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SOME REMARKS ON THE DISCRETIZATION OF THE PREISACH OPERATOR

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1.Introduction. The Preisach operator W by definition maps a given function $u : [0,T] \to \mathbb{R}$ to the function $w = Wu$,

$$(1) \qquad w(t) = \int_P (W_\rho u)(t) d\mu(\rho) ,$$

where μ is a finite Borel measure on the Preisach halfplane

$$P = \{\rho = (\rho_1,\rho_2) \in \mathbb{R}^2 : \rho_1 \leq \rho_2\}$$

and W_ρ denotes the action of an elementary switch with hysteresis, i.e. the function $W_\rho u : [0,T] \to \mathbb{R}$ switches to the value 1 at some t if $u(t) = \rho_2$ resp. to the value -1 if $u(t) = \rho_1$. We refer to [1,3,6] for a mathematical analysis regarding the Preisach operator.

If one uses the Preisach operator as a modeling tool in dynamical systems, one would like to incorporate it into algorithms for the numerical solution of such systems. This has been done in [5] for the situation where the Preisach operator is coupled to the diffusion equation, namely,

$$(2) \qquad (u + w)_t - \Delta u = f , \qquad w = Wu.$$

In particular, in [5] a certain discretization scheme is shown to converge to a solution of (2) . However, the question of convergence order seems to be more delicate even in the light of uniqueness result of [2].

As it is apparent from formula (1), the discretization of the Preisach operator involves the discretization of the integral as well as the discretization of the memory structure which arises from the family of switches with hysteresis. In this paper, we show how the non-smoothness of W limits the order to $\mathcal{O}(\Delta t)$ in general, although one may alter the memory discretization in [5] to improve the "memory order". We discuss these questions within the simplest possible setting, so we consider the coupling with an ordinary differential equation

$$(3) \qquad u' = f(u,w) , \qquad w = Wu ,$$

which is discretized by the explicit Euler scheme.

2. The memory discretization. The memory stored in the Preisach model at any given time t consists of knowing which switches are on $+1$ respectively on -1. Let us denote the corresponding sets of switches in the Preisach halfplane P by $A_+(t)$ and $A_-(t)$. As Preisach [4] already observed, the dividing line $\psi(t)$ between A_+t and $A_-(t)$ is piecewise linear with slope ± 1, if we use the coordinates

$$r = \frac{\rho_2 - \rho_1}{2}, \quad s = \frac{\rho_1 + p_2}{2}.$$

(insert figure 1)

(For large $r, \psi(t)$ coincides with the r-axis if we assume that $\psi(0)$ has this property.) One may therefore describe the memory mechanism of the Preisach model through analyzing the time evolution of the dividing line $\psi(t)$, as it was done in [1]. As we shall see now, this is also a convenient way to look at discretizations. Let us consider a piecewise linear input $u : [0,T] \to \mathbb{R}$ characterized by the values $u(t_n) = u_n$, where $0 = t_0 < t_1 < ... < t_N = T$ denotes some partition of the interval $[0,T]$. The corresponding dividing line $\psi(t_n)$ is given by

$$\psi(t_n) = G(u_n, \psi(t_{n-1})), \quad \psi(0) \text{ given},$$

where G maps $v \in \mathbb{R}$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ to the function $G(v,\varphi) : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$G(v,\varphi)(r) = \max\{v\text{-}r, \min\{v+r, \varphi(r)\}\}. \tag{4}$$

The values of the Preisach operator are then obtained from

$$w(t_n) = (Wu)(t_n) = E(\psi(t_n)),$$

where $E(\psi(t_n)) = \mu(A_+(t_n)) - \mu(A_-(t_n))$, i.e.

$$E(\varphi) = \mu(\{(r,s) : r \geq 0, \varphi(r) \leq s\}) - \mu(\{(r,s) : r \geq 0, \varphi(r) > s\}), \tag{5}$$

if we assume that $\mu(\text{graph } \varphi) = 0$.

