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ADAPTIVE SYNCHRONIZATION OF INTERCONNECTED LINEAR SYSTEMS

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März 1990

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Abstract

In this paper we introduce the concept of an adaptive synchronization controller. Synchronization is modelled as an adaptive tracking problem for families of interconnected linear systems. Stabilization and tracking results are obtained for minimum phase systems.

1. INTRODUCTION

Synchronization phenomena, or entrainment as they are sometimes referred to, appear in different areas like mechanical and electrical engineering, chemistry, biology and macrophysiology (cf. [4] – [11]). A further domain, which attracted a lot of research activities in recent years is the explanation of brain functions. Memory, speech recognition and vision as well as their beginning computer implementations are far away from being completely understood in the context of neural networks. However it seems to be clear that parallel processes in highly interconnected networks of neurons and particularly synchronization phenomena will play a role in the explanation of those phenomena.

We believe that complex collective behavior often can be explained as a result of synchronization of many interconnected simple subsystems (frequently harmonic oscillators). Furthermore we believe that feedback and adaptation mechanisms are key concepts in the modelling of synchronization in particular when learning processes are involved. In this paper we propose a control theoretic setup in which certain mathematical models for synchronization processes can be embedded.

In quite general terms, the task of adaptively synchronizing a family of coupled dynamical systems is to find an adaptive controller which forces the outputs of the various subsystems to asymptotically match each other, or even more generally, to track a given set of reference signals. Thus the adaptive synchronization task can be viewed as an adaptive tracking problem for families of interconnected dynamical systems. In more precise terms we consider the following concept of an adaptive synchronization scheme.

An adaptive synchronization scheme (ASS) for finite–dimensional linear systems consists of:

(i) A family of finitely many (N) linear systems

$$\left.\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) \\ y_i(t) &= C_i x_i(t) \end{aligned}\right\} , \ t \geq 0$$

$x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, $y_i(t) \in \mathbb{R}^{p_i}$ belonging to system classes $\Sigma_i(n_i,m_i,p_i)$, $i = 1,\dots, N$, with fixed numbers of inputs, m_i, outputs, p_i, and possibly unknown numbers n_i of states.

(ii) Prescribed classes

$$R_i \subset C_{pc}([0,\infty), \mathbb{R}^{p_i})$$

of admissible (piecewise continuous) reference (synchronized) signals $r_i(\cdot)$.

(iii) An $N \times N$ – interconnection matrix

$$F = \begin{bmatrix} f_{11} & \cdots & f_{1N} \\ \vdots & & \vdots \\ f_{N1} & \cdots & f_{NN} \end{bmatrix}$$

of C^∞–coupling functions $f_{ij}: \mathbb{R}^{p_i} \longrightarrow \mathbb{R}^{m_i}$, $i,j = 1,\dots,N$.

(iv) An <u>adaptive synchronization controller</u> (ASyCo) for $((\Sigma_i \,|\, i \in \underline{N}), (r_i \,|\, i \in \underline{N}), F)$ consisting of

a) Finite dimensional parameter spaces $\mathbb{R}^{q_i}$ for the feedback gains $k_i(t)$,

b) N smooth local control laws

$$v_i(t) = f_i(k_i(t), y_i(t), r_i(t))$$

with C^∞–functions

$$f_i : \mathbb{R}^{q_i+2p_i} \longrightarrow \mathbb{R}^{m_i},$$

c) N parameter adaptation laws

$$\dot{k}_i(t) = g_i(k_i(t), y_i(t), r_i(t))$$

with C^∞–functions

$$g_i : \mathbb{R}^{q_i+2p_i} \longrightarrow \mathbb{R}^{q_i}$$

which satisfy the <u>synchronization task:</u>

(v) For any $(A_i, B_i, C_i) \in \Sigma_i$, $i = 1,\dots,N$, any initial data $x_i(0)$, $k_i(0)$, $i = 1,..,N$ and any sequences $(r_i(\cdot) \,|\, i \in \underline{N})$, $r_i(\cdot) \in R_i$, there exists a unique solution of the interconnected closed loop system

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t)$$

$$y_i(t) = C_i x_i(t)$$

$$u_i(t) = \sum_{j=1}^{N} f_{ij}(y_j(t)) + f_i(k_i(t), y_i(t), r_i(t))$$

$$\dot{k}_i(t) = g_i(k_i(t), y_i(t), r_i(t), \quad i = 1,\dots,N,$$

for all $t \geq 0$ and satisfies

$$\cdot \lim_{t\to\infty} \left[y_i(t)-r_i(t)\right] = 0 \quad \text{for } i \in \underline{N}$$

$$\cdot \exists \lim_{t\to\infty} k_i(t) = k_{i,\infty} < \infty \quad \text{for } i \in \underline{N}.$$

In this paper we consider the special case, where all systems (A_i,b_i,c_i), $i = 1,\ldots,N$, are scalar (m=p=1), linear, minimum phase systems of relative degree 1, the interconnections are linear and the reference signals $r_i(\cdot)$ are solutions of ordinary differential equations of the form $p_i(D)r_i(\cdot) \equiv 0$, where the $p_i(s)$, $i = 1,\ldots,N$, are polynomials with real coefficients. In a recent paper [13] a synchronization problem has been studied for N simple–integral plants of the form $\frac{K_i}{s}$, $i = 1,\ldots,N$ with the objective to make all the N steady–state outputs identical to one another.

2. ADAPTIVE STABILIZATION FOR COUPLED LINEAR SYSTEMS

We consider N scalar time invariant linear systems:

$$\begin{aligned} \dot{x}_i &= A_i x_i + b_i u_i \\ y_i &= c_i x_i \\ A_i &\in \mathbb{R}^{n_i \times n_i} \quad \text{for } i = 1,\ldots,N \end{aligned} \tag{2.1a}$$

coupled via linear output–input connections:

$$u(t) = F\, y(t) + v(t) \tag{2.1b}$$

where:

$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \mathbb{R}^N$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \in \mathbb{R}^N$, $v(t) \in \mathbb{R}^N$ and $F \in \mathbb{R}^{N\times N}$ a possibly time varying matrix.

