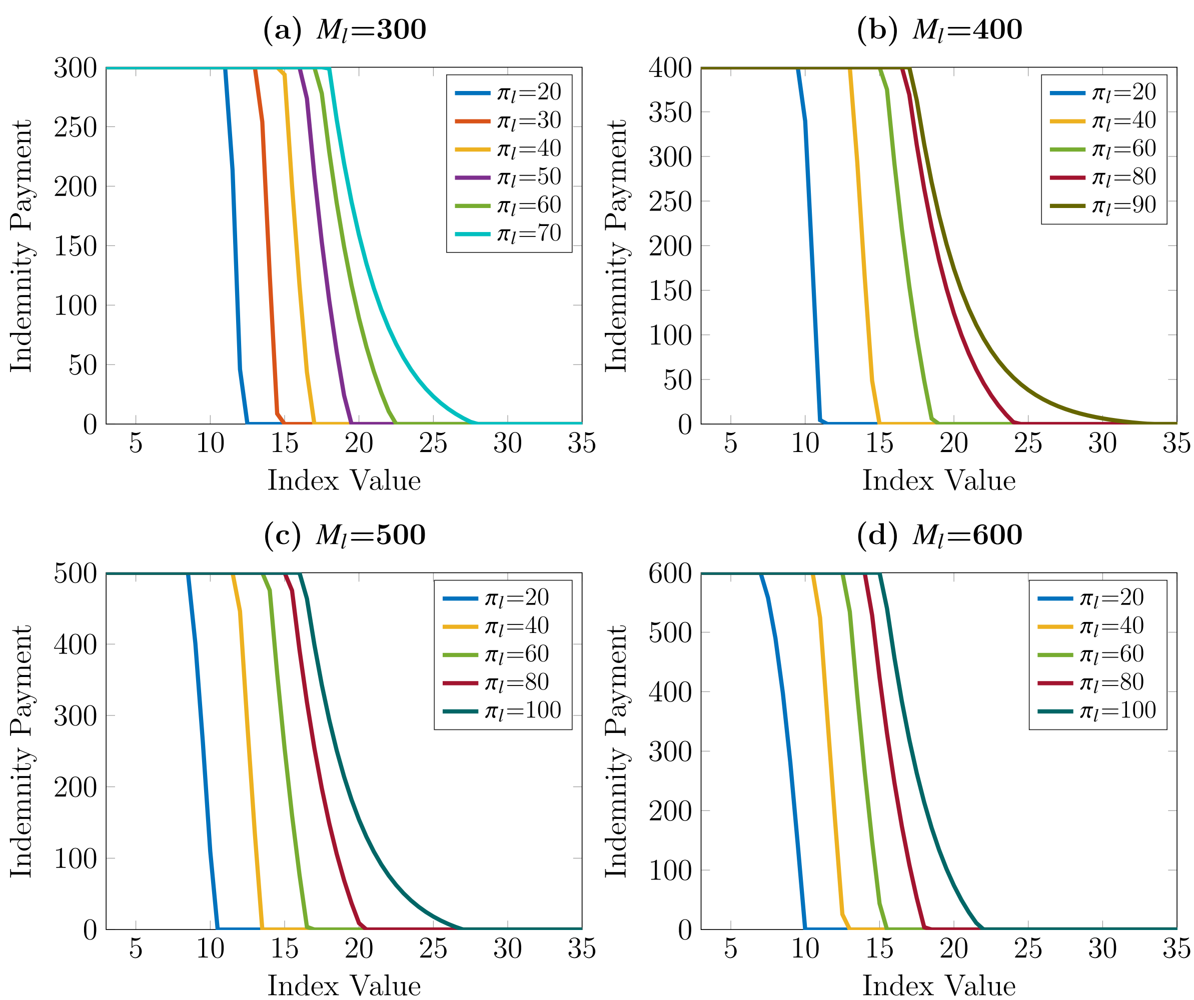

Figure 4.8: Optimal Indemnity of Lender for $\rho = 0.15$

We numerically compute the optimal indemnity of the lender level index insurance for different values of M_l and π_l. Let $N = 100$ and $l = 25$. The numerical results are given in Figure 4.8.

In order to compute the optimal indemnity, for different levels of M_l we consider different levels of premiums. When $M_l = 300$, we choose the premiums from 20 to 70 but when $M_l = 500$, we choose premiums up to 100. When $M_l = 300$ we cannot consider the high premium levels like 100 as the premium of the index insurance. Because in this case maximum payout is not large enough to impose high premium. As shown in Figure 4.8 in all four cases for all premiums the optimal indemnity is first equal to the maximum payout M_l and then it decreases to zero as the index value increases and finally no payout is provided for the rest of the index values. By paying attention to the decreasing part it is clear that for small levels of premiums it decreases linearly (approximately) and as premium increases the decreasing part becomes non linear. Therefore we can suggest that a linear indemnity is not always the best indemnity function for the lender level index insurance.

4.9.3 Wealth of the Lender

Here we discuss the wealth of the lender for the cases where the borrowers take index insurance and the lender takes index insurance. When the borrowers take index insurance it may increase the loan repayment rate and as a result of that, the lender's income may be increased. On the other hand when the lender takes an index insurance even the loan repayment rate is lower but the lender's wealth may be increased due to the insurance payment. In order to analyze these situations we compare the expected utility of the lender's wealth under the following four cases.

1. The borrowers insured ($EU1$).
2. The lender insured ($EU2$).
3. The both borrowers and lender insured ($EU3$).
4. Neither borrowers no lenders insured ($EU0$).

Let I_b and I_l are the optimal indemnity functions of the index insurance of the borrowers and the lender respectively. ϵ_1 and ϵ_2 are the loan repayment rates when the borrowers are insured and uninsured, respectively. A borrower pays the premium π_b and the lender pays the premium π_l for the corresponding index insurance. Where

$$\epsilon_1(x) = P(\mu g_h x y + I_b(x) > l(1 + \rho))$$

and

$$\epsilon_2(x) = P(\mu g_h xy > l(1+\rho)).$$

Then the expected utility of the lender's wealth under above four cases can be given as

$$\begin{aligned}
EU1 &= \mathbb{E}[u(w_0(1+\delta) + Nl(1+\rho)\epsilon_1(X))],\\
EU2 &= \mathbb{E}[u((w_0-\pi_l)(1+\delta) + Nl(1+\rho)\epsilon_2(X) + I_l(X))],\\
EU3 &= \mathbb{E}[u((w_0-\pi_l)(1+\delta) + Nl(1+\rho)\epsilon_1(X) + I_l(X))],\\
EU0 &= \mathbb{E}[u(w_0(1+\delta) + Nl(1+\rho)\epsilon_2(X))].
\end{aligned}$$

The expected utility of the lender's wealth for all four cases for different interest rates on the loan are given in Figure 4.9.

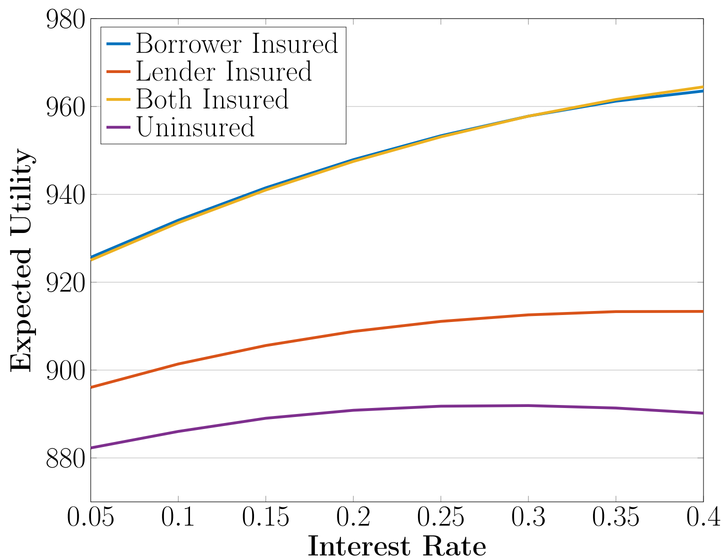

Figure 4.9: Expected Utility of the Lender's Wealth

According to Figure 4.9 it is clear that the expected utility of the lender's wealth is higher when the borrower or lender have an index insurance compared to the case when both are uninsured. Also the expected utility is higher when the borrower insured than the lender insured. Another important results is the expected utility in the cases, where only the borrower is insured and where the both are insured, are very close to each other. This reveals that if the lender can bundle the loan with a mandatory index insurance for the borrower, then the lender does not need to take an index insurance, because the lender's insurance does not help to increase the expected utility further.

4.10 Summary

The main insights of this chapter are:

- An index insurance with optimal payout function has a lower basis risk than the index insurance with a linear payout function.
- It is important to choose the proper level of premium and maximum payout for the index insurance to reach small basis risk.
- A lower loan default probability can be obtained when the agriculture loan is interlinked with index insurance with optimal payout function instead of a linear payout function.
- An optimal index insurance for a lender who issues agriculture loans helps to increase the lender's wealth.
- But when the loans are interlinked with index insurance, index insurance for the lender is not needed.

Chapter 5

Equilibrium Pricing Model for Index Insurance

There are many studies about different aspects of index insurance for agriculture risk. But many of those works only consider either the side of the insurance buyer or the insured. There are several studies evaluating both the demand and supply of the insurance market in an equilibrium framework (Lin (2005), Rothschild and Stiglitz (1978)). Also, there are some works on market equilibrium of traditional crop insurance (Aase (1999), Duncan and Myers (2000)). But only a few studies discuss the index insurance in an equilibrium framework. Shen and Odening (2013) discuss the market equilibrium by considering both demand and supply of an area based index insurance. Duncan and Myers (2000) discuss the existence of the crop insurance market equilibrium. Their model is similar to the model in Rothschild and Stiglitz (1978). They consider risk averse farmers and risk averse insurance firms and the demand and supply are characterized by mean variance preference functions.

Shen and Odening (2013) analyze the feasibility of an area based index insurance by means of the equilibrium model. In addition to that they analyze two strategies to manage the systemic risk of the area based crop insurance, where systemic risk occurs when some natural disaster affects a large number of farmers simultaneously. Those two strategies are first by regional diversification and second by catastrophe (CAT) bonds. They develop an equilibrium pricing model based on the model in Duncan and Myers (2000) for a single region and later extend it to multiple regions. Risk averse farmers and an insurer are considered and the effects of basis risk and systemic risk are investigated. They also address premium subsidies to increase the participation for the index insurance.

Motivated by Shen and Odening (2013), we investigate both demand and supply sides of a weather based index insurance by means of an equilibrium pricing model. We also consider the case where the index insurance is interlinked with an agriculture loan. Then we investigate the market equilibrium in that model. In addition to that we build up an equilibrium model by considering two groups of farmers. This is motivated by the idea behind the model in Sass and Seifried (2014).

5.1 Basic Equilibrium Pricing Model

We develop an equilibrium pricing model by considering both demand and supply sides of a weather index insurance. The model is related to the model in Shen and Odening (2013).

5.1.1 Demand Side of the Market: Insured

In addition to the characteristics of the model in Shen and Odening (2013) we also adopt some characteristics of the weather index insurance model in Clarke (2016). We consider a region with N farmers. We assume that a weather index insurance is available for the farmers to cover their agriculture risk. Since all the farmers are in the same region the same weather event affects all of them similarly. Therefore, the same weather index insurance is available for everyone. Then the revenue of farmer $i \in \{1, 2, \ldots, N\}$ is given by

$$R_i = W_i + \alpha_i P(I) + (w_i - \alpha_i \pi)(1 + \delta), \tag{5.1}$$

where W_i is the income from yield of farmer i. I is the weather index and $P(I)$ is the indemnity payment or the payout of the full cover index insurance, which depends on the weather index. π is the premium of the full cover index insurance. Further, α_i is the index insurance cover or the amount of insurance which is purchased by farmer i and w_i is the initial wealth of farmer i after spent money for farming, where $w_i \geq \pi$ by assuming the farmer's initial wealth is large enough to pay the insurance premium. Finally, δ is the periodic interest rate.

Even though the same weather event affects all the farmers, the other agriculture risks are different for each. As a result of that the relationship between the actual loss and the index insurance payout is also different for each. Then the farmers choose different amounts of the index insurance according to their situation.

Based on Clarke (2016), for the simplicity of the model it is assumed that there are only two possible income levels and only two possible weather conditions. Those are

$$W_i = \begin{cases} w_l & \text{due to low yield with probability } p_i \\ w_h & \text{due to high yield with probability } 1 - p_i, \end{cases}$$

for all $i \in \{1, 2, \ldots, N\}$ and

$$I = \begin{cases} 1 & \text{bad weather with probability } q \\ 0 & \text{good weather with probability } 1 - q, \end{cases}$$

Let $l = w_h - w_l$ and we refer l as the loss of farming. Then the indemnity payment is $P(I) = lI$. r_i is the probability that the loss occurs to the farmer i but the index indicates good weather. We consider it as the basis risk of the index insurance. The four possible levels of farmer i's revenue with the corresponding probabilities are given in Table 5.1.

Table 5.1: Revenue of four state

State s	Probability p_s	Revenue
$l0$	r_i	$w_l + (w_i - \alpha_i \pi)(1 + \delta)$
$l1$	$p_i - r_i$	$w_l + (w_i - \alpha_i \pi)(1 + \delta) + \alpha_i l$
$h0$	$1 - q - r_i$	$w_h + (w_i - \alpha_i \pi)(1 + \delta)$
$h1$	$q + r_i - p_i$	$w_h + (w_i - \alpha_i \pi)(1 + \delta) + \alpha_i l$

We assume that the farmers are strictly risk averse over wealth, with utility function u_d satisfying $u_d' > 0$ and $u_d'' < 0$. As an expected utility maximizer farmer i wants to choose the amount of insurance which maximizes his expected utility. Then the optimal insurance cover of farmer i at a given premium π is given by

$$\alpha_i^* = \underset{\alpha_i \geq 0}{\operatorname{argmax}}\, \mathbb{E}[u_d(W_i + \alpha_i P(I) + (w_i - \alpha_i \pi)(1 + \delta))]. \tag{5.2}$$

Since there are N farmers, the total optimal demand or the total optimal amount of insurance which is purchased by all the farmers at price π is $\alpha^* = \sum_{i=1}^{N} \alpha_i^*$.

5.1.2 Supply Side of the Market: Insurer

In order to discuss the market equilibrium, we need to consider the side of the insurer or the insurance provider. We assume that there is only one insurance provider. We also assume that the insurer is strictly risk averse with utility function u_s satisfying $u_s' > 0$ and $u_s'' < 0$. We discuss the behavior of the insurer in an market equilibrium framework under two methods. Those two methods use two different concepts to find the market equilibrium. We explain those methods below.

Method 1: The Method based on the Amount of Contracts

This method is based on the method of computing an equilibrium in Shen and Odening (2013). In this method the insurance company decides the number of contracts that they want to sell at a given price. If the insurer sells the number β of insurance contacts each at premium π, then the profit of the insurer is given by

$$\beta\pi(1+\delta) - \beta l I. \tag{5.3}$$

Now the optimal amount of the insurance contracts which maximizes the expected utility of the profit of the insurer is given by

$$\beta^* = \operatorname*{argmax}_{\beta \geq 0} \mathbb{E}[u_s(\beta\pi(1+\delta) - \beta l I)]. \tag{5.4}$$

Now we can derive the market equilibrium under Method 1. This is similar to computing the equilibrium from demand and supply curves. In this method α^* as a function of π behaves as the demand curve of the insurance market and β^* as a function of π behaves as the supply curve of the insurance market. The insurance market is at an equilibrium when the demand is equal to the supply. That is

$$\sum_{i=1}^{N} \alpha_i^* = \beta^*. \tag{5.5}$$

The equilibrium premium π^* is the premium which satisfies Equation (5.5) and the equilibrium demand is α^* (or β^*) at π^*.

Method 2: The Method based on the Premium

This method is based on the method of computing the market equilibrium in Sass and Seifried (2014). Here the insurer decides the premium which maximizes its expected utility of profit if it supplies the optimal demand of the insureds. This is in contrast to Method 1, because here the insurer does not decide the amount it is willing to supply. Instead it supplies the amount which is asked by the farmers and the insurer decides the premium. If the insurer supplies the optimal demand of the farmers α^* at premium π then the profit of the insurer is given by

$$\alpha^*\pi(1+\delta) - \alpha^* l I. \tag{5.6}$$

The premium which maximizes the expected utility of the profit of the insurer is given by

$$\pi^* = \operatorname*{argmax}_{\pi} \mathbb{E}[u_s(\alpha^*\pi(1+\delta) - \alpha^* lI)]. \tag{5.7}$$

Here α^* is a function of π. This π^* is the equilibrium premium and α^* at π^* is the equilibrium demand.

5.1.3 Insureds and Insurer with Exponential Utility

Now we assume that both u_d and u_s are exponential utility functions in order to derive the market equilibrium.

Proposition 5.1. *If the insured's utility function u_d is an exponential utility function with coefficient of risk aversion $\gamma_d > 0$, the optimal insurance demand of farmer i at premium π is given by*

$$\alpha_i^* = max\left[0, \frac{1}{\gamma_d l} ln\left(\frac{(l-\pi(1+\delta))}{\pi(1+\delta)} \frac{((p_i - r_i)e^{-\gamma_d w_l} + (q + r_i - p_i)e^{-\gamma_d w_h})}{(r_i e^{-\gamma_d w_l} + (1 - q - r_i)e^{-\gamma_d w_h})}\right)\right]. \tag{5.8}$$

Equilibrium by Method 1

Here we derive the equilibrium demand/supply and premium under Method 1.

Proposition 5.2. *If the insurer's utility function u_s is an exponential utility function with coefficient of risk aversion $\gamma_s > 0$, under Method 1 the optimal amount of insurance contracts the insurer is willing to supply at premium π is given by*

$$\beta^* = \frac{1}{\gamma_s l} \ln\left(\frac{(1-q)\pi(1+\delta)}{q(l-\pi(1+\delta))}\right). \tag{5.9}$$

Now we know the optimal demand and supply at a given premium. Then by market equilibrium condition in Equation (5.5) and by Equation (5.9) we derive the equilibrium premium as a function of α^*. It is given by

$$\pi^* = \frac{q l e^{\gamma_s l \alpha^*(\pi^*)}}{(1+\delta)(q e^{\gamma_s l \alpha^*(\pi^*)} + 1 - q)}. \tag{5.10}$$

Proposition 5.3. *If the insureds' and insurer's utility functions u_d and u_s are exponential utility functions with coefficients of risk aversion $\gamma_d > 0$ and $\gamma_s > 0$ respectively, Under Method 1* *equilibrium premium and demand are given by*

$$\pi^* = \frac{l(\prod_{i(\alpha_i>0)} A_i)^{(\frac{\gamma_s}{\gamma_d+N^*\gamma_s})}(\frac{q}{1-q})^{(\frac{\gamma_d}{\gamma_d+N^*\gamma_s})}}{(1+\delta)(1+(\prod_{i(\alpha_i>0)} A_i)^{(\frac{\gamma_s}{\gamma_d+N^*\gamma_s})}(\frac{q}{1-q})^{(\frac{\gamma_d}{\gamma_d+N^*\gamma_s})})}, \tag{5.11}$$

and

$$\alpha^* = \frac{1}{l(\gamma_d + N^*\gamma_s)} \ln\left(\left(\prod_{i(\alpha_i>0)} A_i\right)\left(\frac{1-q}{q}\right)^{N^*}\right) \tag{5.12}$$

respectively, where $A_i = \frac{(p_i-r_i)e^{-\gamma_d w_l}+(q+r_i-p_i)e^{-\gamma_d w_h}}{r_i e^{-\gamma_d w_l}+(1-q-r_i)e^{-\gamma_d w_h}}$ *and* $N^*(\leq N)$ *is the number of farmers who purchase positive amount of insurance at* π^*.

Equilibrium by Method 2

Now we derive the equilibrium price under Method 2.

Proposition 5.4. *If the insurer's utility function* u_s *is an exponential utility function with coefficient of risk aversion* $\gamma_s > 0$*, under Method 2 the equilibrium premium is given by*

$$\pi^* = \frac{qle^{\gamma_s l\alpha^*(\pi^*)}}{(1+\delta)(qe^{\gamma_s l\alpha^*(\pi^*)}+1-q)} - \frac{\alpha^*(\pi^*)}{(\alpha^*(\pi^*))'}. \tag{5.13}$$

Here $\alpha^*(\pi^*)$ *is the value of* α^* *at* π^* *and* $(\alpha^*(\pi^*))'$ *is the value of the derivative of* α^* *with respect to* π *at* π^*.

Here, we can not derive the equilibrium premium and demand as closed form expressions. But numerically, we can compute the equilibrium premium and demand by Equations (5.7) and (5.8). By comparing Equation (5.10) and (5.13), it is clear that the equilibrium premiums of the two methods are different to each other. We explain the relationship between market equilibrium under the two methods in Proposition 5.5.

Proposition 5.5. *Let us assume that preference on wealth for all the farmers is given by exponential utility with coefficient of risk aversion* γ_d *and the preference on wealth of the insured is given by exponential utility with coefficient of risk aversion* γ_s*. Under Method 1 the equilibrium premium and demand are* π_1^* *and* α_1^* *respectively. Under Method 2 the equilibrium premium and demand are* π_2^* *and* α_2^* *respectively. If* $\alpha_1^* > 0$ *and* $\alpha_2^* > 0$ *then* $\pi_2^* > \pi_1^*$ *and then* $\alpha_2^* < \alpha_1^*$.

Proof. Let $\pi_1^* = \pi_2^*$. That is $\alpha_1^* = \alpha_2^*$. Then by Equation (5.10) and (5.13), we can show that $\alpha_2^*/(\alpha_2^*)' = 0$. Since $\alpha_2^* > 0$ and $(\alpha_2^*)' < 0$, $\alpha_2^*/(\alpha_2^*)' < 0$. This is a contradiction. Therefore, $\pi_1^* \neq \pi_2^*$.

Now let $\pi_1^* > \pi_2^*$. Since the demand increases as premium decreases, we have $\alpha_1^* < \alpha_2^*$. Then by Equation (5.10) and (5.13)

$$\frac{qle^{\gamma_s l\alpha_1^*}}{(1+\delta)(qe^{\gamma_s l\alpha_1^*}+1-q)} > \frac{qle^{\gamma_s l\alpha_2^*}}{(1+\delta)(qe^{\gamma_s l\alpha_2^*}+1-q)} - \frac{\alpha_2^*}{(\alpha_2^*)'}.$$

Since $\frac{\alpha_2^*}{(\alpha_2^*)'} < 0$

$$\frac{qle^{\gamma_s l\alpha_1^*}}{(1+\delta)(qe^{\gamma_s l\alpha_1^*}+1-q)} > \frac{qle^{\gamma_s l\alpha_2^*}}{(1+\delta)(qe^{\gamma_s l\alpha_2^*}+1-q)}$$

$$qe^{\gamma_s l(\alpha_1^*+\alpha_2^*)} + (1-q)e^{\gamma_s l\alpha_1^*} > qe^{\gamma_s l(\alpha_1^*+\alpha_2^*)} + (1-q)e^{\gamma_s l\alpha_2^*}$$

$$e^{\gamma_s l\alpha_1^*} > e^{\gamma_s l\alpha_2^*}$$

$$\alpha_1^* > \alpha_2^*.$$

This contradict the fact that demand increases as premium decreases. Therefore, $\pi_2^* > \pi_1^*$. □

5.1.4 Insured with Power Utility and Insurer with Exponential Utility

Since all possible wealth levels of the farmer is non-negative it is possible to assume that the insured's preference on his wealth is given by the power utility function with $u_d'(w) = w^{-\gamma_d}$, $\gamma_d > 0$. Then the first order condition of the expected utility of the farmer i's revenue with respect to α_i is given by

$$\begin{aligned}
&- r_i\pi(1+\delta)(w_l + (w_i - \alpha_i\pi)(1+\delta))^{-\gamma_d} \\
&+ (p_i - r_i)(l - \pi(1+\delta))(w_l + (w_i - \alpha_i\pi)(1+\delta) + \alpha_i l)^{-\gamma_d} \\
&- (1 - q - r_i)\pi(1+\delta)(w_h + (w_i - \alpha_i\pi)(1+\delta))^{-\gamma_d} \\
&+ (q + r_i - p_i)(l - \pi(1+\delta))(w_h + (w_i - \alpha_i\pi)(1+\delta) + \alpha_i l)^{-\gamma_d} = 0. \qquad (5.14)
\end{aligned}$$

From this first order condition we can not derive the explicit form of the optimal α_i. But it is still possible to solve the first order condition numerically. The utility function of the insurer is unchanged. Therefore, the insurer still has exponential utility function. The power utility function is not a good choice to represent the preference of the insurer. Because according to the model the profit of the insurer can be negative. But only positive values can be used for the power utility function.

5.1.5 Numerical Results of the Basic Model

Table 5.2: Insurance Demand

r_i	Exponential Utility			Power Utility		
	α_i^* at Fair Premium	α_i^* at Equilibrium Premium Method 1	Method 2	α_i^* at Fair Premium	α_i^* at Equilibrium Premium Method 1	Method 2
1/1000	0.9763	0.2426	0.2107	0.9695	0.1239	0.1127
1/750	0.9743	0.2406	0.2087	0.9678	0.1225	0.1113
1/500	0.9702	0.2365	0.2047	0.9645	0.1196	0.1083
1/250	0.9582	0.2245	0.1926	0.9545	0.1108	0.0996
1/100	0.9230	0.1893	0.1575	0.9249	0.0847	0.0735
1/75	0.9040	0.1703	0.1385	0.9087	0.0702	0.0591
1/50	0.8671	0.1334	0.1016	0.8764	0.0415	0.0304
1/25	0.7633	0.0296	0	0.7820	0	0
1/10	0.4899	0	0	0.5114	0	0
1/5	0.0617	0	0	0.0632	0	0
α^*	78.8803	14.6690	12.1430	79.2294	6.3778	5.9520
π^*	0.3236	0.6645	0.6777	0.3236	0.4808	0.483

We consider a setting with only 10 possible levels of basis risk, r_i. For each level of the basis risk there are ten farmers who face that basis risk. Then there are 100 farmers in the region. Let $p_i = 0.32$ for all $i \in \{1, 2, \ldots, 100\}$, $q = 1/3$, $l = 1$, $\gamma_d = 2$ and $\gamma_s = 0.1$. We compute the optimal amount of the index insurance which is purchased by a farmer in each basis risk group at actuarial fair premium and the equilibrium prices under both methods. We compute those results for both exponential and power utility functions. Those results are given in Table 5.2. The last two rows of the table show the total optimal/equilibrium demand and actuarial fair/equilibrium premium respectively.

According to the results in Table 5.2 it is clear that the the equilibrium price is greater than the corresponding actuarial fair premium for all the cases. As a result of that the demand at actuarial fair price is always greater than the demand at equilibrium premium. At any given price and given utility function the demand decreases as the basis risk increases. As given in Proposition 5.5, the equilibrium premium under Method 2 is greater than the equilibrium premium under Method 1 for both utility functions. As a results of this there is a higher demand at market equilibrium under Method 2 compared to Method 1. These results verify that the two methods refer to two different market equilibriums.

Participation Ratio and Premium Subsidy

Motivated by Shen and Odening (2013) we analyze the participation ratio and premium subsidy in this equilibrium framework. First we discuss about the participation ratio (PR). The participation ratio is the ratio of farmers who purchase positive amount of insurance (N^*/N) at the equilibrium premium. The analysis is done numerically. We obtain the results only by considering a group of farmers whose preference on wealth is given by exponential utility function. We consider eight different cases with different combinations of γ_d and γ_s. For each case we compute the equilibrium demand and the equilibrium premium under both methods. Also we count the number of farmers who take some positive amount of the index insurance in order to find the participation ratio. Those results are given in Table 5.3.

Table 5.3: Equilibrium Premium, Demand and Participation Ratio

Case	γ_1	γ_2	Method 1			Method 2		
			α^*	π^*	PR	α^*	π^*	PR
Case 1	1	0.05	14.7662	0.4964	0.80	12.3903	0.5036	0.80
Case 2	1	0.1	8.2436	0.5170	0.70	7.3453	0.5201	0.70
Case 3	1	0.5	1.8426	0.5405	0.50	1.8008	0.5407	0.50
Case 4	1	1	0.9405	0.5454	0.40	0.9070	0.5456	0.40
Case 5	2	0.05	24.4556	0.61105	0.80	18.6054	0.6435	0.80
Case 6	2	0.1	14.6690	0.6645	0.80	12.1430	0.6777	0.70
Case 7	2	0.5	3.5680	0.7266	0.60	3.3822	0.7277	0.60
Case 8	2	1.5	1.2540	0.7433	0.40	1.2316	0.7435	0.40

The equilibrium premium and demand are different under two methods for all the cases. But the values are close to each other. In both methods the highest equilibrium premium is recorded in Case 8, it is a case with high risk averse insurer and insureds. The highest equilibrium demand is recorded in Case 4, where we have high risk averse insureds and an insurer with low risk aversion.

The participation ratio of a given case under Method 1 is equal to the participation ration of the corresponding case under Method 2 except Case 6. It is clear that under different cases there are different participation ratios. Those are from 0.4 to 0.8. We can clearly see that the cases with high equilibrium demand have a high participation ratio. In those cases the demand is high as a result of most of the farmers purchasing some amount of the insurance. The participation in Case 8 is low, because the premium may be too high for some farmers and they do not purchase the index insurance.

If the premium is low, then there is a possibility that more farmers will purchase the insurance. Also, the farmers who would already purchase the insurance at the equilibrium premium will purchase more. The premium can be reduced by giving a subsidy on the equilibrium premium. If the subsidy is provided by a third party like the government then not only the farmers but also the insurer benefit from this. The subsidy as a proportion of the premium can be explained as follows. For an example let us consider that the subsidy rate is 0.15 of the equilibrium premium π^*, then the insureds should pay only (1-0.15)$\times\pi^*$. The different subsidy rates on the equilibrium premium increase the participation ratio to the different levels. The subsidy rates which are required to increase the participation ratio to the different levels are given by Figure 5.1 for some selected cases.

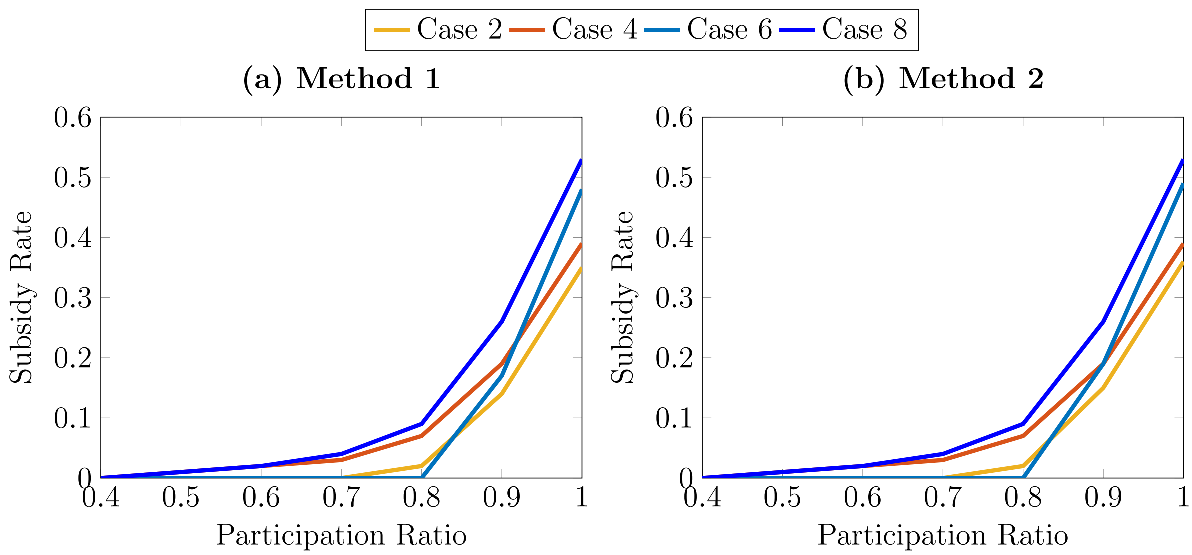

Figure 5.1: Subsidy Rate vs Participation Ratio.

Both figures look similar to each other, because the required subsidy rates under the two methods are equal to each other for most of the participation ratios and only few are different but very close to each other. When comparing two cases with same γ_d and different γ_s (e.g Case 2 and Case 4 or Case 6 and Case 8), it is clear that the case with high γ_s requires a higher subsidy rate for a given level of participation ratio compared to the case with low γ_s. But when considering Case 2 and Case 6, those are two cases with same γ_s and different γ_d, Case 2 requires a higher subsidy rate than Case 6 for participation ratio 0.8, but Case 2 requires a lower subsidy rate than Case 6 for participation ratios 0.9 and 1.

The same participation ratio under two cases does not mean that same happen under both cases. For an example according to the results in Table 5.3 under Method 1 the participation ratio is 0.8 for Case 5 and 6. But the equilibrium demands are considerably different. Therefore, it is important to see how subsidized premiums affect the total demand of the insurance. The total demand at the subsidized premiums to reach participation ratios 0.9 and 1 is given in Table 5.4. The required subsidy rates are also given in the columns labeled as SR. We see that when the participation ratio is 0.9 or 1, in all the cases, the farmers purchase approximately equal amount of insurance.

Table 5.4: Subsidy Rate and Insurance Demand

Case	Method 1				Method 2			
	PR=0.9		PR=1		PR=0.9		PR=1	
	SR	α^*	SR	α^*	SR	α^*	SR	α^*
Case 1	0.1	32.9617	0.32	73.3885	0.12	32.4490	0.33	73.4521
Case 2	0.14	31.9066	0.35	74.0713	0.15	32.8537	0.36	75.5245
Case 3	0.18	32.4333	0.38	74.4994	0.18	32.3720	0.38	74.4429
Case 4	0.19	32.9700	0.39	75.6023	0.19	32.9700	0.39	75.6023
Case 5	0.1	35.0403	0.43	68.3532	0.14	34.3857	0.46	73.4443
Case 6	0.17	34.7401	0.48	73.8818	0.19	35.2305	0.49	73.8623
Case 7	0.23	34.6114	0.52	73.1581	0.24	35.4534	0.52	73.0400
Case 8	0.26	35.0220	0.53	73.0277	0.26	34.9941	0.53	73.0067

5.2 Equilibrium Pricing Model for Interlinked Loan and Index Insurance

We have discussed an interlinked loan and index insurance in our previous chapters from different perspectives. Here we discuss an interlinked loan and index insurance in a market equilibrium framework. Here we assume that the agriculture loan is interlinked with an index insurance, that is when a farmer takes a loan he should purchase an index insurance. At the end of the period the farmer should repay the loan with interest. We consider that, if the farmer's wealth is not enough to repay the loan then he should give whatever amount he has to the lender. Then the amount he repaid is less than the outstanding loan balance. In that case his remaining wealth become 0. We consider two possible situations where farmers want to take loans. Those are

1. Loan for farming: Here we assume that the farmer does not have enough money to go for farming. Therefore, he takes a loan to go for farming.

2. Loan for high technology for farming: Here the farmer has enough money to go for farming the normal way. But by taking a loan he can use high or modern technology for

farming to increase the yield.

5.2.1 Loan for Farming

We consider the same model as in Section 5.1. In addition to that by assuming that all N farmers take loans for farming, we connect an agriculture loan to that model. Here we assume that the farmer uses the full loan amount for farming and he pays the premium of the insurance out of his initial wealth. Section 5.1 assumes that the initial wealth is large enough to pay the premium of the full cover index insurance. It is true for this model too.

Remark 5.6. Since the initial wealth is large enough to pay the premium of the full cover index insurance, the remaining initial wealth can be used for farming. But we assume that the farmer takes a loan to cover the full amount required for farming. The rest of the initial wealth continues as a savings of the farmer.

Again we consider a group of N farmers. For simplicity of the model we assume that all the farmers take the same amount of loan and the same index insurance. Now the remaining wealth of the farmer $i \in \{1, 2, 3, \ldots, N\}$ is given by

$$R_i^c = \max\{0, W_i + \alpha_i P(I) + (w_i - \alpha_i \pi)(1 + \delta) - c(1 + \rho)\}, \tag{5.15}$$

where c is the loan/credit amount and ρ is the interest rate on the loan. All the other parameters are similar to the basic model. Table 5.5 shows the possible remaining wealth levels after loan repayment and the corresponding probabilities.

Table 5.5: Remaining Wealth of Four States-Loan for Farming

State s	Probability p_s	Remaining Wealth w_s
$l0$	r_i	$\max(0, w_l + (w_i - \alpha_i \pi)(1 + \delta) - c(1 + \rho))$
$l1$	$p_i - r_i$	$\max(0, w_l + (w_i - \alpha_i \pi)(1 + \delta) + \alpha_i l - c(1 + \rho))$
$h0$	$1 - q - r_i$	$\max(0, w_h + (w_i - \alpha_i \pi)(1 + \delta) - c(1 + \rho))$
$h1$	$q + r_i - p_i$	$\max(0, w_h + (w_i - \alpha_i \pi)(1 + \delta) + \alpha_i l - c(1 + \rho))$

The loan does not affect the insurer. Therefore the profit function of the insurer is the same as in the basic model. If we can make sure that always the wealth of the farmer is large enough to repay the outstanding loan ($W_i + \alpha_i I + (w_i - \alpha_i \pi)(1 + \delta) - c(1 + \rho) \geq 0$), then we can derive the equilibrium price and demand similar to the basic model. Otherwise the computation of equilibrium is more complicated than the basic model. Below we discuss a procedure to compute the equilibrium price and demand in this model.

In this model it is difficult to find the first order condition and then derive the optimal insurance demand. Because at a given π for some values of α_i, any given w_s can be 0 or positive. That is

$$w_{l0} > 0 \quad \text{if} \quad \alpha_i < \frac{w_l + w_i(1+\delta) - c(1+\rho)}{\pi(1+\delta)} = \alpha_{l0}, \quad w_{l1} > 0 \quad \text{if} \quad \alpha_i > \frac{c(1+\rho) - w_l - w_i(1+\delta)}{l - \pi(1+\delta)} = \alpha_{l1},$$
$$w_{h0} > 0 \quad \text{if} \quad \alpha_i < \frac{w_h + w_i(1+\delta) - c(1+\rho)}{\pi(1+\delta)} = \alpha_{h0}, \quad w_{h1} > 0 \quad \text{if} \quad \alpha_i > \frac{c(1+\rho) - w_h - w_i(1+\delta)}{l - \pi(1+\delta)} = \alpha_{h1}.$$

In the intervals based on $\alpha_{l0}, \alpha_{l1}, \alpha_{h0}$ and α_{h1} some w_s's are positive and some others 0. For example, in the interval from α_{l0} to α_{h0}, $w_{l0} = 0$ and $w_{h0} > 0$. Due to this it is difficult to state the expected utility of the remaining wealth. Therefore, in the different intervals we express the expected utility by identifying the positive w_s's of that interval. Then we build up the expected utility maximization problems for every possible interval and the optimal demand of each maximization problem is calculated. Out of those optimal α_i's the α_i which gives the maximum expected utility is the optimal demand of the original maximization problem of the farmer at the given π. This is the procedure we use to compute the equilibrium. The following remark discusses the conditions under which the farmers are always able to repay the full loan.