A general discrete scheme for the Preisach operator now naturally takes on the form

$$(6)\qquad \begin{aligned} \psi_n &= G_\delta(u_n,\psi_{n-1})\ , \quad \psi_o \text{ given,} \\ w_n &= E_\delta(\psi_n)\ , \end{aligned}$$

where ψ_n lies in some "discrete" space Ψ_δ (usually not a linear space), and $\delta > 0$ is a discretization parameter related to the discrete memory evolution $G_\delta : \mathbb{R} \times \Psi_\delta$ and to the numerical integration procedure $E_\delta : \Psi_\delta \to \mathbb{R}$ used to approximate (5) .

The first specific discrete memory scheme we consider is the one discussed by Verdi and Visintin in [5] . Here, the Preisach halfplane P is replaced by an equidistant grid of horizontal and vertical lines. In (r,s) - coordinates, it is given by

$$(7)\qquad R_\delta = \{(r,k\delta \pm r) : r \geq 0\ , k \in \mathbb{Z}\}\ , \delta > 0\ .$$

To restrict the discrete memory evolution to the grid R_δ , the natural choice (although not stated explicitly in [5]) is

$$(8)\qquad G_\delta(v,\varphi) = G(\pi_\delta v,\varphi)\ ,$$

where $\pi_\delta : \mathbb{R} \to Z_\delta = \{k\delta : k \in \mathbb{Z}\}$ is the projection (the ambiguity in points $k\delta + \delta/2$ is irrelevant). It is obvious from (4) that

$$(r,\varphi(r)) \in R_\delta \Rightarrow (r,G_\delta(v,\varphi)(r)) \in R_\delta\ ,$$

and we omit a formal definition of Ψ_δ since we will not need it.

In an actual implementation as described in [5] , the corners of $\varphi \in \Psi_\delta$ have to be stored. If the measure μ from (1),(5) has bounded support, the maximum number M of corners to be stored is related to the grid parameter δ by

$$(9)\qquad M\delta = \text{const.}$$

Since the corners close to the s-axis are in general less important for the memory evolution than the corners which are farther away, we are led to consider a second discrete memory scheme. For any piecewise linear $\varphi : \mathbb{R}_+ \to \mathbb{R}$, let us denote by $c(\varphi)$ the number of corners of graph(φ). We set

$$(10)\qquad G_\delta(v,\varphi) = \begin{cases} G(v,\varphi)\ , & \text{if } c(G(v,\varphi)) \leq M \\ \varphi & \text{, otherwise}\ , \end{cases}$$

where $M\delta = c_0$, $c_0 > 0$ given. In other words, if we already have M corners, additional ones (which are necessarily closer to the s-axis than the already existing ones) are simply ignored, but in contrast to (8) the first M corners are stored with (up to rounding error) exact values. It is easy to modify the memory procedure of [5] according to (10); the numerical approximation of (5), however, becomes more complicated.

3.Error analysis. We consider the initial value problem

$$(11)\qquad \begin{aligned} u' &= f(u,w)\ ,\ u(0) = u_0 \\ w &= Wu\ ,\ \psi(0) = \psi_0, \end{aligned}$$

where W is the Preisach operator given by (1) .

3.1 Assumptions

(i) $f : \mathbb{R}^2 \to \mathbb{R}$ is globally Lipschitz continuous with Lipschitz constant L_f .

(ii) The measure μ has a bounded measurable density and compact support.

(iii) The initial state ψ_0 belongs to Ψ_0 , where we set

$$\Psi_0 = \{\varphi : \varphi \in W^{1,\infty}(\mathbb{R}_+)\ ,\ \|\varphi'\|_\infty \le 1\ ,\ \mathrm{supp}(\varphi) \text{ is bounded}\}.$$

◇

Since assumption (3.1) implies that $W : C[0,T] \to C[0,T]$ is a Lipschitz continuous Volterra operator and maps $W^{1,\infty}$ to itself [1,3,6] - see also proposition 3.3 below - standard theory yields the following result.