Furthermore we assume that the systems (A_i,b_i,c_i) belong to $\Sigma_+(n_i)$, the class of scalar linear systems (A,b,c) which satisfy:

- $cb > 0$

- $\det \begin{bmatrix} sI-A & b \\ c & 0 \end{bmatrix}$ is a Hurwitz polynomial.

Otherwise, the system parameters, i.e. the entries of (A_i,b_i,c_i), $i = 1,\dots,N$, as well as the system orders n_i are unrestricted and can be assumed to be unknown. We will show that independent of the coupling structure F any subsystem configuration of the form (2.1) is adaptively stabilized by local feedback compensators of the form

$$\begin{aligned} v_i(t) &= -k_i(t)y_i(t) \\ \dot{k}_i(t) &= y_i(t)^2. \end{aligned} \tag{2.1c}$$

As a preparatory result we need the following extension of a result due to Wazewski (cf. [2], chap. 29, Ex. 3).

Lemma 2.1

Let

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$$

be a piecewise continuous matrix function on $[0,\infty)$ such that $A_{12}(t)$, $A_{21}(t)$ are uniformly bounded on $[0,\infty)$ and $\dot{x}_1(t) = A_{11}(t)x_1(t)$ is exponentially stable. Suppose there exist constants $c \geq 0$, $a > 0$ such that the maximal eigenvalue $\lambda_m(t)$ of $A_{22}(t) + A_{22}(t)'$ satisfies for all $t \geq t_o \geq 0$

$$\int_{t_o}^{t} \lambda_m(\tau)d\tau \leq -a(t-t_o) + c$$

(where ' means transposed). Then the system $\dot{x}(t) = A(t)x(t)$ is exponentially stable.

Proof

Consider $A_{21}(t)x_1(t)$ as the output of the system

$$\dot{x}_1(t) = A_{11}(t)x_1(t) + A_{12}(t)x_2(t).$$

By a straightforward extension of a result in [13] to the time–varying case there exist constants M_1, $M_2 > 0$, $\mu > 0$ and exponentially bounded functions $c_1(t)$, $c_2(t)$:

$$0 \leq c_i(t) \leq M_i e^{-\mu(t-t_0)},\ t \in [t_o,\infty),\ i = 1,2$$

such that for all $x_1(t_o) \in \mathbb{R}^{n_1}$ and $t \in [t_o,\infty)$:

$$|\langle A_{21}(t)x_1(t),x_2(t)\rangle| \leq c_1(t)\|x_1(t_o)\|^2 + c_2(t)\|x_2(t)\|^2 \tag{2.2}$$

Let $x(\cdot) = \begin{bmatrix} x_1(\cdot) \\ x_2(\cdot) \end{bmatrix}$ be a solution of $\dot{x}(t) = A(t)x(t)$.

Then

$$\frac{d}{dt}\|x_2(t)\|^2 = 2\langle A_{21}(t)x_1(t),x_2(t)\rangle + \langle (A_{22}(t)+A_{22}(t)')x_2(t),x_2(t)\rangle$$

and thus

$$\frac{d}{dt}\|x_2(t)\|^2 \leq 2c_1(t)\|x_1(t_o)\|^2 + (2c_2(t) + \lambda_m(t))\|x_2(t)\|.$$

Using the above exponential bounds for $c_1(\cdot),c_2(\cdot)$ one concludes that for all $t \in [t_o,\infty)$

$$\|x_2(t)\|^2 \leq K \cdot e^{\int_{t_o}^{t} \lambda_m(\tau)d\tau} \cdot \|x_2(t_o)\|^2$$

for a suitable constant $K \geq 0$. Thus, by the assumption on $\lambda_m(t)$, $\|x_2(t)\|^2$ goes exponentially to zero as $t \longrightarrow \infty$. Let $\phi(t,t_o)$ denote the fundamental matrix for

$\dot{z}(t) = A_{11}(t)z(t)$.

Since

$$x_1(t) = \phi(t,t_o)x_1(t_o) + \int_{t_o}^{t} \phi(t,\tau)\, A_{12}(\tau)x_2(\tau)d\tau$$

it follows that $x_1(t)$ is exponentially stable.

Remark

If $\lambda_m(t)$ satisfies that weaker condition

$$\lim_{t \to \infty} \int_{t_o}^{t} \lambda_m(\tau)d\tau = -\infty,$$

then the above proof shows that $\dot{x}(t) = A(t)x(t)$ is asymptotically stable.

We use Lemma 2.1 to prove the following generalization of the high gain theorem (Tychonov's singular perturbation theorem) (cf. [1]):

<u>Proposition 2.2</u>

Let $(A_i,b_i,c_i) \in \Sigma_+(n_i)$ and $k_i(\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}$ piecewise continuous with $\lim_{t\to\infty} k_i(t) = \infty$, $i = 1,\ldots,N$. Let $M(t)$ be an arbitrary block matrix

$$M(t) = \begin{bmatrix} M_{11}(t) & \cdots & M_{1N}(t) \\ \vdots & & \vdots \\ M_{N1}(t) & \cdots & M_{NN}(t) \end{bmatrix}, \ t \in \mathbb{R}_+$$

satisfying

(a) $M_{ij} : \mathbb{R}_+ \longrightarrow \mathbb{R}^{n_i \times n_j}$ is uniformly bounded and continuous

(b) For all $t \in \mathbb{R}_+$ and $i,j \in \underline{N}$ the image of $M_{ij}(t)$ is contained in the one–dimensional subspace of $\mathbb{R}^{n_i}$ spanned by b_i.

Then the system

$$\dot{x}(t) = [\mathrm{diag}(A_i - k_i(t)b_ic_i) + M(t)]x(t) \tag{2.3}$$

is exponentially stable.

