Good Deal Bounds for Option Prices under VaR and ES Constraints, MAI and SUBMA Risk Measures

Christian Roman Laudagé

1. Gutachter: Prof. Dr. Jörn Sass
   Technische Universität Kaiserslautern
2. Gutachter: Prof. Dr. Pablo Koch-Medina
   Universität Zürich

Datum der Disputation: 28. März 2022

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern zur Verleihung des akademischen Grades Doktor der Naturwissenschaften (Doctor rerum naturalium, Dr. rer. nat.) genehmigte Dissertation.

D386

Abstract

Risk management is an indispensable component of the financial system. In this context, capital requirements are built by financial institutions to avoid future bankruptcy. Their calculation is based on a specific kind of maps, so-called risk measures. There exist several forms and definitions of them. Multi-asset risk measures are the starting point of this dissertation. They determine the capital requirements as the minimal amount of money invested into multiple eligible assets to secure future payoffs.

The dissertation consists of three main contributions: First, multi-asset risk measures are used to calculate pricing bounds for European type options. Second, multi-asset risk measures are combined with recently proposed intrinsic risk measures to obtain a new kind of a risk measure which we call a multi-asset intrinsic (MAI) risk measure. Third, the preferences of an agent are included in the calculation of the capital requirements. This leads to another new risk measure which we call a scalarized utility-based multi-asset (SUBMA) risk measure.

In the introductory chapter, we recall the definition and properties of multi-asset risk measures. Then, each of the aforementioned contributions covers a separate chapter. In the following, the content of these three chapters is explained in more detail:

Chapter 2: Good deal bounds under Value-at-Risk and Expected Shortfall constraints

Risk measures can be used to calculate pricing bounds for financial derivatives. In this chapter, we deal with the pricing of European options in an incomplete financial market model. We use the common risk measures Value-at-Risk and Expected Shortfall to define good deals on a financial market with log-normally distributed rates of return. We show that the pricing bounds obtained from Value-at-Risk may have a non-smooth behavior under parameter changes. Additionally, we find situations in which the seller's bound for a call option is smaller than the buyer's bound. We identify the missing convexity of the Value-at-Risk as main reason for this behavior. Due to the strong connection between the obtained pricing bounds and the theory of risk measures, we further obtain new insights in the finiteness and the continuity of multi-asset risk measures.

Chapter 3: Multi-asset intrinsic risk measures

In this chapter, we construct the MAI risk measure. Therefore, recall that a multi-asset risk measure describes the minimal external capital that has to be raised into multiple eligible assets to make a future financial position acceptable, i.e., that it passes a capital adequacy test. Recently, the alternative methodology of intrinsic risk measures was introduced in the literature. These ask for the minimal proportion of the financial position that has to be reallocated to pass the capital adequacy test, i.e., only internal capital is used. We combine these two concepts and call this new type of risk measure an MAI risk measure. It allows to secure the financial position by external capital as well as reallocating parts of the portfolio as an internal rebooking.

We investigate several properties to demonstrate similarities and differences to the two aforementioned classical types of risk measures. We find out that diversification reduces the capital requirement only in special situations depending on the financial positions. With the help of Sion's minimax theorem we also prove a dual representation for MAI risk measures. Finally, we determine capital requirements in a model motivated by the Solvency II methodology.

Chapter 4: Scalarized utility-based multi-asset risk measures

In this final chapter, we construct the SUBMA risk measure. In doing so, we consider the situation in which a financial institution has to satisfy a capital adequacy test, e.g., by the Basel Accords for banks or by Solvency II for insurers. If the financial situation of this institution is tight, then it can happen that no reallocation of the initial endowment would pass the capital adequacy test. The classical portfolio optimization approach breaks down and a capital increase is needed. We introduce the SUBMA risk measure which optimizes the hedging costs and the expected utility of the institution simultaneously subject to the capital adequacy test.

We find out that the SUBMA risk measure is coherent if the utility function has constant relative risk aversion and the capital adequacy test leads to a coherent acceptance set. In a one-period financial market model we present a sufficient condition for the SUBMA risk measure to be finite-valued and continuous. Under further assumptions on the utility function we obtain existence and uniqueness results for the optimal hedging strategies. Finally, we calculate the SUBMA risk measure in a continuous-time financial market model for two benchmark capital adequacy tests.

Zusammenfassung

Risikomanagement ist ein unverzichtbarer Bestandteil des Finanzsystems. Hierbei werden Kapitalrücklagen von Finanzinstitutionen gebildet, um eine künftige Insolvenz zu vermeiden. Deren Berechnung basiert auf sogenannten Risikomaßen. Für diese existieren verschiedene Definitionen. Der Ausgangspunkt dieser Dissertation sind Multi-Asset-Risikomaße. Sie bestimmen die Kapitalrücklagen als den minimalen Geldbetrag, der in zulässige Anlagen investiert werden muss, um künftige Auszahlungen abzusichern.

Die Dissertation besteht aus drei Beiträgen: Erstens werden Multi-Asset-Risikomaße genutzt um Preisschranken für europäische Optionen zu berechnen. Zweitens werden Multi-Asset-Risikomaße mit den kürzlich eingeführten intrinsischen Risikomaßen kombiniert. Dies führt zu dem neuen Multi-Asset-intrinsischen-Risikomaß (MAI-Risikomaß). Drittens werden die Präferenzen des Marktteilnehmers in der Berechnung der Kapitalrücklagen berücksichtigt. Dies führt zu einem weiteren neuen Risikomaß, dem sogenannten skalarisierten nutzenbasierten Multi-Asset-Risikomaß (SUBMA-Risikomaß).

In dem Einführungskapitel wiederholen wir die Definition und Eigenschaften von Multi-Asset-Risikomaßen. Anschließend widmen wir jedem der genannten Beiträge ein eigenständiges Kapitel. Im Folgenden gehen wir auf diese drei Kapitel im Detail ein:

Kapitel 2: Good deal bounds under Value-at-Risk and Expected Shortfall constraints

Risikomaße können genutzt werden, um Preisschranken für Finanzderivate zu berechnen. In diesem Kapitel behandeln wir die Preisgestaltung von europäischen Optionen in einem unvollständigen Finanzmarktmodell. Wir nutzen die gängigen Risikomaße Value-at-Risk und Expected Shortfall, um sogenannte gute Deals auf einem Finanzmarkt mit log-normalverteilten Renditen zu definieren. Wir zeigen, dass die Preisschranken, die wir aus dem Value-at-Risk erhalten, ein nicht glattes Verhalten bei Parameteränderungen aufweisen. Zusätzlich finden wir Situationen in denen die Schranke des Verkäufers einer Call-Option kleiner ist als die Schranke des Käufers. Wir indentifizieren die fehlende Konvexität des Value-at-Risk als Hauptgrund für dieses Verhalten. Auf Grund der starken Verbindung zwischen den gefundenen Preisschranken und der Theorie der Risikomaße, erhalten wir neue Erkenntnisse über die Endlichkeit und die Stetigkeit von Multi-Asset-Risikomaßen.

Kapitel 3: Multi-asset intrinsic risk measures

In diesem Kapitel konstruieren wir das MAI-Risikomaß. Dafür möchten wir daran erinnern, dass Multi-Asset Risikomaße das minimale externe Kapital beschreiben, welches in mehrere zulässige Anlagen investiert werden muss, so dass die künftige Finanzposition akzeptabel wird, d.h. dass sie die Kapitalanforderungen erfüllt. Kürzlich wurde die alternative Methodik der intrinsischen Risikomaße entwickelt. Diese fragt nach dem minimalen Anteil der Finanzposition, welche reallokiert werden muss, um die Kapitalanforderungen zu erfüllen, d.h. nur internes Kapital wird verwendet. Wir kombinieren diese beiden Konzepte und nennen das neue Risikomaß ein MAI-Risikomaß. Es erlaubt uns, eine Finanzposition sowohl mit externem Kapital als auch durch Reallokieren des Portfolios abzusichern.

Wir untersuchen verschiedene Eigenschaften, um Gemeinsamkeiten und Unterschiede zu den zwei oben genannten klassischen Risikomaßen aufzuzeigen. Dabei reduziert Diversifikation die Kapitalrückstellungen nur in speziellen Situationen, welche von den beteiligten Finanzpositionen abhängen. Mit der Hife von Sion's Min-Max Theorem beweisen wir eine Dualdarstellung für MAI-Risikomaße. Abschließend bestimmen wir die Kapitalrückstellungen in einem Modell, das durch Solvency II motiviert ist.

Kapitel 4: Scalarized utility-based multi-asset risk measures

Im letzten Kapitel konstruieren wir das SUBMA-Risikomaß. Dafür betrachten wir eine Situation, in welcher eine Finanzinstitution bestimmte Kapitalanforderungen erfüllen muss, bspw. die Basel Abkommen für Banken oder Solvency II für Versicherungen. Wenn die finanzielle Situation der Institution angespannt ist, dann kann es passieren, dass keine der möglichen Reallokationen die Kapitalanforderungen erfüllt. Der klassische Portfoliooptimierungsansatz ist nicht anwendbar und eine Kapitalerhöhung ist erforderlich. Dafür führen wir das SUBMA-Risikomaß ein, dass die Hedging Kosten und den erwarteten Nutzen der Institution zeitgleich unter Berücksichtigung der Kapitalanforderungen optimiert.

Wir zeigen, dass das SUBMA-Risikomaß kohärent ist, wenn die Nutzenfunktion konstante relative Risikoaversion besitzt und die Kapitalanforderungen zu einer kohärenten Akzeptanzmenge führen. In einem Ein-Perioden-Modell präsentieren wir eine hinreichende Bedingung, so dass das SUBMA-Risikomaß reellwertig und stetig ist. Durch weitere Annahmen an die Nutzenfunktion erhalten wir Existenz- und Eindeutigkeitsaussagen für die optimalen Hedging-Strategien. Abschließend berechnen wir das SUBMA-Risikomaß in einem zeitstetigen Finanzmarktmodell für zwei standardmäßige Kapitalanforderungen.

Acknowledgements

First and foremost, I would like to thank my supervisor Prof. Dr. Jörn Sass for his professional, constant and friendly support during the past years. Any discussion with him had a huge impact on the final results of this dissertation. I learned a lot from his way of looking at problems in financial mathematics. He showed me how helpful a good intuition can be in finding a suitable solution path.

I would also thank Prof. Dr. Pablo Koch-Medina for accepting the invitation to act as a referee for this dissertation.

I also thank the Fraunhofer Institute for Industrial Mathematics ITWM for its financial support. I am especially grateful for the opportunity to participate on several international conferences. Further, the financial mathematics department gave me a professional and familiar working environment. In particular, it was always a pleasure to be a member of the department.

I also would like to thank Assist.-Prof. Dr. Sascha Desmettre. I am very grateful for his continuous interest in my work and I enjoyed the time of writing several papers together very much. His consistently positive attitude towards difficult tasks was very inspiring for me. Furthermore, he made a research stay at the Johannes Kepler University Linz possible. I would like to thank the whole Institute of Financial Mathematics and Applied Number Theory for the kind hospitality. This time was full of instructive and beautiful moments.

Special thanks goes to Dr. Roman Horsky for proofreading this dissertation. His comments helped me to improve the readability and clarity of many aspects.

Finally, I would like to thank my whole family for supporting me throughout my entire life. Special thanks goes to my parents, Claudia and Werner, and my girlfriend, Christina, who helped me during the difficult times of my doctoral thesis. Without them, this dissertation would not have been possible.

Contents

1 Multi-asset risk measures **1**

1.1 Introduction 1

1.1.1 Motivation 1

1.1.2 Bibliographic notes 2

1.1.3 Structure of this chapter 2

1.1.4 Standard notations and concepts 3

1.2 Financial positions 3

1.2.1 Model space 4

1.2.2 Marketed space 5

1.2.3 Pricing functional 6

1.3 Definition of multi-asset risk measures 6

1.3.1 Acceptance sets 7

1.3.2 Monetary risk measures 8

1.3.3 Multi-asset risk measures 10

1.4 Properties of multi-asset risk measures 11

1.4.1 Convexity and coherence 12

1.4.2 Finiteness and continuity 13

1.4.3 Value-at-Risk and Expected Shortfall 14

2 Good deal bounds under Value-at-Risk and Expected Shortfall constraints **15**

2.1 Introduction 15

2.1.1 Motivation 15

2.1.2 Bibliographic notes 15

2.1.3 Contributions of this chapter 16

2.1.4 Structure of this chapter 17

2.1.5 Standard notations and concepts 17

2.2 One-period financial market model for multi-asset risk measures 17

2.2.1 Financial market model 18

2.2.2 Multi-asset risk measures 19

2.3 Hedging with multi-asset risk measures 19

2.3.1 Representation and acceptability arbitrage 19

2.3.2 Finiteness and continuity for VaR and ES 22

2.3.3 European options 25

2.4 Pricing with multi-asset risk measures 30

2.4.1 Good deals of the first kind 30

2.4.2 Good deal bounds for option prices 31
2.4.3 Limiting behavior 33
2.5 Extension of the basis market 35
2.5.1 Duality relations 36
2.5.2 Failure of the extension theorem 37

3 Multi-asset intrinsic risk measures 41

3.1 Introduction 41
3.1.1 Motivation 41
3.1.2 Bibliographic notes 42
3.1.3 Path to a new risk measure 43
3.1.4 Contributions of this chapter 45
3.1.5 Structure of this chapter 45
3.1.6 Standard notations and concepts 46
3.2 Introducing multi-asset intrinsic risk measures 46
3.2.1 The financial market model 46
3.2.2 External and internal management actions 47
3.2.3 Multi-asset intrinsic risk measures 47
3.3 MAI and multi-asset risk measures 48
3.3.1 Representation by multi-asset risk measures 49
3.3.2 $\mathcal{S}$-additivity, monotonicity and positive homogeneity 50
3.4 Diversification 52
3.4.1 Types of diversification 53
3.4.2 Diversification within the balance sheet 54
3.4.3 Diversification over business lines 57
3.5 Dual representations 58
3.5.1 Some preliminaries 59
3.5.2 Duality of MAI risk measures 59
3.6 Numerical examples 61
3.6.1 Balance sheet model and acceptability criterion 62
3.6.2 Risk measure calculations 63
3.6.3 Numerical results 64
3.6.4 Heavy-tailed distributed liabilities 67

4 Scalarized utility-based multi-asset risk measures 73

4.1 Introduction 73
4.1.1 Motivation 73
4.1.2 Bibliographic notes 76
4.1.3 Path to a new risk measure 76
4.1.4 Contributions of this chapter 77
4.1.5 Structure of this chapter 77
4.1.6 Standard notations and concepts 78
4.2 Utility functions and financial positions 78
4.2.1 Utility functions 78
4.2.2 Orlicz spaces 79

4.2.3 Financial positions 80
4.3 Scalarized utility-based multi-asset risk measures 82
4.3.1 Link to multi-asset risk measures 83
4.3.2 Link to OCE and OEU 83
4.3.3 Link to multi-objective optimization 84
4.4 Desirable properties: Convexity and coherence 84
4.4.1 Monotonicity, $\mathcal{M}$-invariance and convexity 85
4.4.2 Coherence 86
4.5 One-period trading: Model and absence of good deals 88
4.5.1 Standard one-period model 88
4.5.2 Absence of good deals 88
4.5.3 Karush-Kuhn-Tucker conditions 91
4.5.4 Numerical example 92
4.6 One-period trading: Finiteness, continuity and optimal payoffs 94
4.6.1 Finiteness 94
4.6.2 Continuity 101
4.6.3 Optimal payoffs 103
4.7 Continuous-time trading: Solution technique 104
4.7.1 Primal assets and admissible trading strategies 105
4.7.2 SUBMA representation result 106
4.7.3 CRRA 107
4.7.4 CARA 112

5 Conclusion and outlook **115**

Bibliography **119**

Appendix **125**
A Concepts and results from functional analysis 125
A.1 Topological vector space 125
A.2 Topological dual space 126
A.3 Separation theorems 126
A.4 Continuity and coerciveness 127
B Results from utility theory 129
B.1 Integral functional 129
B.2 Certainty equivalent 130

Chapter 1

Multi-asset risk measures

1.1 Introduction

This chapter is an introduction to the mathematical concepts used for risk assessment. The main task in this field is the calculation of a capital requirement for a financial position. Such a financial position could be the asset portfolio of an investor or the equity capital of a financial institution. The characteristic of a financial position changes when eligible assets are bought or sold. Certain functionals make use of this fact to estimate the capital requirement. These functionals are risk measures using multiple eligible assets. To introduce them is the main concern of this chapter. They are then the starting point for the new contributions in the subsequent chapters.

1.1.1 Motivation

In order to prepare against unfavorable scenarios in the future, investors and companies are willing to build capital reserves. Their calculation is often prescribed by legal requirements. They should prevent investors and companies to become insolvent and thus to stabilize the financial system. Summarizing, regulatory agencies require companies to pass capital adequacy tests by building adequate capital reserves.

Capital requirements are calculated by so-called risk measures. The main idea behind risk measures is summarized as follows: *Risk measures ask for the minimal amount of capital that has to be invested into eligible assets to pass the capital adequacy test.*

For the sake of illustration, assume that a company faces a payoff X at some finite future time T. Further, assume a single eligible asset with actual price S_0 and future price S_T for which we write $S = (S_0, S_T)$. Future payoffs which would be ***accepted*** by the regulator are modeled by a subset $\mathcal{A}$, i.e., every element in $\mathcal{A}$ passes the capital adequacy test. The risk measure is then given by the following functional:

$$\rho_{\mathcal{A},S}(X) = \inf\{\lambda S_0 \mid \lambda \in \mathbb{R}, X + \lambda S_T \in \mathcal{A}\}.$$

If $X \notin \mathcal{A}$, then the risk measure suggests to invest an additional amount of money into the eligible asset S to pass the capital adequacy test. It is a restriction to focus on a single eligible asset S as a hedging instrument. For example, Baes, Koch-Medina, and Munari (2020, page 129) point out that "*[...] limiting the investment choice to a single eligible asset instead of allowing investments in portfolios of multiple eligible assets is inefficient, in that, it generally leads to higher capital requirements*". Hence, we are interested in risk measures using multiple eligible assets. This is a by now well established concept that we repeat in this chapter. We start by giving an overview of the literature.

1.1.2 Bibliographic notes

The seminal paper of Artzner, Delbaen, Eber, and Heath (1999) introduces risk measures using a single eligible asset. If it is possible to use the single eligible asset as a numéraire, then one can focus on the simpler concept of so-called ***monetary risk measures***, see e.g., Föllmer and Schied (2016), Delbaen (2002), Cheridito and Li (2009). But not every single eligible asset can be used as a numéraire. For instance, if the eligible asset can default, then it is not possible to divide by its payoff. Hence, Farkas, Koch-Medina, and Munari (2014), Munari (2015) analysis risk measures with respect to an eligible asset under default risk.

It is also critical to limit the hedging actions to a single eligible asset. Normally, there are various possible assets available, e.g., all stocks at a stock exchange. To this end, Scandolo (2004) and Frittelli and Scandolo (2006) introduce risk measures using multiple eligible assets. Artzner, Delbaen, and Koch-Medina (2009) further investigate this concept. Farkas, Koch-Medina, and Munari (2015) name them ***multi-asset risk measures*** and develop finiteness and continuity results as well as dual representations for them. Results for the existence, uniqueness and stability of optimal payoffs with respect to multi-asset risk measures can be found in Baes et al. (2020), Baes and Munari (2020).

As mentioned above, in this chapter we repeat the definition and properties of multi-asset risk measures. Hence, the concepts and results in this chapter are not ours. They should serve as a preparation for the subsequent chapters. This chapter is mainly based on the explanations and results in Farkas et al. (2014, 2015), Munari (2015).

1.1.3 Structure of this chapter

In Section 1.2, we introduce the model for the eligible assets, i.e., we introduce the model for the financial market. In Section 1.3, we describe how to model capital adequacy tests. This allows us to state the definition of multi-asset risk measures. Section 1.4 summarizes well-known results for multi-asset risk measures which we need in the upcoming chapters. We start with the conditions on the capital adequacy test such that the multi-asset risk measure is convex or satisfies even the desirable property of coherence. Then we introduce different sufficient conditions such that the multi-asset risk measure is finite-valued and continuous. A special focus is put on the Value-at-Risk and the Expected Shortfall as prominent examples for capital adequacy tests.

For the sake of completeness, we first recall some terminology which is used throughout the entire dissertation.

1.1.4 Standard notations and concepts

For two sets A, B the expression $A \subset B$ means that A is a subset of B and $A \subsetneq B$ means that A is a proper subset of B. The linear span of a set of vectors $S \subset \mathcal{X}$ is written as $\operatorname{span} S$. For a topological space $(\mathcal{X}, \tau)$ we denote the interior, respectively the closure, of a subset $A \subset \mathcal{X}$ by $\operatorname{int}_\tau(A)$, respectively $\operatorname{cl}_\tau(A)$. We omit the topology τ in these notations if the context is clear.

The epigraph of a map $f : \mathcal{X} \to [-\infty, \infty]$ is $\operatorname{epi}(f) := \{(X, \alpha) \in \mathcal{X} \times \mathbb{R} \,|\, f(X) \leq \alpha\}$. The kernel of a linear map f on a linear space $\mathcal{X}$ is $\ker(f) := \{X \in \mathcal{X} \,|\, f(X) = 0\}$.

To achieve a quite general definition of a risk measure, we use ordered topological vector spaces, compare also Appendix A. An ordered topological vector space $(\mathcal{X}, \tau, \preceq)$, or $\mathcal{X}$ for short, means that $(\mathcal{X}, \tau)$ is a topological vector space and $(\mathcal{X}, \preceq)$ is an ordered vector space and its positive cone $\mathcal{X}_+ := \{X \in \mathcal{X} \,|\, 0 \preceq X\}$ is τ-closed.

Assume a probability space $(\Omega, \mathscr{F}, P)$. The linear space of equivalence classes of random variables on it is denoted by $L^0(\Omega, \mathscr{F}, P)$. For a number $p \in (0, \infty)$ the linear space of equivalence classes of p-integrable random variables is denoted by $L^p(\Omega, \mathscr{F}, P)$, or L^p for short. For $p = \infty$ we use the analogous symbols to denote the linear space of equivalence classes of essentially bounded random variables. We always equip these linear spaces with the usual L^p-norm and the P-almost sure order. The latter is denoted by $\geq$. Note, every space L^p with $p \in [1, \infty]$ is a Banach lattice and therefore an ordered topological vector space, see Aliprantis and Border (2006, Theorem 13.5).

The upper quantile function of a random variable X is denoted by q_X^+.

1.2 Financial positions

Assume an ***agent*** who is interested in the risk of her daily business. Such an agent could be a private investor or a financial institution, e.g., an insurance company or a bank. This agent interacts with her environment. For instance, a private investor or a financial institution can trade on a stock exchange. In this section, we introduce a framework for such interactions.

1.2.1 Model space

The agent deals with unforeseeable future developments. For instance, a private investor does not know the future stock prices or an insurance company does not know the number and severity of damages in an upcoming year, e.g., think of a private liability insurance.

We describe these unforeseeable developments in a mathematical way, i.e., we describe random outcomes in the future. To do so, we have to clarify the objects of interest. We name the following two possibilities:

(i) Fix a future time and consider the value of positions at it, e.g., the value of a stock in one month or the equity capital of a financial institution in one year. This is a *static* point of view. The objects of interest are *random variables*.

(ii) We could also consider the development of a position over time, e.g., the wealth of a portfolio over one month or the equity capital over one year. This is a *dynamic* point of view. The objects of interest are *stochastic processes*.

Remark 1.2.1. Additionally, we could consider different fields of business simultaneously, e.g., the balance sheets of subsidiaries of a financial institution. We could either use a static or a dynamic point of view. The objects of interest in the static case are *vectors* of random variables and in the dynamic case are *multi-dimensional* stochastic processes.

Baes et al. (2020, page 129) point out that one uses "*[...] typically a space of random variables*". Motivated by this quote, most of the time we focus on random variables. This is reflected by the choice of our examples. Notwithstanding, especially the results in Chapter 3 are stated in a general framework. Hence, we start by assuming an ordered topological vector space $(\mathcal{X}, \tau, \preceq)$. For the sake of brevity, we call it a ***model space***. The elements of $\mathcal{X}$ are called ***financial positions*** or ***payoffs***.

We give economic interpretations for the topological vector space $\mathcal{X}$:

- The vector addition and the scalar multiplication of $\mathcal{X}$ allows us to sum up and scale payoffs. For example, the agent can buy stocks which are not yet part of her portfolio (summation) or she only holds shares of a single company and buys more shares of this company (scaling).

- Among others, the topology τ is used to analyze if a small change in the financial position leads to a small change in the capital requirement (continuity). Continuity is a desirable property for risk assessment.

- The partial order $\preceq$ says if a payoff is strictly larger than another one. This should be reflected by the capital requirements. Further, it allows us to introduce the set of positive, respectively strictly positive, elements $\mathcal{X}_+$, respectively $\mathcal{X}_{++}$. For the formal definition of the latter set, see Appendix A.

1.2.2 Marketed space

A capital requirement is the minimal amount of money that needs to be invested in eligible assets to pass the capital adequacy test. The term eligible assets covers every asset that the agent can reallocate in a short time period. Typical examples of eligible assets are a bank account, bonds or stocks. But not every payoff in $\mathcal{X}$ arises from trading in eligible assets, e.g., the equity capital of a company. So, we have to specify the payoffs that can be created by trading in eligible assets. To this end, we introduce a linear subspace of $\mathcal{X}$ that represents all of these payoffs.

We denote this linear subspace by $\mathcal{M}$. It contains every payoff that can be created by trading in the eligible assets that represents $\mathcal{M}$. For instance, if the agent solely buys stocks, then the corresponding payoff is an element of $\mathcal{M}$. We call $\mathcal{M}$ the ***marketed space*** or for illustration sometimes a ***stock exchange***. An element of it is a ***marketed portfolio***, ***marketed payoff*** or a ***portfolio*** for short.

A financial position in $\mathcal{X}\backslash\mathcal{M}$ can not be traded on the stock exchange $\mathcal{M}$, e.g., think of the damage to an insurance company from its operative business. But it can also be a combination of different financial positions, e.g., an insurance company holds different assets, e.g., stocks or real estates, but has also to pay for future insurance claims. Some of these assets could be traded at a stock exchange, but for example, real estates as illiquid assets are not traded.

To obtain desirable properties of risk measures, we state the following permanent assumption:

Assumption 1.2.2. $\mathcal{M}$ *contains a non-zero positive element.*

Finally, in the following remark we discuss the condition that the marketed space is a linear space in view of trading.

Remark 1.2.3. A discrete-time financial market model only allows for trading at finitely many time points. In this case, the assumption of a linear subspace $\mathcal{M}$ is satisfied in general. For continuous-time financial market models we have to be more careful in defining the trading strategies that are admissible, because we have to exclude so-called doubling strategies. By executing a doubling strategy, the agent creates "gain out of nothing". The usual trick to exclude doubling strategies is the restriction to trading strategies such that the corresponding wealth process is uniformly bounded from below, see e.g., Delbaen and Schachermayer (2006). But this leads not any longer to a linear space $\mathcal{M}$ in general.

1.2.3 Pricing functional

Recall that $\mathcal{M}$ consists of the payoffs with respect to trading eligible assets. These assets are liquid and hence, the agent knows the actual price of the resulting payoffs. We model these prices by a map $\pi : \mathcal{M} \to \mathbb{R}$ which we call a ***pricing functional***. Note, the pricing functional takes values on the real line, i.e., we work under the assumption that prices are finite-valued.

Sometimes it is helpful to consider only the marketed payoffs at a certain price $x \in \mathbb{R}$. For this purpose, we write

$$\mathcal{M}_x := \{X \in \mathcal{M} \,|\, \pi(Z) = x\}\,.$$

Hence, for price zero it holds that $\mathcal{M}_0 = \ker(\pi)$.

We would like to work in an efficient market, i.e., it should not be possible to create gain without any risk. To describe this situation for the marketed space $\mathcal{M}$ we introduce the notion of an arbitrage opportunity.

Definition 1.2.4 (Arbitrage opportunity)**.** *A non-zero positive element $M \in \mathcal{M}$ with $\pi(M) \leq 0$ is called an arbitrage opportunity.*

To work in an arbitrage-free market, we make the following standing assumption:

Assumption 1.2.5 (Free of arbitrage)**.** *The marketed space is free of arbitrage opportunities, i.e., the pricing functional is a strictly positive functional which means that for all non-zero positive $M \in \mathcal{M}$ it holds that $\pi(M) > 0$.*

Finally, we would like to work in a liquid market. This is described by the linearity of the pricing functional, i.e., the price of sums of scaled portfolios is equal to the sums of the scaled prices of the portfolios.

Assumption 1.2.6 (Liquidity)**.** *The pricing functional π is a linear functional, i.e., for each $\alpha, \beta \in \mathbb{R}$ and $Z, U \in \mathcal{M}$ it holds that $\pi(\alpha Z + \beta U) = \alpha\pi(Z) + \beta\pi(U)$.*

1.3 Definition of multi-asset risk measures

In this section, we explain the calculation of capital requirements. We use the classical motivation of a capital adequacy test imposed by some regulatory agency as it is done in the seminal paper of Artzner et al. (1999). In a first step, we model the capital adequacy test and present the most prominent examples based on the Value-at-Risk and the Expected Shortfall. Then we define multi-asset risk measures.

1.3.1 Acceptance sets

For the following motivation, we assume that the agent is a bank or an insurance company. Hence, the agent has to pass a capital adequacy test. In doing so, the agent is restricted to payoffs that pass this test. We say that such payoffs are ***acceptable*** from the regulator's perspective.

The following concept describes the capital adequacy test stipulated by the regulator. To introduce special kinds of such tests, we make use of two properties for sets. First, note that a set $\mathcal{C} \subset \mathcal{X}$ is ***convex***, if for each $\alpha \in (0,1)$ it holds that $\alpha\mathcal{C} + (1-\alpha)\,\mathcal{C} \subset \mathcal{C}$. Second, $\mathcal{C}$ is a ***cone***, if for each $\alpha \in [0,\infty)$ it holds that $\alpha\mathcal{C} \subset \mathcal{C}$.

Definition 1.3.1 (Acceptance sets)**.** *An **acceptance set** $\mathcal{A}$ is a subset $\mathcal{A} \subset \mathcal{X}$ satisfying the following properties:*

(i) Non-triviality: $\mathcal{A} \neq \emptyset$, $\mathcal{A} \neq \mathcal{X}$.

(ii) Monotonicity: For $X \in \mathcal{X}$, $Y \in \mathcal{A}$ with $Y \preceq X$ it follows that $X \in \mathcal{A}$.

If $\mathcal{A}$ is additionally convex, respectively a convex cone, then we call it a convex, respectively coherent, acceptance set.

Non-triviality means on the one hand that there exist financial positions, that are accepted by the regulator, and on the other hand that there exist financial positions, that the regulator does not tolerate. Monotonicity states that financial positions, that are better than already acceptable financial positions, are also acceptable.

Convexity says that mixtures between acceptable financial positions remain acceptable. In Chapter 3 this will be important to guarantee that diversification between financial positions is rewarded. This means that diversifying between two financial positions does not destroy acceptability and the capital reserve is reduced.

Conicity describes a kind of liquidity. If the agent is able to scale her acceptable payoffs arbitrarily and they remain acceptable, then this scaling property carries over to the multi-asset risk measures.

In Chapters 2 and 4 we need another property. It describes a link between the marketed space, the pricing functional and an acceptance set. We state this link in the following definition which goes back to Jaschke and Küchler (2001).

Definition 1.3.2 (Good deals of the first kind)**.** *Assume an acceptance set $\mathcal{A}$. We call a payoff $X \in \mathcal{X}$ a **good deal of the first kind,** if $X \in (\mathcal{A} \cap \ker(\pi)) \setminus \{0\}$. The marketed space $\mathcal{M}$ is **free of good deals of the first kind,** whenever the following condition is* *satisfied:*

$$(\mathcal{A} \cap \ker(\pi)) \backslash \{0\} = \varnothing.$$

Remark 1.3.3. A good deal is a marketed payoff that is too profitable for an agent. As stated in Baes et al. (2020, page 138), "*[...] there is no reason to expect any consistency between the financial market, where eligible assets are traded, and the capital adequacy test*", as it is supposed by the absence of good deals of the first kind condition. Nevertheless, one could expect that the government strives for a capital adequacy test that is consistent with the actual financial market. Hence, the government would like to ensure that the market does not allow for good deals of the first kind.

1.3.2 Monetary risk measures

In this section, we introduce the most important acceptance sets in practice. Many of them are given by an appropriate function $f : \mathcal{X} \to \mathbb{R}$ which leads to an acceptance set of the following form:

$$\mathcal{A} = \{X \in \mathcal{X} \mid f(X) \leq 0\}.$$

Functions f with desirable and economically meaningful properties are so-called monetary risk measures. We define them as follows:

Definition 1.3.4 (Monetary risk measures)**.** *A function $\rho : \mathcal{X} \to (-\infty, \infty]$ is called a **monetary risk measure**, if for all $X, Y \in \mathcal{X}$ the following properties hold:*

(i) Finiteness at 0*:* $\rho(0) \in \mathbb{R}$.

(ii) Monotonicity: $X \preceq Y$ *implies* $\rho(X) \geq \rho(Y)$.

(iii) Cash invariance: For all $m \in \mathbb{R}$ *we have* $\rho(X + m) = \rho(X) - m$.

A monetary risk measure is called convex, if for all $X, Y \in \mathcal{X}$ *it holds:*

(iv) Convexity: For $\alpha \in (0, 1)$ *we have* $\rho(\alpha X + (1 - \alpha)Y) \leq \alpha\rho(X) + (1 - \alpha)\rho(Y)$.

A convex monetary risk measure is called coherent, if for all $X \in \mathcal{X}$ *it holds:*

(v) Positive homogeneity: For $\alpha \geq 0$ *we have* $\rho(\alpha X) = \alpha\rho(X)$.

Next, we formally introduce the idea of an acceptance set specified via a monetary risk measure.

Definition 1.3.5 (Acceptance sets based on monetary risk measures)**.** *The acceptance set based on a monetary risk measure* ρ *is defined by*

$$\mathcal{A}_\rho := \{X \in \mathcal{X} \mid \rho(X) \leq 0\}.$$

It is easy to verify that $\mathcal{A}_\rho$ is indeed an acceptance set in the sense of Definition 1.3.1. The following examples are prominent monetary risk measures which are often used in practice.

Example 1.3.6 (Value-at-Risk and Expected Shortfall)**.** Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\lambda \in (0, 1)$.

(i) Assume $\mathcal{X} \subset L^0(\Omega, \mathscr{F}, P)$ and $X \in \mathcal{X}$. The ***Value-at-Risk (VaR)*** at level λ of X is $\mathrm{VaR}_\lambda(X) := -q_X^+(\lambda)$. Recall that $q_X^+(\lambda)$ denotes the upper quantile function of X at level λ. VaR is a monetary risk measure in the sense of Definition 1.3.4. The VaR acceptance set is $\mathcal{A}_{\mathrm{VaR}_\lambda} = \{X \in \mathcal{X} \mid \mathrm{VaR}_\lambda(X) \leq 0\}$. It is a closed cone which is not convex in general.

(ii) Assume $\mathcal{X} \subset L^1(\Omega, \mathscr{F}, P)$ and $X \in \mathcal{X}$. The ***Expected Shortfall (ES)*** of X at level λ is $\mathrm{ES}_\lambda(X) := -\frac{1}{\lambda} \int_0^\lambda q_X^+(u)\,\mathrm{d}u$. ES is a coherent monetary risk measure in the sense of Definition 1.3.4. The ES acceptance set $\mathcal{A}_{\mathrm{ES}_\lambda} = \{X \in \mathcal{X} \mid \mathrm{ES}_\lambda(X) \leq 0\}$ is also coherent.

Remark 1.3.7. VaR and ES are used in several regulatory frameworks, e.g., the Basel Accords or Solvency II. There is an ongoing debate which of these monetary risk measures should be preferred. For instance, Wang and Zitikis (2020) introduce an intuitive set of four axioms to describe economic intentions in a risk assessment framework, namely monotonicity, law-invariance, prudence and no reward for concentration (NRC). The main intention of the regulator is reflected by the NRC axiom. It states the following: If two payoffs realize large losses in the case of a specific stress scenario, then no diversification effects occur and the capital requirement of the sum of the payoffs is equal to the sum of the individual capital requirements, i.e., no capital reduction is possible. A functional $\rho : L^1 \to \mathbb{R}$ satisfies the four axioms if and only if it is the ES at a specific level, see e.g., Wang and Zitikis (2020, Theorem 1). Hence, the ES is the only monetary risk measure satisfying these economic desirable axioms. This is one advantage of using ES instead of VaR to define a capital adequacy test. For the model used in Sections 2.4 and 2.5 we obtain other desirable properties of ES when pricing financial derivatives.

But there are also disadvantages by using ES instead of VaR. A bunch of them are given in Koch-Medina and Munari (2016). In particular, they argue that the use of ES instead of VaR can lead to more extreme risk profiles.

In Chapter 2 we apply monetary risk measures that are comonotonic. This property is introduced in the following definition. For more information about comonotonic risk measures we refer to Föllmer and Schied (2016, Section 4.7).

Definition 1.3.8 (Comonotonicity)**.** *Assume a probability space* $(\Omega, \mathscr{F}, P)$*. We say that* $X, Y \in L^0(\Omega, \mathscr{F}, P)$ *are* ***comonotone*** *if there exists a* $P \otimes P$*-null set* $N \subset \mathscr{F} \otimes \mathscr{F}$ *such that for all* $(\omega, \omega') \in \Omega \times \Omega \backslash N$ *it holds that*

$$\left(X(\omega) - X(\omega')\right)\left(Y(\omega) - Y(\omega')\right) \geq 0.$$

If $\mathcal{X} \subset L^0(\Omega, \mathscr{F}, P)$*, then we call a monetary risk measure* $\rho : \mathcal{X} \to \mathbb{R}$ ***comonotonic*** *if*

$$\rho(X + Y) = \rho(X) + \rho(Y)$$

whenever $X, Y \in \mathcal{X}$ *are comonotone.*

1.3.3 Multi-asset risk measures

Now we incorporate the concepts introduced so far and calculate capital requirements. The classical definition of a risk measure is based on a single eligible asset. But in this dissertation we state new results for risk measures using multiple eligible assets.

Definition 1.3.9 (Multi-asset and single-asset risk measures)**.** *Assume an acceptance set* $\mathcal{A}$*. The* ***multi-asset risk measure*** *for* $X \in \mathcal{X}$ *is defined by*

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) \coloneqq \inf\{\pi(Z) \,|\, Z \in \mathcal{M}, X + Z \in \mathcal{A}\}.$$

If $Z \in \mathcal{X}$ *such that* $\mathcal{M} = \operatorname{span}\{Z\}$*, then we write* $\rho_{\mathcal{A},Z,\pi} \coloneqq \rho_{\mathcal{A},\mathcal{M},\pi}$ *for short and call it a* ***single-asset risk measure****.*

A multi-asset risk measure determines the capital requirement as the minimal costs with respect to marketed payoffs such that the sum of the actual payoff and a marketed payoff becomes acceptable.

Note that the multi-asset risk measure is not finite-valued in general, i.e., it can attain $-\infty$ or $+\infty$. Hence, a common assumption is the one of a finite-valued multi-asset risk measure. For example, Liebrich and Svindland (2017) assume that $\rho_{\mathcal{A},\mathcal{M},\pi}(X) > -\infty$ for all $X \in \mathcal{X}$. Such an assumption avoids economically implausible situations, as Farkas et al. (2014, page 148) point out [1]:

[1] Farkas et al. (2014) denote by $\rho_{\mathcal{A},S}$ the single-asset risk measure with respect to an eligible asset $S = (S_0, S_T)$ with actual price S_0 and payoff S_T at future time T.

> *[...] if $\rho_{\mathcal{A},S}(X) = -\infty$, then we could extract arbitrary amounts of capital without compromising the acceptability of X, which is financially implausible.*

In the opposite case, i.e., the risk measure attains $+\infty$, it is not possible to pass the capital adequacy test, see Farkas et al. (2014, page 148):

> *[...] if $\rho_{\mathcal{A},S}(X) = +\infty$ [...], then X cannot be made acceptable by raising any amount of capital and investing it in the reference asset S.*

Hence, sufficient and necessary conditions for risk measures to be finite-valued are important. In addition, finiteness is often a prerequisite to prove continuity of a risk measure. Continuity deserves a similar degree of attention as finiteness, as the following quote by Farkas et al. (2014, page 148) tells us:

> *[...] if $\rho_{\mathcal{A},S}$ fails to be continuous at some position X, then a slight change or misstatement of X might lead to a dramatically different capital requirement.*

Continuity is also important to obtain so-called dual or robust representations of risk measures. We also would like to motivate this property by a quote, see Farkas et al. (2014, page 148):

> *[...] continuity is also a useful property in the context of dual representations, which play an important role in optimization problems, for instance arising in connection to portfolio selection.*

The above discussion motivates us to develop new ***finiteness*** and ***continuity*** results for risk measures. For the new risk measure in Chapter 3 we also develop a ***dual representation*** which allows for an interesting economic interpretation.

In the upcoming section, we state well-known finiteness and continuity results for multi-asset risk measures. We need them in the subsequent chapters.

1.4 Properties of multi-asset risk measures

In this section, we collect well-known results for multi-asset risk measures. We start in Section 1.4.1 with direct consequences from Definition 1.3.1 of the acceptance set. They allow for important economic interpretations pointing out their relevance in risk assessment. In Section 1.4.2, we state sufficient conditions for a multi-asset risk measure to be finite-valued and continuous. Finally, in Section 1.4.2, we give tailor-made results for the special cases of VaR and ES acceptance sets.

1.4.1 Convexity and coherence

In the theory of single-asset risk measures, the coherence property (compare Definition 1.3.4) is of major importance and is extensively studied. This property is introduced and examined by Artzner et al. (1999). The authors argued that coherence is a desirable property in the context of risk management. The next well-known result characterizes the coherence of a multi-asset risk measure, see e.g., Farkas et al. (2015, Lemma 2).

Lemma 1.4.1. *For an arbitrary acceptance set $\mathcal{A} \subset \mathcal{X}$ the corresponding multi-asset risk measure $\rho_{\mathcal{A},\mathcal{M},\pi}$ satisfies the following properties:*

(i) Monotonicity: $\rho_{\mathcal{A},\mathcal{M},\pi}$ is decreasing.

(ii) $\mathcal{M}$-invariance: For all $X \in \mathcal{X}, Z \in \mathcal{M}$ the following translation property holds:

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X + Z) = \rho_{\mathcal{A},\mathcal{M},\pi}(X) - \pi(Z).$$

(iii) Convexity: If $\mathcal{A}$ is convex, then the epigraph $\mathrm{epi}(\rho_{\mathcal{A},\mathcal{M},\pi})$ *is convex.*

(iv) Positive homogeneity: If $\mathcal{A}$ is a cone, then the epigraph $\mathrm{epi}(\rho_{\mathcal{A},\mathcal{M},\pi})$ *is a cone.*

Remark 1.4.2. For consistency note that the convexity, respectively positive homogeneity, in Definition 1.3.4 is equivalent to the convexity, respectively conicity, of the epigraph $\mathrm{epi}(\rho)$ of the monetary risk measure ρ.

By monotonicity, a larger payoff leads to a smaller capital requirement. For the $\mathcal{M}$-invariance, remember that marketed portfolios are traded on the stock exchange $\mathcal{M}$ and they are used to hedge the financial position. Hence, if the agent buys new shares of assets on this stock exchange, then the value of the multi-asset risk measure can be reduced by the costs of this portfolio. Convexity states that diversification, i.e., building a mixture of different financial positions, should not increase the capital requirement. Positive homogeneity says that risk scales with the size of the financial position. This reflects that the market $\mathcal{M}$ is liquid. For large scaling factors this assumption is often questionable.

Based on the properties in Lemma 1.4.1 we can introduce the coherence property for multi-asset risk measures:

Definition 1.4.3 (Coherent multi-asset risk measures)**.** *For a coherent acceptance set $\mathcal{A}$ we say that $\rho_{\mathcal{A},\mathcal{M},\pi}$ is coherent.*

1.4.2 Finiteness and continuity

We recall important results from Farkas et al. (2015) describing the finitness and continuity of multi-asset risk measures. In the first result, a convex acceptance set is used.

Proposition 1.4.4 (Farkas et al. (2015, Proposition 2))**.** *Let $\mathcal{A}$ be a convex acceptance set with non-empty interior. Assume $\mathcal{M}$ contains a strictly positive element U. The following statements hold:*

(i) For all $X \in \mathcal{X}$ it holds that $\rho_{\mathcal{A},\mathcal{M},\pi}(X) < \infty$.

(ii) The following assertions are equivalent:

(a) For all $X \in \mathcal{X}$ it holds that $\rho_{\mathcal{A},\mathcal{M},\pi}(X) > -\infty$.

(b) $\rho_{\mathcal{A},\mathcal{M},\pi}(0) > -\infty$.

(c) For some $x \in \mathbb{R}$ we have $\text{int}(\mathcal{A}) \cap \mathcal{M}_x = \varnothing$.

(d) $\mathcal{A} + \ker(\pi) \neq \mathcal{X}$.

In particular, $\rho_{\mathcal{A},\mathcal{M},\pi}$ is finite-valued if and only if $\rho_{\mathcal{A},\mathcal{M},\pi}(0) > -\infty$. In this case, $\rho_{\mathcal{A},\mathcal{M},\pi}$ is also continuous.

The next result states equivalent conditions for the semicontinuity of a multi-asset risk measure.

Proposition 1.4.5 (Farkas et al. (2015, Proposition 4))**.** *Let $\mathcal{A}$ be an acceptance set and fix $X \in \mathcal{X}$. Then the following statements hold:*

(i) $\rho_{\mathcal{A},\mathcal{M},\pi}$ is lower semicontinuous at X if and only if $X + Z \notin \text{cl}(\mathcal{A} + \ker(\pi))$ for any $Z \in \mathcal{M}$ with $\pi(Z) < \rho_{\mathcal{A},\mathcal{M},\pi}(X)$.

(ii) $\rho_{\mathcal{A},\mathcal{M},\pi}$ is upper semicontinuous at X if and only if $X + Z \in \text{int}(\mathcal{A} + \ker(\pi))$ for any $Z \in \mathcal{M}$ with $\pi(Z) > \rho_{\mathcal{A},\mathcal{M},\pi}(X)$.

In particular, if $\mathcal{A}+\ker(\pi)$ is closed, respectively open, then $\rho_{\mathcal{A},\mathcal{M},\pi}$ is lower, respectively upper, semicontinuous on $\mathcal{X}$.

The following result is helpful to apply Proposition 1.4.5.

Proposition 1.4.6 (Farkas et al. (2015, Proposition 5))**.** *Let $\mathcal{A}$ be a closed acceptance set with $0 \in \mathcal{A}$. Assume that $\mathcal{A}$ is either a cone or convex and let $\mathcal{M}$ be finite-dimensional. If $\mathcal{A} \cap \ker(\pi) = \{0\}$, then $\mathcal{A} + \ker(\pi)$ is closed.*

1.4.3 Value-at-Risk and Expected Shortfall

For the results in Chapter 2, we characterize the interior of the VaR and ES acceptance sets. These results can be found in Farkas et al. (2014). They focus on a static point of view (compare the discussion in Section 1.2.1). Hence, for the rest of this section we fix a probability space $(\Omega, \mathscr{F}, P)$. We start with the VaR acceptance set.