3.2 Theorem

If assumption (3.1) holds, then for any $T > 0$ the initial value problem (11) has a unique solution

$$u \in W^{2,\infty}(0,T)\ ,\ w \in W^{2,\infty}(0,T)\ .$$

◇

We now consider the discrete scheme

$$
(12) \qquad
\begin{aligned}
u_n &= u_{n-1} + hf(u_{n-1}, w_{n-1}) \\
\psi_n &= G_\delta(u_n, \psi_{n-1}) \\
w_n &= E_\delta(\psi_n) ,
\end{aligned}
$$

where h and δ are fixed positive numbers.

To obtain error estimates, we have to compare the discrete evolution (12) to the continuous evolution (11). We therefore have ro recall some properties of the latter from [1]. On Ψ_0, we use the metrics d_1 and d_∞, generated by

$$\|\varphi\|_\infty = \sup_{r\geq 0} |\varphi(r)| , \ \|\varphi\|_1 = \int_0^\infty |\varphi(r)|dr ,$$

respectively. Moreover, for $\varphi \in \Psi_0$ let us denote by $b(\varphi)$ the lowest upper bound for $supp(\varphi)$, i.e.

$$b(\varphi) = \inf\{r \in \mathbb{R}_+ : \varphi|[r,\infty) = 0\}.$$

<u>3.3 Proposition</u>

(i) For G defined by (4), we have $G : \mathbb{R} \times \Psi_0 \to \Psi_0$ and

$$b(G(v,\varphi)) = \max\{|v| , b(\varphi)\},$$

$$\|G(v_1,\varphi_1) - G(v_2,\varphi_2)\|_\infty \leq \max\{|v_1 - v_2| , \|\varphi_1 - \varphi_2\|_\infty\}$$

for any $v,v_1,v_2 \in \mathbb{R}$ and any $\varphi,\varphi_1,\varphi_2 \in \Psi_0$.

(ii) If for a given piecewise linear funtion $u : [0,T] \to \mathbb{R}$ and $\psi_0 \in \Psi_0$ we define $\psi(t) = F(u,\psi)(t) \in \Psi_0$ inductively by

$$F(u,\psi_0)(t) = G(u(t), F(u,\psi_0)(\tau_i)) , t \in [\tau_i,\tau_{i+1}] ,$$

u being linear on $[\tau_i,\tau_{i+1}]$, then its unique Lipschitz continuous extension $F : C[0,T] \times \Psi_0 \to C(0,T;\Psi_0)$ has the properties

$$b(F(u,\psi_0)(t)) = \max\{b(\psi_0), \max_{\tau\le t}|u(\tau)|\}$$

$$\|F(u_1,\psi_{01})(t) - F(u_2,\psi_{02})(t)\|_\infty \le \max\{\|\psi_{01} - \psi_{02}\|_\infty , \max_{\tau\le t}|u_1(\tau) - u_2(\tau)|\}$$

$$\|F(u_1,\psi_{01})(t) - F(u_2,\psi_{02})(t)\|_1 \le L_F\|F(u_1,\psi_{01})(t) - F(u_2,\psi_{02})(t)\|_\infty$$

for any $u_1,u_2 \in C[0,T]$ and any $\psi_{01},\psi_{02} \in \Psi_0$, where

$$L_F = \max\{b(\psi_{01}),b(\psi_{02}), \|u\|_\infty\} .$$

(iii) Under the assumption (3.1ii) , the map $E : \Psi_0 \to \mathbb{R}$ satisfies

$$|E(\varphi_1) - E(\varphi_2)| \le L_E\|\varphi_1 - \varphi_2\|_\Psi \quad \forall\, \varphi_1 , \varphi_2 \in \Psi_0$$

for some $L_E > 0$, where $\|.\|_\Psi$ can be either $\|.\|_1$ or $\|.\|_\infty$.

Proof: All assertions concerning $\|.\|_\infty$ are proved in [1] , the others are slight modifications whose proof is immediate.