Remark 5.7. Since $l - \pi(1+\delta) > 0$, if $c < \frac{w_l + w_i(1+\delta)}{1+\rho}$ then $\alpha_{l0} > 0$. At a given π, if farmer i chooses $\alpha_i \in (0, \alpha_{l0})$, his revenue is always large enough to repay the total outstanding loan, because any given π and $\alpha_i \in (0, \alpha_{l0})$ satisfy all four inequalities above. If $c \geq \frac{w_l + w_i(1+\delta)}{1+\rho}$, then there is no insurance cover which satisfies all the above inequalities. Therefore, at any given insurance cover the farmer is unable to repay the full loan amount at one or more wealth states.

Exponential Utility

Let us assume that the farmer's preference on wealth is given by an exponential utility function with risk aversion parameter γ_d. If $c < \frac{w_l + w_i(1+\delta)}{1+\rho}$ then at premium π the insurance cover in the interval $[0, \alpha_{l0}]$ which maximizes the expected utility of farmer i's remaining wealth is given by

$$\begin{aligned}
\alpha_i^c &= \operatorname*{argmax}_{\alpha_i \in (0, \alpha_{l0})} \mathbb{E}[u_d(R_i^c)] \\
&= \operatorname*{argmax}_{\alpha_i \in (0, \alpha_{l0})} \mathbb{E}[u_d(W_i + \alpha_i P(I) + (w_i - \alpha_i \pi)(1+\delta) - c(1+\rho))] \\
&= \min\left[\alpha_{l0}, \max\left[0, \frac{1}{\gamma_d l} \ln\left(\frac{(l - \pi(1+\delta))}{\pi(1+\delta)} \frac{((p_i - r_i)e^{-\gamma_d w_l} + (q + r_i - p_i)e^{-\gamma_d w_h})}{(r_i e^{-\gamma_d w_l} + (1 - q - r_i)e^{-\gamma_d w_h})}\right)\right]\right].
\end{aligned}$$

Now consider the following two remarks which explain the conditions which leads to the same optimal demand and market equilibrium as the basic model.

Remark 5.8. If $c < \frac{w_l + w_i(1+\delta)}{1+\rho}$ and if $\alpha_i^c \in (0, \alpha_{l0})$ is the optimal cover of the whole interval [0,1] then it is the same as the optimal cover in the basic model in Equation (5.8), because due to the nature of the exponential utility function, $\alpha_i^c \in (0, \alpha_{l0})$ has no effect on the loan amount and the interest rate on the loan (they cancel out when deriving the optimal cover).

Remark 5.9. By Remarks 5.7 and 5.8 the optimal cover which satisfies all four inequalities has the same form as the optimal cover in the basic model. Also there is no effect on the profit functions of the insurer in Equations (5.3) and (5.6) from the interlinkage between the loan for farming and the index insurance. As a result of that if the equilibrium premium (π^*) and equilibrium demand of all the farmers ($\alpha_i^*(\pi^*)$ for all i) of this model satisfy all four inequalities above (all remaining wealth levels are positive), then that equilibrium demand and premium are the same as the equilibrium demand and the premium in the basic model. If at least one wealth level of at least one farmer is 0 at the equilibrium, then that equilibrium is different to the equilibrium in the basic model.

Power Utility

We can apply the power utility function to measure the preference only over positive values. Therefore it is problematic to use the power utility function to measure the farmer's preference over remaining wealth in this model. But in order to use the power utility we should ignore the conditions which leads to 0 wealth level. The optimal index insurance cover under the power utility changes due to any change of the wealth levels. Therefore, at a given level of the premium the optimal insurance demand with loan is always different to the optimal insurance demand without loan. Then the equilibrium of this model is always different to the equilibrium in the basic model. Since the optimal cover depends on the loan amount and interest rate on the loan, there are different equilibrium for different interest rates and loan amounts.

5.2.2 Loan for High Technology for Farming

The farmers use high technology for farming to increase the yield. This results in an increment of the income from farming. Now $\tilde{W}_i$ is the income of farmer i from farming under high technology. It also has two states given by

$$\tilde{W}_i = \begin{cases} \tilde{w}_l (= a w_l) & \text{due to low yield with probability } p_i \\ \tilde{w}_h (= a w_h) & \text{due to high yield with probability } 1 - p_i, \end{cases}$$

where $a > 1$. The loss of farming is denoted by $\tilde{l} = \tilde{w}_h - \tilde{w}_l = al$. Now the index insurance is purchased to fully or partially cover this loss. Since the loss under high technology is higher, probably the premium is higher. Now let $h(> w_i)$ denote the cost of the high technology. Still the farmer has the same initial wealth to pay the premium, but that amount may be not enough to pay the premium of an insurance which covers al fully. The amount which is required to pay the premium in addition to the initial wealth can be also covered by the loan. But there is a possibility that the initial wealth is large enough to pay the premium. In contrast to the previous model, here we assume that the farmer invests the remaining amount of his initial wealth for high technology. Then the farmer can go for a loan which is less than h. We consider it like this in order to impose a considerable difference between the two models. The total loan amount of farmer i is now given by $c_i = h + \alpha_i \pi - w_i$. So it can be different from farmer to farmer. Now, the remaining wealth of farmer i is given by

$$R_i = \max\{0, \tilde{W}_i + \alpha_i \tilde{P}(I) - (h + \alpha_i \pi - w_i)(1 + \rho)\}, \tag{5.16}$$

where $\tilde{P}(I) = \tilde{l} I$. Table 5.6 shows the possible remaining wealth levels after the loan repayment and the corresponding probabilities.

Table 5.6: Remaining Wealth of Four States-Loan for High Technology

State s	Probability p_s	Remaining Wealth w_s
$l0$	r_i	$\max(0, \tilde{w}_l - (h + \alpha_i \pi - w_i)(1 + \rho))$
$l1$	$p_i - r_i$	$\max(0, \tilde{w}_l + \alpha_i \tilde{l} - (h + \alpha_i \pi - w_i)(1 + \rho))$
$h0$	$1 - q - r_i$	$\max(0, \tilde{w}_h - (h + \alpha_i \pi - w_i)(1 + \rho))$
$h1$	$q + r_i - p_i$	$\max(0, \tilde{w}_h + \alpha_i \tilde{l} - (h + \alpha_i \pi - w_i)(1 + \rho))$

Under exponential utility 0 wealth is possible and then we can compute the equilibrium as explained in Section 5.2.1. But in contrast to the previous model, the optimal insurance cover of a farmer in this model depends on the loan and the interest rate of the loan, because the loan amount depends on the insurance premium. Therefore, the equilibriums under different interest rates on the loan are different to each other even though all the wealth levels are positive.

5.2.3 Numerical Results for the Model with Loan

Loan for Farming

For the numerical computations let $c = 1.25$ and all the other parameter values are similar to the basic model. At some interest rates on the loan and at any given insurance cover from 0 to 1, the farmer is able to repay the full loan amount under all the possible levels of his revenue. But at some other interest rates the farmer is able to repay the full loan amount under all the possible levels of his revenue at some insurance covers only. Therefore, we consider different interest rates for the calculations. Even if it is less realistic, we use very high interest rates as $\rho = 0.4$ and $\rho = 0.55$ under exponential utility for illustration, because small interest rates do not make a considerable difference compared to the basic model.

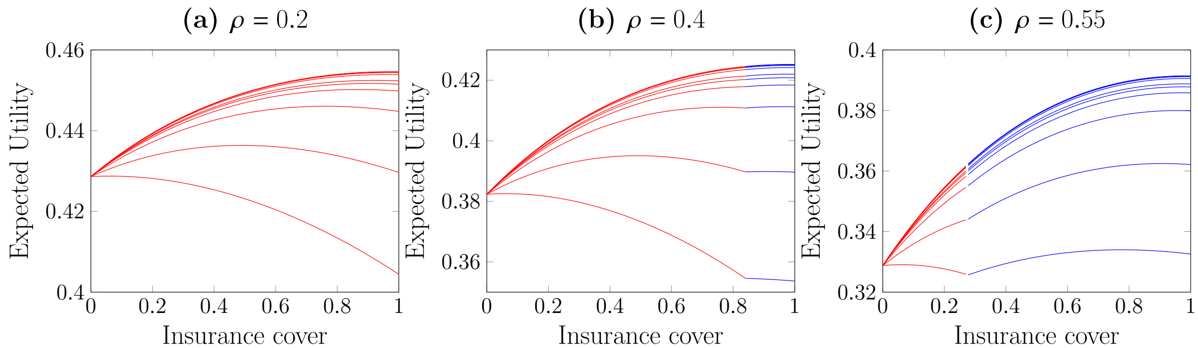

Figure 5.2: Expected Utility with Exponential Utility for Actuarial Fair Price.

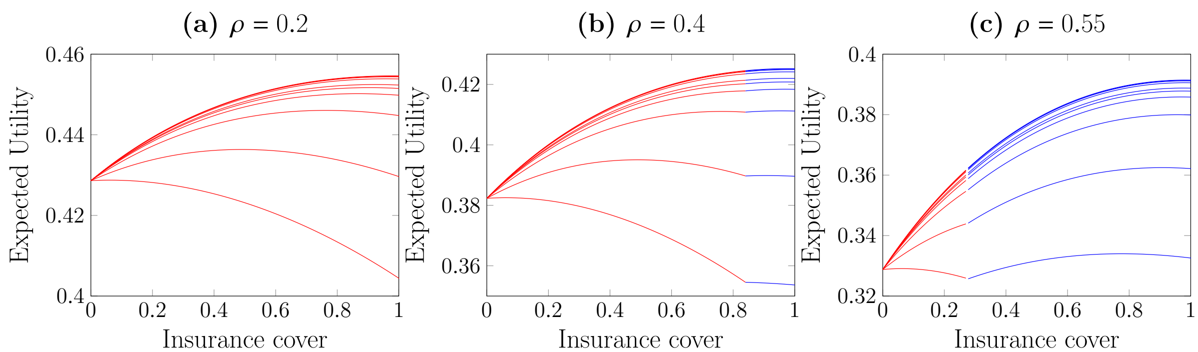

Figure 5.3: Expected Utility with Exponential Utility for Equilibrium Price-Method 1.

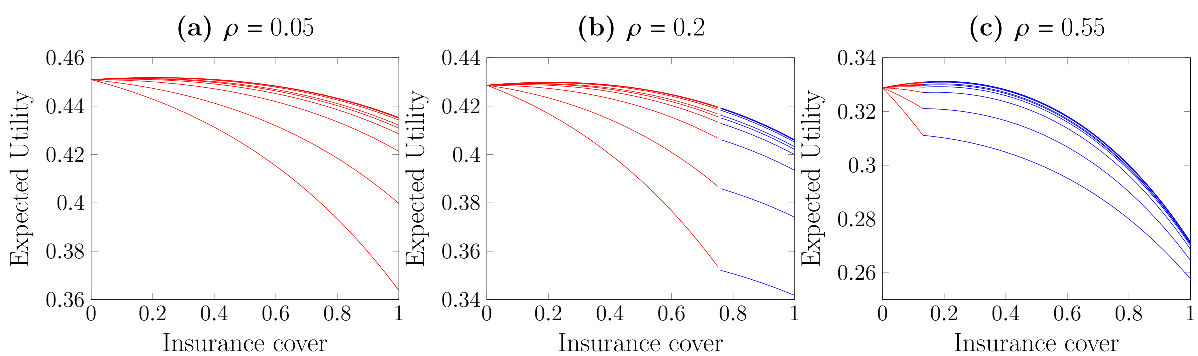

Figure 5.4: Expected Utility with Exponential Utility for Equilibrium Price-Method 2.

Figures 5.2, 5.3 and 5.4 show the expected utility of the remaining wealth of the farmer against insurance cover for different interest rates at actuarial fair price and equilibrium premium under Method 1 and 2, respectively. Those figures help to understand the computation of the optimal insurance cover according to the above explained procedure.

In each of the above figures there are 10 lines to represent the expected utility under 10 different basis risks (see Table 5.7). If the farmer i chooses α_i from the region of the insurance cover with red curves, then he is able to repay the full loan under all possible levels of his revenue. According to the selected parameters if the farmer i choose α_i from the region of insurance cover with blue curves, then he does not have enough money to repay the total loan at his smallest level of revenue, that is $w_{l0} = 0$. According to the figure under actuarial fair premium when $\rho = 0.2$ at any insurance cover from 0 to 1 the revenue of the farmer is always enough to repay the full loan amount. As a result of that the optimal index cover of each farmer is similar to the optimal index cover of the basic model. We observe the similar behavior at equilibrium premiums under both methods when $\rho = 0.05$. Then the equilibrium is similar to the equilibrium of the basic model. But when $\rho = 0.2$, at equilibrium prices under both methods for high levels of α_i ($> \alpha_{l0}$), the farmers are able to repay the full loan only at some revenue levels ($w_{h1} > 0, w_{h0} > 0, w_{l1} > 0$ and $w_{l0} = 0$). The same happens at the actuarial fair premium when $\rho = 0.4$. When $\rho = 0.55$ at actuarial fair and equilibrium prices the farmers are able to repay the total loan only if they choose small insurance covers. However, in some cases (e.g. actuarial fair price with $\rho = 0.4, \rho = 0.55$ and equilibrium price with $\rho = 0.55$) the optimal cover of some farmers comes from the red region, that is they are always able to repay the total loan. For the rest of the farmers the optimal cover comes from blue region. Then $w_{l0} = 0$ for those farmers, but that optimal cover maximizes their expected utility. The demand under different interest rates of this model are given in Table 5.7. Figures 5.2, 5.3, 5.4 and Remarks 5.8, 5.9 help to understand the results in Table 5.7. We explain the results in Table 5.7 as follows.

$\rho = 0.2$**:** At actuarial fair premium as per Figure 5.2 (a) at any insurance cover all the farmers are always able to repay the full loan. Then obviously at their optimal demands the farmers are always able to repay the full loan. By Remark 5.8 the optimal demand of this model is similar to the optimal demand in the basic model. But at equilibrium premiums (Method 1 and 2) as per Figure 5.3 (b) and 5.4 (b) at some insurance covers all the farmers are always able to repay the full loan. At other insurance covers all the farmers are not able to repay the full loan at their lowest wealth level. However the optimal demands of the farmers are come from the region where all the wealth levels are positive. By Remark 5.9 we then obtain the same equilibrium premium and demand. (see Table 5.2 for comparison).

$\rho = 0.4$**:** At actuarial fair price as per Figure 5.2 (b) at some insurance covers all the farmers are always able to repay the full loan. At other insurance covers all the farmers are not able to repay the full loan at their lowest revenue level. Some farmers choose their optimal demand from the region in which all the wealth levels are positive and their optimal demand are same as the basic model. But others choose their optimal demand from the other region (when $r_i = 1/10$ and $r_i = 1/5$). This leads to different optimal demand than in the basic model. At equilibrium premium the situation under $\rho = 0.4$ is the same as for $\rho = 0.2$. Therefore, the demands at the equilibrium of all the farmers are still same as in the basic model.

Table 5.7: Insurance Demand with Loan for Farming – Exponential Utility

r_i	α_i^* at Fair Premium			α_i^* at Equilibrium Premium					
				Method 1			Method 2		
	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.55$	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.55$	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.55$
1/1000	0.9763	0.9818	0.9818	0.2426	0.2426	0.2209	0.2107	0.2107	0.1953
1/750	0.9743	0.9816	0.9816	0.2406	0.2406	0.2207	0.2087	0.2087	0.1951
1/500	0.9702	0.9812	0.9812	0.2365	0.2365	0.2203	0.2047	0.2047	0.1947
1/250	0.9582	0.9800	0.9800	0.2245	0.2245	0.2191	0.1926	0.1926	0.1935
1/100	0.9230	0.9763	0.9763	0.1893	0.1893	0.2154	0.1575	0.1575	0.1899
1/75	0.9040	0.9743	0.9743	0.1703	0.1703	0.2134	0.1385	0.1385	0.1878
1/50	0.8671	0.9700	0.9700	0.1334	0.1334	0.2091	0.1016	0.1016	0.1835
1/25	0.7633	0.9565	0.9565	0.0296	0.0296	0.0024	0	0	0
1/10	0.4899	0.4899	0.9072	0	0	0	0	0	0
1/5	0.0617	0.0617	0.7753	0	0	0	0	0	0
α^*	78.8803	85.5335	94.8423	14.6690	14.6690	15.2126	12.1430	12.1430	13.3984
π^*	0.3236	0.3236	0.3236	0.6645	0.6645	0.6758	0.6777	0.6777	0.6862

$\rho = 0.55$**:** At the actuarial fair premium as per Figure 5.2 (c) all the farmers choose their optimal covers from the region in which they are not able to repay the full loan at their lowest wealth level. Then the optimal demand of all the farmers are different to the optimal demands in the basic model. At the equilibrium premiums the situation is similar to the situation of the actuarial fair premium at $\rho = 0.4$. Then the demands at the equilibrium of some farmers are different to their demand at the equilibrium of the basic model and others take the same amount as in the basic model.

In addition to the above results from Table 5.7 it is clear that the total optimal demand for the fair premium and for the equilibrium premiums increase as the interest rate on the loan increases. In order to properly understand how interest rate and the basis risk affect the optimal demand and equilibrium we perform another numerical analysis. For all the above numerical analysis we used only ten levels of basis risk and there are ten farmers with each basis risk. But now we consider a group of 100 farmers with 100 different levels of basis risk in the interval [1/1000, 1/5]. Because then we can clearly see how the demand changes with the basis risk. The obtained results are given in Figure 5.5.

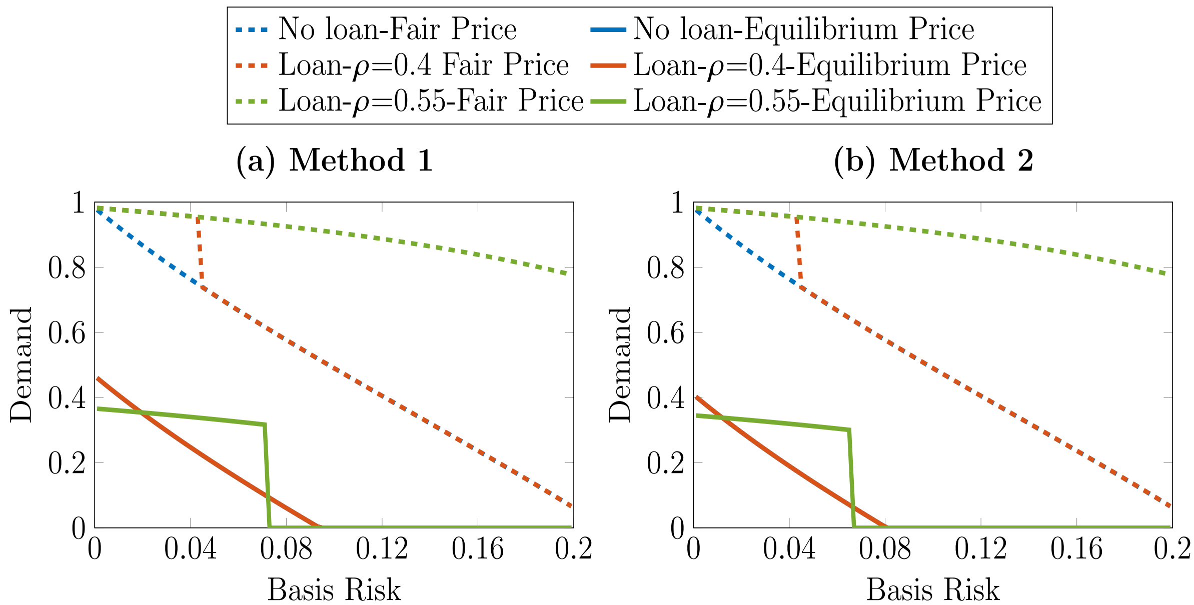

Figure 5.5: Demand Comparison – Exponential Utility.

Figure 5.5 (a) and (b) for Method 1 and Method 2 respectively have many features in common and there are only very slight differences. In all the cases the demand monotonically decreases as the basis risk increases. This is not a surprising results because basis risk is an unfavorable feature of an index insurance. At the actuarial fair price the farmers with low basis risk purchase the same amount of insurance under no loan and no loan with $\rho = 0.4$. But the other farmers purchase the same amount under loan with $\rho = 0.4$ and $\rho = 0.55$. At the equilibrium prices the farmers with low basis risk (not very low) purchase more insurance when $\rho = 0.55$ than in the case $\rho = 0.4$. But the other farmer's purchase more when $\rho = 0.4$.

We perform a similar analysis by assuming that the farmers preference on wealth is given by a power utility function. Let $\rho = 0.05$ and $\rho = 0.1$ for the numerical computations. The demand at different interest rates on the loan under the actuarial fair premium and the equilibrium premiums are given in Table 5.8.

By comparing the results in Table 5.2 and Table 5.8, we can easily see that the demand of the index insurance with loan at actuarial fair price is lower than the demand without loan. The equilibrium premiums of this model (at both interest rates) are greater than the corresponding equilibrium premiums of the basic model. But the equilibrium demands are also higher than in the basic model. Also at the actuarial fair premium there is a high demand at low interest rate, but in contrast to that at the equilibrium premiums there is a high demand at high interest rate.

Table 5.8: Insurance Demand with Loan for Farming – Power Utility

r_i	α_i^* at Fair Premium		α_i^* at Equilibrium Premium			
			Method 1		Method 2	
	$\rho = 0.05$	$\rho = 0.15$	$\rho = 0.05$	$\rho = 0.15$	$\rho = 0.05$	$\rho = 0.15$
1/1000	0.9682	0.9617	0.2248	0.2496	0.1959	0.2134
1/750	0.9639	0.9553	0.2224	0.2466	0.1936	0.2105
1/500	0.9555	0.9430	0.2177	0.2407	0.1890	0.2048
1/250	0.9319	0.9101	0.2041	0.2239	0.1757	0.1887
1/100	0.8702	0.8330	0.1666	0.1801	0.1389	0.1464
1/75	0.8404	0.7985	0.1475	0.1589	0.1201	0.1256
1/50	0.7873	0.7400	0.1119	0.1208	0.0849	0.0882
1/25	0.6598	0.6091	0.0196	0.0272	0	0
1/10	0.3936	0.3562	0	0	0	0
1/5	0.0473	0.0424	0	0	0	0
α^*	74.1801	71.4917	13.1465	14.4781	10.9817	11.7754
π^*	0.3236	0.3236	0.6317	0.6602	0.6432	0.676

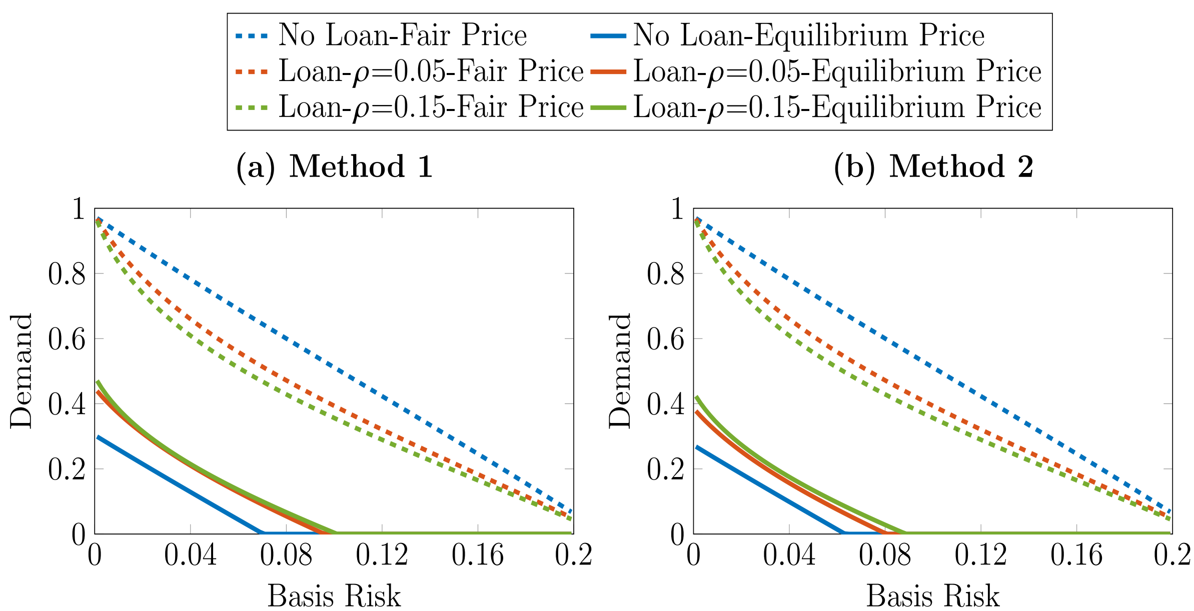

Figure 5.6: Demand Comparison – Power Utility.

A further comparison between demand with and without loan under power utility is given in Figure 5.6. At the actuarial fair premium the demand with loan at any given level of basis risk is below the corresponding demand of no loan case. Also, the demand at any given level of the basis risk decreases as the interest increases. But at the equilibrium premium the demand of the farmers with low basis risks increases as the interest rate increases. When the basis risks is high the farmers do not prefer index insurance at equilibrium premium and as a result of that the demand is 0.

Loan for High Technology for Farming

Now let the cost of high technology be $h = 1.2$. We consider three interest rates, $\rho = 0.05$, $\rho = 0.15$ and $\rho = 0.25$. We numerically compute the equilibrium premium and demand for these 3 interest rates under exponential and power utility function by Method 1 and Method 2. Also, the optimal demands at the actuarial fair premium under all above interest rates are computed. The obtained results are given in Table 5.9 and 5.10.

Table 5.9: Insurance Demand with Loan for High Technology – Exponential Utility

r_i	α_i^* at Fair Premium			α_i^* at Equilibrium Premium					
				Method 1			Method 2		
	$\rho = 0.05$	$\rho = 0.15$	$\rho = 0.25$	$\rho = 0.05$	$\rho = 0.15$	$\rho = 0.25$	$\rho = 0.05$	$\rho = 0.15$	$\rho = 0.25$
1/1000	0.9626	0.9273	0.8933	0.2756	0.2409	0.2140	0.2492	0.2224	0.2005
1/750	0.9563	0.9209	0.8869	0.2692	0.2345	0.2076	0.2428	0.2160	0.1941
1/500	0.9439	0.9086	0.8745	0.2569	0.2222	0.1952	0.2304	0.2037	0.1818
1/250	0.9099	0.8746	0.8406	0.2229	0.1882	0.1613	0.1965	0.1697	0.1478
1/100	0.8275	0.7921	0.7581	0.1404	0.1057	0.0788	0.1140	0.0872	0.0653
1/75	0.7902	0.7549	0.7208	0.1031	0.0684	0.0415	0.0767	0.0499	0.0281
1/50	0.7273	0.6920	0.6579	0.0402	0.0055	0	0.0138	0	0
1/25	0.5903	0.5550	0.5209	0	0	0	0	0	0
1/10	0.3408	0.3055	0.2714	0	0	0	0	0	0
1/5	0.0359	0.0006	0	0	0	0	0	0	0
α^*	70.8470	67.3162	64.2436	13.0820	10.6527	8.9835	11.2335	9.4890	8.1766
π^*	0.6472	0.6472	0.6472	1.694	1.5692	1.4582	1.713	1.5802	1.465

The results for both exponential and power utility functions in Table 5.9 and 5.10 show similar features. Therefore we interpret the results in both tables together. At the actuarial fair premium the optimal demand decreases as interest rate increases. Also, under Method 1 and Method 2 the equilibrium demand and the premium decrease as interest rate increases. The opposite happens in the model with loan for farming. In that model as interest rate increases the equilibrium demand and premium increase or keep unchanged. In the model with loan for farming when interest rate increases the farmers would like to purchase more even at higher premium, because more insurance helps them to repay the increasing loan. But in this model more insurance at a higher premium results in more loan, because in this model the loan increases as the premium increases. Then as the interest rate increases farmers may like to purchase less at a low premium in order to reduce the amount of loan they should repay. The total optimal demand at the actuarial fair premium and the equilibrium demand under Method 1 and 2 at three given interest rates are below the demand of the corresponding cases of the basic model. That is when the insurance is interlinked with a loan for high technology the farmers purchase less insurance than the amount of insurance they purchase when they do farming under normal way without a loan.

Table 5.10: Insurance Demand with Loan for High Technology – Power Utility

r_i	α_i^* at Fair Premium			α_i^* at Equilibrium Premium					
				Method 1			Method 2		
	$\rho = 0.05$	$\rho = 0.15$	$\rho = 0.25$	$\rho = 0.05$	$\rho = 0.15$	$\rho = 0.25$	$\rho = 0.05$	$\rho = 0.15$	$\rho = 0.25$
1/1000	0.9486	0.8444	0.7505	0.1212	0.1068	0.0945	0.11422	0.1015	0.0891
1/750	0.9456	0.8415	0.7477	0.1193	0.1050	0.0926	0.1124	0.0996	0.0873
1/500	0.9397	0.8358	0.7423	0.1156	0.1012	0.0889	0.1087	0.0959	0.0836
1/250	0.9225	0.8193	0.7263	0.1046	0.0903	0.0779	0.0977	0.0850	0.0726
1/100	0.8747	0.7732	0.6820	0.0730	0.0587	0.0464	0.0662	0.0535	0.0412
1/75	0.8501	0.7496	0.6592	0.0563	0.0420	0.0296	0.0495	0.0367	0.0244
1/50	0.8043	0.7057	0.6169	0.0240	0.0098	0	0.0173	0.0046	0
1/25	0.6860	0.5922	0.5075	0	0	0	0	0	0
1/10	0.4142	0.3303	0.2540	0	0	0	0	0	0
1/5	0.0367	0	0	0	0	0	0	0	0
α^*	74.2236	64.9195	56.8635	6.1409	5.1373	4.2986	5.6589	4.7676	3.9827
π^*	0.6472	0.6472	0.6472	1.2252	1.1315	1.0517	1.2298	1.1347	1.0546

5.3 Equilibrium Pricing Model for Two Groups of Farmers

The equilibrium models in the previous sections consider a single group of farmers and discuss the market equilibrium in that group. But in reality a society consists of different groups of people based on their characteristics. As mentioned earlier Sass and Seifried (2014) build an equilibrium model for two risk groups. They build up an equilibrium model for two scenarios, first by assuming the same insurance contract for both risk groups and second considering two different insurance contract for two risk groups. Also the social welfare under both scenarios are discussed. Motivated by Sass and Seifried (2014) in this section we build an equilibrium model for the index insurance by assuming two group of farmers with different characteristics.

In the equilibrium pricing models in the previous sections, the farmers are different to each other by the probability of loss and the basis risk (only by basis risk in the numerical simulations). We assume that the same weather event affects all the farmers in the group and therefore all the farmers have the same probability of bad weather q. In this model we consider that there are two different groups of farmers where the effect of the corresponding weather event is independent across the groups. For an example this situation is similar to two groups of farmers living in two regions with different effects from the corresponding weather event. As a result of that the probability of bad weather of two groups are different to each other. According to the point of view of the insurance company the group (or the farmers in the region) with high probability of bad weather is the high risk group and the other group is the low risk group. Because the payment to the farmer depends on the corresponding weather event. In addition to that similar to the previous model, it is also possible to have farmers with different probability of loss and different basis risk within each group. The individuals in the high risk group are $\oplus$ farmers and the individuals in the low risk group are $\ominus$ farmers. The probability of bad weather is denoted by $q_\oplus$ and $q_\ominus(< q_\oplus)$ for high risk and low risk groups respectively.

5.3.1 Demand Side

We assume that there are N farmers in each group. For the farmers in each group a specific weather index based insurance against agriculture risk is available. The farmers can decide the preferred level of the insurance cover under the offered contract. As expected utility maximizer any farmer wants to choose the amount of insurance which maximizes his expected utility of his wealth. We assume that the farmers are strictly risk averse, with utility function u_d satisfying $u_d' > 0$ and $u_d'' < 0$. Then the optimal insurance cover of farmer $i \in \{1, 2, 3, \ldots, N\}$ in group $\oplus/\ominus$ at premium π is given by

$$\alpha^*_{\oplus/\ominus,i} = \underset{(\alpha_{\oplus/\ominus,i} \geq 0)}{\text{argmax}}\ \mathbb{E}[u_d(W_{\oplus/\ominus,i} + \alpha_{\oplus/\ominus,i} P(I_{\oplus/\ominus}) + (w_{\oplus/\ominus,i} - \alpha_{\oplus/\ominus,i}\pi)(1+\delta))], \tag{5.17}$$

where $W_{\oplus/\ominus,i}$ and $I_{\oplus/\ominus}$ are defined similar to previous models. That is

$$W_{\oplus/\ominus,i} = \begin{cases} w_l & \text{due to low yield with probability } p_{\oplus/\ominus,i} \\ w_h & \text{due to high yield with probability } 1 - p_{\oplus/\ominus,i}, \end{cases}$$

for all $i \in \{1, 2, \ldots, N\}$ and

$$I_{\oplus/\ominus} = \begin{cases} 1 & \text{bad weather with probability } q_{\oplus/\ominus} \\ 0 & \text{good weather with probability } 1 - q_{\oplus/\ominus}. \end{cases}$$

Let $l = w_h - w_l$ and we refer to l as the loss of farming. Then the indemnity payment $P(I_{\oplus/\ominus}) = lI_{\oplus/\ominus}$. $w_{\oplus/\ominus,i}$ is the initial wealth of farmer i in group $\oplus/\ominus$ and δ is the periodic interest rate. According to this setting there are four possible wealth states for farmer $i \in \{1, 2, 3, \dots, N\}$ in group $\oplus/\ominus$. Those wealth states with the corresponding probabilities are given in Table 5.11.

Table 5.11: Wealth States of Farmer i in Group $\oplus/\ominus$

State s	Probability p_s	Wealth
$l0$	$r_{\oplus/\ominus,i}$	$w_l + (w_{\oplus/\ominus,i} - \alpha_{\oplus/\ominus,i}\pi)(1+\delta)$
$l1$	$p_{\oplus/\ominus,i} - r_{\oplus/\ominus,i}$	$w_l + (w_{\oplus/\ominus,i} - \alpha_{\oplus/\ominus,i}\pi)(1+\delta) + \alpha_{\oplus/\ominus,i}l$
$h0$	$1 - q_{\oplus/\ominus} - r_{\oplus/\ominus,i}$	$w_h + (w_{\oplus/\ominus,i} - \alpha_{\oplus/\ominus,i}\pi)(1+\delta)$
$h1$	$q_{\oplus/\ominus} + r_{\oplus/\ominus,i} - p_{\oplus/\ominus,i}$	$w_h + (w_{\oplus/\ominus,i} - \alpha_{\oplus/\ominus,i}\pi)(1+\delta) + \alpha_{\oplus/\ominus,i}l$

Here $r_{\oplus/\ominus,i}$ is the probability that the loss occurs to the farmer i in group $\oplus/\ominus$ but the index indicates good weather. Since there are N farmers, the total optimal demand or the total optimal amount of insurance are purchased by all the farmers in group $\oplus/\ominus$ at price π is $\alpha^*_{\oplus/\ominus} = \sum_{i=1}^{N} \alpha^*_{\oplus/\ominus,i}$.

Proposition 5.10. *If the insured's utility function u_d is an exponential utility function with coefficient of risk aversion $\gamma_d > 0$, the optimal insurance demand of farmer i in group $\oplus/\ominus$ at premium π is given by*

$$\alpha^*_{\oplus/\ominus,i} = max\left[0, \frac{1}{\gamma_d l}\ln\left(\frac{(l-\pi(1+\delta))}{\pi(1+\delta)}\frac{(A_{\oplus/\ominus,i}e^{-\gamma_d w_l} + B_{\oplus/\ominus,i}e^{-\gamma_d w_h})}{(r_{\oplus/\ominus,i}e^{-\gamma_d w_l} + C_{\oplus/\ominus,i}e^{-\gamma_d w_h})}\right)\right], \tag{5.18}$$

where

$A_{\oplus/\ominus,i} = p_{\oplus/\ominus,i} - r_{\oplus/\ominus,i}$,
$B_{\oplus/\ominus,i} = q_{\oplus/\ominus} + r_{\oplus/\ominus,i} - p_{\oplus/\ominus,i}$,
$C_{\oplus/\ominus,i} = 1 - q_{\oplus/\ominus} - r_{\oplus/\ominus,i}$.

5.3.2 Supply Side

We assume that there is only one insurer in the market. The insurer sells contracts to both groups of farmers. Similar to Sass and Seifried (2014) we consider the following two scenarios.

1. Mandatory equal contracts
2. Free contract design

Under scenario 1 the insurance company should offer the contracts with equal premium $\pi_{\odot}$ to both groups. Under scenario 2 the insurance company can offer contracts with two different premia to two groups. Let the premium for the farmers in group $\oplus$ and group $\ominus$ be $\pi_\oplus$ and $\pi_\ominus$ respectively. In contrast to the single group model, here at a given premium and demand there are four possible profit levels of the insurer. Those profit levels under scenarios 1 and 2 are given in Table 5.12 and 5.13, respectively.

Table 5.12: Four Levels of Profit – Scenario 1

Probability	Profit
$q_\oplus q_\ominus$	$(\alpha_\oplus^* + \alpha_\ominus^*)\pi_\odot(1+\delta) - (\alpha_\oplus^* + \alpha_\ominus^*)l$
$q_\oplus(1-q_\ominus)$	$(\alpha_\oplus^* + \alpha_\ominus^*)\pi_\odot(1+\delta) - \alpha_\oplus^* l$
$(1-q_\oplus)q_\ominus$	$(\alpha_\oplus^* + \alpha_\ominus^*)\pi_\odot(1+\delta) - \alpha_\ominus^* l$
$(1-q_\oplus)(1-q_\ominus)$	$(\alpha_\oplus^* + \alpha_\ominus^*)\pi_\odot(1+\delta)$

Table 5.13: Four Levels of Profit – Scenario 2

Probability	Profit
$q_\oplus q_\ominus$	$\alpha_\oplus^*\pi_\oplus(1+\delta) + \alpha_\ominus^*\pi_\ominus(1+\delta) - (\alpha_\oplus^* + \alpha_\ominus^*)l$
$q_\oplus(1-q_\ominus)$	$\alpha_\oplus^*\pi_\oplus(1+\delta) + \alpha_\ominus^*\pi_\ominus(1+\delta) - \alpha_\oplus^* l$
$(1-q_\oplus)q_\ominus$	$\alpha_\oplus^*\pi_\oplus(1+\delta) + \alpha_\ominus^*\pi_\ominus(1+\delta) - \alpha_\ominus^* l$
$(1-q_\oplus)(1-q_\ominus)$	$\alpha_\oplus^*\pi_\oplus(1+\delta) + \alpha_\ominus^*\pi_\ominus(1+\delta)$

5.3.3 Market Equilibrium

We derive the market equilibrium by a similar method as in Sass and Seifried (2014). We already discussed that method as Method 2. We assume that the insurer is strictly risk averse, with utility function u_s satisfying $u_s' > 0$ and $u_s'' < 0$. Now the equilibrium premium under scenario 1 is given by

$$\pi_\odot^* = \underset{\pi_\odot}{\operatorname{argmax}}\, \mathbb{E}[u_s(\alpha_\oplus^*\pi_\odot(1+\delta) - \alpha_\oplus^* l I_\oplus + \alpha_\ominus^*\pi_\odot(1+\delta) - \alpha_\ominus^* l I_\ominus)], \tag{5.19}$$

where $\alpha_{\oplus/\ominus}^*$ is a function of $\pi_\odot$.

