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ON THE MOVING PREISACH MODEL

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Abstract

It is shown that the moving model, which is a variant of the Preisach model for hysteresis, possesses the wiping out property and is continuous in $C[0,T]$ under natural assumptions.

1 Introduction

The Preisach model [8] for ferromagnetism postulates that the dependence of the scalar magnetization $M = M(t)$ on a scalar magnetic field $H = H(t)$, $t \in [0,T]$ can be described approximately by

$$M(t) = (W(H))(t) = \int_{\rho_1 \le \rho_2} (W_{\rho_1\rho_2}H)(t)d\mu(\rho_1,\rho_2), \tag{1}$$

where μ is a measure and $W_{\rho_1\rho_2}$ denotes the action of an elementary switch with hysteresis, switching to $+1$ if $H(t) = \rho_2$ and to -1 if $H(t) = \rho_1$. This model has been used in electrical engineering for some time [7,10]. It has the so-called congruency property [6]: If one considers a field simply oscillating between values H_1 and H_2, the shape of the resulting hysteresis loop in the (H,M)-plane does not depend upon the history, in particular not upon the current value of the magnetization. This however contradicts experimental evidence. Backed up by a statistical analysis of the interaction field between elementary particles, Della Torre proposed in [3] a modification of (1) which he calls the moving model. Written as an equation in function space, it is defined by

$$M = W(H + \alpha M), \tag{2}$$

where α is a given (nonnegative) scalar. This implicit equation can be formally rewritten as

$$M = W(I - \alpha W)^{-1}H. \tag{3}$$

In general, the operator $M = M(H)$ defined by (2), (3) does not have the congruency property. On the other hand, it turns out that it shares with the Preisach operator (1) the so-called [6] wiping out property, which states that any change of the field from H_1 to H_2 erases any memory due to previous variations of the field between H_1 to H_2. In addition, the moving model is well posed in the sense that the operator $M = M(H)$ is continuous in $C[0,T]$. Both assertions are proved in this paper under minimal assumptions. Our analysis is based upon the formalism developed in [2]; although the present paper is self contained in the sense that all essential points are covered, we recommend [2] as a more detailed exposition of some of the arguments.

In [5] it is shown how the wiping out property of the moving model follows from general results on the composition and inverse of hysteresis operators, under somewhat stronger assumptions.

2 The moving model.

Let us denote by M_∞ the set of all regular locally finite Borel measures (not necessarily nonnegative) on the Preisach halfplane $P = \{(\rho_1, \rho_2) \in \mathbf{R}^2 : \rho_1 \leq \rho_2\}$, and by $h : P \to \mathbf{R}$ the height function, also called Everett integral in the literature,

$$h(\rho_1, \rho_2) = 2\mu(\Delta(\rho_1, \rho_2)), \tag{4}$$

where

$$\Delta(\rho_1, \rho_2) = \{(r_1, r_2) : \rho_1 \leq r_1 \leq r_2 \leq \rho_2\}. \tag{5}$$

The value $h(\rho_1, \rho_2)$ is equal to the height of the hysteresis loop generated in (1) by a field oscillating between ρ_1 and ρ_2. Throughout this paper, we assume partial continuity of the function h, i.e.

$$h(\rho_1.\cdot) \quad \text{and} \quad h(\cdot, \rho_2) \quad \text{are continous for any} \quad \rho_1, \rho_2 \in \mathbf{R}. \tag{6}$$

For the convenience of the reader, we explicitly state the connection to a wellknown continuity result.

2.1 Proposition

Let $\mu \in M_\infty$, let W be the Preisach operator defined by (1). Then the following assertions are equivalent:

(i) $h : P \to \mathbf{R}$ is partially continuous.

(ii) $|\mu|(L) = 0$ for any horizontal and vertical line $L \subset P$.

(iii) $h : P \to \mathbf{R}$ is (jointly) continuous.

(iv) W maps $C[0, T]$ into $C[0, T]$.

(v) $W : C[0, T] \to C[0, T]$ is continuous.

Proof: For the equivalence of (ii), (iv) and (v) see [2,4,9]. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow easily from the assumption that $\mu \in M_\infty$. The reverse implications are trivial. □

We now turn our attention to the moving model. Introducing a function $x : [0, T] \to \mathbf{R}$ by

$$x = H + \alpha M, \tag{7}$$

we may write (2) in the form

$$x = H + \alpha W x, \quad M = W x. \tag{8}$$

For any $\mu \in M_\infty$ and any $\alpha \in \mathbf{R}$, the operator $W_\alpha = I - \alpha W$ is again a Preisach operator with height function h_α given by

$$h_\alpha(\rho_1, \rho_2) = \rho_2 - \rho_1 - \alpha h(\rho_1, \rho_2). \tag{9}$$

The corresponding measure μ_α has the form

$$\mu_\alpha = \frac{1}{2}\lambda_D - \alpha\mu, \tag{10}$$

where λ_D denotes the one-dimensional Lebesgue measure along the main diagonal $\rho_1 = \rho_2$. Now for the moving model to make sense, equation (8) has to define a unique function $x : [0,T] \to \mathbf{R}$ at least for monotone functions $H \in C[0,T]$. It is easy to see that for this to be true, the measure μ_α has to be strictly monotone in the following sense.