◇

We also recall that $\psi(t) = F(u,\psi_0)(t)$ in fact describes the dividing line between $A_+(t)$ and $A_-(t)$, and therefore the Preisach operator W can be written as

$$(W\,u)(t) = E(F(u,\psi_0)(t)) .$$

For the discrete scheme (12) , we also formulate some assumptions.

3.4 Assumptions

Let $\Psi_\delta \subset \Psi_0$ for any $\delta > 0$, let

$$G_\delta : \mathbb{R} \times \Psi_\delta \to \Psi_\delta , \quad E_\delta : \Psi_\delta \to \mathbb{R} ,$$

and assume that there exists $\eta : \mathbb{R}_+ \to \mathbb{R}$ with

$$|E(\varphi) - E_\delta(\varphi)| \le \eta(\delta) \qquad \text{for all } \varphi \in \Psi_\delta .$$

◇

We now start to derive step - by - step estimates for the errors $u(t_n) - u_n$, $w(t_n) - w_n$,

$\psi(t_n) - \psi_n$, $t_n = nh$, where we denote by u , $w : [0,T] \to \mathbb{R}$ the unique solution of (11) and we have set

$$(13) \qquad \psi(t) = F(u,\psi_0)(t) \ , \qquad w(t) = E(\psi(t))$$

Using assumptions (3.1),(3.4) and proposition (3.3) we immediately obtain from (12) and (13)

$$(14) \qquad |w(t_n) - w_n| \leq L_E \|\psi(t_n) - \psi_n\|_\Psi + \eta(\delta)$$

Here and in the following we write $\|.\|_\Psi$ to indicate that either $\|.\|_1$ or $\|.\|_\infty$ can be used in its place. Moreover, the standard analysis of Euler's method yields

$$(15) \qquad |u(t_n) - u_n| \leq \lambda(h) + (1 + h L_f) |u(t_{n-1}) - u_{n-1}| + h L_f |w(t_{n-1}) - w_{n-1}| \ ,$$

where

$$(16) \qquad \lambda(h) = L_f (\| u' \|_\infty + \| w' \|_\infty) h^2$$

is a bound for the local error.

We next turn to the error estimate for the memory evolution. Let us denote by u_L , $u_D : [0,T] \to \mathbb{R}^n$ the linear interpolate for the values $(t_n,u(t_n))$ of the continuous respectively for the values (t_n,u_n) of the discrete solution, and by ψ_L , ψ_D the corresponding memory evolution, i.e.

$$(17) \qquad \psi_L = F(u_L,\psi_0) \ , \qquad \psi_D = F(u_D,\psi_0) \ .$$

We split the memory error as follows:

$$(18) \qquad \psi(t_n) - \psi_n = [\psi(t_n) - \psi_L(t_n)] + [\psi_L(t_n) - \psi_D(t_n)] + [\psi_D(t_n) - \psi_n]$$

To estimate the middle bracket, we apply (3.3ii) to (17) and obtain

$$(19) \qquad \|\psi_L(t_n) - \psi_D(t_n)\|_\Psi \leq L_F \max |u(t_i) - u_i| \ ,$$

L_F being given by

$$L_F = 1 \quad \text{if } \|\cdot\|_\Psi = \|\cdot\|_\infty \tag{20}$$

$$L_F = \max\{b(\psi_0), \|u\|_\infty, \|u_D\|_\infty\}, \quad \text{if } \|\cdot\|_\Psi = \|\cdot\|_1 .$$

The first bracket in (18) represents what we call the "monotonicity error". It is estimated again with the aid of (3.3ii) to yield

$$\|\psi(t_n) - \psi_L(t_n)\|_\Psi \le L_F\|u - u_L\|_\infty \le L_F\|u''\|_\infty h^2 . \tag{21}$$

The third bracket in (18) can be written as

$$\psi_D(t_n) - \psi_n = G(u_n, \psi_D(t_{n-1})) - G_\delta(u_n, \psi_{n-1}) . \tag{22}$$

For the memory discretization scheme (8), we estimate this as

$$\|\psi_D(t_n) - \psi_n\|_\infty = G(u_n, \psi_D(t_{n-1})) - G(\pi_\delta u_n, \psi_{n-1})\|_\infty$$
$$\le \max\{|u_n - \pi_\delta u_n|, \|\psi_D(t_{n-1}) - \psi_{n-1}\|_\infty\}$$

so by induction on n we have

$$\|\psi_D(t_n) - \psi_n\|_\infty \le \frac{\delta}{2} . \tag{23}$$

All this can be combined to yield our first result concerning the order of convergence.