Proposition 5.11. *If the insurer's utility function u_s is an exponential utility function with coefficient of risk aversion $\gamma_s > 0$, the equilibrium premium under scenario 1 is given by*

$$\pi_\odot^* = \frac{l(A(\alpha_\oplus^* + \alpha_\ominus^*)' + B(\alpha_\oplus^*)' + C(\alpha_\ominus^*)')}{(1+\delta)(\alpha_\oplus^* + \alpha_\ominus^*)'(A+B+C+D)} - \frac{(\alpha_\oplus^* + \alpha_\ominus^*)}{(\alpha_\oplus^* + \alpha_\ominus^*)'}, \tag{5.20}$$

where
$A = q_{\oplus} q_{\ominus} e^{\gamma_s l(\alpha^*_{\oplus}+\alpha^*_{\ominus})}$,
$B = q_{\oplus}(1 - q_{\ominus}) e^{\gamma_s l \alpha^*_{\oplus}}$,
$C = (1 - q_{\oplus}) q_{\ominus} e^{\gamma_s l \alpha^*_{\ominus}}$,
$D = (1 - q_{\oplus})(1 - q_{\ominus})$.
Here $\alpha^*_{\oplus}$ *and* $\alpha^*_{\ominus}$ *are functions of* $\pi^*_{\odot}$ *and* $(\alpha^*_{\oplus/\ominus})'$ *is the derivative of* $(\alpha^*_{\oplus/\ominus})$ *w.r.t to* $\pi_{\odot}$.

Under scenario 2 the equilibrium premiums $\pi^*_{\oplus}$ and $\pi^*_{\ominus}$ are given by

$$(\pi^*_{\oplus}, \pi^*_{\ominus}) = \underset{\pi_{\oplus}, \pi_{\ominus}}{\operatorname{argmax}} \mathbb{E}[u_s(\alpha^*_{\oplus}\pi_{\oplus}(1+\delta) - \alpha^*_{\oplus} l I_{\oplus} + \alpha^*_{\ominus}\pi_{\ominus}(1+\delta) - \alpha^*_{\ominus} l I_{\ominus})], \tag{5.21}$$

where $\alpha^*_{\oplus/\ominus}$ is a function of $\pi_{\oplus/\ominus}$.

Proposition 5.12. *If the insurer's utility function* u_s *is an exponential utility function with coefficient of risk aversion* $\gamma_s > 0$*, the equilibrium premium of group* $\oplus/\ominus$ *under scenario 2 is given by*

$$\pi^*_{\oplus/\ominus} = \frac{q_{\oplus/\ominus} l e^{\gamma_s l \alpha^*_{\oplus/\ominus}}}{(1+\delta)(q_{\oplus/\ominus} e^{\gamma_s l \alpha^*_{\oplus/\ominus}} + 1 - q_{\oplus/\ominus})} - \frac{\alpha^*_{\oplus/\ominus}}{(\alpha^*_{\oplus/\ominus})'}. \tag{5.22}$$

Here $\alpha^*_{\oplus/\ominus}$ *is a functions of* $\pi^*_{\oplus/\ominus}$ *and* $(\alpha^*_{\oplus/\ominus})'$ *is the derivative of* $(\alpha^*_{\oplus/\ominus})$ *w.r.t to* $\pi_{\oplus/\ominus}$.

Now we can derive the equilibrium demand under scenarios 1 and 2 via Equation (5.18) by substituting $\pi^*_{\odot}$ and $\pi^*_{\oplus/\ominus}$ respectively. When the farmer's preference on wealth is given by power utility function it is difficult to solve the optimization problem of the farmer. But it is possible to solve the optimization problem numerically and obtain the optimal cover.

5.3.4 Numerical Results: Two Risk Types

We consider two groups of farmers and there are $N = 100$ farmers in each group. Also similar to the single group model, each group consist of farmers with 10 different basis risks. The probability of loss is same for all the farmers within the group. Let $p_{\oplus,i} = 0.18$, $p_{\ominus,i} = 0.15$, $w_{\oplus/\ominus,i} = 1$ for all i, $0.2 \leq q_{\oplus} \leq 0.4$, $q_{\ominus} = 0.18$, $w_l = 1$, $w_h = 2$, $\delta = 0.03$, $\gamma_d = 2$ and $\gamma_s = 0.1$. We compute the equilibrium prices and demand for both scenarios by assuming exponential utility and th e obtained results are given in Figure 5.7.

According to the results in Figure 5.7, as $q_{\oplus}$ increases the low risk farmers drop out from the insurance market under scenario 1. For $q_{\oplus}$, where the demand of low risk farmers becomes 0, the demand of high risk farmers shift from scenario 1 to scenario 2. That is after that $q_{\oplus}$ the high risk farmers purchase the same amount of insurance under both scenarios. Under scenario 1, when $q_{\oplus} < 0.31$ the low risk farmers subsidize high risk farmers and when $q_{\oplus} \geq 0.31$ the low

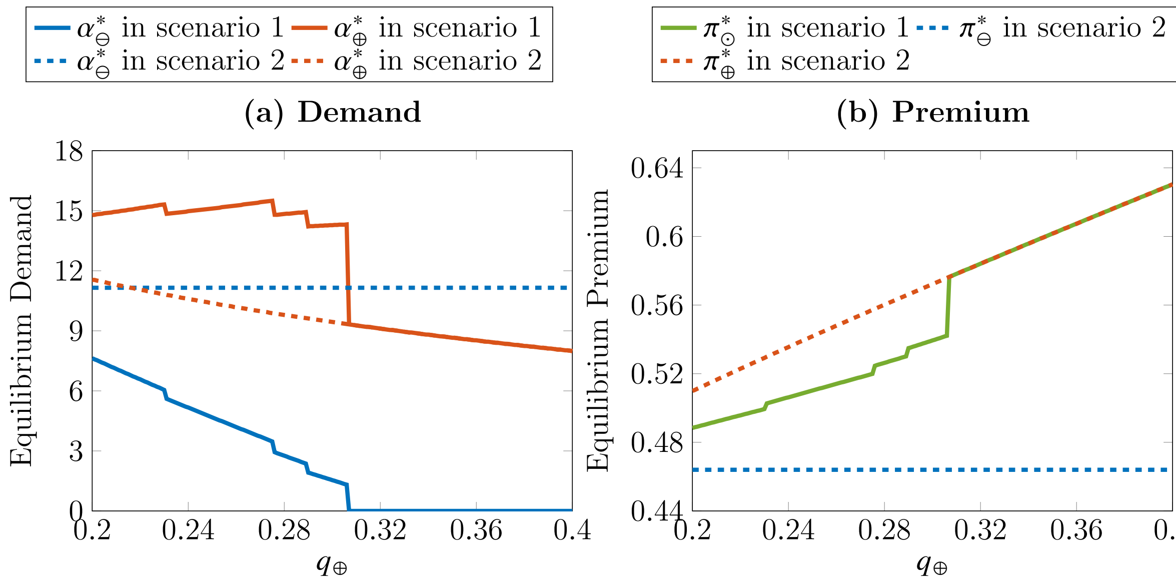

Figure 5.7: Equilibrium Demand and Equilibrium Premium – Exponential Utility.

risk farmers do not purchase the insurance. Then the high risk farmers pay the same amount as scenario 2 and purchase the same amount of the insurance. It is clear that there are jumps in the demand curve and the price curve under scenario 1. Since in any given risk group there are farmers with 10 different basis risks. That is in any given risk group there are 10 sub groups based on the basis risk. As $q_\oplus$ increases, farmers with high basis risk drop out from the market and it leads to the jumps in the demand and price curves.

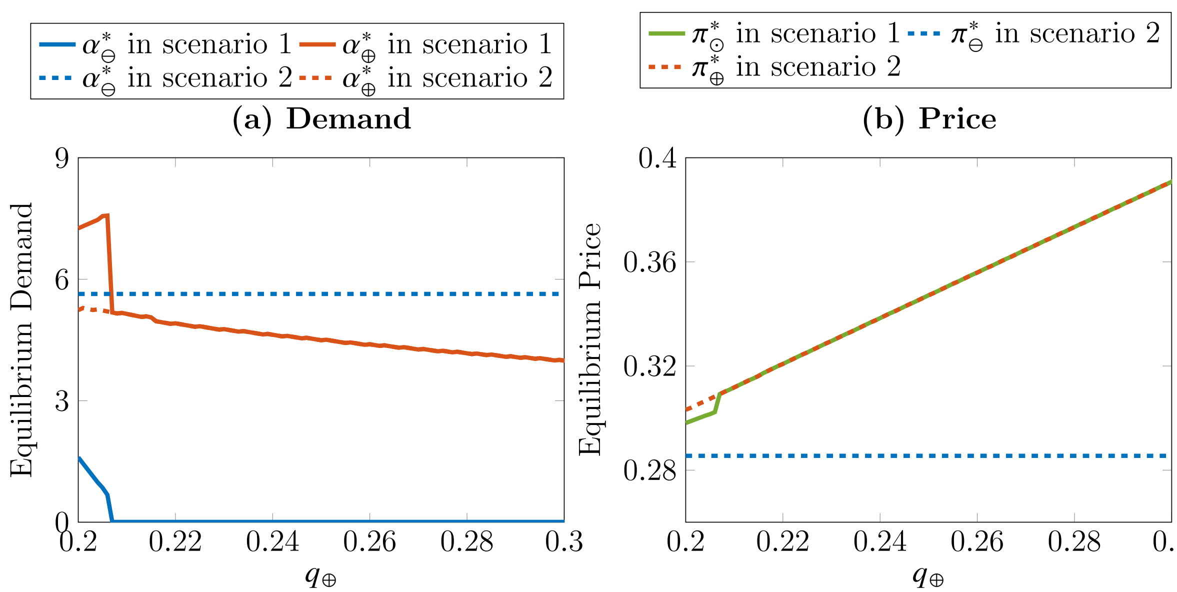

Figure 5.8: Equilibrium Demand and Equilibrium Price – Power Utility

Equilibrium demand and price under both scenarios by assuming u_d is a power utility function and u_s is exponential utility function are given in Figure 5.8. We consider the same set of parameter values as above. But here, we choose $p_{\oplus,i} = 0.15$ for all i. We can see the same behavior as for exponential utility. But in contrast to that case the low risk farmers purchase insurance only when $q_\oplus$ is very close to $q_\ominus$, that is, low risk farmers drop out from the market sooner. Therefore, low risk farmers subsidize high risk farmers only when $q_\oplus$ is very close to $q_\ominus$.

5.3.5 Welfare

Here we perform a comparison of welfare under scenarios 1 and 2. The welfare is computed as explained in Sass and Seifried (2014). We compute the welfare under scenario 2 as follows. The welfare of farmer i in group $\oplus/\ominus$ with index insurance is given by

$$WF_{\oplus/\ominus,i} = u_d^{-1}\mathbb{E}[u_d(W_{\oplus/\ominus,i} + \alpha_{\oplus/\ominus,i}P(I_{\oplus/\ominus}) + (w_{\oplus/\ominus,i} - \alpha_{\oplus/\ominus,i}\pi_{\oplus/\ominus})(1+\delta))].$$

Then the welfare of farmer i in group $\oplus/\ominus$ if there is no access to insurance, is given by

$$WF^0_{\oplus/\ominus,i} = u_1^{-1}\mathbb{E}[u_d(W_{\oplus/\ominus,i} + w_{\oplus/\ominus,i}(1+\delta))].$$

Now the social benefit of farmer i as a result of being a part of risk sharing society is given by

$$WF_{\oplus/\ominus,i} - WF^0_{\oplus/\ominus,i}.$$

The aggregated welfare with and without insurance is given by

$$WF = \sum_{i=1}^{N} WF_{\oplus,i} + \sum_{i=1}^{N} WF_{\ominus,i} \tag{5.23}$$

and

$$WF^0 = \sum_{i=1}^{N} WF^0_{\oplus,i} + \sum_{i=1}^{N} WF^0_{\ominus,i}$$

respectively. Then the aggregated social benefit is $WF - WF^0$. The benefit to the individual farmer in the society by risk sharing is

$$\frac{(WF - WF^0)}{2N}. \tag{5.24}$$

The total expected profit of the insurance company is

$$P = \alpha_{\oplus}^{*}\pi_{\oplus}^{*}(1+\delta) - \alpha_{\oplus}^{*} l q_{\oplus} + \alpha_{\ominus}^{*}\pi_{\ominus}(1+\delta) - \alpha_{\ominus}^{*} l q_{\ominus}.$$

If this profit is distributed among the individuals then it must be added to the aggregated welfare in Equation (5.23). Then the social benefit to an individual farmer is $(WF + P - WF^{0})/2N$. We can compute the social benefit under scenario 1 in a similar manner. The social benefit to individual farmer with and without insurance profit by assuming farmers preference on wealth given by exponential utility and power utility are given in Figure 5.9 and 5.10 respectively.

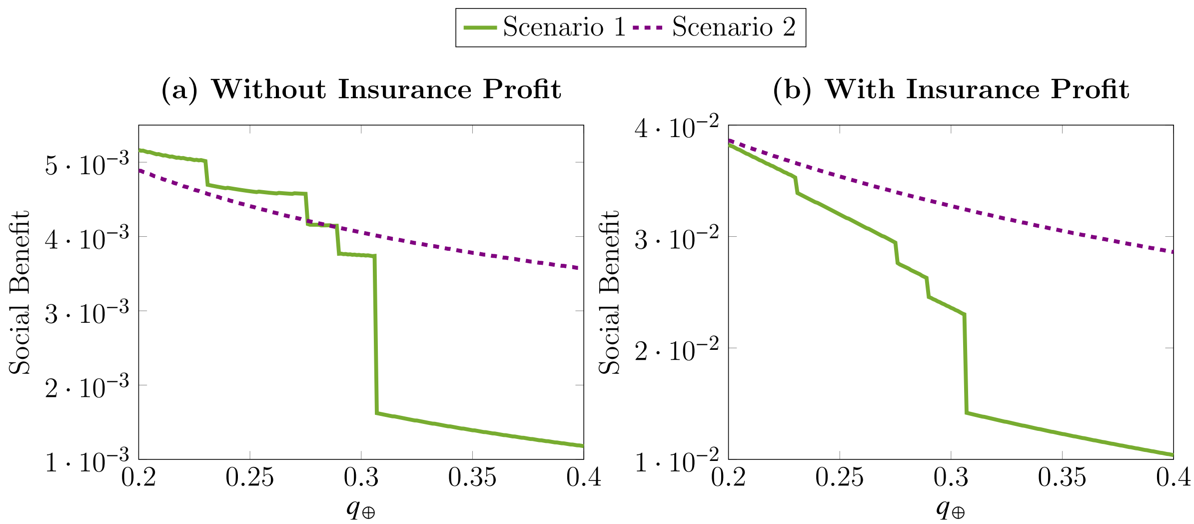

Figure 5.9: Social Benefit – Exponential Utility

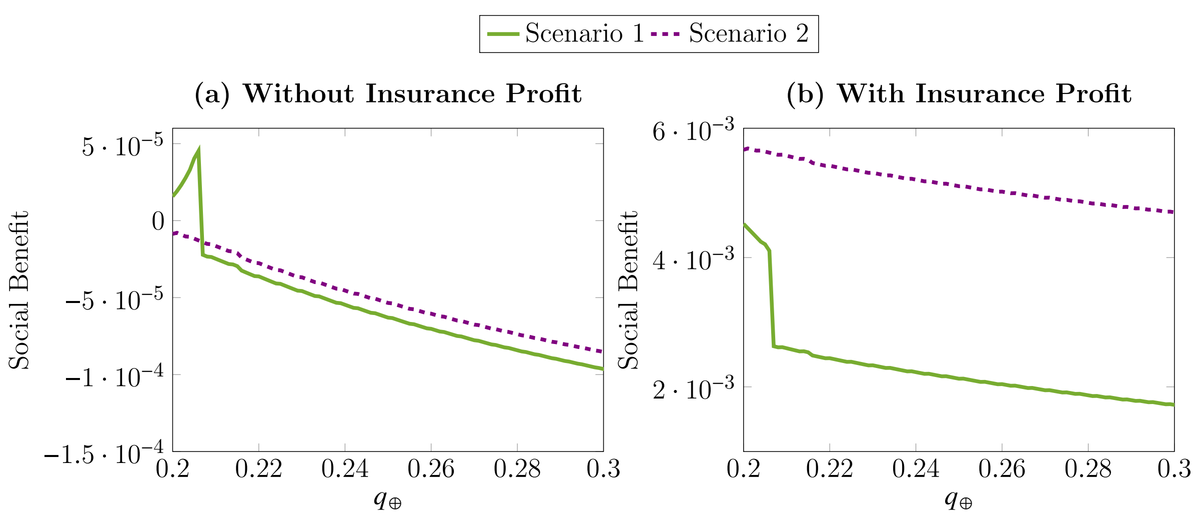

Figure 5.10: Social Benefit – Power Utility

According to Figures 5.9 and 5.10, it is clear that without insurance profit the social benefit can be improved by shifting from scenario 2 to 1 when $q_{\oplus}$ is close to $q_{\ominus}$. But with insurance profit scenario 2 is always better than scenario 1 in terms of social benefits.

5.4 Summary

The main findings of this chapter are:

- We have to distinguish Method 1 and Method 2 which yield two different market equilibriums.
- According to our equilibrium model a group of low risk averse farmers does not always requires a higher subsidy rate than a group of high risk averse farmers in order to reach a given level of participation ratio.
- When an agriculture loan is interlinked with index insurance, even though the equilibrium price is higher, some farmers purchase more insurance and vice versa.
- According to the equilibrium model for two risk groups, the low risk group subsidizes the high risk group when both groups pay the same premium.

Chapter 6

Informal Risk Sharing and Index Insurance

Informal risk sharing is an important mechanism, specifically for a community without formal insurance. In an informal risk sharing group the individuals provide mutual protection to each other by voluntarily sharing risk. The informal risk sharing is a widely discussed topic in the literature. There are several works which discuss informal risk sharing connected to index insurance (Boucher and Delpierre (2014), Dercon et al. (2014), Santos et al. (2021)). Those works consider different ways of informal risk sharing and how it affects the index insurance. It may be problematic for the usefulness of an index insurance for an individual in an informal risk sharing group and vice versa, since then there is another way to cover the risk. But as a result of the basis risk of the index insurance there is a possibility that the actual risk may not be covered. In such a situations both index insurance and informal risk sharing at the same time may be useful. Therefore we are interested to analyze index insurance in an informal risk sharing environment. In this chapter we build up three models based on three different informal risk sharing methods. The ideas behind those methods are taken from Boucher and Delpierre (2014), Dercon et al. (2014) and Santos et al. (2021). We build those three models to understand the similarities and differences between the difference risk sharing methods. In addition to that we analyze how informal risk sharing affects index insurance.

6.1 The Model Setup

We consider a group of farmers who are willing to share risk informally and also consider the possibility to take a weather index insurance to cover agriculture risk. As explained above we develop three models based on the nature of informal risk sharing. Those are

- Model 1: Based on the idiosyncratic income shock (Boucher and Delpierre (2014)).
- Model 2: Based on the loss of farming (Dercon et al. (2014)).
- Model 3: Based on the insurance payout (Santos et al. (2021)).

We discuss those models in detail in the following sections.

6.1.1 Model 1

We consider a group of farmers belonging to the same community with n members. The income of farmer $i \in \{1, 2, 3, \ldots, n\}$ is given by

$$Z_i = \mu_i + \Theta_i,$$

where μ_i is the income which only depends on the level of risk taking, such as the amount of investment and technology. Θ_i is the random income shock, which depends on factors such as weather, pests and illnesses. We can divide this income shock into two parts. Those are the covariate shock, which is common for all the members of the group and the idiosyncratic shocks, which are independent across individuals in the group. Then $\Theta_i = \theta_g + \theta_i$. Where θ_g is the covariate shock and θ_i is the idiosyncratic shock of farmer i. There are n i.i.d. random variables θ_i representing the idiosyncratic shock of n farmers in the group. Also $\theta_g, \theta_1, \ldots, \theta_n$ are independent. In this model the informal risk sharing depends on the idiosyncratic shock. It is explained in Figure 6.1.

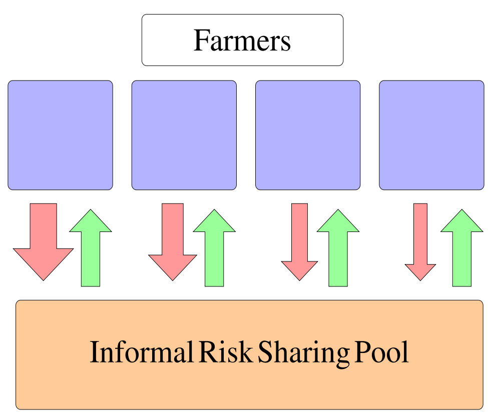

Figure 6.1: Informal Risk Sharing of Model 1.

As shown in Figure 6.1 all the farmers in the group contribute a fraction of their idiosyncratic income in to the risk sharing pool. Then the total amount is equally distributed among all the farmers in the group. Since the idiosyncratic shock is independent across the farmers, different farmers contribute different amounts to the risk sharing pool. As a results of that the farmers with high idiosyncratic income receive less than they contribute and the farmers with low idiosyncratic income receive more than they contribute from the informal risk sharing. This is the way the group of farmers informally share the risk.

Let $\beta \in [0, 1]$ be the fraction of the idiosyncratic income shock that any given member of the group transfers to the group. Then the total amount in the pool of idiosyncratic shocks that is shared within the group is $\sum_{j=1}^{n} \beta_j \theta_j$. Now we assume that an index insurance is available for farmers to reduce a bad effect from the covariate shock. The payout of that full cover index insurance is $P(\theta_g)$ and the premium is π. The insured has the option to choose the level of index insurance coverage. Let $\alpha \in [0, 1]$ be the index insurance cover of any given farmer. Then the level of consumption of farmer i with risk sharing and index insurance is given by

$$C_1^i = \mu_i + \theta_g + \theta_i - \beta\theta_i + \frac{\beta}{n}\sum_{j=1}^{n}\theta_j - \alpha\pi + \alpha P(\theta_g). \tag{6.1}$$

We assume that θ_g and θ_i are binary variables with

$$\theta_g = \begin{cases} \underline{\theta}_g & \text{with probability } q \\ \overline{\theta}_g & \text{with probability } 1-q, \end{cases}$$

$$\theta_i = \begin{cases} \underline{\theta} & \text{with probability } p \\ \overline{\theta} & \text{with probability } 1-p. \end{cases}$$

for $\overline{\theta}_g > \underline{\theta}_g$ and $\overline{\theta} > \underline{\theta}$. Then we define the payout of the index insurance as

$$P(\theta_g) = \overline{\theta}_g - \theta_g = \begin{cases} \overline{\theta}_g - \underline{\theta}_g & \text{with probability } q \\ 0 & \text{with probability } 1-q, \end{cases}$$

and then

$$\pi = m\mathbb{E}[P(\theta_g)] = m(\overline{\theta}_g - \underline{\theta}_g)q,$$

where m is the premium load of the index insurance. Let $S = \sum_{j=1, j\neq i}^{n} \theta_j$. When k farmers get $\overline{\theta}$ and $n-1-k$ get $\underline{\theta}$. Then $S = k\overline{\theta} + (n-1-k)\underline{\theta}$ with probability $\rho_k = \binom{n-1}{k}(1-p)^k p^{(n-1-k)}$. We state the possible values of C_1^i with the corresponding probabilities in Table 6.1.

Table 6.1: Possible Values of C_1^i – Model 1

State	Probability	C_i^{SI}
$\overline{\theta}_g, \overline{\theta}, s_k$	$(1-q)(1-p)\rho_k$	$\mu_i + \overline{\theta}_g + \overline{\theta} - \alpha\pi - \beta\overline{\theta} + \beta(\overline{\theta} + s_k)/n$
$\overline{\theta}_g, \underline{\theta}, s_k$	$(1-q)p]\rho_k$	$\mu_i + \overline{\theta}_g + \underline{\theta} - \alpha\pi - \beta\underline{\theta} + \beta(\underline{\theta} + s_k)/n$
$\underline{\theta}_g, \overline{\theta}, s_k$	$q(1-p)\rho_k$	$\mu_i + \underline{\theta}_g + \overline{\theta} - \alpha\pi + \alpha(\overline{\theta}_g - \underline{\theta}_g) - \beta\overline{\theta} + \beta(\overline{\theta} + s_k)/n$
$\underline{\theta}_g, \underline{\theta}, s_k$	$qp\rho_k$	$\mu_i + \underline{\theta}_g + \underline{\theta} - \alpha\pi + \alpha(\overline{\theta}_g - \underline{\theta}_g) - \beta\underline{\theta} + \beta(\underline{\theta} + s_k)/n$

where $s_k = k\overline{\theta} + (n-1-k)\underline{\theta}$ for $k = 0, 1, 2, \ldots, n-1$,

6.1.2 Model 2

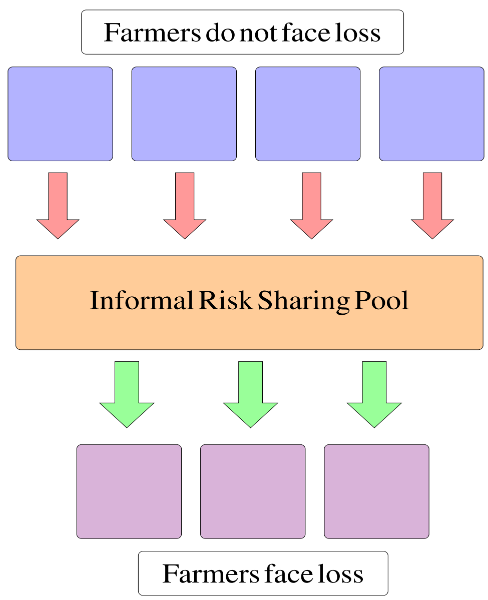

Figure 6.2: Informal Risk Sharing of Model 2.

According to the above model the income shock of a farmer is split into the covariate shock and the idiosyncratic shock. But in some works the income/loss is given by one variable. That kind of models is easier to handle in reality, because it is difficult to identify the amount of income/loss which occurs due to common reasons and reasons which due to specific for the farmer. The models in Dercon et al. (2014) and Clarke (2016) are examples for models which denote the loss by one variable. Based on that, the income of farmer $i \in \{1, 2, 3, \ldots, n\}$ is given by

$$Z_i = w_i - L_i, \tag{6.2}$$

where w_i is the initial wealth of farmer i and L_i is the loss variable of farmer i. There are n i.i.d random variables L_i representing the loss of n farmers. We assume that L_i is a binary variable and defined as

$$L_i = \begin{cases} 0 & \text{with probability } p \\ \ell & \text{with probability } 1-p. \end{cases}$$

The informal risk sharing of this model depends on the loss and it is explained in Figure 6.2.

Dercon et al. (2014) consider a group of only two farmers and they assume that when one farmer faces a loss and the other does not, then the farmer who does not face a loss covers a portion of the loss of the other farmer. In this model we generalize that informal risk sharing method by considering a group of n farmers. As shown in Figure 6.2, here we consider that the farmers in the group who do not face loss transfer fraction β of the loss ℓ (only possible loss of any given farmer) to the informal risk sharing pool and that total amount is equally distributed among the farmers who face a loss to cover a part of their losses. Then the level of consumption of farmer i with index insurance and informal risk sharing is given by

$$C_2^i = w_i - L_i - \beta\ell\mathbb{1}_{\{L_i=0\}} + S\mathbb{1}_{\{L_i=\ell\}} - \alpha\pi + \alpha P(I), \tag{6.3}$$

where I is the weather index attached to the index insurance and it is given by

$$I = \begin{cases} 1 & \text{bad weather with probability } q \\ 0 & \text{good weather with probability } 1-q. \end{cases}$$

Insurance payout $P(I) = \ell I$ and premium $\pi = m\ell q$. Since the loss L_i and index I are correlated, let r be the probability that the farmer faces a loss but does not receive any insurance payout. S is the amount transferred to farmer i as a result of informal risk sharing. When there are k farmers without loss it is given by

$$S = \frac{\beta k l}{n-k}$$

with probability $\rho_k = \binom{n-1}{k}(1-p)^k p^{(n-1-k)}$ for $k = 0, 1, 2, \ldots, n-1$. We state the possible values of C_2^i of this model and the corresponding probabilities in Table 6.2.

Table 6.2: Possible Values of C_2^i – Model 2

State	Probability	C_i^{SI}
$0, 0$	$(1 - q - r)$	$w_i - \beta\ell - \alpha\pi$
$0, \ell, s_k$	$r\rho_k$	$w_i - \ell + s_k - \alpha\pi$
$1, 0$	$(q + r - p)$	$w_i - \beta\ell - \alpha\pi + \alpha\ell$
$1, \ell, s_k$	$(p - r)\rho_k$	$w_i - \ell + s_k - \alpha\pi + \alpha\ell$

where $s_k = \beta kl/(n - k)$ for $k = 0, 1, 2, \ldots, n - 1$. The consumption of farmer i at state $(0, \ell, s_k)$ and $(1, \ell, s_k)$ have different values for each $k = 0, 1, 2, \ldots, n - 1$.

6.1.3 Model 3

We build this model by considering the model in Santos et al. (2021) and Clarke (2016). The informal risk sharing of this model depends on the insurance payout. Therefore we consider a group of n farmers with an index insurance. We assume that the income of farmer $i \in \{1, 2, 3, \ldots, n\}$ is the same as in Model 2 and it is given by Equation (6.2). The informal risk sharing is explained in Figure 6.3.

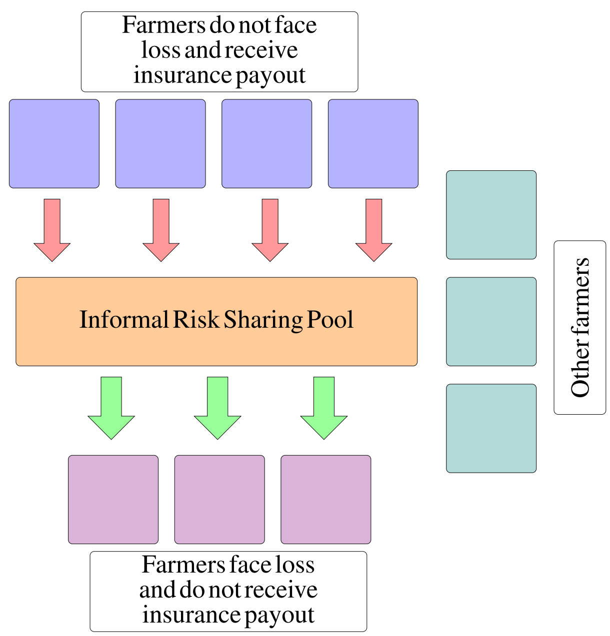

Figure 6.3: Informal Risk Sharing of Model 3.

According to the nature of index insurance there is a possibility that a loss occurs to the farmer but he does not receive a payout. Also there is a possibility that no loss occurs but the farmer receives an insurance payout. Within the group of n farmers, the farmers who receive payout without loss contribute fraction β of their insurance payout to the informal risk sharing pool and then that total amount is equally distributed among the farmers who face a loss and do not receive payout. In Figure 6.3 the group named as other farmers means the farmers who face a loss and receive insurance payout and the farmers who do not face a loss and do not receive insurance payout. That group does not participate in the informal risk sharing. Since we consider a group of farmers with index insurance, there are n i.i.d. random variables I_i representing the weather index attached to the index insurance of n farmers in the group, where

$$I_i = \begin{cases} 1 & \text{bad weather with probability } q \\ 0 & \text{good weather with probability } 1-q. \end{cases}$$

Then the level of consumption of farmer i with risk sharing and index insurance is given by

$$C_i^{SI} = w_i - L_i - \beta\alpha P(I_i)\mathbb{1}_{\{L_i=0\}} + S\mathbb{1}_{\{L_i=\ell, I_i=0\}} - \alpha\pi + \alpha P(I_i), \tag{6.4}$$

where $P(I_i) = \ell I_i$ and $\pi = m\ell q$. Since the loss L_i and index I_i are correlated, let r be the probability that the farmer faces a loss but does not receive any insurance payout. S is the amount transferred to farmer i as a result of informal risk sharing. By assuming all the farmers take cover α of the index insurance, when there are k farmers receive insurance payout without loss and j farmers (in addition to farmer i) face loss but do not receive insurance payout S is given by

$$S = \frac{\beta\alpha kl}{j+1}$$

with probability $\rho_{kj} = \binom{n-1}{k}\binom{n-1-k}{j}(q+r-p)^k r^j (1+p-q-2r)^{(n-k-j)}$ for $k = 0,1,2,\ldots,n-1$ and $j = 0,1,2,\ldots,n-1-k$. The possible values of C_3^i of this model is slightly different to Model 2. We state those possible values of C_3^i of this model and the corresponding probabilities in Table 6.3.

Table 6.3: Possible Values of C_3^i – Model 3

State	Probability	C_3^i
$0,0$	$(1-q-r)$	$w_i - \alpha\pi$
$0,\ell,s_{kj}$	$r\rho_{kj}$	$w_i - \ell + S_{k,j} - \alpha\pi$
$1,0$	$(q+r-p)$	$w_i - \beta\alpha\ell - \alpha\pi + \alpha\ell$
$1,\ell$	$(p-r)$	$w_i - \ell - \alpha\pi + \alpha\ell$

where $s_k = \frac{\beta\alpha kl}{j+1}$ for $k = 0,1,2,\ldots,n-1$ and $j = 0,1,2,\ldots,n-1-k$. The consumption of farmer i at state $(0,\ell,s_{kj})$ has different values for each $k = 0,1,2,\ldots,n-1$ and $j = 0,1,2,\ldots,n-1-k$.

Relaxing an Assumption

Model 2 is based on the model in Dercon et al. (2014). But as we mentioned above they consider a group of only two farmers. Then it is easy to explain informal risk sharing. But once we generalize that model by considering n farmers, it becomes problematic to define the informal risk sharing. Then we consider the idea behind the informal risk sharing in Santos et al. (2021). Our Model 3 is also based on the same work. They use the following assumption.

Assumption 6.1. The individuals contribute to the informal risk sharing pool even if nobody suffers a catastrophe.

We consider the same assumption for our Models 2 and 3. Because of this assumption the contribution to the risk sharing pool only depends on the state of the corresponding farmer and it does not depend on the states of the other farmers in the group. Therefore when there is no one in the state to receive money from the risk sharing pool still the farmers in good state transfer money to the pool. Since we consider only one period, then the money in the pool is useless. We relax that assumption to make the models more realistic. Now we consider the following assumption.

Assumption 6.2. Any farmer in the group contributes to the risk sharing pool only if there is at least one farmer in the state to receive money from the risk sharing pool (Model 2: At least one farmer face loss. Model 3: At least one farmer faces a loss does not receive insurance payout).

We modify Models 2 and 3 based on the above assumption and name those models Model 2* and Model 3*.

6.1.4 Model 2*

This model is different to Model 2 by the condition in which the farmer contributes to the risk sharing pool. Based on Assumption 6.2, farmer $i \in \{1, 2, 3, \ldots, n\}$ contributes to the risk sharing pool if there are one or more farmers with loss. Then the level of consumption of farmer i is given by

$$C_{2^*}^i = w_i - L_i - \beta\ell\mathbb{1}_{\{L_i=0, \sum_{j=1, j\neq i}^n L_j \neq 0\}} + S\mathbb{1}_{\{L_i=\ell\}} - \alpha\pi + \alpha P(I). \tag{6.5}$$

All the terms are defined in analogy to Model 2. The condition $\sum_{j=1,j\neq i}^n L_j \neq 0$ implies that there is at least one farmer who faces loss. We state the possible values of $C_{2^*}^i$ of this model and the corresponding probabilities in Table 6.4.

Table 6.4: Possible Values of $C^i_{2^*}$ – Model 2*

State	Probability	$C^i_{2^*}$
$0, 0, \sum_{j=1,j\neq i}^{n} L_j \neq 0$	$(1-q-r)(1-(1-p)^{n-1})$	$w_i - \beta\ell - \alpha\pi$
$0, 0, \sum_{j=1,j\neq i}^{n} L_j = 0$	$(1-q-r)(1-p)^{n-1}$	$w_i - \alpha\pi$
$0, \ell, s_k$	$r\rho_k$	$w_i - \ell + s_k - \alpha\pi$
$1, 0, \sum_{j=1,j\neq i}^{n} L_j \neq 0$	$(q+r-p)(1-(1-p)^{n-1})$	$w_i - \beta\ell - \alpha\pi + \alpha\ell$
$1, 0, \sum_{j=1,j\neq i}^{n} L_j = 0$	$(q+r-p)(1-p)^{n-1}$	$w_i - \alpha\pi + \alpha\ell$
$1, \ell, s_k$	$(p-r)\rho_k$	$w_i - \ell + s_k - \alpha\pi + \alpha\ell$

Due to the new assumption we divide state $(0, 0)$ and $(1, 0)$ of Model 2 in two states based on whether farmer i contributes or does not contribute to the risk sharing pool.

6.1.5 Model 3*

Now we describe the modification of Model 3 based on Assumption 6.2. Here farmer $i \in \{1, 2, 3, \ldots, n\}$ contributes to the risk sharing pool if there is one or more farmers with loss who do not receive insurance payout. Then the level of consumption of farmer i is given by

$$C^i_{3^*} = w_i - L_i - \beta\alpha P(I_i)\mathbb{1}_{\{L_i=0, \sum_{j=1,j\neq i}^{n} L_j I_j < \sum_{j=1,j\neq i}^{n} L_j\}} + S\mathbb{1}_{\{L_i=\ell, I_i=0\}} - \alpha\pi + \alpha P(I_i). \tag{6.6}$$

All the terms are defined similar to Model 3. The condition $\sum_{j=1,j\neq i}^{n} L_j I_j < \sum_{j=1,j\neq i}^{n} L_j$ implies that there is at least one farmer who faces loss and there is at least one farmer who does not get insurance payout out of the farmers who face loss. We state the possible values of $C^i_{3^*}$ of this model and the corresponding probabilities in Table 6.5.

Table 6.5: Possible Values of $C^i_{3^*}$ – Model 3*

State	Probability	$C^i_{3^*}$
$0, 0$	$(1-q-r)$	$w_i - \alpha\pi$
$0, \ell, s_{kj}$	$r\rho_k$	$w_i - \ell + s_{kj} - \alpha\pi$
$1, 0, \sum_{j=1,j\neq i}^{n} L_j I_j < \sum_{j=1,j\neq i}^{n} L_j$	$(q+r-p)(1-(1-r)^{n-1})$	$w_i - \beta\alpha\ell - \alpha\pi + \alpha\ell$
$1, 0, \sum_{j=1,j\neq i}^{n} L_j I_j = \sum_{j=1,j\neq i}^{n} L_j$	$(q+r-p)(1-r)^{n-1}$	$w_i - \alpha\pi + \alpha\ell$
$1, \ell$	$(p-r)$	$w_i - \ell - \alpha\pi + \alpha\ell$

Due to the new assumption we divide state $(1, 0)$ of Model 3 in two states based on whether farmer i contributes or does not contribute to the risk sharing pool.

6.1.6 Expected Utility of the Level of Consumption

We compare the expected utility of farmer i's level of consumption for different cases under all five models. The farmer's preference over the wealth is given by exponential utility, $u(x) = \frac{1-e^{-\gamma x}}{\gamma}$, γ is the coefficient of risk aversion. We consider the level of consumption under four different situations. We obtain them by substituting the appropriate values of α and β in Equations (6.1), (6.3), (6.4), (6.5) and (6.6) for Model 1, 2, 3, 2* and 3*, respectively. Below we mention those 4 situations with the appropriate values of α and β.

