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SOME BV PROPERTIES OF THE PREISACH HYSTERESIS OPERATOR

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Abstract.

We discuss mapping and Lipschitz continuity properties of the Preisach model for hysteresis in the spaces BV[0,1] and $W^{1,1}[0,1]$.

1. Introduction

The Preisach model for hysteresis [10] essentially consists of a continuous family of thermostats (i.e. relays with hysteresis). It is customarily formalized as an operator W defined by

$$(Wx)(t) = \int_P (W^\rho x)(t)\, d\mu(\rho) \ ,$$

mapping an "input function" $x: [0,1] \to \mathbb{R}$ to an "output function" $Wx: [0,1] \to \mathbb{R}$; here,

$$P = \{\rho: \rho=(\rho_1,\rho_2) \in \mathbb{R}^2, \ \rho_1 < \rho_2\}$$

is the so-called Preisach plane, μ is a measure on P, and W^ρ denotes, for any $\rho \in P$, the thermostat switching from -1 to 1 if x(t) reaches ρ_2 and from 1 to -1 if x(t) reaches ρ_1. This model is of special interest in the description of ferromagnetic materials [9,13]; moreover, it includes several other formalizations of hysteresis as a special case.

The mathematical properties of the Preisach hysteresis operator have been studied extensively in [2,4,5,9,12]. In particular, the Preisach operator is, with the exception of degenerate cases, not differentiable. It is therefore of interest to find out the situations in which it is Lipschitz continuous. In the present paper, we give sufficient conditions for Lipschitz continuity with respect to the norm of BV[0,1]. Roughly speaking, the Preisach operator maps BV[0,1] and $W^{1,1}[0,1]$ into itself if the measure μ has a bounded density, and is Lipschitz continuous on $W^{1,1}[0,1]$ if the measure μ has a Lipschitz continuous density; it is Lipschitz continuous on BV[0,1] if the measure μ has a density e of the form $e(\rho_1,\rho_2) = e(\rho_1-\rho_2)$.

This paper is not intended as an introduction to the Preisach operator; we refer to [2,5,12] for that matter. We just remark that the memory or internal state of the Preisach model given at any time t by

$$\{(\rho,(W^{\rho}x)(t)) \; : \; \rho \in P\}$$

can be represented by the curve γ separating the regions where $(W^{\rho}x)(t) = +1$ resp. -1 in the Preisach plane as in figure 1.

Figure 1

The time evolution of this separating curve is studied in [2]; we use it here, too, but mainly in the slightly different approach of [7], since the latter reduces the "hysteresis part" of the Preisach operator to the hysteresis operator defined by figure 2, sometimes called "shaft", which is much easier to deal with. Therefore, we consider the latter operator first, namely in section 2 of this paper, and extend the results to the Preisach operator in section 3.

2. BV properties of the shaft

Consider the diagram in figure 2. It shows two boundary lines parallel to the main diagonal with abscissae $\pm u$, the space in between being filled continuously with horizontal segments. The direction of possible movement is indicated by the arrows.

Figure 2

The corresponding hysteresis operator W_u has been studied in detail (and in more generality) in [5], chapter 1. In particular, it is shown there how to define W_u on the space Reg[0,1] of regulated functions (= completion of the step functions in the sup norm), and that W_u is Lipschitz continuous on this space. Moreover, several theorems on mapping and continuity properties of W_u in different function spaces are stated, among them the Lipschitz continuity of W_u in $W^{1,1}[0,1]$ and BV[0,1], the former also being a special case of a result in [11], elaborated in chapter 6 of [5]. The aim of this section is to provide a simple proof (which seems to be new) of the $W^{1,1}$ result and a full proof of the BV result, along with some lemmata which might be of interest.

Returning to figure 2, we may look upon it as defining a family $g_u(\cdot,y)$ of piecewise linear curves by

$$g_u(x,y) = \min\{x+u, \max\{x-u,y\}\}.$$

Here, $g_u(x,y)$ is the output value corresponding to an input value x, if we start at the level y, which is the old output value.

It is then clear that the following definition gives the correct output, if the input is a step function.

2.1 Definition

Let $x: [0,1] \to \mathbb{R}$ be a step function with values x_i on successive intervals I_i, $1 \le i \le n$, and let $y_o \in \mathbb{R}$. For $u \ge 0$ we define $W_u(x,y_o)$ to be the step function on $[0,1]$ with values y_i on I_i, where

$$y_i = g_u(x_i, y_{i-1}) \ , \quad 1 \le i \le n,$$

$$g_u(x,y) = \min\{x+u, \max\{x-u,y\}\}. \qquad \square$$

Obviously, this definition does not depend upon the choice of the partition.

Let now Reg[0,1], the space of regulated functions, denote the closure of the space of step functions in the space of bounded measurable functions on [0,1], endowed with the sup norm. One immediately obtains the Lipschitz continuity of W_u in Reg[0,1):

2.2 Proposition

Definition 2.1 yields on operator

$$W_u: \mathrm{Reg}[0,1] \times \mathbb{R} \to \mathrm{Reg}[0,1]$$

with

$$\|W_u(x_1,y_{01}) - W_u(x_2,y_{02})\|_\infty \le \max\{\|x_1-x_2\|_\infty, |y_{01}-y_{02}|\}$$

for any arguments in the space above.

Proof: The function g_u has the property that

$$|g_u(z_1,y_1) - g_u(z_2,y_2)| \le \max\{|z_1-z_2|, |y_1-y_2|\}.$$

The result now follows for step functios x_1 and x_2 by induction and for functions in Reg[0,1] by continuous extension.

$\square$

For a continuous and piecewise monotone input $x: [0,1] \to \mathbb{R}$, the output $W_u x$ is explicitly given by

$$W_u(x,y_o)(0) = g_u(x(0),y_o)$$

$$W_u(x,y_o)(t) = g_u(x(t),\ W_u(x,y_o)(t_i)),\quad t \in (t_i,t_{i+1}],$$

if $\Delta = \{t_i\}$ is a monotonicity partition for x. Therefore, W_u maps C[0,1] into itself.

We now want to obtain an estimate of $W_u x_1 - W_u x_2$ in the norm defined by the total variation. It turns out to be convenient to consider, along with the function g_u, the function

$$h_u(x,y) = x - g_u(x,y) \ ,$$

which, if used instead of g_u in definition 2.1, yields the elastic-plastic element studied in [6] and also in [5,8].

The basic lemma is the following.