Lemma 1.4.7 (Farkas et al. (2014, Lemma 4.1))**.** *Let* $\lambda \in (0,1)$ *and* $p \in [0,\infty)$. *Assume* $\mathcal{X} = L^p(\Omega, \mathscr{F}, P)$. *The acceptance set* $\mathcal{A}_{\mathrm{VaR}_\lambda}$ *has non-empty interior in* L^p. *Moreover,*

$$\mathrm{int}(\mathcal{A}_{\mathrm{VaR}_\lambda}) = \{X \in L^p \mid P(X \leq 0) < \lambda\}.$$

In particular, for $X \in L^p_+$, *we have* $X \in \mathrm{int}(\mathcal{A}_{\mathrm{VaR}_\lambda})$ *if and only if* $P(X = 0) < \lambda$.

Now we consider the ES acceptance set.

Lemma 1.4.8 (Farkas et al. (2014, Lemma 4.3))**.** *Let* $\lambda \in (0,1)$ *and* $p \in [1,\infty)$. *Assume* $\mathcal{X} = L^p(\Omega, \mathscr{F}, P)$. *The following holds:*

$$\mathrm{int}(\mathcal{A}_{\mathrm{ES}_\lambda}) = \{X \in L^p \mid \mathrm{ES}_\lambda(X) < 0\} \subset \{X \in L^p \mid P(X \leq 0) < \lambda\}.$$

For $X \in L^p_+$, *we have* $X \in \mathrm{int}(\mathcal{A}_{\mathrm{ES}_\lambda})$ *if and only if* $P(X = 0) < \lambda$.

Chapter 2

Good deal bounds under Value-at-Risk and Expected Shortfall constraints

2.1 Introduction

We apply multi-asset risk measures to calculate pricing bounds for European options in a one-period financial market model. This leads to so-called good deal bounds. This chapter is based on Desmettre, Laudagé, and Sass (2020).

2.1.1 Motivation

Pricing in incomplete financial markets is a demanding task for which various approaches exist in the literature. On the one hand, no arbitrage pricing theory can lead to a wide price range. On the other hand, equilibrium theory could give unique prices, but these are not very robust with respect to changes of the model parameters. The theory of ***good deal bounds*** is a compromise between these two concepts. Essentially, ***good deals*** provide portfolios, which are acceptable in terms of the risk measure under consideration, at zero price. Concluding, a good deal bound excludes prices that would lead to arbitrage opportunities and in addition, it encourages a fair trade-off between supply and demand.

An important point is the way of quantifying a good deal. We give an overview of established methods in the literature.

2.1.2 Bibliographic notes

The seminal paper by Cochrane and Saa-Requejo (2000) uses Sharpe ratios to define good deals. This approach is extended in Černý (2003) and Björk and Slinko (2006). Bernardo and Ledoit (2000) look at the gain-loss ratio. Test measures and floors are another approach, see e.g., Carr, Geman, and Madan (2001). This idea is extended to a dynamic framework in Larsen, Pirvu, Shreve, and Tütüncü (2005). Černý and Hodges (2002) bring together all these ideas and present a unifying theory for good deal bounds. They also draw parallels to the classical arbitrage pricing theory. Arai (2011) uses risk measures based on shortfall risk to determine good deal bounds. The strong connection between good deal bounds and risk measures is presented for the first time by Jaschke and Küchler (2001). The extension of their results to a non-coherent set of good deals can be found in Staum (2004). Results for different representations of convex risk measures in a pricing setup are given in Arai and Fukasawa (2014) and Farkas et al. (2015).

2.1.3 Contributions of this chapter

To the best of our knowledge, there is no comprehensive analysis of good deal bounds characterized by acceptance sets based on Value-at-Risk and Expected Shortfall. We would like to fill this gap in an exemplary but standard market setting. To do so, we use multi-asset risk measures.

Complete financial markets lead to unique prices for derivatives. The most prominent example is the classical Black-Scholes model. We are interested in situations in which prices are not unique. This is the case for incomplete financial markets. Note that for discrete-time models the problem of pricing in an incomplete market is rather the rule than the exception. So, we would like to analyze static trading strategies. Inspired by Cochrane and Saa-Requejo (2000), our focus lies on the pricing of European type options in a Black-Scholes market setup without intermediate trading. This one-period model is a notable setup to test pricing rules for derivatives and at the same time it allows to compare our results with the unique price of the Black-Scholes model based on continuous-time trading. Moreover, it leads to formulas for the relevant multi-asset risk measures with good tractability. An interesting future direction of research could be the extension of our results to a setup with intermediate trading.

Our contributions are divided into three main parts, corresponding to Sections 2.3 up to 2.5. In Section 2.3, we specify the marketed space for the multi-asset risk measure. We find an equivalent condition to the so-called absence of acceptability arbitrage condition in Artzner et al. (2009). If this condition is fulfilled, then the multi-asset risk measure of a European option is finite. For its calculation we develop numerical approaches to determine the value and the optimal solution of these risk measures. The benefits of fast algorithms is pointed out in the literature, e.g., in Björk and Slinko (2006, page 253):

> *[. . .] the problem of numerically determining good deal bounds is relatively complex in terms of CPU time, so there is a very clear need to develop fast, approximative good deal pricing algorithms.*

In Section 2.4, we find new relations between the absence of acceptability arbitrage condition and the absence of good deals of the first kind in Jaschke and Küchler (2001). Working under the latter condition, the consideration of the Value-at-Risk and the Expected Shortfall allows us to demonstrate the differences between non-convex and convex pricing bounds. From this we obtain that the seller's good deal bound based on the Value-at-Risk does not behave smooth if we vary the strike of an option. This behavior is triggered by jumps in the optimal solution of the underlying optimization problem. Additionally, in the Value-at-Risk case the seller's bound could become smaller than the buyer's bound even if there are no good deals of the first kind on the market. This is not the case for the Expected Shortfall due to a separation argument, see Section 2.5.

Our results allow to conclude that pricing with respect to the non-convex bounds is problematic, whereas the good deal bounds based on Expected Shortfall provide a reasonable way for option pricing in incomplete markets.

2.1.4 Structure of this chapter

This chapter is organized as follows: In Section 2.2.1, we introduce the financial market and repeat some well-known results within it. In Section 2.2.2, we specify the multi-asset risk measures for this specific market model. Sections 2.3, 2.4 and 2.5 include our findings for good deal bounds based on Value-at-Risk and Expected Shortfall as outlined above.

2.1.5 Standard notations and concepts

In addition to Section 1.1.4, we use the following standard notations and concepts: The complement of a set A is written as $A^{\complement}$. For $n \in \mathbb{N}$ the inner product of two points $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is written as $\|x\|_2$. The unit sphere is defined by $\partial\, \mathrm{B}_{\mathbb{R}^n} := \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$.

The distribution function of a random variable X is denoted by F_X. We denote with ϕ and Φ the density and the distribution function of a standard normal distributed random variable.

2.2 One-period financial market model for multi-asset risk measures

In this section, we introduce the financial market in which we calculate pricing bounds.

2.2.1 Financial market model

We use a Black-Scholes market setup with one risk-free and one risky asset. For the classical Black-Scholes model we refer to Black and Scholes (1973). We consider financial positions up to a finite time $T > 0$. A filtered probability space $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P\right)$ with $\mathscr{F}_T = \mathscr{F}$ is used. The financial market consists of a bank account B and a stock price process S. The bank account has initial price $B_0 > 0$ and interest rate $r \in \mathbb{R}$. The stock price is modeled via a geometric Brownian motion with initial price $S_0 > 0$, drift $b \in \mathbb{R}$, volatility $\sigma > 0$ and Brownian motion W. The values of the bank account and the stock price at time $t \in [0, T]$ are

$$B_t = B_0 \exp(rt), \; S_t = S_0 \exp\left(\left(b - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right).$$

Remark 2.2.1. The option pricing results in this chapter rely on the use of this financial market model. Nevertheless, some of our results could be extended to a financial market model with multiple stocks modeled as a multi-dimensional geometric Brownian motion. When discussing these results in Remarks 2.3.3 and 2.4.2 we denote by $\hat{S}$ a d-dimensional geometric Brownian motion with $d \in \mathbb{N}$.

To describe the relevant spaces for our pricing problem, we make the following standing assumption throughout this whole chapter:

Assumption 2.2.2. *The model space is given by $\mathcal{X} = L^1(\Omega, \mathscr{F}, P)$. The marketed space is given by $\mathcal{M} = \operatorname{span}\{B_T, S_T\}$. The pricing functional $\pi : \mathcal{M} \to \mathbb{R}$ is defined such that for each $\varphi^B, \varphi^S \in \mathbb{R}$ it holds that $\pi\left(\varphi^B B_T + \varphi^S S_T\right) = \varphi^B B_0 + \varphi^S S_0$.*

Remark 2.2.3. Our results hold for every choice of an L^p-space with $p \in [1, \infty)$. For the sake of simplicity, we use the concrete choice $p = 1$.

As mentioned before, we only allow for trading at time 0, i.e., no intermediate trading is allowed. We call this the ***one-period Black-Scholes model (one-period BSM)***. In a discrete-time setting, an incomplete market is the rule rather than the exception. Based on an additional homogeneity assumption the binomial model in Cox, Ross, and Rubinstein (1979) is the only complete financial market model in discrete time, as pointed out in Föllmer and Schied (2016, Section 5.5). We repeat the well-known finding that the one-period BSM is incomplete, and hence the problem of pricing an option is a demanding task.

Lemma 2.2.4. *The one-period BSM is incomplete and a European call option with strike $K > 0$ is not attainable.*

Proof: The probability space $(\Omega, \mathscr{F}, P)$ supports the random variable S_T which has a continuous distribution. By Föllmer and Schied (2016, Proposition A.31) the probability space $(\Omega, \mathscr{F}, P)$ is atomless. By Föllmer and Schied (2016, Corollary 1.42) it follows that the one-period BSM is incomplete.

The arbitrage-free price range of the European call option is given by the interval $\left(\max\left\{0, S_0 - e^{-rT}K\right\}, S_0\right)$. The second result is then a consequence of Föllmer and Schied (2016, Corollary 1.35). □

2.2.2 Multi-asset risk measures

Throughout the whole chapter, we are interested in acceptance sets based on monetary risk measures. For a monetary risk measure ρ, the agent in the one-period BSM would search for a marketed portfolio $Z \in \mathcal{M}$ such that together with the financial position $X \in \mathcal{X}$ it holds that

$$\rho(X + Z) \leq 0.$$

Recall, the agent wants to achieve the acceptability constraint at minimal costs. This leads to the capital requirement:

$$\begin{aligned}\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(X) &= \inf\{\pi(Z) \,|\, Z \in \mathcal{M}, X + Z \in \mathcal{A}_\rho\} \\ &= \inf\left\{\varphi^B B_0 + \varphi^S S_0 \,\middle|\, \varphi^B, \varphi^S \in \mathbb{R}, \rho(X + \varphi^B B_T + \varphi^S S_T) \leq 0\right\}.\end{aligned}$$

Hence, the agent deposits money in the bank account and buys stock shares. This portfolio remains the same until the end of the trading period. The deterministic interest rate then leads to an analytical tractable calculation of the capital requirement.

2.3 Hedging with multi-asset risk measures

We present several properties of the multi-asset risk measures in our concrete market framework and provide explicit formulas for measuring the risk of European options.

2.3.1 Representation and acceptability arbitrage

We start by showing a new representation for multi-asset risk measures. This is used to determine the risk measures in our examples.

Proposition 2.3.1 (Representation in terms of stock holdings)**.** *Assume a monetary* *risk measure ρ. For every $X \in \mathcal{X}$ we obtain:*

$$\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(X) = \inf_{\varphi\in\mathbb{R}} \left(\varphi S_0 + \frac{\rho(\varphi S_T + X)}{B_T} B_0\right). \tag{2.1}$$

Proof: The claim follows from the cash invariance of the monetary risk measure and the relation

$$\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(X) = \inf\left\{\varphi^B B_0 + \varphi^S S_0 \,\middle|\, \varphi^B, \varphi^S \in \mathbb{R}, \rho(\varphi^S S_T + X) \le \varphi^B B_T\right\}.$$

Hence, the minimum for given $\varphi^S = \varphi$ is attained for $\varphi^B = \rho(\varphi S_T + X)/B_T$. □

Before we focus on acceptance sets based on VaR and ES, we examine an expression that is helpful to determine finiteness and continuity properties of multi-asset risk measures. It is a characterization of the so-called absence of acceptability arbitrage condition in Farkas et al. (2015, Section 3). If this condition is not satisfied, then it is possible to construct acceptable positions with arbitrary negative costs, see Example 2.3.4. This would destroy any reasonable pricing and lead to infinite capital requirements. In the following, we discuss the important concept of the absence of acceptability arbitrage opportunities, which is closely related to the absence of good deals of the first kind. The latter was already introduced in Definition 1.3.2 above.

Theorem 2.3.2 (Absence of acceptability arbitrage)**.** *Assume a positive homogeneous monetary risk measure ρ such that $\rho(S_T)$ and $\rho(-S_T)$ are finite. In the one-period BSM it holds that*

$$\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(0) = \begin{cases} 0 & , \textit{if } \frac{B_T}{B_0} \in \left[\frac{-\rho(S_T)}{S_0}, \frac{\rho(-S_T)}{S_0}\right], \\ -\infty & , \textit{otherwise}. \end{cases} \tag{2.2}$$

The condition (2.2) is equivalent to the existence of a value $m \in \mathbb{R}$ such that

$$\mathcal{A}_\rho \cap \{Z \in \mathcal{M} \,|\, \pi(Z) \le m\} = \varnothing$$

which is the classical definition for absence of acceptability arbitrage.

Proof: By positive homogeneity, we have

$$\varphi S_0 + \frac{\rho(\varphi S_T)}{B_T} B_0 = \begin{cases} \varphi\left(S_0 + \rho(S_T)\frac{B_0}{B_T}\right) & , \varphi > 0, \\ 0 & , \varphi = 0, \\ \varphi\left(S_0 - \rho(-S_T)\frac{B_0}{B_T}\right) & , \varphi < 0. \end{cases}$$

This yields for

$$\rho^{*,1} = \inf_{\varphi>0} \left(\varphi S_0 + \frac{\rho(\varphi S_T)}{B_T} B_0 \right), \ \rho^{*,2} = \inf_{\varphi<0} \left(\varphi S_0 + \frac{\rho(\varphi S_T)}{B_T} B_0 \right),$$

that $\rho^{*,1} = -\infty$ if $\frac{B_T}{B_0} < \frac{-\rho(S_T)}{S_0}$ and 0 otherwise, as well as, $\rho^{*,2} = -\infty$ if $\frac{B_T}{B_0} > \frac{\rho(-S_T)}{S_0}$ and 0 otherwise. This yields the claim, because by Proposition 2.3.1 we get

$$\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(0) = \min \left\{ 0, \rho^{*,1}, \rho^{*,2} \right\}.$$

The second part is a direct consequence of Farkas et al. (2015, Lemma 4). □

Remark 2.3.3. (i) Consider a level $\lambda \in (0, 0.5)$, initial prices $S_0 = B_0 = 1$, zero interest rate and $-\operatorname{ES}_\lambda(S_T) \leq B_T < -\operatorname{VaR}_\lambda(S_T)$. In such a model, VaR leads to acceptability arbitrage opportunities while ES does not.

(ii) Assume that there are more risky assets $\hat{S}$ in the market. In an analogous manner to the proof of Theorem 2.3.2, we could show that the multi-asset risk measure of the zero payoff is zero if and only if for every point in the unit sphere $\varphi \in \partial \mathrm{B}_{\mathbb{R}^d}$ it holds that $\left\langle \frac{B_T}{B_0} \hat{S}_0, \varphi \right\rangle \leq \rho \left(- \left\langle \hat{S}_T, \varphi \right\rangle \right)$. Furthermore, we obtain a representation result by taking the infimum over all elements in $\mathbb{R}^d$ and replacing products with the scalar product in (2.1) if necessary.

Example 2.3.4 (Acceptability arbitrage opportunities)**.** If acceptability arbitrage exists, i.e., for all $m \in \mathbb{R}$, $\mathcal{A}_\rho \cap \{Z \in \mathcal{M} \,|\, \pi(Z) \leq m\} \neq \varnothing$ or equivalently

$$\frac{B_T}{B_0} \notin \left[\frac{-\rho(S_T)}{S_0}, \frac{\rho(-S_T)}{S_0} \right]$$

by Theorem 2.3.2, then there exists a sequence of acceptable marketed portfolios which falls below every negative bound of costs. For instance, if $\frac{B_T}{B_0} < \frac{-\rho(S_T)}{S_0}$, then we build long positions in the stock and short positions in the bank account such that we obtain the sequence of buy and hold trading strategies

$$\left(\varphi_n^B, \varphi_n^S \right)_{n\in\mathbb{N}} = \left(\frac{-n\rho(S_T)}{B_0\rho(S_T) + S_0 B_T}, \frac{-nB_T}{B_0\rho(S_T) + S_0 B_T} \right)_{n\in\mathbb{N}}.$$

By positive homogeneity these strategies lead to $\rho \left(\varphi_n^B B_T + \varphi_n^S S_T \right) = 0$ with prices $\pi \left(\varphi_n^B B_T + \varphi_n^S S_T \right) = -n$.

The previous example shows that if there exists an acceptability arbitrage opportunity, then one can construct portfolio payoffs that have an arbitrary low price but are still acceptable. If there are no such opportunities one can hope for reasonable prices following the idea of acceptability. This is also strongly related with finiteness and continuity of the associated multi-asset risk measures as we analyze in detail in the following.

2.3.2 Finiteness and continuity for VaR and ES

In this subsection, we give basic finiteness and continuity properties for multi-asset risk measures based on VaR and ES acceptance sets. The analysis in this section is inspired by Farkas et al. (2015). Not all of their results are applicable in our concrete setting in combination with VaR acceptance sets. Hence, Proposition 2.3.6 and Theorem 2.3.9 show that the risk measure based on the VaR acceptance set is finite and not globally upper semicontinuous.

We start with the easier case of an ES acceptance set. The coherence of the acceptance set allows us to apply some of the results in Sections 1.4.2 and 1.4.3 in order to show that the multi-asset risk measure is finite and continuous.

Proposition 2.3.5 (Finiteness and continuity of ES)**.** *Assume a level* $\lambda \in (0, 0.5)$ *and let the absence of acceptability arbitrage condition be fulfilled, that is,* $\rho_{\mathcal{A}_{\mathrm{ES}_\lambda},\mathcal{M},\pi}(0) > -\infty$*. Then* $\rho_{\mathcal{A}_{\mathrm{ES}_\lambda},\mathcal{M},\pi}$ *is finite and continuous.*

Proof: Note that the strictly positive elements of L^1 are all random variables that are P-a.s. strictly positive. Hence, S_T is a strictly positive element of $\mathcal{M}$. Further, by Lemma 1.4.8, we see that $S_T \in \operatorname{int}(\mathcal{A}_{\mathrm{ES}_\lambda})$, that is, $\operatorname{int}(\mathcal{A}_{\mathrm{ES}_\lambda}) \neq \varnothing$. Hence, all conditions in Proposition 1.4.4 are satisfied. This leads to the claim. □

In our setup, we can not employ the results of Farkas et al. (2015), to say something about finiteness and continuity of the multi-asset risk measure with respect to VaR. Due to the missing convexity, we could not apply Proposition 1.4.4 for convex or Farkas et al. (2015, Proposition 3) for coherent acceptance sets. Moreover, since the space L^1 does not admit any order unit, we are also not allowed to apply Farkas et al. (2015, Proposition 1). Thus, we present new results for finiteness and continuity of VaR in our concrete setting. For its proof remember that $q_X^+(\lambda)$ denotes the upper quantile of a random variable X at level λ.

Proposition 2.3.6 (Finiteness and lower semicontinuity of VaR)**.** *Let* $\lambda \in (0, 0.5)$*. If* $\frac{B_T}{B_0} \in \left(\frac{-\mathrm{VaR}_\lambda(S_T)}{S_0}, \frac{\mathrm{VaR}_\lambda(-S_T)}{S_0}\right)$*, then the following two statements hold:*

(i) $\rho_{\mathcal{A}_{\mathrm{VaR}_\lambda},\mathcal{M},\pi}$ *is finite.*

(ii) $\rho_{\mathcal{A}_{\mathrm{VaR}_\lambda},\mathcal{M},\pi}$ *is lower semicontinuous.*

Proof: (i) As a direct consequence of Proposition 2.3.1 we have $\rho_{\mathcal{A}_{\mathrm{VaR}_\lambda},\mathcal{M},\pi} < \infty$. Without loss of generality, assume that $S_0 = B_0 = 1$. Let $X \in L^1$ and $\epsilon > 0$ such that $B_T \in \left(q^+_{S_T}(\lambda) + \epsilon, q^+_{S_T}(1-\lambda) - \epsilon\right)$. Choose an arbitrary sequence of real numbers $(\varphi_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$ we obtain:

$$P(X \leq nS_T + \varphi_n(B_T - S_T)) \geq \begin{cases} P\left(\{X \leq nS_T\} \cap \left\{S_T < q^+_{S_T}(\lambda) + \epsilon\right\}\right) & ,\varphi_n \geq 0, \\ P\left(\{X \leq nS_T\} \cap \left\{S_T > q^+_{S_T}(1-\lambda) - \epsilon\right\}\right) & ,\varphi_n < 0. \end{cases}$$

This implies that there exists $n^* \in \mathbb{N}$ such that for every sequence $(\varphi_n)_{n\in\mathbb{N}}$ and every $n > n^*$ we obtain $P(X \leq nS_T + \varphi_n(B_T - S_T)) > \lambda$. This implies $\rho_{\mathcal{A}_{\mathrm{VaR}_\lambda},\mathcal{M},\pi} > -\infty$.

(ii) It holds that $\mathcal{A}_{\mathrm{VaR}_\lambda}$ is closed (see Farkas et al. (2014, Section 4.1)), $0 \in \mathcal{A}_{\mathrm{VaR}_\lambda}$ and $\mathcal{M}$ is finite-dimensional. Further, from our assumption we obtain $\mathcal{A}_{\mathrm{VaR}_\lambda} \cap \ker(\pi) = \{0\}$. Hence, every condition in Proposition 1.4.6 is fulfilled. This implies that $\mathcal{A}_{\mathrm{VaR}_\lambda} + \ker(\pi)$ is a closed set. The lower semicontinuity then follows from Proposition 1.4.5. □

Remark 2.3.7. The condition $\frac{B_T}{B_0} \in \left(\frac{-\mathrm{VaR}_\lambda(S_T)}{S_0}, \frac{\mathrm{VaR}_\lambda(-S_T)}{S_0}\right)$ is further characterized in Section 2.4.

To show that $\rho_{\mathcal{A}_{\mathrm{VaR}_\lambda},\mathcal{M},\pi}$ is not upper semicontinuous, we start by identifying the interior of the set $\mathcal{A}_{\mathrm{VaR}_\lambda} + \ker(\pi)$. In view of Farkas et al. (2014, Section 4.1), we see that

$$\mathcal{A}_{\mathrm{VaR}_\lambda} + \ker(\pi) = \left\{X \in L^1 \,\middle|\, \exists \varphi \in \mathbb{R} : P\left(X + \varphi\frac{S_0}{B_0}B_T - \varphi S_T < 0\right) \leq \lambda\right\}.$$

The following result characterizes $\mathrm{int}(\mathcal{A}_{\mathrm{VaR}_\lambda} + \ker(\pi))$, the interior of $\mathcal{A}_{\mathrm{VaR}_\lambda} + \ker(\pi)$. We omit the proof, because it is analogous to the one of Lemma 1.4.7 for which we refer to Farkas et al. (2014, Lemma 4.1).

Lemma 2.3.8 (Interior of augmented acceptance set)**.** *For a level* $\lambda \in (0, 0.5)$ *we obtain:*

$$\mathrm{int}(\mathcal{A}_{\mathrm{VaR}_\lambda} + \ker(\pi)) = \left\{X \in L^1 \,\middle|\, \exists \varphi \in \mathbb{R} : P\left(X + \varphi\frac{S_0}{B_0}B_T - \varphi S_T \leq 0\right) < \lambda\right\}.$$

To complete the analysis, we show that VaR is not globally continuous in our setting. To do so, we use Proposition 1.4.5 to create a counterexample for upper semicontinuity, i.e., we try to find $X \in \mathcal{X}$ and $Z \in \mathcal{M}$ with $X + Z \notin \mathrm{int}(\mathcal{A}_{\mathrm{VaR}_\lambda} + \ker(\pi))$ but $\rho_{\mathcal{A}_{\mathrm{VaR}_\lambda},\mathcal{M},\pi}(X) < \pi(Z)$.

The proof then demonstrates the difficulty in characterizing continuity properties, if we use a non-convex acceptance set in combination with multiple eligible assets. The reason for this is the complex structure of the augmented acceptance set $\mathcal{A}_{\mathrm{VaR}_\lambda} + \ker(\pi)$

which makes it harder to characterize the points in it.

Theorem 2.3.9 (Absence of upper semicontinuity of VaR)**.** *Let $\lambda \in (0, 0.5)$ and assume that there are no acceptability arbitrage opportunities, i.e., $\frac{B_T}{B_0} \in \left[\frac{-\operatorname{VaR}_\lambda(S_T)}{S_0}, \frac{\operatorname{VaR}_\lambda(-S_T)}{S_0}\right]$. This implies that $\rho_{\mathcal{A}_{\operatorname{VaR}_\lambda},\mathcal{M},\pi}$ is not (globally) upper semicontinuous.*

Proof: Without loss of generality, we assume $S_0 = B_0 = 1$. We use a constructive proof, i.e., we construct $X \in L^1$ and $\varphi^S, \varphi^B \in \mathbb{R}$ with $\varphi^S + \varphi^B > 0$ such that

$$X + \varphi^S S_T + \varphi^B B_T \notin \operatorname{int}(\mathcal{A}_{\operatorname{VaR}_\lambda} + \ker(\pi)).$$

In view of Lemma 2.3.8 this means that for every $\varphi \in \mathbb{R}$ we have that

$$P\left(X + \varphi^S S_T + \varphi^B B_T + \varphi(B_T - S_T) \le 0\right) \ge \lambda.$$

This is equivalent to the validity of the following three cases:

$$\begin{aligned}
\varphi = \varphi^S: &\quad P\left(\left(\varphi^B + \varphi^S\right) B_T + X \le 0\right) && \ge \lambda,\\
\varphi < \varphi^S: &\quad P\left(S_T \le \frac{1}{\varphi - \varphi^S}\left(\left(\varphi^B + \varphi\right) B_T + X\right)\right) && \ge \lambda,\\
\varphi > \varphi^S: &\quad P\left(S_T \ge \frac{1}{\varphi - \varphi^S}\left(\left(\varphi^B + \varphi\right) B_T + X\right)\right) && \ge \lambda.
\end{aligned}$$

For the sake of brevity, we use the following function depending on a scenario $\omega \in \Omega$:

$$f(\varphi;\omega) = \frac{1}{\varphi - \varphi^S}\left(\left(\varphi^B + \varphi\right) B_T + X(\omega)\right).$$

Now we construct a concrete payoff and to do so, we set $C = \left\{S_T < q^+_{S_T}(\lambda)\right\}$ and $D = \left\{S_T > q^+_{S_T}(1-\lambda)\right\}$. The payoff is defined in the following way:

$$X = -\left(\varphi^B + \varphi^S\right)(1_C + 1_D) B_T.$$

For every $\omega \in C \cup D$ it holds that $\left(\varphi^B + \varphi^S\right) B_T + X(\omega) = 0$, that is, $\varphi^B + \frac{X(\omega)}{B_T} = -\varphi^S$. Hence, for each such ω we obtain

$$f(\varphi;\omega) = \frac{\left(\varphi^B + X(\omega)/B_T\right) + \varphi}{\varphi - \varphi^S} B_T = \frac{-\varphi^S + \varphi}{\varphi - \varphi^S} B_T = B_T.$$

Together with the law of total probability this implies the following expressions for our three cases:

$$\begin{aligned}
\varphi = \varphi^S: &\quad P\left(\left(\varphi^B + \varphi^S\right) B_T + X \le 0\right) = P(C \cup D) = 2\lambda \ge \lambda, \\
\varphi < \varphi^S: &\quad P\left(S_T \le f(\varphi; .)\right) \ge P(C)P(S_T \le B_T \,|\, C) \qquad = \lambda, \\
\varphi > \varphi^S: &\quad P\left(S_T \ge f(\varphi; .)\right) \ge P(D)P(S_T \ge B_T \,|\, D) \qquad = \lambda.
\end{aligned}$$

This proves that $X + \varphi^S S_T + \varphi^B B_T \notin \operatorname{int}(\mathcal{A}_{\mathrm{VaR}_\lambda} + \ker(\pi))$. It remains to show that $\rho_{\mathcal{A}_{\mathrm{VaR}_\lambda},\mathcal{M},\pi}(X) < \varphi^S + \varphi^B$. First, assume that $B_T \in \left[q^+_{S_T}(\lambda), q^+_{S_T}(1-\lambda)\right)$ and set $\varphi = -\varphi^B$. Choose $\epsilon \in \left(0, \varphi^S + \varphi^B\right)$. With $\varphi^*(\epsilon) = \left(\varphi^S + \varphi^B\right) / \left(\varphi^S + \varphi^B - \epsilon\right) > 1$ we get for $P^1_\epsilon = P\left(X + \varphi^B B_T + \varphi^S S_T + \varphi(B_T - S_T) - \epsilon S_T < 0\right)$:

$$\begin{aligned}
P^1_\epsilon &= P(S_T < \varphi^*(\epsilon)(1_C + 1_D)B_T) \\
&= P(C)P(S_T < \varphi^*(\epsilon)B_T \,|\, C) + P(D)P(S_T < \varphi^*(\epsilon)B_T \,|\, D) \\
&\quad + P\left(C^\complement \cap D^\complement\right) P\left(S_T < 0 \,\middle|\, C^\complement \cap D^\complement\right).
\end{aligned}$$

The claim $\rho_{\mathcal{A}_{\mathrm{VaR}_\lambda},\mathcal{M},\pi}(X) < \varphi^S + \varphi^B$ follows by the fact that we are able to find $\epsilon > 0$ small enough such that $P(S_T < \varphi^*(\epsilon)B_T \,|\, D) = 0$. This implies that the previous probability is equal to λ.

Secondly, if $B_T = q^+_{S_T}(1-\lambda)$ choose $\epsilon \in \left(0, \varphi^S + \varphi^B\right)$ and $\delta > 0$ arbitrary. Set $\varphi = \varphi^S + \delta$. For the probability $P^2_\delta = P\left(X + \left(\varphi^B + \varphi^S + \delta\right) B_T - (\delta + \epsilon)S_T < 0\right)$, this gives us

$$\begin{aligned}
P^2_\delta &= P\left(S_T > \frac{\varphi^B + \varphi^S + \delta}{\delta + \epsilon} B_T + \frac{1}{\delta + \epsilon} X\right) \\
&= P(C)P\left(S_T > \frac{\delta}{\delta + \epsilon} B_T \,\middle|\, C\right) + P(D)P\left(S_T > \frac{\delta}{\delta + \epsilon} B_T \,\middle|\, D\right) \\
&\quad + P\left(C^\complement \cap D^\complement\right) P\left(S_T > \frac{\varphi^B + \varphi^S + \delta}{\delta + \epsilon} B_T \,\middle|\, C^\complement \cap D^\complement\right).
\end{aligned}$$

Due to the fact that $q^+_{S_T}(\lambda) < q^+_{S_T}(1-\lambda) = B_T$ we are able to choose $\delta > 0$ such that $q^+_{S_T}(\lambda) < \frac{\delta}{\delta+\epsilon} B_T$. This implies that the first term is equal to zero. This completes the proof. □

2.3.3 European options

Now we introduce a new formulation to calculate multi-asset risk measures for European call options. The result relies on the comonotone behavior between the payoffs of the stock and a corresponding call option. To make use of this fact, an acceptance set based on a comonotonic monetary risk measure is assumed. Remember that the concept of a comonotonic monetary risk measure was introduced in Section 1.3.2, recall Definition 1.3.8.

Theorem 2.3.10. *Assume a monetary risk measure ρ that is finite-valued, positive homogeneous and comonotonic. Let $K > 0$. In the one-period BSM the seller's risk measure for the payoff of a European call option $X = (S_T - K)^+$ can be represented by*

$$\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(-X) = \begin{cases} \inf\limits_{\varphi\in[0,1]} \left(\varphi S_0 + \frac{\rho(\varphi S_T - X)}{B_T} B_0\right) & , \frac{B_T}{B_0} \in \left[\frac{-\rho(S_T)}{S_0}, \frac{\rho(-S_T)}{S_0}\right], \\ -\infty & , \textit{otherwise.} \end{cases} \tag{2.3}$$

The corresponding buyer's risk measure can be represented by

$$\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(X) = \begin{cases} \inf\limits_{\varphi\in[-1,0]} \left(\varphi S_0 + \frac{\rho(\varphi S_T + X)}{B_T} B_0\right) & , \frac{B_T}{B_0} \in \left[\frac{-\rho(S_T)}{S_0}, \frac{\rho(-S_T)}{S_0}\right], \\ -\infty & , \textit{otherwise.} \end{cases}$$

Proof: The proof works in the same manner as the proof of Theorem 2.3.2. In the seller's case we use the representation in Proposition 2.3.1 and distinguish the cases $\varphi < 0$, $\varphi > 1$ as well as $\varphi \in [0, 1]$. The risk measure admits the value $-\infty$ if one of the cases $\varphi < 0$ or $\varphi > 1$ becomes relevant. For the case $\varphi < 0$ we use the comonotonicity between $-X$ and φS_T. For the case $\varphi > 1$ the comonotonicity between X and $\varphi S_T - X$ is applied. If there is no acceptability arbitrage in the market, then the expression follows from the case $\varphi \in [0, 1]$. The case of the buyer is analogous to the case of the seller, but now the cases $\varphi < -1$, $\varphi > 0$ and $\varphi \in [-1, 0]$ have to be distinguished. □

Remark 2.3.11. (i) The expression in the infimum in (2.3) admits a clear interpretation. For a number φ of stocks, we end up with a new payoff which is influenced by the risk of S_T, more precisely this payoff is $\varphi S_T - X$. The discounted monetary risk measure of this payoff is the investment into the bank account.

(ii) The condition of comonotonicity is among others satisfied for VaR and ES risk measures.

(iii) The risk measures for a European put option could be determined by using the put-call parity and Theorem 2.3.10.

Finally, we would like to present our first numerical example comparing the multi-asset risk measures based on VaR and ES acceptance sets. The calculation of these risk measures is based on Proposition 2.3.1. The core of the calculation of the seller's risk measure is the following distribution function for $\varphi \in (0,1)$:

$$P\left(\varphi S_T - (S_T - K)^+ \leq x\right) = \begin{cases} 1 - F_{S_T}\left(\frac{K-x}{1-\varphi}\right) & , x < 0, \\ 1 - F_{S_T}\left(\frac{K-x}{1-\varphi}\right) + F_{S_T}\left(\frac{x}{\varphi}\right) & , x \in [0, \varphi K), \\ 1 & , x \geq \varphi K. \end{cases}$$

The VaR in the representation of Proposition 2.3.1 is calculated as quantile with respect to this distribution function. If $\lambda \in \left(1 - F_{S_T}\left(\frac{K}{1-\varphi}\right), 1\right)$ we determine the inverse function numerically using a bisection algorithm.

Example 2.3.12. Assume that the seller and the buyer of a call option agree on a price $p > 0$. Assume that the buyer goes short and the seller goes long in the bank account with respect to this value. Let the bank account have interest rate zero. The initial endowments of both are zero. By $\mathcal{M}$-additivity of the multi-asset risk measures we obtain:

$$\begin{aligned} \textbf{Seller: } & \rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(-X + p1_\Omega) = \rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(-X) - p, \\ \textbf{Buyer: } & \rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(X - p1_\Omega) \quad = \rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(X) + p. \end{aligned}$$

We use the Black-Scholes formula to determine the call price and use the following parameter specification:

$$T = 1,\ B_0 = 1,\ S_0 = 1,\ \lambda = 2.5\%,\ b = 5\%,\ \sigma = 20\%,\ K = 1.05.$$

First, we compare the multi-asset risk measure with the single-asset risk measures based on the bank account and the stock. We plot them for different levels in Figure 2.1. The points correspond to the level $\lambda = 2.5\%$. The plots for the VaR and the ES are quite similar. For the seller, there are significant differences between the three risk measures, i.e., the multi-asset risk measure leads to a significantly smaller value. In contrast, the buyer's multi-asset risk measure is in nearly every case equal to the single-asset risk measure with respect to the bank account. This is due to the fact that the long call position is already acceptable. If we would go short in the bank account, then the VaR and ES of the final payoff would be positive and therefore not acceptable. This means, the single-asset risk measure (dotted line) is equal to p.

For the multi-asset risk measure a short position in the stock is not meaningful, because the option is out-of-the money, that is, it is more likely that the option is not executed and the buyer relies on the loss of the short position. On the opposite, for lower strike values, short positions in the stock are hedged by the executed option with higher probability. This means, a short position in the stock would not necessarily lead to an unacceptable position and it is therefore also possible to reduce the capital requirement under the value of the single-asset risk measure with respect to the bank account.

The single-asset risk measure with respect to the stock (dashed line) is significantly larger than the other two risk measures. This is due to the fact that there is no kind of translation invariance property for the short position of the value p, i.e., there is a high effort to secure the negative amount $-p$.

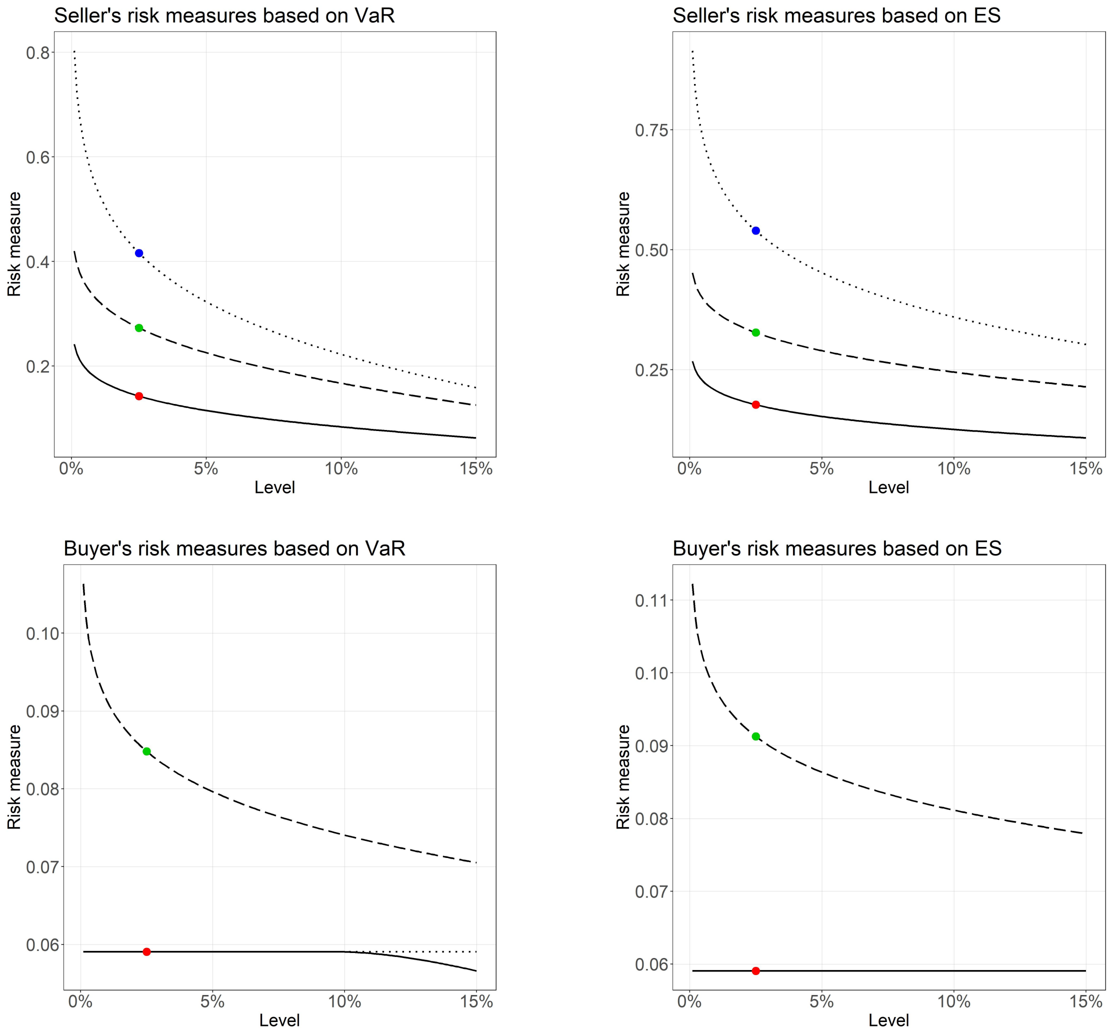


Figure 2.1: Single-asset and multi-asset risk measures depending on the level λ. The dotted (respectively dashed) line corresponds to the single-asset risk measure for the bank account (respectively stock) as eligible asset. The solid line corresponds to the multi-asset risk measure.

To get an intuition of the corresponding hedging strategies, we plot in Figure 2.2 the objective function of the infimum in (2.1) in dependence on the number of stock shares, e.g., for the seller we plot the following function:

$$\mathbb{R} \to \mathbb{R},\ \varphi \mapsto \varphi S_0 + \frac{\rho(\varphi S_T - X)}{B_T} B_0,$$

with $\rho \in \{\mathrm{VaR}_\lambda, \mathrm{ES}_\lambda\}$ and level $\lambda = 2.5\%$. The red point corresponds to the multi-asset risk measure. We see that there exists a unique optimal solution. The blue and the green points are the single-asset risk measures with respect to bank account and stock.

For the seller, hedging with the bank account is most expensive. Further, the multi-asset risk measure attains the minimum in the interior of the interval $[0, 1]$. In contrast, the buyer does not need the additional hedging opportunity given by the stock, that is, the red point is overlapping the blue point.

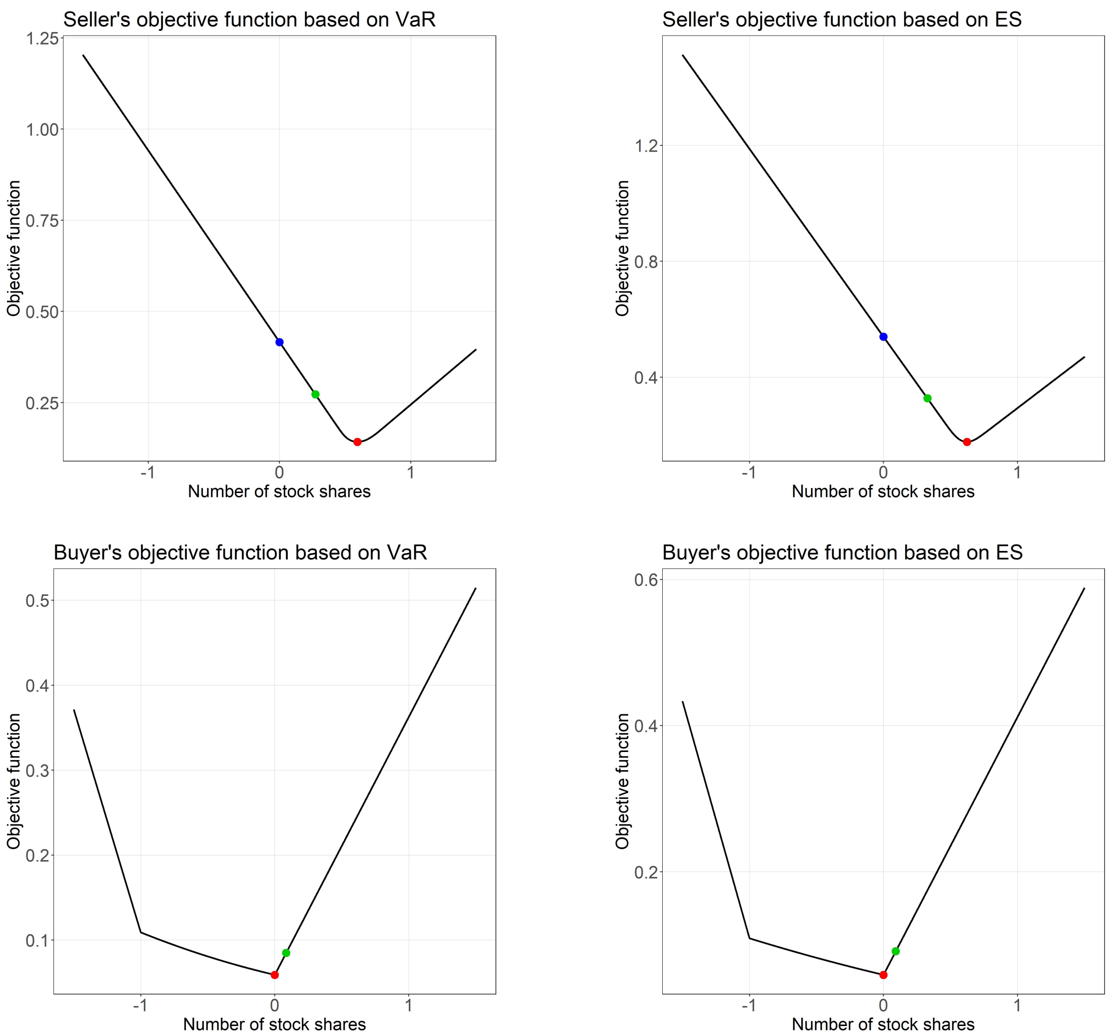


Figure 2.2: Objective function of the infimum in (2.1) depending on the number of stock shares.

2.4 Pricing with multi-asset risk measures

We use the previous results for multi-asset risk measures to obtain pricing bounds for European options. If a price is chosen within these bounds it is guaranteed that there are no trading strategies at zero costs such that the resulting portfolio would be acceptable.

In a first step, we define good deals and illustrate pricing bounds based on VaR and ES acceptance sets. In a second step, in Section 2.5, we show a duality relation. If this relation holds, there is an option price such that there are no good deals on the extended market. The latter is the market that allows for trading of the option.

2.4.1 Good deals of the first kind

Recall the concept of a good deal, see Definition 1.3.2. The following result presents an equivalent condition for the absence of good deals of the first kind in the one-period BSM.

Lemma 2.4.1. *Let ρ be a positive homogeneous monetary risk measure. The absence of good deals of the first kind with respect to the acceptance set $\mathcal{A}_\rho$ is equivalent to the condition that*

$$\frac{B_T}{B_0} \in \left(\frac{-\rho(S_T)}{S_0}, \frac{\rho(-S_T)}{S_0} \right). \tag{2.4}$$

Proof: The absence of good deals of the first kind is equivalent to the condition that for each $\varphi \in \mathbb{R}\backslash\{0\}$ it holds that $\rho(-\frac{\varphi S_0}{B_0} B_T + \varphi S_T) > 0$. By this statement we are able to consider the cases of $\varphi > 0$ and $\varphi < 0$ such that we obtain the required equivalent statement of $\frac{B_T}{B_0} \in \left(\frac{-\rho(S_T)}{S_0}, \frac{\rho(-S_T)}{S_0} \right)$. □

Remark 2.4.2. (i) In a financial market model with multiple stocks $\hat{S}$ the absence of good deals of the first kind is equivalent to the condition that for each $\varphi \in \partial \mathrm{B}_{\mathbb{R}^d}$ it holds that $\left\langle \frac{B_T}{B_0} \hat{S}_0, \varphi \right\rangle < \rho\left(- \left\langle \hat{S}_T, \varphi \right\rangle \right)$, see Proposition 4.5.2.

(ii) Lemma 2.4.1 shows that in our model the absence of good deals of the first kind implies the absence of acceptability arbitrage, since (2.4) implies (2.2) which by Theorem 2.3.2 is equivalent to the absence of acceptability arbitrage opportunities. But this is not true in general, as the subsequent counterexample shows.