1. No insurance and no risk sharing $\longrightarrow \alpha = 0, \beta = 0$.

2. Risk sharing and no index insurance $\longrightarrow \alpha = 0, \beta > 0$.

3. Index insurance and no risk sharing $\longrightarrow \alpha > 0, \beta = 0$.

4. Index insurance and risk sharing $\longrightarrow \alpha > 0, \beta > 0$.

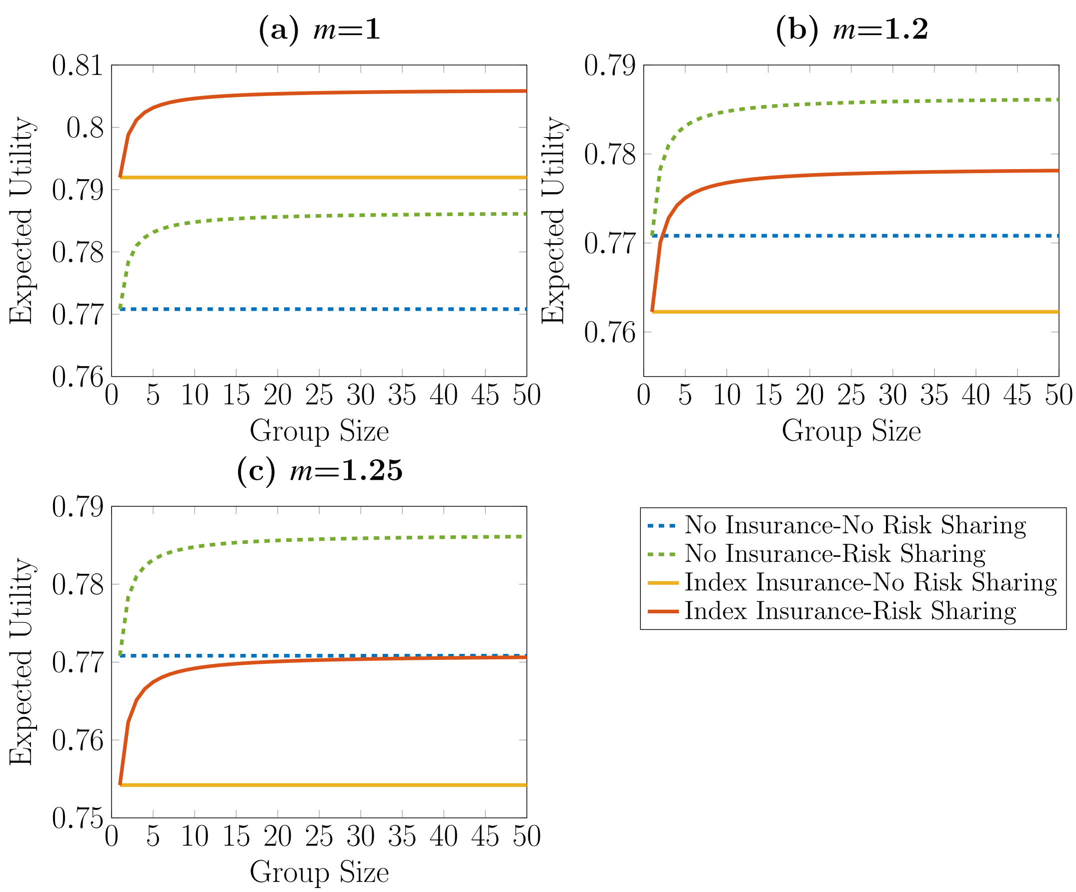

Figure 6.4: Expected Utility of Consumption – Model 1.

We numerically compute the expected utility of farmer i's level of consumption for the above four cases under all five models. The expected utilities of the four cases are plotted against the group size n. We choose different group sizes from 1 to 50. It is clear that when there is no informal risk sharing the expected utility of consumption does not depend on the group size. Therefore the expected utilities of the farmer for situations 1 and 3 do not change with the group size.

Let $\gamma = 1$, $\alpha = 1$, $\beta = 0.5$, $w_i = 1.5$ for all $i = 1, 2, \ldots, n$, $p = 1/3$, $q = 1/3$. $\mu_i = 1$ for all $i = 1, 2, \ldots, n$, $\overline{\theta}_g = 1$, $\underline{\theta}_g = 0$, $\overline{\theta} = 1$, $\underline{\theta} = 0$, $r = 1/9$ and $\ell = 1$. The obtained results are given in Figures 6.4, 6.5, 6.6, 6.7 and 6.8. We consider different levels of premium loads m to see the effect of the premium.

According to Figure 6.4 at actuarial fair price ($m = 1$) index insurance together with informal risk sharing is the best option. For the other two premium loads informal risk sharing without index insurance is the best option. Therefore, when the farmer is in an informal risk sharing group he does not prefer an index insurance with high premium.

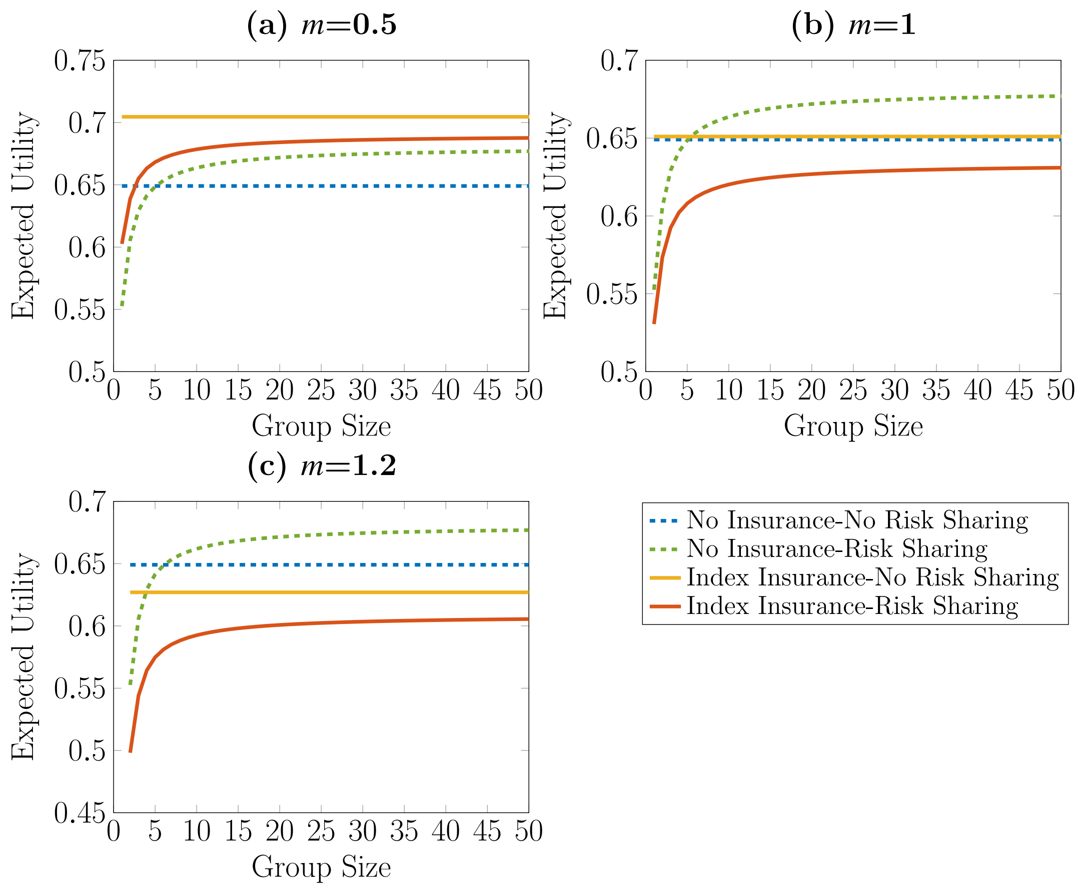

Figure 6.5: Expected Utility of Consumption – Model 2.

According to Figure 6.5 at a very small premium ($m = 0.5$) index insurance without informal risk sharing is the best option. But when the premium is high ($m = 1$, $m = 1.2$) informal risk sharing without index insurance is the best option when the group size is not very small. When the group size is very small at actuarial fair price the farmer prefer index insurance without informal risk sharing. But when group size is very small and the premium is high the farmer does not prefer index insurance or informal risk sharing. Also it is clear that index insurance without informal risk sharing is always better than both together. Therefore insured farmers do not prefer informal risk sharing.

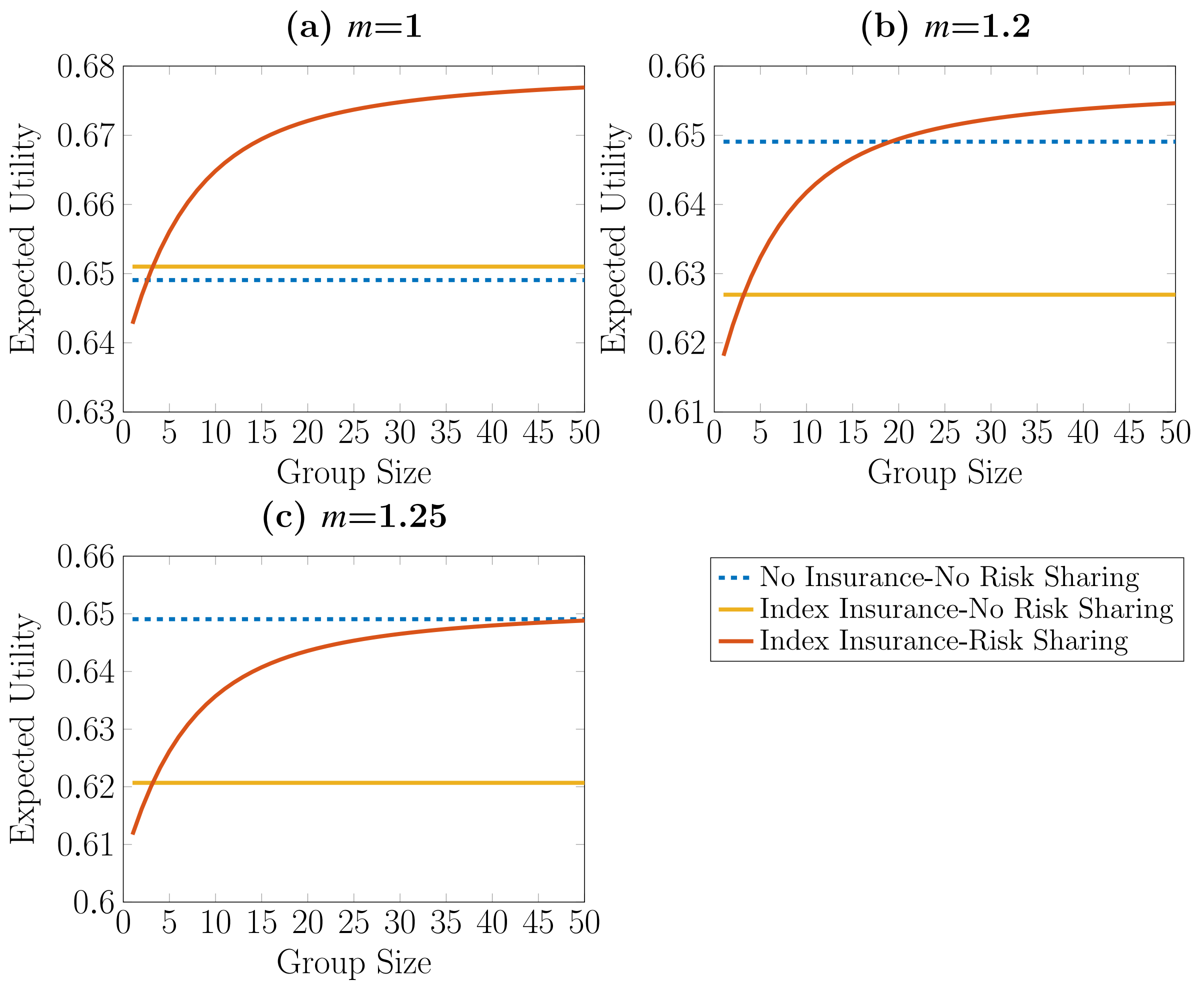

Figure 6.6: Expected Utility of Consumption – Model 3.

For Model 3 we consider only three cases. Because in Model 3 the informal risk sharing depends on the insurance payout. Therefore we can not consider the case of informal risk sharing without index insurance. According to Figure 6.6 at actuarial fair price the index insurance together with informal risk sharing is the best option under all the group sizes except very small group sizes ($n = 1, n = 2$). If the farmer is in those very small groups then he prefers the index insurance without informal risk sharing. At $m = 1.2$ when the group size is small the farmer does not prefer index insurance with or without informal risk sharing. But when the group size is high the farmer prefers index insurance together with informal risk sharing. This implies an index insurance which is not attractive to the farmers becomes attractive if they are in a big group in which farmers are willing to share risk informally. When the premium is slightly increased the farmer does not prefer index insurance with or without informal risk sharing under any given group size.

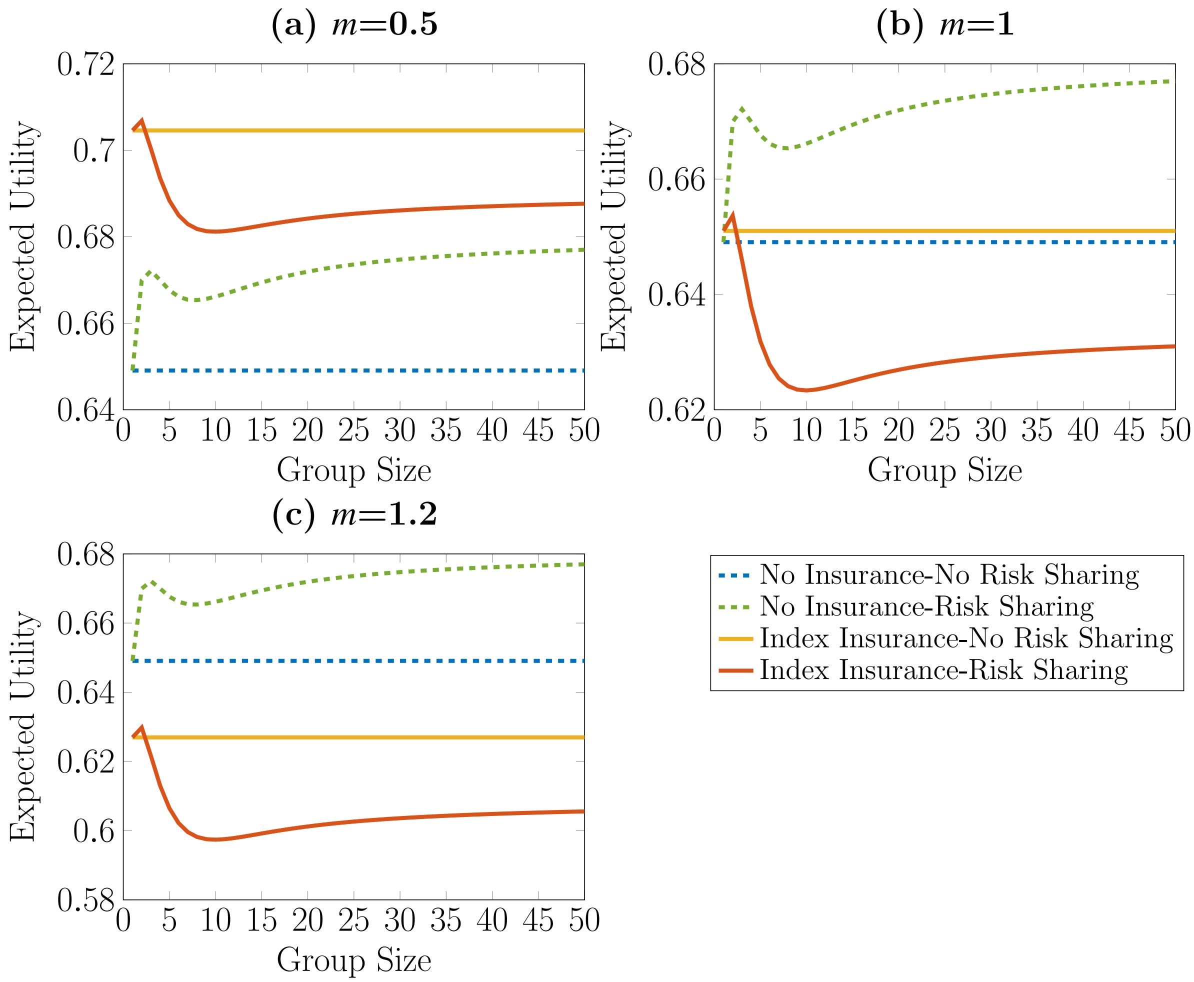

Figure 6.7: Expected Utility of Consumption – Model 2*.

In the above results we considered group sizes from 1 to 50. When the group size is 1 the informal risk sharing is inappropriate. But the risk sharing method under Model 1 still works for a group of 1 farmer, because the amount that the farmer contributes to the risk sharing pool returns to the same farmer. Then there is no effect from the informal risk sharing. As a result of that, when the group size is 1, the farmer's level of consumption with informal risk sharing is the same as without informal risk sharing. We can clearly see that in Figure 6.4. In that figure the expected utility of the farmer's consumption with and without informal risk sharing is the same when the group size is 1 (see the staring point of the red and yellow curves and green and blue curves of Figure 6.4). We can not see that feature in Figures 6.5 and 6.6. The possible reason is Assumption 6.1. According to that assumption the farmers contribute to the risk sharing pool even when nobody is in a bad state. Then regardless of the group size this is unfavorable for the farmers. We can clearly see this negative effect in Figure 6.5 and 6.6. In those figures the staring points of red and yellow curves are different due to the unnecessary contribution to the risk sharing pool. Even if it is not visible for the other group sizes, the farmers in any size of group sometimes inappropriately contribute to the risk sharing pool due to Assumption 6.1. We compute the expected utility of farmer's level of consumption for the models based on Assumption 6.2. That assumption avoids the unnecessary contributions. The obtained results are given in Figures 6.7 and 6.8.

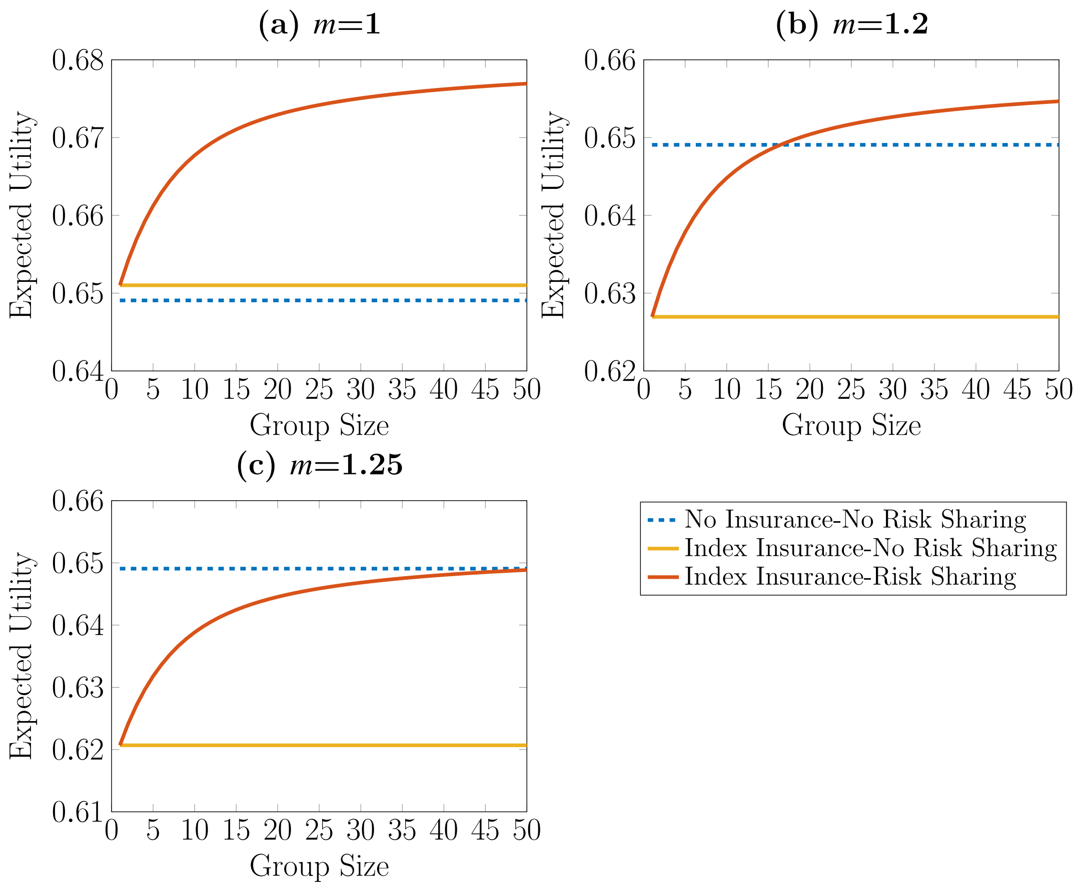

Figure 6.8: Expected Utility of Consumption – Model 3*.

Figure 6.7 and 6.8 clearly show that when the group size is 1 the expected utility with and without informal risk sharing is the same. According to Figure 6.7, when there are 2 farmers in the group they prefer index insurance with informal risk sharing over the index insurance without informal risk sharing. When the farmers informally share risk then the expected utility first increases then decreases and again increases as the group size increases. At actuarial fair or high premiums even when the group size is very small the farmer prefers informal risk sharing without index insurance. But in Model 2, when the group size is very small, the farmers do not prefer informal risk sharing. This may occur because there is no unnecessary contribution due to the new assumption. However, similar to Model 2 in this model also the preference to the index insurance together with informal risk sharing is very low.

Figure 6.8 is mostly similar to Figure 6.6. But in Model 3 as a result of Assumption 6.1 when the group size is very small the farmer prefers index insurance without risk sharing over the index insurance with informal risk sharing. Since we avoid that situation in Model 3* the farmer always prefers index insurance together with informal risk sharing to index insurance alone.

By these results we see that, when the group size is very small, the index insurance together with informal risk sharing performs better when we consider Assumption 6.2 instead of Assumption 6.1. We make a proper comparison to see the effect of replacement of Assumption 6.1 by Assumption 6.2. We compare the expected utility of the farmer's level of consumption with informal risk sharing under Model 2 and 2* as well as under Model 3 and 3*. We consider the actuarial fair price for the numerical computations. The comparison is given in Figure 6.9.

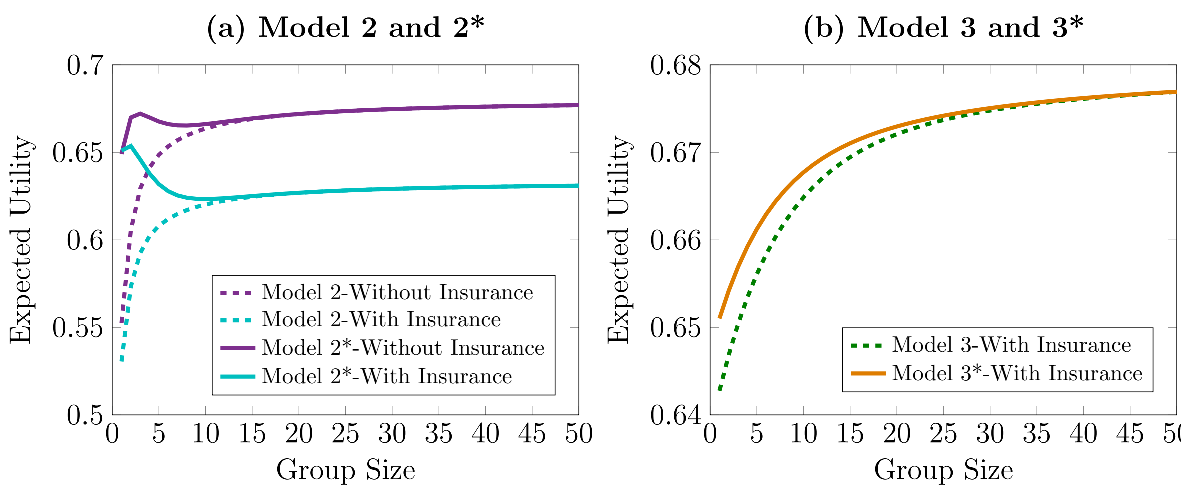

Figure 6.9: Model 2 vs 2* and Model 3 vs 3*.

According to Figure 6.9 (a) and (b), when the group size is small informal risk sharing as per Assumption 6.2 is more productive than the risk sharing as per Assumption 6.1. We can see the same behavior at moderate group sizes when comparing Models 3 and 3*. When the group size is high the expected utilities under the models based on Assumptions 6.1 and 6.2 are very close to each other. It is clear that when n is high $(1-p)^{n-1} \approx 0$ and $(1-r)^{n-1} \approx 0$. Then the probabilities of occurrence of the new wealth states due to Assumption 6.2 become very small. Even though it is not properly visible at high group sizes the expected utilities under the models based on Assumption 6.2 is slightly higher than the corresponding values under the models based on Assumption 6.1. Therefore according to the results we can say that the informal risk sharing based on Assumption 6.2 is better than the informal risk sharing based on Assumption 6.1 from the farmer's perspective.

6.2 Demand for the Index Insurance

The numerical results in the previous section show that the informal risk sharing affects the preference for the index insurance. Here we discuss the demand for index insurance in an informal risk sharing environment. We derive the optimal index insurance cover which maximizes expected utility of the farmer's level of consumption at a given fraction of risk sharing. We assume that the farmer is strictly risk averse over wealth, with utility function u satisfying $u' > 0$ and $u'' < 0$. Since Models 2* and 3* are the more realistic versions of Model 2 and 3, respectively, here we only consider Models 1, 2* and 3*. Now the optimal insurance cover under Models 1, 2* and 3* are given by

$$\alpha_1^* = \underset{\alpha \geq 0}{\text{argmax}}\, \mathbb{E}[u(\mu_i + \theta_g + \theta_i - \beta\theta_i + \frac{\beta}{n}\sum_{j=1}^{n}\theta_j - \alpha\pi + \alpha P(\theta_g))],$$

$$\alpha_{2^*}^* = \underset{\alpha \geq 0}{\text{argmax}}\, \mathbb{E}[u(w_i - L_i - \beta\ell\mathbb{1}_{\{L_i=0, \sum_{j=1, j\neq i}^{n} L_j \neq 0\}} + S\mathbb{1}_{\{L_i=\ell\}} - \alpha\pi + \alpha P(I))], \tag{6.7}$$

and

$$\alpha_{3^*}^* = \underset{\alpha \geq 0}{\text{argmax}}\, \mathbb{E}[u(w_i - L_i - \beta\alpha P(I_i)\mathbb{1}_{\{L_i=0, \sum_{j=1, j\neq i}^{n} L_j I_j < \sum_{j=1, j\neq i}^{n} L_j\}} + S\mathbb{1}_{\{L_i=\ell, I_i=0\}} - \alpha\pi + \alpha P(I_i))]. \tag{6.8}$$

respectively.

Proposition 6.3. *If the farmer's preference over wealth is given by exponential utility,* $u(x) = \frac{1-e^{-\gamma x}}{\gamma}$*, where* γ *is the coefficient of risk aversion, then the optimal insurance cover under Model 1 is given by*

$$\alpha_1^* = min\left[1, max\left[0, \frac{1}{\gamma(\overline{\theta}_g - \underline{\theta}_g)} \ln\left(\frac{(1-mq)}{m(1-q)} e^{\gamma(\overline{\theta}_g - \underline{\theta}_g)}\right)\right]\right]. \tag{6.9}$$

Here it is clear that the optimal insurance cover does not have any dependence on the fraction of risk sharing and the idiosyncratic shock. Therefore, at any fraction of informal risk sharing the farmer optimally choose the same insurance cover. As we have discussed, in

Model 1 we split the income of farming into covariate shock and idiosyncratic shock. The index insurance only depends on the covariate shock and the informal risk sharing only depends on the idiosyncratic shock. That may be the reason that there is no link between the optimal insurance cover and informal risk sharing. We obtain a totally different optimal insurance cover under Model 2*.

Proposition 6.4. *If the farmer's preference over wealth is given by exponential utility, $u(x) = \frac{1-e^{-\gamma x}}{\gamma}$, where γ is the coefficient of risk aversion, then the optimal insurance cover under Model 2* is given by*

$$\alpha^*_{2^*} = min\left[1, max\left[0, \frac{1}{\gamma\ell}\ln\left(\frac{(1-mq)}{mq}\frac{(q+r-p)((1-p)^{n-1}+(1-(1-p)^{n-1})e^{\gamma\beta\ell})+(p-r)e^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k}}{(1-q-r)((1-p)^{n-1}+(1-(1-p)^{n-1})e^{\gamma\beta\ell})+re^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k}}\right)\right]\right]. \tag{6.10}$$

Here the optimal insurance cover depends on the fraction of risk sharing. Therefore at different fractions of risk sharing the farmer optimally choose different insurance covers. We know that in Model 2* both index insurance and informal risk sharing depend on the loss. In Model 3* the informal risk sharing depends on the insurance payout. Therefore the insurance covers of all farmers are affected by informal risk sharing. In Model 3* the farmers differ to each other by the initial wealth. Due to the nature of the exponential utility function the initial wealth of the farmer does not affect to the optimal cover. But the level of risk aversion affects the optimal cover. Therefore in order to make sure any given farmer optimally choose the same insurance cover, we consider a group of farmers with the same level of risk aversion. In order to find the optimal insurance cover under Model 3*, we assume that all the farmers in the group choose the optimal insurance cover. But then it is difficult to solve the first order condition to express the optimal insurance cover as an explicit function. That first order condition is given by

$$\begin{aligned}\frac{\partial\mathbb{E}[u(C_i^{SI})]}{\partial\alpha} =& re^{\gamma\ell}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1-k}\left[e^{-\frac{\gamma\ell k\alpha\beta}{j+1}}\rho_{kj}\left(\frac{k\beta}{j+1}-mq\right)\right] + e^{-\gamma\ell(\alpha-1)}(p-r)(1-mq) \\ &+ e^{-\gamma\alpha\ell(1-\beta)}(q+r-p)(1-(1-r)^{n-1})(1-\beta-mq) \\ &+ e^{-\gamma\alpha l}(q+r-p)(1-r)^{n-1}(1-mq)-(1-q-r)mq = 0.\end{aligned} \tag{6.11}$$

An α which satisfy above equation is the optimal insurance cover. Due to the α inside the two summations, it is difficult to derive the optimal insurance cover.

6.2.1 Numerical Computations of the Optimal Insurance Cover

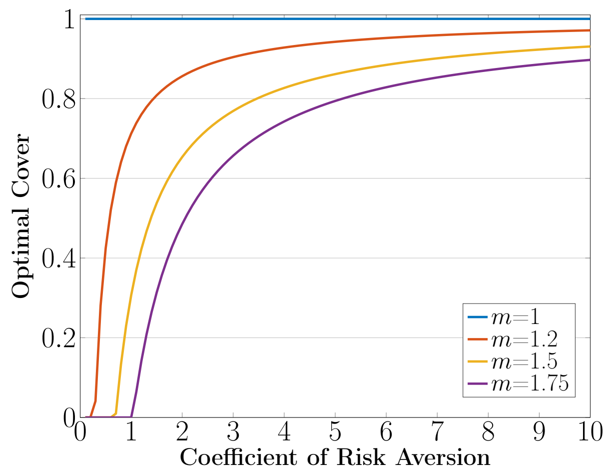

Figure 6.10: Optimal Cover – Model 1.

We numerically compute the optimal insurance cover under Model 1, 2* and 3*. Since the optimal insurance cover under Model 1 does not depend on the fraction of risk sharing, we compute the optimal cover at different premiums. We use the same parameter values as earlier. The optimal insurance covers are plotted against the coefficient of risk aversion, see Figure 6.10. According to the figure at actuarial fair price ($m = 1$) the farmer optimally chooses the full insurance cover. At the other three premium loads the farmer optimally chooses some partial cover and that optimal cover increases as the risk aversion increases. We can observe similar characteristics in a tradition indemnity insurance. Therefore the index insurance we consider in Model 1 behaves similar to a traditional indemnity insurance. Again the reason may be that in Model 1 we split the income of farming into covariate shock and idiosyncratic shock. The index insurance only depends on the covariate shock and the informal risk sharing only depends on the idiosyncratic shock.

Then we compute the optimal insurance cover under Model 2* and 3*. Since in these two models the optimal covers depend on the fraction of risk sharing, we consider three levels of β. In addition to that we consider three levels of m. Other parameter values are similar to the values of the numerical computations of Model 2* and 3* in Section 6.1.6. In addition to that we let $n = 50$. The optimal insurance covers are plotted against the coefficient of risk aversion and those are given in Figure 6.11 and 6.12. At the three different premium loads the optimal covers behave in different manners under both models. According to Figure 6.11 when $m = 0.5$ the optimal insurance cover decreases as the risk aversion increases. But when $m = 1$ or $m = 1.2$ at some β the optimal cover first increases and then decreases as the risk aversion increases. At $m = 1$ and $m = 1.2$ high risk averse farmers purchase approximately the

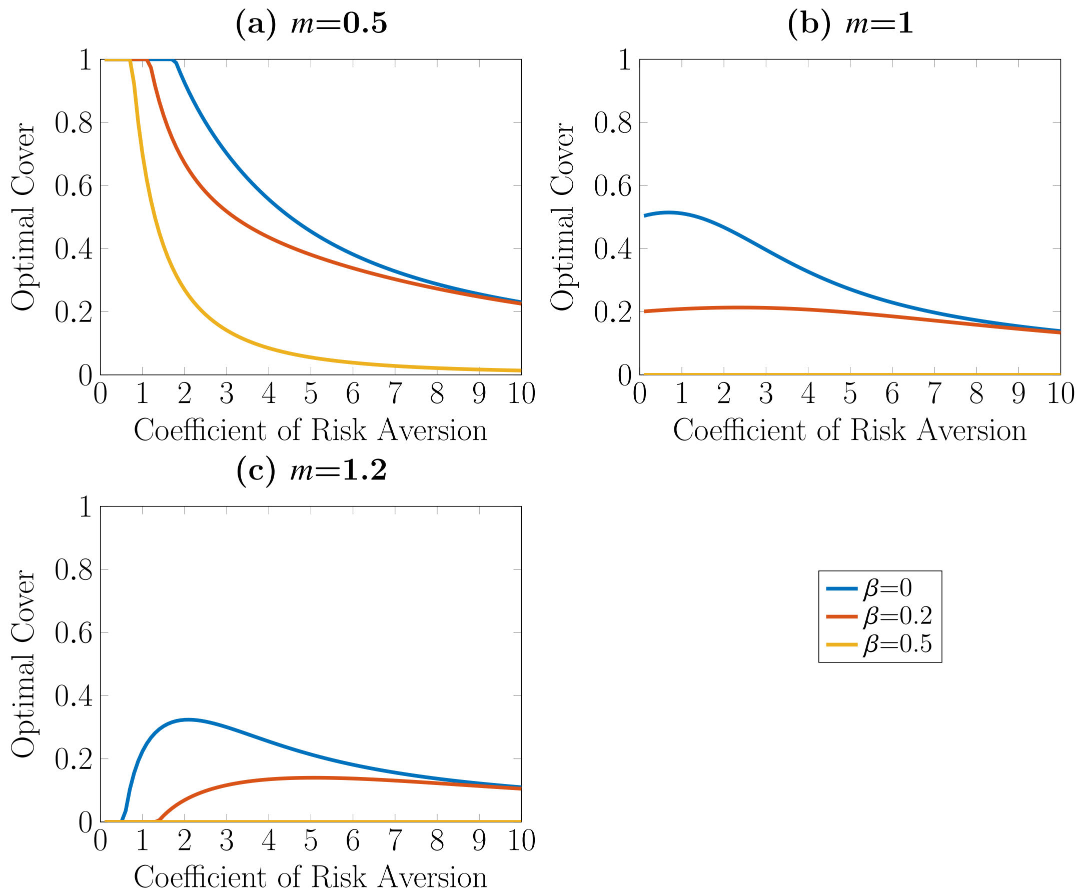

Figure 6.11: Optimal Cover – Model 2*.

same amount of the insurance when $\beta = 0$ and $\beta = 0.2$. Also there is no demand for the index insurance when $\beta = 0.5$. However at all three levels of m the red curve ($\beta = 0.2$) is below the blue curve ($\beta = 0$) and the yellow curve ($\beta = 0.5$) is below the red curve. Therefore, according to this numerical example, at a given level of risk aversion the demand of the index insurance decreases as the fraction of risk sharing increases. Then the insurer can expect higher demand by selling the insurance to the farmers who do not informally share risk than selling the same insurance to a risk sharing group.

According to Figure 6.12 in Model 3*, similar to Model 2*, when $m = 0.5$ the optimal demand decreases as the risk aversion increases. Also when $m = 1$ or $m = 1.2$ the optimal demand first increases and then decreases as the risk aversion increases. But in contrast to Model 2*, here the red curve is below the yellow curve and the blue curve is below the red curve. This implies that at a given level of risk aversion the demand of the index insurance increases as the fraction of risk sharing increases. Therefore, the informal risk sharing positively affects the demand for the index insurance. According to our numerical example an informal risk sharing group purchases more insurance than a group without informal risk sharing. In

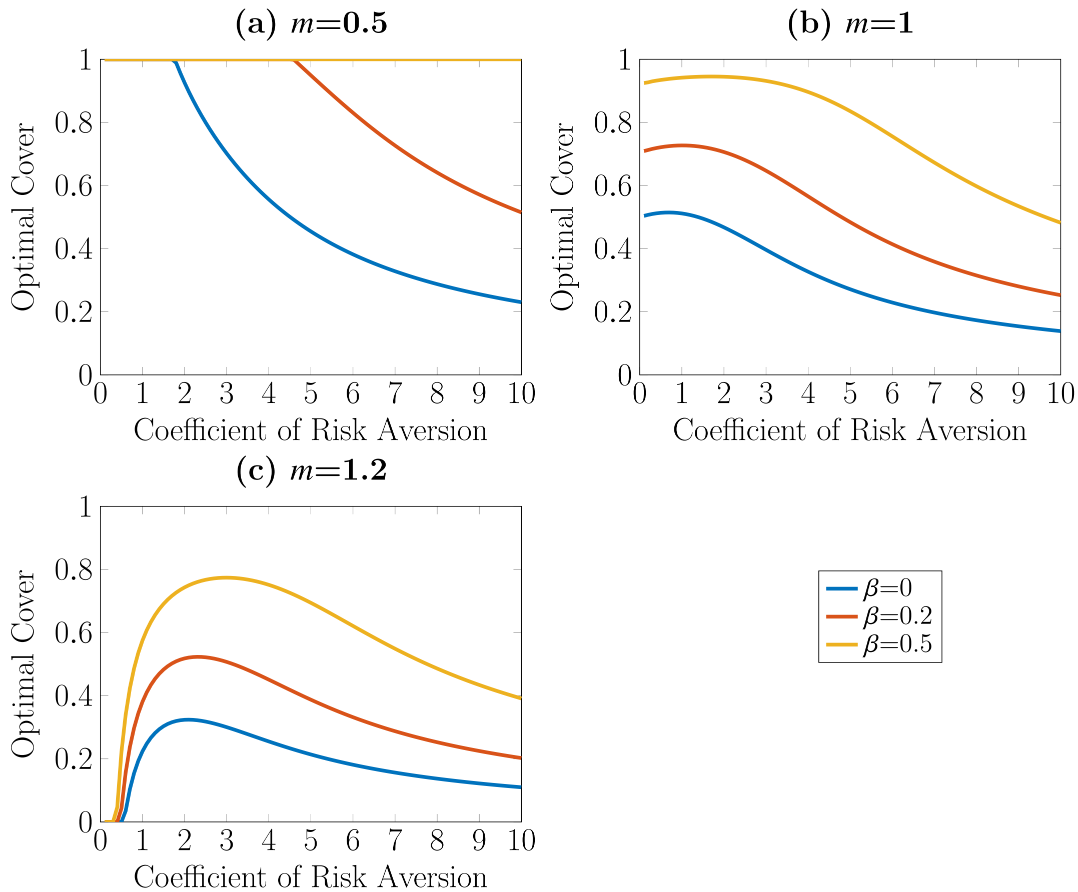

Figure 6.12: Optimal Cover – Model 3*.

this model the informal risk sharing depends on the insurance payout. That may lead to the above behavior.