2.3 Lemma

We have

$$|[g_u(x_1,y_1)-g_u(x_2,y_2)]-[y_1-y_2]| + |h_u(x_1,y_1)-h_u(x_2,y_2)|$$
$$= |(x_1-x_2)-(y_1-y_2)|$$

for any $x_1,x_2,y_1,y_2 \in \mathbb{R}$, where

$$h_u(x,y) = x - g_u(x,y).$$

Proof: One easily computes that

$$g_u(x,y)-y = \min\{x-y+u,\max\{x-y-u,0\}\},$$

$$h_u(x,y) \quad = \max\{-u,\min\{u,x-y\}\}.$$

Both left hand sides are, therefore, nondecreasing functions of x-y, so

$$|(g_u(x_1,y_1)-y_1)-(g_u(x_2,y_2)-y_2)| + |h_u(x_1,y_1)-h_u(x_2,y_2)|$$
$$= |(g_u(x_1,y_1)-y_1)-(g_u(x_2,y_2)-y_2)+h_u(x_1,y_1)-h_u(x_2,y_2)|$$
$$= |(x_1-x_2)-(y_1-y_2)|. \qquad \square$$

2.4 Definition

Let $x\colon [0,1] \to \mathbb{R}$ be piecewise monotone. A partition $\Delta = \{t_i\}$ of $[0,1]$ is said to be compatible with x, if $x|I$ is monotone for every interval $I = (t_i,t_{i+1})$ of Δ. $\square$

We introduce some notation, mostly standard, for the BV setting. For $x\colon [0,1] \to \mathbb{R}$ and any partition $\Delta = \{t_o,\dots,t_n\}$ of $[0,1]$ we denote

$$\mathrm{Var}_\Delta(x) = \sum_{i=1}^{n-1} |x(t_{i+1})-x(t_i)| ,$$

$$\mathrm{Var}(x) = \sup\{\mathrm{Var}_\Delta(x) : \Delta \text{ partition of } [0,1]\}.$$

We define

$$BV[0,1] = \{x|x\colon [0,1] \to \mathbb{R},\ \mathrm{Var}(x) < +\infty\} .$$

It is well known that $BV[0,1]$ is a Banach space with the norm

$$\|x\|_{BV} = |x(0)| + \mathrm{Var}(x) ,$$

and we also have

$$BV[0,1] \subset \mathrm{Reg}[0,1] ,$$

since every monotone function can be uniformly approximated by step functions. Furthermore,

$$W^{1,1}[0,1] = \{x|x \in L^1(0,1),\ \dot{x} \in L^1(0,1)\}$$

is a closed subspace of $(BV[0,1], \|\cdot\|_{BV})$. The piecewise linear functions form a dense subset of $W^{1,1}[0,1]$, and for any piecewise linear function $x\colon [0,1] \to \mathbb{R}$ we have

$$\int_0^1 |\dot{x}(t)|dt = \mathrm{Var}(x) = \mathrm{Var}_\Delta(x) ,$$

if the partition $\Delta = \{t_i\}$ is compatible with x, so the BV norm coincides with the standard norm on $W^{1,1}[0,1]$.
We first estimate the variation of $W_u x_1 - W_u x_2$ for piecewise linear inputs.

<u>2.5 Lemma</u>

Let $x_1, x_2 \in C[0,1]$ be piecewise linear, let $y_{01}, y_{02} \in \mathbb{R}$ and let the partition Δ be compatible with x_1 and x_2. Then

$$\mathrm{Var}_\Delta(W_u(x_1,y_{01}) - W_u(x_2,y_{02})) \leq$$
$$\leq \mathrm{Var}_\Delta(x_1-x_2) + |h_u(x_1,y_{01}) - h_u(x_2,y_{02})| \ .$$

<u>Proof:</u> We apply lemma 2.3 with the arguments

$$(x_1,x_2,y_1,y_2) \hat{=}$$
$$\hat{=} (x_1(t_{i+1}), x_2(t_{i+1}), W_u(x_1,y_{01})(t_i), W_u(x_2,y_{02})(t_i)).$$

Since Δ is compatible with x_1 and x_2, for j=1,2 and any i we have

$$g_u(x_j(t_{i+1}), W_u(x_j,y_{0j})(t_i)) = W_u(x_j,y_{0j})(t_{i+1}),$$

and consequently

$$h_u(x_j(t_{i+1}), W_u(x_j,y_{0j})(t_i)) = x_j(t_{i+1}) - W_u(x_j,y_{0j})(t_{i+1}).$$

We obtain from lemma 2.3

$$|[W_u(x_1,y_{01}) - W_u(x_2,y_{02})](t_{i+1}) - [W_u(x_1,y_{01}) - W_u(x_2,y_{02})](t_i)|$$

$$= |(x_1-x_2)(t_{i+1}) - [W_u(x_1,y_{01}) - W_u(x_2,y_{02})](t_i)| -$$
$$- |h_u(x_1(t_{i+1}), W_u(x_1,y_{01})(t_i)) - h_u(x_2(t_{i+1}), W_u(x_2,y_{02})(t_i))|$$

$$\leq |(x_1-x_2)(t_{i+1}) - (x_1-x_2)(t_i)| +$$
$$+ |(x_1-x_2)(t_i) - [W_u(x_1,y_{01}) - W_u(x_2,y_{02})](t_i)| -$$
$$- |(x_1-x_2)(t_{i+1}) - [W_u(x_1,y_{01}) - W_u(x_2,y_{02})](t_{i+1})| \ .$$

Summation over i now yields the assertion, since

$$|(x_1-x_2)(0)-[W_u(x_1,y_{01})-W_u(x_2,y_{02})](0)|$$
$$= |(x_1(0)-g_u(x_1(0),y_{01}))-(x_2(0)-g_u(x_2(0),y_{02}))|$$
$$= |h_u(x_1(0),y_{01})-h_u(x_2(0),y_{02})|. \qquad \square$$

From this, we will now deduce the Lipschitz continuity of W_u on $W^{1,1}[0,1]$.

2.6 Proposition

The operator W_u: $W^{1,1}[0,1] \times \mathbb{R} \rightarrow W^{1,1}[0,1]$ is Lipschitz continuous, and for any $x_1,x_2 \in W^{1,1}[0,1]$ and $y_{01},y_{02} \in \mathbb{R}$ we have

$$\|W_u(x_1,y_{01})-W_u(x_2,y_{02})\|_{BV} \leq \|x_1-x_2\|_{BV} + 2\cdot|y_{01}-y_{02}|.$$

Proof: It suffices to prove the inequality for piecewise linear functions x_1,x_2. We have

$$|W_u(x_1,y_{01})(0)-W_u(x_2,y_{02})(0)| = |g_u(x_1(0),y_{01})-g_u(x_2(0),y_{02})|,$$

and for any partition Δ compatible with x_1 and x_2 we conclude from lemmata 2.5 and 2.3 that

$$\mathrm{Var}_\Delta(W_u(x_1,y_{01})-W_u(x_2,y_{02}))+|W_u(x_1,y_{01})(0)-W_u(x_2,y_{02})(0)| \leq$$
$$\leq \mathrm{Var}_\Delta(x_1-x_2)+|(x_1(0)-x_2(0))-(y_{01}-y_{02})|+|y_{01}-y_{02}|$$
$$\leq \|x_1-x_2\|_{BV}+2|y_{01}-y_{02}| .$$

This proves the result, since an arbitrary partition $\tilde{\Delta}$ can always be refined to a partition compatible with x_1 and x_2. $\square$

The Lipschitz constants in proposition 2.7 are best possible, as simple examples show; of course, they depend upon the way one treats the constant functions in the BV norm.

If we want to know the output $W_u(x,y_o)$ only at finitely many points, we may replace x by a suitable chosen interpolate $\tilde{x}$:

2.7 Proposition

Let $x \in C[0,1]$ and $y_o \in \mathbb{R}$, let $\Delta = \{t_i\}$ be a partition of $[0,1]$. Then there exists a refinement Δ_o of Δ such that, for all refinements $\tilde{\Delta}$ of Δ_o, the piecewise linear interpolate $\tilde{x}$ of x on $\tilde{\Delta}$ satisfies

$$W_u(\tilde{x},y_o)(t_i) = W_u(x,y_o)(t_i)$$

for all grid points t_i of Δ.