Example 2.4.3. We present a setup in which the following implication does not hold in general:

$$(\mathcal{A} \cap \ker(\pi))\backslash\{0\} = \varnothing \Rightarrow \mathcal{A} \cap \mathcal{M}_x = \varnothing \text{ for some } x \in \mathbb{R}.$$

The only case in which this implication could fail is that $\mathcal{A} \cap \ker(\pi) = \{0\}$. We use a one-period BSM with marketed space $\mathcal{M} = \operatorname{span}\{1_\Omega\}$, the pricing functional given by $\pi(\varphi) = \varphi$ and the acceptance set $\mathcal{A} = \bigcup_{m \in \mathbb{R}}\{X \in L^1 \mid X \geq m \ P\text{-a.s.}\}$. Then $\ker(\pi) = \{0\}$ implies that there are no good deals of the first kind, because of

$$\mathcal{A} \cap \ker(\pi) = \mathcal{A} \cap \{0\} = \{0\}\,.$$

But the absence of acceptability arbitrage condition is not fulfilled, because for every $x \in \mathbb{R}$ we have

$$\mathcal{A} \cap \mathcal{M}_x = \mathcal{A} \cap \{x 1_\Omega\} = \{x 1_\Omega\} \neq \varnothing,$$

i.e., the pricing functional π is unbounded on $\mathcal{A} \cap \mathcal{M}$.

2.4.2 Good deal bounds for option prices

In the following, we work under the assumption that the one-period BSM does not admit good deals of the first kind. By Remark 2.4.2 (ii), then also the absence of acceptability arbitrage condition is satisfied. Let $\rho \in \{\mathrm{VaR}_\lambda, \mathrm{ES}_\lambda\}$ with level $\lambda \in (0, 0.5)$. As described in Jaschke and Küchler (2001), the good deal bounds for a call option are then given by $\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(-X)$ and $-\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(X)$. From our previous results we obtain:

$$\begin{aligned}
\textbf{Seller:}\quad \rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(-X) &= \inf_{\varphi \in [0,1]} \left(\varphi S_0 + \rho\left(\varphi S_T - (S_T - K)^+\right)\frac{B_0}{B_T}\right),\\
\textbf{Buyer:}\quad -\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(X) &= -\inf_{\varphi \in [-1,0]} \left(\varphi S_0 + \rho\left(\varphi S_T + (S_T - K)^+\right)\frac{B_0}{B_T}\right).
\end{aligned}$$

Remark 2.4.4. Sub- and superhedging prices could be obtained by using the positive cone L^1_+ as acceptance set. It is possible to rewrite the buyer good deal bound to emphasize the analogy to the subhedging price. For this note that for the VaR it holds that $\mathrm{VaR}_\lambda(X) = -\mathrm{VaR}_{1-\lambda}(-X)$. Then, we get the following expression for the buyer's good deal bound:

$$\begin{aligned}
-\rho_{\mathcal{A}_{\mathrm{VaR}_\lambda},\mathcal{M},\pi}(X) &= -\inf_{\varphi \in [-1,0]} \left(\varphi S_0 + \mathrm{VaR}_\lambda\left(\varphi S_T + (S_T - K)^+\right)\frac{B_0}{B_T}\right)\\
&= -\inf_{\varphi \in [-1,0]} \left(\varphi S_0 - \mathrm{VaR}_{1-\lambda}\left(-\varphi S_T - (S_T - K)^+\right)\frac{B_0}{B_T}\right)\\
&= \sup_{\varphi \in [0,1]} \left(\varphi S_0 + \mathrm{VaR}_{1-\lambda}\left(\varphi S_T - (S_T - K)^+\right)\frac{B_0}{B_T}\right).
\end{aligned}$$

In the following, we calculate good deal bounds for VaR and ES acceptance sets. For the sake of illustration, we choose a large level λ. This leads to plots demonstrating the advantage of the Expected Shortfall over the Value-at-Risk.

Example 2.4.5. In Figure 2.3, we illustrate the good deal bounds for the following parameters:

$$T = 1, \; B_0 = 1, \; r = 0, \; \lambda = 25\%, \; b = 5\%, \; \sigma = 20\%, \; K = 1.$$

The plots show the good deal bounds as a function of the initial price of the underlying stock. Additionally, the prices using the Black-Scholes formula for instantaneous trading and the subhedging prices are given. Note that the no arbitrage price range is $(\max\{0, S_0 - K\}, S_0)$. The plots are inspired by Cochrane and Saa-Requejo (2000, Figure 1).

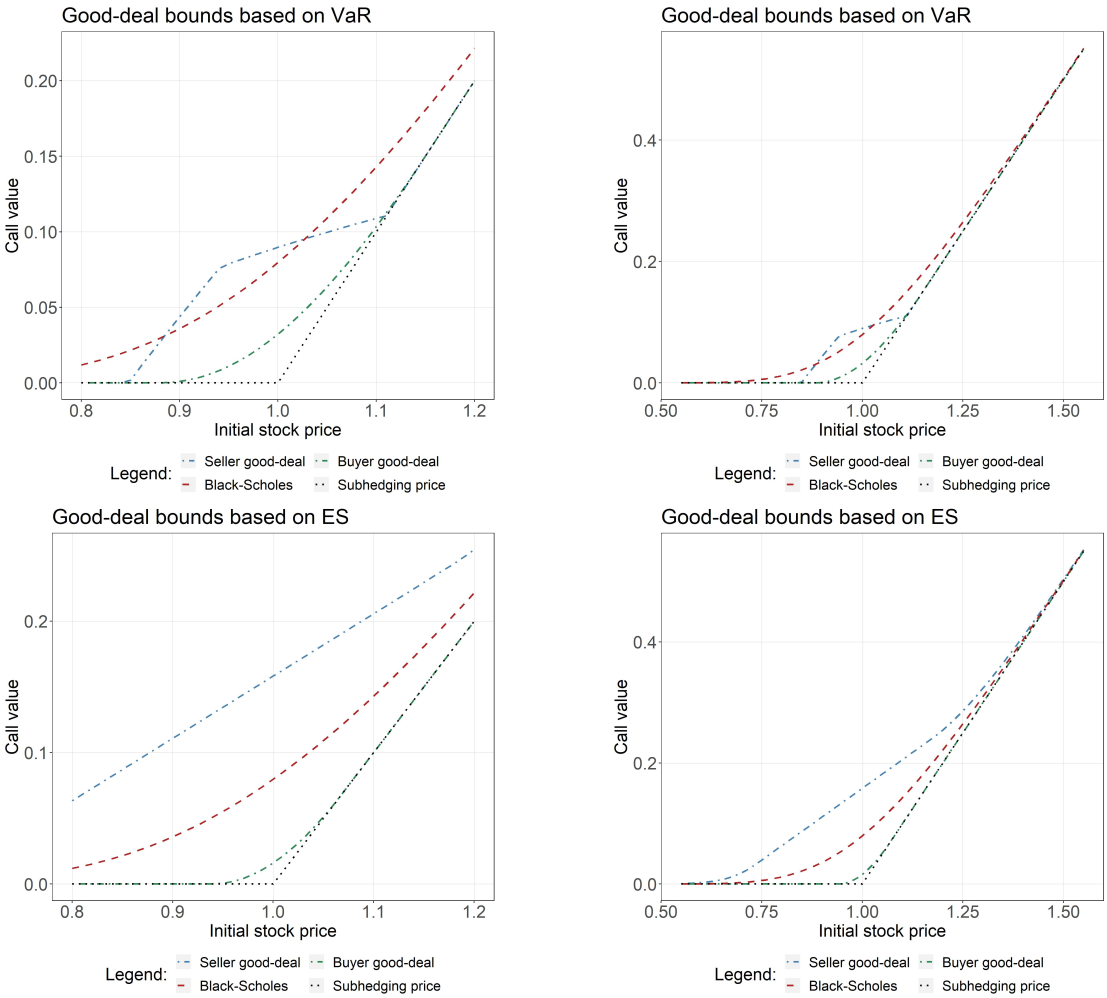


Figure 2.3: Good deal bounds depending on the stock price.

Compared with the Black-Scholes and subhedging price the good deal bounds with respect to the ES do not admit an unexpected behavior. For the VaR the buyer good deal bounds seem plausible. But it is not directly intuitive why the seller's bound is sometimes lower than the buyer's bound. For instance, this is the case for stock prices in the interval $[1.11, 1.15]$. For a specific $\varphi \in [0,1]$ and any $\lambda \in (0, 0.5)$ we obtain:

$$\varphi S_0 + \mathrm{VaR}_{1-\lambda}\left(\varphi S_1 - (S_1 - K)^+\right) \leq \varphi S_0 + \mathrm{VaR}_{\lambda}\left(\varphi S_1 - (S_1 - K)^+\right).$$

Nevertheless, we are not able to conclude that the seller's bound is always greater or equal than the buyer's bound as our counterexample shows. This undesirable behavior occurs if there does not exist an equivalent martingale measure such that the extended market is free of good deals. We will discuss this fact in detail in Section 2.5.2 below.

To provide more intuition on the shape of the seller's good deal bound based on VaR in Figure 2.3, in Figure 2.4 we plot the objective function from the optimization problem of the seller using VaR as preference criterion, that is, we plot the function $\varphi \mapsto \varphi S_0 + \mathrm{VaR}_{\lambda}\left(\varphi S_1 - (S_1 - K)^+\right)$. Each plot corresponds to a specific stock price S_0. We see that the minimum for rising stock prices goes hand in hand with a rising number of stock shares. For the stock prices of 0.9 and 0.95 there are one local and one global minimum. For increasing stock prices in the interval $(0.9, 0.95)$ the stock position belonging to the global minimum changes in a discontinuous manner, because at some stock price local and global minima change their roles. This is the reason for the kink of the seller good deal bound in Figure 2.3.

Finally, we would like to elaborate on the level λ. To this end, we alter the previous parameters in the following way:

$$T = 1,\ B_0 = 1,\ \mathbf{S_0 = 1.12},\ r = 0,\ b = 5\%,\ \sigma = 20\%,\ K = 1.$$

In Figure 2.5, we illustrate the good deal bounds depending on the level λ. For the VaR acceptance set we see that only for high levels the seller's price is lower than the buyer's price.

2.4.3 Limiting behavior

For increasing stock prices the seller's good deal bound in Figure 2.3 tends to the sub-hedging price. In this short subsection, we give the reason for this behavior in case of the VaR. There is an intuitive explanation. If the stock price goes up, then the call option is deeper in-the-money. This results in the following two effects:

- It becomes more likely that the option is exercised, resulting in a loss for the seller.
- Higher losses for the seller become more probable.

Hence, it is more expensive to hedge the short call position. As we have seen in Figure 2.4, the seller buys one stock share for high enough stock prices. Even if the seller goes short in the bank account, the future loss is bounded from below. The resulting

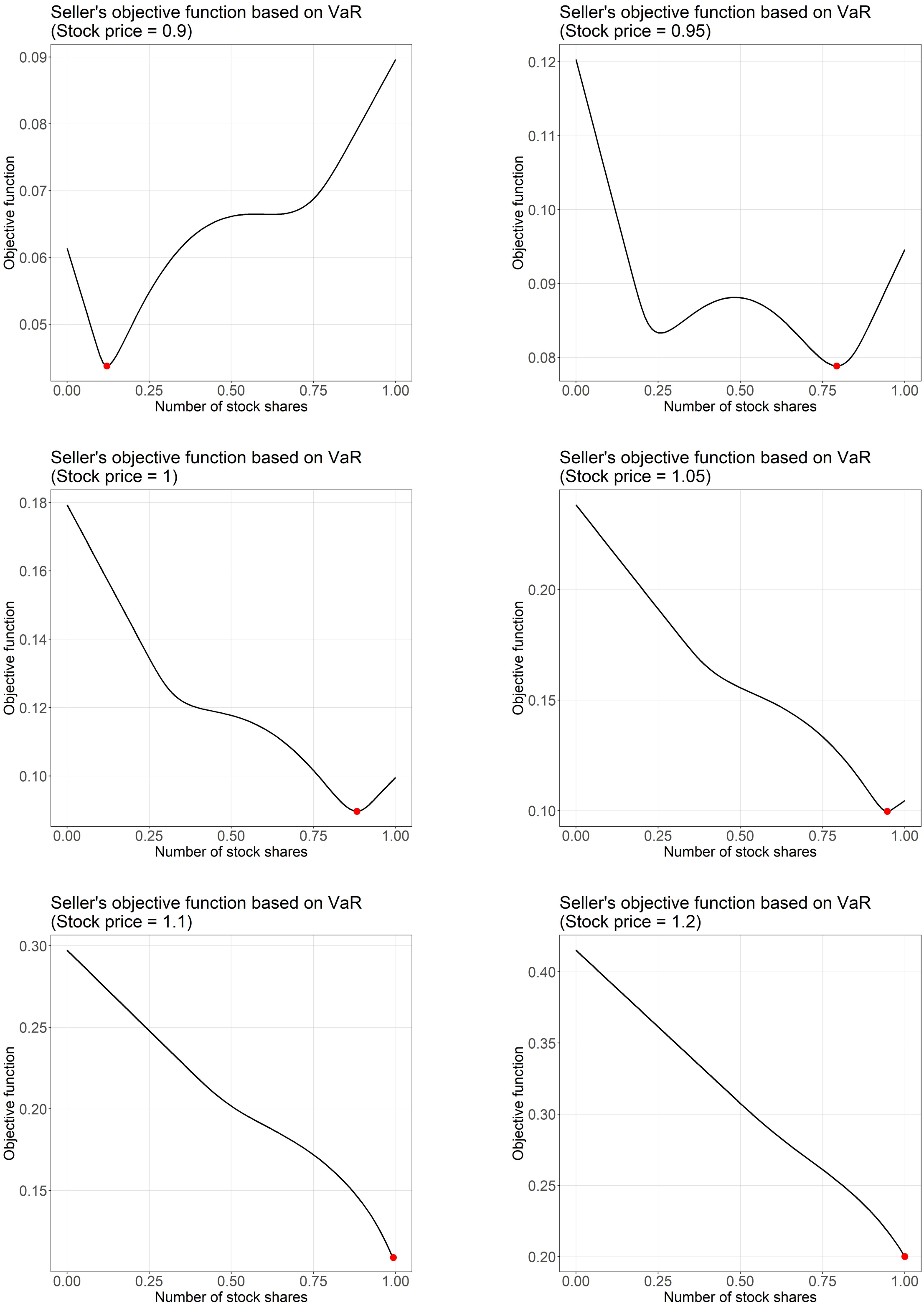


Figure 2.4: Seller's objective function for different stock prices.

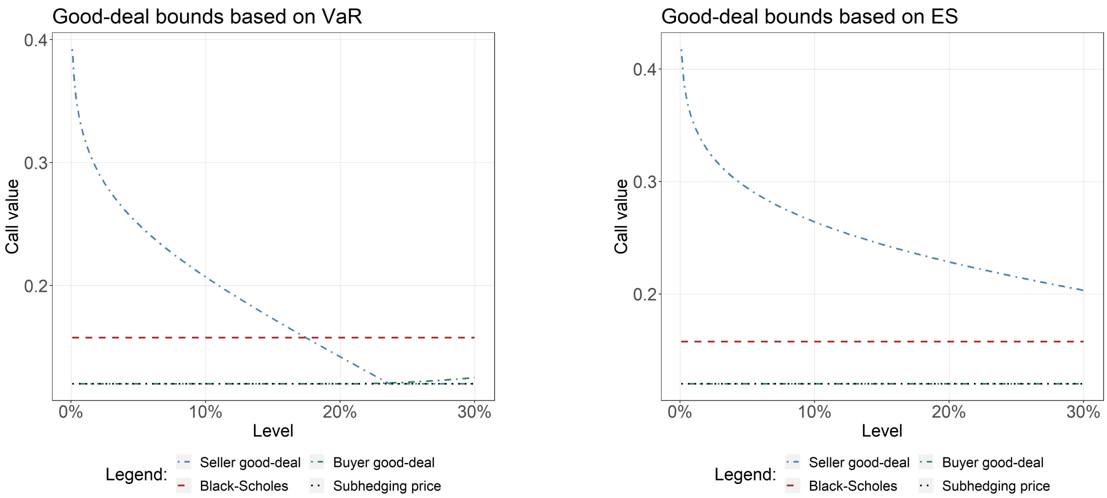


Figure 2.5: Good deal bounds depending on the level λ.

seller good deal bound is

$$S_0 + \mathrm{VaR}_\lambda \left(S_1 - (S_1 - K)^+ \right).$$

If $\lambda > F_{S_1}(K)$, then

$$S_0 + \mathrm{VaR}_\lambda \left(S_1 - (S_1 - K)^+ \right) = S_0 - K,$$

i.e., the seller's bound is equal to the subhedging price. For large enough stock prices the condition is fulfilled, because of

$$\lim_{S_0 \to \infty} F_{S_1}(K) = \lim_{S_0 \to \infty} \Phi \left(\frac{\ln(K/S_0) - (b - \sigma^2/2)}{\sigma} \right) = 0.$$

An analogous explanation holds for the case if the stock price tends to zero.

2.5 Extension of the basis market

In this section, we discuss in more detail the not intuitive behavior that the seller's good deal bound can become smaller than the buyer's good deal bound when using a VaR acceptance set. This is due to the failure of the so-called extension theorem for VaR. Subsequently, we have to prove the validity of the extension theorem for an ES acceptance set in our concrete setup, to show that the seller's good deal bound is always larger than the buyer's good deal bound for ES.

2.5.1 Duality relations

We develop a duality relation for the seller's good deal bound with respect to an acceptance set based on a positive homogeneous monetary risk measure ρ. Let $\mathcal{P}_\rho(C)$ be the set of prices for a payoff $C \in \mathcal{X}$ such that there are no good deals of the first kind in the extended market, i.e., for $p \in \mathcal{P}_\rho(C)$ there do not exist $\varphi^S, \varphi^C \in \mathbb{R}$ excluding the case $\varphi^S = \varphi^C = 0$ such that

$$\rho\left(\varphi^S\left(S_T - S_0\frac{B_T}{B_0}\right) + \varphi^C\left(C - p\frac{B_T}{B_0}\right)\right) \leq 0.$$

By positive homogeneity of ρ we obtain that $L^1_+ \subset \mathcal{A}$. This implies that every arbitrage opportunity is also a good deal. Hence, the set $\mathcal{P}_\rho(C)$ is a subset of the no arbitrage price range of C. Therefore, the set of no good deal prices in the extended market can be characterized as a subset of all equivalent martingale measures. This subset of equivalent martingale measures is denoted by $\mathcal{P}^*_\rho(C)$.

In the following, we assume that there exists at least one price such that there are no good deals in the extended market, i.e., we work under the assumption

$$\mathcal{P}^*_\rho(C) \neq \varnothing.$$

Remark 2.5.1. This condition implies that the basis market $\mathcal{M}$ satisfies the condition of the absence of good deals of the first kind.

Now we are able to present the duality result. It is an analogous result as for the superhedging price, see Föllmer and Schied (2016, Theorem 1.32).

Proposition 2.5.2. *The seller's good deal bound for a P-a.s. non-negative random variable C in the one-period BSM is given by*

$$\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(-C) = \inf_{\varphi\in\mathbb{R}}\left(\varphi S_0 + \frac{\rho(\varphi S_T - C)}{B_T}B_0\right) = \sup_{P^*\in\mathcal{P}^*_\rho(C)} E_{P^*}[C].$$

Proof: From the previous discussion we obtain that

$$\sup_{P^*\in\mathcal{P}^*_\rho(C)} E_{P^*}[C] = \sup \mathcal{P}_\rho(C).$$

Now we assume that there exists $\varphi \in \mathbb{R}$ such that for a value $m \in \mathbb{R}$ the following inequality holds:

$$\rho\left(m\frac{B_T}{B_0} + \varphi\left(S_T - S_0\frac{B_T}{B_0}\right) - C\right) \leq 0.$$

From cash invariance we obtain $\frac{B_0}{B_T}\rho\left(\varphi\left(S_T - S_0\frac{B_T}{B_0}\right) - C\right) \leq m$. Hence, every $p \geq m$ can not be a price in the extended market such that there does not exist a good deal. This implies $\rho(-C) \geq \sup \mathcal{P}_\rho(C)$.

It is left to show that for every $m > \sup \mathcal{P}_\rho(C)$ it holds that $m \geq \rho(-C)$. To do so, assume $m > \sup \mathcal{P}_\rho(C)$. There are $\varphi^S, \varphi^C \in \mathbb{R}$ excluding the case $\varphi^S = \varphi^C = 0$ such that

$$\rho\left(\varphi^S\left(S_T - S_0\frac{B_T}{B_0}\right) + \varphi^C\left(C - m\frac{B_T}{B_0}\right)\right) \leq 0.$$

The basis market does not admit good deals of the first kind. This implies $\varphi^C \neq 0$. If $\varphi^C < 0$, we obtain from the positive homogeneity of ρ that

$$\rho\left(m\frac{B_T}{B_0} - \frac{\varphi^S}{\varphi^C}\left(S_T - S_0\frac{B_T}{B_0}\right) - C\right) \leq 0.$$

It remains to show that the case $\varphi^C > 0$ is not feasible. We know that there exists a price $p \in \mathcal{P}_\rho(C)$ for each $x, y \in \mathbb{R}$ excluding the case $x = y = 0$ and we have

$$\rho\left(x\left(S_T - S_0\frac{B_T}{B_0}\right) + y\left(C - p\frac{B_T}{B_0}\right)\right) > 0.$$

At the same time positive homogeneity of ρ implies

$$\rho\left(\frac{\varphi^S}{\varphi^C}\left(S_T - S_0\frac{B_T}{B_0}\right) + C - m\frac{B_T}{B_0}\right) \leq 0.$$

These two statements together with monotonicity of ρ and $m > p$ imply

$$0 \geq \rho\left(\frac{\varphi^S}{\varphi^C}\left(S_T - S_0\frac{B_T}{B_0}\right) + C - m\frac{B_T}{B_0}\right) > \rho\left(\frac{\varphi^S}{\varphi^C}\left(S_T - S_0\frac{B_T}{B_0}\right) + C - p\frac{B_T}{B_0}\right) > 0$$

which is a contradiction. □

Remark 2.5.3. A corresponding result holds for the buyer's good deal bound:

$$\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(-C) = \sup_{P^* \in \mathcal{P}^*_\rho(C)} E_{P^*}[C] \geq \inf_{P^* \in \mathcal{P}^*_\rho(C)} E_{P^*}[C] = -\rho_{\mathcal{A}_\rho,\mathcal{M},\pi}(C).$$

2.5.2 Failure of the extension theorem

If we go back to Figure 2.3, we are now able to answer the question why the seller's bound with respect to the VaR is lower than the buyer's bound. If we recall Remark 2.5.3, then we know that this is only possible if

$$\mathcal{P}^*_{\mathrm{VaR}_\lambda}\left((S_T - K)^+\right) = \varnothing.$$

This means that there is no price such that the extended market is free of good deals of the first kind. This must also hold if there are no good deals of the first kind on the basis market $\mathcal{M}$. From Lemma 2.4.1, we obtain that the absence of good deals on the basis market is equivalent to

$$\frac{B_T}{B_0} \in \left(\frac{q^+_{S_T}(\lambda)}{S_0}, \frac{q^+_{S_T}(1-\lambda)}{S_0}\right).$$

This condition is independent of the stock price and is fulfilled in particular in Example 2.4.5. The situation that every possible option price leads to an extended market that is not free of good deals means that for every price $p \in \mathbb{R}$ there exists $\theta \in \mathbb{R}$ such that

$$\theta S_0 + \mathrm{VaR}_\lambda\left(\theta S_T + (S_T - K)^+\right)\frac{B_T}{B_0} \leq -p,$$

or

$$\theta S_0 + \mathrm{VaR}_{1-\lambda}\left(\theta S_T + (S_T - K)^+\right)\frac{B_T}{B_0} \geq -p.$$

In contrast to the basis market the absence of good deals of the first kind depend on the stock price. For $\theta = 1$, the earlier argument that the seller's good deal bound converges to the subhedging price, could be used in an analogous way to show that for high enough stock prices both VaR expressions in the previous inequalities are equal to the subhedging price $S_0 - K$. Hence, one of the above conditions is fulfilled for large enough stock prices.

Remark 2.5.4. The extension theorem as described in Černý and Hodges (2002, Section 2.1) states, that a basis market which is free of good deals could be extended such that the extended market is free of good deals of the first kind. But this would be a contradiction to our VaR example, i.e., a contradiction to $\mathcal{P}^*_{\mathrm{VaR}_\lambda}\left((S_T - K)^+\right) = \varnothing$. Hence, a sufficient condition in the extension theorem is not satisfied. In fact, in our setting the VaR acceptance set is not boundedly generated [1,2]. This is intuitive, because the proof of the extension theorem is based on a separation argument. Such a separation

[1] Let K be a convex set of claims disjoint from the origin. It is called boundedly generated if there exists a closed bounded subset $B \subset K$ such that any point in K can be regarded as a scalar multiple of a point in B, see Černý and Hodges (2002, Definition 2.2).

[2] Nevertheless, the VaR acceptance set is closed in L^p, see Munari (2015, Proposition 2.4.5).

is likely to fail if the acceptance set is not convex, like for the VaR.

The previous remark raises the question if the extension theorem holds for the ES acceptance set. This is indeed the case. The reason is that the convexity of the acceptance set allows for a separation argument to extend the pricing functional to the whole space of contingent claims. The arguments in the following proof are in line with the ones in the proof of Theorem 2.3 in Černý and Hodges (2002). Nevertheless, our setup differs slightly from the one in Černý and Hodges (2002), e.g., the intersection of $\ker(\pi)$ and $\mathcal{A}$ could contain the zero vector which is not the case in Černý and Hodges (2002). Furthermore, our proof does not explicitly use the condition of a boundedly generated set and it is more in the style of a classical no arbitrage price proof, compare e.g., with the proofs of Propositions 2.1 and 3.1 in Kreps (2019).

Theorem 2.5.5 (Expected Shortfall extension theorem)**.** *Assume the one-period BSM model and good deals defined by the ES acceptance set with respect to a level $\lambda \in (0, 0.5)$. Assume that there are no good deals of the first kind on the basis market $\mathcal{M}$. In this setup, there exists an extension of the pricing functional $\pi : \mathcal{M} \to \mathbb{R}$ to the whole space of financial payoffs $\mathcal{X}$ such that there are no good deals in the extended market.*

Proof: From Lemma 1.4.8 we obtain

$$\operatorname{int}(\mathcal{A}_{\mathrm{ES}_\lambda}) = \{X \in \mathcal{X} \,|\, \mathrm{ES}_\lambda(X) < 0\}\,.$$

This shows that $0 \notin \operatorname{int}(\mathcal{A}_{\mathrm{ES}_\lambda})$ which in turn implies that $\operatorname{int}(\mathcal{A}_{\mathrm{ES}_\lambda} \backslash \{0\}) = \operatorname{int}(\mathcal{A}_{\mathrm{ES}_\lambda})$. Again, from Lemma 1.4.8 we obtain $S_T \in \operatorname{int}(\mathcal{A}_{\mathrm{ES}_\lambda})$. A straightforward calculation gives that $\mathcal{A}_{\mathrm{ES}_\lambda} \backslash \{0\}$ is convex. The assumption of no good deals of the first kind on the basis market states that $(\mathcal{A}_{\mathrm{ES}_\lambda} \cap \ker(\pi)) \backslash \{0\} = \varnothing$. Summarizing, $\mathcal{A}_{\mathrm{ES}_\lambda} \backslash \{0\}$ and $\ker(\pi)$ are two non-empty, disjoint, convex sets for which $\mathcal{A}_{\mathrm{ES}_\lambda} \backslash \{0\}$ contains an interior point. Hence, by the Interior Separating Hyperplane Theorem (see Theorem A.6) there exists a non-zero continuous functional p which properly separates these two sets.

Since $\ker(\pi)$ is a subspace, we obtain for each $Z \in \ker(\pi)$ that $p(Z) = 0$ and additionally we could assume that for each $X \in \mathcal{A}_{\mathrm{ES}_\lambda} \backslash \{0\}$ it holds that $p(X) \geq 0$. Since the separation is proper, there exists $Y \in \mathcal{A}_{\mathrm{ES}_\lambda} \backslash \{0\}$ with $p(Y) > 0$.

Note that every $X \in \mathcal{A}_{\mathrm{ES}_\lambda} \backslash \{0\}$ obtains a representation with $\alpha \in \mathbb{R}, U \in [p = 0]$ (for the definition of the latter set see Definition A.4) such that $X = \alpha Y + U$. This implies $p(X) = \alpha p(Y)$. The scalar α can not be zero ($X \notin \ker(\pi)$) and also not negative ($p(X) \geq 0$). Hence, it holds that $p\,(X) > 0$ and the usage of p as pricing functional in the extended market would exclude good deals. It also would exclude arbitrage opportunities, because of $L^1_+ \subset \mathcal{A}_{\mathrm{ES}_\lambda}$. But, we are not allowed to use p directly as extended pricing functional. This is due to the fact that p could price the basis assets in a different way than before, i.e., $p\left(\varphi^B B_T + \varphi^S S_T\right) \neq \varphi^B B_0 + \varphi^S S_0$. We obtain a suitable functional $\tilde{p}$ by rescaling p with $p(B_T)/B_0$, i.e., for each $X \in L^1$ we set $\tilde{p}(X) = \frac{p(X)}{p(B_T)} B_0$. $\square$

Chapter 3

Multi-asset intrinsic risk measures

3.1 Introduction

Multi-asset risk measures are used to calculate capital reserves to avoid future bankruptcy. This is done by determining the minimal capital that has to be invested in eligible assets such that a capital adequacy test is passed. Alternatively to using only external capital to hedge a financial position, we combine this with internal restructuring. Pure internal restructuring would lead to the recently proposed intrinsic risk measures. Both approaches are based on management actions that could be taken by the company. In this chapter, we combine external and internal management actions in the capital calculation and derive properties of the resulting new risk measure. This chapter is based on Laudagé, Sass, and Wenzel (2021).

3.1.1 Motivation

We repeat the idea behind single-asset risk measures to illustrate an external management action. To do so, we use a random variable X_T, modeling the value of a financial position at a future time T. Then, the company invests an additional amount in a single eligible asset with current value $B_0 > 0$ and future value B_T. The combined position of X_T and B_T should pass a capital adequacy test, i.e., it should be an element of the acceptance set $\mathcal{A}$. The risk measure describes the minimal *external* amount invested in the eligible asset such that the combined position is acceptable:

$$\rho(X_T) = \inf\left\{ m \in \mathbb{R} \,\middle|\, X_T + \frac{m}{B_0} B_T \in \mathcal{A} \right\}. \tag{3.1}$$

The idea behind this concept is illustrated in Figure 3.1. It shows the influence of an external management action on the balance sheet of a company. The financial position corresponds to the future equity capital. The external management action leads to a capital increase and in order to offset the balance sheet, the additional capital is invested into traded assets on a financial market.

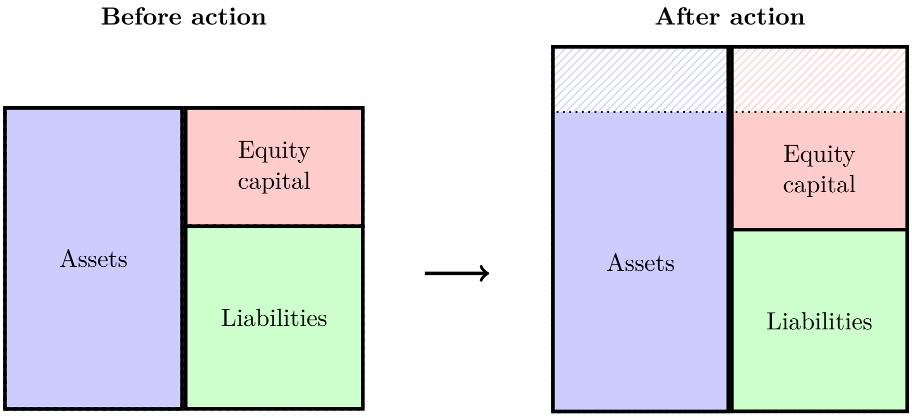


Figure 3.1: *Left-hand side:* Simplified version of the balance sheet of a company. *Right-hand side:* Changes in the balance sheet after performing an external management action, i.e., a capital increase.

Recently, the new concept of intrinsic risk measures was introduced in Farkas and Smirnow (2018). Their motivation is the following quote in Artzner et al. (1999, page 205):

> *For an unacceptable position [...] one remedy may be to alter the position.*

Farkas and Smirnow (2018) assume that the financial position is liquidly traded. The risk is measured by the minimal proportion of the financial position that has to be sold and reinvested in the eligible asset to reach acceptability, i.e., no external capital is required. The operation corresponds to an *internal* rebooking. If X_0 is today's value of the financial position, the *intrinsic risk measure* is defined as follows:

$$R(X_T) = \inf \left\{ \kappa \in [0,1] \,\middle|\, (1-\kappa)X_T + \kappa \frac{X_0}{B_0} B_T \in \mathcal{A} \right\}.$$

If we consider again a balance sheet of a company, then an internal management action is a rebooking of some liquid assets, i.e., it changes the proportions of these liquid assets in the balance sheet. This is illustrated in Figure 3.2.

Our aim is to combine multi-asset risk measures with the intrinsic risk measures in Farkas and Smirnow (2018). In doing so, the resulting new risk measure combines external and internal management actions. For more details on the handling of these actions in practice, we refer to Rockel, Helten, Ott, and Sauer (2012).

3.1.2 Bibliographic notes

For the new risk measure that we construct, we develop analytical results. In the following, we provide references for properties of multi-asset and intrinsic risk measures

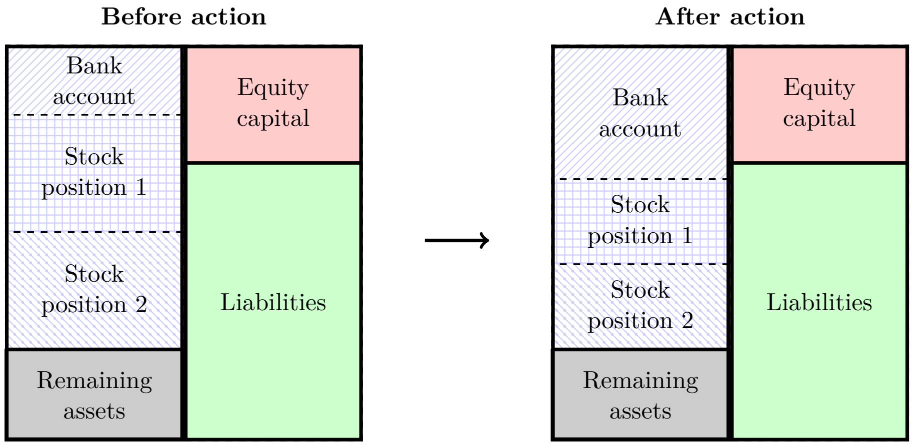


Figure 3.2: *Left-hand side:* Simplified version of the balance sheet of a company. *Right-hand side:* Changes in the balance sheet after performing an internal management action, i.e., a reallocation of some liquid assets.

directly related to the findings in this chapter. Among others, we examine the monotonicity, the additivity and the positive homogeneity of the new risk measure. Analog properties for multi-asset risk measures can be found in Scandolo (2004), Frittelli and Scandolo (2006) and Artzner et al. (2009). We also discuss the impact of diversification on the capital requirement. For similar discussions with respect to multi-asset and intrinsic risk measures we refer to Karoui and Ravanelli (2009), Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011), Farkas et al. (2014), Farkas and Smirnow (2018). We also develop a dual representation which can be compared to the one for multi-asset risk measures in Farkas et al. (2015).

3.1.3 Path to a new risk measure

Now we illustrate how we combine multi-asset with intrinsic risk measures. The idea is motivated by considering the equity capital of an institution as a financial position as it is the case for common capital adequacy tests in practice, e.g., Solvency II or the Basel Accords. Then we subdivide the assets of a company into two groups. The first group consists of the assets, the company is willing to reallocate. Note that such a reallocation is done immediately and therefore the chosen assets have to be liquid, e.g., cash in a bank account, bonds, stocks or derivatives. The second group are the remaining assets. These could be illiquid assets, e.g., real estates or liquid assets that can not be reallocated. An example for the latter are stocks used to secure a loan and hence they can not be sold. It could also be the case that the assets are liquid, but the firm is not willing to reallocate them, e.g., if they are needed to hold a certain proportion of another firm. Selling them would reduce the impact on this firm.

Now we are interested in the capital requirement if the risk is measured in terms of the external capital that has to be invested in the eligible assets such that the combined position is acceptable taking into account reallocations of liquid assets. This concept is illustrated in Figure 3.3. In this illustration, the proportions of the stock portfolios are reduced and additionally the equity capital increases.

For instance, let X_T^L be the liquid assets in the balance sheet that can be sold with value X_0^L today, and X_T^R the remaining part. The combined risk measure is given by

$$\varrho(X_T^R, X_T^L) = \inf\left\{ m \in \mathbb{R} \,\middle|\, \exists \kappa \in [0,1] : (1-\kappa)X_T^L + X_T^R + \frac{m + \kappa X_0^L}{B_0} B_T \in \mathcal{A} \right\}. \quad (3.2)$$

We used a single eligible asset B to illustrate the idea behind our new concept. But there is no reason in doing so. Hence, in the main part of this chapter we allow for multiple eligible assets. To emphasize that the idea of our new risk measure is inspired by Farkas et al. (2015) and Farkas and Smirnow (2018) we call it a ***multi-asset intrinsic (MAI) risk measure***. This risk measure can additionally be seen as a multi-asset risk measure under trading constraints, compare the discussion in Section 3.3.2.

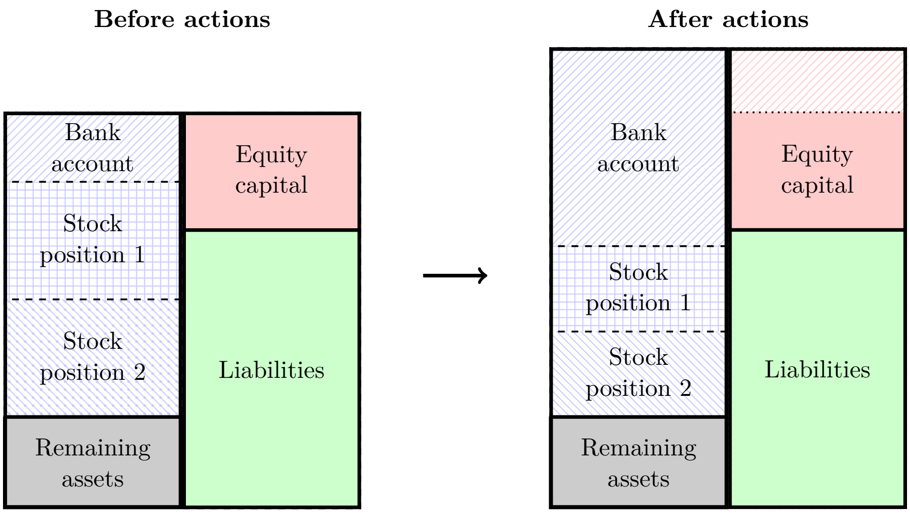


Figure 3.3: *Left-hand side:* Simplified version of the balance sheet of a company. *Right-hand side:* Changes in the balance sheet after performing external and internal management actions, i.e., a capital increase and a reallocation of liquid assets.

Finally, we discuss a connection of MAI and classical risk measures. To this end, note that $\varrho(X_T^R, B_T) = \rho(X_T^R) - B_0$, i.e., if the financial position X_T^L is given by the eligible asset itself, then the capital requirement is given by the classical risk measure from Equation (3.1) minus the actual price of X_T^L. Hence, the more demanding situations are the ones in which the portfolio X_T^L consists of other financial positions than cash in the bank account. In these situations, MAI and classical risk measures can differ, compare the discussion in Section 3.3.2.

3.1.4 Contributions of this chapter

MAI risk measures put the focus on the impact of two specific management actions available in practice. This makes it indispensable to split the total payoff into the objects X_T^R and X_T^L used in Equation (3.2). They are the input for the MAI risk measure. This is a new feature, because multi-asset and intrinsic risk measures get a single object, namely the total payoff, as input. Another innovative feature, similar to intrinsic risk measures, but different to multi-asset risk measures, is that the MAI risk measure depends on the actual price of X_T^L. The new features make the analysis of the MAI risk measure a sophisticated task from an economic as well as a mathematical point of view. In the following, we summarize our key findings:

(i) **Representation result:** Theorem 3.3.2 enables an efficient computation of the MAI risk measure by building an infimum over multi-asset risk measures. Furthermore, it allows to transfer the properties of additivity, monotonicity and positive homogeneity of multi-asset risk measures to MAI risk measures.

(ii) **Diversification:** If we only diversify the assets available for reallocation and if they have the same price, then the MAI risk measure rewards diversification. We present two examples demonstrating that diversification is not rewarded in general. We discuss the reasons for this behavior.

(iii) **Dual representation:** Using Sion's minimax theorem and convex duality methods, we develop a robust representation for MAI risk measures. This gives new insights into the structure of MAI risk measures, especially for the case of bounded random variables, see Theorem 3.5.2 and Example 3.5.3.

(iv) **Numerical examples:** We apply MAI risk measures to extended models motivated by the one in Floreani (2013). These models are used to discuss effects in the Solvency II methodology. We find situations in which multi-asset and MAI risk measures coincide. This demonstrates that the MAI risk measure could be considered as a multi-asset risk measure under trading restrictions.

3.1.5 Structure of this chapter

In Section 3.2, we describe our model and give a formal definition of MAI risk measures. In Section 3.3, we represent MAI risk measures using multi-asset risk measures and state conditions such that the MAI risk measure is additive, monotone and positive homogeneous. In Section 3.4, we discuss whether and how diversification is rewarded. In Section 3.5, we state a dual representation for MAI risk measures. Section 3.6 closes with examples to demonstrate differences between multi-asset and MAI risk measures.

3.1.6 Standard notations and concepts

In addition to Sections 1.1.4 and 2.1.5, we use the following standard notations and concepts: Assume a non-empty set $\mathcal{C}$ and a map $f : \mathcal{C} \to [-\infty, \infty]$. If f is convex, respectively concave, then the effective domain of f is given by $\operatorname{dom}(f) := \{X \in \mathcal{C} \mid f(X) < \infty\}$, respectively $\operatorname{dom}(f) := \{X \in \mathcal{C} \mid f(X) > -\infty\}$.

Assume an ordered topological vector space $(\mathcal{X}, \tau, \preceq)$. The negative cone of $\mathcal{X}$ is defined by $\mathcal{X}_- := \{X \in \mathcal{X} \mid X \preceq 0\}$. The topological dual is denoted by $\mathcal{X}'$ and the positive cone $\mathcal{X}'_+$ consists of each $\psi \in \mathcal{X}'$ such that for every $X \in \mathcal{X}_+$ it holds that $\psi(X) \geq 0$, see also Appendix A.2. If necessary we equip a subset $\mathcal{C} \subset \mathcal{X}$ with the relative topology $\tau|_{\mathcal{C}} := \{V \cap \mathcal{C} \mid V \in \tau\}$. Cartesian products of topological spaces are equipped with the product topology.

3.2 Introducing multi-asset intrinsic risk measures

In this section, we specify the model space, the marketed space and the pricing functional. Further, we differ between external and internal management actions. These model assumptions straightforwardly lead to the definition of MAI risk measures.

3.2.1 The financial market model

We do not have to restrict our attention to random variables. For example, we could also consider the entire wealth process of a portfolio, instead of using its payoff at some terminal time. Hence, as already stated in Section 1.2.1, we aim at a unifying approach and follow Farkas et al. (2014) who use an ordered topological vector space $(\mathcal{X}, \tau, \preceq)$ as model space. This choice is a common one in the theory of risk measures, see Farkas et al. (2014), Liebrich and Svindland (2017), Baes et al. (2020). Remember, we call a point in $\mathcal{X}$ a financial position or a payoff.

We assume that any investment for hedging purposes is listed in a financial market, like e.g., a stock exchange, modeled by the marketed space $\mathcal{M}$ (a linear subspace of $\mathcal{X}$). Remember, a point in $\mathcal{M}$ is called a marketed portfolio, a marketed payoff or a portfolio and it could, e.g., be the payoff at the end of the trading period with respect to a stock portfolio. A financial position in $\mathcal{X} \backslash \mathcal{M}$ is a position that is not traded in the financial market $\mathcal{M}$, e.g., it can be the equity capital of a company. The idea is that this company tries to improve the structure of its balance sheet by investing on the stock exchange.

For every marketed portfolio we know the current value by the pricing functional $\pi : \mathcal{M} \to \mathbb{R}$ which is a strictly positive linear functional, i.e., for each $M \in (\mathcal{M} \cap \mathcal{X}_+) \backslash \{0\}$ it holds that $\pi(M) > 0$.

For the sake of brevity, we often omit the information on the ordered topological vector space $\mathcal{X}$ and the corresponding marketed space $\mathcal{M} \subset \mathcal{X}$ and take them for granted.

3.2.2 External and internal management actions

In the following, we describe the hedging opportunities. In the previous chapters, the agent has full market access. If this is the case and there are no restrictions by the regulator, all positions in the marketed space $\mathcal{M}$ could be traded. Otherwise, the available positions build a proper subset of $\mathcal{M}$.

Definition 3.2.1 (Security space)**.** *A* ***security space*** *is a linear subspace $\mathcal{S} \subset \mathcal{M}$ with $(\mathcal{S} \cap \mathcal{X}_+)\backslash\{0\} \neq \varnothing$.*

For a security space $\mathcal{S}$, a number $\kappa \in [0,1]$ and a portfolio $M \in \mathcal{M}$ we define the set $\mathcal{S}_{\kappa,M} := \{U \in \mathcal{S} \,|\, \pi(U) = \kappa\pi(M)\}$.

Remark 3.2.2. The set $\mathcal{S}_{\kappa,M}$ is introduced for the sake of brevity. We use such sets in Lemma 3.3.1.

To obtain a general formulation for MAI risk measures, we distinguish between investments with respect to different management actions. Hence, we use two security spaces $\mathcal{S}^E, \mathcal{S}^I \subset \mathcal{M}$ to model external and internal management actions, i.e., different marketed portfolios could be allowed for different administrative actions. The indexes for the security spaces stand for management actions financed by *external* (E) capital and reallocated *internal* (I) capital. The separate handling of management actions could be used for an individually adjusted risk management. For instance, an institution could reallocate its stock portfolio M only with other stocks and ask for the minimal additional cash in a bank account to reach acceptability.

3.2.3 Multi-asset intrinsic risk measures

For the sake of brevity, we name the model components introduced so far.

Definition 3.2.3 (Intrinsic risk measurement regimes)**.** *For an acceptance set $\mathcal{A}$, the pricing functional $\pi : \mathcal{M} \to \mathbb{R}$ and two security spaces $\mathcal{S}^E, \mathcal{S}^I \subset \mathcal{M}$ we call the tuple $\mathcal{R} = \left(\mathcal{A}, \mathcal{S}^E, \mathcal{S}^I, \pi\right)$ an* ***intrinsic risk measurement regime****. If $\mathcal{S}^E = \mathcal{S}^I$, then we write $\mathcal{R} = \left(\mathcal{A}, \mathcal{S}^E, \pi\right)$ for short*[1]*.*

Now we are ready to give the desired definition of MAI risk measures.