3.5 Theorem

Let assumptions (3.1) and (3.4) hold. Then the global error

$$U_n = \max_{k \le n} |u(t_k) - u_k|$$

of the discretization (12) of the initial value problem (11), using the memory discretization (8), satisfies

$$U_N \le c_1 h + c_2 \delta + c_3 \eta(\delta) , \quad h N = T ,$$

where the c_i do not depend on h, δ.

Proof: From (13) - (23) above one readily obtains, using $\|\cdot\|_\Psi = \|\cdot\|_\infty$,

$$|u(t_n) - u_n| \leq \lambda(h) + (1 + h L_1) U_{n-1} + h L_2 \eta(\delta) + h L_3 m(h) + h L_4 \delta$$

with some constants L_i , where $m(h) = L_F \| u'' \|_\infty h^2$ represents the monotonicity error. The discrete form of Gronwall's inequality now yields the result.

◇

Let us now consider the memory discretization scheme (10) , which we repeat for convenience:

$$(10) \qquad G_\delta(v,\varphi) = \begin{cases} G(v,\varphi) & , \text{ if } c(G(v,W)) \leq M \\ \varphi & , \text{ otherwise} \end{cases} ,$$

where $c(\varphi)$ denotes the number of corners of graph(φ) . The analogue of estimate (23) will be given in the following lemma.

3.6 Lemma

Assume that $\psi_0 \in \Psi_0$ is piecewise linear and that $c(\psi_0) < M$. Then for the scheme (12) with G_δ given by (10) and ψ_0 defined by (17) we have

$$\|\psi_D(t_n) - \psi_n\|_1 \leq \frac{\mathrm{Var}(u_D)^2}{4} \cdot \frac{1}{(M - c(\psi_0))^2} ,$$

where as usual $\mathrm{Var}(u_D) = \|u_D'\|_1 = \sum_{n=1}^{N} |u_n - u_{n-1}|$, $hN = T$.

Proof: First we claim that for any m , $1 \leq m \leq N$,

$$(24) \qquad c(\psi_D(t_n)) \leq M \quad \Rightarrow \quad \psi_D(t_m) = \psi_m .$$

To prove (24) , it is sufficient to consider the situation

$$(25) \qquad c(\psi_D(t_k)) = M , \; c(\psi_D(t_n)) > M \text{ for } k < n < m , \; c(\psi_D(t_m)) \leq M$$

and to show that $\psi_D(t_k) = \psi_k$ implies $\psi_D(t_m) = \psi_m$. For this, let us look at figure 2, where we have drawn $\psi_D(t_k) = \psi_k$.

(insert figure 2)

It is not difficult to see, although awkward to derive formally, that (25) implies

$$
\begin{aligned}
&u_n \in [u_k, u_k + 2r_m) \quad \text{if } k < n < m , \\
(26) \qquad & \\
&u_m \notin [u_k, u_k + 2r_M) ,
\end{aligned}
$$

since the only way to have more than M corners is to move within the triangle Δ. In addition, one has

$$\psi_k(t_m) = G(u_m, \psi_k(t_{m-1})) = G(u_m, \psi_k) = \psi_m ,$$

so (24) is proved. To prove the lemma, we again look at figure 2 and (26) to obtain

(27) $$\|\psi_D(t_n) - \psi_n\|_1 \leq \text{area}(\Delta) = r_M^2 .$$

Now the corners P_j, $c(\psi_0) < j \leq M$, were not present at time $t = 0$, so they must have been formed by the discrete input u_D. This implies

$$\sum_{j=n_0}^{M} 2r_j \leq \text{Var}(u_D) \quad , \quad n_0 = c(\psi_0) + 1$$

and in particular $2\, r_M\, (M - c(\psi_0)) \leq \text{Var}(u_D)$, so the assertion follows from (27).