Proof: Assume $u>0$; otherwise W_u is the identity operator. We also may assume that

$$osc(x;I) < 2u$$

for all $I = [t_i,t_{i+1}]$, $t_i \in \Delta$, where $osc(x;I)$ denotes the oscillation of x on the set I. To obtain Δ_o from Δ, for each $I = [t_i,t_{i+1}]$ adjoin $r_i,s_i \in I$ to Δ, where

$$x(r_i) = \min_I x \ , \qquad x(s_i) = \max_I x .$$

We now prove inductively that, for any refinement $\tilde{\Delta}$ of Δ_o with piecewise linear interpolate $\tilde{x}$ of x on $\tilde{\Delta}$, the output values

$$y_i(\tilde{\Delta}) = W_u(\tilde{x},y_o)(t_i) \quad , \qquad t_i \in \Delta \ ,$$

do not depend upon the special choice of $\tilde{\Delta}$. Firstly, we have

$$y_o(\tilde{\Delta}) = W_u(\tilde{x},y_o)(0) = g_u(\tilde{x}(0),y_o) = g_u(x(0),y_o).$$

On the interval $I = [t_i,t_{i+1}]$, two cases may occur:

Case 1. We have

$$y_i(\tilde{\Delta})-u < x(r_i) \leq x(s_i) < y_i(\tilde{\Delta})+u \ .$$

The the curve $(\tilde{x}(t),\tilde{y}(t))$ with $\tilde{y}(t) = W_u(\tilde{x},y_o)(t)$ never hits a boundary line for $t \in I$ (compare figure 2), and

$$y_{i+1}(\tilde{\Delta}) = y_i(\tilde{\Delta}) \ .$$

Case 2. The curve $(\tilde{x}(t),\tilde{y}(t))$ hits a boundary line for some $t \in I$. Since $osc(x;I) < 2u$, the curve can hit only one boundary line, and it is easy to check that

$$y_{i+1}(\tilde{\Delta}) = x(s_i) - u \ , \quad \text{or}$$

$$y_{i+1}(\tilde{\Delta}) = x(r_i) + u \ ,$$

depending on whether the curve hits at the right or at the left. This completes the induction step and proves the lemma, since we may choose a sequence $(\tilde{x}_n,\tilde{\Delta}_n)$ having the properties above, with $\tilde{x}_n \to x$ uniformly, and apply proposition 2.2.

□

2.8 Lemma

For any $x_1,x_2 \in C[0,1] \cap BV[0,1]$ and any $y_{01},y_{02} \in \mathbb{R}$ we have

$$Var(W_u(x_1,y_{01})-W_u(x_2,y_{02})) \leq$$

$$\leq Var(x_2-x_2) + |h_u(x_1(0),y_{01}) - h_u(x_2(0),y_{02})| \ .$$

Proof: Let $\Delta = \{t_i\}$ be a partition of $[0,1]$. According to proposition 2.7 we may choose a refinement $\tilde{\Delta}$ of Δ such that

$$W_u(\tilde{x}_j,y_{0j})(t_i) = W_u(x_j,y_{0j})(t_i)$$

for $j=1,2$ and any $t_i \in \Delta$, where $\tilde{x}_j$ is the piecewise linear interpolate for x_j on $\tilde{\Delta}$. Applying lemma 2.5 to $(\tilde{x}_1,\tilde{x}_2)$ on $\tilde{\Delta}$, we obtain

$$Var_{\Delta}(W_u(x_1,y_{01})-W_u(x_2,y_{02})) =$$

$$= Var_{\Delta}(W_u(\tilde{x}_1,y_{01})-W_u(\tilde{x}_2,y_{02}))$$

$$\leq Var_{\tilde{\Delta}}(W_u(\tilde{x}_1,y_{01})-W_u(\tilde{x}_2,y_{02}))$$

$$\leq Var_{\tilde{\Delta}}(\tilde{x}_1-\tilde{x}_2) + |h_u(\tilde{x}_1(0),y_{01})-h_u(\tilde{x}_2(0),y_{02})|$$

$$\leq Var(x_1-x_2) + |h_u(x_1(0),y_{01})-h_u(x_2(0),y_{02})| \ .$$

□

In order to generalize a result concerning W_u on C[0,1] to Reg[0,1], the following device has been developed in [5], p. 44. For $x \in$ Reg[0,1], one interprets the output $W_u x$ as restriction of the output $W_u x^A$ for some continuous input x^A, which interpolates the discontinuities of x in a suitable way. This is possible since x has an at most countable number of discontinuities, and the right resp. left sided limits x(t+) and x(t-) exist for all $t \in [0,1]$, see [3], p. 16ff.

<u>2.10 Definition</u>

Let $A = \{a_j\}_{j \in J}$ be an at most countable subset of [0,1]. Set

$$b = 1 + \sum_{j \in J} 2^{-j} ,$$

and choose the transformation $T: [0,b] \to [0,1]$, defined below, with the following properties:

(i) T is absolutely continuous and nondecreasing,

(ii) $T(0) = 0$, $T(b) = 1$,

(iii) $I_j := T^{-1}(\{a_j\})$ is an interval of length 2^{-j},

(iv) $T^{-1}(\{t\})$ is a single point for $t \notin A$.

To achieve this, define the left endpoint of I_j to be the real number

$$a_j + \sum_{k \in J, a_k < a_j} 2^{-k}$$

and set

$$T(s) = \int_0^s v(\tau) d\tau ,$$

where v=0 on the union of the I_j and v=1 elsewhere. □

2.11 Definition

Let $A = \{a_j\}_{j \in J}$ and T be as in definition 2.10. For any $x \in Reg[0,1]$ we define $x^A : [0,b] \to \mathbb{R}$ by

(i) $x^A | I_j$ is the linear interpolant for the values $x(a_j-)$ and $x(a_j+)$ at the endpoints of I_j, $x(a)$ at the midpoint of I_j,

(ii) $x^A(s) = x(Ts)$ otherwise.

For any $z \in Reg[0,b]$ we define $z^R: [0,1] \to \mathbb{R}$ by

(i') $z^R(a_j) = z(r)$, if $j \in J$ and r is the midpoint of I_j,

(ii') $z^R(t) = z(T^{-1}t)$, otherwise. □

As stated in [5] in a slightly different way, the following properties hold.

2.12 Lemma

In the situation of definitions 2.10 and 2.11, the following assertions are true for any $x \in Reg[0,1]$ and $z \in Reg[0,b]$:

(i) We have $x^A \in Reg[0,b]$ and $z^R \in Reg[0,1]$.

(ii) If A includes all discontinuity points of x, then $x^A \in C[0,b]$.

(iii) We have

$$Var(x^A) = Var(x) \ , \quad \text{if} \quad x \in BV[0,1]$$

$$Var(z^R) \le Var(z) \ , \quad \text{if} \quad z \in BV[0,b].$$

(iv) We have for any $y_o \in \mathbb{R}$

$$W_u(x,y_o) = [W_u(x^A,y_o)]^R \ .$$

Proof: The mappings $x \longmapsto x^A$ and $z \longmapsto z^R$ are linear and continuous on Reg[0,1] resp. Reg[0,b] and map step functions to regulated functions. This proves (i) and also (iv), since (iv) holds for step functions, and W_u is continuous by proposition 2.2. The proofs of (ii) and (iii) are elementary.