According to the numerical results in Figure 6.9, when the group size is high, the expected utility of farmer's level of consumption under Model 2 and 2* are very close to each other. We see the same behavior under Model 3 and 3*. Therefore, when $n = 50$ we expect the same optimal insurance covers under Model 2 and 2* and also under Model 3 and 3*. In order to see the effect of Assumption 6.2 over Assumption 6.1 on the optimal demand of the index insurance, we should consider small group sizes. We compare the optimal insurance covers under Model 2 and 2* at $n = 5$ and $n = 10$. We do a same comparison between Model 3 and 3*. We consider actuarial fair premium for the computations. According to out comparison between Model 2 vs 2* and Model 3 vs 3* at small group sizes, we realize that replacement of Assumption 6.1 by Assumption 6.2 is effective in terms of insurance demand specially when the group size is small, premium is small and the farmers are less risk averse (See Figure B.4 and B.5 in Appendix B).

6.2.2 Optimal Insurance Demand vs Fraction of Risk Sharing

As we discussed earlier, according to our numerical example in Models 2 and 2* the optimal demand of the index insurance deceases as the fraction of risk sharing increases. But in Models 3 and 3* the optimal demand of the index insurance increases as the fraction of risk sharing increases. But in our numerical example we consider only three levels of β. Therefore we need further analysis to identify the relationship between the optimal insurance demand and fraction of risk sharing. Here we discuss how the fraction of risk sharing affects the optimal insurance demand. The following proposition show relationship between the fraction of risk sharing and the optimal insurance demand of Model 2*.

Proposition 6.5. *In Model 2* the optimal insurance demand decreases as the fraction of risk sharing increases.*

Proof. Let

$$A = \frac{(1-mq)}{mq}\frac{(q+r-p)((1-p)^{n-1}+(1-(1-p)^{n-1})e^{\gamma\beta\ell})+(p-r)e^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k}}{(1-q-r)((1-p)^{n-1}+(1-(1-p)^{n-1})e^{\gamma\beta\ell})+re^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k}}.$$

Then as given in Equation (6.10)

$$\alpha_{2^*}^* = \min\left[1, \max\left[0, \frac{1}{\gamma\ell}\ln(A)\right]\right].$$

Then we consider the first derivative of A with respect to β. It is given by

$$\frac{\partial A}{\partial \beta} = -\frac{(1-mq)}{mq}\left(\frac{(p(1-q)-r)C}{((1-q-r)((1-p)^{n-1}+(1-(1-p)^{n-1})e^{\gamma\beta\ell})+re^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k})^2}\right),$$

where
$$C = \gamma\ell e^{\gamma\ell}\left[(1-(1-p)^{n-1})e^{\gamma\beta\ell}\left(\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k} + \sum_{k=0}^{n-1}\frac{\rho_k e^{-\gamma s_k}}{(n-k)}\right) + (1-p)^{n-1}\sum_{k=0}^{n-1}\frac{\rho_k e^{-\gamma s_k}}{(n-k)}\right].$$

As discussed by Clarke (2016) an index realization to be a good signal of loss, we require $\frac{p-r}{q+r-p} > \frac{r}{1-q-r}$. This implies $p(1-q)-r > 0$. Therefore $\frac{\partial A}{\partial \beta} < 0$. Now it is clear that in Model 2* the optimal insurance demand decreases as the fraction of risk sharing increases. □

Note that similarly we can show that in Model 2 the optimal insurance demand decreases as β increases (see Appendix B).

In order to see this behavior we plot the optimal insurance cover against β. We consider the same parameter values as in Section 6.1.6. The obtained numerical results are given in Figure 6.13. According to Figure 6.13 it is clear that the optimal insurance demand of Model 2*

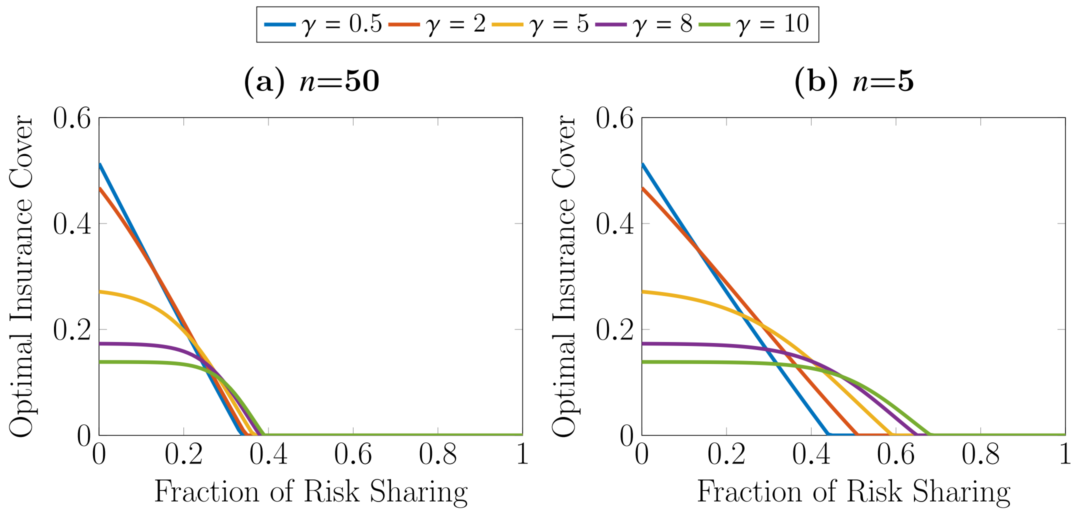

Figure 6.13: Optimal Cover vs β – Model 2*.

decreases as the fraction of risk sharing increases. When $n = 50$ under Model 2* the farmers do not take index insurance when the fraction of risk sharing is greater than 0.4. But when $n = 5$ the farmers, specially the high risk averse farmers purchase index insurance even at β's greater than 0.4. This results implies that when the group size is high and β is moderate or high farmers are not interested in the index insurance. But when the group size is small and β is moderate in size, the farmers still take some partial cover of the index insurance and the farmers do not take insurance at high β's. A possible reason may be that, when the group size is high the farmers get a proper protection against risk even at a small β, but when the group size is small high β may be required for a proper protection.

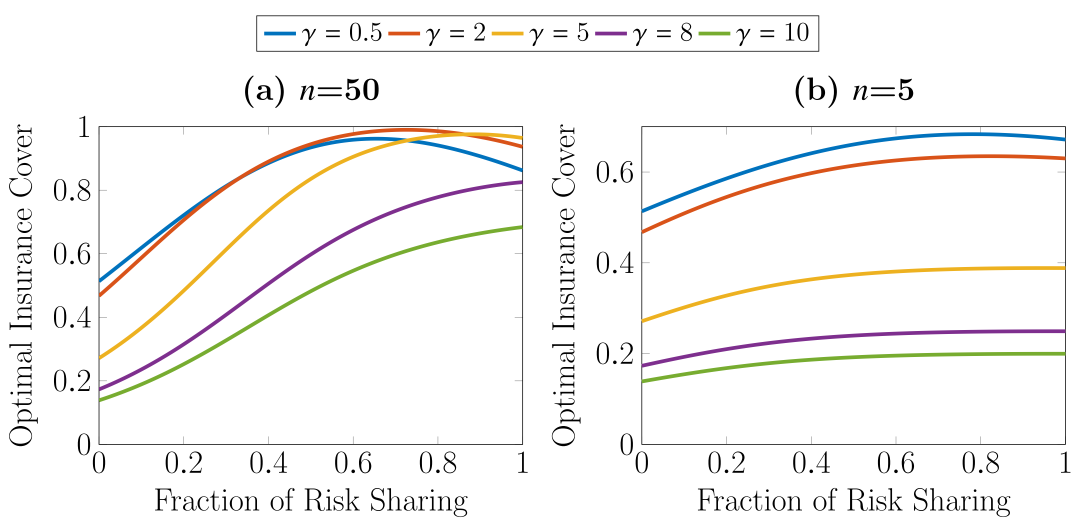

Figure 6.14: Optimal Cover vs β – Model 3*.

As we discussed above we are unable to derive the optimal insurance cover in Model 3* as explicit functions. Therefore it is difficult to show the relationship between the optimal insurance cover and the fraction of informal risk sharing of Model 3 and 3*. Since it is still possible to numerically compute the optimal insurance cover at a given level of β, we plot the optimal insurance cover of Model 3 and 3* against the fraction of risk sharing. The obtained results are given in Figure 6.14. According to Figure 6.14 it is clear that the optimal insurance cover of Model 3* behaves totally different to the optimal insurance cover of Model 2*. Here in most of the considered cases the optimal insurance cover first increases and then decreases as the fraction of risk sharing increases.

6.2.3 Optimally Cover Risk

In the previous section we discussed the optimal choice of the index at a given level of informal risk sharing. Now we assume that the group of farmers choose the level of insurance cover and the fraction of informal risk sharing in order to optimally cover their risk. Here again we consider model by model. As we showed earlier in Model 1, β does not affect to the optimal insurance cover. Similarly, insurance cover does not affect the optimal β. We can easily see that by the first order condition of farmer's level of consumption of Model 1 with respect to β. That first order condition is given by

$$\begin{aligned}(1-p)e^{-\gamma\overline{\theta}}\sum_{k=0}^{n-1}\rho_k e^{-\gamma\beta\left(\frac{s_k-\overline{\theta}(n-1)}{n}\right)}\left(\frac{s_k-\overline{\theta}(n-1)}{n}\right)\\+pe^{-\gamma\underline{\theta}}\sum_{k=0}^{n-1}\rho_k e^{-\gamma\beta\left(\frac{s_k-\underline{\theta}(n-1)}{n}\right)}\left(\frac{s_k-\underline{\theta}(n-1)}{n}\right)=0.\end{aligned}$$

A β which satisfies the above equation is the optimal fraction of informal risk sharing. Due to the nature of that equation we can not derive the optimal β explicitly. It is more complicated for other models to derive the optimal β at optimal α, because in those models α depends on β. Optimal β of Model 2* and Model 3* at $\alpha^*_{2^*}$ and $\alpha^*_{3^*}$ are given by

$$\beta^*_{2^*}=\underset{\beta\geq 0}{\operatorname{argmax}}\,\mathbb{E}[u(w_i-L_i-\beta\ell\mathbb{1}_{\{L_i=0,\sum_{j=1,j\neq i}^n L_j\neq 0\}}+S\mathbb{1}_{\{L_i=\ell\}}-\alpha^*_{2^*}\pi+\alpha^*_{2^*}P(I))],$$

and

$$\beta^*_{3^*}=\underset{\beta\geq 0}{\operatorname{argmax}}\,\mathbb{E}[u(w_i-L_i-\beta\alpha^*_{3^*}P(I_i)\mathbb{1}_{\{L_i=0,\sum_{j=1,j\neq i}^n L_jI_j<\sum_{j=1,j\neq i}^n L_j\}}+S\mathbb{1}_{\{L_i=\ell,I_i=0\}}-\alpha^*_{3^*}\pi+\alpha^*_{3^*}P(I_i))].$$

respectively.

Here $\alpha^*_{2^*}$ is a function of β and given in Equation (6.10) and Here $\alpha^*_{3^*}$ is a function of β which satisfies Equation (6.11). We numerically compute the optimal insurance cover and the optimal level of informal risk sharing for Model 2* and 3*. We show them in Figure 6.15 and 6.16.

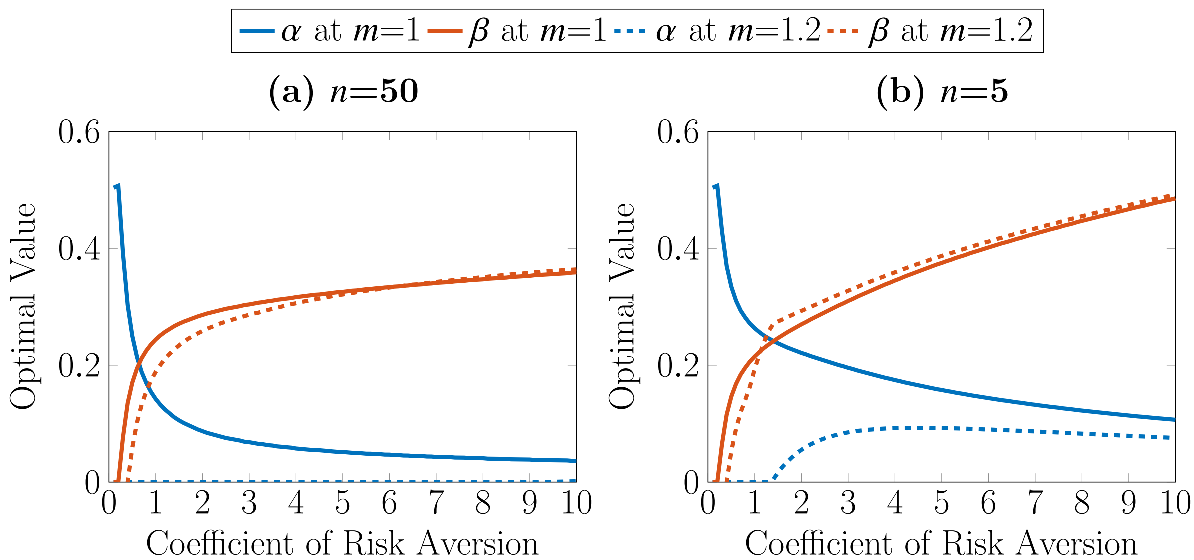

Figure 6.15: Optimal Insurance Cover and Level of Informal Risk Sharing – Model 2*.

According to the numerical results in Figure 6.15 it is clear that at the actuarial fair price under both group sizes the optimal insurance cover decreases as the risk aversion increases. But in contrast to that the optimal β increases as the risk aversion increases. Therefore, we can say that at actuarial fair price when the group of farmers is more risk averse the farmers prefer less insurance and more informal risk sharing compared to a less risk averse group. When $m = 1.2$ the optimal β still increases as the risk aversion increases. But when the group size is small the optimal insurance cover first increases and then slightly decreases as the risk aversion increases. When the group size is large the farmers do not purchase index insurance at $m = 1.2$. This implies that a farmer in a small risk sharing group purchases some partial cover of the index insurance even at a high price. But a farmer in a large group does not purchases index insurance at high price and uses only informal risk sharing. It is also clear that the optimal β does not have a considerable effect from the insurance premium.

According to Figure 6.16 the optimal insurance cover at all the cases first increases and then decreases as the risk aversion increases. However a partial cover is always optimal. But once we focus on the optimal β, at actuarial fair price the optimal fraction of risk sharing is 1 for both group sizes and all the considered levels of risk aversion. This implies that, the farmers prefer to share the full amount of the unnecessarily received insurance payout within the group.

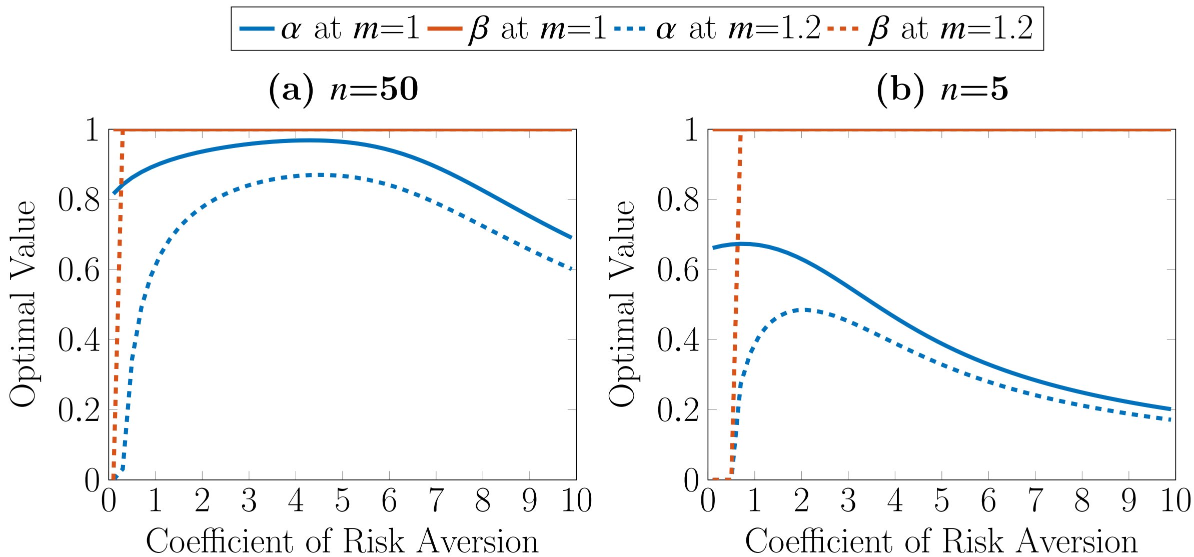

Figure 6.16: Optimal Insurance Cover and Level of Informal Risk Sharing – Model 3*.

Also when $m = 1.2$ only the group of very low risk averse farmers does not prefer to informally share risk. At the same time at other levels of risk aversion the farmers optimally share the full amount of the unnecessarily received insurance payout.

6.3 Summary

The main results of this chapter are:

- The performance of index insurance together with informal risk sharing depends on several factors, such as nature of informal risk sharing, group size and insurance premium.
- In Model 1 the optimal insurance cover and fraction of informal risk sharing are independent.
- In Model 2 the optimal insurance cover decreases as the fraction of informal risk sharing increases.
- In Model 2* the optimal insurance cover decreases as the fraction of informal risk sharing increases.
- In Model 3* we numerically show that the optimal insurance cover first increases and then decreases as the fraction of informal risk sharing increases.

Chapter 7

Gap Insurance and Reinsurance

In this chapter we consider two further aspects of insurance and discuss them in the index insurance framework. Those aspects are gap insurance for index insurance and index insurance as a reinsurance. Here a gap insurance is an insurance which covers the gap between the actual loss and the payout of the index insurance. As suggested in Clarke (2016), a combined index and gap insurance may increase the demand of the index insurance. Doherty and Richter (2002) discuss the efficiency of combining an index insurance with a gap insurance. Motivated by those works we analyze the influence and effectiveness of a gap insurance on an index insurance.

In addition to that, we also discuss an index insurance which is used as a reinsurance. Zeng (2005) has done similar work and we are motivated by that to study the index insurance as a reinsurance. Zeng (2005) addresses how index insurances enhance the efficiency of reinsurance. Also they derive optimal proportional reinsurance in a mean-variance framework. The second part of this chapter discusses an index insurance as a proportional and excess loss reinsurance. In contrast to Zeng (2005), in order to derive optimal reinsurance contracts we consider the expected utility theory.

Part 1: Gap Insurance for Index Insurance

7.1 Gap Insurance Model

Here we consider the index insurance model of Clarke (2016) and a gap insurance connected to that model. The farmer face loss L, where $L = l$ with probability p and $L = 0$ with probability

$1-p$. The payout of the index insurance depends on the weather index I where

$$I = \begin{cases} 1 & \text{bad weather with probability } q \\ 0 & \text{good weather with probability } 1-q. \end{cases}$$

Then the payout of the full cover index insurance is given by $P_I = lI$. Let r be the probability that a loss occurs but the index indicates good weather. w_0 is the initial wealth of the farmer. The premium of the index insurance is given by $\pi_i = \alpha m_i \mathbb{E}[P_I] = \alpha m_i q l$, where α is the index insurance cover purchased by the farmer and m_i is the premium load of the index insurance.

Since we consider a gap insurance to cover the gap between index insurance payout and the actual loss of the farmer, the payout to the farmer from the gap insurance is given by $P_G = \max\{L - \alpha P_I, 0\}$. Now the premium of the gap insurance is given by

$$\begin{aligned} \pi_g &= \beta m_g \mathbb{E}[P_G] \\ &= \beta m_g r l + \beta m_g (p-r)(l - \alpha l) \\ &= \beta m_g p l - \beta \alpha m_g (p-r) l, \end{aligned}$$

where β is the gap insurance cover purchased by the farmer and m_g is the premium load of the gap insurance. We ignore the cases in which $q m_i \geq 1$ and $p m_g \geq 1$ because in those cases the premium paid is higher than the maximum possible indemnity payment of the corresponding insurance. There are four possible wealth levels for a farmer with the index insurance and gap insurance as introduced above. Those are given in Table 7.1 with the corresponding probabilities.

Table 7.1: Wealth Levels and Probabilities

State s	Probability p_s	Wealth
$l0$	r	$w_0 - \pi_i - \pi_g - l + \beta l$
$l1$	$p - r$	$w_0 - \pi_i - \pi_g - l + \alpha l + \beta(l - \alpha l)$
00	$1 - q - r$	$w_0 - \pi_i - \pi_g$
01	$q + r - p$	$w_0 - \pi_i - \pi_g + \alpha l$

As explained in Clarke (2016) the index realization to be a signal of loss, it is required $\frac{p_{l1}}{p_{01}} > \frac{p_{l0}}{p_{00}}$. This implies $r < p(1-q)$. We also consider that r is positive and the probabilities of all four states are non negative. Therefore $0 < r < p(1-q)$, $p - q \leq r$ and $q + r \leq 1$.

7.2 Optimal Index and Gap Insurance Covers

As an expected utility maximizer the farmer's objective is to choose the levels of α and β which maximize his expected utility. We assume that the farmer is strictly risk averse over his wealth with utility function u which satisfies $u' > 0$ and $u'' < 0$. Now we state the optimization problem of the farmer as

$$\max_{\alpha,\beta} \mathbb{E}[u(w_0 - \pi_i - \pi_g - L + \alpha P_I + \beta P_G)]. \tag{7.1}$$

According to the four possible wealth levels in Table 7.1, we state the expected utility in Equation (7.1) ($\mathbb{E}U$) of the farmer is

$$\begin{aligned}\mathbb{E}U =& ru(w_0 - \pi_i - \pi_g - l + \beta l) + (p - r)u(w_0 - \pi_i - \pi_g - l + \alpha l + \beta(l - \alpha l)) \\ &+ (1 - q - r)u(w_0 - \pi_i - \pi_g) + (q + r - p)u(w_0 - \pi_i - \pi_g + \alpha l).\end{aligned} \tag{7.2}$$

First we focus on deriving the optimal index insurance cover at a given level of gap insurance cover and vice versa. According to the expected utility of the insured's wealth in Equation (7.2) we can find the optimal index insurance cover without solving the problem (7.1) under a special condition on the gap insurance cover. We explain it in the following proposition.

Proposition 7.1. *If the gap insurance cover satisfies $\beta \geq \frac{qm_i}{(p-r)m_g}$, then the full index insurance cover ($\alpha = 1$) is optimal.*

Proof. Let $w_{l0} = w_0 - \pi_i - \pi_g - l + \beta l$, $w_{l1} = w_0 - \pi_i - \pi_g - l + \alpha l + \beta(l - \alpha l)$, $w_{00} = w_0 - \pi_i - \pi_g$, $w_{01} = w_0 - \pi_i - \pi_g + \alpha l$.
Suppose $\beta \geq \frac{qm_i}{(p-r)m_g}$. Then $qm_i - \beta(p - r)m_g \leq 0$. Since $\frac{\partial(\pi_i+\pi_g)}{\partial\alpha} = qm_i - \beta(p - r)m_g$, $\frac{\partial(\pi_i+\pi_g)}{\partial\alpha} \leq 0$. That is $(\pi_i + \pi_g)$ decreases as α increases or $(-\pi_i - \pi_g)$ increases as α increases. Then w_{l0}, w_{l1}, w_{00} and w_{01} increase as α increases. Since u is an increasing function, $\mathbb{E}U$ increases as α increases. Then the maximum possible level of α is the index insurance cover which maximize the expected utility. This result in choosing $\alpha = 1$ as the optimal cover. □

We can use Proposition 7.1 to find the optimal index insurance cover only if $\beta \geq \frac{qm_i}{(p-r)m_g}$, But we are also interested in finding the optimal index insurance cover for the rest of the gap insurance covers. For any given $\beta < \frac{qm_i}{(p-r)m_g}$ we can compute the optimal α by the first order condition of the expected utility in Equation (7.2) with respect to α. That first order condition is given by

$$\begin{aligned}\frac{\partial \mathbb{E}U}{\partial \alpha} =& r u'_{l0}(-q m_i l + \beta m_g (p-r) l) \\ &+ (p-r) u'_{l1}(-q m_i l + \beta m_g (p-r) l + l - \beta l) \\ &+ (1-q-r) u'_{00}(-q m_i l + \beta m_g (p-r) l) \\ &+ (q+r-p) u'_{01}(-q m_i l + \beta m_g (p-r) l + l) = 0,\end{aligned} \tag{7.3}$$

where u'_s is the marginal utility of state s. e.g. $u'_{l0} = u'(w_0 - \pi_i - \pi_g - l + \beta l)$. The α which satisfies Equation (7.3) is the optimal index insurance cover and it is a function of β.

Now for a given α we can compute the optimal β by the first order condition of the expected utility in Equation (7.2) with respect to β, which is given by

$$\begin{aligned}\frac{\partial \mathbb{E}U}{\partial \beta} =& r u'_{l0}(-m_g p l + \alpha m_g (p-r) l + l) \\ &+ (p-r) u'_{l1}(-m_g p l + \alpha m_g (p-r) l + l - \alpha l) \\ &+ (1-q-r) u'_{00}(-m_g p l + \alpha m_g (p-r) l) \\ &+ (q+r-p) u'_{01}(-m_g p l + \alpha m_g (p-r) l) = 0.\end{aligned} \tag{7.4}$$

The β which satisfies Equation (7.4) is the optimal gap insurance cover. It is a function of α. Now we consider a utility function with constant absolute risk aversion for which $u' \propto e^{-\gamma x}$, where γ is the coefficient of absolute risk aversion. Then we restructure the first order conditions in Equation (7.3) and (7.4) as

$$\begin{aligned}e^{\gamma(w_0 - \pi_i - \pi_g)} \frac{\partial \mathbb{E}U}{\partial \alpha} =& r e^{-\gamma l(\beta - 1)}(-q m_i l + \beta m_g (p-r) l) \\ &+ (p-r) e^{-\gamma l(\alpha + \beta(1-\alpha) - 1)}(-q m_i l + \beta m_g (p-r) l + l - \beta l) \\ &+ (1-q-r)(-q m_i l + \beta m_g (p-r) l) \\ &+ (q+r-p) e^{-\gamma l \alpha}(-q m_i l + \beta m_g (p-r) l + l) = 0,\end{aligned} \tag{7.5}$$

$$\begin{aligned}e^{\gamma(w_0 - \pi_i - \pi_g)} \frac{\partial \mathbb{E}U}{\partial \beta} =& r e^{-\gamma l(\beta - 1)}(-m_g p l + \alpha m_g (p-r) l - l) \\ &+ (p-r) e^{-\gamma l(\alpha + \beta(1-\alpha) - 1)}(-m_g p l + \alpha m_g (p-r) l + l - \alpha l) \\ &+ (1-q-r)(-m_g p l + \alpha m_g (p-r) l) \\ &+ (q+r-p) e^{-\gamma l \alpha}(-m_g p l + \alpha m_g (p-r) l) = 0.\end{aligned} \tag{7.6}$$

We can compute the optimal α by Equation (7.5) for a given β. Similarly we can compute the optimal β by Equation (7.6) for a given α.

7.2.1 Numerical Computation of Optimal Insurance Covers

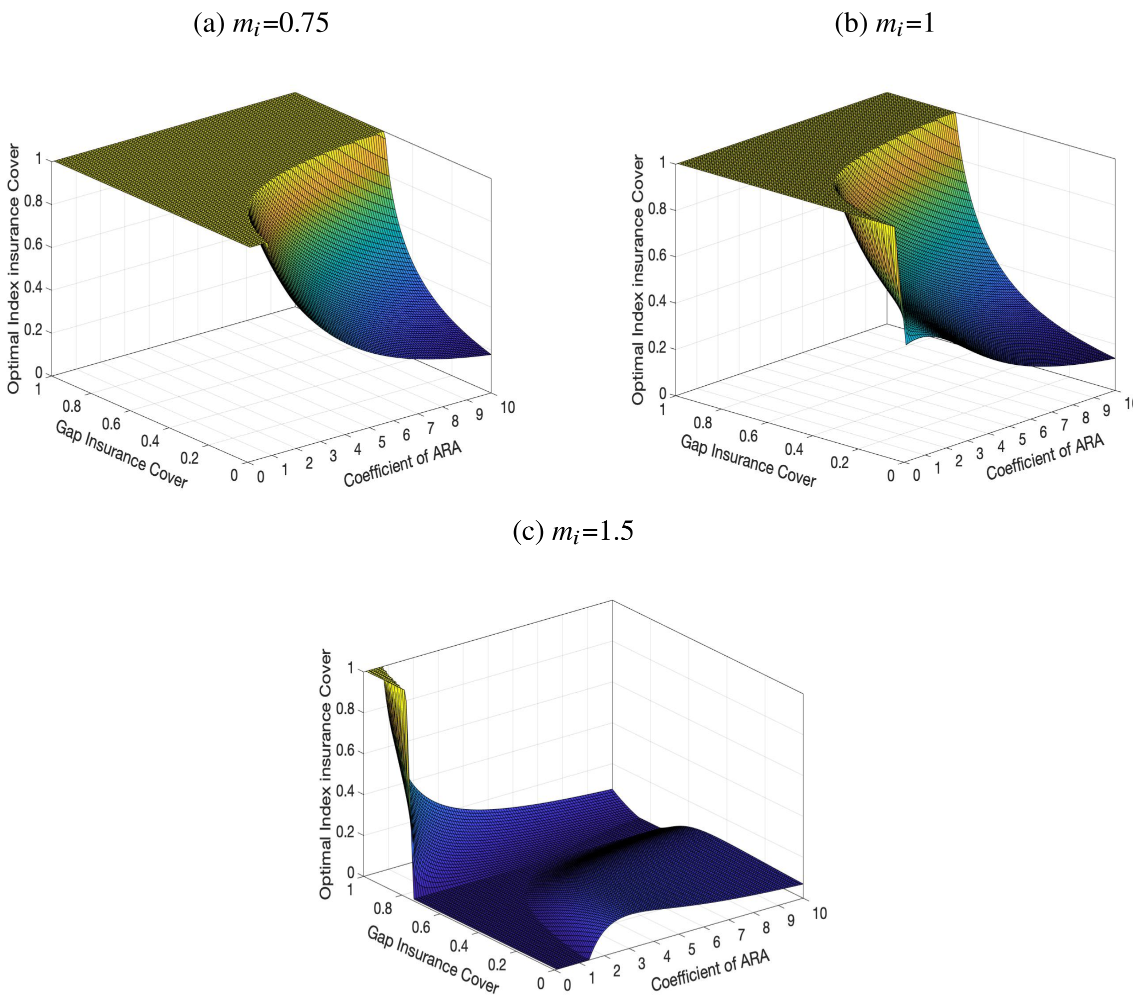

Figure 7.1: Optimal Index Insurance Cover – Case 1

Similar to Clarke (2016) we consider two cases to compute the optimal insurance covers. Those are Case 1 ($p = q = 1/3$ and $r = 1/9$) and Case 2 ($p = 1/20, q = 1/5$ and $r = 1/1000$). Let $l = 1$ and we consider three levels of m_i ($m_i = 0.75, 1, 1.5$). We choose $m_g = 2$, by considering the fact that the gap insurance should be costly, because the gap insurance provider needs to estimate the actual loss before making a payment. Here we discuss the numerical results only for the Case 1 and some important numerical results of Case 2 are given in Appendix B. The optimal αs are given in Figure 7.1 for different levels of γ and β. The optimal βs are given in Figure 7.2 for different levels of γ and α. In Figure 7.1 there are 3 sub plots for each case which represent three different levels of m_i. But in Figure 7.2 there is only one plot. Because according to the first order condition in Equation (7.6) it is clear that the optimal β does not have any effect from m_i.

According to Figure 7.1 the demand for the index insurance with $m_i = 0.75$ increases as the gap insurance cover increases. Individuals with very low risk aversion purchase full cover index insurance at any given level of gap insurance cover. When β is greater than 0.6 the insured with any given level of risk aversion purchases full index insurance cover, because $\beta = 0.6$ satisfies the condition in Proposition 7.1. We can see similar features with $m_i = 1$. But we observe quite different features with $m_i = 1.5$. In that case, for individuals with very low risk aversion, first there is no demand for the index insurance but then the demand increases dramatically as gap insurance cover increases. Full index insurance cover is optimal for individuals with very low risk aversion when the gap insurance cover is close to 1. For the rest of the levels of risk aversion there is a low demand for the index insurance at any given level of gap insurance. The optimal α slightly fluctuates as the gap insurance cover increases. By considering the results under all three premium levels, it is clear that at some positive gap insurance covers there is a high demand for the index insurance compared to the no gap insurance case. This means, that sometimes gap insurance can make a positive impact on the demand of the index insurance.

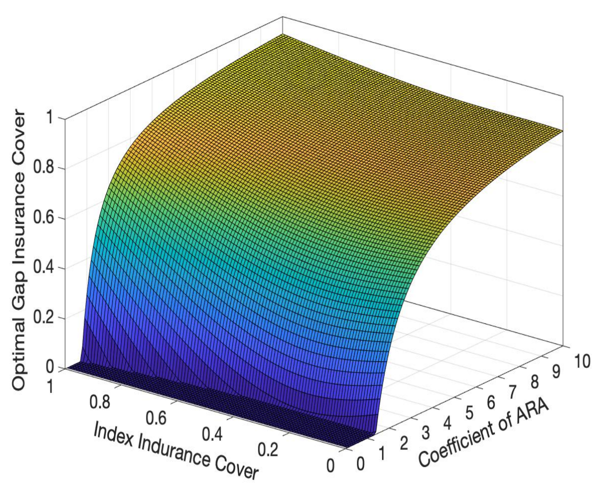

Figure 7.2: Optimal Gap Insurance Cover – Case 1

According to Figure 7.2 at any given level of α the optimal gap insurance cover monotonically increases as γ increases. At any given level of γ the optimal gap insurance cover changes slightly for different levels of α.

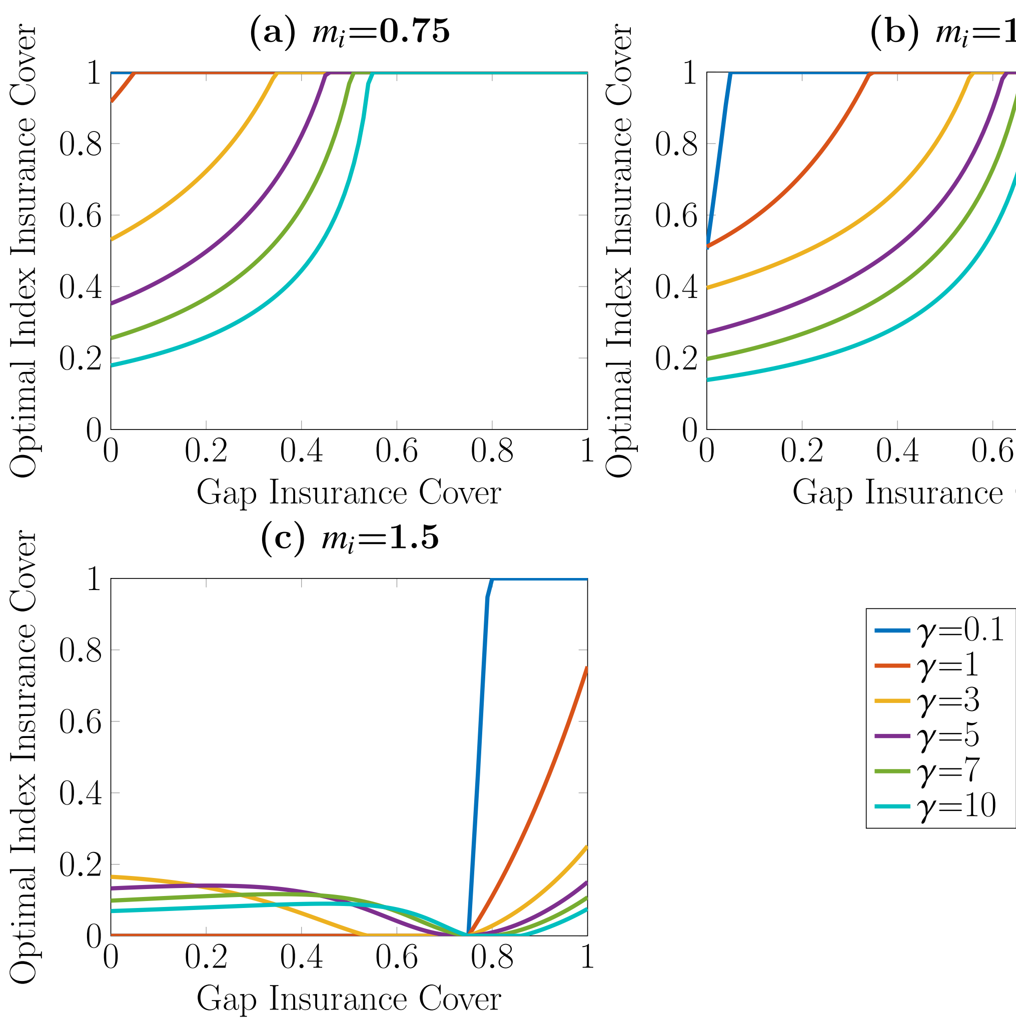

Figure 7.3: Optimal Index Insurance Cover vs Gap Insurance Cover – Case 1

Due to the complicated nature of Figure 7.1 it is not easy to see how the optimal index cover behaves at different levels of gap insurance cover and risk aversion. In order to see the behavior of the optimal index insurance more clearly, we represent it in a simple way in Figure 7.3. We plot the optimal α against gap insurance cover in Figure 7.3 for some selected levels of risk aversions in order to see the above explained shapes more clearly. Here we consider Case 1 while results for Case 2 are given in the Appendix. According to Figures 7.3 (a) and (b) it is clear that when $m_i = 0.75$ and $m_i = 1$ any level of gap insurance makes a favorable effect on the demand for the index insurance of the insureds for all the considered level of risk aversion. But when $m_i = 1.5$ only some levels of gap insurance makes a good effect on the demand of the index insurance.