◇

We now have the second result on convergence order.

<u>3.7 Theorem</u>

If in theorem (3.5) we use the memory discretization (10) instead of (8) and assume that $M\,\delta = \text{const}$, the global error estimate becomes

$$U_N \leq c_1 h + c_2\, \delta^2 + c_3\, \eta(\delta) \quad , \quad h N = T ,$$

for sufficiently small δ.

<u>Proof:</u> Without loss of generality, we may assume that the right hand side f of the differential equation $u' = f(u,w)$ is bounded. This implies that $\text{Var}(u_D)$ and $\| u_D \|_\infty$ are

bounded uniformly in h , therefore from lemma (3.6) we conclude

$$\|\psi_D(t_n) - \psi_n\|_1 \leq c_0 \, \delta^2$$

for some constant c_0 , if δ is sufficiently small, since $M\,\delta$ is constant. Also, the constant L_F in (20) can be chosen independently from h . Using $\| \; \|_\Psi = \| \; \|_1$, the proof is now analogous to the proof of the theorem (3.5) , since we obtain the inequality

$$|u(t_n) - u_n| \leq \lambda(h) + (1 + h\,L_1)\,U_{n-1} + h\,L_2\,\eta(\delta) + h\,L_3\,m(h) + h\,L_4\,\delta^2 .$$

◇

3.8 Remarks

(i) The memory erasure mechanism of the Preisach model also serves to erase discretization errors to some extent, so there is no buildup in time in estimate (23) respectively lemma (3.6) . In fact, this is crucial for the results in (3.5) and (3.7) since the naive proof would only yield the order $\frac{\delta}{h}$ respectively $\frac{\delta^2}{h}$. The author wants to thank Augusto Visintin for focusing on this point.

(ii) If one uses the scheme of Verdi and Visintin [5] for the ODE problem, one gets a result similar to theorem (3.5). Since their numerical integration procedure is not of the form $w_n = E_\delta(\psi_n)$, it cannot be subsumed formally under (3.5).

(iii) Since $w \in W^{1,\infty}$ is the optimal regularity in the general case, it is not possible to improve the $\mathcal{O}(h)$ term in general. However, if one assumes that u and w are piecewise smooth, then with a second order ODE scheme one has $\lambda(h) = \mathcal{O}(h^3)$ except at a finite number of points, where it is $\mathcal{O}(h^2)$. Since the monotonicity error m(h) is $\mathcal{O}(h^2)$, one gets a second order method. To obtain an order greater than 2 , it is necessary to track the discontinuities due to the creation and erasue of corners in graph($\psi(t)$) more accurately; in particular, a variable stepsize h is needed.

[1] Brokate,M., Visintin,A.: Properties of the Preisach model for hysteresis. : Reine Angew. Math. 402 (1989), 1-40

[2] Hilpert,M.: On uniqueness for evolution problems with hysteresis. In: Mathematical models for phase change problems (ed. J.F. Rodrigues), Birkhäuser, Basel 1989.

[3] Krasnosel'skii,M.A., Pokrovskii,A.V.: Systems with hysteresis. Nauka, Moscow 1983 (in Russian). English translation: Springer, Berlin 1989.

[4] Preisach,F.: Über die magnetische Nachwirkung. Zeitschrift für Physik 94 (1935), 277-302.

[5] Verdi,C., Visintin,A.: Numerical approximation of the Preisach model for hysteresis. Math. Modelling and Numer. Anal. 23 (1989), 335-356.

[6] Visintin,A.: On the Preisach model for hysteresis. Nonlinear Anal. 9 (1984), 977-996.