2.2 Definition

Let $J \subset \mathbf{R}$ be a closed interval. We say that $\mu \in M_\infty$ is strictly monotone on J, if its associated height function h is strictly increasing in ρ_2 for any fixed ρ_1, and strictly decreasing in ρ_1 for any fixed ρ_2, where $\rho_1, \rho_2 \in J$.

□

To prove the continuity of the moving model using representation (3), we want to have a continuous inverse of the operator $W_\alpha = I - \alpha W$. Since typically $\alpha \geq 0$ and $\mu \geq 0$, the measure μ_α has no sign. In [2, theorem 5.8] the existence and continuity of the inverse of a Preisach operator is characterized in the case of nonnegative measures. In section 3 of this paper, we extend this result to strictly monotone measures. This covers the moving model as the following special case.

2.3 Theorem

Let $\mu \in M_\infty$ be finite with bounded support, let $\alpha \in \mathbf{R}$ be given. Assume that (6) holds and that μ_α is strictly monotone on $\mathbf{R}$. Moreover, assume negative saturation at $t = 0$, i.e. initially all switches are on -1. Then

$$W_\alpha = I - \alpha W \tag{11}$$

has a continuous inverse on $C[0,T]$, and the moving model (2), (3) defines a continuous operator on $C[0,T]$.

Proof: The first assertion will be proved in section 3. The second follows from lemma 2.1.

□

We remark that the additional assumption in 2.3 concerning the finiteness of μ and the initial condition are inessential and are made only to avoid complicated notation.

We now discuss the memory properties of the moving model. Since the moving model operator is rate independent (i.e. it commutes with bijective transformations of the time scale [0,T]), it is enough to consider piecewise linear inputs $H = H(t)$ on [0,T], represented by its local extrema $(H_0, H_1, \cdots, H_n)$. Let us denote the corresponding values of the magnetization by $(M_0, M_1, \cdots, M_n)$. These values also depend on the initial state ψ_0 of the system at time $t = 0$, which we assume to be fixed. Firstly, the moving model has the monotone erasure property, i.e. if $H_{i-1} \leq H_i \leq H_{i+1}$ for some i, and if we delete H_i from the input string, we only have to delete M_i to obtain the corresponding output string - this is an obvious consequence of the rate independence property. Secondly, the moving model is nonanticipative, i.e. M_i does not depend on H_j for $j > i$. Thirdly, the moving model has the wiping out property as stated in the next theorem.

2.4 Theorem

Let the assumptions of theorem 2.3 hold, let $(M_0, \cdots, M_n)$ be the output string generated by the moving model for the input string $(H_0, \cdots, H_n)$ where $n \geq 3$. Assume that, for some i with $1 \leq i \leq n-2$, we have either $H_{i+2} \leq H_i \leq H_{i+1}$ or $H_{i+1} \leq H_i \leq H_{i+2}$. Then if we delete H_i from the inputs, the corresponding outputs of the moving model are

$$(M_0, \cdots, M_{i-1}, M'_{i+1}, M_{i+2}, \cdots, M_n),$$

i.e. we only have to delete M_i and possibly alter M_{i+1}.

Proof: We refer to section 4 of this paper.

□

As a link to the terminology of [1], the above results show that the moving model defines a hysteresis operator with the memory erasure rule (E).

3 Continuity of the moving model.

The aim of this section is to prove a theorem on the continuous invertibility of the Preisach operator without the assumption that its defining measure μ is nonnegative. Following [2], we use the rotated coordinates

$$r = \frac{\rho_2 - \rho_1}{2} \quad , \quad s = \frac{\rho_2 + \rho_1}{2}, \tag{12}$$

so that the Preisach plane P is now represented by the right halfplane $R = \{(r, s) \in \mathbf{R}^2 : r \geq 0\}$. For the corresponding measure on R, we use the same letter μ. At any given time t, the two sets of switches being on +1 or −1 respectively are separated by a broken line $\psi = \psi(t)$, a typical one being drawn in figure 1. The formal definition of the set of admissible broken lines is

$$\begin{aligned}\Psi = \quad & \{\varphi | \varphi : \mathbf{R}_+ \to \mathbf{R} \quad \text{there exist} \quad P_k = (r_k, s_k) \in R \quad \text{such that} \\ & \lim_{k\to\infty} r_k = 0, r_{k+1} < r_k \quad \text{if} \quad r_{k+1} < 0, \varphi|[r_1, \infty) = 0, \phi(r_k) = s_k, \\ & \phi|[r_{k+1}, r_k] \text{ is a straight line of slope either +1 or -1}\}.\end{aligned} \tag{13}$$

Figure 1 also includes a typical triangle $\Delta(a, b)$, as used in the definition of the height function (4) , and a cone

$$C(r_0, s_0) = \{(r, s) : r \geq r_0, |s - s_0| \leq r - r_0\} \tag{14}$$

in order to visualize the following requirements concerning the measure μ and the initial condition ψ_0.

3.1 Assumption

Let $\mu \in M_\infty, \psi_0 \in \Psi, (r_0, s_0) \in R$, and $J \subset \mathbf{R}$ be a closed interval. We assume that

(i) μ is strictly monotone on J.

(ii) The height function h satisfies (6); if J is unbounded from above resp. below, then

$$\lim_{\rho_2 \to \infty} h(\rho_1, \rho_2) = +\infty \quad \text{resp.} \quad \lim_{\rho_1 \to -\infty} h(\rho_1, \rho_2) = +\infty$$

for any fixed ρ_1 resp. ρ_2 in J.