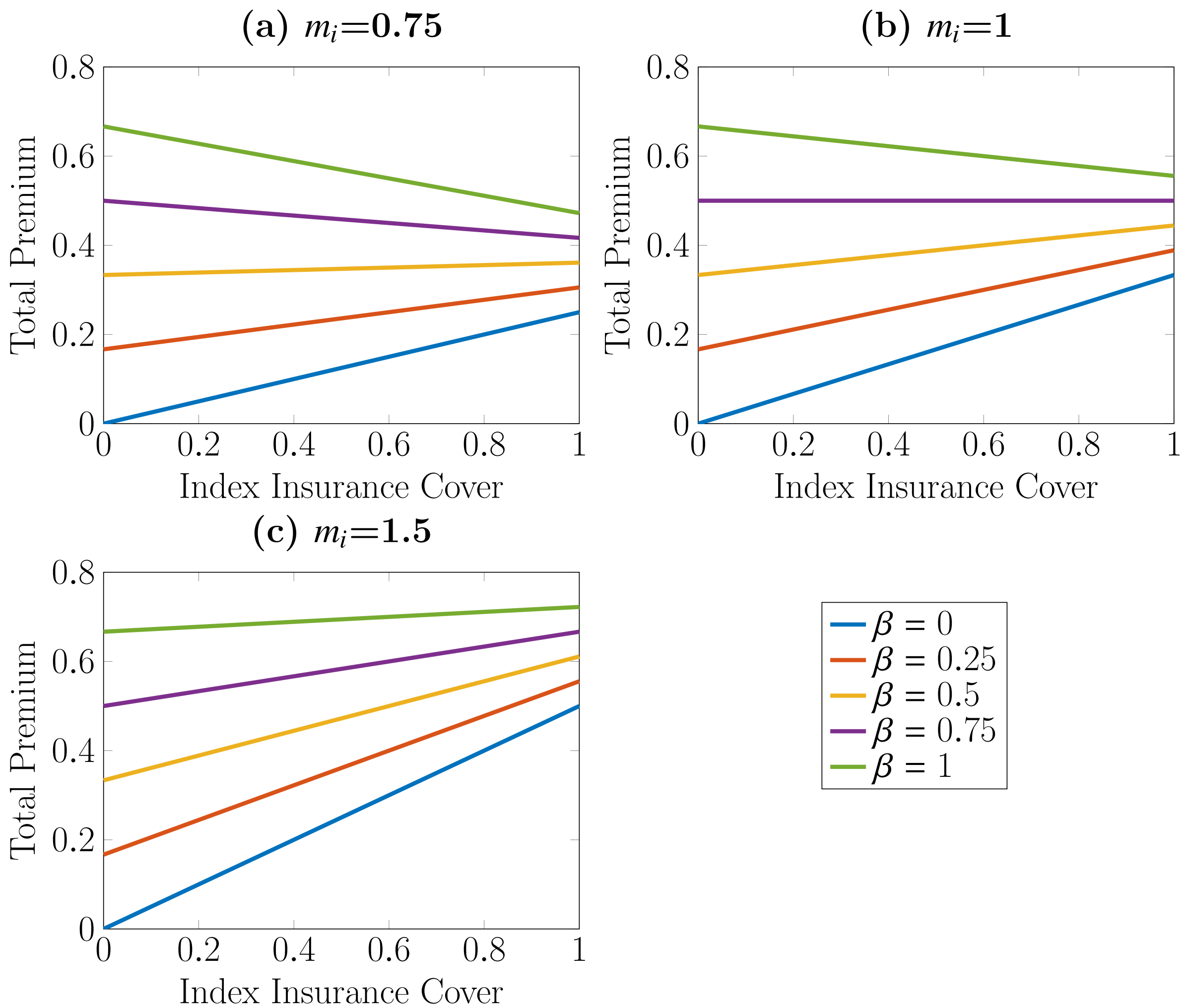

Figure 7.4: Total Premium – Case 1

As we discuss in the proof of Proposition 7.1 when $\beta \geq \frac{qm_i}{(p-r)m_g}$, the total premium $(\pi_i + \pi_g)$ decreases as α increases. Then this results in optimal full cover index insurance. In order to show this behavior properly we compute the total premium against index insurance cover and the obtained results are given in Figure 7.4. Again we consider Case 1 and five different levels of β. Only in Figures 7.4 (a) and (b), for the high gap insurance covers, the total premium decreases as the index insurance cover increases. For all the other cases the total premium increases as the index cover increases. It implies that in those cases $\beta < \frac{qm_i}{(p-r)m_g}$. But it does not imply that in those cases there is no possibility of optimal full index insurance cover. Because there is only a oneside implication in Proposition 7.1.

7.3 Full Cover Gap Insurance vs No Gap Insurance

Due to the nature of the first order condition in Equation (7.5) we cannot explicitly derive the optimal index insurance cover. But for two special cases with $\beta = 0$ and $\beta = 1$ we can derive the optimal index insurance cover as an explicit function of $p, q, r, \gamma, m_i, m_g$ and l. Where $\beta = 0$ represents no gap insurance and $\beta = 1$ represents full cover gap insurance. Let α_0^* and α_1^* be the optimal index insurance covers when $\beta = 0$ and $\beta = 1$ respectively. Also note that by Proposition 7.1 if $\frac{qm_i}{(p-r)m_g} \leq 1$ then $\alpha_1^* = 1$ and otherwise we can derive α_1^* from the first order condition in Equation (7.5) by setting $\beta = 1$. We can directly derive α_0^* from Equation (7.5) by setting $\beta = 0$. α_0^* and α_1^* which are derived by the first order condition in Equation (7.5) are given by

$$\alpha_0^* = \min\left[1, \max\left[0, \frac{1}{\gamma l}\ln\left(\frac{(1-qm_i)((p-r)e^{\gamma l}+q+r-p)}{qm_i(re^{\gamma l}+1-q-r)}\right)\right]\right], \tag{7.7}$$

$$\alpha_1^* = \min\left[1, \max\left[0, \frac{1}{\gamma l}\ln\left(\frac{(1-qm_i+m_g(p-r))(q+r-p)}{(qm_i-m_g(p-r))(1-q-r+p)}\right)\right]\right]. \tag{7.8}$$

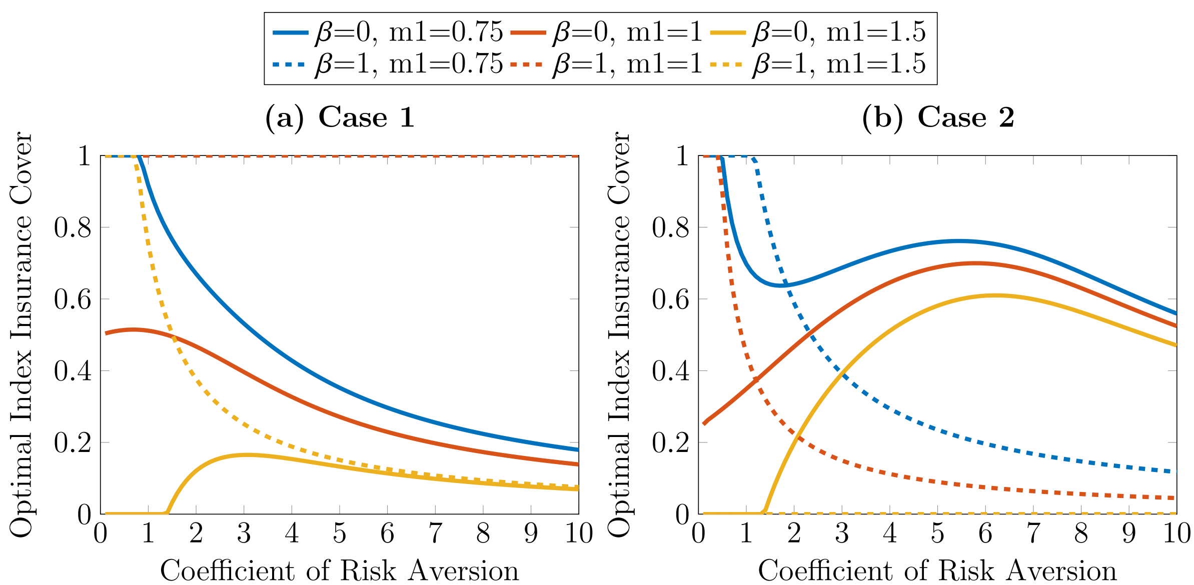

Figure 7.5: Optimal Index Insurance Cover for $\beta = 0$ and $\beta = 1$

Figure 7.5 shows the comparison between optimal covers against level of risk aversion for $\beta = 0$ and $\beta = 1$. We consider the same two cases and same three levels of m_i as above. The behavior of the optimal index insurance cover against level of risk aversion under Case 1 is already given in Figure 7.1 for β from 0 to 1. But Figure 7.1 is not compatible to make a comparison between the optimal index insurance covers under no gap insurance and full cover gap insurance. Therefore Figure 7.5 is important to make a proper comparison.

According to Figure 7.5 (a) when $\beta = 1$ it is clear that full index cover is optimal at any given level of risk aversion when $m_i = 0.75$ and $m_i = 1$. Because for these two levels of premium loads, the parameter values of Case 1 satisfy the condition $\frac{qm_i}{(p-r)m_g} \leq 1$. Then by Proposition 7.1 it leads to the optimal full cover index insurance. Since rest of the cases under full cover gap insurance do not satisfy this condition, we compute the optimal index insurance cover by Equation (7.8). Under Case 1 the demand for the index insurance is always higher when the insured has full cover gap insurance compared to the case with no gap insurance. But under Case 2 only low risk averse individuals purchase more index insurance with $m_i = 0.75$ and $m_i = 1$ when there is a full cover gap insurance compared to the case with no gap insurance. When $m_i = 1.5$ there is no demand for index insurance when the individual has full cover gap insurance.

7.4 Optimal Index Insurance at Optimal Gap Insurance

We already discuss the nature of the optimal index insurance cover and gap insurance cover. But we compute the optimal α for a given β and vice versa. Since choosing the level of both index and gap insurance is a decision of the insured, it is important to find the optimal pair of α and β. As we discussed before, due to the nature of the first order conditions the optimal α and β can not be derived explicitly. But still it is possible to compute the optimal pair of α and β numerically.

We use the same parameter values as above for the numerical computations. The obtained numerical results under Case 1 are given in Figure 7.6. In addition to the optimal pair of α and β, the optimal α without gap insurance is also given in Figure 7.6. It is given to make a comparison between the optimal α with and without gap insurance.

According to Figure 7.6 there is a common feature in all three cases. That is the optimal β is initially 0 and then increases as the level of risk aversion increases. As we can see in the figure it is obvious that, when the optimal β is 0, the optimal α with and without gap insurance are the same. When $m_i = 0.75$ and $m_i = 1$ the individual optimally purchases full index cover when some positive level of β is optimal. But when there is no gap insurance the individual optimally purchases partial cover under the corresponding case. This implies that the demand for the index insurance is higher when there is gap insurance compared to the case with no gap insurance. When $m_i = 1.5$ there is no demand for the index insurance when it is combined with a gap insurance. Then it converts to a traditional indemnity insurance. In the same case, an insured with any given level of risk aversion other than very small levels optimally purchases partial cover of the gap insurance and also a partial cover of the index insurance without gap insurance. Also it is clear that the demand of gap insurance is higher than the demand of index

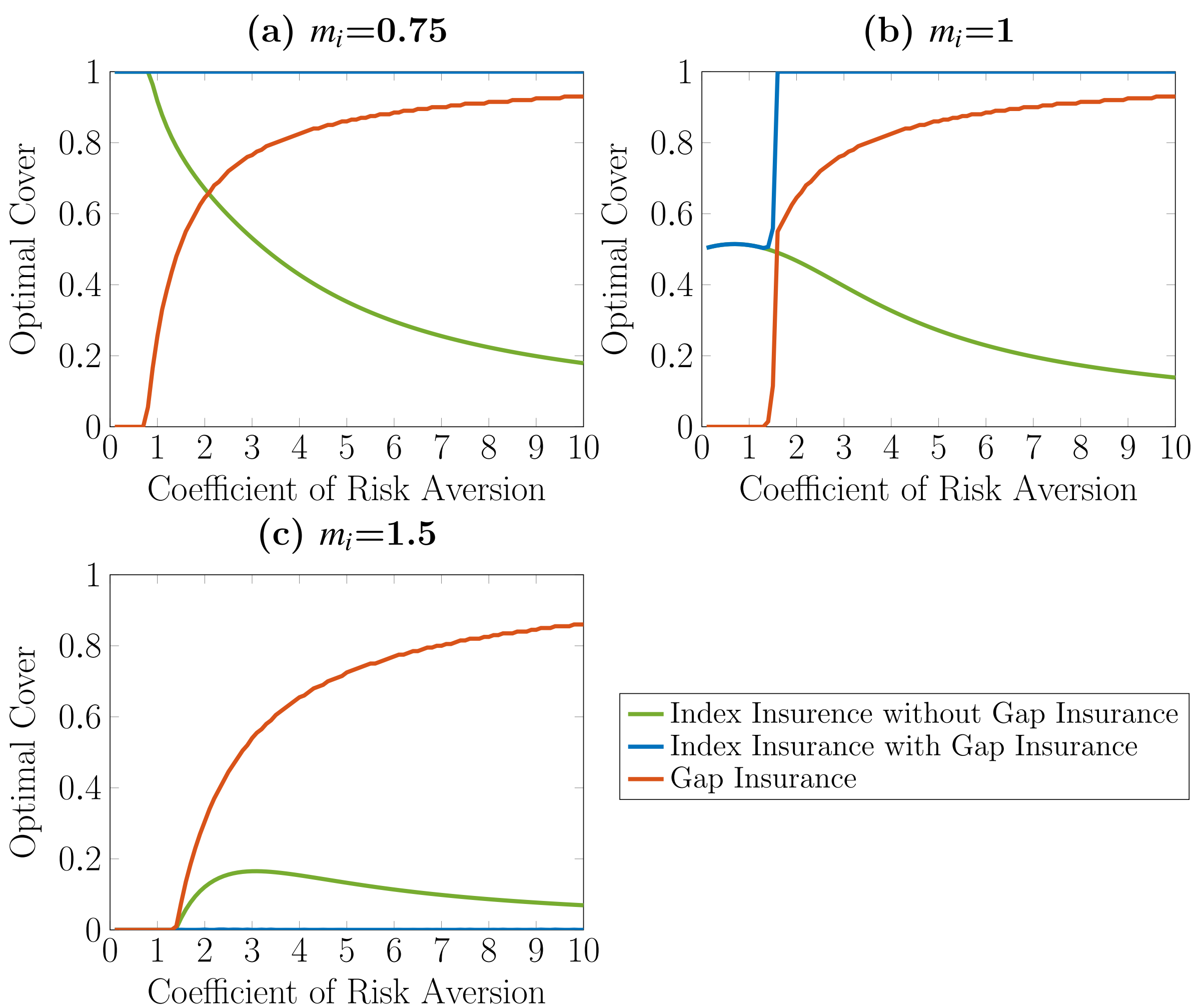

Figure 7.6: Optimal Pair of Index and Gap Insurance Covers – Case 1

insurance without gap insurance. This is an indication of the well known fact of low demand for index insurance.

7.5 The Effectiveness of the Combined Index and Gap Insurance

It is well known that the basis risk is one of the major disadvantages of an index insurance. The basis risk arises when the index measures and actual losses are not identical. According to our model when the index insurance pays more than the actual loss (then there is no payout from the gap insurance), then there is an excess gain to the insured. When the index insurance pays less than the actual loss, then the gap insurance covers the shortage fully or partially. But still there is a possibility of shortage to cover the actual loss due to the partial cover of the gap insurance. Here we discuss the effectiveness of the combined index and gap insurance in terms of these shortages and excess gains of the insured. Zhang et al. (2018) compute the basis risk of an index insurance by the standard deviation of the shortages and the excess gains of the insured. In that way the basis risk of our model is given by

$$\begin{aligned}\text{Basis Risk} &= \text{Std. Dev}(L - \alpha P_I - \beta P_G) \\ &= \sqrt{\mathbb{E}[(L - \alpha P_I - \beta P_G)^2] - \mathbb{E}[(L - \alpha P_I - \beta P_G)]^2},\end{aligned} \tag{7.9}$$

where

$$\mathbb{E}[(L - \alpha P_I - \beta P_G)^2] = l^2(1-\beta)^2(r + (1-\alpha)^2(p-r)) + \alpha^2 l^2(q + r - p),$$

$$\mathbb{E}[(L - \alpha P_I - \beta P_G)] = lp(1-\beta) + \alpha l(\beta(p-r) - q).$$

We compare the basis risk of the combined index and gap insurance with the basis risk of the index insurance without gap insurance. We compute the basis risk of the index insurance without gap insurance similarly by setting $\beta = 0$ in Equation (7.9). For the numerical computations we use the same set of parameter values as above. Also we consider the optimal levels of index and gap insurance covers. Since the gap insurance is introduced to cover the part of the loss which is not covered by the index insurance, we can expect lower basis risk under the combined index and gap insurance compared to the basis risk of the index insurance without gap insurance. But according to the results in Figure 7.6 we can see that optimal index insurance cover with and without gap insurance are different to each other and also a partial cover of gap insurance is optimal. Therefore without proper investigation we can not say which insurance has lower basis risk. But by observing the nature of the optimal covers in Figure 7.6 we can find which insurance has the lower basis risk when $m_i = 1.5$. We explain it as follows.

When $m_i = 1.5$ and when there is a gap insurance then there is no demand for the index insurance. Then the combination of index and gap insurance converts to a traditional indemnity insurance. Now comparing the demand of that indemnity insurance (gap insurance) and index insurance without gap insurance (red and green lines in Figure 7.6 (c)) it is clear that the demand of the indemnity insurance is higher than the demand of the index insurance. Therefore when there is a loss the indemnity insurance covers a higher portion of the loss compared to the portion of loss covered by the index insurance. Also sometimes index insurance does not cover the loss. As a result of that the shortages under the indemnity insurance can be lower than the shortages under the index insurance. Also it is obvious that there is no excess gain under the indemnity insurance (because it does not pay when there is no actual loss). These facts result in lower basis risk under the indemnity insurance than the index insurance when $m_i = 1.5$. The basis risk of the index insurance with gap insurance is computed by Equation (7.9). The basis risk of the index insurance without gap insurance is computed by the same equation by setting $\beta = 0$. The obtained results under Case 1 are given in Figure 7.7. According to the figure it is clear that for all the cases the combined index and gap insurance is better than index insurance in terms of reducing basis risk. The basis risk under Case 2 is given in the Appendix B and it shows similar results as Case 1.

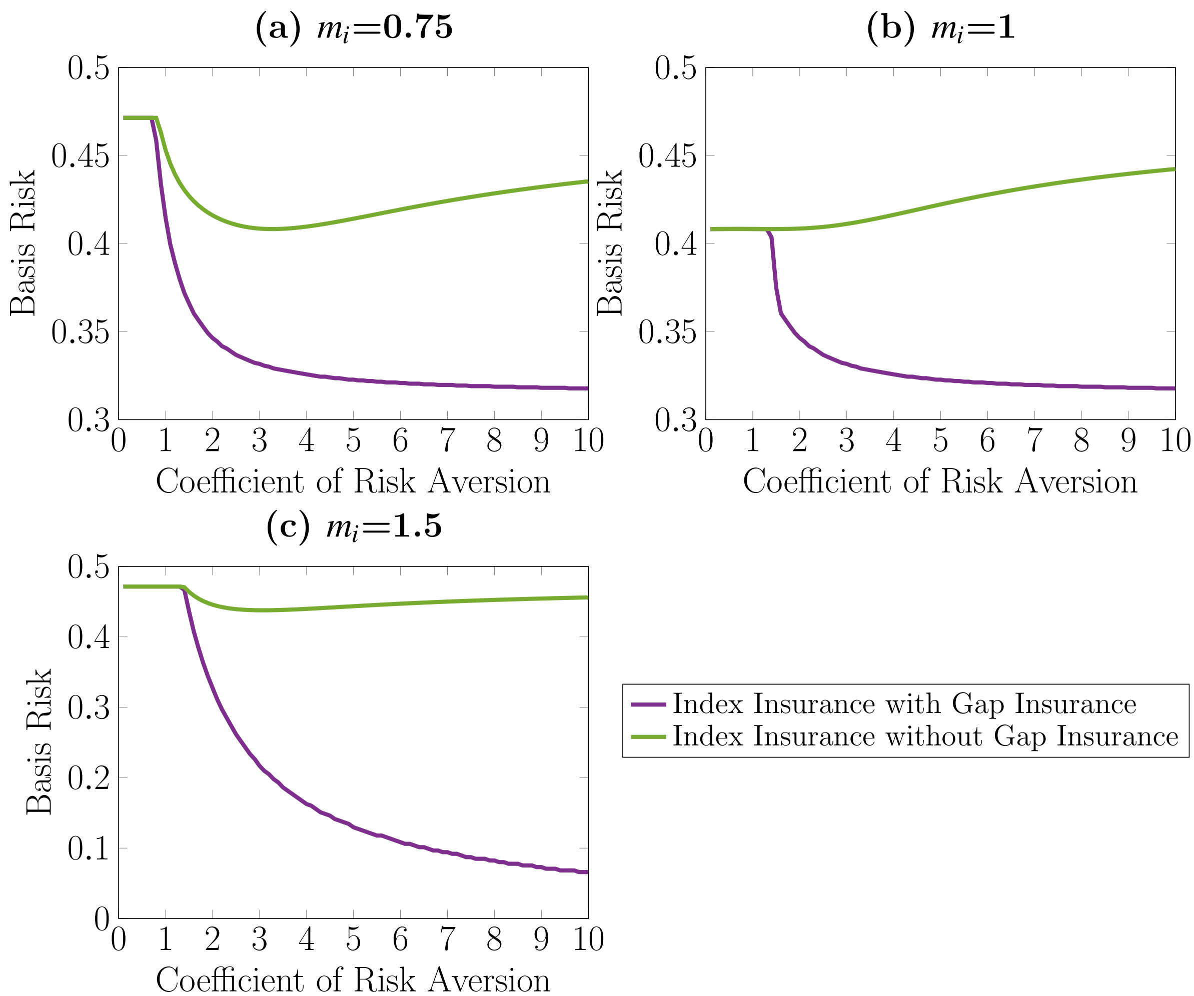

Figure 7.7: Basis Risk – Case 1

The expectation of shortages and excess gains, $\mathbb{E}[(L - \alpha P_I - \beta P_G)]$ or the expectation of square of $\mathbb{E}[(L - \alpha P_I - \beta P_G)^2]$ are alternative ways to represent the basis risk. Therefore we compute $\mathbb{E}[(L - \alpha P_I - \beta P_G)]$ and $\mathbb{E}[(L - \alpha P_I - \beta P_G)^2]$. The obtained results are given in the Appendix B. Those results also show that index insurance together with gap insurance has lower basis risk.

Part 2: Index Insurance as a Reinsurance

7.6 Proportional Reinsurance

In proportional reinsurance the direct insurer and the reinsurer divide the original premiums and the original losses at a fixed ratio (Antal and Re (2007)). We consider an insurer who issues an indemnity insurance to cover loss L, where $L = l$ with probability p and $L = 0$ with probability $(1 - p)$. We assume that the insurer issues a full cover insurance with premium load m. Then the premium of that insurance is given by

$$\pi = m\mathbb{E}L = mpl.$$

Then the profit of the insurer is given by

$$X = mpl - L.$$

The following two reinsurance options are available.

- Proportional reinsurance
- Index based reinsurance

First we consider that the insurer chooses the proportional reinsurance option. Let α_P and m_P be the cover and the premium load of the proportional reinsurance respectively. Then $\alpha_P L$ is the payoff of the proportional reinsurance. The premium of the proportional reinsurance is given by

$$\pi_P = \alpha_P m_P \mathbb{E}L = \alpha_P m_P pl.$$

The profit of the insurer with proportional reinsurance is given by

$$\begin{aligned} X_P &= X - \pi_P + \alpha_P L \\ &= mpl - L - \alpha_P m_P pl + \alpha_P L. \end{aligned} \tag{7.10}$$

Now we consider that the insurer choose the index based reinsurance option. Let α_I and m_I be the cover and the premium load of the index based reinsurance, respectively. Let I be the loss based on the corresponding index. Also similar to the model in Clarke (2016) it is considered that the index indicates bad weather (Index=1) with probability q and then $I = l$ and the index indicates good weather (Index=0) with probability $1 - q$ and then $I = 0$. Let r be the probability that loss occurs but the index indicates good weather. Now $\alpha_I I$ is the payoff of the index based reinsurance. The premium of that index based reinsurance is given by

$$\pi_I = \alpha_I m_I \mathbb{E}I = \alpha_I m_I ql.$$

The profit of the insurer with the index based reinsurance is given by

$$\begin{aligned} X_I &= X - \pi_I + \alpha_I I \\ &= mpl - L - \alpha_I m_I ql + \alpha_I I. \end{aligned} \tag{7.11}$$

By considering all the possibilities it is clear that there are two possible profit levels of the insurer when he has a proportional reinsurance and there are four possible profit levels when he has an index based reinsurance. Those are given in Table 7.2 and 7.3 respectively.

Table 7.2: Profit Levels – Proportional Reinsurance

State s	Probability p_s	Profit
l	p	$mpl - l - \alpha_P m_P pl + \alpha_P l$
0	$1 - p$	$mpl - \alpha_P m_P pl$

Table 7.3: Profit Levels-Index Based Reinsurance

State s	Probability p_s	Profit
$l0$	r	$mpl - l - \alpha_I m_I ql$
$l1$	$p - r$	$mpl - l - \alpha_I m_I ql + \alpha_I l$
00	$1 - q - r$	$mpl - \alpha_I m_I ql$
01	$q + r - p$	$mpl - \alpha_I m_I ql + \alpha_I l$

Zeng (2005) assumes that the insurer's risk management objective of taking reinsurance is to reduce the standard deviation of his profit. Here we consider the same risk management objective. That is, similar to Zeng (2005) we assume that the insurer wants to reduce the standard deviation of the profit at least by a factor b by taking a reinsurance. More clearly we express this condition as $\sigma_r - (1 - b)\sigma \leq 0$, where $0 < b < 1$, σ and σ_r are the standard deviations of the profit before and after reinsurance, respectively.

By assuming that the insurer is an expected utility maximizer with utility function u, the goal of the insurer is to choose the level of reinsurance cover which maximizes his expected profit. Now let σ_P and σ_I be the standard deviations of the profit with proportional reinsurance and index based reinsurance, respectively. The optimization problems for both types of reinsurance with the corresponding risk management constraints are given by

$$\begin{aligned} \max_{\alpha_P} \quad & \mathbb{E}[u(mpl - L - \alpha_P m_P pl + \alpha_P L)] \\ \text{s.t} \quad & \sigma_P - (1-b)\sigma \leq 0, \end{aligned} \tag{7.12}$$

$$\begin{aligned} \max_{\alpha_I} \quad & \mathbb{E}[u(mpl - L - \alpha_I m_I ql + \alpha_I L)] \\ \text{s.t} \quad & \sigma_I - (1-b)\sigma \leq 0. \end{aligned} \tag{7.13}$$

According to the information in Table 7.2 we state the expected utility of insurer's profit with proportional reinsurance as

$$\mathbb{E}[u(X_P)] = pu(mpl - l - \alpha_P m_P pl + \alpha_P l) + (1-p)u(mpl - \alpha_P m_P pl).$$

Now according to the information in Table 7.3 we state the expected utility of insurer's profit with index based reinsurance as

$$\begin{aligned} \mathbb{E}[u(X_I)] =& ru(mpl - l - \alpha_I m_I ql) + (p-r)u(mpl - l - \alpha_I m_I ql + \alpha_I l) \\ &+ (1-q-r)u(mpl - \alpha_I m_I ql) + (q+r-p)u(mpl - \alpha_I m_I ql + \alpha_I l). \end{aligned} \tag{7.14}$$

In order to solve the optimization problems first we focus on the constraints. We analyze the constraints similar to Zeng (2005). First we consider the constraint of the optimization problem with proportional reinsurance. By Equation (7.10) it is clear that

$$\sigma_P = (1-\alpha_P)\sigma.$$

Then we rewrite the constraint as

$$(1-\alpha_P)\sigma - (1-b)\sigma \leq 0.$$

This results in

$$\alpha_P \geq b. \tag{7.15}$$

Then we consider the constraint of the optimization problem with index based reinsurance. By Equation (7.11) we express σ_I as

$$\sigma_I = \sqrt{\sigma^2 + \alpha_I^2 \sigma_i^2 - 2\alpha_I \sigma \sigma_i \rho}.$$

Where σ_i is the standard deviation of the loss based on the index (I) and ρ is the correlation coefficient between L and I. Now we rewrite the constraint as

$$\sqrt{\sigma^2 + \alpha_I^2\sigma_i^2 - 2\alpha_I\sigma\sigma_i\rho} - (1-b)\sigma \leq 0.$$

It can be restructured as

$$\alpha_I^2 - 2\rho\frac{\sigma}{\sigma_i}\alpha_I + (2b - b^2)\frac{\sigma^2}{\sigma_i^2} \leq 0. \tag{7.16}$$

Inequality (7.16) holds if and only if the following two conditions are fulfilled.

The first condition is that the equation

$$\alpha_I^2 - 2\rho\frac{\sigma}{\sigma_i}\alpha_I + (2b - b^2)\frac{\sigma^2}{\sigma_i^2} = 0$$

has two real solutions (α_1 and α_2). The discriminant of the above quadratic equation should be non negative in order to have two real solutions. That is

$$\rho^2 \geq 2b - b^2.$$

The second condition is $\alpha_1 < \alpha_I < \alpha_2$, where

$$\alpha_1 = \frac{\sigma}{\sigma_i}\Big(\rho - \sqrt{\rho^2 - 2b + b^2}\Big),$$

$$\alpha_2 = \frac{\sigma}{\sigma_i}\Big(\rho + \sqrt{\rho^2 - 2b + b^2}\Big), \tag{7.17}$$

since only the α_I's which lie between α_1 and α_2 satisfy the inequality in (7.16). Now it is clear that the optimal proportional reinsurance cover should be greater than b and the optimal index based reinsurance cover should be between α_1 and α_2. We note that $\alpha_1 > 0$ since $\rho > \sqrt{\rho^2 - 2b + b^2}$.

We can derive the optimal insurance covers by the first order conditions of the expected utility with respect to the corresponding insurance cover. The first order condition of the expected utility of the insurer with proportional reinsurance and index based reinsurance are given by Equations (7.18) and (7.19), respectively.

$$\frac{\partial \mathbb{E}[u(X_P)]}{\partial \alpha_P} = pu_l'(-m_P pl + l) + (1-p)u_0'(-m_P pl) = 0, \tag{7.18}$$

$$\frac{\partial \mathbb{E}[u(X_I)]}{\partial \alpha_I} = ru'_{l0}(-m_I ql) + (p-r)u'_{l1}(-m_I ql + l) + (1-q-r)u'_{00}(-m_I ql) + (q+r-p)u'_{01}(-m_I ql + l) = 0, \tag{7.19}$$

where u'_s is the marginal utility of state s, α_P which satisfies Equation (7.18) is the optimal proportional reinsurance cover and α_I which satisfies Equation (7.19) is the optimal index insurance cover. Now we consider a constant absolute risk averse utility function for which $u' \propto e^{-\gamma x}$, where γ is the coefficient of absolute risk aversion. Then we can explicitly derive the optimal proportional reinsurance cover α_P^* and optimal index based reinsurance cover α_I^* from Equations (7.18) and (7.19), respectively by considering the corresponding risk management constraints. We state those α_P^* and α_I^* in the following two propositions.

Proposition 7.2. *For an individual with constant absolute risk aversion of $\gamma > 0$ the solution to the optimization problem (7.12) is*

$$\alpha_P^* = max\left[b, \frac{1}{\gamma l} ln\left(\frac{(1-pm_P)pe^{\gamma l}}{pm_P(1-p)}\right)\right]. \tag{7.20}$$

Proposition 7.3. *For an individual with constant absolute risk aversion of $\gamma > 0$ the solution to the optimization problem (7.13) is*

$$\alpha_I^* = min\left[\alpha_2, max\left[\alpha_I, \frac{1}{\gamma l} ln\left(\frac{(1-qm_I)((p-r)e^{\gamma l} + q + r - p)}{qm_I(re^{\gamma l} + 1 - q - r)}\right)\right]\right]. \tag{7.21}$$

We can easily prove Proposition 7.2 and 7.3 by solving Equation (7.18) and (7.19) respectively. In order to evaluate the efficiency of index insurance as a reinsurance instrument we can compare the maximum expected utility of the profit with index based reinsurance ($\mathbb{E}U_I^*$) with maximum expected utility of the profit with proportional reinsurance ($\mathbb{E}U_P^*$).

7.6.1 Numerical Results for Proportional Reinsurance

Similar to Clarke (2016) we consider two cases with different probabilities for the numerical results, where Case 1: $p = 1/3, q = 1/3, r = 1/9$ and Case 2: $p = 1/20, q = 1/5, r = 1/1000$. By considering the fact that traditional indemnity insurance is costly, let $m = 2$. Also since the proportional reinsurance is an indemnity insurance, it is more realistic to consider $m_P \geq m_I$. But for the comparison we consider three levels of premium loads for each type of reinsurance. The obtained results are given in Figure 7.8.

First we suppose that the premium load of the available proportional reinsurance is twice the premium load of the available index based reinsurance. Then according to Figure 7.8 (a) when

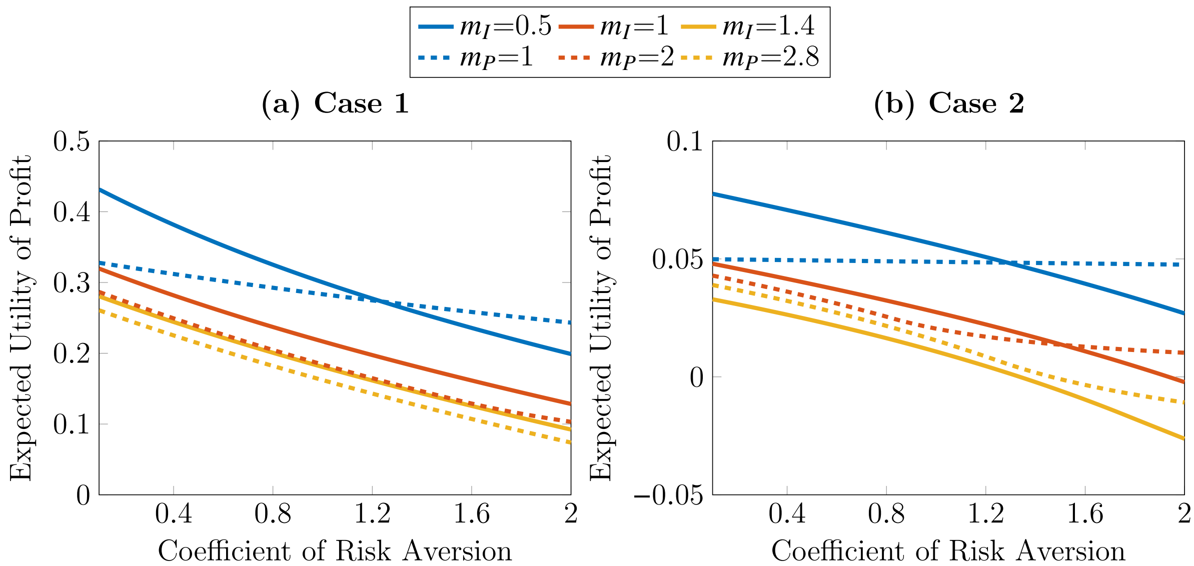

Figure 7.8: Maximum Expected Utility of Profit – Proportional Reinsurance

$m_I = 0.5$ and $m_P = 1$ the less risk averse insurers optimally choose index based reinsurance and more risk averse insurers optimally choose proportional reinsurance. By comparison for the other two pairs ($m_I = 1, m_P = 2$ and $m_I = 1.4, m_P = 2.8$) it is clear that the insurers optimally choose the index based reinsurance. According to Case 2, when $m_I = 0.5, m_P = 1$ and $m_I = 1, m_P = 2$, the less risk averse insurers chooses the index based reinsurance and the others choose the proportional reinsurance. But the insurers choose the proportional reinsurance with $m_P = 2.8$ over the index based reinsurance with $m_I = 1.4$. However by comparing the expected utility of the profit under both reinsurances with actuarial fair premium (red solid line and blue dashed line) it is clear that the proportional reinsurance is the better reinsurance option. Also the proportional reinsurance with $m_P = 2$ is better than the index insurance with $m_I = 1.4$ in terms of expected utility of profit. Therefore, according to the numerical example, the insurers choose the index based reinsurance if the premium load of the index based reinsurance is considerably lower than the premium load of the given proportional reinsurance.

7.7 Excess Loss Reinsurance

Excess loss reinsurance is another commonly used reinsurance instrument. It covers the losses that exceed some given trigger (t) and usually up to a given cap (c) (Cummins et al. (1999)). That is if the loss is below the trigger there is no payout, if the loss is in between trigger and cap then the payout is the amount of loss exceeding the trigger (loss-t) and if the loss exceeds cap then the payout is $c - t$. Again we consider an insurer who issue an indemnity insurance to cover the loss L. But a two state loss variable is not a good choice to investigate the excess loss reinsurance. Therefore, we consider that L is a continuous random variable. Then similar to the previous model the profit of the insurer is given by

$$X = m\mathbb{E}[L] - L.$$

We assume that the following two reinsurance options are available.

- Excess loss reinsurance based on the actual loss,
- Excess loss reinsurance based on a weather index.

The payout of the excess loss insurance based on the actual loss is given by

$$P_L = \max[0, \min[L - t, c - t]].$$

More clearly the payout P_L is given by

$$P_L = \begin{cases} 0 & L < t \\ L - t & t \leq L < c \\ c - t & L \geq c. \end{cases}$$

Then the premium π_L of the excess loss reinsurance based on the actual loss with premium load m_L and the profit of the insurer after the reinsurance are given by

$$\pi_L = m_L\mathbb{E}[P_L]$$

and

$$X_L = m\mathbb{E}[L] - L - m_L\mathbb{E}[P_L] + P_L.$$

respectively.

We consider the excess loss reinsurance based on a weather index. So, now let I be the loss based on the corresponding weather index. It is a continuous random variable. Then the payout of this reinsurance is given by

$$P_I = \begin{cases} 0 & I < t \\ I - t & t \leq I < c \\ c - t & I \geq c. \end{cases}$$

The premium π_I of the index based excess loss reinsurance with the premium load m_I and the profit after reinsurance are given by

$$\pi_I = m_I \mathbb{E}[P_I]$$

and

$$X_I = m\mathbb{E}[L] - L - m_I\mathbb{E}[P_I] + P_I.$$

respectively.