Definition 3.2.4 (MAI risk measures)**.** *The* ***multi-asset intrinsic (MAI) risk mea-***

[1] In Liebrich and Svindland (2017, 2019) such a triple is called a risk measurement regime. For our purposes it is more convenient to use it as a shorthand notation for an intrinsic risk measurement regime.

sure *for an intrinsic risk measurement regime* $\mathcal{R} = \left(\mathcal{A}, \mathcal{S}^E, \mathcal{S}^I, \pi\right)$ *is given by*

$$\begin{aligned} \varrho_{\mathcal{R}} : \mathcal{X} \times \mathcal{M} &\to [-\infty, \infty]\,, \\ (X, M) &\mapsto \inf\{\pi(Z) \,|\, Z \in \mathcal{S}^E, \kappa \in [0,1], U \in \mathcal{S}^I \text{ with } \pi(U) = \kappa\pi(M) \\ &\qquad \text{and } X + (1-\kappa)M + Z + U \in \mathcal{A}\}. \end{aligned} \tag{3.3}$$

We further write $\varrho_{\mathcal{A},\mathcal{S}^E,\mathcal{S}^I,\pi} := \varrho_{\mathcal{R}}$ *or in the case of* $\mathcal{S}^E = \mathcal{S}^I$ *simply* $\varrho_{\mathcal{A},\mathcal{S}^E,\pi} := \varrho_{\mathcal{R}}$*.*

Remark 3.2.5. For completeness, we recall the intuition behind Equation (3.3): The Cartesian product $\mathcal{X} \times \mathcal{M}$ describes that the financial position consists of a part in $\mathcal{X}$ which can not be reallocated and some liquid assets in $\mathcal{M}$. In a first step, the agent sells a proportion $\kappa \in [0,1]$ of the marketed portfolio M and reinvest it in another marketed portfolio $U \in \mathcal{S}^I_{\kappa,M}$. Second, the agent uses an additional amount of money to buy a marketed portfolio $Z \in \mathcal{S}^E$ such that the new position $X + (1-\kappa)M + Z + U$ becomes acceptable. The agent chooses the acceptable payoff that can be generated at minimal costs. Remember, having a Cartesian product as a domain for a risk measure is a new feature. Further, the MAI risk measure also depends on the actual price of the marketed portfolio M. We have no such dependence for multi-asset risk measures.

Finally, we repeat the definition of multi-asset risk measures based on a security space as a proper subset of the marketed space. This definition is analogous to Definition 1.3.9. We mentioned it here for the sake of completeness.

Definition 3.2.6 (Single-asset and multi-asset risk measures)**.** *For an acceptance set* $\mathcal{A}$*, the pricing functional* $\pi : \mathcal{M} \to \mathbb{R}$ *and a subset* $\mathcal{S} \subset \mathcal{M}$ *we define the following map:*

$$\rho_{\mathcal{A},\mathcal{S},\pi} : \mathcal{X} \to [-\infty, \infty],\ X \mapsto \inf\{\pi(Z) \,|\, Z \in \mathcal{S}, X + Z \in \mathcal{A}\}.$$

If $\mathcal{S}$ *is a security space, we call the map* $\rho_{\mathcal{A},\mathcal{S},\pi}$ *a* ***multi-asset risk measure****. If* $M \in \mathcal{M}$ *such that* $\mathcal{S} = \operatorname{span}\{M\}$ *we write* $\rho_{\mathcal{A},M,\pi} := \rho_{\mathcal{A},\mathcal{S},\pi}$ *for short and call it a* ***single-asset risk measure****.*

3.3 MAI and multi-asset risk measures

In Section 3.3.1, we represent the MAI risk measure in terms of multi-asset risk measures. In Section 3.3.2, we use this representation to prove economically desirable properties motivated by the ones for multi-asset risk measures, namely $\mathcal{S}$-additivity, monotonicity and positive homogeneity.

3.3.1 Representation by multi-asset risk measures

The following result is a representation of the MAI risk measure as an infimum with respect to the map in Definition 1.3.9.

Lemma 3.3.1. *Let $\mathcal{R} = \left(\mathcal{A}, \mathcal{S}^E, \mathcal{S}^I, \pi\right)$ be an intrinsic risk measurement regime. For each $X \in \mathcal{X}$ and $M \in \mathcal{M}$ it holds that*

$$\varrho_{\mathcal{R}}(X, M) = \inf_{\kappa \in [0,1]} \left(\rho_{\mathcal{A}, \mathcal{S}^E + \mathcal{S}^I_{\kappa,M}, \pi}(X + (1-\kappa)M) - \kappa\pi(M)\right). \tag{3.4}$$

Proof: Since the extended real line $[-\infty, \infty]$ is a complete lattice, for an arbitrary indexed family $\mathcal{C}$ of sets contained in the extended real line we obtain

$$\inf \bigcup \mathcal{C} = \inf\{\inf C \mid C \in \mathcal{C}\}.$$

Making use of this result we obtain that

$$\begin{aligned}
\varrho_{\mathcal{R}}(X, M) &= \inf \left\{\pi(Z) \,\middle|\, Z \in \mathcal{S}^E, \kappa \in [0,1], U \in \mathcal{S}^I_{\kappa,M} : X + (1-\kappa)M + Z + U \in \mathcal{A}\right\} \\
&= \inf_{\kappa \in [0,1]} \left(\inf \left\{\pi(Z^*) - \kappa\pi(M) \,\middle|\, Z^* \in \mathcal{S}^E + \mathcal{S}^I_{\kappa,M} : X + (1-\kappa)M + Z^* \in \mathcal{A}\right\}\right) \\
&= \inf_{\kappa \in [0,1]} \left(\rho_{\mathcal{A}, \mathcal{S}^E + \mathcal{S}^I_{\kappa,M}, \pi}(X + (1-\kappa)M) - \kappa\pi(M)\right).
\end{aligned}$$

□

The Minkowski sum $\mathcal{S}^E + \mathcal{S}^I_{\kappa,M}$ in Equation (3.4) is not a linear subspace of the marketed space $\mathcal{M}$ in general, i.e., we do not obtain an infimum with respect to a multi-asset risk measure. As shown next, this undesirable effect disappears, if internal hedging opportunities could also be used as external hedging opportunities, i.e., $\mathcal{S}^I \subset \mathcal{S}^E$. Even in this case, the MAI risk measure is a strict extension of multi-asset risk measures. In particular, even in the case of $\mathcal{S}^I = \mathcal{S}^E$, the MAI risk measure and the multi-asset risk measure do not coincide in general, compare the results of the numerical examples in Section 3.6.

Theorem 3.3.2 (MAI representation)**.** *Let $\mathcal{R} = \left(\mathcal{A}, \mathcal{S}^E, \mathcal{S}^I, \pi\right)$ be an intrinsic risk measurement regime. If $\mathcal{S}^I \subset \mathcal{S}^E$, then we obtain*

$$\varrho_{\mathcal{R}}(X, M) = \inf_{\kappa \in [0,1]} \left(\rho_{\mathcal{A}, \mathcal{S}^E, \pi}(X + (1-\kappa)M) - \kappa\pi(M)\right). \tag{3.5}$$

In particular, $\varrho_{\mathcal{R}}(X, 0) = \rho_{\mathcal{A}, \mathcal{S}^E, \pi}(X)$.

Proof: We directly see that $\mathcal{S}^I_{\kappa,M} + \mathcal{S}^E \subset \mathcal{S}^E$. For the inverse inclusion note that

$\left(\mathcal{S}^I \cap \mathcal{X}_+\right) \setminus \{0\} \neq \varnothing$. Together with the strict positivity of π this guarantees the existence of an element $U \in \mathcal{S}^I_{\kappa,M}$. So, for each $Z \in \mathcal{S}^E$ we have $(Z-U)+U \in \mathcal{S}^E + \mathcal{S}^I_{\kappa,M}$. The result then follows by Lemma 3.3.1. □

It is also worth noting that $\mathcal{S}^I$ no longer appears in Equation (3.5) due to the following economical reason: Selling a proportion κ of the liquid assets leads to the new portfolio $X + (1-\kappa)M$ plus the additional amount $\kappa\pi(M)$ invested into a payoff Z contained in $\mathcal{S}^I$. Now every payoff in $\mathcal{S}^I$ is also contained in $\mathcal{S}^E$ and hence, it is possible to add a short position of the same payoff, $-Z$, via an external management action. By this cumbersome restructuring it is possible to invest the amount $\kappa\pi(M)$ in every payoff in $\mathcal{S}^E$ at same costs, i.e., the intrinsic capital can be used for any external management action. Motivated by the previous discussion we assume from now on $\mathcal{S}^I = \mathcal{S}^E$.

In this situation, the following result states lower and upper bounds for the capital requirement.

Corollary 3.3.3. *Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \pi)$ be an intrinsic risk measurement regime. For every $M \in \mathcal{M} \cap \mathcal{X}_+$ and $X \in \mathcal{X}$ we obtain*

$$\rho_{\mathcal{A},\mathcal{S},\pi}(X+M) - \pi(M) \leq \varrho_{\mathcal{R}}(X,M) \leq \rho_{\mathcal{A},\mathcal{S},\pi}(X) - \pi(M).$$

Proof: Follows from Theorem 3.3.2 and the monotonicity of $\mathcal{A}$. □

Remark 3.3.4. The condition $M \in \mathcal{M} \cap \mathcal{X}_+$ is, e.g., satisfied by a portfolio of only long stock investments. An analogous result holds for each $M \in \mathcal{M} \cap \mathcal{X}_-$.

3.3.2 $\mathcal{S}$-additivity, monotonicity and positive homogeneity

Now we use Theorem 3.3.2 to obtain desirable properties of the MAI risk measure from an economic point of view. We start with the $\mathcal{S}$-additivity of multi-asset risk measures, see e.g., Farkas et al. (2015, Equation (8)). We omit the proof, because it is a direct application of Theorem 3.3.2.

Corollary 3.3.5 ($\mathcal{S}$-additivity)**.** *Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \pi)$ be an intrinsic risk measurement regime. For all $(X,M) \in \mathcal{X} \times \mathcal{M}$ and $Z, U \in \mathcal{S}$ we obtain that*

$$\varrho_{\mathcal{R}}(X+Z, M+U) = \varrho_{\mathcal{R}}(X,M) - \pi(Z+U).$$

If the company has full market access, i.e., $\mathcal{S} = \mathcal{M}$, then Corollary 3.3.5 states that for every $(X,M) \in \mathcal{X} \times \mathcal{M}$ we have

$$\varrho_{\mathcal{R}}(X,M) = \varrho_{\mathcal{R}}(X,0) - \pi(M) = \rho_{\mathcal{A},\mathcal{M},\pi}(X) - \pi(M).$$

Hence, the MAI risk measure simplifies to the calculation of a multi-asset risk measure with respect to X, the price of the marketed portfolio M being deducted. The interpretation of this result is as follows: First, the company sells the marketed portfolio M. Second, the company buys new assets to secure the remaining assets and liabilities X.

This situation is due to full market access. In the case of $\mathcal{S} \subsetneq \mathcal{M}$, the MAI risk measure does not simplify to the calculation of a multi-asset risk measure. This situation can occur if the company is not willing to trade arbitrary financial instruments. For instance, the company could rate specific stocks as too risky. Another case is that the regulator forbids the usage of specific assets to pass the capital adequacy test, e.g., highly risky derivatives.

So, in the case of $\mathcal{S} \subsetneq \mathcal{M}$ not every asset in the marketed portfolio M is also available as hedging instrument in general. Such assets in M lead to trading restrictions in the sense that it is only possible to sell them, but not to buy new shares of them. Hence, the MAI risk measure can be interpreted as a multi-asset risk measure under trading constraints.

Now we state an analogous property to the monotonicity for multi-asset risk measures.

Proposition 3.3.6 (Monotonicity). *Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \pi)$ be an intrinsic risk measurement regime and $(X_1, M_1), (X_2, M_2) \in \mathcal{X} \times \mathcal{M}$ such that one of the following conditions holds:*

(i) $X_1 \preceq X_2$ and $M_1 \preceq M_2$.

(ii) For each $\kappa_1 \in [0,1]$, $U_1 \in \mathcal{S}_{\kappa_1, M_1}$ there exists $\kappa_2 \in [0,1]$, $U_2 \in \mathcal{S}_{\kappa_2, M_2}$ such that

$$X_1 + (1 - \kappa_1)M_1 + U_1 \preceq X_2 + (1 - \kappa_2)M_2 + U_2.$$

Then it holds that $\varrho_{\mathcal{R}}(X_1, M_1) \geq \varrho_{\mathcal{R}}(X_2, M_2)$.

Proof: The result is a direct consequence of the definition of an MAI risk measure as well as Theorem 3.3.2. □

Remark 3.3.7. Condition (i) is the element-wise order of the financial positions as used in Farkas and Smirnow (2018, Proposition 3.1) for intrinsic risk measures. Condition (ii) states that every reallocation of the first portfolio is dominated by a reallocation of the second portfolio.

The following result treats the case of a cone as acceptance set $\mathcal{A}$, i.e., for every $\lambda \geq 0$ it holds that $\lambda\mathcal{A} \subset \mathcal{A}$. It shows that the MAI risk measure is positive homogeneous, i.e., its epigraph is a cone.

Proposition 3.3.8 (Positive homogeneity)**.** *Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \pi)$ be an intrinsic risk measurement regime such that $\mathcal{A}$ is a cone. Then for each $M \in \mathcal{M}$ we have that*

$$\varrho_{\mathcal{R}}(0, M) = \begin{cases} \rho_{\mathcal{A},\mathcal{S},\pi}(M), & \rho_{\mathcal{A},\mathcal{S},\pi}(M) \leq -\pi(M), \\ -\pi(M), & \text{otherwise.} \end{cases}$$

Further, the epigraph of the MAI risk measure is a cone.

Proof: The first claim is a consequence of Theorem 3.3.2. For the second claim we prove for each $\lambda \geq 0$ that $\lambda \operatorname{epi}(\varrho_{\mathcal{R}}) \subset \operatorname{epi}(\varrho_{\mathcal{R}})$. Assume an arbitrary point $(X, M) \in \mathcal{X} \times \mathcal{M}$ and a scalar $\alpha \in \mathbb{R}$ such that $\varrho_{\mathcal{R}}(X, M) \leq \alpha$. For $\lambda = 0$ we have $\varrho_{\mathcal{R}}(0, 0) \leq \rho_{\mathcal{A},\mathcal{S},\pi}(0) \leq 0$. For $\lambda > 0$ we obtain for each $U, Z \in \mathcal{S}$ with $\pi(U) = \kappa\lambda\pi(M)$ and

$$\lambda X + (1 - \kappa)\, \lambda M + Z + U \in \mathcal{A}$$

that

$$\lambda \left(X + (1 - \kappa) M + \frac{1}{\lambda} Z + \frac{1}{\lambda} U \right) \in \mathcal{A} = \lambda \mathcal{A}.$$

Together with the definition of the MAI risk measure we obtain that

$$\varrho_{\mathcal{R}}(\lambda X, \lambda M) = \lambda \varrho_{\mathcal{R}}(X, M) \leq \lambda \alpha$$

and hence, $\operatorname{epi}(\varrho_{\mathcal{R}})$ is a cone. □

3.4 Diversification

In this section, we analyze if diversification between financial positions is rewarded by the MAI risk measure, i.e., the MAI risk measure of a diversified portfolio is smaller than the maximum of the MAI risk measures of the individual portfolios.

We clarify in Section 3.4.1 what we mean by diversification between financial positions. We present two different concepts: *Diversification within the balance sheet* and *diversification over business lines.* We show in Section 3.4.2 that diversification within the balance sheet is satisfied if we use marketed portfolios with equal prices. We identify this as the most relevant situation for a company. Further, we present a counterexample if the prices are different.

This example also implies that the MAI risk measure is not quasi-convex. Now, diversification over business lines refers to the quasi-convexity of the MAI risk measure. Hence, the MAI risk measure does not reward diversification over business lines. At the first glance, this is counter-intuitive, because the intrinsic risk measure is quasi-convex (but not convex), see Farkas and Smirnow (2018, Section 3.1). Since quasi-convexity reflects the diversification principle, the intrinsic risk measure rewards for diversification.

Finally, in Section 3.4.3 we present another example in which diversification over business lines is not satisfied. The example is different to the one in Section 3.4.2, because we use marketed portfolios with the same price.

3.4.1 Types of diversification

We start to recall the meaning of diversification for classical risk measures. In doing so, assuming a functional ρ describing the risk of a financial position, the diversification principle states the following: If for three payoffs X, Y, Z we have

$$\max\{\rho(X), \rho(Y)\} \leq \rho(Z),$$

then for any diversified position $\alpha X + (1-\alpha)Y$ with $\alpha \in (0,1)$ it also holds that

$$\rho(\alpha X + (1-\alpha)Y) \leq \rho(Z).$$

As pointed out in Karoui and Ravanelli (2009) and Cerreia-Vioglio et al. (2011), the diversification principle is equivalent to quasi-convexity of the underlying risk measure which is not equivalent to convexity in general.

Now an MAI risk measure depends on two random variables, one for the assets which can be reallocated and one for the remaining assets and liabilities. This is different to classical risk measures and hence, we have to clarify what diversification means in the context of an MAI risk measure. We distinguish two kinds of diversification:

First, if we have two marketed portfolios $M_1, M_2 \in \mathcal{M}$ and remaining assets and liabilities $X \in \mathcal{X}$, then we are interested in the MAI risk measure for the tuple

$$(X, \alpha M_1 + (1-\alpha)M_2)$$

with $\alpha \in (0,1)$. This means, we diversify between the parts in the balance sheet that are available for reallocation. This is a common situation, because the company has several possibilities to adjust liquid capital in the asset portfolio. In doing so, it is desirable that diversification between the resulting marketed portfolios reduces the capital requirement. Concluding, we consider diversification with respect to some liquid parts in the asset portfolio. This analysis answers the question whether diversification between marketed portfolios leads to a lower capital requirement. We call this situation *diversification within the balance sheet.*

For the second kind of diversification, we use the classical notion of convexity for maps. In this case, we look at tuples $(X, M_1), (Y, M_2) \in \mathcal{X} \times \mathcal{M}$ and we are interested in the MAI risk measure for

$$\alpha(X, M_1) + (1-\alpha)(Y, M_2).$$

If $X = Y$, then we have diversification within the balance sheet. Hence, the interesting case is the one of $X \neq Y$. In this case, we obtain a mixture of two balance sheets which motivates the following question: Does a splitting of a company into two business lines (e.g., subsidiaries) increase the capital requirement? We refer to this situation as *diversification over business lines.*

3.4.2 Diversification within the balance sheet

We start by analyzing diversification within the balance sheet. To this end, the balance sheet of a company is given by the tuple $(X, M_1) \in \mathcal{X} \times \mathcal{M}$. Another marketed portfolio would lead to the modified balance sheet $(X, M_2) \in \mathcal{X} \times \mathcal{M}$. Note that the relevant situation is the one in which the company sells the marketed portfolio M_1 and buys the marketed portfolio M_2. Hence, the prices of M_1 and M_2 are equal. The following result states that the MAI risk measure in this case reduces the capital requirement if the company diversifies between M_1 and M_2.

Theorem 3.4.1. *Assume a convex acceptance set $\mathcal{A}$ and a security space $\mathcal{S} = \operatorname{span}\{U\}$ with $U \in (\mathcal{M} \cap \mathcal{X}_+)\backslash\{0\}$ and $\pi(U) = 1$. This gives the intrinsic risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \pi)$. For marketed portfolios $M_1, M_2 \in \mathcal{M}$ with $\pi(M_1) = \pi(M_2) \neq 0$ and $\alpha \in (0,1)$ we obtain that*

$$\varrho_{\mathcal{R}}(X, \alpha M_1 + (1-\alpha)M_2) \leq \max\{\varrho_{\mathcal{R}}(X, M_1), \varrho_{\mathcal{R}}(X, M_2)\}.$$

Proof: Without loss of generality, we assume that $\varrho_{\mathcal{R}}(X, M_2) \leq \varrho_{\mathcal{R}}(X, M_1)$. Further, assume real numbers $y \leq x$ and $\kappa_1, \kappa_2 \in [0,1]$ with

$$X + xU + (1-\kappa_1)M_1 + \kappa_1\pi(M_1)U \in \mathcal{A},$$
$$X + yU + (1-\kappa_2)M_2 + \kappa_2\pi(M_2)U \in \mathcal{A}.$$

By monotonicity of $\mathcal{A}$ we also have that

$$X + xU + (1-\kappa_2)M_2 + \kappa_2\pi(M_2)U \in \mathcal{A}.$$

By convexity of $\mathcal{A}$ and since $\pi(M_1) = \pi(M_2)$ we know that for all $\beta \in [0,1]$ it holds that

$$X + xU + \beta(1-\kappa_1)M_1 + (1-\beta)(1-\kappa_2)M_2 + (\beta\kappa_1 + (1-\beta)\kappa_2)\pi(M_1)U \in \mathcal{A}.$$

The claim follows, if there exist $\beta \in [0,1]$ and $\kappa \in [0,1]$ such that

$$\begin{aligned} &X + xU + \beta(1-\kappa_1)M_1 + (1-\beta)(1-\kappa_2)M_2 + (\beta\kappa_1 + (1-\beta)\kappa_2)\pi(M_1)U \\ &= X + xU + (1-\kappa)(\alpha M_1 + (1-\alpha)M_2) + \kappa\pi(M_1)U. \end{aligned}$$

If $\kappa_1 = \kappa_2$, then the solution is $\beta = \alpha$ and $\kappa = \kappa_1$. Hence, we can assume that $\kappa_1 \neq \kappa_2$. Then, by comparing the coefficients of M_1, M_2 and U we obtain the following system of linear equations:

$$\begin{aligned} \beta(1-\kappa_1) &= \alpha(1-\kappa), \\ (1-\beta)(1-\kappa_2) &= (1-\alpha)(1-\kappa), \\ \beta\kappa_1 + (1-\beta)\kappa_2 &= \kappa. \end{aligned}$$

This system admits the following unique solution:

$$\kappa^* = \frac{\kappa_1 - \kappa_2 + \frac{1}{\alpha}(1-\kappa_1)\kappa_2}{\kappa_1 - \kappa_2 + \frac{1}{\alpha}(1-\kappa_1)}, \quad \beta^* = \frac{1-\kappa_2}{\kappa_1 - \kappa_2 + \frac{1}{\alpha}(1-\kappa_1)}.$$

Note that the denominator can not be zero, since $\alpha \in (0,1)$. Otherwise, we would obtain $\alpha = \frac{1-\kappa_1}{\kappa_2-\kappa_1}$ which gives us $\alpha < 0$ if $\kappa_2 < \kappa_1$ and $\alpha > 1$ if $\kappa_2 > \kappa_1$.

It remains to show that $\kappa^*, \beta^* \in [0,1]$. We only prove the case of β^*. The argument for κ^* works analogously. The numerator of β^* is greater or equal zero, i.e., $1-\kappa_2 \geq 0$. If $\kappa_2 < \kappa_1$, then by $1-\kappa_1 \geq 0$ we directly obtain a positive denominator, i.e.,

$$\kappa_1 - \kappa_2 + \frac{1}{\alpha}(1-\kappa_1) > 0.$$

In the remaining case of $\kappa_2 > \kappa_1$, the conditions $0 < \kappa_2 - \kappa_1 \leq 1 - \kappa_1$ and $\frac{1}{\alpha} > 0$ imply that $\kappa_1 - \kappa_2 + \frac{1}{\alpha}(1-\kappa_1) > 0$. This give us that $\beta^* \geq 0$.

Now, we compare the numerator and the denominator:

$$\left(\kappa_1 - \kappa_2 + \frac{1}{\alpha}(1-\kappa_1)\right) - (1-\kappa_2) = \left(\frac{1}{\alpha} - 1\right)(1-\kappa_1) \geq 0.$$

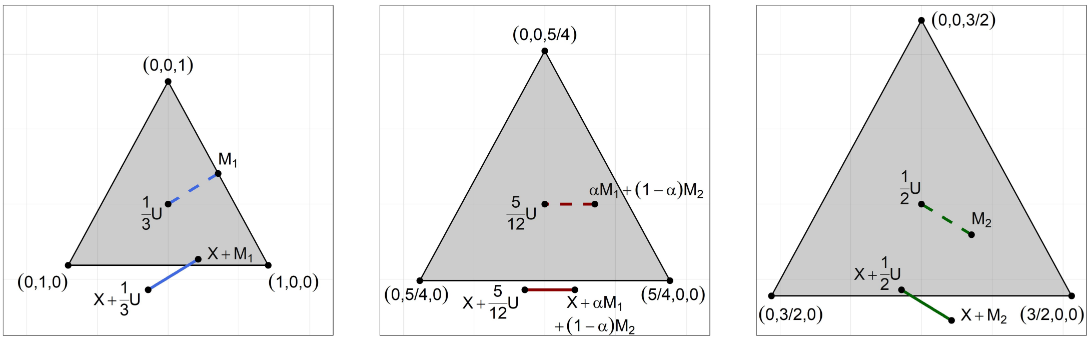
Hence, $\beta^* \leq 1$. □

The previous theorem works for marketed portfolios with the same price. The following example states that diversification is not rewarded for different prices in general.

Example 3.4.2. We use a sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, i.e., the linear space of financial positions $\mathcal{X}$ is 3-dimensional and can be identified with $\mathcal{X} = \mathbb{R}^3$. This is for example the situation in a one-period trinomial model in which the underlying asset can have 3 potential values at a fixed future time. In our example, the market consists of a risk-free bank account and two defaultable bonds:

$$U = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; S^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; S^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Hence, the marketed space is given as $\mathcal{M} = \operatorname{span}\{U, S^1, S^2\} = \mathbb{R}^3$ and we define the pricing functional $\pi : \mathcal{M} \to \mathbb{R}$ by $\pi(U) = 1$ and $\pi(S^1) = \pi(S^2) = \frac{1}{3}$. The bank account is used as the security asset, i.e., $\mathcal{S} = \operatorname{span}\{U\}$. The acceptance set $\mathcal{A}$ is given by the positive cone $\mathbb{R}^3_+$. Hence, $\mathcal{A}$ is convex. The intrinsic risk measurement regime is $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \pi)$. We use the following liquid positions: $M_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)^\intercal$ and $M_2 = \left(\frac{5}{6}, \frac{1}{3}, \frac{1}{3}\right)^\intercal$. M_1 has price $\frac{1}{3}$ and M_2 has price $\frac{1}{2}$. Hence, the intrinsic management actions allow to reallocate capital between M_1 and $\frac{1}{3}U$ and between M_2 and $\frac{1}{2}U$. These reallocations are illustrated by the dashed lines in Figures 3.4a and 3.4c. Further, the remaining assets and liabilities are given by $X = \left(\frac{2}{15}, \frac{1}{3}, -\frac{7}{15}\right)^\intercal$. The shifted line segments for the reallocations are illustrated by the solid lines in Figures 3.4a and 3.4c. In both cases, it is possible to create a payoff contained in the acceptance set. This implies that $\varrho_{\mathcal{R}}(X, M_1) \leq 0$ and $\varrho_{\mathcal{R}}(X, M_2) \leq 0$. But if we diversify the marketed portfolios with a mixture weight $\alpha = \frac{1}{2}$, then we obtain Figure 3.4b. Note, that $\alpha\frac{1}{3}U + (1-\alpha)\frac{1}{2}U = \frac{5}{12}U$. The solid line segment does not admit points contained in the acceptance set. Hence, the company has to invest additional capital in the security asset U to reach acceptability. This implies that $\varrho_{\mathcal{R}}(X, \alpha M_1 + (1-\alpha)M_2) > 0$. So, the MAI risk measure does not reward diversification between (X, M_1) and (X, M_2).



(a) Hyperplane containing M_1 and $X + M_1$.

(b) Hyperplane containing the diversified position.

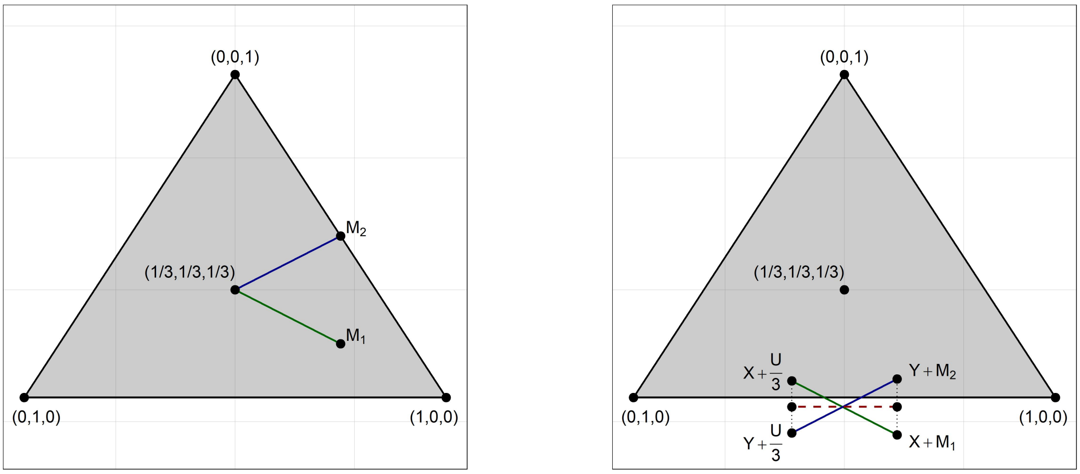
(c) Hyperplane containing M_2 and $X + M_2$.

Figure 3.4: All figures show a hyperplane from the direction $U = (1, 1, 1)^\intercal$. All points on such a hyperplane admit the same price. The gray area is the slice of the acceptance set $\mathcal{A} = \mathbb{R}^3_+$. We also demonstrate the line segments for reallocations between the liquid portfolio and the bank account U.

Remark 3.4.3. The previous example shows that the MAI risk measure is not quasi-convex, if we fix the remaining assets and liabilities. This can be explained as follows: By concentrating on the tuple (X, M_2), we see that a huge amount of the marketed portfolio M_2 needs to be reallocated to reach acceptability. For the tuple (X, M_1) we have vice versa that only a small amount of the marketed portfolio M_1 can be reallocated to stay acceptable. Now, if we consider the mixture $\alpha(X, M_1) + (1 - \alpha)(X, M_2)$, then reallocation always leads to an unacceptable position, because it is only possible to apply the same proportion for reallocating M_1 and M_2. As we already argued, this is not a disadvantage of the MAI risk measure, because the situation in Theorem 3.4.1 is the one of interest for a company.

3.4.3 Diversification over business lines

In the previous section, we showed that the MAI risk measure does not reward diversification within the balance sheet if the prices of the marketed portfolios differ. This implies in addition that the MAI risk measure is not quasi-convex. Hence, it also does not reward diversification over business lines.



(a) Before translation. (b) After translation.

Figure 3.5: Both figures show the hyperplane $\{x \in \mathbb{R}^3 \,|\, x_1 + x_2 + x_3 = 1\}$ from the direction given by $U = (1, 1, 1)^\intercal$. The gray area is the slice of the acceptance set $\mathcal{A} = \mathbb{R}^3_+$. All portfolios in this hyperplane have the price $\frac{1}{3}$. *Left-hand side:* Line segments between the security asset $\frac{1}{3}U$ and the liquid portfolios M_1 and M_2. *Right-hand side:* Line segments translated by the remaining assets and liabilities X and Y. The dashed red line is the line segment between the diversified portfolios $\alpha X + (1-\alpha)Y + \frac{1}{3}U$ and $\alpha X + (1-\alpha)Y + \alpha M_1 + (1-\alpha)M_2$.

But, we also know from Theorem 3.4.1 that the MAI risk measure rewards diversification within the balance sheet if the prices of the marketed portfolios are equal. Hence, the main finding of this section is Example 3.4.4 which shows in particular that the MAI risk measure does not reward diversification over business lines even if the marketed portfolios have the same price. We discuss the reason for this behavior in Remark 3.4.5.

Example 3.4.4. We use the same setting as in Example 3.4.2, i.e., $\mathcal{X} = \mathbb{R}^3$ and the market consists of the same risk-free bank account and the same defaultable bonds as before. The pricing functional is again defined by $\pi(U) = 1$ and $\pi(S^1) = \pi(S^2) = \frac{1}{3}$. The security space is $\mathcal{S} = \operatorname{span}\{U\}$. The acceptance set $\mathcal{A}$ is given by the positive cone $\mathbb{R}^3_+$ which leads to the intrinsic risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \pi)$. We modify the marketed portfolios as follows: $M_1 = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)^\intercal$ and $M_2 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)^\intercal$. They are illustrated on the left-hand side in Figure 3.5. The corresponding remaining assets and liabilities are given by: $X = (0.05, 0.3, -0.25)^\intercal$ and $Y = (0.15, 0.4, -0.39)^\intercal$. If we diversify the two positions $X + M_1$ and $Y + M_2$ with respect to a weight $\alpha = \frac{1}{2}$, the right-hand side in Figure 3.5 illustrates that $\varrho_{\mathcal{R}}(X, M_1) \leq 0$ and $\varrho_{\mathcal{R}}(Y, M_2) \leq 0$, but $\varrho_{\mathcal{R}}(\alpha X + (1-\alpha)\,Y, \alpha M_1 + (1-\alpha)\,M_2) > 0$. Hence, $\varrho_{\mathcal{R}}$ is not quasi-convex and it especially does not reward diversification between (X, M_1) and (Y, M_2).

Remark 3.4.5. Concluding, the MAI risk measure does not reward diversification between business lines in general. The reason for this behavior is analogous to the one in the previous section, compare Remark 3.4.3. It can be explained with the right-hand side of Figure 3.5. For the tuple (X, M_1), a huge amount of the marketed portfolio M_1 is reallocated to be acceptable. For the tuple (Y, M_2), only a small amount of the marketed portfolio M_2 is reallocated to be acceptable. The mixture $\alpha(X, M_1) + (1-\alpha)(Y, M_2)$ is unacceptable for every reallocation, because the MAI risk measure is not flexible enough to apply two different proportions for reallocating the marketed portfolios. Hence, an extension would be the usage of different intrinsic levels. But this requires a new construction of a risk measure which is out of the scope of this dissertation.

3.5 Dual representations

In this section, we present a dual representation of MAI risk measures, showing that an MAI risk measure is the maximum of two values, determined by extensions of the restricted pricing functional $\pi|_{\mathcal{S}}$. Each of these extensions can be interpreted as a possible probabilistic model. The agent does not know which of these models is correct. As a result, the capital requirement is calculated in a robust way, i.e., as the maximum capital requirement over all of these models, compare also Remark 3.5.4.

3.5.1 Some preliminaries

To perform a conjugate approach we assume for the rest of this section that the ordered topological vector space $\mathcal{X}$ is a locally convex space. Next, we recall terms and notations from functional analysis. The (lower) support function of a non-empty subset $\mathcal{C} \subset \mathcal{X}$ is defined as the map

$$h_{\mathcal{C}} : \mathcal{X}' \to [-\infty, \infty)\,,\ \psi \mapsto \inf_{C \in \mathcal{C}} \psi(C).$$

The barrier cone of $\mathcal{C}$ is defined by $\mathcal{B}(\mathcal{C}) := \operatorname{dom}(-h_{\mathcal{C}})$. For a pricing functional $\pi : \mathcal{M} \to \mathbb{R}$ and a security space $\mathcal{S} \subset \mathcal{M}$ we define the set of all positive, continuous and linear extensions of the restricted pricing functional $\pi|_{\mathcal{S}}$ as

$$\mathcal{E}_\pi(\mathcal{S}) := \left\{ \psi \in \mathcal{X}'_+ \,\middle|\, \psi|_{\mathcal{S}} = \pi|_{\mathcal{S}} \right\}.$$

For the proof of our main result we need Sion's minimax theorem which is a generalization of von Neumann's minimax theorem, see Sion (1958, Corollary 3.3). For completeness, we recall this result in the form given in Komiya (1988, Section 1).

Theorem 3.5.1 (Sion's minimax theorem)**.** *Let X be a compact convex subset of a linear topological space and Y a convex subset of a linear topological space. Let f be a real-valued function on $X \times Y$ such that*

(i) $f(x, .)$ is upper semicontinuous and quasi-concave on Y for each $x \in X$,

(ii) $f(., y)$ is lower semicontinuous and quasi-convex on X for each $y \in Y$.

Then

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y).$$

3.5.2 Duality of MAI risk measures

Now we are ready to present the main result of this section in which we require that $\mathcal{B}(\mathcal{A}) \cap \mathcal{E}_\pi(\mathcal{S}) \neq \varnothing$. For equivalent statements to this condition we refer to Theorem 2 and Corollary 1 in Farkas et al. (2015).

Theorem 3.5.2 (Dual representation)**.** *Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \pi)$ be an intrinsic risk measurement regime such that the acceptance set $\mathcal{A}$ is convex, the multi-asset risk measure $\rho_{\mathcal{A},\mathcal{S},\pi}$ is lower semicontinuous and $\mathcal{B}(\mathcal{A}) \cap \mathcal{E}_\pi(\mathcal{S}) \neq \varnothing$. Then for each $(X, M) \in \mathcal{X} \times \mathcal{M}$ we* *obtain that*

$$\varrho_{\mathcal{R}}(X,M) = \max\left\{ \sup_{\substack{\psi\in\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S})\\ \psi(M)\geq\pi(M)}} \Big(h_{\mathcal{A}}(\psi) - \psi(X+M)\Big), \right.$$
$$\left. \sup_{\substack{\psi\in\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S})\\ \psi(M)<\pi(M)}} \Big(h_{\mathcal{A}}(\psi) - \psi(X) - \pi(M)\Big)\right\}.$$

Proof: From Theorem 3.3.2 and Farkas et al. (2015, Theorem 3) we get

$$\varrho_{\mathcal{R}}(X,M) = \inf_{\kappa\in[0,1]} \sup_{\psi\in\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S})} \Big(h_{\mathcal{A}}(\psi) - \psi(X+M) + \kappa(\psi(M)-\pi(M))\Big).$$

For the sake of brevity, we use the following map:

$$f:[0,1]\times(\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S}))\to\mathbb{R},$$
$$(\kappa,\psi)\mapsto h_{\mathcal{A}}(\psi)-\psi(X+M)+\kappa(\psi(M)-\pi(M)).$$

To interchange the infimum and supremum in the previous equation we prove the conditions in Theorem 3.5.1:

- The unit interval $[0,1]$ is compact and convex. It is straightforward to see that $\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S})$ is also convex.
- For each $\psi\in\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S})$ the function $\kappa\mapsto f(\kappa,\psi)$ is lower semicontinuous and quasi-convex. This is a consequence of the linearity of this map.
- For each $\kappa\in[0,1]$ the function $g(\psi):=f(\kappa,\psi)$ is quasi-concave. Indeed, with $h_{\mathcal{A}}(\alpha\psi_1+(1-\alpha)\psi_2)\geq\alpha h_{\mathcal{A}}(\psi_1)+(1-\alpha)h_{\mathcal{A}}(\psi_2)$ we obtain that $h_{\mathcal{A}}$ is concave. This implies quasi-concavity of g. To prove the upper semicontinuity of g we equip the set $\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S})$ with the relative weak*-topology defined by

  $$\sigma\left(\mathcal{X}',\mathcal{X}\right)_{\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S})} := \left\{V\cap\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S}) \,\middle|\, V\in\sigma\left(\mathcal{X}',\mathcal{X}\right)\right\}.$$

  Let $\mathcal{C}_{\mathcal{A}}$ be the set of all $\sigma\left(\mathcal{X}',\mathcal{X}\right)_{\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S})}$-continuous functions of the form

  $$\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S})\to\mathbb{R},\ \psi\mapsto\psi(A)-\psi(X+M)+\kappa(\psi(M)-\pi(M))$$

  with $A\in\mathcal{A}$. So, $g(\psi)=\inf_{l\in\mathcal{C}_{\mathcal{A}}} l(\psi)$ as the pointwise infimum of $\sigma\left(\mathcal{X}',\mathcal{X}\right)_{\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S})}$-continuous functions is $\sigma\left(\mathcal{X}',\mathcal{X}\right)_{\mathcal{B}(\mathcal{A})\cap\mathcal{E}_\pi(\mathcal{S})}$-upper semicontinuous.

After interchanging the infimum and supremum the claim is a consequence of distinguishing the cases $\psi(M)\geq\pi(M)$ and $\psi(M)<\pi(M)$. □

As an example, we investigate the special case of monetary risk measures on the Banach space of essentially bounded random variables.

Example 3.5.3. Consider a probability space $(\Omega, \mathscr{F}, P)$. As an ordered topological vector space use $L^\infty(\Omega, \mathscr{F}, P)$ together with the essential supremum norm and the P-a.s. order. Assume a suitable marketed space $\mathcal{M} \subset L^\infty(P)$ such that the security space can be defined by $\mathcal{S} = \operatorname{span}\{1_\Omega\}$. For the pricing functional $\pi : \mathcal{M} \to \mathbb{R}$ it holds that $\pi(1_\Omega) = 1$. Assume a *convex* monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$ which is continuous from above, i.e., $X_n \downarrow X$ implies $\rho(X_n) \uparrow \rho(X)$. The acceptance set is defined by $\mathcal{A} = \{X \in L^\infty(P) \mid \rho(X) \leq 0\}$. Given the penalty function $\alpha(Q) := \sup_{A \in \mathcal{A}} E_Q[-A]$ and denoting by $M(P)$ the set consisting of probability measures Q such that Q is absolutely continuous with respect to P and $\alpha(Q) < \infty$, we obtain for every $X \in L^\infty(P)$ and $M \in \mathcal{M}$ that

$$\varrho_{\mathcal{R}}(X, M) = \max\left\{ \sup_{\substack{Q \in M(P) \\ E_Q[M] \geq \pi(M)}} \Big(E_Q[-X - M] - \alpha(Q) \Big), \sup_{\substack{Q \in M(P) \\ E_Q[M] < \pi(M)}} \Big(E_Q[-X - \pi(M)] - \alpha(Q) \Big) \right\}. \tag{3.6}$$

Remark 3.5.4. (i) Continuity from above ensures that the restriction to countably additive measures is sufficient, see e.g., Föllmer and Schied (2016, Theorem 4.33).

(ii) Equation (3.6) allows for the following interpretation: Each $Q \in M(P)$ corresponds to a probabilistic model. Given such a model, the intrinsic action compares the value $E_Q[M]$ of maintaining the liquid part M with the amount $\pi(M)$ by reallocating the whole portfolio M into the bank account 1_Ω. These two situations lead to the objective values of $E_Q[-X-M]-\alpha(Q)$ and $E_Q[-X-\pi(M)]-\alpha(Q)$. Hence, the supremum over all costs with respect to the possible probabilistic models yields the value of the MAI risk measure.

3.6 Numerical examples

In this section, we illustrate the behavior of the MAI risk measure with respect to concrete balance sheet models. The original idea for the chosen models stems from Floreani (2013). Their model is used to demonstrate effects in the Solvency II framework and solely relies on normal distributed assumptions. We use different distributions by applying a Black-Scholes model and common assumptions for insurance portfolios.

3.6.1 Balance sheet model and acceptability criterion

We consider a simplified balance sheet model. We bundle assets and liabilities to describe them by a low number of random variables. To do so, we have to assume a probability space $(\Omega, \mathscr{F}, P)$. The corresponding ordered topological vector space is chosen to be large enough to cover all relevant random variables, e.g., we could use $\mathcal{X} = L^1(P)$ together with the P-a.s. order and the usual L^1-norm.

We consider a time interval from 0 up to a finite future time $T > 0$. Further, we use a Black-Scholes model with two stocks. In doing so, we equip the probability space with a filtration $(\mathscr{F}_t)_{t\in[0,T]}$. The assets are then described as follows:

Black-Scholes model: The bank account with interest rate $r \in \mathbb{R}$ admits the starting value $B_0 > 0$ and thus has the future value $B_T := B_0 \exp(rT)$. The two stocks are based on a two-dimensional standard Brownian motion $(W_t^1, W_t^2)_{t\in[0,T]}$. The stock $(S_t^i)_{t\in[0,T]}$ with $i \in \{1,2\}$ is based on the excess return $\lambda_i \in \mathbb{R}$, the volatility $\sigma_i > 0$ and the initial value $S_0^i > 0$. Further, we assume a correlation $\rho_{1,2} \in [-1,1]$. This leads to the following stock values at T:

$$S_T^1 = S_0^1 e^{\left(r+\lambda_1-\frac{1}{2}\sigma_1^2\right)T+\sigma_1 W_T^1}, \; S_T^2 = S_0^2 e^{\left(r+\lambda_2-\frac{1}{2}\sigma_2^2\right)T+\sigma_2\left(\rho_{1,2}W_T^1+\sqrt{1-\rho_{1,2}^2}W_T^2\right)}.$$

The pricing functional π is defined by $\pi(xB_T + yS_T^1 + zS_T^2) = xB_0 + yS_0^1 + zS_0^2$ for all $x, y, z \in \mathbb{R}$.

We restrict attention to one-period trading. The institution holds the number of shares $\varphi^B \geq 0$ in the bank account, $\varphi_1 \geq 0$ in stock 1 and $\varphi_2 \geq 0$ in stock 2. Concluding, the assets of the institution at T are given by $\varphi^B B_T + \varphi_1 S_T^1 + \varphi_2 S_T^2$.

Liabilities: It remains to model the liabilities. We apply a gamma distribution as a typical example used for claim size modeling in insurance mathematics, see e.g., Ohlsson and Johansson (2010). We denote the random variable of the liabilities at time T by L_T and we assume that it is gamma distributed with shape $\alpha > 0$ and rate $\beta > 0$, i.e., the probability density function for $x > 0$ is given by

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

where Γ denotes the Gamma function. We assume that the liabilities are uncorrelated to the two stocks. Finally, in our simplified model we determine the equity capital X_T at time T as assets minus liabilities,

$$X_T = \varphi^B B_T + \varphi_1 S_T^1 + \varphi_2 S_T^2 - L_T.$$

Reallocation: We assume that the investment in stock 1 can be reallocated, but the investment in stock 2 is not available for reallocation. For the MAI risk measure, we use the linear span of the bank account as security space, i.e., reallocating capital between stock 1 and the bank account is allowed.

This situation can be interpreted as follows: To pass the capital adequacy test, the institution would like to increase the amount in the bank account, as it is the case for classical risk measures. On the one hand, the institution is not sure, if stock 1 is any longer an attractive investment, e.g., due to a bad performance in the past. On the other hand, the institution is confident in stock 2. Hence, the institution solely allows for reallocating capital from stock 1 into the bank account.

Acceptability criterion: For a specific confidence level $\lambda \in (0,1)$, we employ the Expected Shortfall acceptance set $\mathcal{A}_{\mathrm{ES}_\lambda}$.

3.6.2 Risk measure calculations

In our study, we compare a single-asset, a multi asset and an MAI risk measure. The formulas for the risk measures are as follows:

- **Single-asset risk measure:**

$$\rho_{\mathcal{A}_{\mathrm{ES}_\lambda},B_T,\pi}(X_T) = \frac{B_0}{B_T}\,\mathrm{ES}_\lambda\left(\varphi_1 S_T^1 + \varphi_2 S_T^2 - L_T\right) - \varphi^B B_0.$$

- **MAI risk measure:** Theorem 3.3.2 gives us the representation for the MAI risk measure based on the intrinsic risk measurement regime $\mathcal{R} = (\mathcal{A}_{\mathrm{ES}_\lambda}, \mathrm{span}\{B_T\}, \pi)$:

$$\begin{aligned}&\varrho_{\mathcal{R}}\left(\varphi^B B_T + \varphi_2 S_T^2 - L_T, \varphi_1 S_T^1\right)\\&= \inf_{\kappa\in[0,1]}\left(\frac{B_0}{B_T}\,\mathrm{ES}_\lambda\left(\varphi_1(1-\kappa)S_T^1 + \varphi_2 S_T^2 - L_T\right) - \kappa\varphi_1 S_0^1\right) - \varphi^B B_0.\end{aligned} \tag{3.7}$$

- **Multi-asset risk measure:** Similar to the MAI risk measure, we obtain the following representation for the multi-asset risk measure with respect to the security space $\mathcal{S} = \mathrm{span}\{B_T, S_T^1\}$:

$$\rho_{\mathcal{A}_{\mathrm{ES}_\lambda},\mathcal{S},\pi}(X_T) = \inf_{x\in\mathbb{R}}\left(\frac{B_0}{B_T}\,\mathrm{ES}_\lambda\left(x S_T^1 + \varphi_2 S_T^2 - L_T\right) + x S_0^1\right) - \varphi^B B_0 - \varphi_1 S_0^1.$$

It is not possible to solve these risk measures analytically, because our distributional assumptions prevent from an explicit calculation of the Expected Shortfall. Hence, we calculate it by Monte-Carlo simulations based on 1 000 000 scenarios. In addition, for the MAI risk measure and the multi-asset risk measure we perform a numerical optimization algorithm to determine the infimum.