□

As remarked in [5], p. 46, the output $W_u x$ has bounded variation if we exclude the case u=0 where W_u is the identity operator. This is a byproduct of proposition 2.7, compare also proposition 2.2 in [2].

2.13 Lemma

The operator W_u maps $Reg[0,1] \times \mathbb{R}$ into BV[0,1], if $u>0$.

Proof: Let $y_o \in \mathbb{R}$. Consider the case where $x \in C[0,1]$ is piecewise linear. The number of changes of the monotonicity direction of $W_u x$ is obviously bounded by $1+\omega(x;2u)^{-1}$, where

$$\omega(x;h) = \sup\{r: |t-s| < r \Rightarrow |x(t)-x(s)| < h\}.$$

Therefore we have

$$\mathrm{Var}(W_u(x,y_o)) \leq 2\|x\|_\infty \cdot (2+\omega(x;2u)^{-1}).$$

Applying proposition 2.2, this inequality extends to any $x \in C[0,1]$, since

$$\omega(x;h) \leq \omega(\tilde{x};h+2\|x-\tilde{x}\|_\infty).$$

For arbitrary $x \in Reg[0,1]$, lemma 2.12 yields the assertion.

□

We conclude this section with the statement of Lipschitz continuity of W_u in BV[0,1].

2.14 Theorem

The operator W_u: $BV[0,1] \times \mathbb{R} \rightarrow BV[0,1]$ is Lipschitz continuous, and for any $x_1, x_2 \in BV[0,1]$ and any $y_{01}, y_{02} \in \mathbb{R}$ we have

$$\|W_u(x_1,y_{01})-W_u(x_2,y_{02})\|_{BV} \leq \|x_1-x_2\|_{BV} + 2|y_{01}-y_{02}| .$$

Proof: Let A consist of all points where x_1 or x_2 are discontinuous. Applying lemma 2.8 to x_1^A and x_2^A, repeated use of lemma 2.12 yields

$$\mathrm{Var}(W_u(x_1,y_{01})-W_u(x_2,y_{02})) \leq$$

$$\leq \mathrm{Var}(x_1-x_2) + |h_u(x_1(0),y_{01})-h_u(x_2(0),y_{02})|.$$

From this, the assertion follows, using lemma 2.3 in the same way as in the proof of proposition 2.6. □

3. BV properties of the Preisach operator

As already mentioned in the introduction, there are various ways to define the Preisach operator. Nevertheless, one always has to consider the time evolution of the boundary curve depicted in figure 1. If one takes the natural coordinates

$$u = \frac{\rho_2 - \rho_1}{2} \quad , \qquad v = \frac{\rho_2 + \rho_1}{2} \quad ,$$

then a particular boundary curve may be written as $v = \Phi(u)$, and it turns out that the boundary curve $\Phi = \psi(t)$ at time t is related to the operator W_u from section 2 by

$$\psi(t)u = W_u(x, \psi_o(u))(t) \quad ,$$

if ψ_o is the initial boundary at time t=0. This motivates the following definition of the Preisach operator.

3.1 Definition

Let μ be a finite Borel measure on $[0,\infty) \times \mathbb{R}$, let ψ_o: $[0,\infty) \to \mathbb{R}$ be Lipschitz continuous with Lipschitz constant not greater than 1 and with compact support, let $c_+, c_- \in \mathbb{R}$. Then for any $x \in \mathrm{Reg}[0,1]$ we define the Preisach operator W by

$$(Wx)(t) = W(x,\mu,\psi_o)(t) = E(\psi(t),\mu) \quad ,$$

where

$$(\psi(t))u = W_u(x,\psi_o(u))(t) \quad ,$$

$$E(\Phi,\mu) = c_+\mu(\{v<\Phi(u)\}) + c_-\mu(\{v\geq\Phi(u)\}) \quad ,$$

if Φ: $[0,\infty) \to \mathbb{R}$ is Borel measurable. □

Several comments on this definition are in order. First, for continuous inputs it yields the operator considered in [2], section 3. This can be checked from the definitions and basic continuity properties. In particular, for any fixed t, the boundary curve $\psi(t)$ is then Lipschitz continuous, a fact which carries over to inputs $x \in \text{Reg}[0,1]$ because of lemma 2.12. Therefore, $E(\psi(t),\mu)$ is well defined. Second, the operator W coincides with the operator defined via elementary thermostats if and only if the measure μ, seen in (u,v)-coordinates, satisfies $|\mu|(L) = 0$ for every straight line L of slope 1 or -1; this condition also characterizes the continuity of W on C[0,1].

If one writes the Preisach operator W as in definition 3.1, it is natural to split the measure μ in the following way.

3.2 Proposition

Let μ be a finite nonnegative Borel measure on $\mathbb{R}_+ \times \mathbb{R}$. Then we have for any measurable set $Q \subset \mathbb{R}_+ \times \mathbb{R}$

$$\mu(Q) = \int \nu(u,Q_u)\, d\pi(u) ,$$

where π is the projection of μ onto $\mathbb{R}_+$,

$$\pi(A) = \mu(A \times \mathbb{R}) ,$$

if $A \subset \mathbb{R}_+$ is a Borel set,

$$Q_u = \{v \in \mathbb{R}: (u,v) \in Q\} ,$$

and $\nu = \nu(u,B)$ is a certain real-valued function defined for any $u \in \mathbb{R}_+$ and any Borel set $B \subset \mathbb{R}$ with the properties

$$u \longmapsto \nu(u,B) \quad \text{is } \pi\text{-measurable}$$

$$B \longmapsto \nu(u,B) \quad \text{is a Borel measure.}$$

Proof: This is a basic result in probability theory, related to the existence of a conditional probability distribution, so we sketch the arguments briefly. For a Borel measurable set $B \subset \mathbb{R}$, one obtains $u \longmapsto \nu(u,B)$ as the Radon-Nikodym derivative of the measure $\mu_B(A) := \mu(A \times B)$ with respect to π. Thus, $\nu(u,B)$ is determined π-a.e. for fixed B. Defining a countable generator G of the Borel σ-algebra on $\mathbb{R}$ by finite unions, intersections and complements of open intervals with rational endpoints, one obtains a finitely additive set function $B \longmapsto \nu(u,B)$ on G for any u except from a π-null set. This can be shown to be a measure $B \longmapsto \nu(u,B)$ on the Borel algebra, yielding in turn a measure $\tilde{\mu}$ on $\mathbb{R}_+ \times \mathbb{R}$, if we set

$$\tilde{\mu}(Q) = \int \nu(u,Q_u)\, d\pi(u) ,$$

which must be equal to μ since it coincides with μ on sets of the form $Q = A \times B$. □

Proposition 3.2 enables us to write the Preisach operator in the form

$$(Wx)(t) = \int_0^\infty c_+\nu(u,\{v<\psi(t)u\}) + c_-\nu(u,\{v\geq\psi(t)u\})\, d\pi(u) ,$$

and if we define $f\colon \mathbb{R}_+ \times \mathbb{R}$ by

$$f(u,z) = \nu(u,\{v<z\}) ,$$

we obtain the following result, a generalization of the representation in [7].