(iii) The support of μ does not intersect the interior of $C(r_0, s_0)$.

(iv) The slope of graph ψ_0 is nowhere zero outside $C(r_0, s_0)$.

□

The assumptions (3.1 iii, iv) are made to avoid cumbersome notation due to the influence of the initial condition.

We will prove that the Preisach operator $W : X \to Y$ has a continuous inverse on subsets X, Y of $C[0, T]$ defined as follows.

3.2 Definition

Let assumption 3.1 hold, let W be the Preisach operator defined by the measure μ. We set

$$\begin{aligned} X &= \{x : x \in C[0,T]; x(t) \in J \quad \text{for all} \quad t\} \\ X_{pmon} &= \{x : x \in X, \quad x \quad \text{is piecewise monotone}\} \\ Y &= cl(W(X_{pmon})), \end{aligned}$$

the closure being taken in the sup norm.

□

As in [2], our analysis will be based mainly on the study of the evolution of the broken line or boundary curve $\psi = \psi(t)$. As a starting point, we recall that a given boundary curve $\phi \in \Psi$ subject to an input $\xi \in \mathbf{R}$ yields a new boundary curve $G(\xi, \phi) \in \Psi$ defined by

$$G(\xi, \phi) = \phi \wedge g_\xi^+ \vee g_\xi^- . \tag{15}$$

Here,

$$g_\xi^+(r) = \xi + r, \quad g_\xi^-(r) = \xi - r \tag{16}$$

are straight lines, and "$\wedge$" respectively "$\vee$" denotes the pointwise min respectively max operation. Any piecewise monotone input represented by the string $(x_1, \cdots, x_n) \in \mathbf{R}^n$ then defines a string $(\psi_1, \cdots, \psi_n)$ of boundary curves through successive application of formula (15), starting with the initial state ψ_0. The corresponding output string $(y_1, \cdots, y_n) \in \mathbf{R}^n$ representing the values of the Preisach operator is given by $y_i = E(\psi_i)$, where $E : \Psi \to \mathbf{R}$ is defined by

$$E(\phi) = \mu(\{(r,s) \in R : s < \phi(r)\}) - \mu(\{(r,s) \in R : s \geq \phi(r)\}) \tag{17}$$

In this manner, we obtain for any piecewise monotone input $x \in C[0,T]$ the corresponding boundary curves $\psi : [0,T] \to \Psi$, and we have

$$(Wx)(t) = E(\psi(t)). \tag{18}$$

This can be extended to arbitrary $x \in C[0,T]$, see [2] for details. We also remark that formula (17) only makes sense for a finite measure μ. For a locally finite but infinite measure μ one fixes a reference curve $\psi_\mu \in \Psi$ and a reference output value $y_\mu = E(\psi_\mu)$, for example $\psi_\mu \equiv 0$ and $y_\mu = 0$, and modifies (17) to

$$\begin{aligned} E(\phi) = y_\mu \; &+ \; \mu(\{(r,s) : \psi_\mu(r) \leq s < \phi(r)\}) - \\ &- \; \mu(\{(r,s) : \phi(r) \leq s < \psi_\mu(r)\}). \end{aligned} \tag{19}$$

Up to an additive constant the operators defined by (1) respectively by (18) and (19) are identical. We now formally subsume the moving model under assumption 3.1.

3.3 Lemma

Let the assumptions of theorem 2.3 hold. Then the measure μ_α satisfies assumption 3.1 with $J = \mathbf{R}$ and r_0, s_0, ψ_0 suitably defined. Moreover, in definition 3.2 we obtain $X = Y = C[0,T]$.

Proof: Assuming supp$(\mu) = \Delta(a,b)$ in figure 1 yields the first assertion. For the second, we note the any piecewise monotone $y \in C[0,T]$ can be obtained as $y = W_\alpha x$ with $x \in X_{pmon}$. This is true, because the strict monotonicity of μ_α implies that on intervals $I \subset [0,T]$, where x is strictly increasing resp. decreasing, so is $W_\alpha x$, and because moreover the output can attain arbitrarily large positive or negative values due to the identity operator in the definition of W_α.

□

Theorem 2.3 is therefore a special case of the following theorem.

3.4 Theorem

Let assumption 3.1 hold. Then $W : X \to Y$ is continuous and has a continuous inverse.

□

We will prove theorem 3.4 with arguments extending the ones used in [2, section 5].

3.5 Lemma

Let assumption 3.1 hold. Then $W : X \to Y$ is injective and continuous.