3.6.3 Numerical results

We illustrate similarities and differences between the MAI risk measure and the single-asset as well as the multi-asset risk measure by calculating the risk measures for three different parameter settings, see Table 3.1.

Parameter	**Setting 1**	**Setting 2**	**Setting 3**
λ	0.5%	0.5%	0.5%
r	0	0	0
T	1	1	1
φ^B	0.50	0.50	0.50
φ_1	2.00	2.00	2.00
φ_2	1.50	1.50	1.50
$B_0 = S_0^1 = S_0^2$	1.00	1.00	1.00
λ_1	**0.02**	**0.04**	**0.06**
λ_2	0.08	0.08	0.08
σ_1	0.20	0.20	0.20
σ_2	0.40	0.40	0.40
$\rho_{1,2}$	**0.75**	**0.00**	**-0.75**
α	35	35	35
β	10	10	10

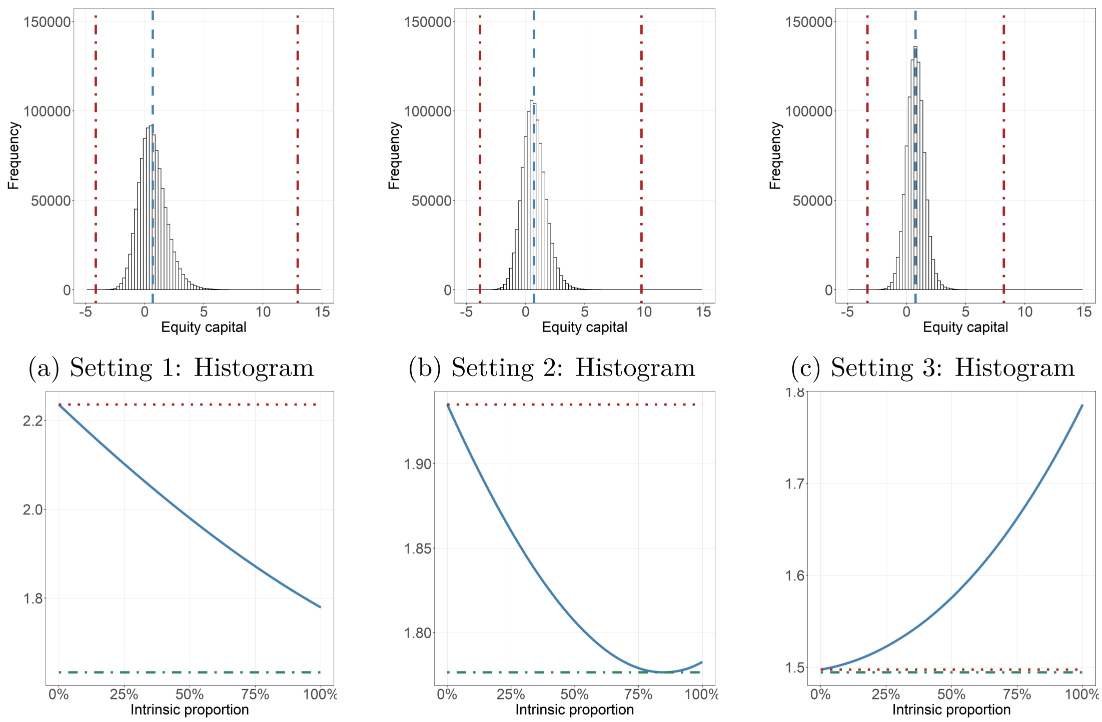
Table 3.1: Parameter values for the generic analysis of the model.

The settings differ in the excess return of stock 1 and the correlation between the stocks. The large positive correlation in setting 1 may make diversification between the two stocks less attractive. In the other two settings, this behavior changes and stock 1 becomes more attractive.

In Figure 3.6, we show the histograms of the equity capital X_T for the 1 000 000 Monte-Carlo simulations. Further, it shows the objective functions in the infimum of (3.7), i.e., the minimum of this objective function is the value of the MAI risk measure. In the following, we interpret the histograms and the objective functions:

Histogram: The largest range of values occurs in the first setting. In contrast, in the third setting, the distribution is more centered, because of the higher excess return of stock 1 and the negative correlation between the stocks. Hence, a decrease in a stock is compensated by an increase in the other stock.

Further, in the first setting the distribution is right-skewed and in the third setting the distribution is more symmetric. The characteristics of the second setting are in between. The minimum values and the means are almost the same in all settings. In contrast, the maximum value is the highest in the first setting and the lowest in the



(a) Setting 1: Histogram (b) Setting 2: Histogram (c) Setting 3: Histogram

(d) Setting 1: MAI objective (e) Setting 2: MAI objective (f) Setting 3: MAI objective

Figure 3.6: *First row:* Histograms of the equity capital X_T. The minimum and the maximum values (red dotted dashed) and the means (blue dashed) of the simulated values are given. *Second row:* Objectives of the MAI risk measures depending on the intrinsic level, see (3.7) (blue solid line). The single-asset (red dotted) and the multi-asset (green dotted dashed) risk measures correspond to the horizontal lines.

third setting.

Objective function: The three settings demonstrate three possible behaviors of the MAI risk measure in comparison to a single-asset and a multi-asset risk measure. In the first setting, the value of the MAI risk measure is attained on the boundary of the interval at 100%. Hence, all shares of stock 1 are sold and reinvested into the bank account. This trading constraint leads to a higher capital requirement than for the multi-asset risk measure.

If the trading constraint does not matter, then both risk measures are equal. This is the case for the second setting. Then, stock 1 performs much better than in the first setting and the institution keeps shares of this stock in order to benefit from its future performance.

In the third setting, the stocks admit a high negative correlation and therefore, from a risk management perspective it is desirable to keep all shares of stock 1 to benefit from diversification effects between the stocks. In this case, the MAI risk measure and the single-asset risk measure are identical.

The MAI risk measure is different from the multi-asset risk measure when the transaction constraint is active, i.e., the intrinsic level is attained on the boundary of the unit interval. To illustrate their differences, we concentrate on the first setting and stress the capital requirement by changing certain parameters, see Figure 3.7. We interpret the plots in the following:

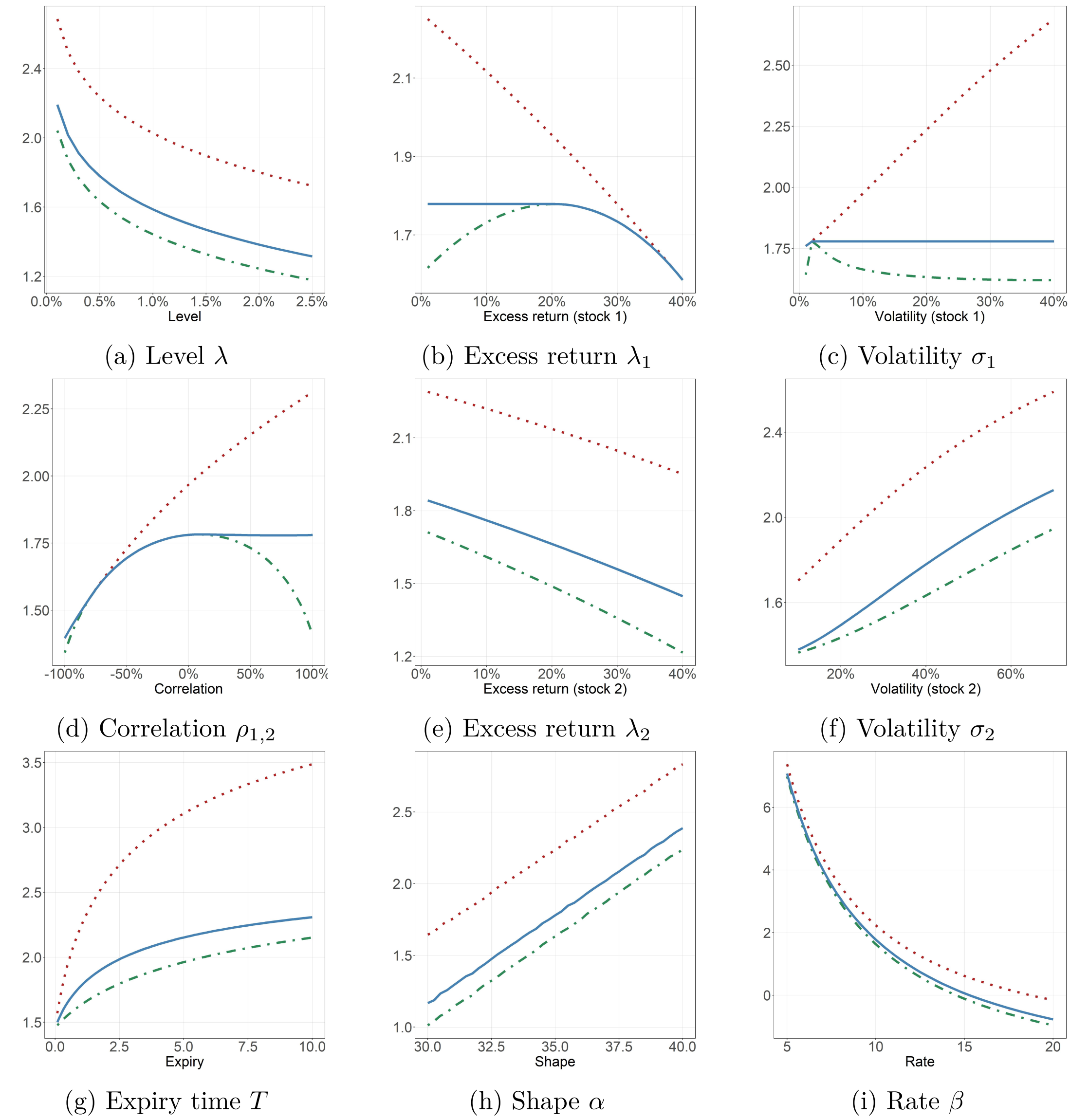
- **Figure 3.9a (level λ):** If the level increases, then the risk measures decrease. The graphs are convex, i.e., the higher the confidence level, the smaller the decrease in the risk measure.
- **Figure 3.9b (excess return λ_1):** For a small excess return the MAI risk measure leads to a sale of all shares of stock 1. Hence, the MAI risk measure is constant for small excess returns. By increasing the excess return further, some shares of stock 1 are maintained in the portfolio. In this case, the MAI risk measure coincides with the multi-asset risk measure. If the excess return is large, then no shares are sold and the MAI risk measure coincides with the single-asset risk measure.
- **Figure 3.9c (volatility σ_1):** If the volatility is low, then the probability of a downfall of stock 1 is low and it becomes an alternative to the classical bank account. Hence, no shares of stock 1 are sold and the MAI risk measure coincides with the single-asset risk measure. We also see that the multi-asset risk measure is lower than the MAI risk measure, because the capital requirement can be further reduced by buying new shares of stock 1. This trading action is not allowed by the MAI risk measure.

If the volatility increases, then stock 1 becomes less attractive and all shares are sold. Hence, the MAI risk measure is constant for large enough values of σ_1. We see that the multi-asset risk measure decreases, because it can create short positions in stock 1. In contrast, the single-asset risk measure completely relies on the original stock portfolio. Hence, a huge additional amount in the bank account is required to pass the capital adequacy test.

- **Figure 3.9d (correlation $\rho_{1,2}$):** If the correlation admits a large negative value, then the MAI risk measure tries to benefit from diversification effects between the stocks and no shares of stock 1 are sold. In this case, the MAI and the single-asset risk measure are equal. If the correlation is around zero, then the MAI risk measure suggests to sell a proportion between 0 and 1 of the stock 1 shares. In this case, the MAI and the multi-asset risk measure coincide. If the correlation admits a large positive value, then stock 1 is not an attractive investment and all shares are sold. The MAI risk measure is constant and differs from the multi-asset risk measure which creates short positions in stock 1.
- **Figure 3.9e and 3.9f (excess return λ_2 and volatility σ_2):** If the excess return increases, then all three risk measures decrease almost linearly. The large excess return makes stock 2 more attractive and its simulated values increase. Hence, less capital requirement is needed. If the volatility increases, then all three risk measures increase. The large volatility leads to more uncertainty in stock 2. Hence, more capital is required to pass the capital adequacy test.
- **Figure 3.9g (expiry T):** For higher T, the single-asset risk measure moves further away from the others. All risk measures increase in T. This means, the higher uncertainty in stock 1 is stronger than the opposite effect with respect to the discount factor and the larger excess return.
- **Figure 3.9h and 3.7i (shape α and rate β):** If we increase the shape or decrease the rate, then the risk measures increase. In both situations, the mean and the variance of the liabilities increase. So, we obtain higher liabilities on average and also the range of the values is larger. Furthermore, we see that the changes in the shape α have a significant influence on the numerical stability of our procedure.

3.6.4 Heavy-tailed distributed liabilities

We also tested the use of a generalized Pareto distribution (GPD) to model heavy-tailed distributed liabilities. Of course, this is a more dangerous situation instead of using a gamma distribution and consequently the capital requirements increase significantly.



(a) Level λ (b) Excess return λ_1 (c) Volatility σ_1

(d) Correlation $\rho_{1,2}$ (e) Excess return λ_2 (f) Volatility σ_2

(g) Expiry time T (h) Shape α (i) Rate β

Figure 3.7: Single-asset (red dotted), multi-asset (green dotted dashed) and MAI (blue solid) risk measures under parameter changes.

The probability density function of a GPD with shape $\xi \in \mathbb{R}$ and scale $\beta > 0$ is given by

$$f(x) = \frac{1}{\beta}\left(1 + \xi\frac{x}{\beta}\right)^{-\frac{1}{\xi+1}}.$$

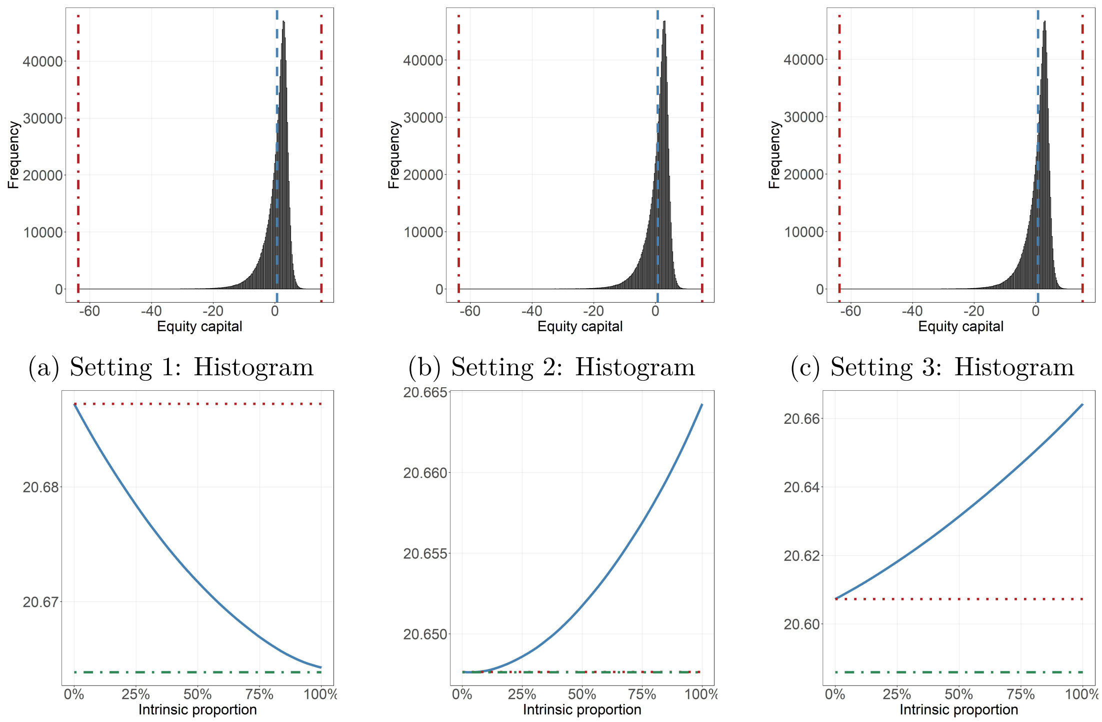
The rest of the model is analogous to the one in Section 3.6.1. For our first results, we use the three parameter settings in Table 3.2. We use the same parameters as in Table 3.1. Only the correlation $\rho_{1,2}$ is the same for all settings to illustrate three different behaviors of the MAI risk measure, see Figure 3.8. The forms of these curves are analogous to the ones in Section 3.6.3.

Parameter	**Setting 1**	**Setting 2**	**Setting 3**
λ	0.5%	0.5%	0.5%
r	0	0	0
T	1	1	1
φ^B	0.50	0.50	0.50
φ_1	2.00	2.00	2.00
φ_2	1.50	1.50	1.50
$B_0 = S_0^1 = S_0^2$	1.00	1.00	1.00
λ_1	**0.02**	**0.04**	**0.06**
λ_2	0.08	0.08	0.08
σ_1	0.20	0.20	0.20
σ_2	0.40	0.40	0.40
$\rho_{1,2}$	0.75	**0.75**	**0.75**
ξ	0.05	0.05	0.05
β	3.325	3.325	3.325

Table 3.2: Parameter values for the generic analysis of the model with liabilities following a GPD.

Finally, the shape and the scale of the GPD are chosen such that the mean is equal to the one for the gamma distribution, namely 3.5. The histograms in Figure 3.8 look quite similar, because the distribution is now significantly influenced by the liabilities and they follow the same distribution in all three examples. Further, the minimum value of the simulated data is much lower than in the example based on the gamma distribution. This explains why the values of the risk measures are now significantly larger.

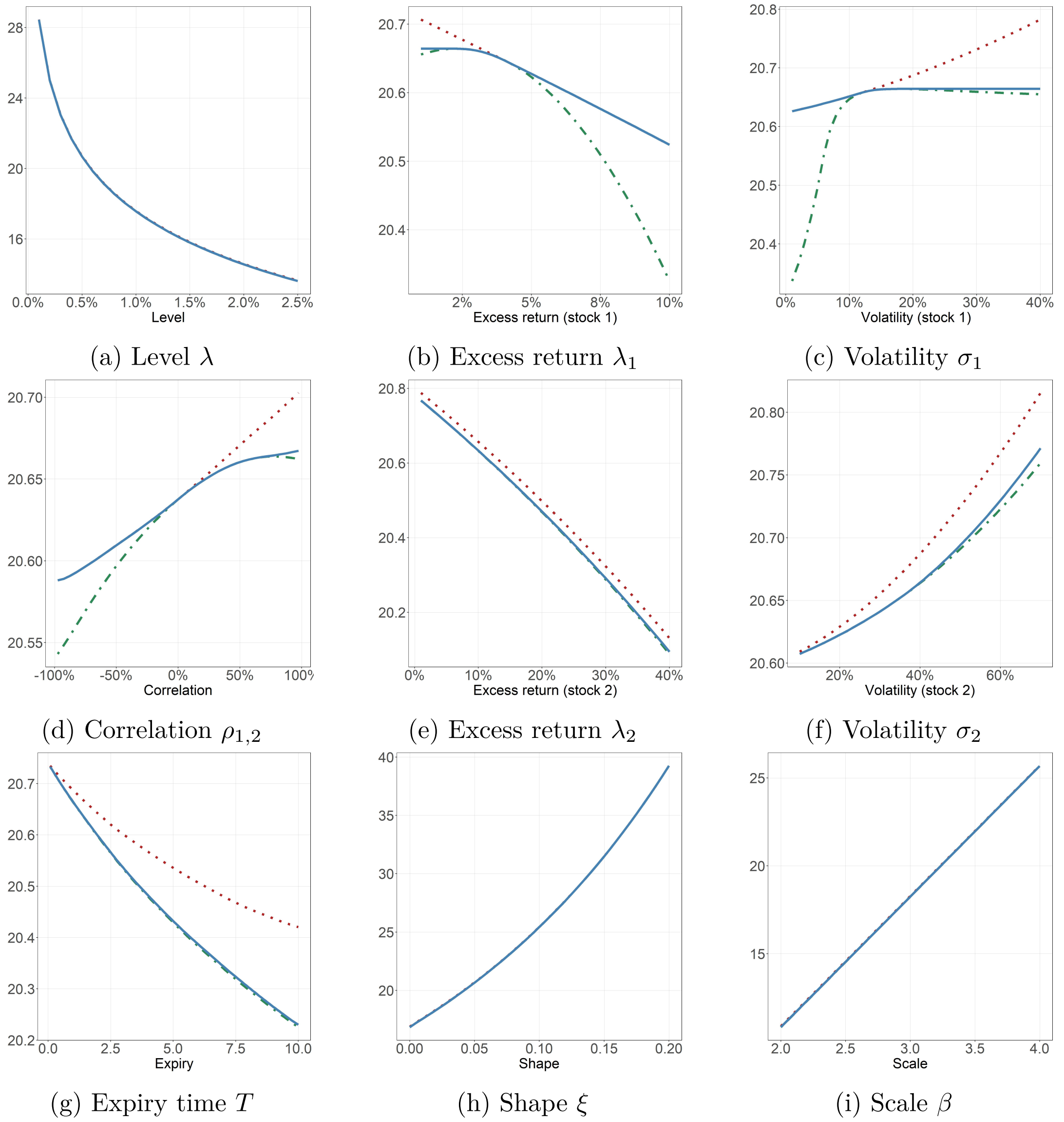
For the plots in Figure 3.9 we apply the first setting and change for each plot a single parameter. We observe in most of the plots the same forms as for the plots in Figure 3.7. Solely, for the expiry time T the curves are now decreasing instead of increasing. This is explained as follows: By increasing the expiry we have two opposite effects. On the one hand, the discount factor decreases and the excess returns of the stocks increase. On the other hand, it increases the variance of the stock values. In the gamma distributed example, the second effect is stronger and hence, the capital requirement increases. In the GPD example, the first effect is stronger and the capital requirement decreases.



(a) Setting 1: Histogram (b) Setting 2: Histogram (c) Setting 3: Histogram

(d) Setting 1: MAI objective (e) Setting 2: MAI objective (f) Setting 3: MAI objective

Figure 3.8: *First row:* Histograms of the equity capital. The red dotted dashed lines are the minimum and maximum values and the blue dashed lines are the means of the simulated values. *Second row:* Objectives of the MAI risk measure depending on the intrinsic level (blue solid line). The single-asset (red dotted) and the multi-asset (green dotted dashed) risk measures correspond to the horizontal lines.



(a) Level λ (b) Excess return λ_1 (c) Volatility σ_1

(d) Correlation $\rho_{1,2}$ (e) Excess return λ_2 (f) Volatility σ_2

(g) Expiry time T (h) Shape ξ (i) Scale β

Figure 3.9: Single-asset (red dotted), multi-asset (green dotted dashed) and MAI (blue solid) risk measures under parameter changes.

Chapter 4

Scalarized utility-based multi-asset risk measures

4.1 Introduction

Banks and insurance companies have to satisfy ambitious legal requirements, like the Basel Accords for banks or Solvency II for insurers. But at the same time their main concern is to maximize their own benefits, i.e., they aim for maximizing their expected utility. If the initial endowment of the company is not sufficient to pass the capital adequacy test, classical portfolio optimization methods fail. As a consequence, an increase of capital is necessary. In this chapter we develop a methodology that enables banks and insurance companies to optimize their expected utility in accordance with legal requirements, even if their financial circumstances are tight. This leads to a new kind of a multi-asset risk measure. This chapter is based on Desmettre, Laudagé, and Sass (2021).

4.1.1 Motivation

We start by presenting the classical portfolio optimization problem, where the legal requirements appear as a constraint and show that this methodology is not an expedient solution in a tight financial situation. Such kind of portfolio optimization problems are, e.g., discussed in Basak and Shapiro (2001), Cuoco, He, and Isaenko (2008) and regarding to a Solvency II framework in Nguyen and Stadje (2020, Section 4). Their approaches rely on the initial endowment. In doing so, a constraint states that only reallocation with respect to this initial endowment is allowed. This is meaningful if the financial situation of the bank or insurance company is good enough. Otherwise, there is no guarantee for the existence of a reallocation that satisfies the capital adequacy test. This means, the initial endowment is insufficient to create a financial position that satisfies the legal requirements and an increase of capital is indispensable. The latter is the situation we are interested in and what we called earlier a tight financial situation. Before we go in this direction, we provide an example in which no reallocation of the initial endowment satisfies a capital adequacy test.

Consider a balance sheet model as illustrated in Figure 4.1. Reallocation is only possible for assets traded in a financial market, i.e., only for liquid assets. We assume that $E_0 = 0.3$ and $F_0 = 1$, in particular that $A_0 - L_0 = E_0 - F_0 = -0.7$. The financial position of interest is the random equity capital in one year. The liquid assets in one year are given as a portfolio of a deterministic bank account B_1 and a single stock S_1, both with actual price 1. The interest rate is zero, i.e., the bank account is constant over time. Say, the illiquid assets minus liabilities $A_1 - L_1$ are normally distributed with a mean of -0.7 $(= A_0 - L_0)$ and a standard deviation of 0.11.

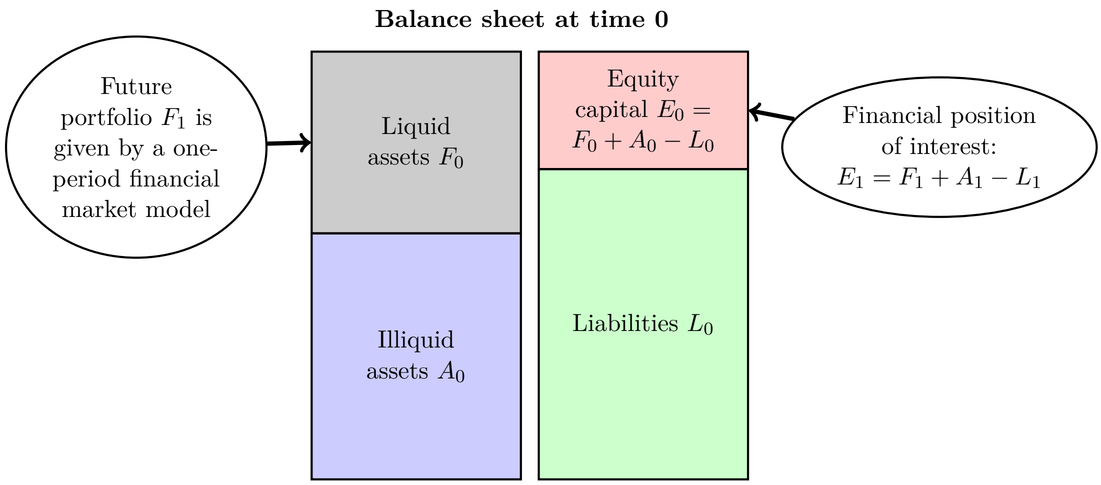


Figure 4.1: Simplified subdivision of a balance sheet.

As two possible legal requirements we consider the 0.5% Value-at-Risk as well as the 0.5% Expected Shortfall of the equity capital. The legal requirement says that the corresponding risk measure should be less or equal zero, i.e., in the Value-at-Risk case the insurance company becomes insolvent in at most 0.5% of the future scenarios. The level of 0.5% corresponds to the Solvency II framework, see the "Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II)" (2009). The equity capital at time 0 is given as

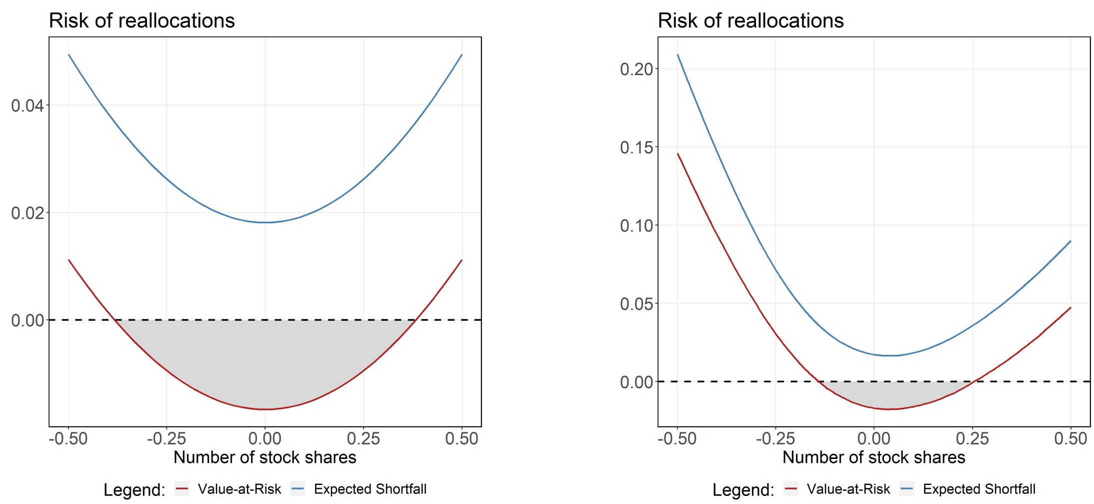
$$F_0 + A_0 - L_0 = 1 + A_0 - L_0 = (1 - \varphi)B_0 + \varphi S_0 + A_0 - L_0,$$

where $\varphi \in \mathbb{R}$ represents the number of stock shares. We do not allow for intermediate trading between time 0 and 1. Hence, the future equity capital position is given as

$$F_1 + A_1 - L_1 = (1 - \varphi)B_1 + \varphi S_1 + A_1 - L_1 = 1 - \varphi + \varphi S_1 + A_1 - L_1.$$

Let us consider the future equity capital as a function of the stock shares φ. If the Value-at-Risk, respectively the Expected Shortfall, is strictly positive for all of these equity capital positions, then it is not possible to satisfy the legal requirement by reallocating actual positions and the classical portfolio optimization approach is not applicable. This issue requires to model the stock reasonably. In Figure 4.2, we illustrate the Value-at-Risk and also the Expected Shortfall at level 0.5% for two different stock models. On the left-hand side we use a Bachelier model (Bachelier, 1900) and on the right-hand side a Black-Scholes model (Black & Scholes, 1973).

The Bachelier model allows for an analytical calculation of the risk measures. In contrast, the risk measures in the Black-Scholes model are calculated numerically, because we can not calculate the convolution of normal and log-normal distributed random variables analytically. Hence, we use a Monte-Carlo simulation to determine the risk measures.



(a) Bachelier model (b) Black-Scholes model

Figure 4.2: Risk measures for different allocations of the initial endowment.

We choose the parameters such that it is possible to satisfy the Value-at-Risk constraint. The more conservative requirement given by the Expected Shortfall can not be satisfied by capital rearrangements and an additional increase of capital is needed. We refer to this task as the ***risk assessment problem***. In the following we discuss classical risk measures to resolve this situation and point out their weaknesses.

4.1.2 Bibliographic notes

Recall from Chapter 1 that multi-asset risk measures are based on the idea of an agent investing money into some primary assets to make a payoff acceptable in the sense that it passes the capital adequacy test. The objective is the cost function. Hence, the institution can satisfy the legal requirements by a capital increase, even in a tight financial situation. This solves the risk assessment problem.

The application of multi-asset risk measures is standard but they completely ignore the actual intention of a bank or insurance company. As pointed out above, the main purpose of these institutions is the creation of profits which increase their expected utility. However, this expected utility is not considered in the objective function.

But this is done in the context of utility-based monetary risk measures: The ***optimized certainty equivalent (OCE)*** in Ben-Tal and Teboulle (1986) and Ben-Tal and Teboulle (2007) measures the investment costs in a single eligible asset minus the expected utility of the final position. Another idea is introduced in Krokhmal (2007) and Vinel and Krokhmal (2017). In their setting it is possible to use the certainty equivalent instead of the expected utility. This systematic is further analyzed in Geissel, Sass, and Seifried (2018). They introduce the so-called ***optimal expected utility (OEU) risk measure***.

OEU risk measures are useful for real world problems. For instance, Fink, Geissel, Sass, and Seifried (2019) rate retail structured products with the help of them. Another field of application is proposed in Fink, Geissel, Herbinger, and Seifried (2019). They use OEU risk measures as risk constraint in a return-risk portfolio optimization problem. In particular, the portfolio selection is constrained to an initial endowment. As already mentioned, in this chapter we do not rely on an initial endowment constraint.

4.1.3 Path to a new risk measure

OCE and OEU are not adequate for our intended application for several reasons. They only allow for investments in a single eligible asset to secure a payoff. But this is a restriction, compare the motivation in Section 1.1.1. Furthermore, OCE and OEU only consider implicitly the positive cone of almost surely bounded random variables as an acceptance set. Hence, it is not possible to consider common legal requirements such as the Value-at-Risk or the Expected Shortfall. Due to the aforementioned limitations, we aim for the following generalizations:

(1) We allow for investments into multiple eligible assets instead of a single risk-free eligible asset. (2) We allow for arbitrary preference restrictions, like e.g., Value-at-Risk or Expected Shortfall. (3) To include a wide range of financial market models and especially to be able to handle unbounded future payoffs, we use model spaces that are larger than the space of almost surely bounded random variables. (4) We punish

the hedging costs via a penalization term. This term can be the expected utility or the certainty equivalent, i.e., we treat OCE and OEU simultaneously. For the sake of brevity, we call the resulting new risk measure a ***scalarized utility-based multi-asset (SUBMA) risk measure***.

4.1.4 Contributions of this chapter

Analyzing SUBMA risk measures leads to demanding tasks from a mathematical as well as an application-oriented point of view. The main findings in this chapter are summarized below:

(i) **Construction:** The definition of the SUBMA risk measure as an instrument in tight financial situations is new, see Definition 4.3.1.

(ii) **Coherence:** OCE risk measures are only coherent for piecewise linear utility functions with a kink at 0, see e.g., Geissel et al. (2018, Example 1.5). In particular, they are not coherent for strictly concave utility functions. But OEU risk measures are coherent for utility functions with constant relative risk aversion. We show that the latter remains valid if we use multiple eligible assets and arbitrary risk constraints, see Theorem 4.4.4.

(iii) **Karush-Kuhn-Tucker (KKT) conditions:** We solve SUBMA risk measures with respect to one-period financial market models by applying the KKT conditions, see Sections 4.5.3 and 4.5.4.

(iv) **Finiteness, continuity and optimal payoffs:** For a one-period financial market model, Proposition 4.6.1 characterizes the cases in which the SUBMA risk measure attains $+\infty$. Proposition 4.6.4 and Theorem 4.6.6 give sufficient conditions such that the SUBMA risk measure does not attain $-\infty$. If the SUBMA risk measure is real-valued, then it is also continuous, see Theorem 4.6.12. Theorem 4.6.17 describes the existence and uniqueness of optimal payoffs.

(v) **Representation result:** Theorem 4.7.5 states a representation result for SUBMA risk measures based on continuous-time trading. It leads to a numerical procedure to determine the SUBMA risk measure. We present two benchmark examples.

4.1.5 Structure of this chapter

In Section 4.2, we explain the notions of utility functions and payoffs. In Section 4.3, we formally introduce the SUBMA risk measure and describe its connections to multi-asset, OCE and OEU risk measures. In Section 4.4, we verify convexity and coherence properties. The aim of Sections 4.5 and 4.6 is to state sufficient finiteness and continuity conditions and to prove the existence and the uniqueness of optimal payoffs. In Section 4.7, we consider SUBMA risk measures in continuous-time financial market models.

4.1.6 Standard notations and concepts

In addition to Sections 1.1.4, 2.1.5 and 3.1.6, we use the following standard notations and concepts: Assume a non-empty set $\mathcal{C}$ and a function $f : \mathcal{C} \to [-\infty, \infty]$. If f is convex, then we say that it is proper if it does not attain $-\infty$ and $\mathrm{dom}(f) \neq \varnothing$. If f is concave, then we say that it is proper if it does not attain ∞ and $\mathrm{dom}(f) \neq \varnothing$. The hypograph of a map $f : \mathcal{X} \to [-\infty, \infty]$ is defined as $\mathrm{hypo}(f) := \{(X, \alpha) \in \mathcal{X} \times \mathbb{R} \,|\, f(X) \geq \alpha\}$.

If $\mathcal{X}$ is a Banach space with norm $\|.\| : \mathcal{X} \to \mathbb{R}$, then the interior of a set $\mathcal{C} \subset \mathcal{X}$ is denoted by $\mathrm{int}_{\|.\|}(\mathcal{C})$ or $\mathrm{int}(\mathcal{C})$ if the context is clear.

We write $X \sim N(\mu, \sigma)$ to refer to a normal distributed random variable X with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$.

4.2 Utility functions and financial positions

In this section, we introduce additional concepts needed for the definition of the new risk measure.

4.2.1 Utility functions

The utility functions in Geissel et al. (2018, Definition 1.7) are restricted to a positive domain. Hence, their OEU risk measures implicitly consider a risk constraint, namely the acceptance set given by the positive cone L^∞_+. We have to use utility functions with a larger domain such that further risk constraints can become relevant.

Definition 4.2.1 (Utility functions). *A concave, proper, increasing and upper semicontinuous function $U : \mathbb{R} \to [-\infty, \infty)$ is called a* ***utility function****.*

Throughout the whole chapter we fix a utility function U. The lower bound of the effective domain of U is denoted by

$$\underline{x}_U := \inf(\mathrm{dom}(U)).$$

We set $U(+\infty) := \lim\limits_{x \to +\infty} U(x)$. The so-called bliss point is given as

$$\overline{x}_U := \inf\{x \in \mathbb{R} \,|\, U(x) = U(+\infty)\}.$$

The only point of discontinuity of U is $\underline{x}_U$. By upper semicontinuity the utility function is right-continuous at $\underline{x}_U$. Further, $\underline{x}_U \le \overline{x}_U$ with equality if and only if U is constant on its effective domain. We exclude this case by the following standing assumption which is inspired by Biagini and Černý (2020, Convention 1.1).

Assumption 4.2.2. $\underline{x}_U \le 0 < \overline{x}_U$ *and there exists a unique* $x \in [0, \infty)$ *with* $U(x) = 0$.

We denote the root of U by x_U^*. Some of our results rely on specific utility functions:

Definition 4.2.3 (Risk aversion)**.** *If the utility function U is twice continuously differentiable on* $\operatorname{int}(\operatorname{dom}(U))$ *and* $U'(x) > 0$, $U''(x) < 0$ *for all* $x \in \operatorname{int}(\operatorname{dom}(U))$, *then we define the absolute risk tolerance for all* $x \in \operatorname{int}(\operatorname{dom}(U))$ *by* $\tau_U(x) := -\frac{U'(x)}{U''(x)}$. *For constant absolute risk aversion (CARA) the map* $x \mapsto \frac{1}{\tau_U(x)}$ *is relevant. So, if* $\frac{1}{\tau_U}$ *is constant, then U is a CARA utility function. For constant relative risk aversion (CRRA) the map* $x \mapsto \frac{x}{\tau_U(x)}$ *is relevant. So, if* $\frac{x}{\tau_U}$ *is constant, then U is a CRRA utility function.*

The most important utility functions in practice are CARA or CRRA utility functions.

Example 4.2.4 (Utility functions)**.**

(i) *Exponential utility:* Let $\delta > 0$. The exponential utility function is defined by $U : \mathbb{R} \to \mathbb{R}$, $x \mapsto 1 - e^{-\delta x}$. Up to affine transformations this is the form of a CARA utility function with $\frac{1}{\tau_U} = \delta$ for all x.

(ii) *Logarithmic utility:* For $x > 0$ given by $U(x) = \ln(x)$. For $x \le 0$ given by $U(x) = -\infty$. Up to affine transformations this is the form of a CRRA utility function with $\frac{x}{\tau_U} = 1$.

(iii) *Power utility:* Let $\delta > 0$ with $\delta \ne 1$. The effective domain of a power utility function is $[0, \infty)$ for $\delta < 1$ and $(0, \infty)$ for $\delta > 1$. On their effective domain the power utility is given by $U(x) = \frac{1}{1-\delta}\left(x^{1-\delta} - 1\right)$. Up to affine transformations this is the form of a CRRA utility function with $\frac{x}{\tau_U} = \delta$.

4.2.2 Orlicz spaces

To prepare the choice of the model space $\mathcal{X}$ in the next section, we introduce Orlicz spaces. They allow for a natural connection between payoffs and utility functions. An Orlicz space is defined via a Young function which we introduce first.

Definition 4.2.5 (Young function)**.** *A Young function* $f : [0, \infty) \to [0, \infty]$ *is a convex and lower semicontinuous function with* $f(0) = 0$, $\lim_{x\to\infty} f(x) = \infty$ *and* $f < \infty$ *on an open neighborhood of* 0.

Definition 4.2.6 (Orlicz space and Luxemburg norm)**.** *Let* $(\Omega, \mathscr{F}, P)$ *be a probability* *space and f a Young function. The Orlicz space with respect to f is defined by*

$$L^f(\Omega,\mathscr{F},P) := \left\{ X \in L^0(\Omega,\mathscr{F},P) \,\middle|\, E[f(\lambda|X|)] < +\infty \text{ for some } \lambda > 0 \right\}.$$

We also write L^f for short. For each $X \in L^f$ the Luxemburg norm with respect to f is defined as

$$\|X\|_f := \inf \left\{ \lambda > 0 \,\middle|\, E\left[f\left(\frac{|X|}{\lambda} \right) \right] \leq 1 \right\}.$$

Remark 4.2.7. The Luxemburg norm is a Riesz norm (Zaanen, 1983, Theorem 131.5). We always equip an Orlicz space L^f with the Luxemburg norm and the P-a.s. order. Then we obtain a Banach lattice (Zaanen, 1983, Theorem 131.6). It satisfies the following continuous embeddings (when applying the corresponding inclusion maps), see e.g., Biagini and Frittelli (2008, Section 2.1):

$$L^\infty \hookrightarrow L^f \hookrightarrow L^1.$$

For further details on Young functions and Orlicz spaces we refer to Zaanen (1983, Chapter 19) and Krasnosel'skii and Rutickii (1961). In the latter reference Young functions are called N-functions.

4.2.3 Financial positions

We use a static point of view to specify the model space $\mathcal{X}$, i.e., payoffs are random variables. To do so, we fix throughout the whole chapter a probability space $(\Omega,\mathscr{F},P)$.

The choice of the vector space $\mathcal{X}$ is crucial. It has to be small enough such that the integrals describing expected utility exist. Jensen's inequality states that for each integrable random variable $X \in L^1$ the expectation of $U(X)$ exists and

$$E[U(X)] \leq U(E[X]) < +\infty,$$

see e.g., Perlman (1974, Proposition 1.1) for a precise statement and Ferguson (1967, Lemma 2.8.1) for a detailed proof. Note that $E[U(X)] = -\infty$ can occur. Nevertheless, the expected utility is well-defined. Hence, L^p-spaces are a natural choice for $\mathcal{X}$.

The earlier mentioned Orlicz spaces are another possible choice for which we introduce the following map:

Definition 4.2.8. *Let $a \in \operatorname{int}(\operatorname{dom}(U))$. The map $\hat{U}_a$ is defined by*

$$\hat{U}_a : [0,\infty) \to [0,\infty],\ x \mapsto U(a) - U(a-x).$$

If $a = 0$, then we write $\hat{U}$ for short.

The next result states that $\hat{U}_a$ is a Young function, i.e., every utility function can be transformed into a Young function which leads to the Orlicz space $L^{\hat{U}_a}$, see Definition 4.2.6.

Lemma 4.2.9. *$\hat{U}_a$ is a Young function.*

Proof: It is straightforward to check the conditions in Definition 4.2.5. □

The map $\hat{U}_a$ performs a translation of the utility function U. Hence, if the random variable $X \in L^0$ should describe the payoff we have to check if $X - a1_\Omega \in L^{\hat{U}_a}$. Note that $X \in L^{\hat{U}_a}$ if and only if $X - a1_\Omega \in L^{\hat{U}_a}$.

Summarizing, unless otherwise stated, we work in this chapter under the following assumption:

Assumption 4.2.10. *Let $\mathcal{X}$ be either L^p with $p \in [1, \infty)$ or $L^{\hat{U}_a}$ with $a \in \operatorname{int}(\operatorname{dom}(U))$.*

For the sake of brevity, we denote the norm on $\mathcal{X}$ by $\|.\|_{\mathcal{X}}$, i.e., if $\mathcal{X} = L^p$ with $p \in [1, \infty)$, then $\|.\|_{\mathcal{X}} = \|.\|_{L^p}$ and if $\mathcal{X} = L^{\hat{U}_a}$, then $\|.\|_{\mathcal{X}} = \|.\|_{\hat{U}_a}$, where $\|.\|_{\hat{U}_a}$ denotes the Luxemburg norm, recall Definition 4.2.6.

Finally, we introduce two maps that play a key role in the risk assessment problem.

Definition 4.2.11 (Integral functional and certainty equivalent)**.** *The generalized inverse of the utility function U is given by*

$$U^{-1} : [-\infty, \infty) \to [-\infty, \infty] \,, \; y \mapsto \inf\{x \in \mathbb{R} \,|\, U(x) \geq y\}.$$

We define the following functions with respect to U:

(i) ***Integral functional:*** $I_U : \mathcal{X} \to [-\infty, \infty) \,, \; X \mapsto E[U(X)]$,

(ii) ***Certainty equivalent:*** $C_U : \mathcal{X} \to [-\infty, \infty) \,, \; X \mapsto U^{-1}(E[U(X)])$.

Remark 4.2.12. Note that we have $U^{-1}(-\infty) = -\infty$ and thus, $E[U(X)] = -\infty$ implies $C_U(X) = -\infty$. To see that $C_U < \infty$ we distinguish two cases:

(i) *U strictly increasing on* $\operatorname{dom}(U)$*:* By Jensen's inequality

$$E[U(X)] \leq U(E[X]) < U(+\infty)$$

and hence, $C_U(X) < \infty$. For instance, $U(x) = 1 - e^{-x}$ or $U(x) = \ln(x)$.

(ii) *U not strictly increasing on* $\mathrm{dom}(U)$*:* Then $U(+\infty) < \infty$ and

$$E[U(X)] = U(+\infty) < \infty$$

is possible and in this case $C_U(X) = \overline{x}_U < \infty$. For instance,

$$U(x) = 1 - e^{-x}1_{(-\infty,1)}(x) - e^{-1}1_{[1,\infty)}(x).$$

4.3 Scalarized utility-based multi-asset risk measures

Based on the preliminaries in the previous section, we define the desired new risk measure.

Definition 4.3.1 (SUBMA risk measures)**.** *Assume an acceptance set $\mathcal{A} \subset \mathcal{X}$. The corresponding* ***scalarized utility-based multi-asset (SUBMA) risk measure*** *for $X \in \mathcal{X}$ with respect to $\Psi \in \{I_U, C_U\}$ and a weighting factor $\alpha > 0$ is defined by*

$$\rho_{\alpha,\Psi,\mathcal{A},\mathcal{M},\pi}(X) := \inf\{\pi(Z) - \alpha\Psi(X+Z) \,|\, Z \in \mathcal{M}, X+Z \in \mathcal{A}\}.$$

Remark 4.3.2. If $\Psi = I_U$, then the objective of the SUBMA risk measure is the sum of two amounts measured in different units, i.e., $\pi(Z)$ is a monetary amount and the expected utility $E[U(X+Z)]$ is not. Further, if we scale the utility function by a factor $c > 0$, then the optimal solution changes. By adjusting the weighting factor to $\frac{\alpha}{c}$ the solution does not change. In contrast, the certainty equivalent is not affected by affine transformations of the utility function. So, if $\Psi = C_U$, then an additional adjustment of the weighting factor is not necessary.