Similar to the previous model we assume that the insurer is an expected utility maximizer with utility function u. In order to compare the performance of the two reinsurance options we use the expected utility of the profit. Let us assume that L is a continuous random variable with probability density function $g(l)$ and I is a continuous random variable with probability density function $h(i)$. Let $l \in [a_l, b_l]$ and $i \in [a_i, b_i]$, where $a_l \le t, a_i \le t, b_l \ge c$ and $b_i \ge c$. The joint probability distribution of L and I is $f(l,i)$. The expected utility of the profit after actual loss based excess loss reinsurance and index based excess loss reinsurance are given by

$$\begin{aligned}\mathbb{E}[u(X_L)] = &\int_{a_l}^{t} u\big(m\mathbb{E}[L] - l - m_L\mathbb{E}[P_L]\big)g(l)\,dl \\ &+ \int_{t}^{c} u\big(m\mathbb{E}[L] - l - m_L\mathbb{E}[P_L] + l - t\big)g(l)\,dl \\ &+ \int_{c}^{b_l} u\big(m\mathbb{E}[L] - l - m_L\mathbb{E}[P_L] + c - t\big)g(l)\,dl\end{aligned} \tag{7.22}$$

and

$$\begin{aligned}\mathbb{E}[u(X_I)] = &\int_{a_l}^{b_l}\int_{a_i}^{t} u\big(m\mathbb{E}[L] - l - m_I\mathbb{E}[P_I]\big)f(l,i)\,di\,dl \\ &+ \int_{a_l}^{b_l}\int_{t}^{c} u\big(m\mathbb{E}[L] - l - m_I\mathbb{E}[P_I] + i - t\big)f(l,i)\,di\,dl \\ &+ \int_{a_l}^{b_l}\int_{c}^{b_i} u\big(m\mathbb{E}[L] - l - m_I\mathbb{E}[P_I] + c - t\big)f(l,i)\,di\,dl\end{aligned} \tag{7.23}$$

respectively, where

$$\mathbb{E}[L] = m_L \int_{a_l}^{b_l} l g(l)\,dl,$$

$$\mathbb{E}[P_L] = \int_t^c (l-t)g(l)\,dl + \int_c^{b_l}(c-t)g(l)\,dl, \qquad \mathbb{E}[P_I] = \int_t^c (i-t)h(i)\,di + \int_c^{b_i}(c-t)h(i)\,di.$$

7.7.1 Numerical Results for Excess Loss Reinsurance

For the numerical computations we consider that $L \sim N(\mu_L, \sigma_L^2)$ and $I \sim N(\mu_I, \sigma_I^2)$. Also we consider that the joint probability distribution of L and I is bivariate normally distribution with parameters $\mu_L, \sigma_L^2, \mu_I, \sigma_I^2$ and ρ, where ρ is the correlation coefficient between L and I. Let $\mu_L = 1, \sigma_L = 0.3, \mu_I = 0.8, \sigma_I = 0.2, \rho = 0.75, t = 0.25, c = 1.25$ and $m = 2$. First we compute the expected utility of the profit under both reinsurance contracts by assuming an insurer with constant absolute risk aversion. We consider five possible levels of m_L and three possible levels of m_I. The obtained results are given in Figure 7.9.

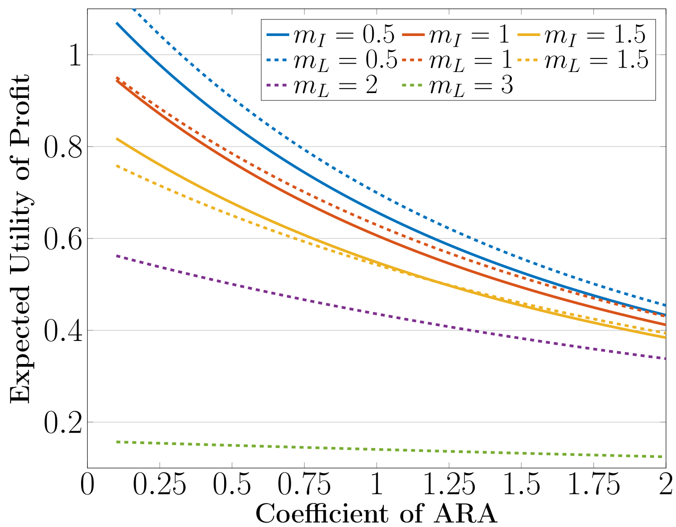

Figure 7.9: Expected Utility of Profit-Excess Loss Reinsurance

In Figure 7.9 the solid lines represent the expected utility of the profit with index based excess loss reinsurance and the dotted lines represent the expected utility of profit with actual loss based excess loss reinsurance. By comparing both reinsurance contracts with same premium loads ($m_L = m_I$) it is clear that the expected utilities of profits are close to each other. When the premium load is 0.5 or 1 the actual loss based excess loss reinsurance is slightly better than the index based excess loss reinsurance. But according to the nature of the index insurance we can assume that the premium load of an index insurance is lower than the premium load of the corresponding traditional insurance. By considering these facts when comparing the expected utility of profit under both insurance contracts with $m_I < m_L$ (For example $m_I = 0.5$ with $m_L = 1, 1.5, 2, 3$ and $m_I = 1.5$ with $m_L = 2, 3$) it is clear that the index based excess loss reinsurance is better than the actual loss based excess loss reinsurance.

We further compare expected utility of the profit between the index based and the actual loss based excess loss reinsurance. We consider the same three levels of m_I and the same five levels of m_L as earlier. Let $\gamma = 1$. Five different excess loss reinsurance products based on t

and c are considered. Those are

- Product 1: $t = 0.1$ and $c = 1.75$
- Product 2: $t = 0.25$ and $c = 1.25$
- Product 3: $t = 0.5$ and $c = 1$
- Product 4: $t = 0.1$ and $c = 1$
- Product 5: $t = 0.8$ and $c = 1.8$

The obtained results are given in Table 7.4. The bold figures in the table are the maximum expected utilities of the profit out of five products for a given type of reinsurance and for a given level of premium. According to the results in Table 7.4 it is clear that for $m_L = m_I$ the expected utility of profit with the index based reinsurance under any given product is close to the expected utility of the profit with the actual loss based reinsurance under the same product. But when $m_L > m_I$ the expected utility of profit with the index based excess loss reinsurance is higher than the expected utility of profit with the actual loss based excess loss reinsurance of the corresponding product.

Table 7.4: Expected Utility of Profit

Type	Premium Load	Product 1	Product 2	Product 3	Product 4	Product 5
Index Based	$m_I = 0.5$	**0.7107**	0.6901	0.6574	0.7106	0.6332
	$m_I = 1$	0.5909	0.5942	0.6065	0.5998	**0.6109**
	$m_I = 1.5$	0.4208	0.4682	0.5481	0.4461	**0.5871**
Actual Loss Based	$m_L = 0.5$	**0.7654**	0.7457	0.6997	0.7541	0.6695
	$m_L = 1$	**0.6321**	0.6318	0.6294	0.6294	0.6317
	$m_L = 1.5$	0.4231	0.4670	0.5427	0.4416	**0.5895**
	$m_L = 2$	0.0952	0.2284	0.4356	0.1585	**0.5425**
	$m_L = 3$	-1.2255	-0.6171	0.1405	-0.9111	**0.4319**

By considering the maximum expected utilities (bold figures) in Table 7.4 we are interest to compare the actuarial fair reinsurance contacts $(m_I = m_L = 1)$. We observe that, if an insurance company wants to purchase an index based actuarial fair excess loss reinsurance Product 5 is the best option. But if the insurance company wants to purchase an actual loss based actuarial fair excess loss reinsurance Product 1 is the best option. However by comparing expected utility of profits under Product 1 and 5 for both types of reinsurance, it is clear that Product 1 is also a good choice for index based actuarial fair excess loss reinsurance and Product 5 is also a good choice for actual loss based actuarial fair excess loss reinsurance. Because when $m_I = m_L = 1$ the expected utility of profit under Product 1 and 5 are close to each other for a given type of reinsurance.

By comparing the five products, generally we can say that when the premium load is small, Product 1 is the best option. That is the product where the reinsurance covers small loss to high loss out of all five products. Therefore, the obtained results are acceptable. But when there is high premium load Product 5 is the best option. That is the product where the reinsurance covers high losses but not the small losses. Since the premium load is high, if the insurance company chooses Product 1 instead of Product 5 the premium may be increased by a considerably high amount. Therefore, an insurer may prefers to get a reinsurance coverage only for high losses with a bearable premium than a reinsurance which covers small to big losses for a high premium.

7.8 Summary

In this chapter we gained some insight on gap insurance and reinsurance:

- The demand for index insurance together with gap insurance changes based on factors such as insurance premiums and gap insurance cover.
- A gap insurance helps to reduce the basis risk of the index insurance.
- Index insurance is a good alternative for proportional reinsurance and excess loss reinsurance when it's premium is low.

Chapter 8

Conclusions

In this thesis we investigated the performance of a weather index insurance in many different contexts. We tried to identify situations and conditions in which the weather index insurance performs as an effective crop insurance.

In Chapter 3, first we considered the model set up in Miranda and Gonzalez-Vega (2011), which depends on the Bellman equation. In our numerical computations we saw that the index insurance is a good choice to combine with agriculture loan to reduce the loan default probability and improve the expected IRR. However the insurer/borrower should choose the proper level of premiums to benefit from the interlinkage of the loan and the index insurance. Because high premiums lead to high loan default probabilities and low expected IRR. We saw mostly similar results under the other two models: Model with savings and Model for excessive amount of the weather event. In addition to that we found out that the combined loan and index insurance increase the wealth of the lender. In that situation also a proper level of premium should be selected.

In Chapter 4, we designed an optimal weather index insurance by following the procedure in Zhang et al. (2018). They included the loss of the farmer in their model. Instead we included the income of the farmers. We also designed the optimal payout function by assuming the income of the farmer has the same form as in Miranda and Gonzalez-Vega (2011). Where the income is represented as a product of idiosyncratic shock and common shock of the farmer. The designed optimal payout function depends on the level of the premium and the maximum indemnity. Our numerical results show that proper combination of premium and maximum indemnity reduce the basis risk. We also observed that the index insurance with optimal payout performs better by reducing basis risk compared to a commonly used linear payout. According to our results, when a loan is interlinked with the optimal index insurance then the loan default probability is smaller than in the case where the loan is interlinked with an index insurance with a linear indemnity function. In that chapter we also studied the optimal lender level index insurance. Basically our results show that if the loan is interlinked with an index insurance then an index insurance for the lender is not required.

In Chapter 5, we developed a market equilibrium model for a weather index insurance. First we realized that a market equilibrium for weather index insurance exists. The demand at the equilibrium price is lower than the demand at the actuarial fair price. Therefore in order to increase the demand premium subsidy may be required. Once we consider a group of farmers whose preferences on wealth are given by a power utility function, we noted that demand at the market equilibrium is higher when the index insurance is interlinked with a loan for farming. As per the equilibrium model with two risk groups, we understood that the low risk group subsidizes the high risk group.

In Chapter 6 we studied the performance of the index insurance in an informal risk sharing environment. We realized that a farmer in an informal risk sharing group does not always prefers index insurance. Nevertheless, the index insurance together with informal risk sharing performs better depending on different factors, such as the method of informal risk sharing, group size and insurance premium. We noted that, in Model 1 the optimal insurance cover and fraction of informal risk sharing are independent. In Model 2*, we proved that the optimal insurance cover decreases as the fraction of informal risk sharing increases. For Model 3* we were unable to prove the dependence theoretically, but in our numerical results we saw that the optimal insurance cover first increases and then decreases as the fraction of informal risk sharing increases.

In Chapter 7 first we analyzed whether a gap insurance will help to increase the demand of an index insurance. Our analysis shows that the index insurance together with a gap insurance performs better only in some circumstances. Basically the performance is based on the insurance premiums, level of risk aversion of the insured and the gap insurance cover. In addition to that we recognized that the index insurance together with a gap insurance as a proper way to reduce the basis risk. In the second part of that chapter we saw that index insurance is a good alternative for proportional or excess loss reinsurance, in particular when it is sold at a low price.

Overall we can say that both insureds and insurers can benefit from the index insurance by imposing relevant conditions and applying it in appropriate situation.

Appendix A

Proofs of Theorems, Propositions and Lemmas

A.1 Proofs of the Theorems, Propositions and Lemmas of Chapter 4

Proof of Proposition 4.1

Since we take Proposition 4.1 from Proposition 1 in Zhang et al. (2018), we state the proof similar to the proof of Proposition 1 in Zhang et al. (2018).

Proof. Let I_1 and I_2 be two optimal solutions to problem (4.2) with $D := \{x \in [x_{min}, x_{max}] | I_1(x) \neq I_2(x)\}$ is non empty. Denote $I_\lambda(x) := \lambda I_1(x) + (1-\lambda)I_2(x)$, $x \in [x_{min}, x_{max}]$ for a constant $\lambda \in (0,1)$. Obviously, I_λ is a feasible indemnity function for problem (4.2) because $0 \leq I_\lambda \leq M$ and it satisfies the constraint in problem (4.2).
Let $v(\pi)$ denote the supremum value for problem (4.2). $v(\pi) < \infty$ because both $I(X)$ and Z are bounded random variables. Thus using the strict concavity, we obtain

$$\begin{aligned}
J(I_\lambda) &= \mathbb{E}[u((w_0 - (1-\theta)\pi)(1+\delta) + \lambda I_1(X) + (1-\lambda)I_2(X) + Z)] \\
&> \lambda\mathbb{E}[u((w_0 - (1-\theta)\pi)(1+\delta) + I_1(X) + Z)] \\
&\quad + (1-\lambda)\mathbb{E}[u((w_0 - (1-\theta)\pi)(1+\delta) + I_2(X) + Z)] \\
&= \lambda v(\pi) + (1-\lambda)v(\pi) \\
&= v(\pi);
\end{aligned}$$

which contradicts the optimality of I_1 and I_2. Thus, the optimal solution to problem (4.2) is unique up to the equality almost everywhere if it exists. □

Proof of Lemma 4.2

Since Lemma 4.2 is a modification of Lemma 1 in Zhang et al. (2018), we state the proof similar to the proof of Lemma 1 in Zhang et al. (2018).

Proof. Let $v(\pi)$ denotes the supremum value of problem (4.2) and let $J(I) = u((w_0 - (1 - \theta)\pi)(1 + \delta) + I(X) + Z)$. Therefore

$$\begin{aligned} v(\pi) &= \sup_{I \text{ s.t } \frac{m}{1+\delta}\mathbb{E}[I]=\pi} \mathbb{E}[J(I)] \\ &= \sup_{I \text{ s.t } \frac{m}{1+\delta}\mathbb{E}[I]=\pi} \{\mathbb{E}[J(I)] + \lambda^*(\pi - \frac{m}{1+\delta}\mathbb{E}[I])\} \\ &\leq \sup_I \{\mathbb{E}[J(I)] + \lambda^*(\pi - \frac{m}{1+\delta}\mathbb{E}[I])\} \\ &= \mathbb{E}[J(I^*)] + \lambda^*(\pi - \frac{m}{1+\delta}\mathbb{E}[I^*]) \\ &= \mathbb{E}[J(I^*)] \\ &\leq \sup_{I \text{ s.t } \frac{m}{1+\delta}\mathbb{E}[I]=\pi} \mathbb{E}[J(I)] \\ &= v(\pi), \end{aligned}$$

which implies that I^* is the solution of problem (4.2) . □

Proof of Lemma 4.3

Since we take Lemma 4.3 from Lemma 2 in Zhang et al. (2018), we state the proof similarly.

Proof. Without loss of generality, it is assumed that S_1^λ and S_2^λ are non empty. It is sufficient to show that $\hat{I}_\lambda(x) \leq 0$ for $x \in S_1^\lambda$ and $\hat{I}_\lambda(x) \geq M$ for $x \in S_2^\lambda$. In fact, if $\hat{I}_\lambda(x_1) > 0$ for some $x_1 \in S_1^\lambda$, then

$$\begin{aligned} 0 &= G(\hat{I}_\lambda(x_1), x_1) - \lambda\frac{m}{1+\delta} \\ &= \int_{z_{min}}^{z_{max}} u'((w_0-(1-\theta)\pi)(1+\delta)+ \hat{I}_\lambda(x_1)+ z)p(z|x)dz - \lambda\frac{m}{1+\delta} \\ &< \int_{z_{min}}^{z_{max}} u'((w_0-(1-\theta)\pi)(1+\delta)+ z)p(z|x)dz - \lambda\frac{m}{1+\delta} \end{aligned}$$

$$= G(0, x_1) - \lambda \frac{m}{1+\delta}$$
$$< 0,$$

where the first inequality is due to the strict concavity of u and the second inequality is due to the fact that $x_1 \in S_1^{\lambda}$. Since $0 < 0$ is a contradiction, $\hat{I}_{\lambda}(x) \leq 0$ for $x \in S_1^{\lambda}$.
Now suppose if $\hat{I}_{\lambda}(x_2) < M$ for some $x_2 \in S_2^{\lambda}$, then

$$\begin{aligned}
0 &= G(\hat{I}_{\lambda}(x_2), x_1) - \lambda \frac{m}{1+\delta} \\
&= \int_{z_{min}}^{z_{max}} u'((w_0-(1-\theta)\pi)(1+\delta)+ \hat{I}_{\lambda}(x_2)+ z)p(z|x)dz - \lambda \frac{m}{1+\delta} \\
&> \int_{z_{min}}^{z_{max}} u'((w_0-(1-\theta)\pi)(1+\delta)+ M + z)p(z|x)dz - \lambda \frac{m}{1+\delta} \\
&= G(M, x_2) - \lambda \frac{m}{1+\delta} \\
&> 0,
\end{aligned}$$

where the first inequality is due to the strict concavity of u and the second inequality is due to the fact that $x_1 \in S_1^{\lambda}$. Since $0 > 0$ is a contradiction, $\hat{I}_{\lambda}(x) \geq M$ for $x \in S_2^{\lambda}$.

According to the proof it is easy to understand that the desired result can be proven when either or both of the sets S_1^{λ} and S_2^{λ} are empty. □

Proof of Proposition 4.4

Since we adopt Proposition 4.4 from Proposition 3 in Zhang et al. (2018), we state the proof similarly.

Proof. The proof of the proposition is done by contradiction. Suppose $S_3^{\lambda^*} = \emptyset$. Then one of the following three scenarios must happen:

- Case 1: $S_1^{\lambda^*} = [x_{min}, x_{max}]$,
- Case 1: $S_2^{\lambda^*} = [x_{min}, x_{max}]$,
- Case 3: $S_1^{\lambda^*} \cup S_2^{\lambda^*} = [x_{min}, x_{max}]$, $S_1^{\lambda^*} \neq \emptyset$ and $S_2^{\lambda^*} \neq \emptyset$.

For Case 1, by Equation (4.11) $I^*(x) = 0$, $\forall\ x \in [x_{min}, x_{max}]$ and thus $\pi = \frac{m}{1+\delta}\mathbb{E}[I_{\lambda^*}(X)] = 0$. Similarly, for Case 2, $I^*(x) = M$, $\forall\ x \in [x_{min}, x_{max}]$ and thus $\pi = \frac{m}{1+\delta}\mathbb{E}[I_{\lambda^*}(X)] = \frac{m}{1+\delta}M$. Since the insurance premium $\pi \in (0, \frac{m}{1+\delta}M)$, both Case 1 and 2 are impossible. Now consider Case 3 and take $x_1 \in S_1^{\lambda^*}$ and $x_2 \in S_2^{\lambda^*}$. By Equations (4.8) and (4.9) and the fact that $G(i, x)$ is strictly decreasing in i, we have

$$G(0, x_1) < \lambda^* \frac{m}{1+\delta} < G(M, x_2) < G(0, x_2). \tag{A.1}$$

Since $p(z|x)$ is continuous on $[x_{min}, x_{max}] \times [z_{min}, z_{max}]$ it must be uniformly continuous on $[z_{min}, z_{max}]$. Therefore $\epsilon > 0$,
$|G(0, x + \epsilon) - G(0, x)|$

$$\begin{aligned}
&= \Big| \int_{z_{min}}^{z_{max}} u'((w_0 - (1-\theta)\pi)(1+\delta) + z)p(z|x+\epsilon)dz \\
&\quad - \int_{z_{min}}^{z_{max}} u'((w_0 - (1-\theta)\pi)(1+\delta) + z)p(z|x)dz \Big| \\
&\geq \max_{z \in [z_{min}, z_{max}]} |u'((w_0 - (1-\theta)\pi)(1+\delta) + z)| \int_{z_{min}}^{z_{max}} |p(z|x+\epsilon) - p(z|x)|dz \\
&\to 0, \quad \text{as } \epsilon \to 0^+, \ \forall\, x \in [x_{min}, x_{max}],
\end{aligned}$$

which implies $G(0, x)$ is a continuous function of x on $[x_{min}, x_{max}]$. Thus the inequalities in (A.1) implies the existence of a constant x_3 between x_1 and x_2 which satisfies $G(0, x_3) = \lambda^* \frac{m}{1+\delta}$. Again by the strictly decreasing property of $G(i, x)$ in i, we have $G(M, x_3) < G(0, x_3) = \lambda^* \frac{m}{1+\delta}$, which means that $x_3 \in S_3^{\lambda^*}$. This contradicts to the assumption of $S_3^{\lambda^*} = \emptyset$, and thus the proof is complete. □

Proof of Theorem 4.5

Since we build up Theorem 4.5 by slightly modifying Theorem 1 in Zhang et al. (2018), we state the proof similar to the proof of Theorem 1 in Zhang et al. (2018).

Proof. By the assumptions u"(.) < 0 and $p(z|x) > 0$ for $(x, z) \in [x_{min}, x_{max}] \times [z_{min}, z_{max}]$, it is clear that

$$\int_{z_{min}}^{z_{max}} u''((w_0 - (1-\theta)\pi)(1+\delta) + L(x) + z)p(z|x)dz < 0, \quad \forall\, x \in [x_{min}, x_{max}],$$

and thus $F(x, L)$ is well defined.
By Proposition 4.4 S_3^{λ} is non empty for any constant λ^* such that $\mathbb{E}[I_{\lambda^*}(X)] = \pi(1+\delta)/m$, where $I_{\lambda^*}(x)$ given in Equation (4.11). It is sufficient to find λ^* which satisfy $\mathbb{E}[I_{\lambda^*}(X)] = \pi(1+\delta)/m$ and show that $\hat{L}(x) = I_{\lambda^*}(x)$ on S_3^{λ}. Then by Lemma 4.3 along with the fact that $\hat{L}(x)$ is well defined on $[x_{min}, x_{max}]$, implies $L^* = ((\hat{L}(x))\vee) \wedge M$ is the optimal solution to problem 4.2.

Since $\hat{L}(x)$ satisfies Equation (4.18), we have

$$\int_{z_{min}}^{z_{max}} \Big[u''((w_0 - (1-\theta)\pi)(1+\delta) + \hat{L}(x) + z)p(z|x)\frac{d\hat{L}(x)}{dx} + u'((w_0 - (1-\theta)\pi)(1+\delta) + \hat{L}(x) + z)\frac{\partial}{\partial x}p(z|x)\Big]dz = 0,$$

that is

$$\frac{d}{dx}\int_{z_{min}}^{z_{max}} u'((w_0 - (1-\theta)\pi)(1+\delta) + \hat{L}(x) + z)p(z|x)dz = 0, \quad x \in [x_{min}, x_{max}].$$

This implies

$$G(\hat{L}(x), x) = \int_{z_{min}}^{z_{max}} u'((w_0 - (1-\theta)\pi)(1+\delta) + \hat{L}(x) + z)p(z|x)dz = \lambda_0 \frac{m}{1+\delta}, \quad x \in [x_{min}, x_{max}], \tag{A.2}$$

where the constant λ_0 is defined as

$$\lambda_0 = \frac{1+\delta}{m}\int_{z_{min}}^{z_{max}} u'((w_0 - (1-\theta)\pi)(1+\delta) + \hat{L}(x_{min}) + z)p(z|x_{min})dz.$$

Equation (A.2) with the fact that Equation (4.12) has a unique solution $\hat{I}_{\lambda_0}(x)$ for every $x \in S_3^{\lambda_0}$, imply that $\hat{L}(x) = \hat{I}_{\lambda^*}(x)$ on $S_3^{\lambda^*}$ for $\lambda^* = \lambda_0$. Comparing Equation (A.2) and (4.12), it is clear $\hat{L}(x)$ and $\hat{I}_{\lambda^*}(x)$ satisfy the same equation. Thus, from the proof of Lemma 4.3, $\hat{L}(x) \leq 0$ for $x \in S_1^{\lambda^*}$ and $\hat{L}(x) \geq M$ for $x \in S_2^{\lambda^*}$. Further, the second part in Equation (4.18) obviously implies $\mathbb{E}[I_{\lambda^*}(X)] = \pi(1+\delta)/m$, and thus the proof is complete. □

Proof of Theorem 4.6 and Theorem 4.7

Proof of Theorem 4.6 and Theorem 4.7 can be done similar to the proof of Theorem 4.5.

Proof of Proposition 4.8

Proof. The function F in Equation (4.19) with exponential utility becomes

$$F(x, L) = -\frac{\int_{z_{min}}^{z_{max}} u'((w_0 - (1-\theta)\pi)(1+\delta) + L + z)\frac{\partial}{\partial x}p(z|x)dz}{\int_{z_{min}}^{z_{max}} u''((w_0 - (1-\theta)\pi)(1+\delta) + L + z)p(z|x)dz}$$

$$
\begin{aligned}
&= -\frac{\int_{z_{min}}^{z_{max}} e^{-\gamma((w_0-(1-\theta)\pi)(1+\delta)+L+z)} \frac{\partial}{\partial x} p(z|x) dz}{\int_{z_{min}}^{z_{max}} -\gamma e^{-\gamma((w_0-(1-\theta)\pi)(1+\delta)+L+z)} p(z|x) dz} \\
&= \frac{e^{-\gamma((w_0-(1-\theta)\pi)(1+\delta)+L)} \int_{z_{min}}^{z_{max}} e^{-\gamma z} \frac{\partial}{\partial x} p(z|x) dz}{\gamma e^{-\gamma((w_0-(1-\theta)\pi)(1+\delta)+L)} \int_{z_{min}}^{z_{max}} e^{-\gamma z} p(z|x) dz} \\
&= \frac{1}{\gamma} \frac{\int_{z_{min}}^{z_{max}} e^{-\gamma z} \frac{\partial}{\partial x} p(z|x) dz}{\int_{z_{min}}^{z_{max}} e^{-\gamma z} p(z|x) dz} \\
&= \frac{1}{\gamma} \frac{\frac{\partial}{\partial x} \mathbb{E}[e^{-\gamma z}|X = x]}{\mathbb{E}[e^{-\gamma z}|X = x]} \\
&= \frac{\partial}{\partial x} \left\{ \frac{1}{\gamma} \ln \mathbb{E}[e^{-\gamma z}|X = x] \right\}
\end{aligned}
$$

The continuity of $\frac{\partial}{\partial x} p(z|x)$ and $p(z|x)$ implies that $\frac{\partial}{\partial x} \mathbb{E}[e^{-\gamma z}|X = x]$ and $\mathbb{E}[e^{-\gamma z}|X = x]$ exist for every $x \in [x_{min}, x_{max}]$, and thus $\frac{\partial}{\partial x} \left\{ \frac{1}{\gamma} \ln \mathbb{E}[e^{-\gamma z}|X = x] \right\}$ exist for every $x \in [x_{min}, x_{max}]$. Then the optimal index insurance is derived as

$$
I^*(x) = \left[\left(\frac{1}{\gamma} \ln \mathbb{E}[e^{-\gamma z}|X = x] + \eta^* \right) \vee 0 \right] \wedge M,
$$

where a constant η^* satisfies $\mathbb{E}[I^*(X)] = \frac{m}{1+\delta}$. Hence the proof is complete. □

Proof of Proposition 4.9

Proof. For Case 1 let $w_h = (w_0 - (1-\theta)\pi)(1+\delta) + L + \mu g_h x y_h$ and $w_l = (w_0 - (1-\theta)\pi)(1+\delta) + L + \mu g_h x y_l$. Then function F in Equation (4.21) with exponential utility becomes

$$
\begin{aligned}
F(x, L) &= -\frac{\mu g_h (u''(w_h)(1-p) y_h + u''(w_l) p y_l)}{u''(w_h)(1-p) + u''(w_l) p} \\
&= -\frac{\gamma \mu g_h (e^{-\gamma w_h}(1-p) y_h + e^{-\gamma w_l} p y_l)}{\gamma (e^{-\gamma w_h}(1-p) + e^{-\gamma w_l} p)} \\
&= -\frac{\gamma \mu g_h e^{-\gamma((w_0-(1-\theta)\pi)(1+\delta)+L)} (e^{-\gamma \mu g_h x y_h}(1-p) y_h + e^{-\gamma \mu g_h x y_l} p y_l)}{\gamma e^{-\gamma((w_0-(1-\theta)\pi)(1+\delta)+L)} (e^{-\gamma \mu g_h x y_h}(1-p) + e^{-\gamma \mu g_h x y_l} p)} \\
&= -\frac{\gamma \mu g_h (e^{-\gamma \mu g_h x y_h}(1-p) y_h + e^{-\gamma \mu g_h x y_l} p y_l)}{\gamma (e^{-\gamma \mu g_h x y_h}(1-p) + e^{-\gamma \mu g_h x y_l} p)} \\
&= \frac{1}{\gamma} \frac{\frac{\partial}{\partial x} \mathbb{E}[e^{-\gamma \mu g_h x Y}]}{\mathbb{E}[e^{-\gamma \mu g_h x Y}]} \\
&= \frac{\partial}{\partial x} \left\{ \frac{1}{\gamma} \ln \mathbb{E}\left[e^{-\gamma \mu g_h x Y} \right] \right\}.
\end{aligned}
$$

For Case 2 the function F in Equation (4.23) with exponential utility becomes

$$\begin{aligned}
F(x,L) &= -\frac{\mu g_h \int_{y_{min}}^{y_{max}} u''((w_0-(1-\theta)\pi)(1+\delta)+L+\mu g_h xy) y g(y)dy}{\int_{y_{min}}^{y_{max}} u''((w_0-(1-\theta)\pi)(1+\delta)+L+\mu g_h xy) g(y)dy} \\
&= -\frac{\gamma\mu g_h \int_{y_{min}}^{y_{max}} e^{-\gamma((w_0-(1-\theta)\pi)(1+\delta)+L+\mu g_h xy)} y g(y)dy}{\gamma \int_{y_{min}}^{y_{max}} e^{-\gamma((w_0-(1-\theta)\pi)(1+\delta)+L+\mu g_h xy)} g(y)dy} \\
&= -\frac{\gamma\mu g_h e^{-\gamma((w_0-(1-\theta)\pi)(1+\delta)+L)} \int_{y_{min}}^{y_{max}} e^{-\gamma\mu g_h xy} y g(y)dy}{\gamma e^{-\gamma((w_0-(1-\theta)\pi)(1+\delta)+L)} \int_{y_{min}}^{y_{max}} e^{-\gamma\mu g_h xy} g(y)dy} \\
&= -\frac{\gamma\mu g_h \int_{y_{min}}^{y_{max}} e^{-\gamma\mu g_h xy} y g(y)dy}{\gamma \int_{y_{min}}^{y_{max}} e^{-\gamma\mu g_h xy} g(y)dy} \\
&= \frac{1}{\gamma}\frac{\frac{\partial}{\partial x}\mathbb{E}[e^{-\gamma\mu g_h xY}]}{\mathbb{E}[e^{-\gamma\mu g_h xY}]} \\
&= \frac{\partial}{\partial x}\left\{\frac{1}{\gamma}\ln\mathbb{E}\left[e^{-\gamma\mu g_h xY}\right]\right\}.
\end{aligned}$$

Then for both Case 1 and 2 the optimal index insurance is derived as

$$I^*(x) = \left[\left(\frac{1}{\gamma}\ln\mathbb{E}\left[e^{-\gamma\mu g_h xY}\right] + \eta^*\right) \vee 0\right] \wedge M,$$

where the constant η^* satisfies $\mathbb{E}[I^*(X)] = \frac{m}{1+\delta}$. Hence the proof is complete. □

Proof of Propositions 4.10, 4.11 and 4.12

We can write the proof of Proposition 4.10, 4.11 and 4.12 by substituting $u'(w) = w^{-\gamma}$ and $u''(w) = -\gamma w^{-\gamma-1}$ to Equations (4.19), (4.21) and (4.23) respectively.

Proof of Theorem 4.14

We can write the proof of Theorem 4.14 similarly to the proof of Theorem 4.5. We state below only the important steps of the proof.

Proof. Since $\hat{I}(x)$ satisfies equation (4.36), we have

$$\frac{d\hat{I}(x)}{dx} + Nl(1+\rho)\frac{\partial\epsilon}{\partial x} = 0,$$

that is

$$\frac{d}{dx}u'((w_0 - \pi_l)(1+\delta) + Nl(1+\rho)\epsilon + \hat{I}(x)) = 0, \quad x \in [x_{min}, x_{max}].$$

This implies

$$G_l(\hat{I}(x), x) = u'((w_0 - \pi_l)(1+\delta) + Nl(1+\rho)\epsilon + \hat{I}(x)) = \lambda_0 \frac{m_l}{1+\delta}, \quad x \in [x_{min}, x_{max}], \tag{A.3}$$

where the constant λ_0 is defined as

$$\lambda_0 = \frac{1+\delta}{m_l} u'((w_0 - \pi_l)(1+\delta) + Nl(1+\rho)\epsilon(x_{min}) + \hat{I}(x_{min})).$$

□

A.2 Proofs of Propositions of Chapter 5

Proof of Proposition 5.1

The optimal insurance demand of farmer i is derived by considering the first order condition of the expected utility of the revenue of farmer i with respect to α_i.

Proof. Let $u_d(x) = \frac{1-e^{-\gamma_d x}}{\gamma_d}$

$$\begin{aligned}\frac{\partial \mathbb{E}[u_d(R_i)]}{\partial \alpha_i} =& e^{-\gamma_d(w_l + (w_i - \alpha_i \pi)(1+\delta))}\gamma_d \pi(1+\delta) r_i \\ &+ e^{-\gamma_d(w_l + (w_i - \alpha_i \pi)(1+\delta) + \alpha_i l)}\gamma_d(\pi(1+\delta) - l)(p_i - r_i) \\ &+ e^{-\gamma_d(w_h + (w_i - \alpha_i \pi)(1+\delta))}\gamma_d \pi(1+\delta)(1 - q - r_i) \\ &+ e^{-\gamma_d(w_h + (w_i - \alpha_i \pi)(1+\delta) + \alpha_i l)}\gamma_d(\pi(1+\delta) - l)(q + r_i - p_i) = 0. \end{aligned} \tag{A.4}$$

Rearranging Equation (A.4) gives

$$e^{-\gamma_d \alpha_i l} = \frac{\pi(1+\delta)[e^{-\gamma_d w_l} r_i + e^{-\gamma_d w_h}(1 - q - r_i)]}{(l - \pi(1+\delta))[e^{-\gamma_d w_l}(p_i - r_i) + e^{-\gamma_d w_h}(q + r_i - p_i)]}.$$

With the condition $\alpha_i \geq 0$, this results in

$$\alpha_i^* = \max\left[0, \frac{1}{\gamma_d l} \ln\left(\frac{(l - \pi(1+\delta))}{\pi(1+\delta)} \frac{((p_i - r_i)e^{-\gamma_d w_l} + (q + r_i - p_i)e^{-\gamma_d w_h})}{(r_i e^{-\gamma_d w_l} + (1 - q - r_i)e^{-\gamma_d w_h})}\right)\right].$$

□

Proof of Proposition 5.2

The optimal amount of insurance contacts is derived by considering the first order condition of the expected utility of insurer's profit in Equation (5.3) with respect to β.

Proof. Let $u_s(x) = \frac{1-e^{-\gamma_s x}}{\gamma_s}$

$$\frac{\partial \mathbb{E}[u_s(P)]}{\partial \beta} = e^{-\gamma_s(\beta\pi(1+\delta)-\beta l)}\gamma_s(l-\pi(1+\delta))q - e^{-\gamma_s(\beta\pi(1+\delta))}\gamma_s\pi(1+\delta)(1-q) = 0. \quad \text{(A.5)}$$

Rearranging Equation (A.5) gives

$$\beta^* = \frac{1}{\gamma_s l}\ln\left(\frac{(1-q)\pi(1+\delta)}{q(l-\pi(1+\delta))}\right).$$

□

Proof of Proposition 5.3

At equilibrium some farmers will not purchase any insurance. Therefore let $N*(\leq N)$ be the number of farmers who purchase some positive amount of insurance at the equilibrium.

Proof. By inserting Equations (5.8) and (5.9) in to Equation (5.5) yields

$$\begin{aligned}\sum_{i=1}^{N}\alpha_i^* &= \sum_{i=1}^{N}\max\left[0, \frac{1}{\gamma_d l}\ln\left(\frac{(l-\pi(1+\delta))}{\pi(1+\delta)}\frac{((p_i-r_i)e^{-\gamma_d w_l}+(q+r_i-p_i)e^{-\gamma_d w_h})}{(r_i e^{-\gamma_d w_l}+(1-q-r_i)e^{-\gamma_d w_h})}\right)\right] \\ &= \frac{1}{\gamma_d l}\ln\left[\left(\frac{1}{\pi(1+\rho)}-1\right)^{N*}\prod_{i(\alpha_i>0)}A_i\right] = \frac{1}{\gamma_s l}\ln\left(\frac{(1-q)\pi(1+\delta)}{q(l-\pi(1+\delta))}\right) = \beta^*. \quad \text{(A.6)}\end{aligned}$$

Rearranging Equation (A.6) gives

$$\left(\frac{l}{\pi(1+\delta)}-1\right)^{\frac{N*}{\gamma_d}}\left(\prod_{i(\alpha_i>0)}A_i\right)^{\frac{1}{\gamma_d}} = \left(\frac{l}{\pi(1+\delta)}-1\right)^{-\frac{1}{\gamma_s}}\left(\frac{1-q}{q}\right)^{\frac{1}{\gamma_s}}.$$

This results in

$$\pi^* = \frac{l(\prod_{i(\alpha_i>0)}A_i)^{(\frac{\gamma_s}{\gamma_d+N^*\gamma_s})}(\frac{q}{1-q})^{(\frac{\gamma_d}{\gamma_d+N^*\gamma_s})}}{(1+\delta)(1+(\prod_{i(\alpha_i>0)}A_i)^{(\frac{\gamma_s}{\gamma_d+N^*\gamma_s})}(\frac{q}{1-q})^{(\frac{\gamma_d}{\gamma_d+N^*\gamma_s})})}.$$

By substituting π in Equation (5.9) by this π^* yields

$$\beta^* = \frac{1}{l(\gamma_d + N^*\gamma_s)} \ln\left(\left(\prod_{i(\alpha_i>0)} A_i\right)\left(\frac{1-q}{q}\right)^{N^*}\right),$$

where $A_i = \frac{(p_i - r_i)e^{-\gamma_d w_l} + (q + r_i - p_i)e^{-\gamma_d w_h}}{r_i e^{-\gamma_d w_l} + (1 - q - r_i)e^{-\gamma_d w_h}}$. □

Proof of Proposition 5.4

The equilibrium premium is derived by considering the first order condition of the expected utility of insurer's profit in Equation (5.6) with respect to π.

Proof.

$$\begin{aligned}\frac{\partial \mathbb{E}[u_s(P)]}{\partial \beta} =& e^{-\gamma_s(\alpha^*\pi(1+\delta)-\alpha^* l)}\gamma_s(-\alpha^*(1+\delta) - \alpha^{*\prime}\pi(1+\delta) + \alpha^{*\prime} l)q \\ &- e^{-\gamma_s(\alpha^*\pi(1+\delta))}\gamma_s(\alpha^*(1+\delta) + \alpha^{*\prime}\pi(1+\delta))(1-q) = 0. \end{aligned} \tag{A.7}$$

Rearranging Equation (A.7) gives

$$\pi^* = \frac{q l e^{\gamma_s l \alpha^*(\pi^*)}}{(1+\delta)(q e^{\gamma_s l \alpha^*(\pi^*)} + 1 - q)} - \frac{\alpha^*(\pi^*)}{(\alpha^*(\pi^*))'}.$$

□

Proof of Proposition 5.10

Proof can be done similar to the proof of 5.1.

Proof of Proposition 5.11

The equilibrium premium is derived by considering the first order condition of the expected utility of insurer's profit under scenario 1 with respect to $\pi_\odot$.