3.3 Proposition

In the setting of definition 3.1, assume that $\mu \geq 0$. Then we have

$$W(x,\mu,\psi_o)(t) = \int_0^\infty (c_+-c_-)f(u,W_u(x,\psi_o(u))(t))\, d\pi(u) +$$

$$+ c_-\mu(\mathbb{R}_+ \times \mathbb{R}) ,$$

where

$$f(u,z) = \nu(u,\{v \in \mathbb{R}: v < z\}) ,$$

and the measures π and $\nu(u,\cdot)$ are obtained from μ as in proposition 3.2.

<u>Proof:</u> This is immediate from definition 3.1 and proposition 3.2. □

For the case of an absolutely continuous measure μ, this representation is due to [7]. It shows the Preisach operator W as a combination of a superposition process and the family of hysteresis operators described in figure 2. If μ is concentrated on $\{u=0\}$, then W is a superposition operator; if μ is the one-dimensional Lebesgue measure concentrated on some line $\{u=u_o\}$ for $u_o>0$, then W is the operator W_u of figure 2. (We tacitly extend definition 3.1 to the σ-finite case.) Some BV properties of W_u are given in the previous section; for BV properties of the superposition operator defined by f, we refer to the survey paper [1], section 8 and 9. The results presented in [1] motivate the following definition.

<u>3.4 Definition</u>

Let μ be a finite nonnegative Borel measure on $\mathbb{R}_+\times\mathbb{R}$, let ν,π,f be as in propositions 3.2 and 3.3. We say that μ has property (L_1), if

$$|f(u,z_1)-f(u,z_2)| \le L_f(u)|z_1-z_2|$$

for any $z_1,z_2 \in \mathbb{R}$ and any $u\ge 0$, where L_f: $\mathbb{R}_+ \to \mathbb{R}_+$ is some function with

$$\int_0^\infty L_f(u)\, d\pi(u) < \infty .$$

We say that μ has property (L_2), if moreover f is differentiable almost everywhere w.r.t. z, and also

$$|D_z f(u,z_1) - D_z f(u,z_2)| \leq L_f(u)|z_1 - z_2|$$

for almost all $z_1, z_2 \in \mathbb{R}$ and any $u \geq 0$. □

We consider the most important special case.

3.5 Lemma

Let μ be a measure with density $e \in L^1(\mathbb{R}_+ \times \mathbb{R})$, $e \geq 0$.

(i) If

$$\int_0^\infty \operatorname*{ess\,sup}_{v \in \mathbb{R}} e(u,v)\, du < \infty ,$$

then μ has property (L_1).

(ii) If moreover the partial derivative $D_v e(u,v)$ exists a.e. and

$$\int_0^\infty \operatorname*{ess\,sup}_{v \in \mathbb{R}} |D_v e(u,v)|\, du < \infty ,$$

then μ has property (L_2).

Proof: It is easy to see that the measure π from proposition 3.2 has density

$$\rho(u) = \int_{-\infty}^{\infty} e(u,v)\, dv ,$$

and that L_f defined by

$$\rho(u) L_f(u) = \operatorname*{ess\,sup}_{v \in \mathbb{R}} e(u,v) ,$$

with $L_f(u) = 0$ if $\rho(u) = 0$, yields property (L_1) for μ, since

$$\rho(u)(f(u,z_1) - f(u,z_2)) = \int_{z_2}^{z_1} e(u,v)\, dv .$$

Similarly, (L_2) is obtained. □

We now present the main result of this section.

3.6 Theorem

Let W be the Preisach operator of definition 3.1.

(i) If the measure μ has property (L_1), then W maps bounded subsets of BV[0,1] into bounded subsets of BV[0,1] for any fixed initial condition ψ_o. Moreover, W maps $W^{1,1}[0,1]$ into itself.

(ii) If the measure μ has property (L_2), then for any bounded subset B of $W^{1,1}[0,1]$ there is an L such that

$$\|W(x_1,\psi_{01})-W(x_2,\psi_{02})\|_{BV} \leq L(\|x_1-x_2\|_{BV} + \|\psi_{01}-\psi_{02}\|_\infty)$$

for any $x_1,x_2 \in B$ and any initial condition ψ_{01},ψ_{02}.

Proof: Using propositions 3.2 and 3.3, we may write W as

$$(Wx)(t) = (c_+-c_-)\int_0^\infty (S_uW_u(x,\psi_o(u)))(t)\, d\pi(u) + c_-\|\mu\| ,$$

where S_u denotes the superposition operator defined by the function f, i.e.

$$(S_uy)(t) = f(u,y(t))$$

for any function $y\colon [0,1] \to \mathbb{R}$. Since the measure μ has property (L_1), for any partition Δ of [0,1] and any $x \in BV[0,1]$ we get

$$\begin{aligned}\mathrm{Var}_\Delta(Wx) &\leq |c_+-c_-| \int_0^\infty \mathrm{Var}_\Delta(S_uW_u(x,\psi_o(u)))\, d\pi(u)\\ &\leq |c_+-c_-| \int_0^\infty L_f(u)\mathrm{Var}_\Delta(W_u(x,\psi_o(u)))\, d\pi(u)\\ &\leq |c_+-c_-| \int_0^\infty L_f(u)\, d\pi(u) \cdot (\|x\|_{BV}+2\cdot\|\psi_o\|_\infty) ,\end{aligned}$$

the latter inequality with the aid of theorem 2.14. This, together with an easy estimate of (Wx)(0), proves the first part of assertion (i). To prove that W maps $W^{1,1}$ into itself, by theorem 4.13 in [2] it is sufficient to show that

$$k(\xi) \leq C\xi$$

with C independent from ξ, where

$$k(\xi) = |c_+ - c_-| \sup\{\mu(R_i(\lambda,\lambda+\xi)) : \lambda \in \mathbb{R},\ i=1 \text{ or } 2\} ,$$

$$R_1(\lambda_1,\lambda_2) = \{(u,v) : u \geq 0,\ \lambda_1 + u \leq v \leq \lambda_2 + u\} ,$$

$$R_2(\lambda_1,\lambda_2) = \{(u,v) : u \geq 0,\ \lambda_1 - u \leq v \leq \lambda_2 - u\} .$$

