TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

FACHBEREICH MATHEMATIK
AG Finanzmathematik

Worst-Case Portfolio Optimization: Stress Scenarios, Crash-/Default-Risk and Ambiguity

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Datum der Disputation: 20. Mai 2022

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern zur Verleihung des akademischen Grades Doktor der Naturwissenschaften (Doctor rerum naturalium, Dr. rer. nat.) genehmigte Dissertation

DE-386

"I learned to embrace risk, as long as it was well thought out and, in a worst-case scenario, I'd still land on my feet. "

Eli Broad

Abstract

In 2002, Korn and Wilmott [37] introduced the worst-case scenario optimal portfolio approach. They extend a Black-Scholes type security market, to include the possibility of a crash. For the modeling of the possible stock price crash they use a Knightian uncertainty approach and thus make no probabilistic assumption on the crash size or the crash time distribution. Based on an indifference argument they determine the optimal portfolio process for an investor who wants to maximize the expected utility from final wealth. In this thesis, the worst-case scenario approach is extended in various directions to enable the consideration of stress scenarios, to include the possibility of asset defaults and to allow for parameter uncertainty.

Insurance companies and banks regularly have to face stress tests performed by regulatory instances. In the first part we model their investment decision problem that includes stress scenarios. This leads to optimal portfolios that are already stress test prone by construction. The solution to this portfolio problem uses the newly introduced concept of minimum constant portfolio processes.

In the second part we formulate an extended worst-case portfolio approach, where asset defaults can occur in addition to asset crashes. In our model, the strictly risk-averse investor does not know which asset is affected by the worst-case scenario. We solve this problem by introducing the so-called worst-case crash/default loss.

In the third part we set up a continuous time portfolio optimization problem that includes the possibility of a crash scenario as well as parameter uncertainty. To do this, we combine the worst-case scenario approach with a model ambiguity approach that is also based on Knightian uncertainty. We solve this portfolio problem and consider two concrete examples with box uncertainty and ellipsoidal drift ambiguity.

Acknowledgements

I would like to express my sincere thanks to Prof. Dr. Ralf Korn for supervising my dissertation and for introducing me to this interesting research topic. He was always available to discuss problems and ideas, and his helpful guidance over the last few years has been indispensable for the development of this work. I am also grateful to Prof. Dr. Mogens Steffensen for accepting to act as a referee for this thesis.

Additionally, I thank all members of the Financial Mathematics Group at the Department of Mathematics for providing such a relaxed working atmosphere. Our joint lunch and coffee breaks were always something I was looking forward to.

Last but not least, I want to thank my whole family, particularly my parents, and my girlfriend Marlene, for their unconditional love and support.

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Introduction

Portfolio optimization is concerned with determining an optimal investment strategy. The mathematical starting point for the theory of portfolio optimization can be traced back to the pioneering work of Markowitz [45]. Here, the objective is to maximize the expected return of the portfolio, subject to the constraint that the portfolio variance does not exceed an upper bound. The aim of the continuous time portfolio problem in this work will be to maximize the expected utility from the terminal wealth at a finite time horizon via finding the optimal admissible trading strategy. In such a continuous-time market model, we mainly distinguish between two basic approaches. On one hand, there is the approach, started by Merton [47], of interpreting the portfolio problem as a stochastic control problem and applying standard methods of stochastic control. On the other hand, there is the martingale method, introduced by Cox and Huang [12], Karatzas et al. [24] and Pliska [53], which consists of a decomposition of the portfolio problem into the determination of the optimal terminal wealth and the subsequent determination of the corresponding portfolio process. There are numerous approaches to extend the traditional portfolio optimization problem to include new, practically relevant aspects in the model. In particular, this includes the stochastic modeling of asset prices with jumps. The application of jump processes to model extreme price changes is used in a large number of publications, including the works of Liu et al. [43] and Aase [1]. A further possibility to integrate price jumps into the portfolio problem is provided by the worst-case scenario approach used in this work and is introduced in more detail in the further course of this section.

The classical Merton problem

Initially, we want to introduce an optimal terminal wealth problem in a Black-Scholes market. This fundamental problem was first solved by Merton [47], [48]. A portfolio problem of an investor with given initial capital x is considered, which consists of determining an optimal investment strategy. The investor's objective is to maximize the expected utility of the terminal wealth at a fixed time horizon T. The challenge is to determine how many shares of which security to hold and when to hold them, in order to achieve this goal. The Black-Scholes type financial market model looked at here consists of a risk-free investment opportunity, the so-called money market account and a risky asset, the stock.

For the mathematical definition of this investment problem, we consider a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. We fix a finite time horizon $T > 0$ and let $(W_t)_{t\in[0,T]}$ be a one-dimensional Brownian motion with respect to $\mathbb{F}$. The money

market account B and the dynamics of the stock price S are defined for $t \in [0, T]$ by

$$\begin{aligned} dB(t) &= B(t)rdt, \quad B(0) = 1, \\ dS(t) &= S(t)\left(bdt + \sigma dW(t)\right), \quad S(0) = s > 0. \end{aligned}$$

Here, $r \in \mathbb{R}$ is the constant interest rate, $b \in \mathbb{R}$ with $b > r$ is the drift term of the risky asset and $\sigma \in \mathbb{R}_{>0}$ is the corresponding stock price volatility.

The investor in our model has an initial capital $x > 0$ and by $\pi(t)$ we denote the fraction of wealth he/she invests into the stock at time t, for $t \in [0, T]$. We call such a portfolio process π admissible if it is progressively measurable, bounded and if the corresponding wealth process X^π solves, for $t \in [0, T]$, the stochastic differential equation (SDE)

$$\begin{aligned} dX^\pi(t) &= X^\pi(t)\left(\left(r + \pi(t)\left(b - r\right)\right)dt + \pi(t)\sigma dW(s)\right), \\ X^\pi(0) &= x, \end{aligned}$$

i.e. we want π to be a self-financing portfolio process. Based on this, we define $\mathbb{A}(x)$ as the set of all admissible portfolio processes.

For the formulation of the portfolio optimization problem we define a utility function $U : (0, \infty) \to \mathbb{R}$ to be a strictly concave, increasing and continuously differentiable function, such that

$$U'(0) := \lim_{x \searrow 0} U'(x) = \infty, \quad U'(\infty) := \lim_{x \to \infty} U'(x) = 0.$$

In this work, we mainly consider an investor with a CRRA (constant relative risk aversion) utility function:

$$U(x) = \frac{1}{1-\gamma} x^{1-\gamma} \text{ for } \gamma > 0,\ \gamma \neq 1.$$

Further, we also look at the log-utility case $U(x) = \ln(x)$, that can be seen as the limit case of $\gamma \to 1$.

We can now state the continuous-time portfolio problem:

$$\sup_{\pi \in \mathbb{A}(x)} \mathbb{E}\left(U\left(X^\pi(T)\right)\right).$$

With the results in [48], it can be shown, that in the case of a power-utility function, the optimal portfolio process is

$$\pi^* = \frac{1}{\gamma} \frac{b - r}{\sigma^2}. \tag{1}$$

For $\gamma = 1$ we obtain the optimal portfolio process in the log-utility case.

Worst-case scenario portfolio optimization

Korn and Wilmott [37] introduced the worst-case scenario approach to portfolio optimization in 2002. Their goal is to determine optimal portfolio processes under the threat of a crash. For this purpose, they use the crash modeling introduced in [22]. In this model, there are no probabilistic assumptions regarding the occurrence of the stress. Instead, only the maximal number of crashes in a given time interval and the biggest possible size of the crashes are given in advance. The crash height, as well as when or whether a crash occurs, is determined with the help of a worst-case approach. We would now like to present the portfolio problem introduced by Korn and Wilmott in an extended Black-Scholes market. More precisely, we consider the case of one stock and one possible crash scenario.

With regard to the underlying probability space, the associated filtration, the time horizon T and the Brownian motion, we adopt the mathematical assumptions from the previous subsection. Also now the investor has the possibility to invest in the risk-free money market account as well as in the risky stock. For a constant interest rate $r \in \mathbb{R}$, the dynamics of the money market account are again given by

$$dB(t) = B(t)rdt, \quad B(0) = 1,$$

for $t \in [0, T]$. For the modeling of the stock price, a distinction is made between normal times and an unknown crash time τ. For a constant drift term $b \in \mathbb{R}$ with $b > r$ and a positive volatility coefficient $\sigma \in \mathbb{R}_{>0}$, the dynamics of the risky stock in normal times are defined as

$$dS(t) = S(t)\left(bdt + \sigma dW(t)\right), \quad S(0) = s > 0.$$

The crash time τ is modeled as a stopping time and the percentage fall in the stock price z at time τ has to be less than or equal to a maximum crash height $z^* \in (0, 1)$, i.e. $0 \leq z \leq z^*$. This means, for a given crash scenario (z, τ), we get the following change in the stock price S at crash time τ:

$$S(\tau) = (1 - z)\, S(\tau-).$$

We decompose each portfolio process $\pi = (\pi_1, \pi_0)$ into the portfolio process π_1, used up to the crash time τ, and the portfolio process π_0, which is used in the post-crash market. Then we extend the set $\mathbb{A}(x)$ of admissible portfolio processes, defined in the previous subsection, by the requirement that all these processes are predictable, π_1 has right-continuous paths and that $\pi_1(t)z^* < 1$ holds for every admissible portfolio process $\pi \in \mathbb{A}(x)$ and all $t \in [0, T]$.