The SUBMA risk measure combines multi-asset risk measures with OCE and OEU risk measures, so that we benefit from the advantages of both concepts. Clarifying the link to these concepts is the purpose of the rest of this section. Hence, in Section 4.3.1 we show that multi-asset risk measures differ from SUBMA risk measures by the expected utility term in the objective. Then, in Section 4.3.2 we show that SUBMA risk measures cover OCE and OEU risk measures.

There is also a strong connection of SUBMA risk measures to the field of multi-criteria optimization. SUBMA risk measures are scalarized versions of multi-objective optimization problems, as we present in Section 4.3.3. The term scalarized then explains our naming convention.

4.3.1 Link to multi-asset risk measures

We recall the definition of a multi-asset risk measure, see Definition 1.3.9. Assume an acceptance set $\mathcal{A}$. The multi-asset risk measure for a payoff $X \in \mathcal{X}$ is defined by

$$\rho_{\mathcal{A},\mathcal{M},\pi}(X) = \inf\{\pi(Z) \,|\, Z \in \mathcal{M}, X + Z \in \mathcal{A}\}.$$

The multi-asset risk measure asks for the minimal costs to create a hedging position such that the total payoff is acceptable. The motivation for the SUBMA risk measure is the fact that the multi-asset risk measure ignores the intention of an agent to maximize her expected utility. Hence, we add the penalty term $-\alpha\Psi(X + Z)$ to the costs $\pi(Z)$. With the help of the penalty factor, the impact of the utility criterion can be controlled. For lower values of α, minimizing costs is more important than the preferences of the agent. Concluding, the agent is willing to create an expensive hedging position, if it increases the expected utility significantly.

4.3.2 Link to OCE and OEU

The previous mentioned penalization of the hedging costs in the SUBMA risk measure is inspired by OCE and OEU risk measures. In the following examples, we explain similarities and differences between SUBMA, OCE and OEU risk measures.

For both examples, we use the same setting: For a finite time $T > 0$, we assume a bank account with initial price $B_0 > 0$ and deterministic future payoff $B_T > 0$. Assume that $\mathcal{M} = \operatorname{span}\{B_T\}$. This implies $\ker(\pi) = \{0\}$. Denote the discount factor by $\beta := B_0/B_T$.

Example 4.3.3 (OCE (Ben-Tal & Teboulle, 1986; Ben-Tal & Teboulle, 2007))**.** Assume $\beta = \alpha = 1$. For an arbitrary acceptance set $\mathcal{A}$ we obtain

$$\rho_{\alpha,I_U,\mathcal{A},\mathcal{M},\pi}(X) = \inf\{\eta - I_U(X + \eta) \,|\, X + \eta \in \mathcal{A}\}.$$

Beside the risk constraint given by the acceptance set ($\mathcal{A} \neq \mathcal{X}$) this is exactly the form of the OCE risk measure in Ben-Tal and Teboulle (2007). If $\operatorname{dom}(U) = [0, \infty]$, then the constraint given by $\mathcal{A} = \mathcal{X}_+$ is implicitly included in their formulation. Further, the payoffs in Ben-Tal and Teboulle (2007) are essentially bounded random variables. So our model space $\mathcal{X}$ is larger. Further, their utility function U satisfies $U(0) = 0$ and $1 \in \partial U(0)$. We use such utility functions in Theorem 4.6.6.

Example 4.3.4 (OEU (Geissel et al., 2018))**.** Let $\mathcal{A} = \mathcal{X}_+$. For $X \in \mathcal{X}$ we then obtain

$$\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}(X) = \inf\left\{m - \alpha C_U\left(X + \frac{m}{B_0}B_T\right) \,\middle|\, m \in \mathbb{R}, X + \frac{m}{B_0}B_T \in \mathcal{X}_+\right\}.$$

With the linear transformation $\eta = \frac{m}{B_0} B_T$ we get

$$\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}(X) = \inf_{\substack{\eta\in\mathbb{R}\\ X+\eta\in\mathcal{X}_+}} \Big(\beta\eta - \alpha C_U(X+\eta)\Big).$$

Further restrictions in the setting of Geissel et al. (2018) are the following: As for the OCE risk measure, the payoffs are essentially bounded random variables. Further, U is strictly increasing, strictly continuous and three times continuously differentiable on the interior of its effective domain given by $\operatorname{int}(\operatorname{dom}(U)) = (0,\infty)$. Finally, τ_U is concave.

Geissel et al. (2018, page 77) note that the weighting factor α "*[...] captures the investor's subjective time preference [...]*" which also holds for the SUBMA risk measure.

Concluding, OCE and OEU risk measures are special cases of SUBMA risk measures which differ in the choice of Ψ.

4.3.3 Link to multi-objective optimization

Let us now shortly sketch the derivation of the SUBMA risk measure by multi-objective optimization techniques. Consider the two objectives of minimizing costs and maximizing expected utility separately. They work in the opposite directions. The optimal solutions lead to a Pareto front. One possibility to calculate this front is the application of a linear scalarization which is a multi-objective optimization approach. This is the idea behind the SUBMA risk measure and it explains the term scalarized in SUBMA.

There is a bunch of literature focusing on multiple criteria for portfolio decisions. For instance, Qi, Steuer, and Wimmer (2017) and Köksalan and Şakar (2016) select portfolios based on three criteria. Dächert, Grindel, Leoff, Mahnkopp, Schirra, and Wenzel (2021) use a multi-criteria approach to support portfolio decisions in practice. All of these approaches rely on the reallocation of an initial endowment and they do not cover the case of a capital increase as the SUBMA risk measure does.

4.4 Desirable properties: Convexity and coherence

In this section, we examine properties often used to define risk measures from an axiomatic point of view. We are interested in monotonicity, invariance with respect to marketed payoffs, convexity and positive homogeneity. These properties are suggested by Artzner et al. (1999). Our main motivation is to generalize a result for OEU risk measures, motivated by the following quote from Geissel et al. (2018, page 79):

> *To the best of our knowledge, OEU is the only non-trivial utility-based risk measure that is coherent for power utility functions [...]*

We achieve this goal in Theorem 4.4.4. It states that the SUBMA risk measure is coherent, if $\Psi = C_U$ with a CRRA utility function U and a convex cone as acceptance set. Throughout the whole section we fix an acceptance set $\mathcal{A}$, $\Psi \in \{I_U, C_U\}$ and $\alpha > 0$.

4.4.1 Monotonicity, $\mathcal{M}$-invariance and convexity

The following result describes the monotonicity, the invariance with respect to marketed payoffs and the convexity of SUBMA risk measures.

Proposition 4.4.1. *The following statements hold:*

(i) Monotonicity: $X \geq Y$ *P-a.s. implies* $\rho_{\alpha,\Psi,\mathcal{A},\mathcal{M},\pi}(X) \leq \rho_{\alpha,\Psi,\mathcal{A},\mathcal{M},\pi}(Y)$.

(ii) $\mathcal{M}$-invariance: $\rho_{\alpha,\Psi,\mathcal{A},\mathcal{M},\pi}(X+Z) = \rho_{\alpha,\Psi,\mathcal{A},\mathcal{M},\pi}(X) - \pi(Z)$ *for every* $Z \in \mathcal{M}$.

(iii) Convexity for I_U: Convex $\mathcal{A}$ implies $\rho_{\alpha,I_U,\mathcal{A},\mathcal{M},\pi}$ *being convex.*

(iv) Convexity for C_U: Convex $\mathcal{A}$ and concave C_U imply $\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}$ *being convex.*

Proof: To prove (i) assume $X, Y \in \mathcal{X}$, $Z \in \mathcal{M}$ with $X \geq Y$ P-a.s. and $Y + Z \in \mathcal{A}$. By monotonicity of $\mathcal{A}$ we obtain that $X + Z \in \mathcal{A}$. By Lemma B.1 and Lemma B.4 the map Ψ is increasing which implies that $\pi(Z) - \alpha\Psi(Y+Z) \geq \pi(Z) - \alpha\Psi(X+Z)$. This proves monotonicity. Assertions (iii) and (iv) are consequences of the concavity of I_U (see Lemma B.1) as well as of C_U and the convexity of $\mathcal{A}$. Statement (ii) is elementary and we omit the proof. □

In general the certainty equivalent is not concave. Ben-Tal and Teboulle (2007, Corollary 5.1) state conditions on the utility function such that it becomes concave. This result holds for utility functions that do not attain $-\infty$, i.e., $U : \mathbb{R} \to \mathbb{R}$. Geissel et al. (2018, Theorem 2.5) give a proof for utility functions with positive effective domain. We introduce a tailor-made result for our needs.

Corollary 4.4.2 (Concavity of C_U)**.** *Assume that the restriction of U to* $\operatorname{int}(\operatorname{dom}(U))$ *is three times continuously differentiable, strictly increasing and strictly concave. If τ_U is concave, then C_U is concave.*

Proof: The proof works analogously to the one of Geissel et al. (2018, Theorem 2.5). Let $(X, \alpha), (Y, \beta) \in \operatorname{hypo}(C_U)$ and $\lambda \in (0, 1)$. Hence,

$$P(X \in \operatorname{dom}(U)) = P(Y \in \operatorname{dom}(U)) = 1.$$

The following map is convex:

$$g : \mathrm{dom}(U) \times \mathrm{dom}(U) \to \mathbb{R},\ (y,z) \mapsto U(\lambda U^{-1}(y) + (1-\lambda)U^{-1}(z)).$$

By Jensen's inequality it follows that

$$E[U(\lambda X + (1-\lambda)Y)] = E[g(U(X), U(Y))] \geq g(E[U(X)], E[U(Y)]).$$

Hence,

$$C_U(\lambda X + (1-\lambda)Y) \geq \lambda C_U(X) + (1-\lambda)C_U(Y) \geq \lambda\alpha + (1-\lambda)\beta$$

and $\mathrm{hypo}(C_U)$ is concave, i.e., C_U is concave. □

In the next section, we analyze the positive homogeneity of the SUBMA risk measure. Together with the convexity of the SUBMA risk measure in the situation of a convex certainty equivalent, we obtain a coherence result by applying a CRRA utility function and a convex cone as acceptance set.

4.4.2 Coherence

The OEU risk measure is positive homogeneous for CRRA utility functions (Geissel et al., 2018, Theorem 2.7). We find out that this result also holds for SUBMA risk measures by using a cone as acceptance set.

Lemma 4.4.3 (Positive homogeneity)**.** *Assume U is a CRRA utility function and there exists $M \in (\mathcal{M} \cap \mathcal{X}_+)\backslash\{0\}$ with $E[U(M)] > -\infty$. Further, let $\mathcal{A}$ be a cone. Then $\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}$ is positive homogeneous, i.e., $\mathrm{epi}(\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi})$ is a cone.*

Proof: Assume that $(X,\gamma) \in \mathrm{epi}(\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi})$. Then, for $\lambda \geq 0$ we have to prove that $\lambda(X,\gamma) \in \mathrm{epi}(\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi})$. We first assume $\lambda > 0$. Up to affine transformations U is a power or logarithmic utility function, because it is of CRRA type. So, we can concentrate on the case of a power or logarithmic utility function. Then we see that $E[U(\lambda X)] = -\infty$ if and only if $E[U(X)] = -\infty$. Together with Pratt (1964, Theorem 6) we have for all $X \in \mathcal{X}$ that $\frac{1}{\lambda}C_U(\lambda X) = C_U(X)$. Hence, we obtain

$$\begin{aligned}
\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}(\lambda X) &= \lambda \inf\left\{\pi\left(\frac{Z}{\lambda}\right) - \alpha\frac{1}{\lambda}C_U(\lambda X + Z)\,\middle|\, Z \in \mathcal{M}, \lambda X + Z \in \mathcal{A}\right\}\\
&= \lambda \inf\left\{\pi(Z^*) - \alpha\frac{1}{\lambda}C_U(\lambda(X + Z^*))\,\middle|\, Z^* \in \mathcal{M}, \lambda(X + Z^*) \in \mathcal{A} = \lambda\mathcal{A}\right\}\\
&= \lambda \inf\{\pi(Z^*) - \alpha C_U(X + Z^*) \,|\, Z^* \in \mathcal{M}, X + Z^* \in \mathcal{A}\}\\
&= \lambda \rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}(X).
\end{aligned}$$

This implies $\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}(\lambda X) \leq \lambda\gamma$ and $\lambda(X,\gamma) \in \text{epi}(\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi})$. It remains the case $\lambda = 0$. Note that it is enough to prove $\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}(0) \in \{-\infty, 0\}$. By conicity of $\mathcal{A}$ we have $0 \in \mathcal{A}$. By monotonicity we get $M \in \mathcal{A}$. Hence, for each $n \in \mathbb{N}$ it holds that $\frac{1}{n}M \in \mathcal{M} \cap \mathcal{A}$. Further,

$$\pi\left(\frac{1}{n}M\right) - \alpha C_U\left(\frac{1}{n}M\right) = \frac{1}{n}\Big(\pi(M) - \alpha C_U(M)\Big) \to 0$$

and hence,

$$\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}(0) \leq \pi\left(\frac{1}{n}M\right) - \alpha C_U\left(\frac{1}{n}M\right) \to 0.$$

To conclude the proof we show that $\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}(0) \in (-\infty, 0)$ can not occur. The latter is equivalent to the existence of $Z^* \in \mathcal{M} \cap \mathcal{X}_+$ with $\pi(Z^*) - \alpha C_U(Z^*) < 0$ and the existence of a constant $c \in (-\infty, 0)$ such that for all $Z \in \mathcal{M} \cap \mathcal{X}_+$ we have $\pi(Z) - \alpha C_U(Z) \geq c$. Assume that there exist a constant $m^* > 0$ and $Z_0^* \in \ker(\pi)$ such that $m^*M + Z_0^* \in \mathcal{X}_+ \backslash \{0\}$ and $m^*\pi(M) - \alpha C_U(m^*M + Z_0^*) < 0$. Without loss of generality, we assume $\pi(M) = 1$. Let U be a logarithmic utility function U. The following argumentation holds analogously for power utility functions. Then, by the previous condition we obtain $-\ln(\alpha) < E\left[\ln\left(M + \frac{Z_0^*}{m^*}\right)\right]$. To show that the objective function is not bounded from below choose an arbitrary constant $c < 0$ such that

$$\ln\left(\frac{m^* - c}{\alpha}\right) \geq E[\ln(m^*M + Z_0^*)] > -\infty.$$

Note that for every $n \in \mathbb{N}$ it holds $nZ_0^* \in \ker(\pi)$. Further, $\mathcal{X}_+$ is a cone. This implies $n(m^*M + Z_0^*) \in \mathcal{X}_+ \backslash \{0\}$. We obtain

$$\ln\left(\frac{nm^* - c}{\alpha}\right) - E[\ln(nm^*M + nZ_0^*)] = \underbrace{-\ln(\alpha) - E\left[\ln\left(M + \frac{Z_0^*}{m^*}\right)\right]}_{<0} + \underbrace{\ln\left(1 - \frac{c}{nm^*}\right)}_{\to 0}.$$

Hence, for large enough n we have

$$\ln\left(\frac{nm^* - c}{\alpha}\right) < E[\ln(nm^*M + nZ_0^*)]$$

which implies that $\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}(0) = -\infty$ and $(0,0) \in \text{epi}(\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi})$. □

Summarizing, we obtain that the SUBMA risk measure is coherent if we use a CRRA utility function and a convex cone as acceptance set.

Theorem 4.4.4 (Coherence)**.** *In the same situation as in Lemma 4.4.3 by assuming that $\mathcal{A}$ is also convex, the SUBMA risk measure $\rho_{\alpha,C_U,\mathcal{A},\mathcal{M},\pi}$ is coherent.*

Proof: This follows by combining Proposition 4.4.1, Corollary 4.4.2 and Lemma 4.4.3. □

4.5 One-period trading: Model and absence of good deals

It is a common assumption for multi-asset risk measures that the marketed space $\mathcal{M}$ is finite-dimensional, see e.g., Liebrich and Svindland (2017), Liebrich and Svindland (2019) and Baes et al. (2020). In the simplest form this corresponds to a single trading period. We have used such a situation already in the previous chapters. Now we assume again a one-period financial market model which we introduce in Section 4.5.1. As a preparation for the next section, we need the absence of good deals of the first kind condition that we discuss in Section 4.5.2. Then in Sections 4.5.3 and 4.5.4 we show how to solve the SUBMA risk measure via a Lagrange ansatz.

4.5.1 Standard one-period model

We assume the following one-period financial market model which we call a ***standard one-period model***:

We use a finite time horizon $T > 0$. Let the market consist of a risk-free asset with price $B_0 > 0$ and future deterministic value $B_T > 0$. Further, there are $d \in \mathbb{N}$ risky assets with price vector $S_0 = \left(S_0^1, \dots, S_0^d\right)^\intercal \in \mathbb{R}^d$ and payoffs $S_T = \left(S_T^1, \dots, S_T^d\right)^\intercal$, a d-dimensional vector of random variables in $\mathcal{X}$. This gives us the following marketed space:

$$\mathcal{M} = \operatorname{span}\{B_T, S_T^1, \dots, S_T^d\}.$$

The pricing functional for shares $\varphi^B \in \mathbb{R}$ and $\varphi \in \mathbb{R}^d$ is defined by

$$\pi\left(\varphi^B B_T + \langle \varphi, S_T \rangle\right) = \varphi^B B_0 + \langle \varphi, S_0 \rangle.$$

Note that every $Z \in \ker(\pi)$ can be represented by a vector $\varphi \in \mathbb{R}^d$ in the form

$$Z = \left\langle \frac{S_0}{B_0} B_T - S_T, \varphi \right\rangle.$$

4.5.2 Absence of good deals

Recall the concept of good deals of the first kind, see Definition 1.3.2. We characterize the absence of good deals of the first kind condition in the standard one-period model using an acceptance set based on a monetary risk measure. In doing so, we make the following standing assumption:

Assumption 4.5.1. *The risk measure ρ is a proper, lower semicontinuous and positive homogeneous monetary risk measure.*

Proposition 4.5.2 (Absence of good deals of the first kind with d risky assets)**.** *In the standard one-period model the following conditions are equivalent:*

(1) $(\mathcal{A}_\rho \cap \ker(\pi))\backslash\{0\} = \varnothing$.

(2) For each $x \in \partial\,\mathrm{B}_d$ *it holds that* $\frac{B_T}{B_0}\langle S_0, x\rangle < \rho(-\langle S_T, x\rangle)$.

Proof: Condition (1) is equivalent to the following statement: For every $\varphi \in \mathbb{R}^d\backslash\{0\}$ it holds that $\frac{B_T}{B_0}\left\langle S_0, \frac{\varphi}{\|\varphi\|_2}\right\rangle < \rho\left(-\left\langle S_T, \frac{\varphi}{\|\varphi\|_2}\right\rangle\right)$. But this is equivalent to (2). □

In the standard one-period model the absence of good deals of the first kind is equivalent to the absence of general good deals. This is shown in the next result.

Proposition 4.5.3. *In the standard one-period model, the following statements are equivalent:*

(1) $(\mathcal{A}_\rho \cap \ker(\pi))\backslash\{0\} = \varnothing$.

(2) For every pair $(m, \varphi) \in \left(\mathbb{R} \times \mathbb{R}^d\right) \setminus \{0\}$ *with* $\pi\left(\frac{B_T}{B_0}m + \left\langle \frac{B_T}{B_0}S_0 - S_T, \varphi\right\rangle\right) \le 0$ *it holds that* $\frac{B_T}{B_0}m + \left\langle \frac{B_T}{B_0}S_0 - S_T, \varphi\right\rangle \notin \mathcal{A}_\rho$.

Proof: The implication (2) $\Rightarrow$ (1) is obvious. For the inverse implication note that for all $\lambda \in (0, \infty)$ we have $\rho(-\lambda B_T) = \lambda B_T > 0$. The claim follows by Baes et al. (2020, Proposition 2.6). □

Remark 4.5.4. The implication (1) $\Rightarrow$ (2) does not hold in general situations, see Example 2.4.3 and Baes et al. (2020, Example 2.9).

We conclude this section by illustrating the absence of good deals of the first kind condition in two concrete financial market models. For the sake of simplicity, we start with a Bachelier model. In the second example, we use a Black-Scholes model.

Example 4.5.5 (Bachelier model)**.** We set $d = 2$. Let $\sigma_1, \sigma_2 > 0$ and assume that W_T^1, W_T^2 are two uncorrelated normally distributed random variables with mean zero and variance T. The interest rate of the risk-free asset is denoted by $r < 0$. For the start prices of the stocks, we assume that $S_0^1, S_0^2 > 0$. The marketed values of the three assets at T are given by

$$B_T = B_0 e^{rT},\ S_T^1 = S_0^1 + \sigma_1 W_T^1,\ S_T^2 = S_0^2 + \sigma_2 W_T^2.$$

Further, we use an acceptance set based on the Expected Shortfall with level $\lambda \in (0, 0.5)$. Now we show that the following parameter constellation leads to a market without good deals of the first kind:

$$T = 1,\ \lambda = 2.5\%,\ r = -1\%,\ S_0^1 = S_0^2 = 1,\ \sigma_1 = 20\%,\ \sigma_2 = 15\%.$$

To see this, we plot on the left-hand side in Figure 4.3 the relation

$$\partial\, \mathrm{B}_2 \to \mathbb{R},\ x \mapsto \rho(-\langle S_T, x\rangle) - e^{rT}\langle S_0, x\rangle.$$

Since every point is strictly positive, by Proposition 4.5.2 the market is free of good deals of the first kind.

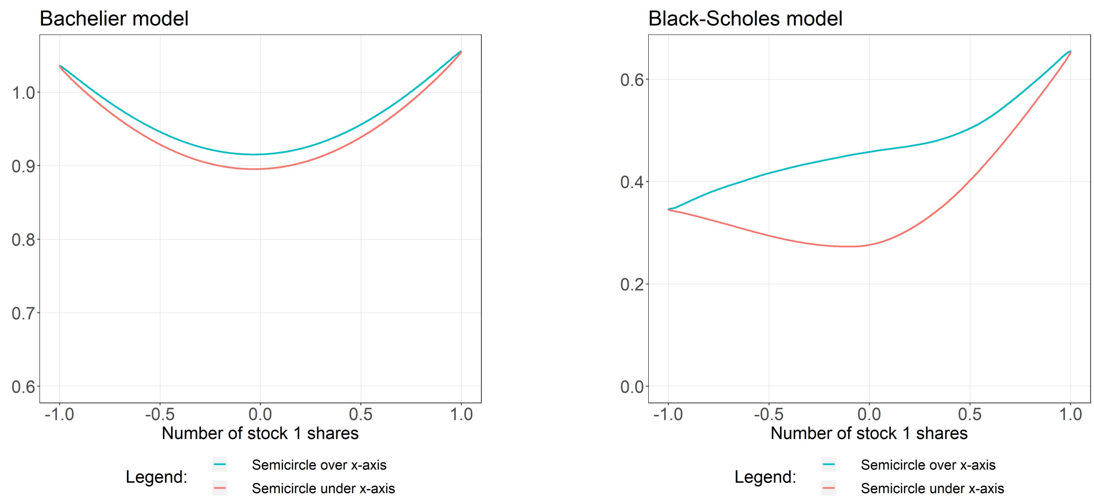


Figure 4.3: Relations $\partial\, \mathrm{B}_2 \to \mathbb{R},\ x \mapsto \rho(-\langle S_T, x\rangle) - e^{rT}\langle S_0, x\rangle$ to prove the absence of good deals of the first kind condition in Examples 4.5.5 and 4.5.6.

Example 4.5.6 (Black-Scholes model)**.** As in the previous example, we set $d = 2$ and model the bank account as in there. The future stock prices are given by geometric Brownian motions. The drift and volatility parameters are $b_1, b_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$. The start prices of the stocks are $S_0^1, S_0^2 > 0$. Finally, let W_T^1, W_T^2 be two uncorrelated normally distributed random variables with mean zero and variance T. The asset values at time T are given by

$$B_T = B_0 e^{rT},\ S_T^1 = S_0^1 e^{\left(b_1 - \frac{1}{2}\sigma_1^2\right)T + \sigma_1 W_T^1},\ S_T^2 = S_0^2 e^{\left(b_2 - \frac{1}{2}\sigma_2^2\right)T + \sigma_2 W_T^2}.$$

We use an acceptance set based on the Expected Shortfall with level $\lambda = 2.5\%$ and the following parameters:

$$T = 1,\ r = -1\%,\ S_0^1 = S_0^2 = 1,\ b_1 = 5\%,\ b_2 = 2\%,\ \sigma_1 = 20\%,\ \sigma_2 = 15\%.$$

The Expected Shortfall of a sum of stocks is calculated via a Monte-Carlo Simulation with 10 000 random variables per stock. On the right-hand side in Figure 4.3, we plot the analog relation as before: $\partial\,\mathrm{B}_2 \to \mathbb{R},\ x \mapsto \rho(-\langle S_T, x\rangle) - e^{rT}\langle S_0, x\rangle$. Every point is strictly positive and hence, the market is free of good deals of the first kind.

4.5.3 Karush-Kuhn-Tucker conditions

A Lagrange ansatz can lead to an optimal payoff. We use an acceptance set based on a convex monetary risk measure ρ. Note that the underlying optimization problem of the SUBMA risk measure satisfies the linear independence constraint qualification (LICQ), because it admits only one inequality constraint. Hence, a local minimum satisfies the Karush–Kuhn–Tucker (KKT) conditions. By introducing a slack variable $s \geq 0$, we obtain the following Lagrangian function:

$$\begin{aligned}\mathcal{L}(m,\varphi,\lambda,s) :=\, & m - \alpha\Psi\left(X + \frac{B_T}{B_0}m + \left\langle \frac{B_T}{B_0}S_0 - S_T, \varphi\right\rangle\right)\\ & - \lambda\left(m - \frac{B_0}{B_T}\rho(X - \langle S_T,\varphi\rangle) + \langle S_0,\varphi\rangle - s^2\right).\end{aligned}$$

The variable λ denotes the multiplier. Let us assume that for each $(m,\varphi) \in \mathbb{R}\times\mathbb{R}^d$ it holds that

$$X + \frac{B_T}{B_0}m + \left\langle \frac{B_T}{B_0}S_0 - S_T, \varphi\right\rangle \in \operatorname{dom}(\Psi).$$

To write down the KKT conditions, we assume that the following two functions are *continuously differentiable*:

$$\begin{aligned} f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R},\ (m,\varphi) &\mapsto \Psi\left(X + \frac{B_T}{B_0}m + \left\langle \frac{B_T}{B_0}S_0 - S_T, \varphi\right\rangle\right),\\ g : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R},\ (m,\varphi) &\to m - \frac{B_0}{B_T}\rho(X - \langle S_T,\varphi\rangle) + \langle S_0,\varphi\rangle.\end{aligned}$$

Then the KKT conditions are as follows:

(i) For all $i \in \{1,\dots,d\}$:

$$\begin{aligned}\frac{\partial}{\partial\varphi_i}\mathcal{L}(m,\varphi,\lambda,s) = & -\alpha\frac{\partial}{\partial\varphi_i}\left(\Psi\left(X + \frac{B_T}{B_0}m + \left\langle \frac{B_T}{B_0}S_0 - S_T, \varphi\right\rangle\right)\right)\\ & + \lambda\frac{B_0}{B_T}\frac{\partial}{\partial\varphi_i}\Big(\rho(X - \langle S_T,\varphi\rangle)\Big) - \lambda S_0^i = 0,\end{aligned}$$

(ii) $\frac{\partial}{\partial m}\mathcal{L}(m,\varphi,\lambda,s) = 1 - \alpha\frac{\partial}{\partial m}\left(\Psi\left(X + \frac{B_T}{B_0}m + \left\langle \frac{B_T}{B_0}S_0 - S_T, \varphi\right\rangle\right)\right) - \lambda = 0,$

(iii) $\frac{\partial}{\partial \lambda}\mathcal{L}(m,\varphi,\lambda,s) = m - \frac{B_0}{B_T}\rho(X - \langle S_T,\varphi\rangle) + \langle S_0,\varphi\rangle - s^2 = 0,$

(iv) $\lambda \geq 0,\ \lambda s = 0.$

If Ψ is concave and ρ is convex, then solving the KKT conditions gives us a minimizer. Next, we state a concrete situation in which the KKT conditions lead to an optimal portfolio, i.e., they are sufficient for a solution to be a minimizer.

4.5.4 Numerical example

Assume a standard one-period model based on two stocks with initial value one. The total return in one year for stock $i \in \{1,2\}$ is normally distributed, i.e., $S_1^i \sim N(\mu_i, \sigma_i)$. The correlation between the stock payoffs S_1^1 and S_1^2 is denoted by $\rho_{1,2} \in [-1,1]$. We assume that the second stock is traded on a stock exchange that is not available to the investor, i.e., $S_1^2 \notin \mathcal{M}$. Hence, the marketed space is assumed to be $\mathcal{M} = \operatorname{span}\{1_\Omega, S_1^1\}$, that is, buy-and-hold strategies with respect to a bank account and a stock. The pricing functional $\pi : \mathcal{M} \to \mathbb{R}$ is defined by $\pi(x1_\Omega + yS_1^1) = x + y$ for all $x, y \in \mathbb{R}$. Further, we assume an ES acceptance set $\mathcal{A}_{\mathrm{ES}_\lambda}$ with level $\lambda \in (0,1)$. The preferences of the agent are modeled by the exponential utility function $U(x) = 1 - e^{-x}$.

For initial numbers of shares $\omega_B, \omega_1, \omega_2 \in \mathbb{R}$ we obtain for the SUBMA risk measure with respect to $\Psi \in \{I_U, C_U\}$ and $\alpha > 0$ that

$$\rho_{\alpha,\Psi,\mathcal{A}_{\mathrm{ES}_\lambda},\mathcal{M},\pi}(\omega_B 1_\Omega + \omega_1 S_1^1 + \omega_2 S_1^2) = \rho_{\alpha,\Psi,\mathcal{A}_{\mathrm{ES}_\lambda},\mathcal{M},\pi}(\omega_2 S_1^2) - \omega_B - \omega_1.$$

Hence, we can focus on calculating $\rho_{\alpha,\Psi,\mathcal{A}_{\mathrm{ES}_\lambda},\mathcal{M},\pi}(\omega_2 S_1^2)$, i.e., $\omega_B = \omega_1 = 0$. Direct calculations lead to the objective functions of the SUBMA risk measures depending on the hedging costs and the negative number of stock shares which we denote by $m \in \mathbb{R}$ and $x \in \mathbb{R}$. For the sake of brevity, we set $\mu(m,x) := m + x - \mu_1 x + \omega_2\mu_2$ and $\sigma(x) := \sqrt{(\sigma_1 x)^2 - 2\rho_{1,2}\omega_2\sigma_1\sigma_2 x + (\omega_2\sigma_2)^2}$. Then, for the certainty equivalent the objective function of the SUBMA risk measure is

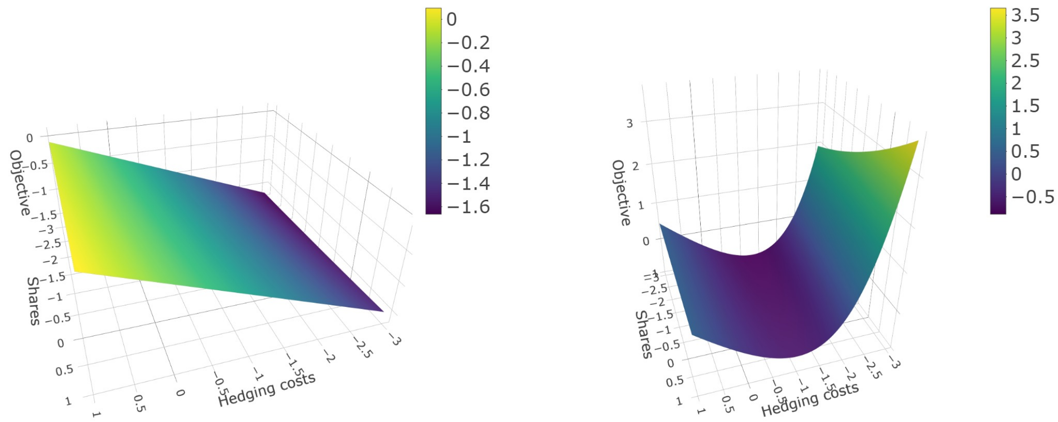
$$(m,x) \mapsto m - \alpha\mu(m,x) + \alpha\frac{(\sigma(x))^2}{2}. \tag{4.1}$$

The parameter μ_2 is not relevant to find the optimal portfolio. If the stocks are uncorrelated, then all parameters of S_1^2 only appear as constant shifts in the objective function. This is not the case for the objective function using the expected utility:

$$(m,x) \mapsto m - \alpha + \alpha\exp\left(-\mu(m,x) + \frac{1}{2}(\sigma(x))^2\right). \tag{4.2}$$

Hence, all parameters of S_1^2 are relevant for the portfolio selection. Especially, μ_2 becomes relevant to find the optimal portfolio. The objective functions (4.1) and (4.2) are continuously differentiable and convex. Both are illustrated in Figure 4.4 for the following parameter constellation:

$$\alpha = 0.6,\ \mu_1 = 1.075,\ \sigma_1 = 0.1,\ \mu_2 = 1.2,\ \sigma_2 = 0.3,\ \rho_{1,2} = -0.1,\ \omega_2 = 0.5.$$



(a) Certainty equivalent (b) Expected utility

Figure 4.4: Objective functions of SUBMA risk measures in dependence of hedging costs m and shares x.

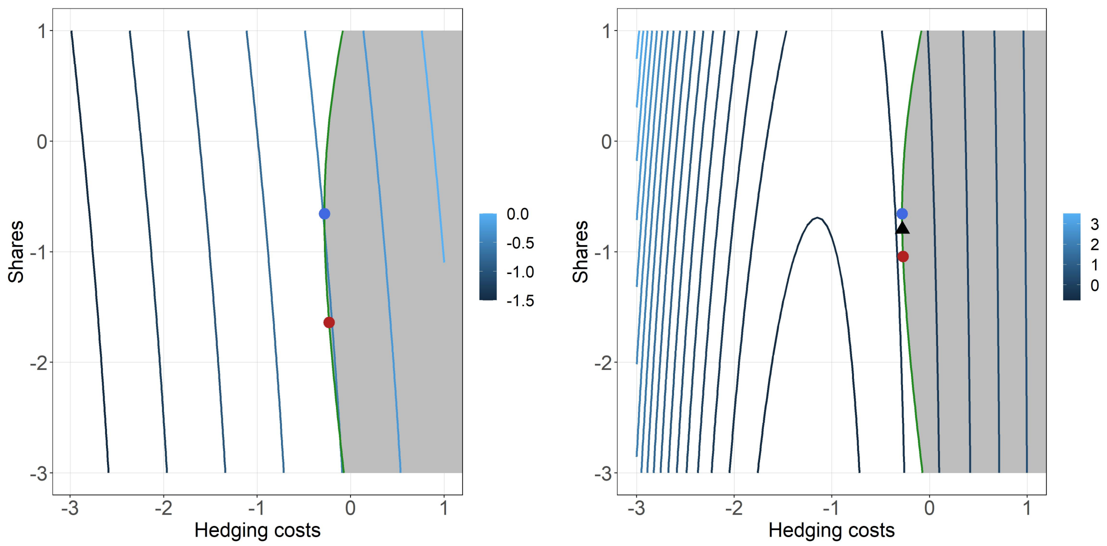
The objective function with respect to the certainty equivalent behaves more linearly then the objective function with respect to the expected utility. The objective for the expected utility admits an exponential growth for short positions ($m < 0$).

The Expected Shortfall leads to the following constraint for the optimization problem:

$$\mathrm{ES}_\lambda(\omega_2 S_1^2 + m + (1-x)S_1^1) = -\mu(m,x) + \frac{\phi(\Phi^{-1}(\lambda))}{\lambda}\sigma(x) \leq 0.$$

The left-hand side of this inequality is continuously differentiable for m and x. Further, the Expected Shortfall is convex. Then solving the KKT conditions leads to an optimal payoff.

The constraint and the contour lines of the objective functions for $\lambda = 0.025$ are illustrated in Figure 4.5. The red points are the optimal solutions of the SUBMA risk measures, calculated by solving the KKT conditions. The blue points correspond to the optimal solution of the classical multi-asset risk measure. We see that all solutions differ. The hedging costs of the solutions are nearly the same. But the allocation of the hedging costs is significantly different. In comparison to the multi-asset risk measure, the SUBMA risk measure leads to a higher stock position. Hence, the agent tries to benefit from the excess return of the stock position to increase her utility. This effect is stronger when using the certainty equivalent instead of the expected utility.



(a) Certainty equivalent (b) Expected utility

Figure 4.5: Contour lines of the SUBMA risk measures in dependence of hedging costs m and shares x for $\alpha = 0.6$. The gray area corresponds to the allowed portfolios out of the Expected Shortfall constraint. The red points are the optimal solutions of the SUBMA risk measures. The blue points are the optimal solution with respect to the multi-asset risk measure. The black triangle is the optimal solution of the SUBMA risk measure with respect to the scaled utility function $U(x) = 0.5(1 - e^{-x})$.

The black triangle in the plot on the right-hand side is the optimal solution of the SUBMA risk measure if we scale the utility function by the factor 0.5. Such a scaling does not affect the optimal solution with respect to the certainty equivalent, compare Remark 4.3.2.

4.6 One-period trading: Finiteness, continuity and optimal payoffs

Now we examine the finiteness and the continuity of the SUBMA risk measure in the standard one-period model. Further, we give sufficient conditions for the existence and uniqueness of optimal payoffs. For this we fix a weighting factor $\alpha > 0$.

4.6.1 Finiteness

For risk measures using a single eligible asset S it is important to validate if the value $+\infty$ is attained, compare the discussion in Section 1.3.3. The following result characterizes the situations in which the SUBMA risk measure attains $+\infty$.

Proposition 4.6.1 ($\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi} = +\infty$)**.** *Let $X \in \mathcal{X}$. Then $(\{X\} + \mathcal{M}) \cap \mathcal{A}_\rho \neq \varnothing$ and the following statements are equivalent:*

(1) $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}(X) = +\infty$.

(2) $\Psi(Y) = -\infty$ for all $Y \in (\{X\} + \mathcal{M}) \cap \mathcal{A}_\rho$.

Proof: Assume that there exists $X \in \mathcal{X}$ such that $(\{X\} + \mathcal{M}) \cap \mathcal{A}_\rho = \varnothing$. This implies that for every $\lambda \in \mathbb{R}$ and every $Z \in \mathcal{A}_\rho$ we have

$$\rho(Z) \leq 0 < \rho(X) + \lambda B_T,$$

i.e., the functional ρ is non-zero and strictly separates the points between the sets $\{X\} + \mathcal{M}$ and $\mathcal{A}_\rho$. Since ρ is finite-valued the above inequality leads to a contradiction and so we have $(\{X\} + \mathcal{M}) \cap \mathcal{A}_\rho \neq \varnothing$. The equivalence between (1) and (2) is then a direct consequence of $(\{X\} + \mathcal{M}) \cap \mathcal{A}_\rho \neq \varnothing$. □

Remark 4.6.2. In contrast to the classical risk measures, the SUBMA risk measure can attain $+\infty$ even if it is possible to pass the capital adequacy test. This is the worst situation for the agent, because the legal requirements can not be satisfied by buying a marketed portfolio such that the expected utility of the final payoff is finite. The agent is not willing to buy such a marketed portfolio.

The following example shows that there are indeed situations such that the condition in Proposition 4.6.1 is satisfied, i.e., the SUBMA risk measure becomes infinite.

Example 4.6.3. Let $d = 1$ and consider the exponential utility function $U(x) = 1 - e^{-x}$. The market consists of a bank account that is constant over time $B_T = B_0 = 1$. The stock S_T is log-normally distributed and $S_0 = 1$. The payoff is $-X$ with X being log-normally distributed and independent of S_T. For an arbitrary monetary risk measure ρ there exist $m, \varphi \in \mathbb{R}$ such that $\rho(-X + m - \varphi + \varphi S_T) \leq 0$, because this is equivalent to $\rho(-X + \varphi S_T) + \varphi \leq m$. Hence, it is possible to hedge the financial position such that it becomes acceptable. But for each combination of m and φ we obtain

$$E[U(-X + m - \varphi + \varphi S_T)] = 1 - e^{-m+\varphi} E\left[e^X\right] E\left[e^{-\varphi S_T}\right] = -\infty,$$

because $E\left[e^X\right] = \infty$. Hence, the payoffs in $\mathcal{M}$ are not suitable to hedge X.

In the following, we develop sufficient conditions such that the SUBMA risk measure does not attain $-\infty$. Note that it attains $-\infty$ for a payoff X, if there exists a sequence $(Z_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ such that $(X+Z_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ and $\pi(Z_n) - \alpha\Psi(X+Z_n) \to -\infty$. This implies that either the hedging costs become arbitrarily small or the value of Ψ becomes arbitrarily large.

If $\Psi = I_U$ and U is bounded from above, then Ψ can not become arbitrarily large. The next result deals with such utility functions.

Proposition 4.6.4 (Utility functions bounded from above)**.** *Assume that the standard one-period model is free of good deals of the first kind and U is bounded from above, i.e., $U(+\infty) < \infty$. Then $\rho_{\alpha,I_U,\mathcal{A}_\rho,\mathcal{M},\pi} > -\infty$.*

Proof: Let $X \in \mathcal{X}$. For the sake of brevity, we set $g(\varphi) = \frac{B_0}{B_T}\rho(X - \langle S_T,\varphi\rangle) - \langle S_0,\varphi\rangle$. Then we obtain

$$\rho_{\alpha,I_U,\mathcal{A}_\rho,\mathcal{M},\pi}(X) \geq \inf_{\varphi\in\mathbb{R}^d}\inf_{m\geq g(\varphi)}\Big(m - \alpha U(+\infty)\Big) = -\alpha U(+\infty) + \inf_{\varphi\in\mathbb{R}^d} g(\varphi). \tag{4.3}$$

Lower semicontinuity of ρ implies lower semicontinuity of g. We show that g is coercive (compare Definition A.10). For this, consider part (3) in Lemma A.13 and choose an arbitrary sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ such that every $\varphi_n \neq 0$, $\|\varphi_n\|_2 \to \infty$ and $\left(\frac{\varphi_n}{\|\varphi_n\|_2}\right)_{n\in\mathbb{N}}$ converges to a point contained in the unit sphere $y \in \partial\,\mathrm{B}_d$. By positive homogeneity of ρ we get

$$\lim_{n\to\infty} g(\varphi_n) = \lim_{n\to\infty}\|\varphi_n\|_2\left(\frac{B_0}{B_T}\rho\left(\frac{X}{\|\varphi_n\|_2} - \left\langle S_T, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right) - \left\langle S_0, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right). \tag{4.4}$$

By absolute homogeneity and subadditivity of the norm $\|.\|_{\mathcal{X}}$ we obtain

$$\begin{aligned}\left\|\frac{X}{\|\varphi_n\|_2} - \left\langle S_T, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle + \langle S_T, y\rangle\right\|_{\mathcal{X}} &\leq \left\|\frac{X}{\|\varphi_n\|_2}\right\|_{\mathcal{X}} + \sum_{i=1}^d\left\|\left(y^i - \frac{\varphi_n^i}{\|\varphi_n\|_2}\right)S_T^i\right\|_{\mathcal{X}} \\ &= \underbrace{\frac{1}{\|\varphi_n\|_2}}_{\to 0}\underbrace{\|X\|_{\mathcal{X}}}_{\geq 0} + \sum_{i=1}^d\underbrace{\left|y^i - \frac{\varphi_n^i}{\|\varphi_n\|_2}\right|}_{\to 0}\underbrace{\left\|S_T^i\right\|_{\mathcal{X}}}_{\geq 0}.\end{aligned}$$

Hence, the sequence $\left(\frac{X}{\|\varphi_n\|_2} - \left\langle S_T, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right)_{n\in\mathbb{N}}$ converges to $-\langle S_T, y\rangle$ with respect to the norm $\|.\|_{\mathcal{X}}$. From lower semicontinuity of ρ we obtain

$$\liminf_{n\to\infty}\left(\frac{B_0}{B_T}\rho\left(\frac{X}{\|\varphi_n\|_2} - \left\langle S_T, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right) - \left\langle S_0, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right) \geq \frac{B_0}{B_T}\rho(-\langle S_T, y\rangle) - \langle S_0, y\rangle.$$

The right-hand side of this inequality is strictly positive by the absence of good deals of the first kind. Together with (4.4), g is coercive. By Lemma A.11 and (4.3) we have that $\rho_{\alpha,I_U,\mathcal{A}_\rho,\mathcal{M},\pi} > -\infty$. □

The reason for the desirable behavior in the previous result is demonstrated in the following example, using an exponential utility function and the zero payoff.

Example 4.6.5 (Exponential utility)**.** Let $U(x) = 1 - e^{-\delta x}$ with $\delta > 0$. Hence, $U < \infty$ and by this upper bound $\rho_{\alpha,I_U,\mathcal{A}_\rho,\mathcal{M},\pi}(0) = -\infty$ is only possible if the price of the hedging portfolio goes to $-\infty$ and at the same time for all $m < 0$ it holds that $\mathcal{A}_\rho \cap \mathcal{M}_m \neq \varnothing$. The latter is not possible if the market is free of good deals of the first kind, because of Proposition 4.5.3. Hence, we obtain $\rho_{\alpha,I_U,\mathcal{A}_\rho,\mathcal{M},\pi}(0) > -\infty$.

Now we present a sufficient condition such that the SUBMA risk measure does not attain $-\infty$, even for utility functions that are unbounded from above. This condition helps the regulator to design a capital adequacy test, see Remark 4.6.7.

To do so, let us denote the collection of all supergradients of a concave function $f : \mathbb{R} \to [-\infty, \infty]$ at a point x by $\partial f(x)$. For more details on these terms we refer to Aliprantis and Border (2006, Section 7.4) and Rockafellar (1970, Sections 23 and 24).

Theorem 4.6.6 ($\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi} > -\infty$)**.** *Assume that the standard one-period model is free of good deals of the first kind and let* $\alpha \in \left(0, \frac{B_0}{B_T}\right)$. *If* $\Psi = I_U$, *then assume in addition that* $1 \in \partial U(x)$ *for some* $x \in \operatorname{dom}(U)$. *If for every* $y \in \partial\, \mathrm{B}_d$ *with* $\frac{B_T}{B_0}\langle S_0, y\rangle > \langle E[S_T], y\rangle$ *it holds that*

$$\alpha < \left(\frac{\rho(-\langle S_T, y\rangle) - \frac{B_T}{B_0}\langle S_0, y\rangle}{\rho(-\langle S_T, y\rangle) - \langle E[S_T], y\rangle}\right)\frac{B_0}{B_T}, \tag{4.5}$$

then the SUBMA risk measure $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}$ *does not attain the value* $-\infty$.