Proof. $\frac{\partial \mathbb{E}[u_s(P)]}{\partial \pi_\odot} =$

$$\begin{aligned}& e^{-\gamma_s((\alpha^*_\oplus+\alpha^*_\ominus)\pi_\odot(1+\delta)-(\alpha^*_\oplus+\alpha^*_\ominus)l)}\gamma_s(-(\alpha^*_\oplus+\alpha^*_\ominus)(1+\delta)-(\alpha^*_\oplus+\alpha^*_\ominus)'\pi_\odot(1+\delta)+(\alpha^*_\oplus+\alpha^*_\ominus)'l)q_\oplus q_\ominus \\ &+ e^{-\gamma_s((\alpha^*_\oplus+\alpha^*_\ominus)\pi_\odot(1+\delta)-\alpha^*_\oplus l)}\gamma_s(-(\alpha^*_\oplus+\alpha^*_\ominus)(1+\delta)-(\alpha^*_\oplus+\alpha^*_\ominus)'\pi_\odot(1+\delta)+(\alpha^*_\oplus)'l)q_\oplus(1-q_\ominus) \\ &+ e^{-\gamma_s((\alpha^*_\oplus+\alpha^*_\ominus)\pi_\odot(1+\delta)-\alpha^*_\ominus l)}\gamma_s(-(\alpha^*_\oplus+\alpha^*_\ominus)(1+\delta)-(\alpha^*_\oplus+\alpha^*_\ominus)'\pi_\odot(1+\delta)+(\alpha^*_\ominus)'l)(1-q_\oplus)q_\ominus \\ &+ e^{-\gamma_s((\alpha^*_\oplus+\alpha^*_\ominus)\pi_\odot(1+\delta))}\gamma_s(-(\alpha^*_\oplus+\alpha^*_\ominus)(1+\delta)-(\alpha^*_\oplus+\alpha^*_\ominus)'\pi_\odot(1+\delta))(1-q_\oplus)(1-q_\ominus) = 0. \end{aligned} \tag{A.8}$$

Rearranging Equation (A.8) gives

$$\pi^*_\odot = \frac{l(A(\alpha^*_\oplus + \alpha^*_\ominus)' + B(\alpha^*_\oplus)' + C(\alpha^*_\ominus)')}{(1+\delta)(\alpha^*_\oplus + \alpha^*_\ominus)'(A+B+C+D)} - \frac{(\alpha^*_\oplus + \alpha^*_\ominus)}{(\alpha^*_\oplus + \alpha^*_\ominus)'},$$

where
$A = q_\oplus q_\ominus e^{\gamma_s l(\alpha^*_\oplus + \alpha^*_\ominus)}$, $B = q_\oplus(1-q_\ominus)e^{\gamma_s l \alpha^*_\oplus}$, $C = (1-q_\oplus)q_\ominus e^{\gamma_s l \alpha^*_\ominus}$, $D=(1-q_\oplus)(1-q_\ominus)$. □

Proof of Proposition 5.12

The equilibrium premium of group $\odot/\ominus$ is derived by considering the first order condition of the expected utility of insurer's profit under scenario 2 with respect to $\pi_{\odot/\ominus}$.

Proof. $\frac{\partial \mathbb{E}[u_s(P)]}{\partial \pi_\oplus} =$

$$\begin{aligned}
&e^{-\gamma_s(\alpha^*_\oplus \pi_\oplus(1+\delta) + \alpha^*_\ominus \pi_\ominus(1+\delta) - (\alpha^*_\oplus + \alpha^*_\ominus)l)}\gamma_s(-\alpha^*_\oplus(1+\delta) - (\alpha^*_\oplus)'\pi_\oplus(1+\delta) + (\alpha^*_\oplus)'l)q_\oplus q_\ominus \\
&+ e^{-\gamma_s(\alpha^*_\oplus \pi_\oplus(1+\delta) + \alpha^*_\ominus \pi_\ominus(1+\delta) - \alpha^*_\oplus l)}\gamma_s(-\alpha^*_\oplus(1+\delta) - (\alpha^*_\oplus)'\pi_\oplus(1+\delta) + (\alpha^*_\oplus)'l)q_\oplus(1-q_\ominus) \\
&+ e^{-\gamma_s(\alpha^*_\oplus \pi_\oplus(1+\delta) + \alpha^*_\ominus \pi_\ominus(1+\delta) - \alpha^*_\ominus l)}\gamma_s(-\alpha^*_\oplus(1+\delta) - (\alpha^*_\oplus)'\pi_\oplus(1+\delta))(1-q_\oplus)q_\ominus \\
&+ e^{-\gamma_s(\alpha^*_\oplus \pi_\oplus(1+\delta) + \alpha^*_\ominus \pi_\ominus(1+\delta))}\gamma_s(-\alpha^*_\oplus(1+\delta) - (\alpha^*_\oplus)'\pi_\oplus(1+\delta))(1-q_\oplus)(1-q_\ominus) = 0. \qquad \text{(A.9)}
\end{aligned}$$

Rearranging Equation (A.9) gives

$$\pi^*_\oplus = \frac{q_\oplus l e^{\gamma_s l \alpha^*_\oplus}}{(1+\delta)(q_\oplus e^{\gamma_s l \alpha^*_\oplus} + 1 - q_\oplus)} - \frac{\alpha^*_\oplus}{(\alpha^*_\oplus)'}.$$

□

A.3 Proofs of Propositions of Chapter 6

Proof of Proposition 6.3

The optimal insurance cover under Model 1 is derived by considering the first order condition of the expected utility of farmer's level of consumption with respect to α.

Proof. Let $u(x) = \frac{1-e^{-\gamma x}}{\gamma}$

$$\begin{aligned}
\frac{\partial \mathbb{E}[u(C^i_1)]}{\partial \alpha} =& e^{-\gamma(\mu_i + \overline{\theta}_g + \overline{\theta} - \alpha\pi - \beta\overline{\theta} + \beta\overline{\theta}/n)}\gamma\pi(1-q)(1-p)\sum_{k=0}^{n} e^{-\gamma\beta s_k/n}\rho_k \\
&+ e^{-\gamma(\mu_i + \overline{\theta}_g + \underline{\theta} - \alpha\pi - \beta\underline{\theta} + \beta\underline{\theta}/n)}\gamma\pi(1-q)p\sum_{k=0}^{n} e^{-\gamma\beta s_k/n}\rho_k
\end{aligned}$$

$$+ e^{-\gamma(\mu_i+\underline{\theta}_g+\overline{\theta}-\alpha\pi+\alpha(\overline{\theta}_g-\underline{\theta}_g)-\beta\overline{\theta}+\beta\overline{\theta}/n)}\gamma(\pi-(\overline{\theta}_g-\underline{\theta}_g))q(1-p)\sum_{k=0}^{n}e^{-\gamma\beta s_k/n}\rho_k$$
$$+ e^{-\gamma(\mu_i+\underline{\theta}_g+\underline{\theta}-\alpha\pi+\alpha(\overline{\theta}_g-\underline{\theta}_g)-\beta\underline{\theta}+\beta\underline{\theta}/n)}\gamma(\pi-(\overline{\theta}_g-\underline{\theta}_g))qp\sum_{k=0}^{n}e^{-\gamma\beta s_k/n}\rho_k = 0. \quad \text{(A.10)}$$

Since $\pi = mq(\overline{\theta}_g - \underline{\theta}_g)$, rearranging Equation (A.10) gives

$$e^{-\gamma\alpha(\overline{\theta}_g-\underline{\theta}_g)} = \frac{mq(1-q)e^{-\gamma(\overline{\theta}_g-\underline{\theta}_g)}}{(1-mq)q}.$$

With the condition $0 \leq \alpha_i \leq 1$, this results in

$$\alpha_1^* = \min\left[1, \max\left[0, \frac{1}{\gamma(\overline{\theta}_g-\underline{\theta}_g)}\ln\left(\frac{(1-mq)}{m(1-q)}e^{\gamma(\overline{\theta}_g-\underline{\theta}_g)}\right)\right]\right].$$

□

Proof of Proposition 6.4

The optimal insurance cover under Model 2* is derived by considering the first order condition of the expected utility of farmer's level of consumption with respect to α.

Proof. Let $u(x) = \frac{1-e^{-\gamma x}}{\gamma}$

$$\frac{\partial\mathbb{E}[u(C^i_{2^*})]}{\partial\alpha} = e^{-\gamma(w_i-\beta\ell-\alpha\pi)}\gamma\pi(1-q-r)(1-(1-p)^{n-1}) + e^{-\gamma(w_i-\alpha\pi)}\gamma\pi(1-q-r)(1-p)^{n-1}$$
$$+ e^{-\gamma(w_i-\ell-\alpha l)}(\sum_{k=0}^{n}e^{-\gamma s_k}\rho_k)\gamma\pi r + e^{-\gamma(w_i-\beta\ell-\alpha\pi+\alpha\ell)}\gamma(\pi-\ell)(q+r-p)(1-(1-p)^{n-1})$$
$$+ e^{-\gamma(w_i-\alpha\pi+\alpha\ell)}\gamma(\pi-\ell)(q+r-p)(1-p)^{n-1}$$
$$+ e^{-\gamma(w_i-\ell-\alpha\pi+\alpha\ell)}(\sum_{k=0}^{n}e^{-\gamma s_k}\rho_k)\gamma(\pi-l)(p-r) = 0. \quad \text{(A.11)}$$

Since $\pi = mq\ell$, rearranging Equation (A.11) gives

$$e^{-\gamma\alpha\ell} = \frac{mq[e^{\gamma\beta\ell}(1-q-r)(1-(1-p)^{n-1})+(1-q-r)(1-p)^{n-1}+e^{\gamma\ell}r(\sum_{k=0}^{n}e^{-\gamma s_k}\rho_k)]}{(1-mq)[e^{\gamma\beta\ell}(q+r-p)(1-(1-p)^{n-1})+(q+r-p)(1-p)^{n-1}+e^{\gamma\ell}(p-r)(\sum_{k=0}^{n}e^{-\gamma s_k}\rho_k)]}.$$

With the condition $0 \leq \alpha_i \leq 1$, this results in

$$\alpha_{2^*}^* = \min\left[1, \max\left[0, \frac{1}{\gamma\ell}\ln\left(\frac{(1-mq)}{mq}\frac{(q+r-p)((1-p)^{n-1}+(1-(1-p)^{n-1})e^{\gamma\beta\ell})+(p-r)e^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k}}{(1-q-r)((1-p)^{n-1}+(1-(1-p)^{n-1})e^{\gamma\beta\ell})+re^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k}}\right)\right]\right].$$

□

Appendix B

Additional Figures and Information

B.1 Additional Information on Chapter 3

Solving the Optimization Problem in the Model without Savings

We go through following steps to find the range of X in which the farmer repay the loan at the end of first period.

- Let $X_1 \cup X_2 = X$, such that $X_1 = \{x|I(x) > 0\}$ and $X_2 = \{x|I(x) = 0\}$. Also $x_1(\in X_1) < x_2(\in X_2)$.

- We know that at a given x and y, $w_1 = \mu g_h xy + \alpha I(x)$.

- At a given $y \in Y$, w_1 is increasing in $x \in X_2$ (because $I(x) = 0$).

- At a given $y \in Y$, w_1 is either increasing or decreasing in $x \in X_1$ and it basically depends on the value of y.
  Eg: Let $I(x) = max\{0, \lambda(k - x)\}$ for $\lambda, k > 0$.
  Then for $x \in X_1$, $w_1 = \mu g_h xy + \alpha\lambda(k - x)$.
  w_1 is increasing in $x \in X_1$ if $\mu g_h y - \alpha\lambda > 0 \rightarrow y > \alpha\lambda/\mu g_h$.
  w_1 is decreasing in $x \in X_1$ if $\mu g_h y - \alpha\lambda < 0 \rightarrow y < \alpha\lambda/\mu g_h$.

- Now let $Y_1 \cup Y_2 = Y$ with $y_1(\in Y_1) < y_2(\in Y_2)$. When $y \in Y_1$, w_1 is decreasing in $x \in X_1$ and $y \in Y_2$, w_1 is increasing in $x \in X_1$.

- Since when $y \in Y_2$, w_1 is increasing in $x \in X_1$ then w_1 is increasing in the whole range of X. Then at given $y \in Y_2$ there exists unique $x^*_{1,y} \in X$, such that $w^*_1 = \mu g_h x^*_{1,y} y + \alpha I(x^*_{1,y})$ and the farmer repays the loan if $x > x^*_{1,y}$. See Figure B.1.

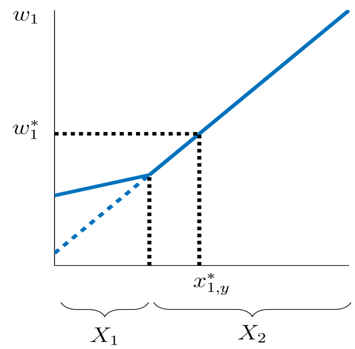

Figure B.1: Illustration of $x_{1,y}^*$ for $y \in Y_2$.

- If $y \in Y_1$, w_1 is decreasing in $x \in X_1$ and w_1 is increasing in $x \in X_2$, then at a given $y \in Y_1$ there exists $x_y^* \in X_2$ such that $w_1^* = sg_h x_y^* y$.
- At a given $y \in Y_1$ if $w_1^* < \mu g_h x_{min} y + \alpha I(x_{min})$ and there exists $x_y^{**} \in X_1$ such that $w_1^* = \mu g_h x_y^{**} y + \alpha I(x_y^{**})$. See Figure B.2.

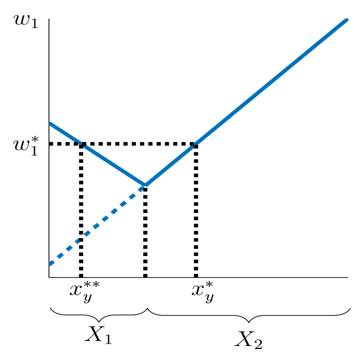

Figure B.2: Illustration of x_y^* and x_y^{**} for $y \in Y_1$ and $w_1^* < \mu g_h x_{min} y + \alpha I(x_{min})$.

- At a given $y \in Y_1$ if $w_1^* \geq \mu g_h x_{min} y + \alpha I(x_{min})$ then let $x_y^{**} = x_{min}$. See Figure B.3.

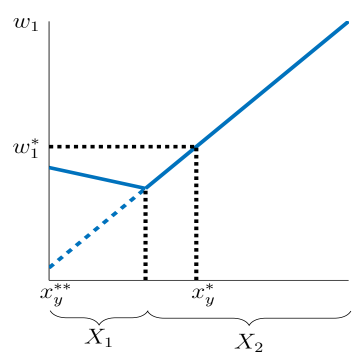

Figure B.3: Illustration of x_y^* and x_y^{**} for $y \in Y_1$ and $w_1^* \geq sg_h x_{min} y + \alpha I(x_{min})$.

- At a given $y \in Y_1$ the farmer repays the loan at the end of first period if $x < x_y^{**}$ or $x > x_y^*$.

Reasons for the Jumps in Loan Default Probability and Expected IRR Curves

Now we explain the reason behind the jumps and shape changes in the loan default probability curves and the expected IRR curves of Figure 3.6. We calculate the loan default probability by finding the range of X (by considering the wealth of the farmer in that range of X) in which the farmer prefers to repay the loan. At 1.1% interest rate, x^*_{t,y_l} for $t = 2, 3, 4, 5$ are very close to x_1 but still less than x_1. Then there exist corresponding x^{**}_{t,y_l} for $t = 2, 3, 4, 5$ such that those are close to x_2 and greater than x_2. Then the farmer repays the loan when $Y = y_l$ if $x \in (x^*_{t,y_l}, x^{**}_{t,y_l})$. This range is wider than the range (x_1, x_2). But at 1.2% interest rate, $x^*_{t,y_l} = x_1$ and then $x^{**}_{t,y_l} = x_2$. Then the farmer is unable to repay the loan at 1.2% interest rate when $Y = y_l$. When $Y = y_l$ at 1.1% interest rate there are possibilities of repaying the loan at the end of period $t = 2, 3, 4, 5$ within a considerably wide range of X. But at 1.2% interest rate the farmer is not able to repay the loan at any level of X. This leads to a significant increase of the loan default probability at 1.2% interest rate compared to 1.1% interest rate. There is a similar reason for the jump at 21.6% interest rate. Here $x^*_{t,y_h} = x_1$ and $x^{**}_{t,y_h} = x_2$ only for $t = 5$ and then the jump is smaller than the jump at 1.2% interest rate. Now it is clear that the jumps of the curves are due to the range of X where the yield does not change as X increases.

There is a jump in the default probability curve of unsubsidized insurance at 20.5% interest rate due to the same reason as explained above. The shape of the loan default probability curve of unsubsidized insurance changes at 20.8% interest rate. When the interest rate is below 20.8% the farmer repays the loan if $x \in (x^*_{1,y_l}, x^{**}_{1,y_l})$, where $x^*_{t,y_l} < k$ and $x^{**}_{t,y_l} > k_2$. That is x^*_{1,y_l} and x^{**}_{1,y_l} are in the range of X in which the farmer receives insurance payment. But for the interest rates equal or above 20.8%, x^*_{1,y_l} and x^{**}_{1,y_l} come from the region where farmer does not get any insurance payment. Then the wealth level of the farmer where the farmer is indifferent between repaying and defaulting loan has two different forms before and after 20.8% interest rate. This leads to different shapes of the default probability curve before and after 20.8% interest rate. Similar jumps and shape changes can be seen in the expected IRR curves due to similar reasons as above.

B.2 Additional Information on Chapter 4

Numerical Scheme for Solving ODE Problems (4.18), (4.20) and (4.22)

Here we state the numerical scheme to solve ODE problems, which is given in Zhang et al. (2018). The boundary value ODE problems (4.18), (4.20) and (4.22) can be express as an initial value problem (B.1) with an algebraic Equation (B.2):

$$
\begin{aligned}
\frac{dL}{dx} &= F(x, L), \\
I(x_{min}) &= I_{x_{min}},
\end{aligned}
\tag{B.1}
$$

with initial value $I_{x_{min}}$ determined by

$$
\pi = \frac{m}{1+\delta}\mathbb{E}[(L(x) \vee 0) \wedge M], \tag{B.2}
$$

Step 1: Find a large enough interval $[L, U]$ such that $I(x_{min}) \in [L, U]$. Check that $(\pi_L - \pi)(\pi_U - \pi) < 0$, where π_L and π_U denote the premium calculated by Equation (B.2) for the contract starting at L and U respectively. Suppose $(\pi_L - \pi) < 0$, $(\pi_U - \pi) > 0$, and define $I_0(x_{min}) = \frac{1}{2}(L + U)$.

Step 2: Apply RK4 with a step-size $h > 0$ to the initial value problem $\frac{dI}{dx} = F(x, I)$, $x \in [x_{min}, x_{max}]$, with $I(x_{min}) = I_0(x_{min})$: For $n = 0, 1, 2, \ldots, [\frac{x_{max}-x_{min}}{h} - 1]$, define

1 $k_1 = F(x_n, I_n)$,

2 $k_2 = F(x_n + \frac{h}{2}, I_n + \frac{h}{2}k_1)$,

3 $k_3 = F(x_n + \frac{h}{2}, I_n + \frac{h}{2}k_2)$,

4 $k_4 = F(x_n + h, I_n + hk_3)$,

5 $x_{n+1} = x_n + h$,

6 $I_{n+1} = I_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

Step 3: Define $I_n^* = (I_n \vee 0) \wedge M$, $n = 0, 1, 2, \ldots, [\frac{x_{max}-x_{min}}{h}]$, where I_n is obtained from the previous step.

Step 4: Approximate the premium constraint $\pi = \frac{m}{1+\delta}\mathbb{E}[I^*]$ numerically using

$$
\pi_0 = \frac{hm}{2(1+\delta)}\Big[2\Big(\sum_{n=0}^{\frac{x_{max}-x_{min}}{h}} I_n^* f(x_{min} + nh)\Big) - I_0^* f(x_{min}) - I^*_{\frac{x_{max}-x_{min}}{h}} f(x_{max})\Big].
$$

Step 5: Verify whether $|\pi_0 - \pi| < \epsilon$ is satisfied by the given tolerance ϵ. If yes, I^* is already an accurate approximation to the solution of ODE (B.1) and (B.2), and we stop the algorithm; otherwise, we go to Step 6.

Step 6: If $\pi_0 < \pi$, then define $I_1(x_{min}) = \frac{1}{2}(I_0(x_{min}) + U)$; $\pi_0 > \pi$, then define $I_1(x_{min}) = \frac{1}{2}(L + I_0(x_{min}))$. Go back to Step 2, replace the initial condition with $I(x_{min}) = I_1(x_{min})$, and repeat Steps 2–6.

B.3 Additional Information and Figures for Chapter 6

Insurance Demand of Model 2 and 3

The optimal insurance cover under Model 2 is given by

$$\alpha_2^* = \min\left[1, \max\left[0, \frac{1}{\gamma\ell}\ln\left(\frac{(1-mq)}{mq}\frac{(q+r-p)e^{\gamma\beta\ell}+(p-r)e^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k}}{(1-q-r)e^{\gamma\beta\ell}+re^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k}}\right)\right]\right]. \tag{B.3}$$

Similarly in Model 3* we can not derive the optimal insurance cover under Model 3. But here we state the first order condition which leads to the optimal cover. It is given by

$$\frac{\partial \mathbb{E}[u(C_3^i)]}{\partial \alpha} = re^{\gamma\ell}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1-k}\left[e^{-\frac{\gamma\ell k\alpha\beta}{j+1}}\rho_{kj}\left(\frac{k\beta}{j+1}-mq\right)\right] + e^{-\gamma\ell(\alpha-1)}(p-r)(1-mq)$$
$$+ e^{-\gamma\alpha\ell(1-\beta)}(q+r-p)(1-\beta-mq) - (1-q-r)mq = 0.$$

The comparison between the optimal insurance covers of Model 2 vs 2* and Models 3 vs 3* for small group sizes are given in Figures B.4 and B.5.

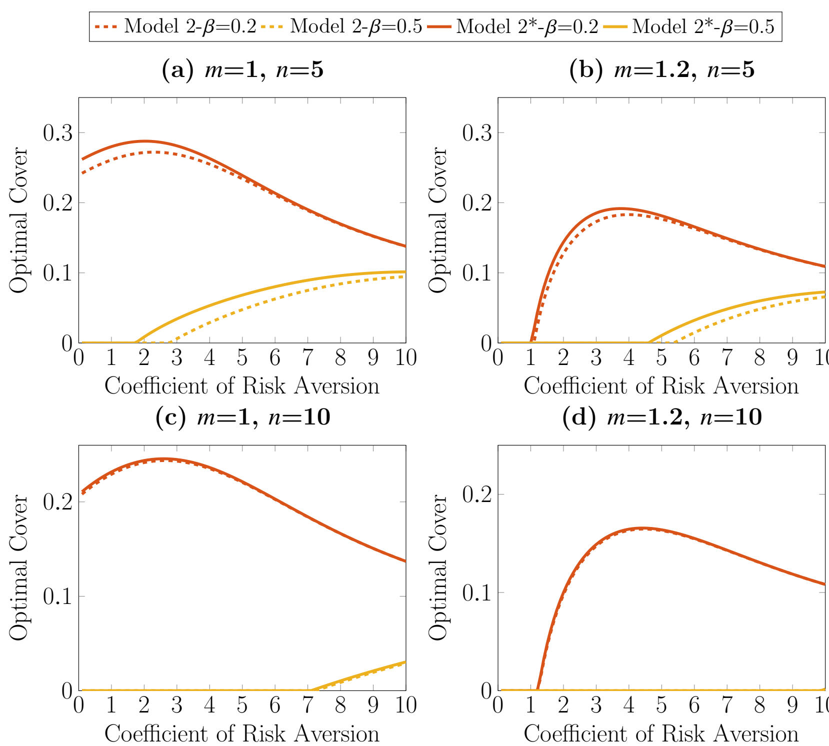

Figure B.4: Optimal Cover Comparison – Model 2 and 2*.

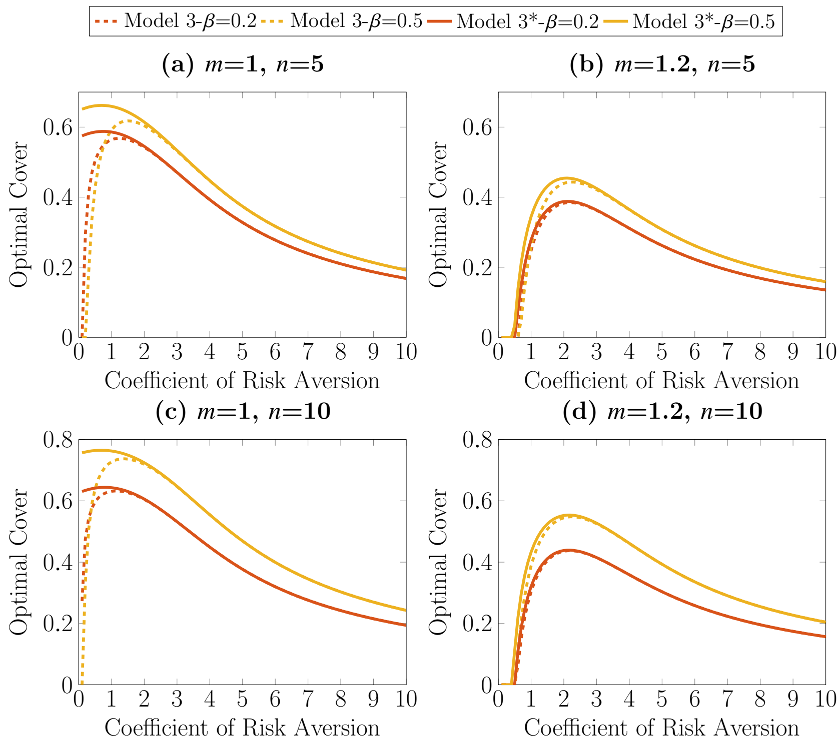

Figure B.5: Optimal Cover Comparison – Model 3 and 3*.

Optimal Insurance Demand vs β for Model 2

The optimal insurance cover under Model 2 is given by Equation (B.3). Now let

$$A = \frac{(1-mq)}{mq}\,\frac{(q+r-p)e^{\gamma\beta\ell}+(p-r)e^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k}}{(1-q-r)e^{\gamma\beta\ell}+re^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k}}.$$

Then as given in Equation (B.3)

$$\alpha_2^* = \min\left[1, \max\left[0, \frac{1}{\gamma\ell}\ln(A)\right]\right].$$

Now we consider the first derivative of A with respect to β. It is given by

$$\frac{\partial A}{\partial \beta} = \frac{(1-mq)}{mq}\left(\frac{-\gamma\ell(p(1-q)-r)e^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k}/(n-k)}{((1-q-r)e^{\gamma\beta\ell}+re^{\gamma\ell}\sum_{k=0}^{n-1}\rho_k e^{-\gamma s_k})^2}\right).$$

Since we assume $p(1-q)-r > 0$ so that the index is a signal of loss (see Clarke (2016)), we can clearly see that $\frac{\partial A}{\partial \beta} < 0$. Therefore when β increases A decreases. Then $\ln(A)$ also decreases. As a result of that when the fraction of informal risk sharing increases the optimal insurance demand decreases

B.4 Additional Information and Figures for Chapter 7

Alternative Ways to Compute Basis Risk

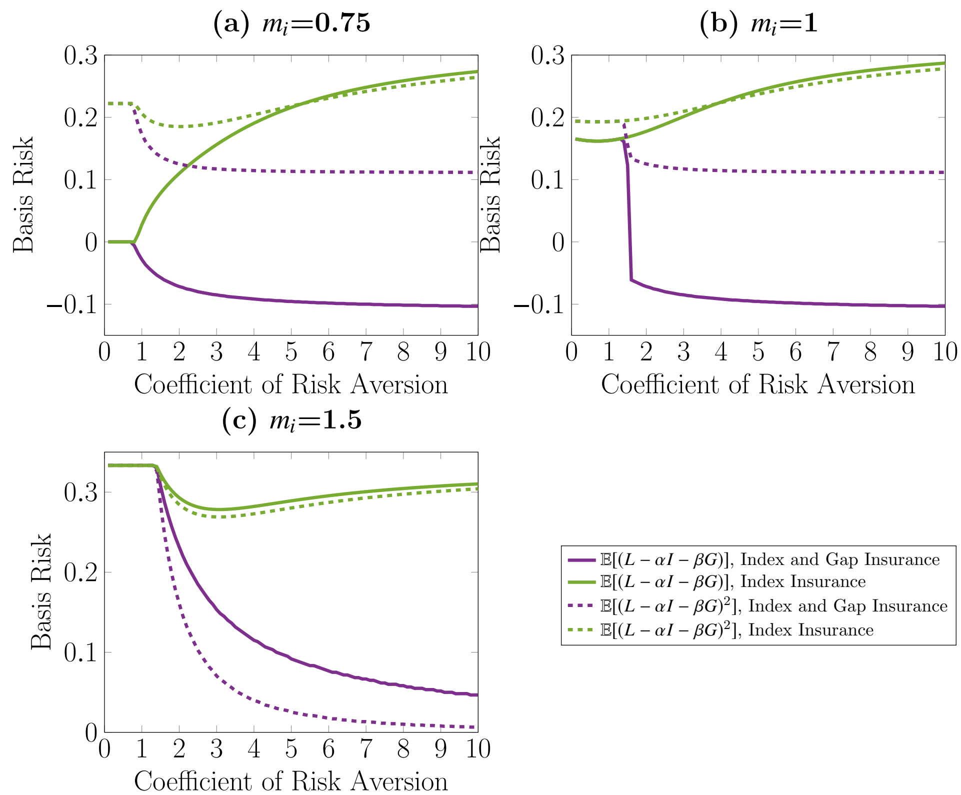

Figure B.6: Basis Risk – Case 1

Gap Insurance for Index Insurance – Case 2

Here we show the numerical results of the index insurance with gap insurance under Case 2 ($p = 1/20, q = 1/5$ and $r = 1/1000$). According to Figure B.7 for some levels of risk aversion (e.g: $\gamma = 3, 5$) the demand of the index insurance decreases as the gap insurance cover increases. Also when $m_i = 1.5$ the gap insurance is not a good instrument to increase the demand of index insurance of most of the insureds. For all the cases in Figure B.8 the total premium increases as the index cover increases. It implies in those cases $\beta < \frac{qm_i}{(p-r)m_g}$. According to Figure B.9 the optimal demand for the index insurance with gap insurance is always less than the demand without gap insurance. Therefore in that case the gap insurance does not perform as an instrument to increase the demand of the index insurance. According to Figure B.10 when the index insurance is purchased with a gap insurance it reduces the basis risk.

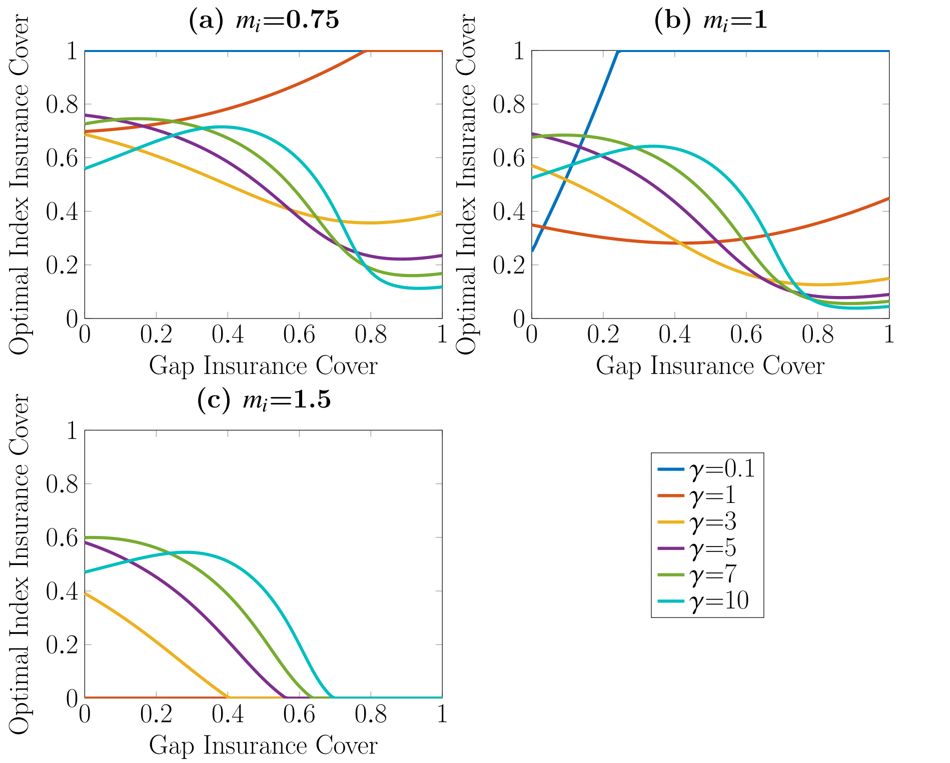

Figure B.7: Optimal Index Insurance Cover Vs Gap Insurance Cover – Case 2

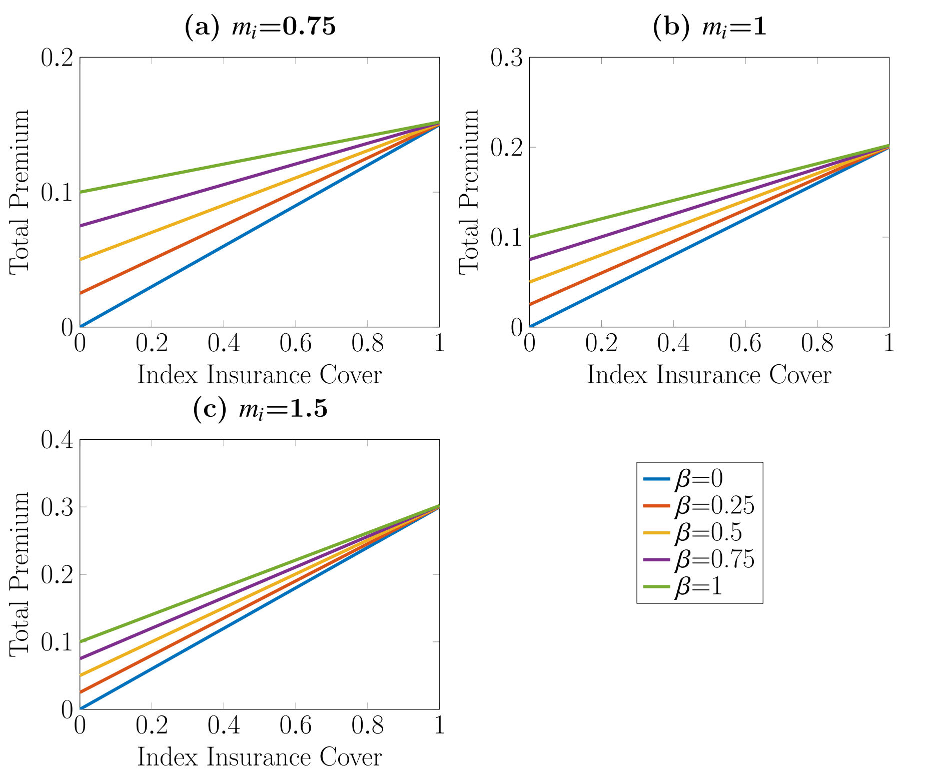

Figure B.8: Total Premium – Case 2

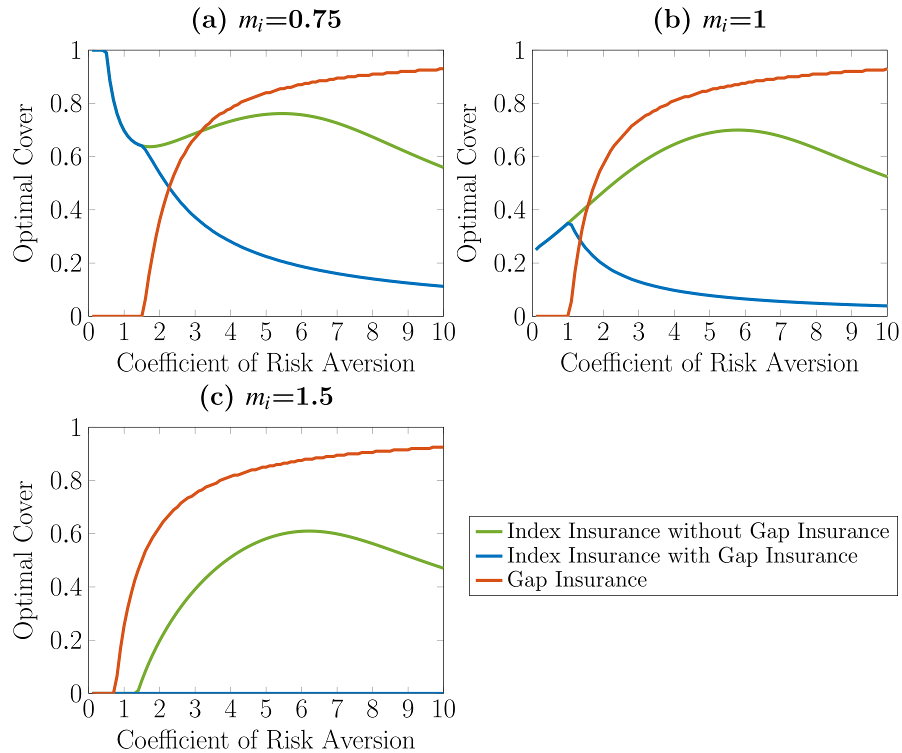

Figure B.9: Optimal Pair of Index and Gap Insurance Covers – Case 2

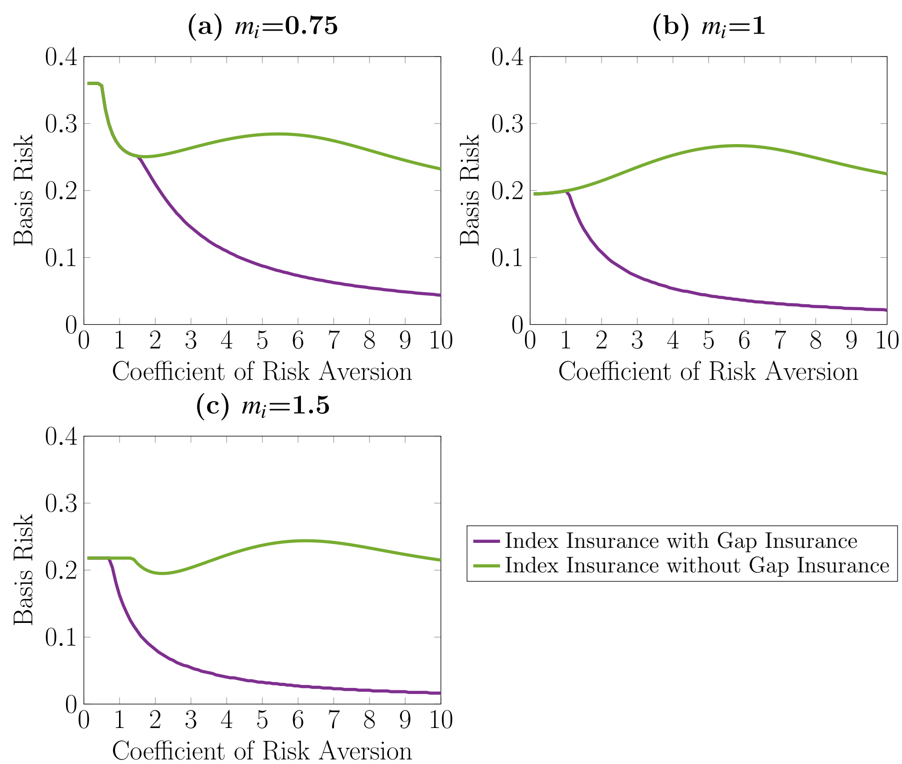

Figure B.10: Basis Risk – Case 2

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