For an initial capital x, an admissible portfolio process $\pi \in \mathbb{A}(x)$ and a possible crash scenario (z, τ), we can now define the dynamics of the investor's wealth process X^π:

$$X^\pi(0) = x,$$

$$
\begin{aligned}
dX^{\pi}(t) &= X^{\pi}(t)\left(\left(r+\pi(t)\left(b-r\right)\right)dt+\pi(t)\sigma dW(s)\right), \text{ on } [0,\tau),\\
X^{\pi}(\tau) &= \left(1-\pi(\tau)z\right)X^{\pi}(\tau-),\\
dX^{\pi}(t) &= X^{\pi}(t)\left(\left(r+\pi(t)\left(b-r\right)\right)dt+\pi(t)\sigma dW(s)\right), \text{ on } (\tau,T].
\end{aligned}
$$

For the formulation of the worst-case portfolio problem it is now assumed that the investor is extremely risk averse, and the crash occurs with the worst possible height at the worst possible time for him/her. This means, that for a utility function U, the worst-case portfolio problem is defined as follows:

$$
\sup_{\pi\in\mathbb{A}(x)}\inf_{\substack{\tau\in[0,T]\\0\le z\le z^*}}\mathbb{E}\left(U\left(X^{\pi}(T)\right)\right).
$$

Korn and Wilmott [37] solved the problem for the case of a logarithmic utility function by going backwards in time. It can be shown directly that the optimal Merton strategy (1) with $\gamma = 1$ is optimal for the investor after the crash has occurred. For solving the pre-crash problem they used indifference arguments. For this, the main idea is to find an admissible portfolio process that makes the investor at every point in time $t \in [0,T]$ indifferent between "the worst crash happens immediately" and "no crash occurs at all". In our one stock, one crash model, Korn and Wilmott have proven the existence of a unique portfolio process $\bar{\pi}_1$, that leads to such an indifference and is optimal for the investor before the crash. This portfolio process $\bar{\pi}_1$ is the solution to the following differential equation in the log-utility case:

$$
\begin{aligned}
\bar{\pi}_1'(t) &= -\frac{\sigma^2}{2z^*}\left(1-\bar{\pi}_1(t)z^*\right)\left(\bar{\pi}(t)-\pi^*\right)^2,\\
\bar{\pi}_1(T) &= 0,
\end{aligned}
$$

for $t \in [0,T]$, where π^* is the optimal Merton strategy (1) with $\gamma = 1$.

The worst-case scenario optimal portfolio approach, which was introduced by Korn and Wilmott [37] in the case of the logarithmic utility function, has been extended to more general utility functions in [30] by using the indifference principle. The investment problem of an insurance company, including a possible market crash, is considered in [27]. In [36] the worst-case portfolio problem is embedded by Korn and Steffensen in a classical HJB-framework and a connection to stochastic control theory is established.

Seifried introduced two concepts in [56], which are also of great importance for this work, namely the indifference frontier and the indifference optimality principle (see also [35]), to generalize the results to more general utility functions and price dynamics.

Korn et al. [31] maximize the expected utility of the surplus process of a non-life insurance company by dynamic proportional reinsurance, under the worst-case claim development. Desmettre et al. [13] investigate the worst-case portfolio problem in case of an infinite-horizon and with the inclusion of a consumption process. In [14], Desmettre et al. consider a robust worst-case scenario problem. They introduce an efficiency criterion to solve the optimization problem when the maximum crash height is not known a priori.

Menkens [46] studies the impact on the worst-case portfolio problem if the probability of a crash occurring is known. In [6] the underlying market model is extended to include transaction costs. Belak et al. [4] allow for crash warnings at random times and thus for a random number of crashes. In [5], the authors use the worst-case approach to investigate an investment problem with asset price bubbles.

Korn and Engler [17] consider the worst-case portfolio problem when the interest rate of the bond is modeled as a Vasicek process. In [29], Korn and Leoff introduce a Lagrangian multiplier approach to analyse the worst-case approach in a multi-asset setting.

Outline of the thesis

In this work, three major extensions of the worst-case scenario approach, introduced by Korn and Wilmott [37], are formulated and solved:

Chapter 1: In the first part, we will take into consideration that insurance companies and banks regularly have to face stress tests performed by regulatory instances. To model their investment decision problems that includes stress scenarios, we include the central issue of the worst-case portfolio approach, that neither the time nor the order of occurrence of the stress scenarios are known. Even more, there are no probabilistic assumptions regarding the occurrence of the stresses.

By defining the relative worst-case loss and introducing the concept of minimum constant portfolio processes, we generalize the traditional concepts of the indifference frontier and the indifference-optimality principle. We prove the existence of a minimum constant portfolio process that is optimal for the multi-stress worst-case problem. As a main result we derive a verification theorem that provides conditions on Lagrange multipliers and non-linear ordinary differential equations that support the construction of optimal worst-case portfolio strategies.

The results of this chapter correspond in large parts to the following article:

Ralf Korn and Lukas Müller. "Optimal portfolios in the presence of stress scenarios A worst-case approach". In: *Mathematics and Financial Economics* (2021), pp. 1–33.

Chapter 2: We consider a multi asset setting, where asset defaults can occur in addition to asset crashes. In our model, the strictly risk-averse investor does not know which asset is affected by the worst-case scenario. Based on a reformulation of the value function we define the set of minimum constant portfolio processes. We prove the existence of such a process that solves the worst-case crash/default portfolio problem. In particular, we prove the existence of a portfolio process that is optimal for the investor and at the same time makes him/her indifferent to the time of occurrence of the worst possible crash or default scenario. The optimal portfolios are derived from solutions of non-linear differential equations. Therefore we construct an algorithmic framework to analyse selected examples. These examples show the importance of adapting the portfolio process when considering such worst-case scenarios in the investment problem.

This chapter is essentially a reprint of

Lukas Müller. "Optimal portfolio choice with crash and default risk". Manuscript submitted for publication. 2021.

Chapter 3: We set up a continuous time portfolio optimization problem that includes the possibility of a crash scenario as well as parameter uncertainty. To do this, we combine the worst-case scenario approach, introduced by Korn and Wilmott [37], with a model ambiguity approach that is also based on Knightian uncertainty. In our model, the crash scenario occurs at the worst possible time for the investor or even not at all. For the modeling of the parameter uncertainty, we choose a general definition of the sets of possible drift and volatility parameters, conditioned by the solution of an optimization problem. In addition, these sets may be different in the pre- and post-crash market. We solve this portfolio problem and then consider two particular examples with box uncertainty and ellipsoidal drift ambiguity.

This chapter is an extended version of

Ralf Korn and Lukas Müller. "Optimal portfolio choice with crash risk and model ambiguity". In: *International Journal of Theoretical and Applied Finance* 25.01 (2022), p. 2250002

Chapter 4: In this concluding chapter, we would like to take a look at extensions of the new results and further research questions. We combine the definition of a stress scenario from chapter 1 with the problem formulation from chapter 2, such that at most one of the $n > 1$ possible stress scenarios can occur. In the future research, the aim may be to integrate further stress scenarios in the optimization problems. Therefore, motivated by the Swiss Solvency Test, we are also looking at an optimal dynamic reinsurance problem with worst-case default of the reinsurer.

The second part of this chapter is largely based on the letter

Ralf Korn and Lukas Müller. "Optimal dynamic reinsurance with worst-case default of the reinsurer". In: *European Actuarial Journal* (2022), pp. 1–7.

Note: Each of the four chapters includes a scientific paper. This means that these chapters can be considered independently of each other. Every chapter is based on its own mathematical assumptions and different notations are used. Nevertheless, the author has tried his best to present the different portfolio problems in such a way, that a consistent view on the work as a whole is possible.

Chapter 1

Optimal portfolios in the presence of stress scenarios

1.1 Introduction

Stress tests for banks and insurance companies performed by regulatory instances such as e.g. the European Central Bank are nowadays even an issue of public news. The usual way how these stresses are considered in finding an optimal investment strategy is to first find a good strategy with respect to a primal goal (such as expected utility). Then, it is checked if the resulting strategy also performs acceptable when stresses occur, possibly has to be modified, and the checking is repeated until all stress criteria are met.

We will in this chapter present an alternative approach which directly includes the effects of possible stresses in the investment problem. This new approach consists of modifying the so-called worst-case scenario approach to portfolio optimization for explicitly considering the occurrence of stress scenarios. Consequently, the trial-and-error character of the usual approach can be avoided.

Our approach is mainly motivated by regulatory texts. Indeed, in their methodological principles of insurance stress testing [19], the European Insurance and Occupational Pensions Authority (EIOPA) gives the following definition of stress scenarios:

> "Stress scenarios are severe but plausible hypothetical situations that can adversely affect the balance sheets and solvency positions of insurance undertakings. Scenarios can be compromise a single shock or a combination of market, demographic, financial and insurance-specific shocks that are expected to affect the resilience of individual undertakings and the insurance sector as a whole."