But because of property (L_1),

$$\mu(R_1(\lambda,\lambda+\xi)) = \int_0^\infty f(u,\lambda+\xi+u) - f(u,\lambda+u)\ d\pi(u)$$

$$\leq \int_0^\infty L_f(u)\ d\pi(u) \cdot \xi ,$$

and with an analogous estimate for $R_2(\lambda,\lambda+\xi)$, the proof of (i) is complete. Now assume that μ has property (L_2). We first consider the superposition operator. Fix any $z_1, z_2 \in W^{1,1}[0,1]$. If $\|\cdot\|_1$ denotes L_1-norm, we have

$$\left\|\frac{d}{dt}(S_u z_1 - S_u z_2)\right\|_1 =$$

$$= \int_0^1 |D_z f(u,z_1(t))\dot{z}_1(t) - D_z f(u,z_2(t))\dot{z}_2(t)|\ dt$$

$$\leq \int_0^1 L_f(u)|z_1(t)-z_2(t)|\ |\dot{z}_1(t)|dt + \int_0^1 |D_z f(u,z_2(t))||\dot{z}_2(t)|dt$$

$$\leq L_f(u)(\|\dot{z}_1\|_1 \|z_1 - z_2\|_\infty + C_1\|\dot{z}_1 - \dot{z}_2\|_1)$$

with some constant C_1, since $D_z f$ has compact support. Therefore,

$$\left\|\frac{d}{dt}(S_u z_1 - S_u z_2)\right\|_1 \leq L_f(u) C_2(\|\dot{z}_1\|_1)\|z_1 - z_2\|_{BV}$$

with some nondecreasing function C_2. Now take any $x_1, x_2 \in B$ and any partition Δ of (0,1). We then have, using the estimate just derived,

$$\mathrm{Var}_\Delta(W(x_1,\psi_{01})-W(x_2,\psi_{02})) \le$$

$$\le |c_+-c_-| \int_0^\infty \mathrm{Var}_\Delta[S_uW_u(x_1,\psi_{01}(u))-S_uW_u(x_2,\psi_{02}(u))]\ d\pi(u)$$

$$\le |c_+-c_-| \int_0^\infty L_f(u)C_2(\mathrm{Var}(W_u(x_1,\psi_{01})))\|W_u(x_1,\psi_{01})-W_u(x_2,\psi_{02})\|_{BV}\cdot$$

$$\cdot\ d\pi(u)$$

$$\le |c_+-c_-|\cdot C_3 \int_0^\infty L_f(u)\ d\pi(u)\cdot(\|x_1-x_2\|_{BV}+\|\psi_{01}-\psi_{02}\|_\infty)\ ,$$

the latter inequality being true for some constant C_3 because of theorem 2.14, since x_1 varies in the bounded set B and we may restrict the initial conditions ψ_{0i} to have values in a bounded set containing the support of μ. This completes the proof of the theorem. □

As it is to be expected, theorem 3.6 links properties of the Preisach operator W to properties of the measure μ. In view of the results included in [1], it seems that the assumptions (L_1) and (L_2) cannot be weakened substantially, but we do not attempt to give a complete characterization. Moreover, theorem 9.3 in [1] implies that the superposition operator S_u generated by $f=f(u,z)$ is Lipschitz continuous in BV[0,1] if and only if f is linear with respect to z; this means that μ has constant density along all lines $\{u=u_o\}$, respectively along all lines parallel to the main diagonal in the (ρ_1,ρ_2)-coordinates. The corresponding Preisach operator W then becomes the so-called Ishlinskii operator. Accordingly, we conjecture that the Preisach operator is Lipschitz continuous in BV[0,1] if and only if it is an Ishlinskii operator, the "if"-part being a direct consequence of theorem 2.14.

We now return to the study of the time evolution of the boundary curve in figure 1. As an immediate corollary of theorem 2.14 we obtain

3.7 Proposition

The mapping $(x,\psi_o) \longmapsto \psi$ defined by

$$\psi(t,u) = W_u(x,\psi_o(u))(t)$$

yields a Lipschitz continuous operator from

$BV[0,1] \times L^\infty(\mathbb{R}_+)$ into $L^\infty(\mathbb{R}_+;BV[0,1])$, mapping

$W^{1,1}[0,1] \times L^\infty(\mathbb{R}_+)$ into $L^\infty(\mathbb{R}_+;W^{1,1}[0,1])$.

□

However, viewing $\psi(t,\cdot)$ as the boundary curve at time t, it is more natural to ask for regularity in spaces of functions ψ: $[0,T] \to \Psi$, where Ψ is a suitable space of boundary curves. Basic properties of ψ and time regularity with respect to the sup norm have been studied extensively in [2], section 3. The remainder of the present section is intended as an addendum to these results. We therefore introduce the relevant notation without discussion; for this we refer to [2], section 3.

3.8 Definition

We set

$$\Psi_1 = \{\phi \mid \phi: [0,\infty) \to \mathbb{R},\ \phi \text{ has compact support and is Lipschitz continuous with Lipschitz constant} \le 1\}.$$

On Ψ_1 we introduce the distances d_1, d_∞ and $d_{1,1}$ which correspond to the norm in L^1, L^∞ and the seminorm

$$|\phi|_{1,1} = \int_0^\infty |\phi'(u)|\ du .$$

We define G: $\mathbb{R} \times \Psi_1 \to \Psi_1$ by

$$G(x,\Phi)(u) = \min\{x+u, \max\{x-u, \Phi(u)\}\},$$

and define $F(x,\psi_o)\colon [0,1] \to \Psi_1$ for any $\psi_o \in \Psi_1$ and any piecewise monotone $x \in C[0,1]$ with monotonicity partition $\Delta = \{t_i\}$ by

$$F(x,\psi_o)(0) = G(x(0),\psi_o)$$

$$F(x,\psi_o)(t) = G(x(t),F(x,\psi_o)(t_i)), \text{ if } t \in [t_i,t_{i+1}].$$

□

Since G is linked to g_u from definition 2.1 by

$$G(x,\Phi)(u) = g_u(x,\Phi(u)),$$

one easily checks that, if one extends F to $C[0,1]\times\Psi_1$,

$$[F(x,\psi_o)(t)](u) = [W_u(x,\psi_o(u))](t).$$

We are now interested in continuity properties of F in the space $W^{1,1}[0,1]$. Again, from theorem 2.14 we obtain the following result.

3.9 Proposition

The definitions in 3.8 yield a mapping

$$F\colon W^{1,1}[0,1] \times \Psi_1 \to W^{1,1}(0,1;(\Psi_1,d_1)).$$

For any bounded subset $B \subset W^{1,1}[0,1]$ and any $M>0$,

$$F\colon B \times (\Psi_{1M},d_\infty) \to W^{1,1}(0,1;(\Psi_1,d_1))$$

is Lipschitz continuous. (Here Ψ_{1M} denotes the subset of Ψ_1 consisting of functions with support in $[0,M]$.)

Proof: For any $\psi_o \in \Psi_1$ and any piecewise linear $x \in C[0,1]$ one checks that

$$\psi(t,u) = [W_u(x,\psi_o(u))](t) = [F(x,\psi_o)(t)](u)$$

gives a Lipschitz continuous function $\psi\colon [0,1] \times \mathbb{R}_+$ with compact support, so $\psi_t \in L^1([0,1] \times \mathbb{R}_+)$ and $F(x,\psi_o) \in W^{1,1}(0,1;(\Psi_1,d_1))$ as claimed. Now let $x_1, x_2 \in C[0,1]$ be piecewise linear and $\psi_{01}, \psi_{02} \in \Psi_{1M}$. For any partition $\Delta = \{t_i\}$ of $[0,1]$ we have

$$\mathrm{Var}_\Delta(F(x_1,\psi_{01})-F(x_2,\psi_{02})) =$$

$$= \sum_{t_i\in\Delta} \|F(x_1,\psi_{01})(t_{i+1})-F(x_2,\psi_{02})(t_i)\|_{L^1(\mathbb{R}_+)}$$

$$= \sum_{t_i\in\Delta} \int_0^\infty |W_u(x_1,\psi_{01}(u))(t_{i+1})-W_u(x_2,\psi_{02}(u))(t_i)|\, du$$

$$= \int_0^\infty \mathrm{Var}_\Delta[W_u(x_1,\psi_{01}(u))-W_u(x_2,\psi_{02}(u))]\, du$$

$$\leq \max\{M, \|x_1\|_\infty, \|x_2\|_\infty\}\cdot(\|x_1-x_2\|_{BV}+2\|\psi_{01}-\psi_{02}\|_\infty),$$

the latter inequality being a consequence of theorem 2.14 and the fact that W =0 for u large enough. For general x_1 and x_2, the assertion now follows with a density argument.