Proof: Fix an arbitrary $X \in \mathcal{X}$. We start with the case of $\Psi = I_U$. Let $x \in \operatorname{dom}(U)$ such that $1 \in \partial U(x)$, i.e., for all $z \in \mathbb{R}$ we have $U(z) \leq z + U(x) - x$. Note that $U(x) - x \in \mathbb{R}$. For the sake of brevity, set $Y = X + U(x) - x$ and $g(\varphi) = \frac{B_0}{B_T}\rho(X - \langle S_T, \varphi\rangle) - \langle S_0, \varphi\rangle$. Then we obtain the following inequality:

$$\begin{aligned}
\rho_{\alpha,I_U,\mathcal{A}_\rho,\mathcal{M},\pi}(X) &= \inf_{\varphi\in\mathbb{R}^d}\inf_{m\geq g(\varphi)}\left(m - \alpha E\left[U\left(X + \frac{B_T}{B_0}m + \left\langle\frac{B_T}{B_0}S_0 - S_T, \varphi\right\rangle\right)\right]\right)\\
&\geq \inf_{\varphi\in\mathbb{R}^d}\inf_{m\geq g(\varphi)}\left(\left(1 - \alpha\frac{B_T}{B_0}\right)m - \alpha E[Y] - \alpha\frac{B_T}{B_0}\langle S_0, \varphi\rangle + \alpha\langle E[S_T], \varphi\rangle\right)\\
&= -\alpha E[Y] + \inf_{\varphi\in\mathbb{R}^d}\left(\left(1 - \alpha\frac{B_T}{B_0}\right)g(\varphi) - \alpha\frac{B_T}{B_0}\langle S_0, \varphi\rangle + \alpha\langle E[S_T], \varphi\rangle\right)\\
&= -\alpha E[Y] + \inf_{\varphi\in\mathbb{R}^d}\left(\left(\frac{B_0}{B_T} - \alpha\right)\rho(X - \langle S_T, \varphi\rangle) + \langle\alpha E[S_T] - S_0, \varphi\rangle\right).
\end{aligned}$$

For the case of $\Psi = C_U$, we develop an analogous result. By applying Jensen's inequality we obtain

$$\rho_{\alpha,C_U,\mathcal{A}_\rho,\mathcal{M},\pi}(X) \geq \inf_{\substack{m\in\mathbb{R},\varphi\in\mathbb{R}^d\\ m\geq g(\varphi)}} \left(-\alpha E[X] + \left(1-\alpha\frac{B_T}{B_0}\right)m - \alpha\frac{B_T}{B_0}\langle S_0,\varphi\rangle + \alpha\langle E[S_T],\varphi\rangle\right).$$

This simplifies to

$$\rho_{\alpha,C_U,\mathcal{A}_\rho,\mathcal{M},\pi}(X) \geq -\alpha E[X] + \inf_{\varphi\in\mathbb{R}^d} \left(\left(\frac{B_0}{B_T}-\alpha\right)\rho(X-\langle S_T,\varphi\rangle) + \langle \alpha E[S_T]-S_0,\varphi\rangle\right).$$

Comparing the inequalities for the cases of expected utility and certainty equivalent, we see that the expressions in the infima correspond to the following function:

$$f:\mathbb{R}^d\to\mathbb{R},\ \varphi\mapsto \left(\frac{B_0}{B_T}-\alpha\right)\rho(X-\langle S_T,\varphi\rangle) + \langle \alpha E[S_T]-S_0,\varphi\rangle.$$

By lower semicontinuity of ρ, we see that f is lower semicontinuous. Now we proceed as in the proof of Proposition 4.6.4 and show that f is coercive. To this end, choose an arbitrary sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ such that every $\varphi_n \neq 0$, $\|\varphi_n\|_2\to\infty$ and $\left(\frac{\varphi_n}{\|\varphi_n\|_2}\right)_{n\in\mathbb{N}}$ converges to a point $y\in\partial\,\mathrm{B}_d$. For the sake of brevity, we also set $\gamma = \left(\frac{B_0}{B_T}-\alpha\right) > 0$. This implies that

$$\lim_{n\to\infty} f(\varphi_n) = \lim_{n\to\infty} \underbrace{\|\varphi_n\|_2}_{\to\infty} \left(\underbrace{\gamma\rho\left(\frac{X}{\|\varphi_n\|_2} - \left\langle S_T,\frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right) + \left\langle \alpha E[S_T]-S_0,\frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle}_{=:x_n}\right).$$

Further, with the same arguments as in the proof of Proposition 4.6.4 it holds that

$$\liminf_{n\to\infty}\rho\left(\frac{X}{\|\varphi_n\|_2} - \left\langle S_T,\frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right) \geq \rho(-\langle S_T,y\rangle)\in\mathbb{R}.$$

For the sequence $(x_n)_{n\in\mathbb{N}}$ we obtain

$$\liminf_{n\to\infty} x_n \geq \gamma\rho(-\langle S_T,y\rangle) + \langle\alpha E[S_T]-S_0,y\rangle =: x(\alpha,y).$$

By a straightforward calculation we see that $x(\alpha,y)>0$ is equivalent to the following condition:

$$\left(\rho(-\langle S_T,y\rangle) - \left\langle \frac{B_T}{B_0}S_0,y\right\rangle\right)\frac{B_0}{B_T} > \alpha\Big(\rho(-\langle S_T,y\rangle) - \langle E[S_T],y\rangle\Big).$$

The right-hand side is strictly larger than zero by the absence of good deals of the first kind. The inequality is satisfied if and only if one of the following statements holds:

(i) $\frac{B_T}{B_0}\langle S_0, y\rangle \leq \langle E[S_T], y\rangle$,

(ii) $\frac{B_T}{B_0}\langle S_0, y\rangle > \langle E[S_T], y\rangle$ and $\alpha < \left(\frac{\rho(-\langle S_T,y\rangle)-\frac{B_T}{B_0}\langle S_0,y\rangle}{\rho(-\langle S_T,y\rangle)-\langle E[S_T],y\rangle}\right)\frac{B_0}{B_T}$.

Point (ii) is satisfied by assumption (4.5). So, f is lower semicontinuous and coercive. By Lemma A.11 it admits at least one global minimizer. This proves the assertion. □

Remark 4.6.7. (i) Theorem 4.6.6 gives a global finiteness condition, i.e., it is independent of the payoff X. Further, it does not depend on the utility function. Hence, the agent has neither to fear changes in the financial position nor in her preferences. Moreover, if every agent acts under a small enough weighting factor, then the government can choose the capital adequacy test ρ such that none of the agents would increase the hedging costs arbitrarily.

(ii) The condition $1 \in \partial U(x)$ for some $x \in \operatorname{dom}(U)$ is a normalization for utility functions, compare Ben-Tal and Teboulle (2007, Section 2.1). It is implied by the mild economic condition that the utility function satisfies the Inada conditions. In particular, it holds for the utility functions in Example 4.2.4.

In the simpler case of one risky asset, i.e., $d = 1$, the boundary of the unit sphere is $\{-1, 1\}$. Hence, by Proposition 4.5.2 we see that the absence of good deals of the first kind is equivalent to the following condition:

$$\frac{B_T}{B_0} \in \left(\frac{-\rho(S_T)}{S_0}, \frac{\rho(-S_T)}{S_0}\right).$$

Furthermore, we obtain the following consequence from Theorem 4.6.6:

Corollary 4.6.8. *Let $d = 1$ and $\alpha \in \left(0, \frac{B_0}{B_T}\right)$. Condition (4.5) states that*

$$\left(\frac{E[S_T]}{S_0} \leq \frac{B_T}{B_0}\right) \text{ and } \left(\alpha < \left(\frac{\rho(-S_T) - S_0\,(B_T/B_0)}{\rho(-S_T) - E[S_T]}\right)\frac{B_0}{B_T}\right),$$

or

$$\left(\frac{E[S_T]}{S_0} \geq \frac{B_T}{B_0}\right) \text{ and } \left(\alpha < \left(\frac{\rho(S_T) + S_0\,(B_T/B_0)}{\rho(S_T) + E[S_T]}\right)\frac{B_0}{B_T}\right).$$

If the remaining conditions in Theorem 4.6.6 are also satisfied, then $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi} > -\infty$.

We graphically illustrate the sufficient condition (4.5) for the Black-Scholes financial market model from Example 4.5.6. We use the same setting and parameter values.

Example 4.6.9 (Black-Scholes model)**.** Figure 4.6 allows to verify the sufficient condition for the weighting factor $\alpha = 0.85$. On the left-hand side we illustrate the points on the unit sphere depending on the sufficient condition. The upper bounds from (4.5) are not relevant for the black points of the graph. The inequality (4.5) is satisfied for the green points, i.e., the upper bounds are strictly larger than the weighting factor. The upper bounds generated by the red points are smaller than the weighting factor, i.e., the sufficient condition is not satisfied.

On the right-hand side we plot the upper bounds for the weighting factor. The dashed line corresponds to the chosen α. If we, e.g., alter the weighting factor to $\alpha < 0.8$ then the sufficient condition in Theorem 4.6.6 would be satisfied, i.e., in an analogous plot to the one on the left-hand side no red points would occur. This means there is still enough space to select a weighting factor such that the sufficient condition is satisfied.

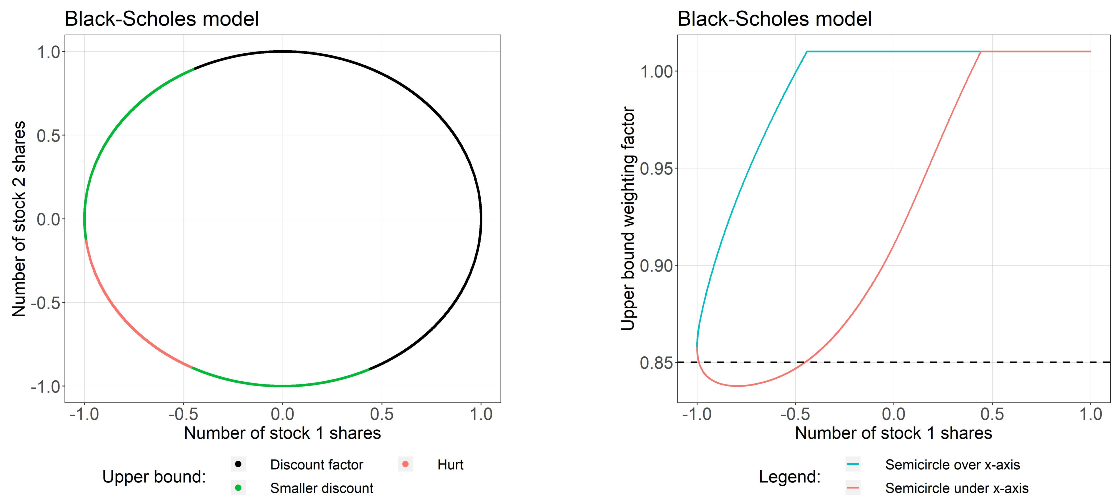


Figure 4.6: *Left-hand side:* Points in the unit sphere characterized by the conditions in Theorem 4.6.6. *Right-hand side:* Minimum upper bound for the weighting factor α, cf. Example 4.6.9. The dashed line corresponds to the weighting factor $\alpha = 0.85$.

It remains the following question: Can the SUBMA risk measure attain $-\infty$, if the sufficient condition in Theorem 4.6.6 is violated? To answer this question let us assume the sufficient condition is violated. Then we distinguish between the cases of $\alpha \geq \frac{B_0}{B_T}$ and $\alpha < \frac{B_0}{B_T}$. If $\alpha \geq \frac{B_0}{B_T}$, then the OEU risk measure does attain $-\infty$, see Geissel et al. (2018, Remark 2.3). Since OEU is a special case of a SUBMA risk measure, the counterexamples are also valid for our setting. In the remaining case of $\alpha < \frac{B_0}{B_T}$ the following example shows that the SUBMA risk measure can indeed attain $-\infty$.

Example 4.6.10 (Sufficient condition is violated)**.** Assume $U(x) = x$ and $\alpha \in \left(0, \frac{B_0}{B_T}\right)$

such that there exists $y \in \partial \mathrm{B}_d$ with

$$\Big(\rho(-\langle S_T, y\rangle) - \langle E[S_T], y\rangle\Big)\alpha > \frac{B_0}{B_T}\rho(-\langle S_T, y\rangle) - \langle S_0, y\rangle.$$

For $X \in \mathcal{X}$ we obtain

$$\rho_{\alpha,I_U,\mathcal{A}_\rho,\mathcal{M},\pi}(X) = -\alpha E[X] + \inf_{\varphi\in\mathbb{R}^d}\left(\left(\frac{B_0}{B_T} - \alpha\right)\rho(X - \langle S_T, \varphi\rangle) - \langle \alpha E[S_T] - S_0, \varphi\rangle\right).$$

Now we choose a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that every $\varphi_n \neq 0$, $\|\varphi_n\|_2 \to \infty$ and $\left(\frac{\varphi_n}{\|\varphi_n\|_2}\right)_{n\in\mathbb{N}}$ converges to y. Then

$$\begin{aligned}\rho_{\alpha,I_U,\mathcal{A}_\rho,\mathcal{M},\pi}(X) \leq -\alpha E[X] + \liminf_{n\to\infty}\|\varphi_n\|_2\Bigg(\left(\frac{B_0}{B_T} - \alpha\right)\rho\left(\frac{X}{\|\varphi_n\|_2} - \left\langle S_T, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right)\\ + \left\langle \alpha E[S_T] - S_0, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\Bigg).\end{aligned}$$

Let ρ be continuous. Then we have

$$\begin{aligned}&\lim_{n\to\infty}\left(\left(\frac{B_0}{B_T} - \alpha\right)\rho\left(\frac{X}{\|\varphi_n\|_2} - \left\langle S_T, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right) + \left\langle \alpha E[S_T] - S_0, \frac{\varphi_n}{\|\varphi_n\|_2}\right\rangle\right)\\ &\quad= \left(\frac{B_0}{B_T} - \alpha\right)\rho(-\langle S_T, y\rangle) + \langle \alpha E[S_T] - S_0, y\rangle < 0,\end{aligned}$$

and hence, $\rho_{\alpha,I_U,\mathcal{A}_\rho,\mathcal{M},\pi}(X) = -\infty$, so $-\infty$ can indeed be attained.

4.6.2 Continuity

We make use of the condition $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi} > -\infty$ to reach the result that the SUBMA risk measure becomes continuous. For this purpose, we need the following auxiliary result which follows immediately from the definition of a SUBMA risk measure and is not restricted to the standard one-period model.

Lemma 4.6.11. *Assume a marketed space $\mathcal{M}$, a pricing functional $\pi : \mathcal{M} \to \mathbb{R}$ and an acceptance set $\mathcal{A}$. Let $\Psi \in \{I_U, C_U\}$ and $\alpha > 0$. Then we obtain*

$$\mathcal{A} \subset \{X \in \mathcal{X} \,|\, \rho_{\alpha,\Psi,\mathcal{A},\mathcal{M},\pi}(X) \leq -\alpha\Psi(X)\}.$$

Now we are able to prove the main result in this section.

Theorem 4.6.12 (Continuity)**.** *Assume the standard one-period model. Further, let $\Psi \in \{I_U, C_U\}$. We assume that $\mathcal{A}_\rho$ is convex, Ψ is concave, $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi} > -\infty$ and* *there exists a subset $\mathcal{B} \subset \mathcal{A}_\rho$ such that*

$$\inf_{X\in\mathcal{B}} \Psi(X) > -\infty \text{ and } \operatorname{int}_{\|.\|_{\mathcal{X}}}(\mathcal{B}) \neq \varnothing. \tag{4.6}$$

Then $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}$ is continuous on $\operatorname{int}_{\|.\|_{\mathcal{X}}}(\operatorname{dom}(\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}))$.

In particular, if $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}$ is finite-valued, then it is continuous.

Proof: Due to Lemma 4.6.11, we have $\operatorname{int}_{\|.\|_{\mathcal{X}}}(\mathcal{B}) \subset \operatorname{int}_{\|.\|_{\mathcal{X}}}(\operatorname{dom}(\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}))$ and hence, $\operatorname{int}_{\|.\|_{\mathcal{X}}}(\operatorname{dom}(\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi})) \neq \varnothing$. Further, $\operatorname{int}_{\|.\|_{\mathcal{X}}}(\operatorname{dom}(\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}))$ is convex. Since the SUBMA risk measure $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}$ is bounded from above on $\operatorname{int}_{\|.\|_{\mathcal{X}}}(\mathcal{B})$ we can apply Theorem A.7 which yields the claim. □

Remark 4.6.13. The existence of a subset of $\mathcal{A}_\rho$ with non-empty interior adapts the assumption in Farkas et al. (2014, Theorem 3.10) and Proposition 1.4.4 of an acceptance set with non-empty interior which is a commonly used assumption to ensure finiteness of risk measures. The additional requirement of $\inf_{X\in\mathcal{B}} \Psi(X) > -\infty$ builds a connection between acceptance set and utility function.

The next two examples differ in the choice of the model space $\mathcal{X}$ and for both examples the condition (4.6) is satisfied.

Example 4.6.14 (Expected Shortfall and $\mathcal{X} = L^p$)**.** Let $\mathcal{X} = L^p$ with $p \in [1, \infty)$. Assume the Expected Shortfall at level $\lambda \in (0,1)$. By Lemma 1.4.8 we obtain

$$\operatorname{int}_{\|.\|_{L^p}}(\mathcal{A}_{\mathrm{ES}_\lambda}) = \{X \in L^p \mid \mathrm{ES}_\lambda(X) < 0\}$$

which shows that the interior of the acceptance set is non-empty, e.g., $\mathrm{ES}_\lambda(1_\Omega) = -1 < 0$. Choose a constant $c \in \mathbb{R}$ such that there exists $x_0 \in \mathbb{R}$ with $U(x_0) \geq c$. Further, assume that $(\Omega, \mathscr{F}, P)$ is atomless. In this setup, Munari (2015, Proposition 2.4.27) shows that the set $\{X \in L^1 \mid E[U(X)] \geq c\}$ has non-empty interior if and only if $U(x_0) > c$ for some $x_0 \in \mathbb{R}$ and

$$\lim_{x\to\infty} \frac{x^p}{U(-x)} < 0. \tag{4.7}$$

If U satisfies these conditions, then $\mathcal{B} = \mathcal{A}_{\mathrm{ES}_\lambda} \cap \{X \in L^p \mid E[U(X)] \geq c\}$ satisfies (4.6).

Remark 4.6.15. Condition (4.7) is crucial, because it excludes the most common utility functions, e.g., for $U(x) = 1 - e^{-x}$ we get

$$\lim_{x\to\infty} \frac{x^p}{U(-x)} = \lim_{x\to\infty} \frac{x^p}{1-e^x} = 0.$$

The reason is the large model space $\mathcal{X} = L^p$. Hence, in the next example we use Orlicz spaces to achieve continuity with respect to a larger class of utility functions.

Example 4.6.16 (Expected Shortfall and $\mathcal{X} = L^{\hat{U}}$)**.** Assume $U(0) = 0$ and $\underline{x}_U < 0$. Then set $\mathcal{X} = L^{\hat{U}}$. Again we use the Expected Shortfall at level $\lambda \in (0,1)$. With Farkas et al. (2014, Proposition 4.4) we obtain that ES_λ is finite. By direct calculations we see that ES_λ is convex and decreasing. Since $L^{\hat{U}}$ is a Banach lattice, we obtain by Proposition A.8 that ES_λ is continuous on $L^{\hat{U}}$. This gives us that

$$\left\{X \in L^{\hat{U}} \,\middle|\, \mathrm{ES}_\lambda(X) < 0\right\} \subset \mathrm{int}_{\|\cdot\|_{\hat{U}}}(\mathcal{A}_{\mathrm{ES}_\lambda})$$

which implies $m1_\Omega \in \mathrm{int}_{\|\cdot\|_{\hat{U}}}(\mathcal{A}_{\mathrm{ES}_\lambda})$ for all $m > 0$. Now assume that there exists a constant $c \in \mathbb{R}$ and $x_0 > 0$ with $U(x_0) > c$. For $(\Omega, \mathscr{F}, P)$ being atomless, this is equivalent to the fact that the set $\left\{X \in L^{\hat{U}} \,\middle|\, E[U(X)] \geq c\right\}$ has non-empty interior, see Munari (2015, Proposition 2.4.31). For instance, it is satisfied for $U(x) = 1-e^{-x}$ and $c = 0$. Under this condition, the set $\mathcal{B} = \mathcal{A}_{\mathrm{ES}_\lambda} \cap \left\{X \in L^{\hat{U}} \,\middle|\, E[U(X)] \geq c\right\}$ satisfies (4.6).

4.6.3 Optimal payoffs

For an investor it is important to know if there exists a marketed payoff for which the value of the SUBMA risk measure is attained. Such a marketed payoff is called an ***optimal payoff***. This leads to the ***optimal payoff map*** of a SUBMA risk measure $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}$ which is for every $X \in \mathcal{X}$ with $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}(X) < \infty$ defined by

$$\mathcal{E}(X) := \{Z \in \mathcal{M} \mid X + Z \in \mathcal{A}_\rho, \rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}(X) = \pi(Z) - \alpha\Psi(X+Z)\}.$$

The following result characterizes optimal payoffs. We denote the cardinality of a set A by $|A|$. For a set C we denote by δ_C the map with $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = \infty$ otherwise.

Theorem 4.6.17 (Existence and uniqueness of optimal payoffs)**.** *Assume the standard one-period model and let* $\Psi \in \{I_U, C_U\}$. *Choose* $X \in \mathcal{X}$ *with* $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}(X) \in \mathbb{R}$. *Then the following statements hold:*

(i) $\mathcal{E}(X) \neq \varnothing$.

(ii) If ρ *and* $-\Psi$ *are convex and the restriction of* Ψ *to* $\mathrm{dom}(\Psi)$ *is strictly concave, then* $|\mathcal{E}(X)| = 1$.

Proof: We choose an arbitrary $X \in \mathcal{X}$. By recalling the arguments in the proof of Theorem 4.6.6 and setting $g(\varphi) = \frac{B_0}{B_T}\rho(X - \langle S_T, \varphi\rangle) - \langle S_0, \varphi\rangle$ we see that the map

$$h : (m, \varphi) \mapsto m - \alpha\Psi\left(X + \frac{B_T}{B_0}m + \left\langle \frac{B_T}{B_0}S_0 - S_T, \varphi\right\rangle\right) + \delta_{[g(\varphi),\infty)}(m),$$

is an upper bound of the map

$$\tilde{f} : (m, \varphi) \mapsto \left(\frac{B_0}{B_T} - \alpha\right)\rho(X - \langle S_T, \varphi\rangle) + \langle \alpha E[S_T] - S_0, \varphi\rangle + b + \delta_{[g(\varphi),\infty)}(m),$$

where $b \in \mathbb{R}$ is an appropriate constant. We also know that $\tilde{f}$ is coercive which in turn implies that h is coercive. Further, h is lower semicontinuous and therefore, by Lemma A.11 we get that h attains a global minimum. This proves (i). To show (ii), consider the set

$$\mathcal{C} = \left\{ (m, \varphi) \in \mathbb{R} \times \mathbb{R}^d \,\middle|\, m \geq \frac{B_0}{B_T}\rho(X - \langle S_T, \varphi\rangle) - \langle S_0, \varphi\rangle, \right.$$
$$\left. X + \frac{B_T}{B_0}m + \left\langle \frac{B_T}{B_0}S_0 - S_T, \varphi\right\rangle \in \operatorname{dom}(\Psi)\right\}.$$

The condition $\rho_{\alpha,\Psi,\mathcal{A}_\rho,\mathcal{M},\pi}(X) < \infty$ implies that $\mathcal{C}$ is non-empty. Further, from ρ being convex and Ψ being concave we know that $\mathcal{C}$ is convex. Finally, $\mathcal{C} \subset \operatorname{dom}(\Psi)$ gives us that the restriction of h to $\mathcal{C}$ is strictly convex. Hence, if $(m, \varphi), (n, \kappa) \in \mathcal{C}$ with $(m, \varphi) \neq (n, \kappa)$ are two minima of h (note, $h(m, \varphi) = h(n, \kappa) < \infty$), then for each $\lambda \in (0, 1)$ we get

$$h(\lambda m + (1 - \lambda)n, \lambda\varphi + (1 - \lambda)\kappa) < \lambda h(m, \varphi) + (1 - \lambda)h(n, \kappa) = h(n, \kappa).$$

But this is a contradiction and hence, $|\mathcal{E}(X)| = 1$. □

Remark 4.6.18. A natural situation to satisfy the conditions in Theorem 4.6.17 (ii) is the one in which the utility function U is strictly concave on $\operatorname{dom}(U)$. Then I_U is strictly concave on $\operatorname{dom}(I_U)$.

4.7 Continuous-time trading: Solution technique

We now consider a continuous-time financial market model. This leads to a set of marketed payoffs which no longer form a linear space in general. Hence, we need to adjust the concept of a SUBMA risk measure slightly. In Section 4.7.1, we introduce the primal assets describing the financial market and introduce the admissible trading strategies. In Section 4.7.2, we define the SUBMA risk measure for continuous-time trading and state a representation result. Then in Sections 4.7.3 and 4.7.4, we apply this representation result to find solutions for a constant payoff with respect to CRRA and CARA utility functions.

4.7.1 Primal assets and admissible trading strategies

We assume a financial market model with finite time horizon $T > 0$. Further, we equip the probability space $(\Omega, \mathscr{F}, P)$ with a filtration $(\mathscr{F}_t)_{t\in[0,T]}$ such that $\mathscr{F}_T = \mathscr{F}$. Then we assume a market consisting of a single risk-free asset with constant value 1 and $d \in \mathbb{N}$ stocks. Their prices are denoted by S and are assumed to be a $\mathbb{R}^d$-valued (càdlàg) semi-martingale.

We formulate the set of trading strategies similarly to Biagini and Černý (2020). It is a unifying Orlicz space framework for portfolio optimization problems that allows a simultaneous description of the model components for CRRA and CARA utility functions. Without loss of generality, we make the following standing assumption:

Assumption 4.7.1. *For the lower bound of the effective domain of the utility function U it holds that $\underline{x}_U < 0$. Further, assume that $U(0) = 0$, i.e., $x^*_U = 0$.*

We define admissible trading strategies with the help of the Orlicz space $L^{\hat{U}}$. We denote by $L(S)$ the set of all predictable, S-integrable stochastic processes. The corresponding stochastic integral for $H \in L(S)$ at time $t \in [0,T]$ is written as $(H \cdot S)_t$.

Definition 4.7.2 (Admissible trading strategies)**.** *The set of admissible trading strategies is*

$$\mathcal{T} := \left\{ H \in L(S) \,\middle|\, \inf_{t\in[0,T]} (H \cdot S)_t \in L^{\hat{U}} \right\}.$$

The set of $L^{\hat{U}}$-tame trading strategies for an initial endowment $B \in L^{\hat{U}}$ is given by

$$\mathcal{T}(B) := \{H \in \mathcal{T} \,|\, E[U(B + (H \cdot S)_T)] \text{ exists}\}. \tag{4.8}$$

The set of payoffs, respectively the set of payoffs for an initial endowment $B \in L^{\hat{U}}$, is defined by $\mathcal{K} := \{(H \cdot S)_T \,|\, H \in \mathcal{T}\}$, respectively $\mathcal{K}(B) := \{(H \cdot S)_T \,|\, H \in \mathcal{T}(B)\}$.

We give two examples for $L^{\hat{U}}$-tame trading strategies.

Example 4.7.3. (i) *Power or logarithmic utility:* For a power or logarithmic utility function U we obtain that $L^{\hat{U}} = L^\infty$ and $L^{\hat{U}}$-tame trading strategies are exactly the stochastic processes in $L(S)$ for which the wealth process is uniformly bounded from below and the relevant expected utility exists, i.e., our framework reproduces the one in Kramkov and Schachermayer (1999).

(ii) *Exponential utility:* Assume $T = 1$, $d = 1$ and S is a Brownian motion. Further, U is an exponential utility function with risk aversion parameter $\delta = 1$. Then the constant trading strategy by investing one unit in S is $L^{\hat{U}}$-tame for an initial endowment $B = 0$, because of

$$E\left[e^{\left|\inf_{t\in[0,1]} S_t\right|}\right] = E\left[e^{|S_1|}\right] = 2E\left[e^{S_1}1_{\{S_1\geq 0\}}\right] \leq 2e^{\frac{1}{2}} < +\infty.$$

Note that the classical approach of a uniform bound on the wealth process does not allow for such a trading strategy.

4.7.2 SUBMA representation result

Assume an acceptance set $\mathcal{A}$ describing the preferences in L^0, i.e., a proper, non-empty and monotone subset of L^0. Then we define the following SUBMA risk measure for continuous-time trading as follows:

Definition 4.7.4 (SUBMA risk measure)**.** *Choose a weighting factor $\alpha > 0$. For $\Psi \in \{I_U, C_U\}$ and a payoff $X \in L^0$ we define the SUBMA risk measure for trading in continuous time by*

$$\nu_{\alpha,\Psi,\mathcal{A}}(X) := \inf\{x - \alpha\Psi(X + x + K) \,|\, x \in \mathbb{R}, K \in \mathcal{K}(X + x), X + x + K \in \mathcal{A}\}.$$

In contrast to Section 4.3, the continuous-time trading does not lead to a marketed space in general, i.e., a linear space of payoffs. The set of hedging opportunities is given by $\mathcal{M}(X) := \bigcup_{x\in\mathbb{R}}\left(\{x\} + \mathcal{K}(X + x)\right)$ and hence it depends on the payoff X. We are interested in calculating the optimal payoff for a fixed X.

To calculate the SUBMA risk measure we use the following representation result:

Theorem 4.7.5 (SUBMA representation)**.** *Let $\alpha > 0$. For $X \in L^0$ and $\Psi \in \{I_U, C_U\}$ it holds that*

$$\nu_{\alpha,\Psi,\mathcal{A}}(X) := \inf_{x\in\mathbb{R}}\left(x - \alpha \sup_{\substack{K\in\mathcal{K}(X+x)\\ X+x+K\in\mathcal{A}}} \Psi(X + x + K)\right). \tag{4.9}$$

Proof: First, let us introduce the following set for $x \in \mathbb{R}$:

$$\mathcal{C}(x) := \{x - \alpha\Psi(X + x + K) \,|\, K \in \mathcal{K}(X + x), X + x + K \in \mathcal{A}\}.$$

Then it holds that

$$\nu_{\alpha,\Psi,\mathcal{A}}(X) = \inf \bigcup_{x\in\mathbb{R}} \mathcal{C}(x) = \inf\{\inf \mathcal{C}(x) \mid x \in \mathbb{R}\}.$$

From this statement we obtain the claim:

$$\begin{aligned}\nu_{\alpha,\Psi,\mathcal{A}}(X) &= \inf_{x\in\mathbb{R}} \inf_{\substack{K\in\mathcal{K}(X+x)\\ X+x+K\in\mathcal{A}}} \Big(x - \alpha\Psi(X+x+K)\Big)\\ &= \inf_{x\in\mathbb{R}} \left(x - \alpha \sup_{\substack{K\in\mathcal{K}(X+x)\\ X+x+K\in\mathcal{A}}} \Psi(X+x+K)\right).\end{aligned}$$

□

Remark 4.7.6. Theorem 4.7.5 shows that the calculation of the SUBMA risk measure relies on various portfolio optimization tasks which differ in the initial endowment.

Theorem 4.7.5 suggests a two step procedure to calculate the SUBMA risk measure:

Step 1: Calculate the following portfolio optimization value in dependence of the initial capital $x \in \mathbb{R}$:

$$u(X+x) := \sup_{\substack{K\in\mathcal{K}(X+x)\\ X+x+K\in\mathcal{A}}} \Psi(X+x+K). \tag{4.10}$$

Step 2: Plug the result of (4.10) in (4.9) and calculate $\nu_{\alpha,\Psi,\mathcal{A}}(X)$.

Remark 4.7.7. In the first step it is necessary to calculate Lagrange multipliers numerically. Further, in the majority of our examples we also perform a numerical calculation to find the minimum of the objective function in the second step.

Now we apply the suggested procedure. To make use of the classical portfolio optimization results under risk constraints as, e.g., in Basak and Shapiro (2001), we consider in the upcoming sections a constant payoff X. To calculate the SUBMA risk measure for non-constant payoffs, one has to solve the portfolio optimization problem under risk constraints for a random initial endowment. This interesting but demanding task is beyond the scope of this dissertation, compare with the outlook in Chapter 5.

4.7.3 CRRA

We use two benchmark acceptance sets: The acceptance set of a portfolio insurer and the acceptance set with respect to the actual expected loss, see Basak and Shapiro (2001).

Setup: Let $z \in \mathbb{R}$, $\epsilon \geq 0, q > -1$. The acceptance set of a portfolio insurer (PI) is

$$\mathcal{A}_z := \{X \in \mathcal{X} \mid X \geq z \ P\text{-a.s.}\}.$$

The acceptance set based on the limited expected loss (LEL) is

$$\mathcal{A}_{q,\epsilon} := \left\{X \in \mathcal{X} \mid E_Q[(X-q)^-] \leq \epsilon\right\}.$$

We use the logarithmic utility function $U(x) = \ln(x+1)$ and the power utility function $U(x) = \frac{1}{1-\delta}\left((x+1)^{1-\delta} - 1\right)$ with $\delta \in (0,\infty)\backslash\{1\}$. We assume a Black-Scholes market model with one bank account and one stock. The unique equivalent martingale measure is denoted by Q and the market price of risk is denoted by γ. The payoff is $X = k$ with constant $k \in (-\infty, 0]$.

Example 4.7.8 (CRRA utility function with $\Psi = I_U$)**.** In this example, we define the penalty term by the expected utility, i.e., $\Psi = I_U$. Based on the logarithmic utility function we would like to calculate the SUBMA risk measure:

$$\nu_{\alpha,I_U,\mathcal{A}}(k) = \inf_{m \geq -k-1}\left(m - \alpha \sup_{\substack{H \in \mathcal{T}(m+k) \\ k+m+(H\cdot S)_T \in \mathcal{A}}} E[\ln(k+m+(H\cdot S)_T+1)]\right).$$

For PI with $z > -1$ we perform a pointwise Lagrange approach as in the proof of Teplá (2001, Proposition 1) such that we obtain the optimal payoff of the portfolio optimization problem in the SUBMA formulation by

$$X_T^* = \begin{cases} \frac{1}{\lambda}\frac{\mathrm{d}P}{\mathrm{d}Q} - 1 & , \frac{\mathrm{d}Q}{\mathrm{d}P} \leq \frac{1}{\lambda(z+1)}, \\ z & , \frac{\mathrm{d}Q}{\mathrm{d}P} > \frac{1}{\lambda(z+1)}, \end{cases} \tag{4.11}$$

where the Lagrange multiplier $\lambda > 0$ satisfies the budget constraint

$$\frac{1}{\lambda}P\left(\frac{\mathrm{d}Q}{\mathrm{d}P} \leq \frac{1}{\lambda(z+1)}\right) + (z+1)E\left[\frac{\mathrm{d}Q}{\mathrm{d}P}1_{\left\{\frac{\mathrm{d}Q}{\mathrm{d}P} > \frac{1}{\lambda(z+1)}\right\}}\right] = m+k+1. \tag{4.12}$$

The latter is solved numerically. By knowing the optimal payoff we can calculate the objective of the SUBMA risk measure and find the minimal hedging costs numerically. For the sake of brevity, we define

$$c(x) := -\frac{1}{\gamma\sqrt{T}}\ln\left(\frac{1}{x(z+1)}\right) - \frac{1}{2}\gamma\sqrt{T},$$

$$d(x) := \left(-\ln(x) + \frac{1}{2}\gamma^2 T\right)\left(1 - \Phi(c(x))\right) + \ln(z+1)\Phi(c(x)) + \gamma\sqrt{\frac{T}{2\pi}}e^{-\frac{1}{2}(c(x))^2}.$$

Then by (4.11) the solution of the corresponding portfolio optimization problem is

$$\sup_{\substack{H\in\mathcal{T}(m+k)\\ k+m+(H\cdot S)_T\in\mathcal{A}_z}} E[\ln(k+m+(H\cdot S)_T+1)] = d(\lambda).$$

For LEL we obtain from Gabih, Sass, and Wunderlich (2009, Theorem 5.1) the optimal payoff. Then we are able to determine the SUBMA risk measure numerically.

In Figure 4.7, we illustrate the SUBMA risk measure and the corresponding hedging costs for the following parameter values:

$$\gamma = 0.04,\ T = 10,\ k = -0.5,\ z = -0.1,\ q = -0.1,\ \epsilon = 0.05. \tag{4.13}$$

Additionally, the Merton case ($z \le -1$) is given, i.e., the risk constraint is redundant.

The PI (dotted red) and LEL (solid blue) SUBMA graphs show a linear behavior for small values of α. The reason is that minimizing costs is more important for the agent than maximizing the expected utility. Hence, she chooses the minimal hedging costs and hence, the graphs on the right-hand side in Figure 4.7 are constant. Since the same costs lead to the same optimal payoffs, the expected utility is the same for all α up to approximately 0.75. For larger α the solution is close to the Merton solution (dashed green). This can be explained as follows: For increasing α the portfolio optimization criterion becomes more relevant than the cost criterion. The agent can increase her expected utility by increasing her hedging costs m. For instance, in the PI case we see from the budget constraint (4.12) that if m increases, then the Lagrange multiplier λ decreases. Further, for $\gamma > 0$ we have $\lim_{x\downarrow 0} c(x) = -\infty$ and hence

$$\lim_{m\to\infty} \sup_{\substack{H\in\mathcal{T}(m+k)\\ k+m+(H\cdot S)_T\in\mathcal{A}_z}} E[\ln(k+m+(H\cdot S)_T+1)] = \ln(m+k+1) + \frac{1}{2}\gamma^2 T.$$

The right-hand side is the value function in the Merton case $z \le -1$, i.e., for large m values the PI case is close to the Merton case.

Inspired by Basak and Shapiro (2001), we illustrate the optimal payoffs in dependence of the pricing density $\frac{\mathrm{d}Q}{\mathrm{d}P}$ for different values of α in Figure 4.8. The interpretation of the three cases is as follows:

(i) $\underline{\alpha = 0.25:}$ As earlier mentioned, the PI and LEL payoffs correspond to the minimal possible hedging costs. Hence, in the PI case, a constant amount is invested into the bank account. In the LEL case, the payoff is less than the shortfall level $q = -0.1$ for bad states (large values of $\frac{\mathrm{d}Q}{\mathrm{d}P}$). Hence, the hedging costs in the LEL case are less than in the PI case.

(ii) $\underline{\alpha = 0.75:}$ For small values of $\frac{\mathrm{d}Q}{\mathrm{d}P}$, the PI and LEL payoffs are larger than the value

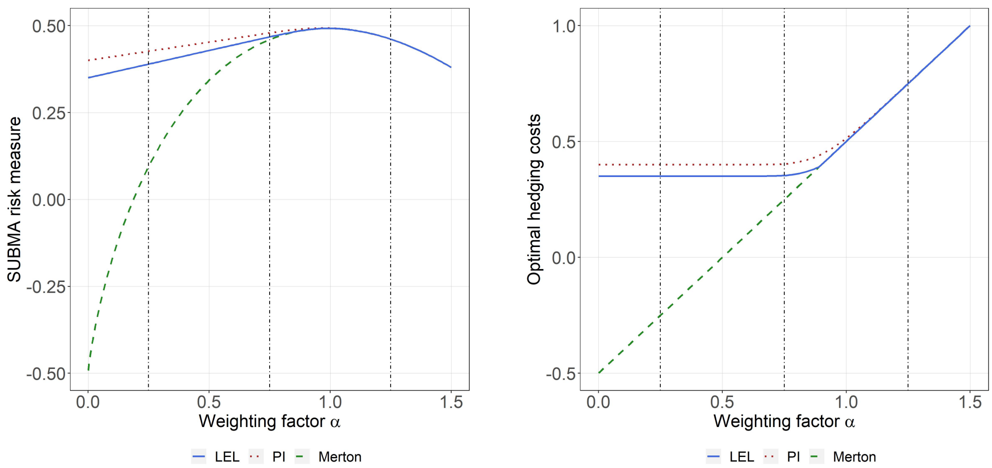


Figure 4.7: Value of SUBMA risk measure (left-hand side) and corresponding hedging costs (right-hand side) for logarithmic utility in dependence of α. The dotted dashed vertical lines correspond to values 0.25, 0.75 and 1.25 for α.

of $z = q = -0.1$. It is notable that the payoffs in these states are close together. This is due to the feature of the SUBMA risk measure that leads to larger hedging costs in the PI case.

(iii) $\underline{\alpha = 1.25:}$ Here the LEL payoff is equal to the Merton payoff. By slightly larger hedging costs, the PI payoff is for good states close to the Merton payoff.

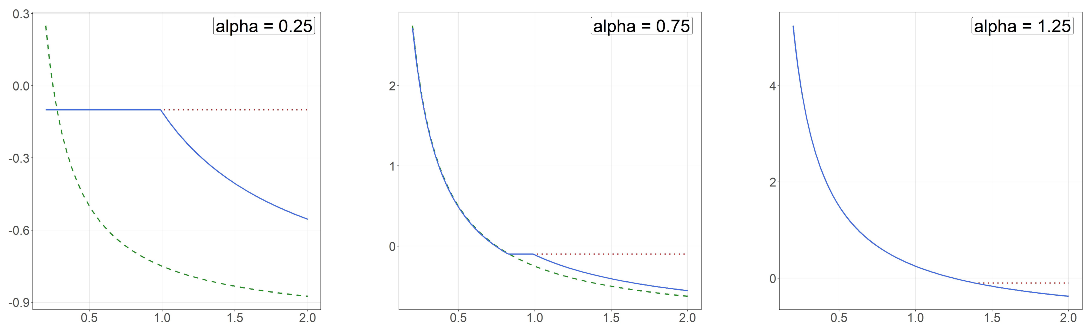
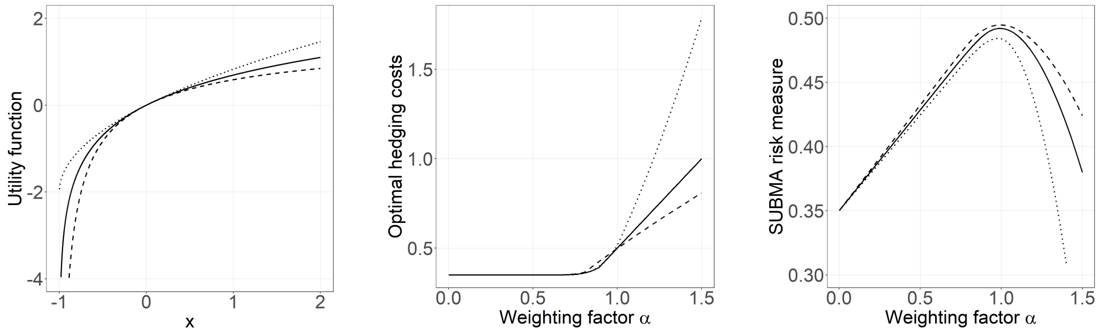


Figure 4.8: Optimal payoffs for different α values. The x-axis corresponds to values of the pricing density $\frac{dQ}{dP}$. The y-axis is the value of the optimal payoff in the corresponding state. We consider the LEL (solid blue), the PI (dotted red) and the Merton (dashed green) case.

Finally, in Figure 4.9 we present the impact of changes in the risk aversion parameter with respect to the LEL acceptance set. For $\delta \in (0, \infty)\backslash\{1\}$ we assume the power utility function $U(x) = \frac{1}{1-\delta}\Big((x+1)^{1-\delta} - 1\Big)$. For $\delta = 1$ we use logarithmic utility.

For small α all three values of δ lead to the same hedging costs and hence, the optimal payoffs are the same. Hence, the linear increase in the SUBMA risk measures is solely driven by increasing α. If α is close below 1, then $\delta = 1.5$ leads to the highest hedging costs. Indeed, for $\delta = 1.5$ the agent is more risk averse as in the remaining cases. This means that the agent with $\delta = 1.5$ reacts first, if the expected utility criterion becomes more relevant, i.e., if α increases. For α values larger than 1 the situation changes. Note that for $\delta > 1$ the power utility function is bounded from above. So there is a natural limit for improving the SUBMA objective by increasing the hedging costs. But for $\delta \leq 1$ the utility functions are unbounded from above. Hence, the hedging costs are larger than for bounded utility functions.



(a) Utility functions (b) Hedging costs (c) SUBMA risk measure

Figure 4.9: Power utility functions, hedging costs and SUBMA risk measures for the LEL acceptance set and the risk aversion parameters $\delta = 0.5$ (dotted line), $\delta = 1$ (solid line) and $\delta = 1.5$ (dashed line).

Now we discuss the results if we use the certainty equivalent, i.e., $\Psi = C_U$.

Example 4.7.9 (CRRA utility function with $\Psi = C_U$)**.** In contrast to the previous example, it is remarkable that the SUBMA risk measure is infinite for large values of α and hence, also the hedging costs are infinite. For the PI and the LEL acceptance sets we plot the SUBMA risk measure and the corresponding hedging costs in Figure 4.10. The underlying parameters are the same as in the previous example, see (4.13). If α is small enough, then the curves for the SUBMA risk measures are similar to the case $\Psi = I_U$. But the hedging costs are different. If $\alpha \uparrow e^{-\frac{1}{2}\gamma^2 T}$, then they do not converge to the costs of the unrestricted Merton case ($z < -1$). But they converge to costs such that the risk measures converge to the same value of $-k - 1 + e^{-\frac{1}{2}\gamma^2 T}$.

Finally, in Figure 4.11 we plot the SUBMA risk measure for the LEL acceptance set and different risk aversions δ. We use the same values for δ as in the previous example. We see that the impact of changes in the risk aversion is less than before, i.e., the SUBMA curves are nearly unchanged. If we increase the weighting factor, then the agent with $\delta = 0.5$ increases the hedging costs earlier than in the other cases. This leads to a reduction in the SUBMA risk measure for $\delta = 0.5$. So, in contrast to the

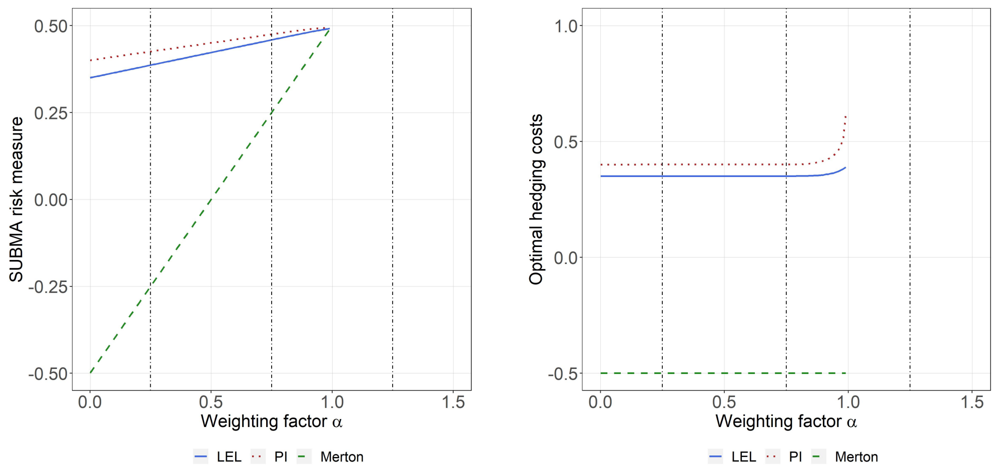
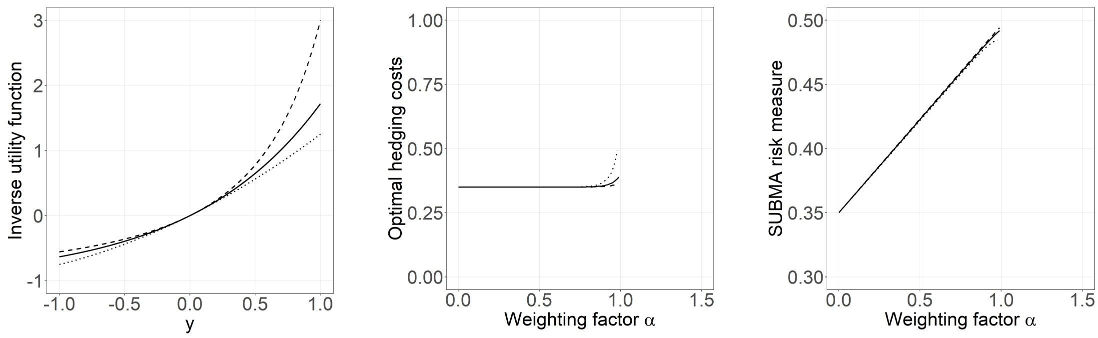


Figure 4.10: Value of SUBMA risk measure (left-hand side) and corresponding hedging costs (right-hand side) for logarithmic utility in dependence of α. The dotted dashed vertical lines correspond to values 0.25, 0.75 and 1.25 for α.

previous example the certainty equivalent does not encourage the more risk averse agent ($\delta = 1.5$) to increase her hedging costs first.