<u>Proof</u>

Because of assumption (b) we can write:

$$M_{ij}(t) = b_i m_{ij}(t)$$

with uniformmly bounded row vectors $m_{ij}(t) \in \mathbb{R}^{1\times n}$. Consider the closed loop system (2.3):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} A_1 + b_1(m_{11}-k_1c_1) & b_1m_{12} & \cdots & b_1m_{1N} \\ b_2m_{21} & A_2+b_2(m_{22}-k_2c_2) & \cdots & b_2m_{2N} \\ \vdots & & \ddots & \vdots \\ b_Nm_{N1} & \cdots & & A_N+b_N(m_{NN}-k_Nc_N) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

Because $(A_i,b_i,c_i) \in \Sigma_+(n_i)$ is time invariant, there exists a (time invariant) change of coordinates of $\mathbb{R}^n$ such that for $i = 1,\ldots,N$,

$$x_i = \begin{bmatrix} z_i \\ y_i \end{bmatrix}, \ \dim z_i = n_i - 1,$$

$$m_{ij} = (m_{ij}^1, m_{ij}^2)$$

$$A_i = \begin{bmatrix} A_i(1,1) & A_i(1,2) \\ A_i(2,1) & A_i(2,2) \end{bmatrix}, \ b_i = \beta_i \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}, \ c_i = [0,\dots,1]$$

with $\beta_i > 0$ constant and $\sigma(A_i(1,1)) \subset \mathbb{C}^-$. By a suitable permutation of the coordinates of $\mathbb{R}^n$, (2.3) is state space equivalent to the system

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12}(t) \\ \hat{A}_{21}(t) & \hat{A}_{22}(t) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{2.4}$$

where

- $$\xi = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}, \quad \eta = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \tag{2.4a}$$

- $\hat{A}_{11} = \operatorname{diag}(A_1(1,1),\dots,A_N(1,1))$ is Hurwitz (2.4b)

- $\hat{A}_{12}(t)$, $\hat{A}_{21}(t)$ are uniformly bounded and continuous on $\mathbb{R}_+$ (2.4c)

- $$\hat{A}_{22} = \begin{bmatrix} A_1(2,2)+\beta_1(m_{11}^2 - k_1) & \beta_1 m_{12}^2 & \cdots & \beta_1 m_{1N}^2 \\ \beta_2 m_{21}^2 & A_2(2,2)+\beta_2(m_{22}^2 - k_2) & \cdots & \beta_2 m_{2N}^2 \\ \vdots & & \ddots & \vdots \\ \beta_N m_{N1}^2 & \cdots & & A_N(2,2)+\beta_N(m_{NN}^2 - k_N) \end{bmatrix} \tag{2.4d}$$

It is an easy exercise to prove that the maximal eigenvalue of $\hat{A}_{22}(t) + \hat{A}_{22}(t)'$ satisfies the condition in Lemma 2.1. Thus the result follows immediately from Lemma 2.1.

Remark

The following example shows that a condition like (b) in Proposition 2.2 is necessary for stability:

Let $N = 2$, $n_1 = n_2 = 2$, $M = \begin{bmatrix} 0 & M_1 \\ M_2 & 0 \end{bmatrix}$ with $M_1 = M_2 = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$, $k(t) = k_1(t) = k_2(t)$ and $(A_1,b_1,c_1) = (A_2,b_2,c_2) = (A,b,c) \in \Sigma_+(2)$ is given by

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ c = (0,1)$$

It is easily verified that the closed loop matrix

$$\hat{A} = \begin{bmatrix} A-kbc & M_1 \\ M_2 & A-kbc \end{bmatrix}$$

is not Hurwitz for any $k \geq 0$.

The announced stabilization result for the interconnected systems is now a corollary of Proposition 2.2.

Theorem 2.3

For every family of systems $(A_i,b_i,c_i) \in \Sigma_+(n_i)$, $i = 1,\dots,N$, and every uniformly bounded time varying feedback matrix $F(t) \in \mathbb{R}^{N\times N}$ the time varying closed loop system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} A_1+b_1(f_{11}-k_1)c_1 & b_1f_{12}c_2 & \cdots & b_1f_{1N}c_N \\ b_2f_{21}c_1 & A_2+b_2(f_{22}-k_2)c_2 & \cdots & b_2f_{2N}c_N \\ & & \ddots & \vdots \\ b_Nf_{N1}c_1 & \cdots & & A_N+b_N(f_{NN}-k_N)c_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \tag{2.5a}$$

$$\dot{k}_i(t) = y_i(t)^2,\ i = 1,\dots,N \tag{2.5b}$$

satisfies for all initial conditions $x_i(t_o)$, $k_i(t_o)$ and $i = 1,\ldots,N$:

$$\cdot \lim_{t\to\infty} y_i(t) = 0 \tag{2.6a}$$

$$\cdot \lim_{t\to\infty} k_i(t) = k_{i,\infty} \in \mathbb{R}. \tag{2.6b}$$

<u>Proof</u>

Consider the closed loop system (2.5). Suppose that $k_i(\cdot)$ is unbounded for some $1 \leq i \leq N$. By a suitable permutation of the coordinates we can assume without loss of generality that:

$$k_i(t) \longrightarrow \infty \quad \text{for } i = 1,\ldots,s \tag{2.7a}$$

$$\lim_{t\to\infty} k_i(t) = k_{i,\infty} \text{ exists for } i = s+1,\ldots,N \tag{2.7b}$$

Then:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_N \end{bmatrix} = \left[\mathrm{diag}(A_1,\ldots,A_s) + \begin{bmatrix} 0 & f_{12}b_1c_1 & \ldots & f_{1s}b_1c_s \\ f_{21}b_2c_1 & 0 & \ldots & f_{2s}b_2c_s \\ & & \ddots & \vdots \\ f_{s1}b_sc_1 & \ldots & & 0 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{bmatrix}$$

$$- \mathrm{diag}((k_1-f_{11})b_1c_1,\ldots,(k_s-f_{ss})b_sc_s) \begin{bmatrix} x_1 \\ x_2 \\ \\ x_s \end{bmatrix}$$

$$+ \begin{bmatrix} b_1f_{1s+1} & \ldots & b_1f_{1N} \\ \vdots & & \vdots \\ b_sf_{ss+1} & \ldots & b_sf_{sN} \end{bmatrix} \begin{bmatrix} y_{s+1} \\ \vdots \\ y_N \end{bmatrix}$$