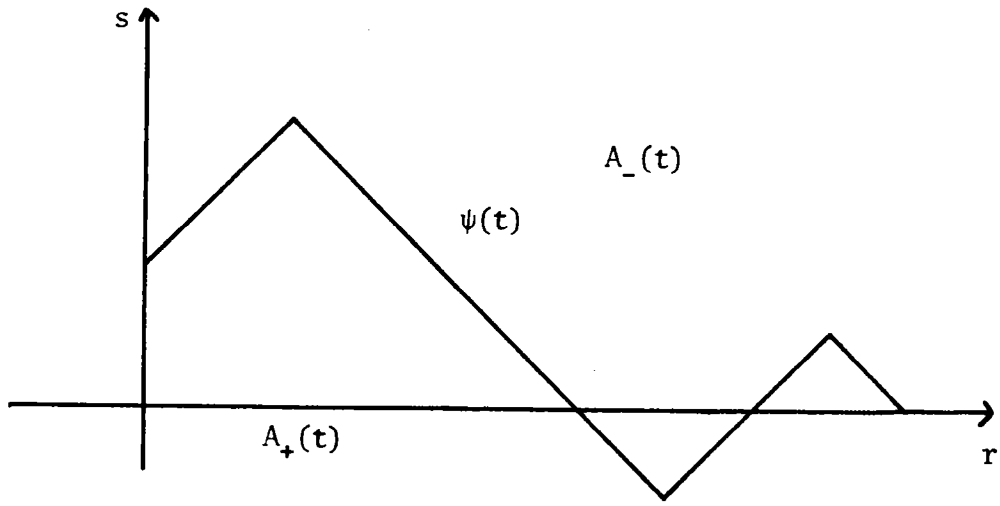

Figure 1.

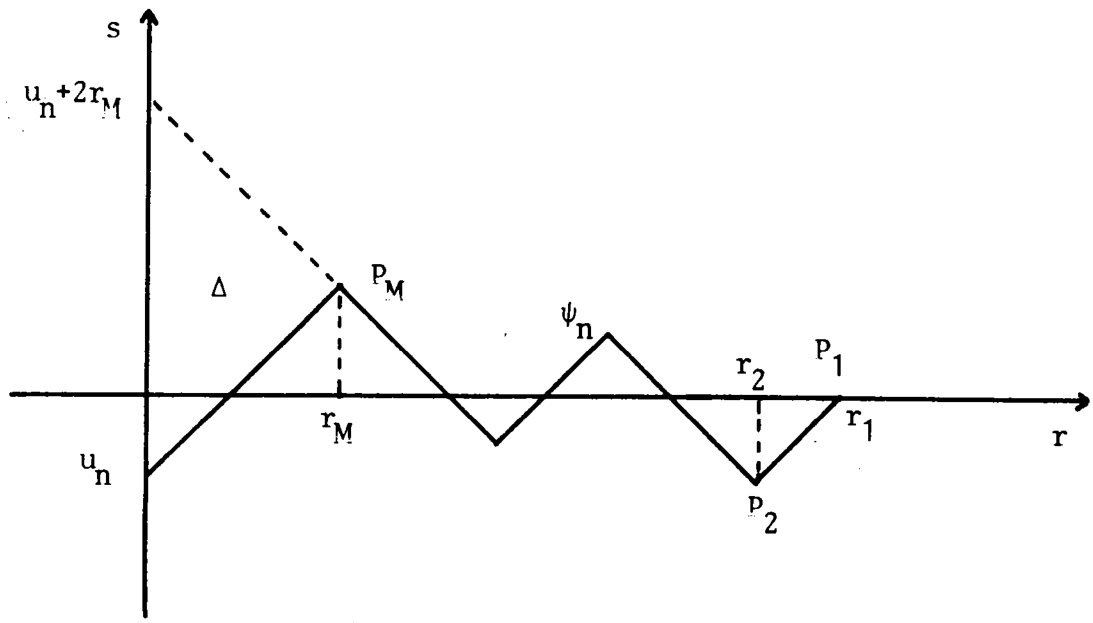

Figure 2.