Proof: Continuity follows from lemma 2.1, therefore we have $W(X) \subset Y$. Let now $x_1, x_2 \in X$ with $x_1 \neq x_2$, so we have $x_1(\hat{t}) \neq x_2(\hat{t})$ for some $\hat{t} \in (0,T)$. Denote by $\psi_1, \psi_2 : [0,T] \to \Psi$ the corresponding boundary curves as functions of time. Since $\psi_1(\hat{t}) \neq \psi_2(\hat{t})$, the two curves branch at some point $P_* = (r_*, s_*)$ with

$$r_* = \sup\{r : r \geq 0, (\psi_1(\hat{t}))(r) \neq (\psi_2(\hat{t}))(r)\}$$

Define t_* to be the last time where P_* is touched by either ψ_1 or ψ_2, namely

$$t_* = \max\{l_1(r_*), l_2(r_*)\},$$

where

$$l_i(r_*) = \sup\{t : \psi_i(t_*)r_* = x(t) + r_* \quad \text{or} \quad \psi_i(t_*)r_* = x(t) - r_*, t < t_*\}.$$

Assume that $t_* = l_1(r_*)$. The function $\psi_1(t_*)$ restricted to $[0, r_*]$ must be a straight line of slope ± 1, assume it to be $+1$. It is proved in [2, prop. 3.21] that $x_1(t_*) < x_2(t_*)$, so the situation at time t_* must look like figure 2, where to the left of P_*, the curve $\psi_2(t_*)$ has finitely or infinitely many corners, in the latter case converging to the point $(0, x_2(t_*))$. Now look at the regions A_i in figure 2. They are obtained as differences of two triangles, and since the measure μ is strictly monotone on J, we must have $\mu(A_i) > 0$ for all these regions. This implies

$$(Wx_1)(t_*) = E(\psi_1(t_*)) < E(\psi_2(t_*)) = (Wx_2)(t_*),$$

so $Wx_1 \neq Wx_2$ as was to be proved.

□

3.6 Lemma

Let assumption 3.1 hold, let K be a bounded subset of $C[0,T]$. Then $X \cap W^{-1}(K)$ is bounded in $C[0,T]$.

Proof: If the interval J is bounded, there is nothing to prove. Let $J = [a, \infty]$ for some $a \in \mathbf{R}$, assume there exists an unbounded sequence $x_n \in X$ with $Wx_n \in K$. We find t_n with

$$x_n(t_n) = \max_{t \in [0,T]} x_n(t) \quad , \quad \lim_{u \to \infty} x_n(t_n) = +\infty.$$

For n large enough, the corresponding boundary curves are

$$(\psi_n(t_n))(r) = \max\{0, x_n(t_n) - r\}$$

and because of assumption 3.1 (ii) and (iii)

$$(Wx_n)(t_n) = E(\psi_n(t_n)) \to +\infty$$

The other cases for J are treated in the same way.

□

3.7 Lemma

Let assumption 3.1 hold, let K be a relatively compact subset of $C[0,T]$, and let $\epsilon > 0$ be given. Then there exists an $N \in \mathbf{N}$ with the following property: For any $x \in X$ with $Wx \in K$ and any $t \in [0,T]$, the corresponding boundary curve $\psi(t)$ has at most N corners $P = (r,s)$ with $r \geq \epsilon$.

Proof:Take any $x \in X$ with $Wx \in K$ and any $t \in [0,T]$. Let $P_i = (r_i, s_i), M_0 < i \leq M$, be the corners of $\psi(t) \in \Psi$ with $r_i \geq \epsilon$, which are not already corners of ψ_0. Let $t_i \in [0,t]$ be the last time at which P_i is touched by ψ. Then

$$|(Wx)(t_{i+1}) - (Wx)(t_i)| = 2\mu(\triangle(s_i - r_i, s_i + r_i)) = h(s_i - r_i, s_i + r_i).$$

We consider the function

$$\chi(r) = \min \{h(s-r, s+r) : s - r \in J, s + r \in J\}.$$

Because of lemma 3.6, we may assume that J is compact. Since h is jointly continuous and μ is strictly monotone, the function χ is nondecreasing, and $\chi(r) > 0$ for $r > 0$, so we have

$$|(Wx)(t_{i+1}) - (Wx)(t_i)| \geq \chi(\epsilon) > 0.$$

Since K is relatively compact, we may find $\eta > 0$ depending only on K, but not on x, such that $t_{i+1} - t_i \geq \eta > 0$. But this implies that $M - M_0 \leq T/\eta$ and the assertion follows.

□

3.8 Lemma

Let assumption 3.1 hold, let K be a relatively compact subset of $C[0,T]$. Then $X \cap W^{-1}(K)$ is relatively compact in $C[0,T]$.

Proof: To show that $X \cap W^{-1}(K)$ is equicontinuous (and therefore relatively compact by lemma 3.6) it is enough to prove the following: For any $\epsilon > 0$ there exists an $\eta > 0$ such that

$$(*) \qquad \operatorname*{osc}_{[t_1,t_2]} x > 4\epsilon \quad \Rightarrow \quad \operatorname*{osc}_{[t_1,t_2]} Wx \geq \eta$$

holds for any $x \in X$ with $Wx \in K$ and any $[t_1, t_2] \subset [0,T]$. For this, take any $x \in X$ with $Wx \in K$ and any $[t_1, t_2]$ with

$$\operatorname*{osc}_{[t_1,t_2]} x > 4\epsilon.$$

Choose $t_M, t_m \in [t_1, t_2]$ such that

$$s_M := x(t_M) = \max_{[t_1,t_2]} x \quad , \quad s_m := x(t_m) = \min_{[t_1,t_2]} x$$