□

The result of proposition 3.9 no longer remains valid if we replace (Ψ_1,d_1) by (Ψ_1,d_∞), even if we do not vary the initial state. This is shown by the following counterexample.

__3.10 Example__

Set $\psi_{01} = \psi_{02} = 0$. Define $x, x_\varepsilon \in C[0,1]$ by

$$x(t) = x_\varepsilon(t) = t\ , \qquad 0 \leq 1 \leq \tfrac{1}{2}$$

$$x(t) = 1-t\ , \qquad \tfrac{1}{2} \leq t \leq 1$$

$$x_\varepsilon(t) = \tfrac{1}{2}\ , \qquad \tfrac{1}{2} \leq 1 \leq \tfrac{1}{2}+\varepsilon$$

$$x_\varepsilon(t) = x(t-\varepsilon)\ , \qquad \tfrac{1}{2}+\varepsilon \leq t \leq 1\ .$$

Then we have

$$\|x-x_\varepsilon\|_\infty = \|x-x_\varepsilon\|_{BV} = \varepsilon$$

and

$$(W_u x)(\tfrac{1}{2}) = (W_u x_\varepsilon)(\tfrac{1}{2}) = \tfrac{1}{2} - u , \qquad \text{if} \quad 0 \le u \le \tfrac{1}{2} .$$

Moreover, for $0 \le u \le \frac{1}{2}$

$$(W_u x)(t) = \min\{1-t+u, \tfrac{1}{2}-u\} \quad , \qquad t \ge \tfrac{1}{2}$$

$$(W_u x_\varepsilon)(t) = (W_u x)(t-\varepsilon) \quad , \qquad t \ge \tfrac{1}{2}+\varepsilon .$$

This can also be written as

$$(W_u x)(t) = \begin{cases} \frac{1}{2}-u , & t-\frac{1}{2} \le 2u \\ 1-t+u , & t-\frac{1}{2} \ge 2u . \end{cases}$$

Now set

$$\delta_\varepsilon(u,t) = (W_u x)(t)-(W_u x_\varepsilon)(t).$$

For $t \ge \frac{1}{2}+\varepsilon$, $2u = t-\frac{1}{2}$ we obtain

$$\delta_\varepsilon(u,t+\varepsilon)-\delta_\varepsilon(u,t) = -\varepsilon ,$$

therefore

$$\sup_{u\ge 0} |\delta_\varepsilon(u,t+\varepsilon)-\delta_\varepsilon(u,t)| \ge \varepsilon .$$

If we partition $[\frac{1}{2},1]$ by $\Delta = \{t_n\}$, $t_n = \frac{1}{2}+n\varepsilon$, the preceding considerations yield

$$\mathrm{Var}_\Delta[\sup_{u\ge 0} (W_u x-W_u x_\varepsilon)] \ge \varepsilon \cdot \frac{1}{2\varepsilon} = \frac{1}{2} ,$$

so $F(x_\varepsilon,0)$ does not converge to $F(x,0)$ in the space $W^{1,1}(0,1;(\Psi_1,d_\infty))$. □

If we allow a loss of time regularity, we obtain Lipschitz continuity of F with respect to the distance $d_{1,1}$ in Ψ_1 generated by the seminorm

$$|\psi|_{1,1} = \int_0^\infty |\psi'(u)| \, du .$$

We need the corresponding property for the mapping G.

3.11 Lemma

For any $x_1, x_2 \in \mathbb{R}$ and any $\psi_1, \psi_2 \in \Psi_1$ we have

$$|G(x_1,\psi_1)-G(x_2,\psi_2)|_{1,1} \leq$$

$$\leq |\psi_1-\psi_2|_{1,1} + |(\psi_1(0)-\psi_2(0))-(x_1-x_2)|$$

Proof: We freely use basic properties of G given in [2], section 3. We first consider the case $x_1 \leq \psi_1(0)$, $x_2 \leq \psi_2(0)$. Setting

$$\phi_i(u) = G(x_i,\psi_i)(u)-\psi_i(u) \quad , \quad u \geq 0,$$

we conclude that

$$\phi_2'(u) \leq 0 \leq \phi_1'(u) \qquad \text{a.e. in } \mathbb{R}_+,$$

so

$$|\phi_1-\phi_2|_{1,1} = \int_0^\infty \phi_1'(u)-\phi_2'(u)\, du = \phi_2(0)-\phi_1(0)$$

$$= (\psi_1(0)-\psi_2(0)) - (x_1-x_2) \ ,$$

and the triangle inequality yields the assertion. Now assume that $x_1 < \psi_1(0)$, $x_2 < \psi_2(0)$. Set

$$z(u) = G(x_1,\psi_1)(u)-G(x_2,\psi_2)(u),$$

and define $u_1, u_2 \in \mathbb{R}$ by

$$u_i = \inf\{u: x_i-u < \psi_i(u) < x_i+u\} \ .$$

If $u_1 \leq u_2$ (the reverse inequality is treated similarly), we have

$$z(u) = x_1-x_2 \qquad , \quad 0 \leq u \leq u_1,$$

$$z'(u) \leq 0 \qquad \text{a.e. in } [u_1,u_2],$$

$$z(u) = \psi_1(u)-\psi_2(u) \quad , \quad u_2 \leq u.$$

Since

$$\int_{u_1}^{u_2} |z'(u)|du = z(u_1)-z(u_2) = (x_1-x_2)-(\psi_1(u_2)-\psi_2(u_2))$$

$$= (x_1-x_2)-(\psi_1(0)-\psi_2(0)) - \int_0^{u_2} \psi_1'(u)-\psi_2'(u)\ du\ ,$$

we obtain

$$|G(x_1,\psi_2)-G(x_2,\psi_2)|_{1,1} = \int_{u_1}^{u_2} |z'(u)| + \int_{u_2}^{\infty} |\psi_1'(u)-\psi_2'(u)|\ du$$

$$\leq |(x_1-x_2)-(\psi_1(0)-\psi_2(0))| + |\psi_1-\psi_2|_{1,1}\ ,$$

which was to be proved. The cases $x_1 \geq \psi_1(0)$, $x_2 \leq \psi_2(0)$ and $x_1 > \psi_1(0)$, $x_2 > \psi_2(0)$ are treated analogously. □

From lemma 3.11, we readily obtain the Lipschitz continuity of F.

3.12 Proposition

The mapping F satisfies

$$F:\ W^{1,1}[0,1] \times \Psi_1 \ \rightarrow\ C(0,1;(\Psi_1,d_{1,1}))\ ,$$

and

$$\sup_{t\in[0,1]} |F(x_1,\psi_{01})(t)-F(x_2,\psi_{02})(t)|_{1,1} \leq$$

$$\leq \|x_1-x_2\|_{BV} + 2|\psi_{01}-\psi_{02}|_{1,1}$$

for any $x_1,x_2 \in W^{1,1}[0,1]$ and any $\psi_{01},\psi_{02} \in \Psi_1$.