However by (2.7b) we have $y_{s+j}(\cdot) \in L_2(\mathbb{R}_+,\mathbb{R})$ for $j = 1,\ldots,N-s$.
Therefore Proposition 2.2 implies that the $x_i(\cdot)$, $i = 1,\ldots,s$, are exponentially decaying and hence the $k_i(\cdot)$, $i = 1,\ldots,s$, are bounded. Contradiction. Thus $y_i(\cdot) \in L_2(\mathbb{R}_+,\mathbb{R})$ for $i = 1,\ldots,N$.
Consider the i–th subsytem:

$$\dot{x}_i = A_i x_i + b_i\left(\sum_{j=1}^{N} f_{ij} y_j - k_i y_i\right), \tag{2.8}$$

Because $(A_i,b_i,c_i) \in \Sigma_+(n_i)$, we can decompose (2.8) as follows ($\dim z_i = n_i-1$):

$$\dot{z}_i = A_i(1,1)z_i + A_i(1,2)y_i$$

$$\dot{y}_i = A_i(2,1)z_i + A_i(2,2)y_i + \beta_i\left[\sum_{j=1}^{N} f_{ij} y_j - k_i y_i\right]$$

with $\beta_i > 0$ constant and $\sigma(A_i(1,1)) \subset \mathbb{C}^-$. But z_i is the response of a stable system to an L_2–input. This implies together with $y_i(\cdot) \in L_2(\mathbb{R}_+,\mathbb{R})$:

$$x_i(\cdot) \in L_2(\mathbb{R}_+,\mathbb{R}^{n_i}) \text{ for } i = 1,\ldots,N.$$

Furthermore $\dot{x}_i(\cdot) \in L_2(\mathbb{R}_+,\mathbb{R}^{n_i})$ and therefore $\dot{y}_i(\cdot) \in L_2(\mathbb{R}_+,\mathbb{R})$, which implies (2.6).

□

Assume now that the systems to be stabilized are given in an input–output differential operator description:

$$p_{ii}(D)y_i = u_i \tag{2.9a}$$

$$p_{ii}(s) = s^{n_i} + p_i^{(n_i-1)} s^{n_i-1} + \ldots + p_i^{(o)} s^o \tag{2.9b}$$

$i = 1,\ldots,N.$

The couplings are in differential operator form too:

$$u_i = \sum_{j \neq i} p_{ij}(D) y_j + v_i \tag{2.10a}$$

$$p_{ij}(s) \in \mathbb{R}[s], \ \deg p_{ij} \leq \deg p_{ii} - 1 \tag{2.10b}$$

and for the local adaptive controller we assume the following full state–feedback control law:

$$v_i = -k_i \cdot \alpha_i(D) y_i \tag{2.11a}$$

$$\dot{k}_i = y_i^2 \tag{2.11b}$$

where we require:

$$\cdot \ \alpha_i(s) = \alpha_i^{(n_i-1)} s^{n_i-1} + \ldots + \alpha_i^{(o)} s^o \ \text{ Hurwitz polynomial} \tag{2.12a}$$

$$\cdot \ \alpha_i^{(n_i-1)} > 0 \quad \text{for } i = 1,\ldots,N \tag{2.12b}$$

Then:

<u>Corollary 2.4</u>

Let $P(s) \in \mathbb{R}[s]^{N \times N}$, $\alpha(s) \in \mathbb{R}[s]^N$ satisfy (2.9b), (2.10b) and (2.12). Then the closed loop system (2.9) – (2.12) is adaptively stabilized in the sense of (2.6).

<u>Proof</u>

With $x_i := (y_i^{(o)},\ldots,y_i^{(n_i-1)})$ as state vector, (2.9) – (2.11) admit the following state space description:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ x_N \end{bmatrix} - k \cdot BC \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

where the A_{ij} are of the form:

$$A_{ii} := \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & \\ 0 & \dots & 0 & 1 \\ * & & \dots & * \end{bmatrix}_{n_i \times n_i}, \quad A_{ij} := \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ * & \dots & * \end{bmatrix}_{n_i \times n_j}, \quad i \neq j$$

B and C are given as:

$$B := \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & b_N \end{bmatrix}, \quad b_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n_i \times 1}$$

$$C := \begin{bmatrix} c_1 & 0 & 0 & & 0 \\ 0 & c_2 & 0 & & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & c_N \end{bmatrix} \qquad c_i := [\alpha_i^{(0)} \dots \alpha_i^{(n_i-1)}]$$

and

$$k := \mathrm{diag}(k_1 I_{n_1 \times n_1}, \dots, k_N I_{n_N \times n_N}).$$

But then

$$c_i b_i = \alpha_i^{(n_i-1)} > 0$$

and

$$\begin{bmatrix} sI - A_{ii} & b_i \\ c_i & 0 \end{bmatrix} = \alpha(s) \quad \text{Hurwitz polynomial}$$

imply that $(A_{ii}, b_i, c_i) \in \Sigma_+(n_i)$. Since image $(A_{ij}) \subset$ image (b_i) we are in the situation of Proposition 2.2 and the result follows.

□

We next consider again N linear systems (2.1a) coupled via linear output – input connections (2.16), however no assumptions on the sign of the high frequency gains $c_i b_i \neq 0$ are made. This means we only require the systems to belong to the class $\Sigma(n_i)$ of scalar, relative degree 1, minimum phase systems, instead of the smaller class $\Sigma_+(n_i)$. Then switching concepts become necessary.