Without loss of generality, assume that $t_M < t_m$.
The corresponding boundary curves $\psi(t_M), \psi(t_m)$ look like figure 3, where we have set $s = (s_M + s_m)/2$ and defined $Q = (r_Q, s_Q)$ as the intersection point of $\psi(t_M)$ with the straight live g_s^+. Now for

$$t_* = \inf\{t : x(t) = s, \quad t_M \le t \le t_m\}$$

we have

$$\psi(t_*) = \begin{cases} g_s^+ & \text{on} \quad [0, r_Q] \\ \psi(t_M) & \text{on} \quad [r_Q, \infty), \end{cases}$$

and therefore

$$(Wx)(t_*) - (Wx)(t_m) = E(\psi(t_*)) - E(\psi(t_m)) = 2\mu(A).$$

If we subdivide A into strips A_i in the same manner as in figure 2, we see that $\mu(A) > 0$ since μ is strictly monotone. It remains to bound $\mu(A)$ from below uniformly with respect to x and to $[t_1, t_2]$. Since $\mu(A)$ depends continuously upon $s, s_m, Q_1, \cdots, Q_n$ which vary in a compact set (recall lemma 3.6 and the extra condition $s - s_m \ge 2\epsilon$), we find a uniform lower bound for a fixed number n of corners. But because of lemma 3.7 and since $r_Q \ge \epsilon$, we also have a uniform upper bound $N = N(\epsilon, K)$ for n. This concludes the proof.

□

3.9 Proof of theorem 3.4

From lemmata 3.5 to 3.8 we obtain that $W : X \longrightarrow Y$ is continuous and injective, and $W^{-1}(K)$ is relatively compact for any relatively compact subset K of Y. Also, $W(X)$ is dense in Y by definition 3.2. But this implies the assertion of theorem 3.4, see e.g. lemma 4.8 in [2].

□

4 The wiping out property

We repeat the implicit definition of the moving model

$$x = H + \alpha Wx \quad , \quad M = Wx. \tag{20}$$

From theorem 2.3 we know that $x, M \in C[0,T]$ if $H \in C[0,T]$. If we have a piecewise monotone input H represented by the string $(H_0.\cdots, H_n)$, we obtain corresponding strings $(x_0, \cdots, x_n)$, $(\psi_0, \cdots, \psi_n)$, $(M_0, \cdots, M_n)$, where $x_i \in \mathbf{R}$ and $\psi_i \in \Psi$ are defined by

$$x_i = H_i + \alpha E(\psi_i), \quad \psi_i = G(x_i, \psi_{i-1}), \quad M_i = E(\psi_i) \tag{21}$$

for $0 \le i \le n$, compare the description of W discussed at the beginning of section 3. The initial condition is denoted here by ψ_{-1}, and under the assumptions of theorem 2.3 the equations (21) have a unique solution.
We want to prove theorem 2.3. Since in (21) all information about the past is included in the boundary curve ψ_{i-1}, it is enough to consider the case $i = 1$. Let us delete H_1 from the input string. We obtain the different evolution

$$\begin{aligned} x_2' &= H_2 + \alpha E(\psi_2') \; , \; \psi_2' = G(x_2, \psi_0) \; ; \; M_2' = E(\psi_2'), \\ x_3' &= H_3 + \alpha E(\psi_3') \; , \psi_3' = G(x_3, \psi_2') \; , \; M_3' = E(\psi_3'). \end{aligned} \tag{22}$$

If we can prove that $\psi_3 = \psi_3'$, the deletion of H_1 does not influence the outputs M_i for $i \ge 3$. By symmetry, we only have to consider the situation $H_3 \le H_1 \le H_2$. Moreover, any input decreasing from H_2 to H_3 must pass through H_1. It is therefore enough to prove the following special case of theorem 2.3.

4.1 Theorem

Let the assumptions of theorem 2.3 hold. Consider the input strings (H_0, H_1, H_2, H_3) and (H_0, H_2, H_3), with the corresponding evolution defined by (21) and (22). Then

$$H_3 = H_1 \leq H_2 \quad \text{implies} \quad \psi_3 = \psi_3'.$$

□

The proof of theorem 4.1 mainly uses order properties. For the mapping $G : \mathbf{R} \times \Psi \to \Psi$ defined by (see (15), (16))

$$G(x, \psi) = \psi \wedge g_x^+ \vee g_x^- \tag{23}$$

we always have

$$x \leq x', \ \psi \leq \psi' \quad \Rightarrow \quad G(x, \psi) \leq G(x', \psi'), \tag{24}$$

where in Ψ we use the standard pointwise ordering. At first, we obtain the piecewise monotonicity for the moving model.

4.2 Lemma

Let the assumption of 4.1 hold. Let $H, H', x, x' \in \mathbf{R}$ and $\psi \in \Psi$. If $H \leq H'$ and if

$$x \leq H + \alpha E(G(x, \psi)) \quad , \quad x' \geq H' + \alpha E(G(x', \psi')),$$

then $x \leq x'$.

Proof: Consider $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = x - \alpha E(G(x, \psi)),$$

then obviously $f(x) \leq f(x')$. Now $f(x)$ is just the output value of the Preisach operator $W_\alpha = I - \alpha W$, if we apply the input value x to the state ψ. Since the measure μ_α is strictly monotone, another look at figure 2 shows that f is a strictly increasing function, hence $x \leq x'$.

□

We now prove theorem 4.1 through the following series of lemmata. We use (21) - (24) and the assumptions of 4.1 without further notice.

4.3 Lemma

We have $x_1 \leq x_2$, $\psi_1 \leq \psi_2$, $x_3 \leq x_2$, $\psi_3 \leq \psi_2$, $x_3' \leq x_2'$, $\psi_3' \leq \psi_2'$. Moreover, we have $x_1 \leq x_2'$.