(a) Utility functions (b) Hedging costs (c) SUBMA risk measure

Figure 4.11: Inverse of power utility functions, hedging costs and SUBMA risk measures for the LEL acceptance set and the risk aversion parameters $\delta = 0.5$ (dotted line), $\delta = 1$ (solid line) and $\delta = 1.5$ (dashed line).

4.7.4 CARA

Now we use the exponential utility function $U(x) = 1 - e^{-\delta x}$ with $\delta > 0$.

Example 4.7.10 (CARA utility function with $\Psi = I_U$)**.** We give a short overview of the techniques used to obtain the solutions:

For the PI (respectively LEL) case, we apply a pointwise Lagrange approach in the style of Basak and Shapiro (2001) (respectively Gabih et al. (2009)) to find a candidate for an optimal payoff and then apply a convex duality approach to find the optimal expected utility. This gives us the objective of the SUBMA risk measure which we then minimize numerically. For the sake of brevity, we omit the formulas.

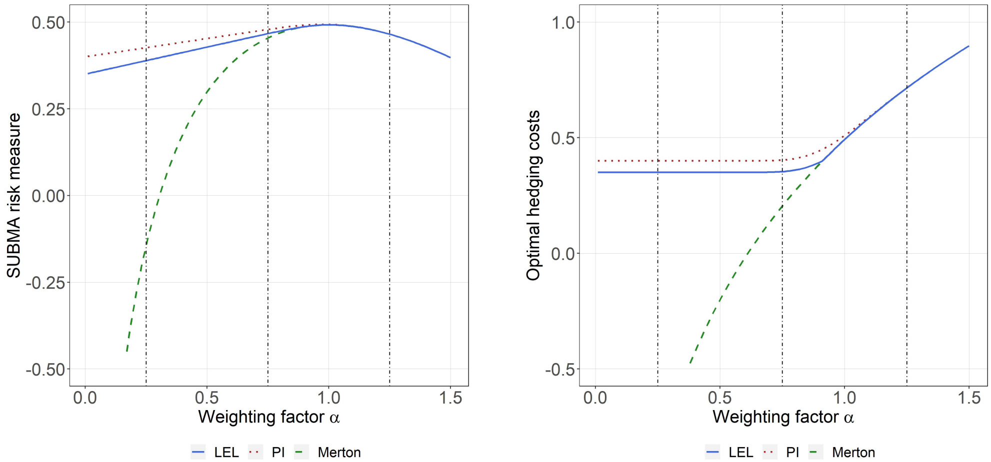
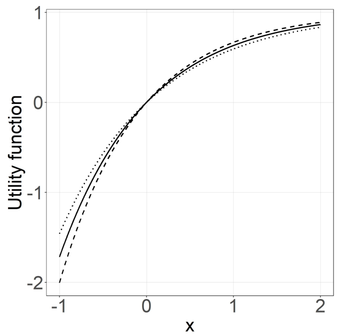
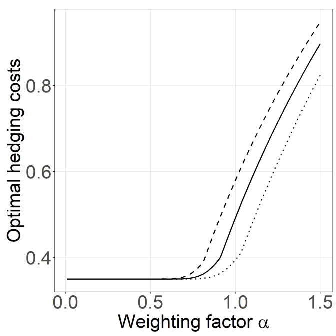


Figure 4.12: Value of SUBMA risk measure (left-hand side) and corresponding hedging costs (right-hand side) for exponential utility in dependence of α. The dotted dashed vertical lines correspond to values 0.25, 0.75 and 1.25 for α.

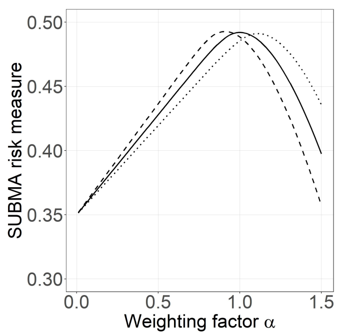
In Figure 4.12, we plot the SUBMA risk measure and the hedging costs in dependence of α for the parameter values already given in (4.13) and $\delta = 1$. At a first glance, the plot seems similar to the one for logarithmic utility. But the hedging costs as the decisive factor for the company behaves not linear for large values of α. The reason is that the expected utility function (in contrast to logarithmic utility) is bounded from above. So, the upper bound limits the benefit in terms of expected utility by increasing the hedging costs.



(a) Utility functions



(b) Hedging costs



(c) SUBMA risk measure

Figure 4.13: Exponential utility functions, hedging costs and SUBMA risk measures for the LEL acceptance set and the risk aversion parameters $\delta = 0.9$ (dotted line), $\delta = 1$ (solid line) and $\delta = 1.1$ (dashed line).

Finally, we take a look at different risk aversion parameters, see Figure 4.13. In contrast to power utility, the exponential utility functions admit a different order of magnitude for negative and positive values. For small α the hedging costs for $\delta = 1.1$ increase first, because for lower payoffs the agent with $\delta = 1.1$ fears losses more than in the remaining cases. So, if the expected utility criterion becomes more important, then this agent has a higher incentive to improve its final payoff. Further, we see that in the cases in which the hedging costs are equal, the linear slope in the SUBMA risk measure is steeper for higher risk aversions.

Remark 4.7.11. We omit the results for the case $\Psi = C_U$, because they are quite similar to the ones in Example 4.7.9. It is only notable, that the SUBMA risk measure for the PI acceptance set with $z \leq -1$ (Merton) is for most values of α infinite.

Chapter 5

Conclusion and outlook

Multi-asset risk measures were the starting point of this dissertation. They determine capital requirements by the minimal amount of money that has to be invested into multiple eligible assets to pass a capital adequacy test. The dissertation contributes in three different ways to the theory of multi-asset risk measures.

In Chapter 2, we used multi-asset risk measures to achieve pricing bounds for European type options in a one-period financial market model by defining good deals via Value-at-Risk and Expected Shortfall constraints. This analysis was motivated by the one in Cochrane and Saa-Requejo (2000) who use the Sharpe ratio to define their pricing bounds.

In Chapter 3 and Chapter 4 we constructed two new risk measures extending the concept of multi-asset risk measures. The first is called an MAI risk measure and it combines multi-asset risk measures with the intrinsic risk measures introduced in Farkas and Smirnow (2018). The second is called a SUBMA risk measure and it includes the preferences of an agent in the calculation of the capital requirement.

In the following, we summarize the content of each chapter in more detail and give an outlook on future research.

Chapter 2: In this chapter, we studied good deal bounds based on Value-at-Risk and Expected Shortfall acceptance sets. Based on a benchmark example, we demonstrated that the Value-at-Risk good deal bounds for a European type call option behave non-smooth under varying the underlying stock price. We showed that the reason for this is a jump in the optimal hedging strategy. The application of the Expected Shortfall does not admit this discontinuous behavior. Additionally, in the Value-at-Risk case, the seller's bound is sometimes smaller than the buyer's bound. To explain this, we showed that the extension theorem in Černý and Hodges (2002) is not applicable in the non-convex Value-at-Risk case. In addition, we presented new finiteness and continuity results for multi-asset risk measures in our concrete setup.

These results depend on the one-period Black-Scholes market model. Consequently, a future direction of research could be to incorporate a market with more than one risky asset for which we gave a short outlook in the Remarks 2.3.3 and 2.4.2. Also allowing for intermediate trading is a natural extension of our setup.

Chapter 3: In this chapter, we studied the impact of internal management actions on capital reserves calculated via multiple eligible assets. In doing so, the financial position is not considered in total. Instead, the company can specify liquid assets that can be reallocated. Hence, in addition to the common management action of investing external capital, this allows for the sale of these liquid positions in order to invest in other eligible assets. The resulting new MAI risk measure describes the minimal additional amount of money that has to be invested into eligible assets to satisfy a predefined acceptability criterion, under consideration of the intrinsic management action.

We provided a representation result for this new kind of risk measure as an infimum over classical risk measures. We showed that the MAI risk measure rewards diversification, if the involved marketed portfolios admit the same price. We identified this as the most relevant situation for an institution. Further, we presented counterexamples to show that diversification is not rewarded in general. We also derived a dual representation from Sion's minimax theorem.

As an example, we analyzed a model motivated by the Solvency II framework with respect to acceptance sets based on the Expected Shortfall. The MAI risk measure was compared to classical risk measures. There are situations in which the MAI risk measure does not match any of the classical risk measures, but there are also situations in which they coincide. In our examples, the MAI risk measure corresponds to trading restrictions of a multi-asset risk measure.

The MAI risk measure combines different kinds of management actions in the risk assessment process. It only considers two of them. But other actions could also be of importance. For example, instead of selling liquid assets and invest the capital into eligible assets, it could also be used to repay liabilities. It is also permissible to transfer specific parts of the liabilities to increase the equity capital. Taking these management actions into account is a challenging but important future task.

Chapter 4: In this chapter, we gave an example in which reallocating capital is not enough to pass a capital adequacy test. This shows that the common approach of maximizing expected utility under a risk constraint is not applicable in a tight financial situation. Hence, we included the expected utility or rather the certainty equivalent in the objective function of multi-asset risk measures. We called this new concept a SUBMA risk measure.

We showed that the SUBMA risk measure with respect to the certainty equivalent is coherent, if the acceptance set is a convex cone and the utility function has constant relative risk aversion. For a one-period financial market model, we developed a sufficient condition such that the SUBMA risk measure is finite-valued. Under this condition, we proved that the SUBMA risk measure satisfies desirable continuity properties. Furthermore, the existence and the uniqueness of optimal payoffs was characterized. Finally, we used two benchmark examples from Basak and Shapiro (2001) to illustrate the behavior of the SUBMA risk measure for continuous-time trading.

In Theorem 4.6.6, we derived a sufficient condition for the finiteness of the SUBMA risk measure in the standard one-period model. Two extensions could be of further interest: First, our proofs rely on the absence of good deals of the first kind. But the situation in which a good deal exists is an open research problem. Second, in the standard one-period model exists a deterministic bank account. Hence, an extension to stochastic interest rate models could lead to new insights.

The calculation of the SUBMA risk measure with respect to continuous-time trading leads to another open and challenging task. We solve the SUBMA risk measure for a constant payoff. So, case studies for other types of payoffs could be of interest. To apply Theorem 4.7.5 we have to solve portfolio optimization problems with one risk constraint and a random initial endowment. On the one hand, portfolio problems without risk constraint and random initial endowment are solved in Cvitanić, Schachermayer, and Wang (2001), Biagini, Frittelli, and Grasselli (2011) and Biagini and Černý (2020). On the other hand, there is a bunch of literature following the seminal paper of Basak and Shapiro (2001) to solve portfolio optimization problems with one risk constraint and a constant initial endowment. Concluding, before we can solve the SUBMA risk measure for arbitrary payoffs, we need to combine the different streams of literature and solve portfolio optimization problems with a risk constraint and random initial endowment.

Bibliography

Aliprantis, C. D., & Border, K. C. (2006). *Infinite Dimensional Analysis: A Hitchhiker's Guide* (3rd ed.). Springer-Verlag, Berlin. https://doi.org/10.1007/3-540-29587-9

Arai, T. (2011). Good deal bounds induced by shortfall risk. *SIAM Journal on Financial Mathematics*, *2*(1), 1–21. https://doi.org/10.1137/090769120

Arai, T., & Fukasawa, M. (2014). Convex risk measures for good deal bounds. *Mathematical Finance*, *24*(3), 464–484. https://doi.org/10.1111/mafi.12020

Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, *9*(3), 203–228. https://doi.org/10.1111/1467-9965.00068

Artzner, P., Delbaen, F., & Koch-Medina, P. (2009). Risk measures and efficient use of capital. *ASTIN Bulletin: The Journal of the IAA*, *39*(1), 101–116. https://doi.org/10.2143/AST.39.1.2038058

Bachelier, L. (1900). Théorie de la spéculation. *Annales scientifiques de l'École Normale Supérieure*, *17*, 21–86. https://doi.org/10.24033/asens.476

Baes, M., Koch-Medina, P., & Munari, C. (2020). Existence, uniqueness, and stability of optimal payoffs of eligible assets. *Mathematical Finance*, *30*(1), 128–166. https://doi.org/10.1111/mafi.12205

Baes, M., & Munari, C. (2020). A continuous selection for optimal portfolios under convex risk measures does not always exist. *Mathematical Methods of Operations Research*, *91*(1), 5–23. https://doi.org/10.1007/s00186-019-00681-x

Basak, S., & Shapiro, A. (2001). Value-at-Risk-based risk management: Optimal policies and asset prices. *The Review of Financial Studies*, *14*(2), 371–405. https://doi.org/10.1093/rfs/14.2.371

Ben-Tal, A., & Teboulle, M. (1986). Expected utility, penalty functions, and duality in stochastic nonlinear programming. *Management Science*, *32*(11), 1445–1466. https://doi.org/10.1287/mnsc.32.11.1445

Ben-Tal, A., & Teboulle, M. (2007). An old-new concept of convex risk measures: The optimized certainty equivalent. *Mathematical Finance*, *17*(3), 449–476. https://doi.org/10.1111/j.1467-9965.2007.00311.x

Bernardo, A. E., & Ledoit, O. (2000). Gain, loss, and asset pricing. *Journal of Political Economy*, *108*(1), 144–172. https://doi.org/10.1086/262114

Biagini, S., & Černý, A. (2020). Convex duality and Orlicz spaces in expected utility maximization. *Mathematical Finance*, *30*(1), 85–127. https://doi.org/10.1111/mafi.12209

Biagini, S., & Frittelli, M. (2008). A unified framework for utility maximization problems: An Orlicz space approach. *Annals of Applied Probability*, *18*(3), 929–966. https://doi.org/10.1214/07-AAP469

Biagini, S., Frittelli, M., & Grasselli, M. (2011). Indifference price with general semimartingales. *Mathematical Finance*, *21*(3), 423–446. https://doi.org/10.1111/j.1467-9965.2010.00443.x

Björk, T., & Slinko, I. (2006). Towards a general theory of good-deal bounds. *Review of Finance*, *10*(2), 221–260. https://doi.org/10.1007/s10679-006-8279-1

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, *81*(3), 637–654. https://doi.org/10.1086/260062

Carr, P., Geman, H., & Madan, D. B. (2001). Pricing and hedging in incomplete markets. *Journal of Financial Economics*, *62*(1), 131–167. https://doi.org/10.1016/S0304-405X(01)00075-7

Černý, A. (2003). Generalised Sharpe ratios and asset pricing in incomplete markets. *Review of Finance*, *7*(2), 191–233. https://doi.org/10.1023/A:1024568429527

Černý, A., & Hodges, S. (2002). The theory of good-deal pricing in financial markets. In H. Geman, D. Madan, S. R. Pliska, & T. Vorst (Eds.), *Mathematical Finance — Bachelier Congress 2000* (pp. 175–202). Springer-Verlag, Berlin. https://doi.org/10.1007/978-3-662-12429-1_9

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., & Montrucchio, L. (2011). Risk measures: Rationality and diversification. *Mathematical Finance*, *21*(4), 743–774. https://doi.org/10.1111/j.1467-9965.2010.00450.x

Cheridito, P., & Li, T. (2009). Risk measures on Orlicz hearts. *Mathematical Finance*, *19*(2), 189–214. https://doi.org/10.1111/j.1467-9965.2009.00364.x

Cochrane, J. H., & Saa-Requejo, J. (2000). Beyond arbitrage: Good-deal asset price bounds in incomplete markets. *Journal of Political Economy*, *108*(1), 79–119. https://doi.org/10.1086/262112

Cox, J. C., Ross, S. A., & Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of Financial Economics*, *7*(3), 229–263. https://doi.org/10.1016/0304-405X(79)90015-1

Cuoco, D., He, H., & Isaenko, S. (2008). Optimal dynamic trading strategies with risk limits. *Operations Research*, *56*(2), 358–368. https://doi.org/10.1287/opre.1070.0433

Cvitanić, J., Schachermayer, W., & Wang, H. (2001). Utility maximization in incomplete markets with random endowment. *Finance and Stochastics*, *5*(2), 259–272. https://doi.org/10.1007/PL00013534

Dächert, K., Grindel, R., Leoff, E., Mahnkopp, J., Schirra, F., & Wenzel, J. (2021). Multicriteria asset allocation in practice. *OR Spectrum*. https://doi.org/10.1007/s00291-021-00641-0

Delbaen, F. (2002). Coherent risk measures on general probability spaces. In K. Sandmann & P. J. Schönbucher (Eds.), *Advances in Finance and Stochastics* (pp. 1–37). Springer-Verlag, Berlin. https://doi.org/10.1007/978-3-662-04790-3_1

Delbaen, F., & Schachermayer, W. (2006). *The Mathematics of Arbitrage*. Springer-Verlag, Berlin. https://doi.org/10.1007/978-3-540-31299-4

Desmettre, S., Laudagé, C., & Sass, J. (2020). Good-deal bounds for option prices under Value-at-Risk and Expected Shortfall constraints. *Risks*, *8*(4), 114. https://doi.org/10.3390/risks8040114

Desmettre, S., Laudagé, C., & Sass, J. (2021). *Scalarized utility-based multi-asset risk measures* (SSRN Scholarly Paper No. ID 3924271). Social Science Research Network. Rochester, NY. https://doi.org/10.2139/ssrn.3924271

Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II). (2009), 155.

Farkas, W., Koch-Medina, P., & Munari, C. (2014). Beyond cash-additive risk measures: When changing the numéraire fails. *Finance and Stochastics*, *18*(1), 145–173. https://doi.org/10.1007/s00780-013-0220-9

Farkas, W., Koch-Medina, P., & Munari, C. (2015). Measuring risk with multiple eligible assets. *Mathematics and Financial Economics*, *9*(1), 3–27. https://doi.org/10.1007/s11579-014-0118-0

Farkas, W., & Smirnow, A. (2018). Intrinsic risk measures. In *Innovations in Insurance, Risk- and Asset Management* (pp. 163–184). World Scientific. https://doi.org/10.1142/9789813272569_0007

Ferguson, T. S. (1967). *Mathematical Statistics: A Decision Theoretic Approach*. Academic Press. https://doi.org/10.1016/C2013-0-07705-5

Fink, H., Geissel, S., Herbinger, J., & Seifried, F. T. (2019). *Portfolio Optimization with Optimal Expected Utility Risk Measures* (SSRN Scholarly Paper No. ID 3412529). Social Science Research Network. Rochester, NY. https://doi.org/10.2139/ssrn.3412529

Fink, H., Geissel, S., Sass, J., & Seifried, F. T. (2019). Implied risk aversion: An alternative rating system for retail structured products. *Review of Derivatives Research*, *22*(3), 357–387. https://doi.org/10.1007/s11147-018-9151-0

Floreani, A. (2013). Risk measures and capital requirements: A critique of the Solvency II approach. *The Geneva Papers on Risk and Insurance - Issues and Practice*, *38*(2), 189–212. https://doi.org/10.1057/gpp.2012.47

Föllmer, H., & Schied, A. (2016). *Stochastic Finance: An Introduction in Discrete Time*. De Gruyter, Berlin. https://doi.org/10.1515/9783110463453

Frittelli, M., & Scandolo, G. (2006). Risk measures and capital requirements for processes. *Mathematical Finance*, *16*(4), 589–612. https://doi.org/10.1111/j.1467-9965.2006.00285.x

Gabih, A., Sass, J., & Wunderlich, R. (2009). Utility maximization under bounded expected loss. *Stochastic Models*, *25*(3), 375–407. https://doi.org/10.1080/15326340903088495

Geissel, S., Sass, J., & Seifried, F. T. (2018). Optimal expected utility risk measures. *Statistics & Risk Modeling*, *35*(1-2), 73–87. https://doi.org/10.1515/strm-2017-0027

Jaschke, S., & Küchler, U. (2001). Coherent risk measures and good-deal bounds. *Finance and Stochastics*, *5*(2), 181–200. https://doi.org/10.1007/PL00013530

Karoui, N. E., & Ravanelli, C. (2009). Cash subadditive risk measures and interest rate ambiguity. *Mathematical Finance*, *19*(4), 561–590. https://doi.org/10.1111/j.1467-9965.2009.00380.x

Koch-Medina, P., & Munari, C. (2016). Unexpected shortfalls of Expected Shortfall: Extreme default profiles and regulatory arbitrage. *Journal of Banking & Finance*, *62*, 141–151. https://doi.org/10.1016/j.jbankfin.2015.11.006

Köksalan, M., & Şakar, C. T. (2016). An interactive approach to stochastic programming-based portfolio optimization. *Annals of Operations Research*, *245*(1), 47–66. https://doi.org/10.1007/s10479-014-1719-y

Komiya, H. (1988). Elementary proof for Sion's minimax theorem. *Kodai Mathematical Journal*, *11*(1), 5–7. https://doi.org/10.2996/kmj/1138038812

Kramkov, D., & Schachermayer, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, *9*(3), 904–950. https://doi.org/10.1214/aoap/1029962818

Krasnosel'skii, M. A., & Rutickii, Y. B. (1961). *Convex Functions and Orlicz Spaces.* (1st edition). Noordhoff, Groningen.

Kreps, D. M. (2019). *The Black–Scholes–Merton Model as an Idealization of Discrete-Time Economies*. Cambridge University Press, Cambridge. https://doi.org/10.1017/9781108626903

Krokhmal, P. A. (2007). Higher moment coherent risk measures. *Quantitative Finance*, *7*(4), 373–387. https://doi.org/10.1080/14697680701458307

Larsen, K., Pirvu, T. A., Shreve, S. E., & Tütüncü, R. (2005). Satisfying convex risk limits by trading. *Finance and Stochastics*, *9*(2), 177–195. https://doi.org/10.1007/s00780-004-0137-4

Laudagé, C., Sass, J., & Wenzel, J. (2021). *Combining multi-asset and intrinsic risk measures* (SSRN Scholarly Paper No. ID 3840009). Social Science Research Network. Rochester, NY. https://doi.org/10.2139/ssrn.3840009

Liebrich, F.-B., & Svindland, G. (2017). Model spaces for risk measures. *Insurance: Mathematics and Economics*, *77*, 150–165. https://doi.org/10.1016/j.insmatheco.2017.09.006

Liebrich, F.-B., & Svindland, G. (2019). Risk sharing for capital requirements with multidimensional security markets. *Finance and Stochastics*, *23*(4), 925–973. https://doi.org/10.1007/s00780-019-00402-6

Munari, C.-A. (2015). *Measuring risk beyond the cash-additive paradigm* (Doctoral Thesis). ETH Zurich. https://doi.org/10.3929/ethz-a-010475509

Nguyen, T., & Stadje, M. (2020). Nonconcave optimal investment with Value-at-Risk constraint: An application to life insurance contracts. *SIAM Journal on Control and Optimization*, *58*(2), 895–936. https://doi.org/10.1137/18M1217322

Ohlsson, E., & Johansson, B. (2010). *Non-Life Insurance Pricing with Generalized Linear Models*. Springer-Verlag, Berlin. https://doi.org/10.1007/978-3-642-10791-7

Peressini, A. L., Sullivan, F. E., & Uhl, J. J. J. (1988). *The Mathematics of Nonlinear Programming*. Springer-Verlag, New York. Retrieved October 6, 2020, from https://www.springer.com/gp/book/9780387966144

Perlman, M. D. (1974). Jensen's inequality for a convex vector-valued function on an infinite-dimensional space. *Journal of Multivariate Analysis*, *4*(1), 52–65. https://doi.org/10.1016/0047-259X(74)90005-0

Pratt, J. W. (1964). Risk aversion in the small and in the large. *Econometrica*, *32*(1/2), 122–136. https://doi.org/10.2307/1913738

Qi, Y., Steuer, R. E., & Wimmer, M. (2017). An analytical derivation of the efficient surface in portfolio selection with three criteria. *Annals of Operations Research*, *251*(1), 161–177. https://doi.org/10.1007/s10479-015-1900-y

Rockafellar, R. T. (1970). *Convex Analysis*. Princeton University Press, Princeton. https://doi.org/10.1515/9781400873173

Rockafellar, R. T. (1974). Conjugate duality and optimization. In *Conjugate Duality and Optimization* (pp. 1–74). Society for Industrial; Applied Mathematics. https://doi.org/10.1137/1.9781611970524.ch1

Rockel, W., Helten, E., Ott, P., & Sauer, R. (2012). *Versicherungsbilanzen. Rechnungslegung nach HGB und IFRS* (3rd ed.). Schäffer-Poeschel Verlag, Stuttgart.

Ruszczyński, A., & Shapiro, A. (2006). Optimization of convex risk functions. *Mathematics of Operations Research*, *31*(3), 433–452. https://doi.org/10.1287/moor.1050.0186

Scandolo, G. (2004). Models of capital requirements in static and dynamic settings. *Economic Notes*, *33*(3), 415–435. https://doi.org/10.1111/j.0391-5026.2004.00139.x

Sion, M. (1958). On general minimax theorems. *Pacific Journal of Mathematics*, *8*(1), 171–176. Retrieved August 28, 2020, from https://projecteuclid.org/euclid.pjm/1103040253

Staum, J. (2004). Fundamental theorems of asset pricing for good deal bounds. *Mathematical Finance*, *14*(2), 141–161. https://doi.org/10.1111/j.0960-1627.2004.00186.x

Teplá, L. (2001). Optimal investment with minimum performance constraints. *Journal of Economic Dynamics and Control*, *25*(10), 1629–1645. https://doi.org/10.1016/S0165-1889(99)00066-4

Vinel, A., & Krokhmal, P. A. (2017). Certainty equivalent measures of risk. *Annals of Operations Research*, *249*(1), 75–95. https://doi.org/10.1007/s10479-015-1801-0

Wang, R., & Zitikis, R. (2020). An axiomatic foundation for the Expected Shortfall. *Management Science*. https://doi.org/10.1287/mnsc.2020.3617

Zaanen, A. C. (1983). *Riesz Spaces, II*. North-Holland Publishing Company, Amsterdam.

Appendix

A Concepts and results from functional analysis

In this section, we provide some concepts and results from functional analysis. They have been moved to the appendix, because they are quite technical and so they do not jeopardize the readability of the main content. This section is orientated on the appendix in Föllmer and Schied (2016). For a more comprehensive reference, the reader is referred to Aliprantis and Border (2006). Some of our notations are motivated by the ones in Farkas et al. (2014) and Farkas et al. (2015).

We state all results without proofs. They can be found in Aliprantis and Border (2006) and Föllmer and Schied (2016).

A.1 Topological vector space

We start by using a vector space notion including a topology.

Definition A.1 (Topological vector spaces and locally convex spaces)**.** *Let τ be a topology on a linear space $\mathcal{X}$ (over $\mathbb{R}$)[1]. It is called a linear topology if addition*

$$\mathcal{X} \times \mathcal{X} \to \mathbb{R},\ (X, Y) \mapsto X + Y,$$

and scalar multiplication

$$\mathbb{R} \times \mathcal{X},\ (\alpha, X) \mapsto \alpha X,$$

*are continuous with respect to the corresponding product topologies. Then $(\mathcal{X}, \tau)$ is called a **topological vector space**. If τ has a base consisting of convex sets, then $(\mathcal{X}, \tau)$ is a **locally convex space**, i.e., there exists a subset $\kappa \subset \tau$ such that every member in κ is convex and every $U \in \tau$ can be obtained as a union of members in κ.*

Remark A.2. Every Banach space $\mathcal{X}$ with norm $\|.\| : \mathcal{X} \to \mathbb{R}$ is a locally convex space,

[1] We restrict our attention to vector spaces over a field $\mathbb{R}$.

because the set of all open balls, given by

$$\{Y \in \mathcal{X} \mid \|Y - X\| < r\},$$

for $X \in \mathcal{X}$ and $r > 0$, is a base for the induced topology consisting of convex sets. In this dissertation, we often work with locally convex spaces, because every L^p-space with $p \in [1, \infty]$ equipped with the L^p-norm is a Banach space.

For our concerns in risk management it is important to equip a topological vector space with a partial order. To do so, we define the positive cone of a partially ordered set $(\mathcal{X}, \preceq)$ by $\mathcal{X}_+ := \{X \in \mathcal{X} \mid X \preceq 0\}$.

Definition A.3 (Ordered topological vector spaces)**.** *A triple $(\mathcal{X}, \tau, \preceq)$, or $\mathcal{X}$ for short, is an* ***ordered topological vector space****, if $(\mathcal{X}, \tau)$ is a topological vector space and $(\mathcal{X}, \tau)$ is an ordered vector space and its positive cone $\mathcal{X}_+$ is τ-closed.*

A.2 Topological dual space

Often we are interested in continuous and linear functionals on a topological vector space $\mathcal{X}$. Recall that a linear functional $f : \mathcal{Y} \to \mathbb{R}$ maps points from a vector space $\mathcal{Y}$ into the real line such that for each $X, Y \in \mathcal{Y}$ and $\alpha, \beta \in \mathbb{R}$ it holds that

$$f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y).$$

The set of all continuous and linear functionals is called the ***(topological) dual space*** and we denote it by $\mathcal{X}'$ or $(\mathcal{X}, \tau)'$. The latter emphasizes that the dual space depends on the underlying topology τ.

The positive cone of $\mathcal{X}'$ is denoted by $\mathcal{X}'_+$. It consists of all $\psi \in \mathcal{X}'$ such that $\psi(X) \geq 0$, all $X \in \mathcal{X}_+$. This allows us to introduce the strictly positive elements in $\mathcal{X}$. We denote the corresponding set by $\mathcal{X}_{++}$ and it shall hold that $X \in \mathcal{X}_{++}$ if we have $\psi(X) > 0$ for all $\psi \in \mathcal{X}'_+$.

A.3 Separation theorems

For some of our results we use some separation techniques. The following definitions clarify what we mean by separating sets.

Definition A.4 (Hyperplanes and half spaces)**.** *Let $\mathcal{X}$ be a topological vector space, $f : \mathcal{X} \to \mathbb{R}$ a non-zero linear functional and $\alpha \in \mathbb{R}$. Then we obtain the following sets:*

Hyperplane:	$[f = \alpha] := \{X \in \mathcal{X} \mid f(X) = \alpha\}$,
Half spaces:	$[f < \alpha] := \{X \in \mathcal{X} \mid f(X) < \alpha\}$ *and*
	$[f > \alpha] := \{X \in \mathcal{X} \mid f(X) > \alpha\}$,
Weak half spaces:	$[f \leq \alpha] := \{X \in \mathcal{X} \mid f(X) \leq \alpha\}$ *and*
	$[f \geq \alpha] := \{X \in \mathcal{X} \mid f(X) \geq \alpha\}$.

Definition A.5 (Kinds of separation)**.** *Let $\mathcal{X}$ be a topological vector space, $f : \mathcal{X} \to \mathbb{R}$ a non-zero linear functional and $\alpha \in \mathbb{R}$. Assume two subsets A and B of $\mathcal{X}$. Then we say that the hyperplane $[f = \alpha]$ ...*

*(i) ...**separates** A and B, if either $A \subset [f \leq \alpha]$ and $B \subset [f \geq \alpha]$ or if $B \subset [f \leq \alpha]$ and $A \subset [f \geq \alpha]$.*

*(ii) ...**properly separates** A and B, if it separates them and $A \cup B$ is not included in $[f = \alpha]$.*

*(iii) ...**strictly separates** A and B, if it separates them and either $A \subset [f > \alpha]$ and $B \subset [f < \alpha]$ or $A \subset [f < \alpha]$ and $B \subset [f > \alpha]$.*

*(iv) ...**strongly separates** A and B, if there exists $\epsilon \in (0, \infty)$ with either $A \subset [f \leq \alpha]$ and $B \subset [f \geq \alpha + \epsilon]$ or $B \subset [f \leq \alpha]$ and $A \subset [f \geq \alpha + \epsilon]$.*

We also say that f separates, properly separates, etc. the sets A and B if there exists a hyperplane that separates, properly separates, etc. them.

Now we state the well-known Interior Separating Hyperplane Theorem, see e.g., Aliprantis and Border (2006, Theorem 5.67).

Theorem A.6 (Interior Separating Hyperplane Theorem)**.** *Assume a topological vector space $(\mathcal{X}, \tau)$. Let A be a convex subset with non-empty interior which is disjoint from another non-empty convex subset B. Then there exists a non-zero continuous linear functional properly separating the τ-closures of A and B.*

A.4 Continuity and coerciveness

We repeat some well-known continuity results.

Theorem A.7 (Aliprantis and Border (2006, Theorem 5.43))**.** *For a convex function $f : \mathcal{C} \to \mathbb{R}$ on an open convex subset of a topological vector space, the following statements are equivalent: (i) f is continuous on $\mathcal{C}$. (ii) f is upper semicontinuous on $\mathcal{C}$. (iii) f is bounded above on a neighborhood of some point in $\mathcal{C}$.*

For the next results we need specific kinds of topological vector spaces. For this recall that an ***ordered vector space*** is a real vector space $\mathcal{X}$ equipped with a partial order $\leq$ which is compatible with the algebraic structure of $\mathcal{X}$, i.e., it satisfies the following two properties for all $X, Y, Z \in \mathcal{X}$ and $\alpha \geq 0$:

(i) $X \leq Y$ implies $X + Z \leq Y + Z$,

(ii) $X \leq Y$ implies $\alpha X \leq \alpha Y$.

Further, a partially ordered set is a ***lattice*** if each pair of elements has a supremum and an infimum. A ***Riesz space*** is then an ordered vector space which is also a lattice. The ***absolute value*** of an element X in a Riesz space is given by the supremum between X and $-X$. We denote it by $|X|$. If we equip a Riesz space $(\mathcal{X}, \leq)$ with a norm $\|.\|$ such that for all $X, Y \in \mathcal{X}$ with $|X| \leq |Y|$ we have $\|X\| \leq \|Y\|$, then we call it a ***normed Riesz space***. A complete normed Riesz space is called a ***Banach lattice***.

Proposition A.8 (Ruszczyński and Shapiro (2006, Proposition 3.1))**.** *Suppose that $\mathcal{X}$ is a Banach lattice and $f : \mathcal{X} \to [-\infty, \infty]$ is a proper, increasing and convex function. Then f is continuous on the interior of its effective domain.*

Corollary A.9 (Rockafellar (1974, Corollary 8B))**.** *Suppose that $\mathcal{X}$ is a Banach lattice and $f : \mathcal{X} \to [-\infty, \infty]$ is lower semicontinuous and convex, then f is continuous on the interior of its effective domain.*

Now we show a sufficient condition under which a real-valued function attains a global minimum. To do so, we repeat the concept of a coercive function.

Definition A.10 (Coercive function)**.** *A map $f : \mathbb{R}^d \to \mathbb{R}$ is called coercive, if for every sequence $(x_n)_{n\in\mathbb{N}}$ with $\|x_n\|_2 \to \infty$ it holds that $f(x_n) \to \infty$. In this situation, we also write $\lim\limits_{\|\varphi\|_2 \to \infty} f(\varphi) = \infty$.*

Now we state the desired result. The proof uses analogous arguments as the proof of Peressini, Sullivan, and Uhl (1988, Theorem 1.4.4) under consideration of Aliprantis and Border (2006, Theorem 2.43).

Lemma A.11 (Existence of global minimum)**.** *Assume a lower semicontinuous and coercive map $f : \mathbb{R}^d \to \mathbb{R}$. Then f admits at least one global minimizer.*

Remark A.12. The result remains valid if f can attain ∞. The reduced version is given to stay in line with Peressini et al. (1988, Theorem 1.4.4).

The following equivalent characterizations for coerciveness are essential for the finiteness results in Section 4.6.1.

Lemma A.13. *Assume a map $f : \mathbb{R}^d \to \mathbb{R}$. The following statements are equivalent:*

(1) f is coercive, i.e., for each $a > 0$ there exists $b > 0$ such that for every $\varphi \in \mathbb{R}^d$ it holds that $\|\varphi\|_2 > b$ implies $f(\varphi) > a$.

(2) For every sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ it holds that $\|\varphi_n\|_2 \to \infty$ implies $f(\varphi_n) \to \infty$.

(3) For every sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ such that for all $n \in \mathbb{N}$, $\varphi_n \neq 0$ and the sequence $\left(\frac{\varphi_n}{\|\varphi_n\|_2}\right)_{n\in\mathbb{N}}$ converges, it holds that $\|\varphi_n\|_2 \to \infty$ implies $f(\varphi_n) \to \infty$.

Proof: The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious. It remains to show that (3) $\Rightarrow$ (1). Assume that there exists $a > 0$ such that for every $b > 0$ there exists a vector $\varphi_b \in \mathbb{R}^d$ with $\|\varphi_b\|_2 > b$, but $f(\varphi_b) \leq a$. This gives us $(\varphi_n)_{n\in\mathbb{N}}$ with $\|\varphi_n\|_2 \to \infty$. Note that for each $n \in \mathbb{N}$ it holds that $\varphi_n \neq 0$, because of the positive definiteness of the euclidean norm. Further, by the compactness of the unit sphere $\partial\,\mathrm{B}_d$ it is possible to find a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ with $\|\varphi_{n_k}\|_2 \to \infty$ such that $\left(\frac{\varphi_{n_k}}{\|\varphi_{n_k}\|_2}\right)_{k\in\mathbb{N}}$ converges. Hence, by (3) we obtain $f(\varphi_{n_k}) \to \infty$, i.e., there exists $k^{*,a} \in \mathbb{N}$ such that for every $k > k^{*,a}$ it holds that $f(\varphi_{n_k}) > a$. This is a contradiction to $f(\varphi_b) \leq a$ for all $b > 0$. □

B Results from utility theory

We give auxiliary results for the functionals in Definition 4.2.11. Remember that the utility function $U : \mathbb{R} \to [-\infty, \infty]$ is a proper, increasing, upper semicontinuous and concave function. Further, we assume $\mathcal{X}$ as in Assumption 4.2.10.

B.1 Integral functional

The following result states properties for the integral functional.

Lemma B.1 (Integral functional)**.** *The integral functional $I_U : \mathcal{X} \to [-\infty, \infty]$ satisfies the following properties: (i) increasing, (ii) concave, (iii) proper, (iv) upper semicontinuous, (v) continuous on the interior of its effective domain.*

Proof: (i) - (iii) are elementary. (v) follows from (ii), (iv) and Corollary A.9 (for $\mathcal{X}$ being an Orlicz space see also Biagini and Černý (2020, Section 2)). It remains to prove (iv). I_U being upper semicontinuous means that for an arbitrary constant $c \in \mathbb{R}$ the set $\mathcal{C} := \{X \in \mathcal{X} \,|\, I_U(X) \geq c\}$ is norm closed. Equivalently, we show for any sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ with $X_n \xrightarrow{\|\cdot\|_{\mathcal{X}}} X$ that $X \in \mathcal{C}$. This is proved in several steps:

Step 1: By Zaanen (1983, Theorem 100.6) we know that every norm-convergent sequence in a Banach lattice admits an order-convergent subsequence. Further, $\mathcal{X}$ is a Banach lattice. Hence, there exists a subsequence $(Y_n) \subset (X_n)$ such that $Y_n \xrightarrow{\text{a.s.}} X$, i.e., there exists a P-null set $N \in \mathscr{F}$ such that for each $\omega \in \Omega \backslash N$ it holds $Y_n(\omega) \to X(\omega)$.

Step 2: By upper semicontinuity of U we obtain $\limsup_{n\to\infty} U(Y_n(\omega)) \leq U(X(\omega))$ for all $\omega \in \Omega \backslash N$, i.e.,

$$\limsup_{n\to\infty} U(Y_n) \leq U(X) \text{ a.s.} \tag{B.1}$$

Step 3: By Assumption 4.2.2 there exists x^* with $U(x^*) = 0$. Since $X \in L^1$ we know that $E\left[X1_{\{X>x^*\}}\right] \in \mathbb{R}$. By Jensen's inequality we obtain

$$E[\max\{0, U(X)\}] = E\left[U\left(X1_{\{X>x^*\}}\right)\right] \leq U\left(E\left[X1_{\{X>x^*\}}\right]\right) < \infty. \tag{B.2}$$

Step 4: Define the non-negative random variable $f_n := (\max\{0, U(X)\} - U(Y_n))\, 1_{\Omega\backslash N}$ for any $n \in \mathbb{N}$. By Fatou's lemma we have $E\left[\liminf_{n\to\infty} f_n\right] \leq \liminf_{n\to\infty} E[f_n]$.

Step 5: By the linearity of the integral operator we obtain

$$E[\max\{0, U(X)\}] - E\left[\limsup_{n\to\infty} U(Y_n)\right] \leq E[\max\{0, U(X)\}] - \limsup_{n\to\infty} E[U(Y_n)]. \tag{B.3}$$

Step 6: By (B.2) we know that $E[\max\{0, U(X)\}] \in [0, \infty)$. Hence, the inequality in (B.3) reduces to $\limsup_{n\to\infty} I_U(Y_n) \leq E\left[\limsup_{n\to\infty} U(Y_n)\right]$.

Step 7: From the previous step, Equation (B.1) and the monotonicity of U we get

$$c \leq \limsup_{n\to\infty} I_U(Y_n) \leq I_U(X),$$

i.e., $X \in \mathcal{C}$. Hence, $\mathcal{C}$ is norm-closed and I_U is an upper semicontinuous map. □

B.2 Certainty equivalent

To obtain analogous results for the certainty equivalent, we first state properties for the generalized inverse U^{-1} from Definition 4.2.11. We omit the proof.

Lemma B.2 (Inverse utility function)**.** U^{-1} *is (i) increasing, (ii) convex, (iii) lower semicontinuous.*

Remark B.3. U^{-1} is not proper, because of $U^{-1}(-\infty) = -\infty$. Further, Assumption 4.2.2 implies that U is not constant and we obtain that the restriction of U^{-1} to $\mathbb{R}$ is proper.

Lemma B.4 (Certainty equivalent)**.** *The certainty equivalent* $C_U : \mathcal{X} \to [-\infty, \infty]$ *is increasing and proper*[2]*. If* $\underline{x}_U > -\infty$ *with* $U(\underline{x}_U) > -\infty$ *or* $\underline{x}_U = -\infty$*, i.e.,* $U > -\infty$*, then* C_U *is upper semicontinuous.*

Proof: By Lemma B.1 and Lemma B.2 we know that I_U and U^{-1} are increasing. Hence, we obtain that C_U is increasing. The discussion in Remark 4.2.12 shows that $C_U < \infty$. By Assumption 4.2.2 there exists a unique point $x_U^* \in \mathbb{R}$ such that $U(x_U^*) = 0$. This implies $C_U(x_U^* 1_\Omega) = U^{-1}(0) = x_U^* > -\infty$ and hence, C_U is proper.

Now we prove upper semicontinuity of C_U. By Lemma B.1 we know that I_U is upper semicontinuous. If $\overline{x}_U = \infty$ let f be the restriction of U^{-1} to $[-\infty, U(+\infty))$ and otherwise, i.e., $\overline{x}_U < \infty$, let f be the restriction of U^{-1} to $[-\infty, U(+\infty)]$. If $U > -\infty$ then f is continuous and increasing. The assertion follows by the fact that the composition of an upper semicontinuous, increasing function and an upper semicontinuous function is upper semicontinuous. If $\underline{x}_U > -\infty$, then the only point of discontinuity of f is $-\infty$. At this point f is left-continuous. Let us assume a sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ with $X_n \xrightarrow{\mathcal{X}} X$. Then $f\left(\limsup_{n\to\infty} E[U(X_n)]\right) \le f(E[U(X)])$ and we have to show that

$$\limsup_{n\to\infty} f(E[U(X_n)]) \le f\left(\limsup_{n\to\infty} E[U(X_n)]\right).$$

It is only possible that $\limsup_{n\to\infty} E[U(X_n)] = -\infty$ if $E[U(X_n)] = -\infty$ for almost all n. Hence, $C_U(X_n) = -\infty$ for almost all n which shows that $\limsup_{n\to\infty} C_U(X_n) = -\infty$. Finally, let $\limsup_{n\to\infty} E[U(X_n)] > -\infty$, i.e., $\limsup_{n\to\infty} E[U(X_n)] \ge U(\underline{x}_U)$. Note, f is increasing and continuous for values strictly larger $-\infty$. Further, $f(y) = \underline{x}_U$ for all $y \le U(\underline{x}_U)$. By these facts we get $\limsup_{n\to\infty} f(E[U(X_n)]) \le f\left(\limsup_{n\to\infty} E[U(X_n)]\right)$. □

If $\underline{x}_U > -\infty$ such that $U(\underline{x}_U) = -\infty$, then the certainty equivalent is not upper semicontinuous in general, as the following example shows.

Example B.5. Let $\mathcal{X} = L^1$ and $U(x) = \ln(x)$ on $(0, \infty)$ and $U(x) = -\infty$ otherwise. The sequence $\left(\frac{1}{n} 1_\Omega\right)_{n\in\mathbb{N}}$ converges to 0 with respect to the L^1-norm. Then we obtain

$$\limsup_{n\to\infty} U^{-1}\left(\ln\left(\frac{1}{n}\right)\right) = \lim_{n\to\infty} \frac{1}{n} = 0 > U^{-1}\left(\lim_{n\to\infty} \ln\left(\frac{1}{n}\right)\right) = U^{-1}(-\infty) = -\infty.$$

Hence, the certainty equivalent is not upper semicontinuous, i.e., we can not omit the additional assumptions on U in Lemma B.4.

[2] Proper in the sense that $C_U < \infty$ and there exists $X \in \mathcal{X}$ such that $C_U(X) > -\infty$.

Scientific career

Since 01/2022	**Project assistant**, Johannes Kepler University Linz.
01/2019 – 12/2021	**PhD Student**, Technische Universität Kaiserslautern. Supervisor: Jörn Sass.
01/2019 – 12/2021	**Research assistant**, Fraunhofer ITWM in Kaiserslautern.
09/2020 – 11/2020	**Research stay**, Johannes Kepler University Linz. Invited by Sascha Desmettre.
01/2017 – 12/2018	**Scientific employee**, Fraunhofer ITWM in Kaiserslautern.
10/2014 – 09/2016	**Master of Science in Applied Mathematics**, Hochschule Koblenz. Grade: 1.0
10/2011 – 10/2014	**Bachelor of Science in Business Mathematics**, Hochschule Koblenz. Grade: 1.0
07/2011	**Abitur**, Commercial high school in Montabaur. Grade: 1.1

Wissenschaftlicher Werdegang

Seit 01/2022	**Projektassistent**, Johannes Kepler Universität Linz.
01/2019 – 12/2021	**Doktorand**, Technische Universität Kaiserslautern. Betreuer: Jörn Sass.
01/2019 – 12/2021	**Wissenschaftliche Hilfskraft**, Fraunhofer ITWM in Kaiserslautern.
09/2020 – 11/2020	**Forschungsaufenthalt**, Johannes Kepler Universität Linz. Eingeladen von Sascha Desmettre.
01/2017 – 12/2018	**Wissenschaftlicher Mitarbeiter**, Fraunhofer ITWM in Kaiserslautern.
10/2014 – 09/2016	**Master of Science in Applied Mathematics**, Hochschule Koblenz. Note: 1.0
10/2011 – 10/2014	**Bachelor of Science in Wirtschaftsmathematik**, Hochschule Koblenz. Note: 1.0
07/2011	**Abitur**, Wirtschaftsgymnasium in Montabaur. Note: 1.1