Further, EIOPA [19] states that so-called equity shocks "are provided in terms of percentage changes". A concrete example of a shock is given in the Solvency II Delegated Act Articles 168 – 173 [18], which deals with the equity risk sub-module of the market risk module of the standard formula. Here equities are divided into Type 1 and Type 2, depending on where they are listed. Apart from symmetrical adjustments and further exceptions, a stress of Type 1 equities is a market decline of 39%, whereas for Type 2 equities the decline is 49%.

As usual in the worst-case scenario approach introduced by Korn and Wilmott [37], we assume that the stresses can all happen, but make no probabilistic assumption if and when they occur, i.e. with regard to this we follow the concept of Knightian uncertainty (see [25]).

The importance to add stress scenarios to the investment decision problem and not to make any probabilistic assumptions regarding the occurrence of the scenarios is justified by the Insurance Regulation Committee of the International Actuarial Association [23]:

> "The likelihood and financial effect of certain scenarios can be extremely uncertain. In these cases it is almost impossible to precisely estimate their small probabilities. In fact, their effect cannot be estimated and even their identification cannot be easily made through the application of a traditional economic capital model.... Yet, such scenarios may lead to the largest financial strain for firms or an entire industry. The use of scenario analysis and stress testing by decision makers and regulators may prove to be the best approach to prioritize their options, whether to add to capital or to use other approaches to mitigate these risks."

In the course of this chapter, we will see in particular that the portfolio optimization problems considered in [36] and [29], among others, are in fact special cases of the problem introduced here. More precisely, we have generalised the problem with one stock and $n \geq 1$ identical stresses stated in [36]. Also, our model includes the multi-asset problem with one stress in [29] as a simple special case.

A key challenge in our model is that the investor is not only faced with the challenge of when or if a stress occurs, rather that there are different types of stresses. Therefore there is the additional uncertainty of not knowing in which order the stresses occur. Due to this additional uncertainty component the existing results cannot simply be adapted, which is why in this chapter we have to introduce new concepts and have to derive new results such as

- the extension of the worst-case concept to different types of stresses,
- allowing for an unknown sequential occurrence of the different stresses,
- the concept of minimum constant portfolio processes,
- the proof of existence of an optimal minimum constant portfolio process,
- a verification theorem for the resulting multi-stress-type multi-asset worst-case problem.

While our definitions and the setting will be given for the general case of $n \geq 1$ stresses, we will in this chapter mostly focus on the case of $n = 2$ stresses. The extension to the general case of $n \geq 2$ is described in Remark 7.

We start with the introduction of the worst-case concept in section 1.2. Then, in section 1.3 and section 1.4, we solve the portfolio optimization problem in the stress-free world and the problem when only one stress is left. In section 1.5, we derive the minimum constant principle and solve the multi-asset multi-stress worst-case problem. The preceding considerations then enable us to formulate a verification theorem in section 1.6. The heuristic construction of the optimal strategy and numerical examples are shown in section 1.7. In section 1.8, we embed the worst-case portfolio problem with stress scenarios in a classical HJB-framework.

1.2 Worst-case portfolio optimization

In this section, we introduce the mathematical framework that allows us to study a portfolio optimization problem in a financial market with stresses.

Basic mathematical assumptions. Throughout this chapter we fix a time horizon $T > 0$. All processes are defined on the complete probability space $(\Omega, \mathcal{F}, P)$.

Asset price dynamics and stress scenarios. We consider a capital market with a money market account with dynamic evolution given by

$$dB(t) = B(t)rdt, \qquad B(0) = 1,$$

and d stocks with prices $S_i(t)$, $i = 1, \ldots, d$. Before introducing the stock price dynamics and the suitable filtration on this probability space, we need to introduce the notion of a stress scenario.

Definition 1. *A vector $z \in \mathbb{R}^d \backslash \{0\}$ with components $z_i \in [0, 1)$ is called a stress. A pair (z, ξ), consisting of a stress z and a stochastic process ξ with values in $[0, T] \cup \{\infty\}$, is called a stress scenario. We will refer to ξ as the time of occurrence of the stress z, where we formally set $\{\xi = \infty\}$ if the stress does not occur. For a stress scenario (z, ξ), we further introduce the (indicator) jump process*

$$J_z(t) := \mathbb{1}_{\{\xi \leq t\}}, \; t \in [0, T].$$

Now we let $(W(t))_{t\in[0,T]}$ be a d-dimensional Brownian motion and consider n stress scenarios $\left(z^{(1)}, \xi_1\right), ..., \left(z^{(n)}, \xi_n\right)$ with corresponding jump processes $J_j(t) := J_{z^{(j)}}(t)$, $j = 1, \ldots, n$. We then assume that our probability space is equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ that satisfies the so-called usual conditions and that includes both, the filtration generated by the Brownian motion $W(t)$ and the one generated by the jump processes $J_1(t), \ldots, J_n(t)$. To be able to include the scenario $\{\xi_j = \infty\}$, for $j \in \{1, \ldots, n\}$, we extend $\mathbb{F}$ to $[0, T] \cup \{\infty\}$ by letting $\mathcal{F}_\infty := \mathcal{F}_T$. Note in particular that then all ξ_j are stopping times with respect to this filtration. We denote this by $\xi_j \in \Theta$, i.e. Θ is defined as the set of $[0, T] \cup \{\infty\}$-valued stopping times.

We are now ready to define the stock price dynamics, for $i = 1, \ldots, d$, via the following jump diffusion process

$$dS_i(t) = S_i(t-) \left(b_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t) - \sum_{j=1}^{n} z_i^{(j)} dJ_j(t) \right), \quad S_i(0) = s_{0,i}.$$

with the drift vector $b = (b_1, \ldots, b_d)^T$ and the volatility matrix $\sigma = \left(\sigma_{ij}\right)$ of full rank.

Our interpretation of Knightian uncertainty about the occurrence of the stresses is, that we can of course observe the jump processes $J_j(t)$ for $j \in \{1, \ldots, n\}$, but have no knowledge about the likelihood of the actual appearance of the jumps. In particular, we have also no knowledge on the dynamic properties of the jump processes, such as their possible intensities.

To be in line with our motivating example of stress tests, we assume that each stress can occur at most once on $[0, T]$.

Portfolio strategies. Let us consider an investor with an initial capital $x > 0$, which he/she can invest in d stocks and the money market account. Furthermore, he/she can observe

a stress scenario and react accordingly after its occurrence. We describe the investor's behaviour by a self-financing portfolio process π which denotes the fractions of his/her wealth invested in the stocks. We assume π to be predictable to model that a reallocation of holdings is not possible during the occurrence of a stress. However, it should also avoid a ruin by using a cautious investment strategy. Technically, we thus define the following set of admissible portfolio processes.

Definition 2. *Let $z^{(1)}, \dots, z^{(n)}$ be n stresses with corresponding times of occurrence $\xi_1, \dots, \xi_n \in \theta$. Then, the set of admissible portfolio processes π corresponding to an initial capital of $x > 0$ at time $t \in [0, T]$, $\mathbb{A}(t, x)$, consists of $\{\mathcal{F}_s, t \leq s \leq T\}$-predictable processes π such that*

1. *the investor's wealth process X^π is given as the unique solution of the stochastic differential equation*

$$\begin{aligned} X^\pi(t) &= x, \\ dX^\pi(s) &= X^\pi(s-)\Bigg(\left(r + \pi(s)^T (b - r\underline{1})\right) ds + \pi(s)^T \sigma dW(s) \\ &\qquad - \sum_{j=1}^{n} \pi(s)^T z^{(j)} dJ_j(s) \Bigg), \end{aligned}$$

   for $s \in [t, T]$.

2. *$\pi(s)$, for $s \in [t, T]$, is of the form*

$$\pi(s) = \sum_{u \in \{0,1\}^n} \mathbb{1}_{\{\bar{u}(s) = u\}} \pi_u(s),$$

   where all $\pi_u(s)$ are assumed to be predictable with right-continuous paths and left limits on $[0, T]$, for all $u = (u_1, \dots, u_n) \in \{0,1\}^n$, and $\bar{u}(s) = (\bar{u}_1(s), \dots, \bar{u}_n(s)) \in \{0,1\}^n$, for $s \in [0, T]$, with $\bar{u}_i(s) = 1$ iff $s \leq \xi_i$, for $i \in \{1, \dots, n\}$.

3. *We further require*

$$\begin{aligned} X^\pi(s) &> 0 \; \forall s \in [t, T], \\ \int_t^T \pi_{u,i}^2(s) ds &< \infty \text{ P-a.s. for } i = 1, \dots, d, \text{ and } u \in \{0,1\}^n. \end{aligned}$$

We introduce the abbreviation $\mathbb{A}(x) := \mathbb{A}(0, x)$.

We would like to use the following remark to draw some conclusions from the above definition and to introduce a new notation.