Proof: If $x \in W^{1,1}[0,1]$ is piecewise monotone, the continuity of $F(x,\psi_o)$: $[0,1] \rightarrow (\Psi_1,d_{1,1})$ with respect to t follows directly from lemma 3.11. If $x_1,x_2 \in W^{1,1}[0,1]$ are piecewise monotone with corresponding common partition $\Delta = \{t_i\}$ of $[0,1]$, we have, using lemma 3.11 again,

$$|F(x_1,\psi_{01})(t_{i+1}) - F(x_2,\psi_{02})(t_{i+1}|_{1,1} =$$

$$= |G(x_1(t_{i+1}),F(x_1,\psi_{01})(t_i)) - G(x_2(t_{i+1}),F(x_2,\psi_2)(t_i))|_{1,1}$$

$$\leq |F(x_1,\psi_{01})(t_i)-F(x_2,\psi_{02})(t_i)|_{1,1} +$$

$$+ |(x_1-x_2)(t_{i+1})-(x_1-x_2)(t_i)| ,$$

therefore for any $t \in [0,1]$

$$|F(x_1,\psi_{01})(t) - F(x_2,\psi_{02})(t)|_{1,1} \leq$$

$$\leq \mathrm{Var}(x_1-x_2)+|G(x_1(0),\psi_{01})-G(x_2(0),\psi_{02})|_{1,1}$$

$$\leq \|x_1-x_2\|_{BV}+|\psi_{01}-\psi_{02}|_{1,1}+|\psi_{01}(0)-\psi_{02}(0)| .$$

Since the unique Lipschitz continuous extension to $W^{1,1}[0,1]$ coincides with F as defined on C[0,1], the proof is complete. □

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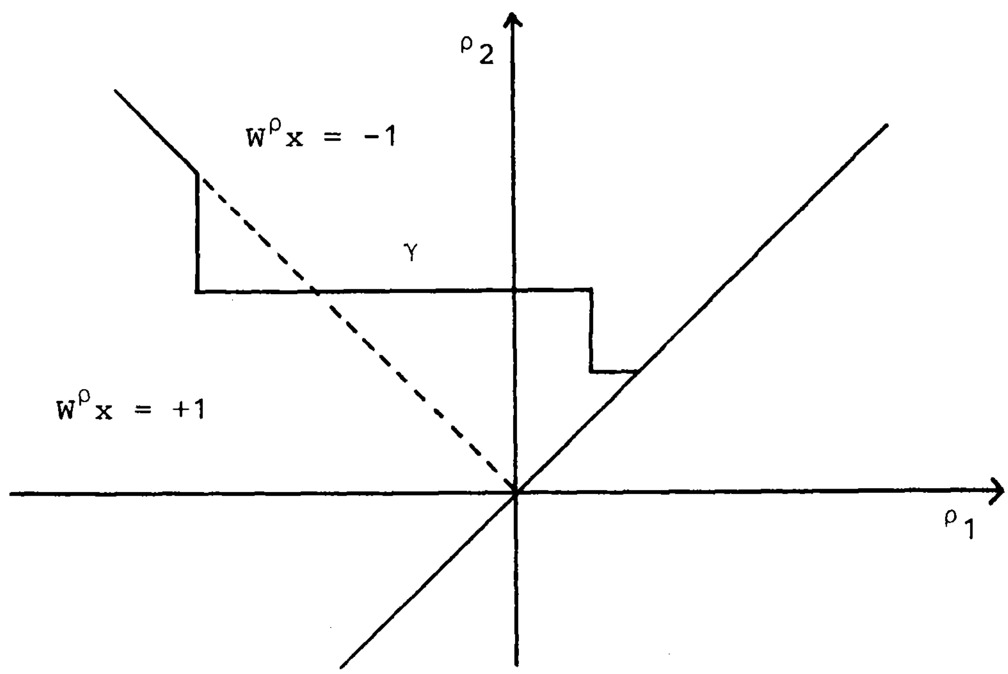

Figure 1.

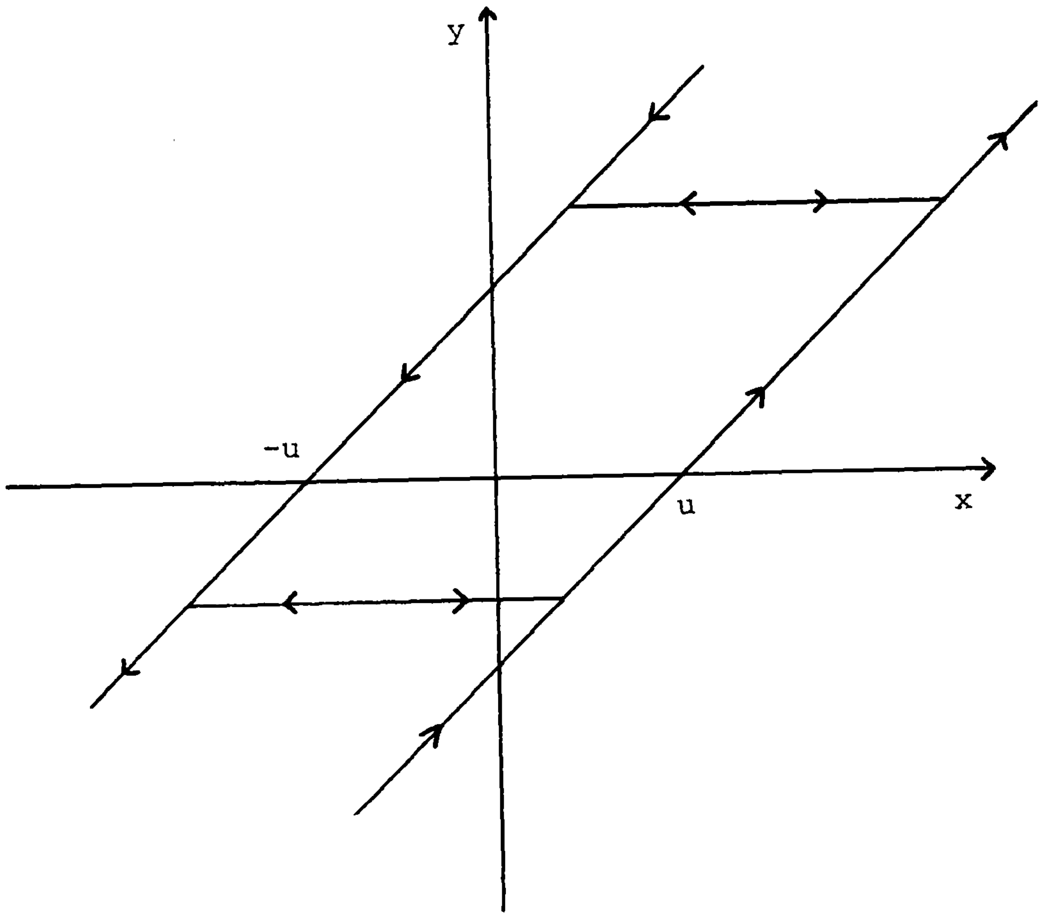

Figure 2.

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