We introduce local adaptive feedback compensators of the form:

$$v_i = N_i(k_i)\, y_i,\ \dot{k}_i = y_i^2 \tag{2.1c'}$$

where the functions $N_i : \mathbb{R} \longrightarrow \mathbb{R}$ are of Nussbaum type, i.e. for

$$F_i(k) := \frac{1}{k} \int_0^k N_i(\sigma) d\sigma$$

we require

$$\sup_{k>0} F_i(k) = +\infty \tag{2.13a}$$

$$\inf_{k>0} F_i(k) = -\infty \tag{2.13b}$$

and additionally the following restriction for the location of the local minima and maxima of $F_i(k)$ holds:

There exist monotonically increasing sequences $(k_{i,n})_{n\in\mathbb{N}}$, $(k'_{i,n})_{n\in\mathbb{N}}$ of local maxima, minima, resp. of F_i, $i = 1,\dots,N$ such that:

– $k_{i,n} \xrightarrow{n\to\infty} +\infty$, $k'_{i,n} \xrightarrow{n\to\infty} +\infty$ (2.14a)

– $F_i(k_{i,n}))_{n\in\mathbb{N}}$ and $(F_i(k'_{i,n}))_{n\in\mathbb{N}}$

are monotonically increasing, decreasing, resp., with

$$\lim_{n\to\infty} F_i(k_{i,n}) = \infty$$

$$\lim_{n\to\infty} F_i(k'_{i,n}) = -\infty \qquad (2.14b)$$

The sequences $(k_{i,n+1}-k_{i,n})_{n\in\mathbb{N}}$, $(k'_{i,n+1}-k_{i,n})_{n\in\mathbb{N}}$ are bounded. (2.14c)

$i = 1,\ldots,N.$

Remarks

- Only the boundedness of the sequences $(k_{i,n+1}-k_{i,n})_{n\in\mathbb{N}}$, $(k'_{i,n+1}-k'_{i,n})_{n\in\mathbb{N}}$ constitutes the additional condition.
- An example of a function satisfying the above requirements is $N(k) = k^2\cos k$.

Theorem 2.5

For every family of systems $(A_i,b_i,c_i) \in \Sigma\,(n_i)$, $i = 1,\ldots,N$ and constant coupling matrix $F \in \mathbb{R}^{N\times N}$ and for any initial data $(x_i(0), k_i(0))$, $i = 1,\ldots,N$, the solutions of the closed loop system configuration (2.1 a,b,c') satisfy:

$$\lim_{t\to\infty} y_i(t) = 0 \qquad (2.6a)$$

$$\lim_{t\to\infty} k_i(t) = k_{i,\infty} < \infty \qquad (2.6b)$$

Proof:

(2.6b) implies (2.6a) by exactly the same reasoning as in the proof of Theorem 2.3. To show that (2.6a) is satisfied, we again decompose the i–th subsystem to obtain:

$$\dot{x}_{i1} = A_{i1}x_{i1} + A_{i2}y_i \tag{2.15a}$$

$$\dot{y}_i = A_{i3}x_{i1} + \alpha_i y_i + \beta_i u_i, \tag{2.15b}$$

with $\sigma(A_{i1}) \subset \mathbb{C}^-$, $\beta_i \neq 0$, and $u_i = \sum_{j=1}^{N} f_{ij}y_j + N_i(k_i)y_i$.

Integration of (2.15b) multiplied by y_i from 0 to t yields:

$$\frac{1}{2}y_i^2(t) = \frac{1}{2}y_i^2(0) + \int_0^t A_{i3}x_{i1}y_i d\tau + \int_0^t (\alpha_i+\beta_i N_i(k_i))y_i^2 d\tau + \sum_{j=1}^{N} \beta_i f_{ij} \int_0^t y_i y_j d\tau$$

Applying a result of ([13]) to the system (2.15a) with output $A_{i3}x_{i1}$, we obtain

$$\int_0^t |A_{i3}x_{i1}y_i| d\tau \leq c_i + M_i \int_0^t y_i^2 d\tau$$

with c_i, M_i constants.

And since $|y_i y_j| \leq \frac{1}{2}(y_i^2+y_j^2)$ we can estimate

$$\begin{aligned} 0 &\leq \frac{1}{2}y_i^2(t) \\ &\leq \frac{1}{2}y_i^2(0) + c_i + \int_0^t (\alpha_i+M_i+\frac{1}{2}\sum_{j=1}^{N} |\beta_i f_{ij}| + \beta_i N_i(k_i)y_i^2)d\tau + \frac{1}{2}\sum_{j=1}^{N} |\beta_i f_{ij}| \int_0^t y_j^2 d\tau. \end{aligned}$$

From this follows by substitution $\sigma = k_i(\tau)$ ($d\sigma = y_i^2 d\tau$):

$$0 \leq \eta_i + \delta_i k_i(t) + \beta_i \int_0^{k_i(t)} N_i(\sigma)d\sigma + \sum_{j=1}^{N} \gamma_j k_j(t) \tag{2.16}$$

with $\gamma_{ij} \geq 0$, $\beta_i \neq 0$, σ_i, η_i constants.

Now suppose (2.6b) were not true. We show that the inequalities (2.16) then lead to a contradiction. We can assume that all $k_i(t)$ are unbounded. For if it were not so we would only consider the inequalities (2.16) for i with unbounded $k_i(t)$ and remove from these the bounded $k_j(t)$ by a suitable modification of the constants η_i.