Proof: Since $G(x_1, \psi_0) = \psi_1 = G(x_1, \psi_1)$, we have

$$x_1 = H_1 + \alpha E(G(x_1, \psi_1)), \quad x_2 = H_2 + \alpha E(G(x_2, \psi_1)),$$

so $x_1 \leq x_2$ by lemma 4.2. Moreover,

$$\psi_2 = G(x_2, \psi_1) \ \geq \ G(x_1, \psi_1) = \psi_1.$$

The other inequalities are obtained in the same way.

□

In the following, we denote the straight lines occuring in the definition of G by

$$g_i^+(r) = x_i + r, \ g_i'^+(r) = x_i' + r, \ g_i^-(r) = x_i - r, \ g_i'^-(r) = x_i' - r.$$

4.4 Lemma

We have $\psi_2 = (\psi_0 \wedge g_1^+) \vee g_2^- = G(x_2, \psi_0 \wedge g_1^+)$.

Proof: Using 4.3, one obtains

$$\psi_2 = G(x_2, G(x_1, \psi_0)) = [\psi_0 \wedge g_1^+ \vee g_1^-] \wedge g_2^+ \vee g_2^- = (\psi_0 \wedge g_1^+) \vee g_2^-.$$

□

We now consider the case where the memory of H_1 is already erased by H_2.

4.5 Lemma

If $\psi_2 = \psi_0 \vee g_2^-$, then theorem 4.1 holds.
Proof: Since $\psi_0 \le \psi_2 \le g_2^+$, we have

$$\psi_2 = \psi_0 \vee g_2^- = (\psi_0 \wedge g_2^+) \vee g_2^- = G(x_2, \psi_0),$$

so

$$x_2 = H_2 + \alpha E(G(x_2, \psi_0)) \quad , \quad x_2' = H_2 + \alpha E(G(x_2', \psi_0)),$$

and from 4.2 we conclude that $x_3 = x_3'$ and $\psi_3 = \psi_3'$.

□

4.6 Lemma

Assume that $\psi_2 \neq \psi_0 \vee g_2^-$. Then the following holds:

(i) $x_1 < x_0$

(ii) $g_2^- \wedge g_1^+ \le \psi_0 \wedge g_1^+$

(iii) $x_3 = x_1$.

Proof: (i) If $x_1 \ge x_0$, we have from 4.4

$$\psi_2 = (\psi_0 \wedge g_1^+) \vee g_2^- = \psi_0 \vee g_2^-.$$

(ii) Since $\psi_2 \le \psi_0 \vee g_2^-$ by 4.4, there exists an $r \ge 0$ with $\psi_2(r) < (\psi_0 \vee g_2^-)(r)$. This implies

$$g_1^+(r) < \psi_0(r) \ , \ g_2^-(r) < \psi_0(r),$$

so $g_2^- \le \psi_0$ in $[r, \infty)$ and $g_1^+ \le \psi_0$ in $[0, r]$, because the slope of ψ_0 is bounded by 1 in absolute value. From this, we get $g_2^- \wedge g_1^+ \le \psi_0$.
(iii) We have

$$\begin{aligned} G(x_1, \psi_2) &= G(x_1, (\psi_0 \wedge g_1^+) \vee g_2^-) \\ &= [(\psi_0 \wedge g_1^+) \vee (g_2^- \wedge g_1^+)] \vee g_1^- \\ &= (\psi_0 \wedge g_1^+) \vee (g_1^- \wedge g_1^+) \\ &= \psi_0 \wedge g_1^+ && \text{by (ii)} \\ &= \psi_1 && \text{by (i)} \end{aligned}$$

The assertion follows from 4.2, since $H_3 = H_1$ and

$$x_1 = H_1 + \alpha E(G(x_1, \psi_2)) \quad , \quad x_3 = H_3 + \alpha E(G(x_3, \psi_2)).$$

4.7 Lemma

Assume that $\psi_2 \neq \psi_0 \vee g_2^-$ and that $x_0 \leq x_2'$. Then we have

$$\psi_0 \vee g_2' \neq (\psi_0 \wedge g_3^+) \vee g_2'^- .$$

Proof: We assume equality and derive a contradiction as follows. We have

$$\begin{aligned} G(x_2', \psi_0 \wedge g_3^+) &= (\psi_0 \wedge g_3^+) \wedge g_2'^+ \wedge g_2'^- \\ &= (\psi_0 \wedge g_3^+) \vee g_2'^- && \text{since} \quad x_3 = x_1 \leq x_2' \\ &= \psi_0 \vee g_2'^- && \text{by assumption} \\ &= \psi_2' && \text{since } x_0 \leq x_2' \end{aligned}$$

and also, by 4.4 and 4.6 (iii)

$$G(x_2, \psi_0 \wedge g_3^+) = G(x_2, \psi_0 \wedge g_1^+) = \psi_2.$$

This yields

$$\begin{aligned} H_2 &= x_2 - \alpha E(\psi_2) = x_2 - \alpha E(G(x_2, \psi_0 \wedge g_3^+)) \\ &= x_2' - \alpha E(\psi_2') = x_2' - \alpha E(G(x_2', \psi_0 \wedge g_3^+)) \end{aligned}$$

and with 4.2 we conclude $x_2 = x_2', \psi_2 = \psi_2'$, leading to the contradiction

$$\psi_2 = \psi_2' = \psi_0 \vee g_2'^- = \psi_0 \vee g_2^- .$$

□

4.8 Lemma

Assume that $\psi_2 \neq \psi_0 \vee g_2^-$. Then we have

$$g_2'^- \wedge g_3^+ \leq \psi_0 \wedge g_3^+ .$$

Proof: If $x_2' \leq x_0$, then $g_2'^- \leq \psi_0$ and we are done. In the other case, we obtain from 4.7 that

$$(\psi_0 \wedge g_3^-) \vee g_2'^- \neq \psi_0 \vee g_2^- .$$

In the same manner as in the proof of 4.6(ii) we then derive that $g_2'^- \wedge g_3^+ \leq \psi_0$.