Remark 1. 1. *If, for $j \in \{1, \dots, n\}$, a stress $z^{(j)}$ occurs at a time $\xi_j \in \Theta$, $\xi_j < \infty$, the wealth process X^π changes at time ξ_j according to*

$$X^\pi(\xi_j) = \left(1 - \pi(\xi_j)^T z^{(j)}\right) X^\pi(\xi_j -).$$

Note that our definition of the portfolio process ensures that a change to the appropriate $\pi_u(.)$ sub strategy follows directly after the jump.

2. *The non-negativity condition in particular implies that an investor does not suffer a total loss due to any stress, i.e. for n remaining stresses $z^{(1)}, \ldots, z^{(n)}$ an admissible portfolio process π needs to fulfil the condition*

$$\max\left\{\pi(t)^T z^{(j)} | j = 1, \ldots, n\right\} < 1,$$

for $t \in [0, T]$.

3. *In the case of n stress scenarios, we will often denote the wealth process X^π before the first stress has occurred by $X^{\pi(1,\ldots,1)}$. This will be of particular interest as we will solve the worst-case problem by building up our solution recursively, starting from the well-known no crash optimal portfolio process.*

The worst-case optimization problem. Let $\left(z^{(1)}, \xi_1\right), \ldots, \left(z^{(n)}, \xi_n\right)$ be n stress scenarios and let $U : (0, \infty) \to \mathbb{R}$ be the investor's utility function, which is assumed to be strictly concave, increasing and continuously differentiable. We call the problem

$$\sup_{\pi \in \mathbb{A}(x)} \inf_{\xi_1, \ldots, \xi_n \in \Theta} \mathbb{E}_{0,x,(u_1,\ldots,u_n)} \left(U\left(X^\pi(T)\right)\right), \; u_1 = \ldots = u_n = 1,$$

the **worst-case portfolio problem under stresses**, where $\mathbb{E}_{t,x,(u_1,\ldots,u_n)}$ for $t \in [0, T]$ denotes the conditional expectation given that $X^\pi(t) = x$ and $u_j \in \{0, 1\}$, $j = 1, \ldots, n$, with $u_j = 1$ if and only if $\xi_j \geq t$. These indices are omitted if $t = 0$ or if $u_1 = \ldots = u_n = 0$.

In the further course of this chapter, we will look at the case of d stocks and two stresses $z^{(1)}$ and $z^{(2)}$ with corresponding times of occurrence $\xi_1, \xi_2 \in \Theta$, since the limitation to two stress scenarios allows for a simpler notation. However, in Remark 7 we will indicate how the results of this chapter can be extended to the general problem with n stress scenarios.

Further, we model the investor's preferences by a CRRA (constant relative risk aversion) utility function, i.e.

$$U(x) = \frac{1}{1-\gamma} x^{1-\gamma} \text{ for } \gamma > 0, \; \gamma \neq 1.$$

In Remark 8 and Remark 9 we explain that the main results can be adapted for the case of a logarithmic utility function $U(x) = \ln(x)$, which can be interpreted as the limit case of $\gamma \to 1$.

We make the additional assumption that only one stress can occur at one time instant. In the following remark we motivate this by a simple example.

Remark 2. *We consider the two-stock portfolio process $\pi(t)^T = (0.25, 0.25)$ and assume that there exist two stresses $z^{(1)}$ and $z^{(2)}$ with*

$$z^{(1)} = (0.5, 0)^T \text{ and } z^{(2)} = (0, 0.5)^T.$$

A simultaneous occurrence of the two stresses at time t would intuitively reduce the wealth by 25%. *On the other hand we have*

$$1 - \left(\left(1 - \pi(t)^T z^{(2)}\right)\left(1 - \pi(t)^T z^{(1)}\right)\right) = 1 - (1 - 0.5 \cdot 0.25)^2 = 23.44\%.$$

Instead, let us now consider the situation in which the two stresses directly follow each other:

$$1 - \left(\left(1 - \pi(t-)^T z^{(2)}\right)\left(1 - \pi(t)^T z^{(1)}\right)\right) \quad \textit{with} \quad \pi(t-)^T = (0.25, 0.25).$$

Assuming that no trading takes place after the first stress, the following applies: $\pi(t)^T = (2/7, 1/7)$, *and therefore:*

$$1 - \left(\left(1 - \pi(t-)^T z^{(2)}\right)\left(1 - \pi(t)^T z^{(1)}\right)\right) = 25\%.$$

Hence, in order to reflect an intuitive understanding of stresses in our model, we must assume that not more than one stress occurs at one time instant.

1.3 Terminal utility decomposition and the post-stress problem

In this section we consider the optimal post-stress strategy, i. e. the optimal strategy after the last possible stress has occurred. For this, we define $\xi_m := \max\{\xi_1, \xi_2\}$. As we are here only interested in the optimal strategy after the last stress, we consider the case $\xi_m < \infty$. For the sake of simplicity, we will refer to the portfolio process $\pi_{(0,0)}$ as $\pi_{(0)}$, for $\pi \in \mathbb{A}(x)$. Thus, we obtain the post-stress problem

$$\max_{\pi \in \mathbb{A}(\xi_m, x)} \mathbb{E}_{\xi_m, x}\left(U\left(X^\pi(T)\right)\right), \tag{1.1}$$

which is the classical Merton problem with random initial time ξ_m.

Decomposition of the terminal utility. To solve the post-stress problem we first look at a decomposition of the terminal utility. This enables us to solve the Merton problem with random initial time as well as the classical Merton problem.

Proposition 1 (Terminal utility decomposition)**.** *Let* $z^{(1)}$ *and* $z^{(2)}$ *be two stresses with corresponding times of occurrences* $\xi_1, \xi_2 < \infty$. *For any admissible portfolio process* $\pi \in \mathbb{A}(x)$ *we get the following decomposition:*

$$U(X^\pi(T)) = U(X^\pi(\xi_m)) \exp\left((1-\gamma)\int_{\xi_m}^T \phi_\gamma(\pi(t))dt\right) Y_T^\gamma(\pi),$$

where the mapping ϕ_γ *is given by*

$$\phi_\gamma(\pi) = r + \pi^T(b - r\underline{1}) - \frac{1}{2}\gamma\pi^T\sigma\sigma^T\pi$$

and $Y^\gamma(\pi) = \left(Y_t^\gamma(\pi)\right)_{t \in [\xi_m, T]}$ *is a positive* $\mathcal{F}_t$*-martingale with* $Y_{\xi_m}^\gamma(\pi) = 1$.

Proof. The proof is analogous to the one of Theorem 5 (Change-of-Measure Device) in [35]. □

We obtain an analogue decomposition for the case that no stress occurs, i.e. if $\xi_1 = \xi_2 = \{\infty\}$. This decomposition becomes important in section 1.5, when we have to consider the no stress scenario while looking at the multi-stress worst-case problem.

The stress-free world. In the stress-free post-stress world we now have to solve the Merton problem with random initial time. With the help of the above decomposition of the terminal utility we prove that this problem can be solved by maximizing ϕ_γ from the utility decomposition, i.e. a portfolio process $\bar{\pi} \in \mathbb{A}(\xi_m, x)$ is optimal for the post-stress problem (1.1) if $\bar{\pi}_{(0)} = \pi^*$, where

$$\pi^* = \arg\max_{\pi \in \mathbb{R}^d} \phi_\gamma(\pi).$$

One can see that the optimal strategy π^* is constant and independent of ξ_m and x.

Proposition 2 (Solution of the post-stress portfolio problem)**.** *For ϕ_γ as in Proposition 1, the optimal strategy π^* and the corresponding value function $v(\xi_m; x)$ in the post-stress problem* (1.1) *with random initial time $\xi_m \in \Theta$, $\xi_m < \infty$, are given by*

$$\pi^* = \frac{1}{\gamma}\left(\sigma\sigma^T\right)^{-1}(b - r\underline{1}), \text{ with}$$

$$v(\xi_m; x) := \max_{\pi \in \mathbb{A}(\xi_m, x)} \mathbb{E}_{\xi_m, x}\left(U\left(X^\pi(T)\right)\right) = \frac{x^{1-\gamma}}{1-\gamma}\exp\left((1-\gamma)\int_{\xi_m}^T \phi_\gamma(\pi^*)ds\right).$$

Proof. See [35]. □

For the interpretation of a stress as a threat we require from now on

$$0 < (\pi^*)^T z \tag{1.2}$$

for every stress z, i.e. the optimal portfolio in the stress-free world should suffer a loss due to a stress z.

1.4 The one-stress worst-case problem

We want to approach the multi-stress scenario step wise. Therefore, we first have a look at the one-stress worst-case problem in the multi asset setting:

$$\sup_{\pi \in \mathbb{A}(x)} \inf_{\xi \in \Theta} \mathbb{E}_{0,x,(1)}\left(U\left(X^\pi(T)\right)\right),$$

where we use the notation (z, ξ) instead of $(z^{(1)}, \xi_1)$. As this problem was discussed in detail in [29], we only recall ideas and results, and introduce essential concepts.

Based on the results of section 1.3, we assume in this section, without loss of generality, that $\pi_{(0)}(t) = \pi^*$, $t \in [0, T]$, for all $\pi \in \mathbb{A}(x)$, with π^* defined as in Proposition 2.