Now choose $T_1 > 0$ such that for $t \geq T_1$:

$$k_i(t) > \frac{1}{N} \tag{2.17}$$

$$i = 1,\ldots N$$

$$\left|\frac{\eta_i}{k_i(t)}\right| < 1 \tag{2.18}$$

For each i choose a monotone sequence $f_i = (k_{i,n})_{n\in\mathbb{N}}$ of local maxima of $\tilde{F}_i(k) := -\delta_i - \frac{\beta_i}{k}\int_0^k N_i(\sigma)d\sigma$, tending to ∞, such that $k_{i,n+1} - k_{i,n} \leq L_i$, $n \in \mathbb{N}$, and $(\tilde{F}_i(k_{i,n}))_{n\in\mathbb{N}}$ is monotone and tends to $+\infty$ (possible because of (2.14)). There are $t_i' \geq T_1$ such that $k_i(t_i') \in f_i$ and

$$\tilde{F}_i(k_i(t_i')) > \sum_j \gamma_{ji} + L + 1 \tag{2.19}$$

where $L := \sum_i \sum_j \gamma_{ji} L_i$. (2.20)

Now for $i = 1,\ldots,N$ define t_i to be the greatest number $\leq T_2 := \max\left\{t_1',\ldots,t_N'\right\}$ such that $k_i(t_i) \in f_i$, $k_i(t_i) = k_{i,n_i} < k_{i,n_i+1}$. Let t_i'' be such that $k_i(t_i'') = k_{i,n_i+1}$. By definition of t_i we have $t_i' \leq t_i \leq T_2 < t_i''$. Monotonicity of $k_i(\cdot)$ therefore implies:

$$k_i(t_j) - k_i(t_i) \leq k_i(t_i'') - k_i(t_i) \leq k_{i,n_i+1} - k_{i,n_i} \leq L_i \tag{2.21}$$

$$i,j = 1,\ldots,N$$

and

$$\tilde{F}_i(k_i(t_i)) \geq \tilde{F}_i(k_i(t_i')), \quad i = 1,\ldots,N. \tag{2.22}$$

Hence,

$$\sum_i \sum_j \gamma_{ji} k_i(t_i) + L \overset{(2.20)}{=} \sum_i \sum_j \gamma_{ji} (k_i(t_i) + L_i) \overset{(2.21)}{\geq} \sum_i \sum_j \gamma_{ji}\, k_i(t_j) = \sum_i \sum_j \gamma_{ij}\, k_j(t_i)$$

$$\overset{(2.8)}{\geq} \sum_i \Big(-\frac{\eta_i}{k_i(t_i)} + \tilde{F}_i(k_i(t_i))\Big)k_i(t_i) \overset{(2.17-2.19),\,(2.22)}{>} \sum_i \Big(\sum_j \gamma_{ji} + L\Big)\, k_i(t_i) \overset{(2.17)}{\geq} \sum_i \sum_j \gamma_{ji} k_i(t_i) + L$$

a contradiction. So we have shown (2.6b)

□

3. ADAPTIVE SYNCHRONIZATION

Now where we have established adaptive stabilization of interconnected systems we can go ahead and apply the tracking results of [3] to this situation. So in (2.1a), (2.1b) we are employing local adaptive tracking controllers of the type presented in [3].

For this N reference signals $\gamma_i(\cdot)$ are given which satisfy the differential equations

$$p_i(D)\gamma_i \equiv 0 \tag{3.1}$$

where

$$p_i(s) = s^{l_i} + p_i^{(l_i-1)} s^{l_i-1} + \ldots + p_i^{(0)} s^0$$

are known real polynomials with zeros in the closed left half plane $\mathbb{C}_- = \{s \in \mathbb{C};\ \mathrm{Re}\, s \leq 0\}$.

The synchronization task is to design local adaptive feedback controllers ensuring asymptotic tracking of these reference signals:

$$\lim_{t \to \infty} (y_i(t) - r_i(t)) = 0 \quad \text{for } i = 1, \ldots, N \tag{3.2}$$

where the system parameters (A_i, b_i, c_i) and initial conditions $x_i(0)$, $i = 1, \ldots, N$ are unknown.

Let first $p(s) = s^{\ell}+p_{\ell-1}s^{\ell-1} + \dots + p_0 s^0$ denote the least common multiple of $p_1(s),\dots,p_N(s)$. Let further $q(s) = s^{\ell}+q_{\ell-1}s^{\ell-1} + \dots + q_0 s^0$ be any Hurwitz polynomial and let $(A_r,b_r,c_r,1)$ be a minimal realization of $g(s) = \frac{q(s)}{p(s)}$. Now we define the synchronization controller as follows:

$$v_i = c_r x_{ri} + N_i(k_i)e_i \tag{3.3a}$$

$$\dot{x}_{ri} = A_r x_{ri} + b_r N_i(k_i)e_i \tag{3.3b}$$

$$\dot{k}_i = e_i^2 \tag{3.3c}$$

with $e_i = y_i - r_i$.

We claim the following is true:

Claim

For any N systems in $\Sigma(n_i)$ (2.1a) coupled via any linear time invariant output–input connections (2.1b):

$$u_i = v_i + \sum_j f_{ij} y_j, \quad i = 1,\dots,N,$$

and for any initial data $(x_i(0),\ x_{ri}(0),\ k_i(0))$ the local feedback compensators (3.3) assure for the closed loop solution

$$\left.\begin{aligned} &\lim_{t\to\infty} e_i(t) = 0 \\ &\lim_{t\to\infty} k_i(t) = k_{i,\infty} < \infty. \end{aligned}\right\} \quad i = 1,\dots,N$$

Remarks

(i) As in the last section the $N_i(\cdot)$ are special switching functions of Nussbaum type. In case sgn $(c_i b_i)$ is known for all i we can choose $N_i(k) = -\operatorname{sgn}(c_i b_i)\, k$.

(ii) In [3] we have proved this claim for series couplings of systems, i.e. if $f_{ij} = 0$ for $j \neq i-1$. The proof for the general situation contains some technical difficulties and will be included in a forthcoming paper.

4. SIMULATIONS

We illustrate the results of this paper by means of simulation examples. They show asymptotic synchronization of interconnected systems to sinussoidal behavior.

First an interconnection of four systems and then an interconnection of eight systems is simulated, each of which is synchronized by controllers (3.3) both of the nonswitching ($N_i(k) = -\text{sgn}\,(c_i b_i)k$) and the switching type ($N_i(k) = k^2 \cos k$).