□

4.9 Completion of the proof of theorem 4.1

Because of 4.5, we only have to consider the case where $\psi_2 \neq \psi_0 \vee g_2^-$. We then have

$$\begin{aligned}
G(x_3,\psi_2) &= G(x_1,\psi_2) = [(\psi_0 \wedge g_1^+) \vee g_2^-] \wedge g_1^+ && \text{by 4.4}\\
&= \psi_0 \wedge g_1^+ && \text{by 4.6}\\
G(x_3,\psi_2') &= (\psi_0 \vee g_2'^-) \wedge g_3^+ && \text{since } x_3 = x_1 \leq x_2'\\
&= \psi_0 \wedge g_3^+ && \text{by 4.8}
\end{aligned}$$

so $G(x_3,\psi_2) = G(x_3,\psi_2')$ and moreover

$$x_3 = H_3 + \alpha E(G(x_3,\psi_2')) \quad , \quad x_3' = H_3 + \alpha E(G(x_3',\psi_2'))$$

Now 4.2 implies $x_3 = x_3'$, and we get

$$\psi_3' = G(x_3',\psi_2') = G(x_3,\psi_2') = G(x_3,\psi_2) = \psi_3.$$

□

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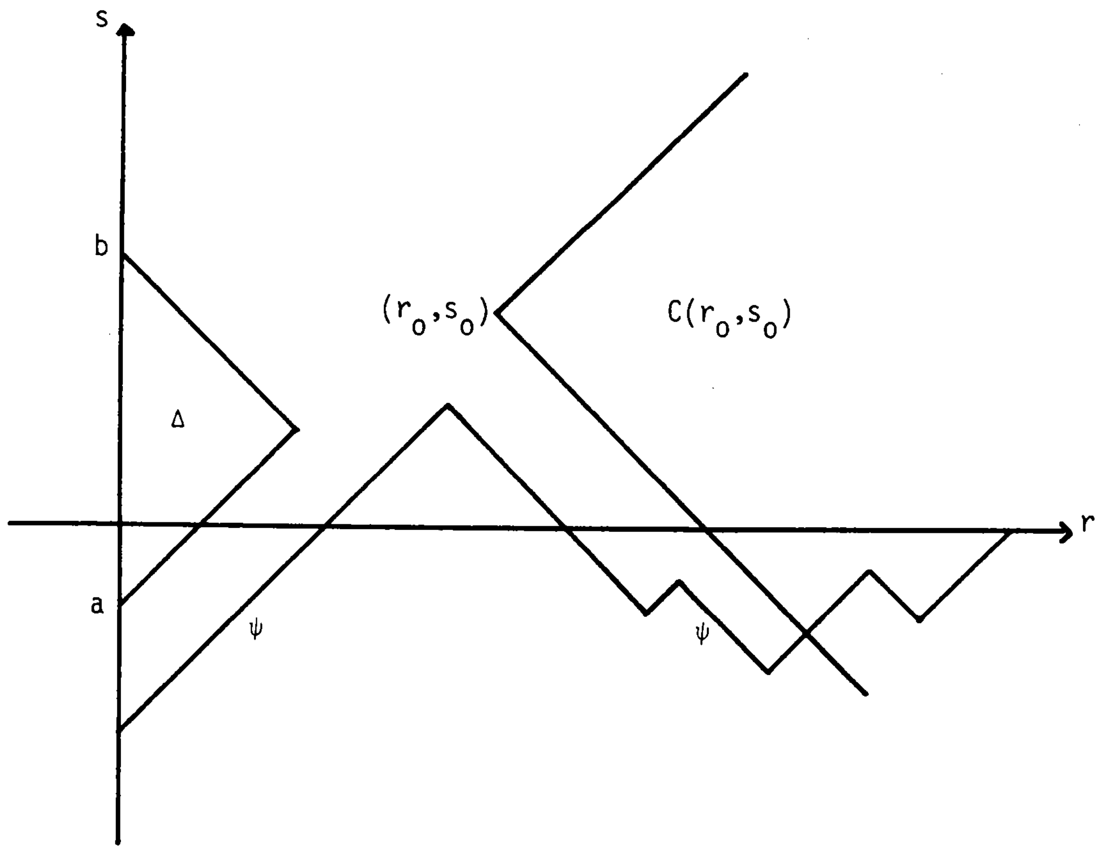