Indifference-optimality and indifference frontier. The following two fundamental concepts for solving worst-case portfolio optimization problems introduced in [56] allow to solve these problems in a very general setting. A central element is the definition of so-called indifference strategies.

Definition 3. *We consider a worst-case portfolio problem with a single stress scenario (z, ξ). Let further $v_0(t; x)$ denote the value function for the corresponding optimal portfolio problem in the post-stress setting. If there exists an admissible portfolio process $\pi \in \mathbb{A}(x)$ such that the process*

$$v_0\left(t; \left(1 - \pi_{(1)}(t)^T z\right) X^{\pi_{(1)}}(t-)\right), \quad t \in [0, T] \cup \{\infty\}$$

is a martingale on $[0, T] \cup \{\infty\}$ for

$$v_0\left(\infty; \left(1 - \pi_{(1)}(\infty)^T z\right) X^{\pi_{(1)}}(\infty-)\right) := U\left(X^{\pi_{(1)}}(T)\right),$$

we call π an indifference strategy for the investor.

This definition has two useful properties. First, it is independent of the concrete form of the utility function. Further, the martingale characterization of an indifference strategy makes its optimality proof independent of the actual market setting. We now cite the indifference-optimality principle from [35].

Proposition 3 (Indifference-optimality principle)**.** *If $\bar{\pi}$ is an indifference strategy, and for all $\pi \in \mathbb{A}(x)$ there exists at least a single time of stress occurrence $\tilde{\xi} \in \Theta$ with*

$$\mathbb{E}\left(v_0\left(t; \left(1 - \pi_{(1)}(\tilde{\xi})^T z\right) X^{\pi_{(1)}}(\tilde{\xi}-)\right)\right) \leq \mathbb{E}\left(v_0\left(t; \left(1 - \bar{\pi}_{(1)}(\tilde{\xi})^T z\right) X^{\bar{\pi}_{(1)}}(\tilde{\xi}-)\right)\right),$$

then $\bar{\pi}$ is a worst-case optimal portfolio process.

In section 1.5, we prove the minimum constant-optimality principle for the multi-stress worst-case problem, which is also based on the idea of worst-case indifference.

From Proposition 2 we know that the value function $v_0(t; x)$ is continuously differentiable with respect to t and twice continuously differentiable in x. Thus, application of Itô's rule to $v_0\left(t; \left(1 - \pi_{(1)}(t)^T z\right) X^{\pi_{(1)}}(t-)\right)$ and setting the resulting integrand in the finite variation part equal to zero, allows to compute an indifference strategy. We will use similar arguments in the proof of Proposition 5.

Now we can introduce the second essential concept, the indifference frontier. For a proof of the following multi-asset statement we refer to [29].

Proposition 4 (Indifference frontier)**.** *Let Assumption (1.2) be valid and let $\pi \in \mathbb{A}(x)$ be an admissible portfolio process for the one-stress worst-case portfolio problem with stress scenario (z, ξ). Further, let $\bar{\pi}$ be an indifference strategy. Set*

$$\tau := \inf\left\{t : \pi_{(1)}(t)^T z > \bar{\pi}_{(1)}(t)^T z\right\}$$

and define

$$\tilde{\pi}_{(1)}(t) := \pi_{(1)}(t) \text{ if } t < \tau \text{ and } \tilde{\pi}_{(1)}(t) := \bar{\pi}_{(1)}(t) \text{ if } t \geq \tau.$$

Then, we have $\tilde{\pi} \in \mathbb{A}(x)$ and the worst-case bound attained by $\tilde{\pi}$ is at least as big as that achieved by π.

Hence, only a portfolio process which does not exceed the indifference frontier of an indifference strategy $\bar{\pi}$, i.e. which does not exceed the indifference level $\bar{\pi}_{(1)}(t)^T z$ for $t \in [0, T]$, can be a worst-case optimal portfolio process.

Solution of the one-stress worst-case problem. In the multi-asset scenario we lose the uniqueness of the indifference strategy of the single stock case. This is overcome in [29] by using Lagrangian-multiplier methods to figure out the worst-case optimal indifference strategy.

Theorem 1 (Solution of the one-stress worst-case problem)**.** *Let Assumption (1.2) be satisfied and assume a single stress scenario (z, ξ). The optimal portfolio process is the indifference strategy $\bar{\pi}$, given by*

$$\bar{\pi}_{(1)}(t) = \pi^* - \frac{z^T \pi^* - N(t)}{z^T \left(\sigma\sigma^T\right)^{-1} z} \left(\sigma\sigma^T\right)^{-1} z,$$

where $N(t)$ is the unique solution of the ODE

$$N'(t) = -\frac{1}{2z^T \left(\sigma\sigma^T\right)^{-1} z} \gamma \left(1 - N(t)\right) \left(z^T \pi^* - N(t)\right)^2 \tag{1.3}$$

with final condition $N(T) = 0$. The corresponding value function $v_1(t; x)$ is

$$\begin{aligned} v_1(t; x) := &\sup_{\pi \in \mathbb{A}(t,x)} \inf_{\xi \in \Theta_t} \mathbb{E}_{t,x,(1)} \left(U\left(X^{\pi}(T)\right)\right) = v_0\left(t; \left(1 - \bar{\pi}_{(1)}(t)^T z\right) x\right) \\ =&\mathbb{E}_{t,x}\left(U\left(X^{\bar{\pi}_{(1)}}(T)\right)\right) \end{aligned}$$

for $t \in [0, T]$, where Θ_t denotes the class of $[t, T] \cup \{\infty\}$-valued stopping times.

Proof. See [29]. □

We would like to prove some more properties of the relative wealth loss $\bar{\pi}_{(1)}^T z$ of the optimal strategy in case that at most one stress can occur.

Lemma 1 (Decreasing wealth loss)**.** *Assume a single stress scenario (z, ξ) and let $\bar{\pi}$ be given as in Theorem 1. Then we obtain*

$$0 \leq \bar{\pi}_{(1)}(t)^T z < \min\{(\pi^*)^T z, 1\}$$

and thus in particular $\left(\bar{\pi}_{(1)}^T z\right)'(t) < 0$ and $\left(\bar{\pi}_{(1)}^T z\right)''(t) < 0$ for all $t \in [0, T]$.

Proof. Note that for N defined in Theorem 1, we obtain $N(t) = \bar{\pi}_{(1)}(t)^T z$ for all $t \in [0,T]$. Therefore it is enough to show that N fulfills the above properties. The inequalities $0 \leq N(t) < 1$ for all $t \in [0,T]$ hold by Lemma 4.2. in [56]. We now use similar arguments to prove $N(t) < (\pi^*)^T z$ for all $t \in [0,T]$. Equation (1.3) implies for $t \in [0,T]$:

$$\frac{\partial}{\partial t} \log\left(z^T \pi^* - N(t)\right) = \frac{1}{2z^T (\sigma\sigma^T)^{-1} z} \gamma (1 - N(t)) \left(z^T \pi^* - N(t)\right)$$

if $z^T \pi^* > N(t)$. Integrating both sides, applying the exponential function and using $N(T) = 0$ leads to

$$\frac{z^T \pi^*}{z^T \pi^* - N(t)} = \exp\left\{\int_t^T \frac{1}{2z^T (\sigma\sigma^T)^{-1} z} \gamma (1 - N(s)) \left(z^T \pi^* - N(s)\right) ds\right\}$$

for $t \in [0,T]$, if $z^T \pi^* > N(s)$ for $s \in [t,T]$. The above integrand is a continuous function in N and N is continuous itself. Therefore there exists an upper bound $M > 0$ such that for $t \in [0,T]$

$$\exp\left\{\int_t^T \frac{1}{2z^T (\sigma\sigma^T)^{-1} z} \gamma (1 - N(s)) \left(z^T \pi^* - N(s)\right) ds\right\} \leq \exp\{M(T-t)\},$$

and therefore

$$z^T \pi^* - N(t) \geq \exp\{-M(T-t)\} z^T \pi^*,$$

if $z^T \pi^* > N(s)$ for $s \in [t,T]$. We define $\delta = \frac{1}{2} \exp\{-MT\} z^T \pi^*$ and since, by assumption (1.2), $z^T \pi^* > 0$ we get $\delta > 0$. Then we obtain for $t \in [0,T]$ that

$$z^T \pi^* - N(t) \geq 2\delta \tag{1.4}$$

if $z^T \pi^* > N(s)$ for $s \in [t,T]$. Now we define $\tilde{t}$ as

$$\tilde{t} := \inf\left\{t \in [0,T] \mid z^T \pi^* - N(s) \geq \delta \text{ for all } s \in [t,T]\right\}.$$

The infimum is attained, since $z^T \pi^* - N(T) = \exp\{MT\} 2\delta \geq \delta$. Now we prove by contradiction that $\tilde{t} = 0$. For this, assume $\tilde{t} > 0$. Inequality (1.4) and the definition of $\tilde{t}$ lead to $z^T \pi^* - N(\tilde{t}) \geq 2\delta$. By continuity, there exists $\bar{t} < \tilde{t}$, with $z^T \pi^* - N(s) \geq \delta$ for all $s \in [\bar{t},T]$, which is a contradiction to the definition of $\tilde{t}$. Therefore we get $\tilde{t} = 0$ and $N(t) < (\pi^*)^T z$ for all $t \in [0,T]$.