The transfer functions of the systems are:

$$\Sigma_1 : g_1(s) = \frac{s+1}{s^2-2s+1}$$

$$\Sigma_2 : g_2(s) = \frac{s^3+4s^2+5s+2}{s^4-5s^3+3s^2+4s-1}$$

$$\Sigma_3 : g_3(s) = \frac{1}{s-1}$$

$$\Sigma_4 : g_4(s) = \frac{s^2+2s+1}{s^3+2s^2+3s-2}$$

$$\Sigma_5 : g_5(s) = \frac{s^4+4s^3+6s^2+4s+1}{s^5-s^4-s^3+s^2-s}$$

$$\Sigma_6 : g_6(s) = \frac{1}{s+1}$$

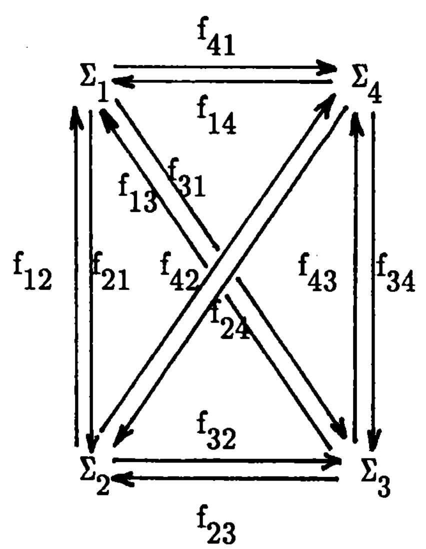
$$\Sigma_7 : g_7(s) = \frac{s+1}{s^2+2s+1}$$

$$\Sigma_8 : g_8(s) = \frac{s^2+4s+4}{s^3+3s^2+2s-1}$$

In the nonswitsching case the sinussoidal reference signals are $r_i(t) = \sin(t+\frac{i-1}{4}\pi)$, such that according to (2.7) $p(s) = s^2+1$, and in the switching case $r_i(t) = \sin(4t-\frac{i-1}{\pi})$ such that $P(s) = s^2+16$. In either case $q(s) = (s+\pi)^2$.

The first configuration to be simulated consists of the four systems Σ_1, Σ_2, Σ_3, Σ_4 interconnected by the matrix

$$F = \begin{bmatrix} 0 & 2 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & 1 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & \frac{3}{2} & 0 \end{bmatrix} :$$

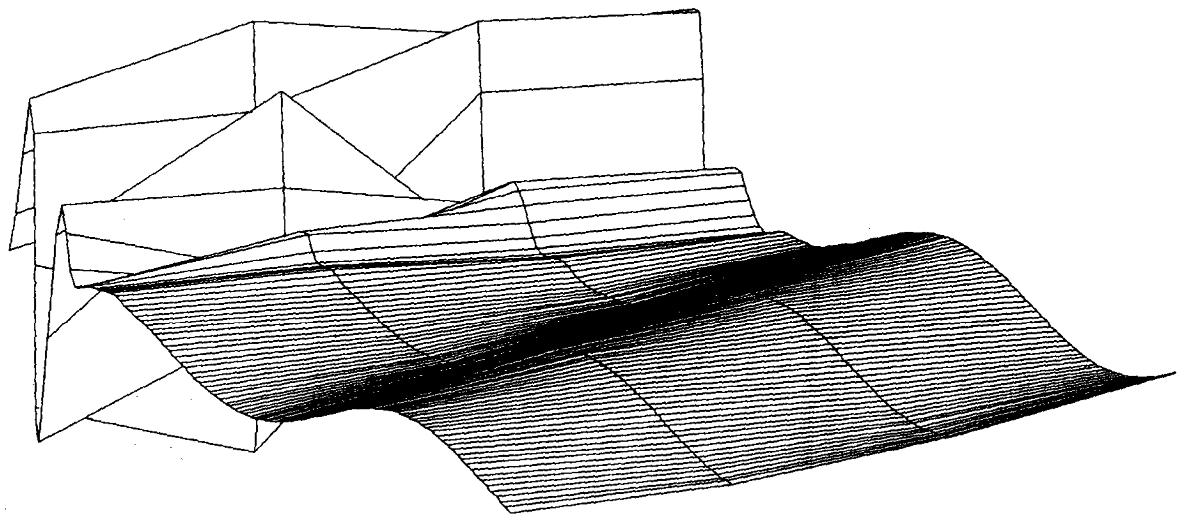


(4.1)

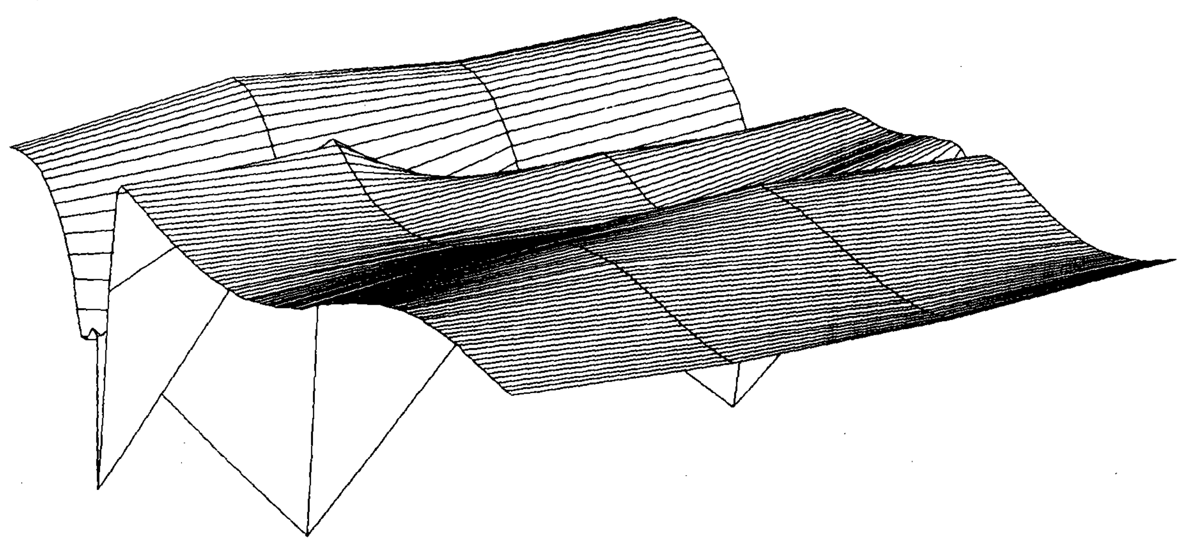
The following 3–D plots visualize the desired synchronization. The output functions $y_1(t)$, $y_2(t)$, $y_3(t)$, $y_4(t)$ are graphed in parallel t–y–planes from left to right; at equidistant points of time the graphs are connected by straight lines.