Figure 1

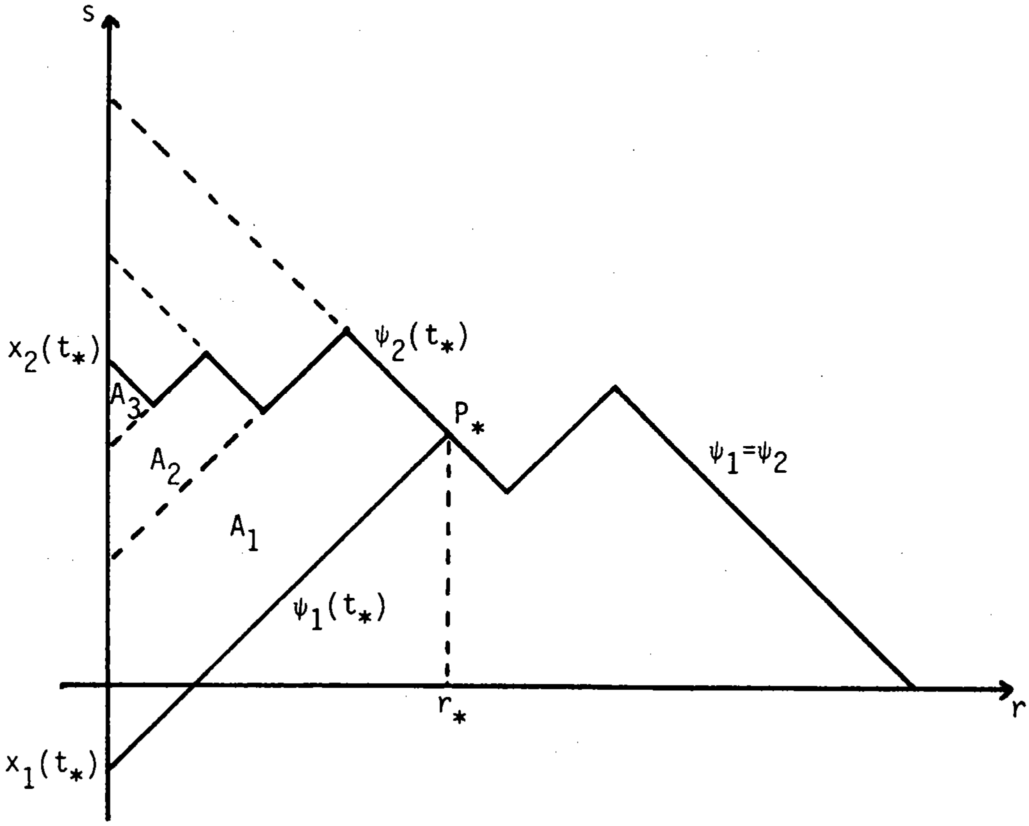

Figure 2

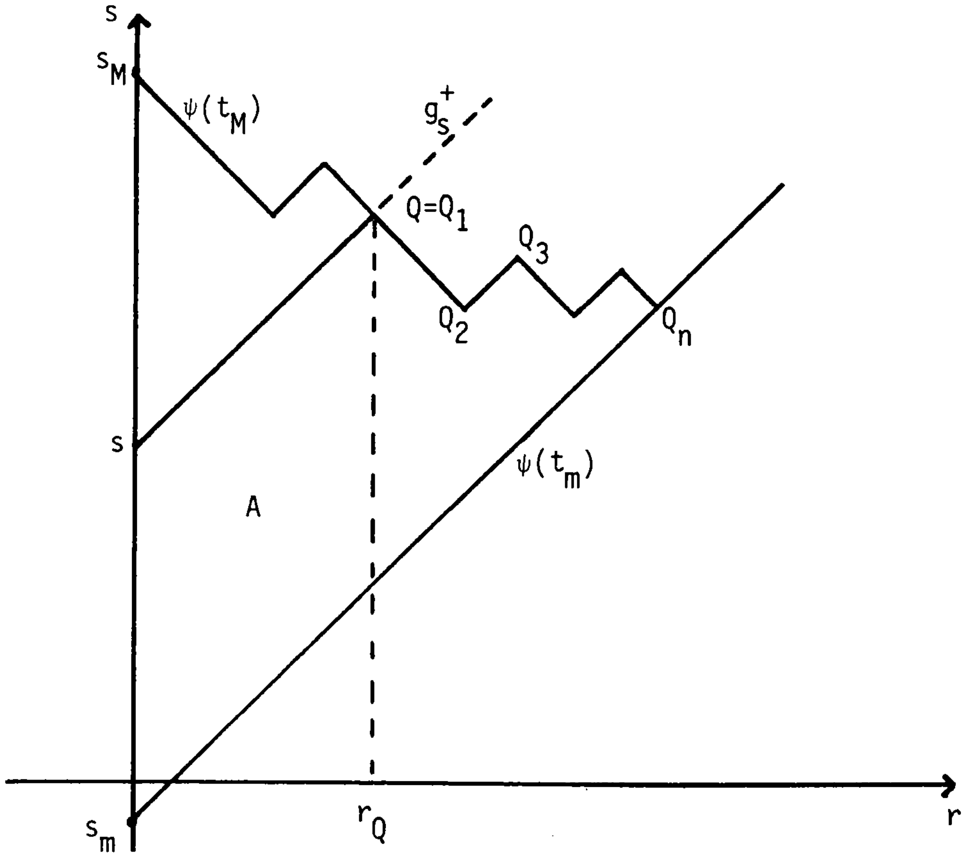

Figure 3