With $\bar{\pi}_{(1)}(t)^T z < \min\{(\pi^*)^T z, 1\}$ and equation (1.3) we get $N'(t) = \left(\bar{\pi}_{(1)}^T z\right)'(t) < 0$ for all $t \in [0,T]$. The second derivative of N is for $t \in [0,T]$ given by

$$N''(t) = KN'(t) \left(z^T \pi^* - N(t)\right) \left(2(1 - N(t)) + \left(z^T \pi^* - N(t)\right)\right),$$

for a constant $K > 0$. With the above considerations it directly follows that $N''(t) = \left(\bar{\pi}_{(1)}^T z\right)''(t) < 0$ for all $t \in [0, T]$. □

Note that $\bar{\pi}$ is admissible in terms of Definition 2 and the optimal strategy is independent of the wealth x. Further, the optimal sub strategy $\bar{\pi}_{(1)}$ from Theorem 1 is deterministic. Therefore, even in the multi-stress worst-case scenario we can assume that we know the optimal portfolio process when only one stress can still occur.

1.5 The multi-stress worst-case problem

Now we are able to consider the worst-case portfolio problem under stresses with two stress scenarios $(z^{(1)}, \xi_1)$ and $(z^{(2)}, \xi_2)$. Hence, we look at the following portfolio optimization problem:

$$\sup_{\pi \in \mathbb{A}(x)} \inf_{\substack{\xi_1, \xi_2 \in \Theta \\ \xi_1 \neq \xi_2}} \mathbb{E}_{0,x,(1,1)} \left(U \left(X^{\pi}(T) \right) \right), \tag{1.5}$$

where the requirement $\xi_1 \neq \xi_2$ is meant in the strong sense that we have the inequality

$$\xi_1(\omega) \neq \xi_2(\omega) \text{ for all } \omega \in \Omega,$$

if at least one of the two values is finite. I.e. there shall not be a simultaneous occurrence of the two stresses, but $\xi_1(\omega) = \xi_2(\omega) = \infty$ should be possible. A central challenge of the multi-type stress worst-case portfolio problem is the unknown sequential occurrence of the different types of stresses. The investor not only has the difficulty that he/she does not know when the next stress will occur, but also which of the remaining stresses will occur. Still, we can draw on previous results and know the optimal portfolio process when at most one stress can occur. Thus, we can assume that we know

- the value function $v_{00}(t; x)$ and the optimal sub strategy $\bar{\pi}_{(0,0)} = \pi^*$ if both stress scenarios have already happened (see section 1.3),
- the value function $v_{10}(t; x)$ and the optimal sub strategy $\bar{\pi}_{(1,0)}$ if only the stress scenario related to $z^{(2)}$ has already happened (see section 1.4 with $(z, \xi) = (z^{(1)}, \xi_1)$),
- the value function $v_{01}(t; x)$ and the optimal sub strategy $\bar{\pi}_{(0,1)}$ if only the stress scenario related to $z^{(1)}$ has already happened (see section 1.4 with $(z, \xi) = (z^{(2)}, \xi_2)$).

Hence, it remains to determine the (worst-case) optimal sub strategy $\bar{\pi}_{(1,1)}$ and the value function $v_{11}(t; x)$ if both stresses can still occur.

Reformulation of the problem. We now reformulate the problem such that we can at least partly use existing concepts. For this, we need to introduce the definition of the relative worst-case loss. This definition takes into account that for an admissible portfolio process at a certain point in time $t \in [0, T]$, the order in which the stresses occur can have different

effects. In our worst-case approach, the order of occurrence that leads to a lower expected utility is of particular importance.

Definition 4. *For a portfolio process π we define the relative losses at time $t \in [0,T]$, $RL^{\pi}_{12}(t)$ and $RL^{\pi}_{21}(t)$, depending on the order of stress occurrences, by*

$$1 - RL^{\pi}_{12}(t) := \left(1 - \pi(t)^T z^{(1)}\right)\left(1 - \bar{\pi}_{(0,1)}(t)^T z^{(2)}\right),$$
$$1 - RL^{\pi}_{21}(t) := \left(1 - \pi(t)^T z^{(2)}\right)\left(1 - \bar{\pi}_{(1,0)}(t)^T z^{(1)}\right).$$

The relative worst-case loss $RL^{\pi}(t)$ of π at time $t \in [0,T]$ is defined as

$$RL^{\pi}(t) := \max\{RL^{\pi}_{12}(t), RL^{\pi}_{21}(t)\}.$$

With Θ_t denoting the class of $[t,T] \cup \{\infty\}$-valued stopping times, we can define the value function $v_{11}(t;x)$ by

$$v_{11}(t;x) := \sup_{\pi \in \mathbb{A}(t,x)} \inf_{\substack{\xi_1,\xi_2 \in \Theta_t \\ \xi_1 \neq \xi_2}} \mathbb{E}_{t,x,(1,1)}\left(U\left(X^{\pi}(T)\right)\right) \text{ for } t \in [0,T], \tag{1.6}$$

where we again use the strong notion of the inequality $\xi_1 \neq \xi_2$ as in (1.5). Definition 4 allows to rewrite the value function $v_{11}(t;x)$ in a suitable way.

Lemma 2 (Reformulation of the value function)**.** *The value function in the two-stress worst case problem can be rewritten as follows:*

$$\begin{aligned} v_{11}(t;x) &= \sup_{\pi \in \mathbb{A}(t,x)} \inf_{\substack{\xi_1,\xi_2 \in \Theta_t \\ \xi_1 \neq \xi_2}} \mathbb{E}_{t,x,(1,1)}\left(U\left(X^{\pi}(T)\right)\right) \\ &= \sup_{\pi \in \mathbb{A}(t,x)} \inf_{\xi \in \Theta_t} \mathbb{E}_{t,x}\left(v_{00}\left(\xi; (1 - RL^{\pi_{(1,1)}}(\xi))X^{\pi_{(1,1)}}(\xi-)\right)\right), \end{aligned}$$

where $v_{00}\left(\infty; (1 - RL^{\pi_{(1,1)}}(\infty))X^{\pi_{(1,1)}}(\infty-)\right) := U\left(X^{\pi_{(1,1)}}(T)\right)$.

Proof. The idea of this proof is to reduce the worst-case problem to a single stopping time ξ, instead of ξ_1 and ξ_2, by taking advantage of the fact that we know the optimal portfolio process after the first stress has occurred. To do this, we first have to introduce some notations:

- We define

$$\begin{aligned} \tilde{\mathbb{A}}(t,x) := \{\tilde{\pi} | \tilde{\pi}(s) &= \pi_{(1,1)}(s)\mathbb{1}_{\{s \leq \min\{\xi_1,\xi_2\}\}} + \bar{\pi}_{(1,0)}(s)\mathbb{1}_{\{\xi_2 < s \leq \xi_1\}} \\ &+ \bar{\pi}_{(0,1)}(s)\mathbb{1}_{\{\xi_1 < s \leq \xi_2\}} + \pi^* \mathbb{1}_{\{s > \max\{\xi_1,\xi_2\}\}} \text{ for } s \in [t,T] \text{ and } \pi \in \mathbb{A}(t,x)\}, \end{aligned}$$

  as a subset of $\mathbb{A}(t,x)$.

- Further, we denote by $\tilde{\xi} = \min\{\xi_1, \xi_2\}$, for $\xi_1, \xi_2 \in \Theta_t$, the time of occurrence of the first stress.

- Then, we define $\mathbb{E}_{\tilde{\xi},x,2}$ as the conditional expectation, given that $X^{\pi}(\tilde{\xi}) = x$ and that the second stress can still occur, i.e. $\mathbb{E}_{\tilde{\xi},x,2} = \mathbb{E}_{\tilde{\xi},x,(0,1)}$ if $\tilde{\xi} = \xi_1$ and $\mathbb{E}_{\tilde{\xi},x,2} = \mathbb{E}_{\tilde{\xi},x,(1,0)}$ otherwise.