Output behaviour of (4.1),

– in the nonswitching case:



– in the switching case:

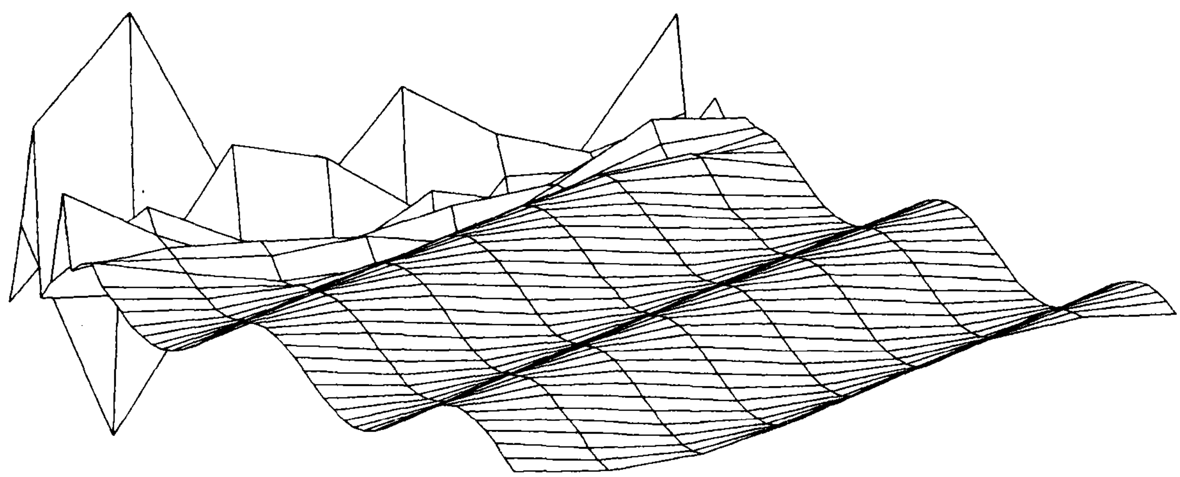


As a second example we consider a series coupling of the eight systems $\Sigma_1,\ldots,\Sigma_8$. This is achieved by the interconnection matrix $F = (f_{ij})$ with $f_{ij} = 1$ if $j=i-1$, $i=2,\ldots,8$, and $f_{ij} = 0$ otherwise:

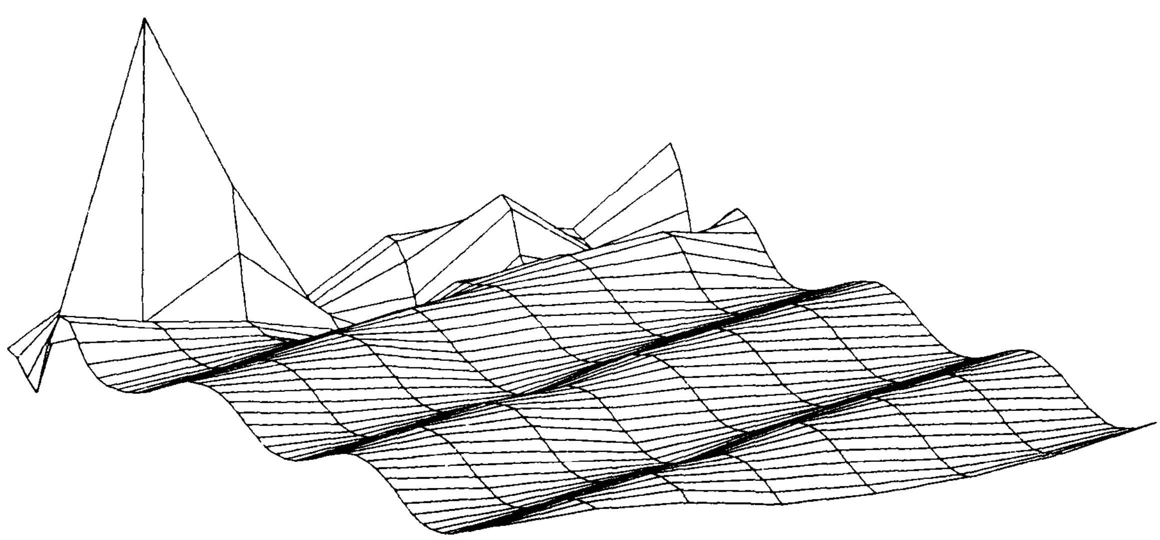
$$\Sigma_1 \xrightarrow{f_{21}} \Sigma_2 \xrightarrow{f_{32}} \Sigma_3 \xrightarrow{f_{43}} \ldots \xrightarrow{f_{87}} \Sigma_8 \tag{4.2}$$

Again, 3–D plots visualize the desired synchronization of (4.2)

– in the nonswitching case:



– in the switching case:



Acknowledgement

The authors would like to thank P.A. Fuhrmann of the University of the Negev, Beer Sheva, who emphasized the synchronization problem and stimulated very much this paper.

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