Due to the definition of $v_{00}\left(\infty; (1 - RL^{\pi_{(1,1)}}(\infty))X^{\pi_{(1,1)}}(\infty-)\right)$, we only need to consider the case $\tilde{\xi} < \infty$. Using the introduced notation leads to

$$
\begin{aligned}
&\sup_{\pi\in\mathbb{A}(t,x)} \inf_{\substack{\xi_1,\xi_2\in\Theta_t\\ \xi_1\neq\xi_2}} \mathbb{E}_{t,x,(1,1)}\left(U\left(X^{\pi}(T)\right)\right)\\
&= \sup_{\pi\in\mathbb{A}(t,x)} \inf_{\substack{\xi_1,\xi_2\in\Theta_t\\ \xi_1\neq\xi_2}} \mathbb{E}_{t,x,(1,1)}\left(\mathbb{E}_{\tilde{\xi},X^{\pi}(\tilde{\xi}),2}\left(U\left(X^{\pi}(T)\right)\right)\right)\\
&= \sup_{\pi\in\tilde{\mathbb{A}}(t,x)} \inf_{\substack{\xi_1,\xi_2\in\Theta_t\\ \xi_1\neq\xi_2}} \mathbb{E}_{t,x,(1,1)}\left(\mathbb{E}_{\tilde{\xi},X^{\pi}(\tilde{\xi}),2}\left(U\left(X^{\pi}(T)\right)\right)\right)\\
&= \sup_{\pi\in\tilde{\mathbb{A}}(t,x)} \inf_{\substack{\xi_1,\xi_2\in\Theta_t\\ \xi_1\neq\xi_2}} \mathbb{E}_{t,x,(1,1)}\Big(\mathbb{1}_{\tilde{\xi}=\xi_1}\mathbb{E}_{\tilde{\xi},X^{\pi}(\tilde{\xi}),(0,1)}\left(U\left(X^{\pi}(T)\right)\right)\\
&\qquad\qquad + \mathbb{1}_{\tilde{\xi}=\xi_2}\mathbb{E}_{\tilde{\xi},X^{\pi}(\tilde{\xi}),(1,0)}\left(U\left(X^{\pi}(T)\right)\right)\Big)\\
&= \sup_{\pi\in\tilde{\mathbb{A}}(t,x)} \inf_{\substack{\xi_1,\xi_2\in\Theta_t\\ \xi_1\neq\xi_2}} \mathbb{E}_{t,x,(1,1)}\Big(\mathbb{1}_{\tilde{\xi}=\xi_1}v_{01}\left(\tilde{\xi}; \left(1-\pi_{(1,1)}(\tilde{\xi})^T z^{(1)}\right)X^{\pi_{(1,1)}}(\tilde{\xi}-)\right)\\
&\qquad\qquad + \mathbb{1}_{\tilde{\xi}=\xi_2}v_{10}\left(\tilde{\xi}; \left(1-\pi_{(1,1)}(\tilde{\xi})^T z^{(2)}\right)X^{\pi_{(1,1)}}(\tilde{\xi}-)\right)\Big).
\end{aligned}
$$

The first equality follows by the tower property. For the second equality note that in section 3 and section 4 we have already determined the optimal strategies for the different cases when at most one or even no more stress can occur. Thus, an optimal strategy in the case when at most two stresses can still occur, must have the form of the strategies in the set $\tilde{\mathbb{A}}(t,x)$, i.e. we can restrict to this set for determining the optimal strategies. The third equality is a direct result of the above introduced notation for the conditional expectation and the assumption $\xi_1 \neq \xi_2$. Theorem 1 and the definition of $\tilde{\mathbb{A}}(t,x)$, as well as the definition of the wealth process at time $\tilde{\xi}$, lead to the fourth equation.

Note that the reformulated value function now only depends on the time of occurrence of the first stress and because of $\xi_1, \xi_2 \in \Theta_t$, this can be any stopping time in Θ_t. Furthermore it follows from the definition of $\tilde{\xi}$ that

$$
\tilde{\xi} = \begin{cases} \xi_1 & \text{if} \quad v_{01}\left(\xi_1; X^{\pi}(\xi_1)\right) \leq v_{10}\left(\xi_2; X^{\pi}(\xi_2)\right), \\ \xi_2 & \text{else.} \end{cases}
$$

Thus we obtain

$$
\begin{aligned}
&\sup_{\pi\in\tilde{\mathbb{A}}(t,x)} \inf_{\substack{\xi_1,\xi_2\in\Theta_t\\ \xi_1\neq\xi_2}} \mathbb{E}_{t,x,(1,1)}\Big(\mathbb{1}_{\tilde{\xi}=\xi_1}v_{01}\left(\tilde{\xi}; \left(1-\pi_{(1,1)}(\tilde{\xi})^T z^{(1)}\right)X^{\pi_{(1,1)}}(\tilde{\xi}-)\right)\\
&\qquad\qquad + \mathbb{1}_{\tilde{\xi}=\xi_2}v_{10}\left(\tilde{\xi}; \left(1-\pi_{(1,1)}(\tilde{\xi})^T z^{(2)}\right)X^{\pi_{(1,1)}}(\tilde{\xi}-)\right)\Big)\\
&= \sup_{\pi\in\tilde{\mathbb{A}}(t,x)} \inf_{\xi\in\Theta_t} \mathbb{E}_{t,x}\Big(\min\Big\{v_{01}\left(\xi; \left(1-\pi_{(1,1)}(\xi)^T z^{(1)}\right)X^{\pi_{(1,1)}}(\xi-)\right),
\end{aligned}
$$

$$
\begin{aligned}
& \qquad\qquad v_{10}\left(\xi;\left(1-\pi_{(1,1)}(\xi)^T z^{(2)}\right) X^{\pi_{(1,1)}}(\xi-)\right)\Big\}\Big) \\
&= \sup_{\pi\in\tilde{\mathbb{A}}(t,x)} \inf_{\xi\in\Theta_t} \mathbb{E}_{t,x}\Big(\min\Big\{v_{00}\left(\xi;\left(1-RL_{12}^{\pi_{(1,1)}}(\xi)\right) X^{\pi_{(1,1)}}(\xi-)\right), \\
& \qquad\qquad v_{00}\left(\xi;\left(1-RL_{21}^{\pi_{(1,1)}}(\xi)\right) X^{\pi_{(1,1)}}(\xi-)\right)\Big\}\Big) \\
&= \sup_{\pi\in\tilde{\mathbb{A}}(t,x)} \inf_{\xi\in\Theta_t} \mathbb{E}_{t,x}\Big(\min\Big\{(1-RL_{12}^{\pi_{(1,1)}}(\xi))^{1-\gamma} v_{00}\left(\xi;X^{\pi_{(1,1)}}(\xi-)\right), \\
& \qquad\qquad (1-RL_{21}^{\pi_{(1,1)}}(\xi))^{1-\gamma} v_{00}\left(\xi;X^{\pi_{(1,1)}}(\xi-)\right)\Big\}\Big) \\
&= \sup_{\pi\in\tilde{\mathbb{A}}(t,x)} \inf_{\xi\in\Theta_t} \mathbb{E}_{t,x}\Big((1-RL^{\pi_{(1,1)}}(\xi))^{1-\gamma} v_{00}\left(\xi;X^{\pi_{(1,1)}}(\xi-)\right)\Big) \\
&= \sup_{\pi\in\tilde{\mathbb{A}}(t,x)} \inf_{\xi\in\Theta_t} \mathbb{E}_{t,x}\Big(v_{00}\left(\xi;(1-RL^{\pi_{(1,1)}}(\xi))X^{\pi_{(1,1)}}(\xi-)\right)\Big) \\
&= \sup_{\pi\in\mathbb{A}(t,x)} \inf_{\xi\in\Theta_t} \mathbb{E}_{t,x}\Big(v_{00}\left(\xi;(1-RL^{\pi_{(1,1)}}(\xi))X^{\pi_{(1,1)}}(\xi-)\right)\Big).
\end{aligned}
$$

Here, the first equality follows with the above considerations on $\tilde{\xi}$ and the second one with Theorem 1 and the definition of the relative losses $RL_{12}^{\pi_{(1,1)}}$ and $RL_{21}^{\pi_{(1,1)}}$. Equation three holds by Proposition 2 and for the fourth equality we have to note that $v_{00}\left(t;X^{\pi_{(1,1)}}(\xi-)\right)$ is negative for $\gamma > 1$ and positive for $\gamma < 1$ and for an admissible portfolio process it holds $RL_{12}^{\pi_{(1,1)}} < 1$ and $RL_{21}^{\pi_{(1,1)}} < 1$. The second but last equality follows again with Proposition 2 and the last one holds by the definition of $\tilde{\mathbb{A}}(t,x)$. □

Remark 3 (Admissible portfolio processes)**.** *From now on, we denote, without loss of generality, by $\mathbb{A}(t,x)$, for $x > 0$, $t \in [0,T]$, the reduced set of admissible portfolio processes $\tilde{\mathbb{A}}(t,x)$, defined in the proof of Lemma 2.*

Remark 4 (Wealth process before the first stress)**.** *At this point we would like to note that the stopping time $\xi \in \Theta$, in the reformulated representation of the value function, is the time of occurrence of the first stress. Thus, the wealth process cannot jump at a time $t < \xi$. This is of particular importance when using Itô's rule in the proofs of Proposition 5 and Proposition 8.*

Minimum constant processes and optimality. The main consequence of Lemma 2 is that we only have to consider the infimum over a single stopping time $\xi \in \Theta$ in the value function $v_{11}(t;x)$. This allows to suitably adapt the indifference optimality principle and the indifference frontier for the two-stress worst-case problem via the introduction of the concept of the minimum process.

Definition 5. *Let $\pi \in \mathbb{A}(x)$ be an admissible portfolio process.*

1. *We define the minimum process $M^\pi(t)$ by*

$$M^\pi(t) := v_{00}\left(t;(1-RL^{\pi_{(1,1)}}(t))X^{\pi_{(1,1)}}(t-)\right) \text{ for } t \in [0,T],$$

with $M^\pi(\infty) := U\left(X^{\pi_{(1,1)}}(T)\right)$.

2. *We call* π minimum constant *if we have*

$$\mathbb{E}\left(M^{\pi}(\tau_1)\right) = \mathbb{E}\left(M^{\pi}(\tau_2)\right) \text{ for all } \tau_1,\ \tau_2 \in \Theta.$$

The unknown order of appearance of the two stresses requires that the investor always has to consider both possible orderings simultaneously when judging a portfolio strategy in terms of the worst-case approach. For this reason, we have introduced the definition of the minimum process. The wording *minimum process* gets its meaning by noting that we have

$$\begin{aligned} M^{\pi}(t) = \min \Big\{ & v_{01}\left(t; (1-\pi_{(1,1)}(t)^T z^{(1)}) X^{\pi_{(1,1)}}(t-)\right), \\ & v_{10}\left(t; (1-\pi_{(1,1)}(t)^T z^{(2)}) X^{\pi_{(1,1)}}(t-)\right) \Big\} \text{ for } t \in [0,T], \end{aligned} \tag{1.7}$$

with the same arguments as in the proof of Lemma 2. To be able to argue similarly as for the indifference optimality, we also need minimum constant portfolio processes.

Remark 5 (Connection to the existing literature)**.** *In the previous definition we have introduced minimum constant portfolio processes. In addition, we will state the minimum constant-optimality principle in Proposition 6 and introduce the minimum indifference frontier in Proposition 7 and Remark 6.*

To solve the worst-case scenario portfolio problem with a single stress, the notions of indifference strategy and indifference frontier have been introduced in [35] and [56]. These indifference concepts are based on an abstract controller-vs-stopper game between the investor (controller) and the market (stopper).

However there exists a fundamental difference, that clearly distinguishes our new concepts of minimum constant portfolio processes and of the minimum constant-optimality principle from the existing literature. For this, consider the one stress scenario, or even the portfolio optimization problem with $n > 1$ *stress scenarios, consisting of* n *equal stresses* $z_1 = \ldots = z_n$*, as solved in [36]. In these problems, the investor chooses the portfolio process and the market decides when the next stress occurs. In our work, the investor still only has the option of choosing the portfolio process. The market, as his/her counterpart, now has the possibility to decide when the next stress will occur as well as which of the remaining stresses occurs. Thus, the market has two controls, namely the time and the order of occurrence of the stresses.*

The existing concepts of indifference strategies and the indifference optimality principle are based on the idea that an optimal investor should be indifferent to the time of occurrence of the next stress, and at the same time presupposes the knowledge of which stress will be the next. It is exactly this knowledge that we can no longer assume in this work with multiple different stresses, which is why the introduction of new concepts is necessary. The new principle of minimum indifference is now intended to take into account an indifference with regard to the time of occurrence of the worst possible order *of stresses. The formal implementation of these new concepts, which finally results in the proof of existence of an optimal portfolio process, is described in the remainder of this section.*

We are already in a position, where we can prove the existence of a minimum constant portfolio process.

Proposition 5 (Existence of a minimum constant portfolio process)**.** *There exists a minimum constant portfolio process $\bar{\pi} \in \mathbb{A}(x)$, such that*

$$\bar{\pi}_{(1,1)}(t) \in \underset{\substack{\pi \in \mathbb{R}^2 \\ RL^{\pi}(t)=N(t)}}{\arg\max} \ \phi_\gamma(\pi) \quad \textit{for } t \in [0,T],$$

where N is defined for $t \in [0,T]$ by the differential equation

$$N'(t) = (1 - N(t)) \left(\max_{\substack{\pi \in \mathbb{R}^2 \\ RL^{\pi}(t)=N(t)}} \phi_\gamma(\pi) - \phi_\gamma(\pi^*) \right), \tag{1.8}$$

with $N(T) = 0$ and ϕ_γ as defined in Proposition 1. Further, for all $t \in [0,T]$ we have $RL^{\bar{\pi}_{(1,1)}}(t) < 1$.

Proof. We give the proof in two steps.

(i) First, we want to show the existence of a deterministic admissible portfolio process $\bar{\pi}$, as claimed in the assertion. We consider the function

$$G(t,N) := \left(\max_{\substack{\pi \in \mathbb{R}^2 \\ RL^{\pi}(t)=N}} \phi_\gamma(\pi) - \phi_\gamma(\pi^*) \right).$$

As $\phi_\gamma(\pi)$ is a strictly concave quadratic function in π and $RL^{\pi}(t)$ is the maximum of two linear functions in π with a range of $\mathbb{R}$, the maximization problem inside $G(t,N)$ can be solved and the maximum is attained. As further $RL^{\pi}(t)$ is continuous, both in t and in π, $G(t,N)$ is also continuous. Thus, the ordinary differential equation (1.8) is well-defined. Now we can apply Lemma 4.2. in [56] and get, for the solution N of (1.8), the inequality $0 \leq N(t) < 1$ for all $t \in [0,T]$. Then it follows directly that a deterministic admissible portfolio process $\bar{\pi} \in \mathbb{A}(x)$, as defined above, exists and in particular $RL^{\bar{\pi}_{(1,1)}} = N$ applies for this portfolio process.

It is admissible as N is continuous and bounded, ϕ_γ is a quadratic function with a unique maximum and therefore there exists a $\bar{\pi}_{(1,1)}$, which is continuous and bounded. Further, we get for every $t \in [0,T]$

$$\begin{aligned} N(t) < 1 \Leftrightarrow & RL^{\bar{\pi}_{(1,1)}}(t) < 1 \Leftrightarrow \max\{RL_{12}^{\bar{\pi}_{(1,1)}}(t), RL_{21}^{\bar{\pi}_{(1,1)}}(t)\} < 1 \\ \Leftrightarrow & 1 - \left(1 - \bar{\pi}_{(1,1)}(t)^T z^{(1)}\right)\left(1 - \bar{\pi}_{(0,1)}(t)^T z^{(2)}\right) < 1 \\ & \wedge \ 1 - \left(1 - \bar{\pi}_{(1,1)}(t)^T z^{(2)}\right)\left(1 - \bar{\pi}_{(1,0)}(t)^T z^{(1)}\right) < 1 \\ \Leftrightarrow & \max\{\bar{\pi}_{(1,1)}(t)^T z^{(1)}, \bar{\pi}_{(1,1)}(t)^T z^{(2)}\} < 1\,, \end{aligned}$$

where the last equivalence is valid since $\bar{\pi}_{(1,0)}$ and $\bar{\pi}_{(0,1)}$ are part of admissible portfolio processes in the one-stress setting and thus $\bar{\pi}_{(1,0)}(t)^T z^{(1)} < 1$ and $\bar{\pi}_{(0,1)}(t)^T z^{(2)} < 1$.

(ii) Now we prove the minimum constant property of an admissible portfolio process $\bar{\pi}$, defined as above. For $t \in [0, T]$, the minimum process $M^{\bar{\pi}}$ is defined as

$$M^{\bar{\pi}}(t) = v_{00}\left(t; (1 - RL^{\bar{\pi}_{(1,1)}}(t))X^{\bar{\pi}_{(1,1)}}(t-)\right).$$

By definition of $\bar{\pi}$ we have

$$\left(RL^{\bar{\pi}_{(1,1)}}\right)'(t) = (1 - RL^{\bar{\pi}_{(1,1)}}(t))\left(\phi_\gamma(\bar{\pi}_{(1,1)}(t)) - \phi_\gamma(\pi^*)\right), \tag{1.9}$$

with $RL^{\bar{\pi}_{(1,1)}}(T) = 0$. Itô's rule for $v_{00}\left(t; (1 - RL^{\bar{\pi}_{(1,1)}}(t))X^{\bar{\pi}_{(1,1)}}(t-)\right)$ leads to

$$\begin{aligned}
&dv_{00}\left(t; (1 - RL^{\bar{\pi}_{(1,1)}}(t))X^{\bar{\pi}_{(1,1)}}(t-)\right)\\
&= (v_{00})_t(t; x(t)))\,dt + (v_{00})_x(t; x(t)))\,x(t)(r + \bar{\pi}_{(1,1)}(t)^T(b - r\underline{1}))dt\\
&\quad - (v_{00})_x(t; x(t)))\,x(t)\frac{\left(RL^{\bar{\pi}_{(1,1)}}\right)'(t)}{1 - RL^{\bar{\pi}_{(1,1)}}(t)}dt\\
&\quad + \frac{1}{2}(v_{00})_{xx}(t; x(t)))\,x(t)^2\bar{\pi}_{(1,1)}(t)^T\sigma\sigma^T\bar{\pi}_{(1,1)}(t)dt\\
&\quad + (v_{00})_x(t; x(t)))\,x(t)\bar{\pi}_{(1,1)}(t)^T\sigma dW(t),
\end{aligned}$$

for $x(t) = \left(1 - RL^{\bar{\pi}_{(1,1)}}(t)\right)X^{\bar{\pi}_{(1,1)}}(t-)$ and $t \in [0, T]$. Using the representation of $v_{00}(t; x)$ from Proposition 2 and equation (1.9) we get

$$dv_{00}(t; x(t)) = v_{00}(t; x(t))\,(1 - \gamma)\,\bar{\pi}_{(1,1)}(t)^T\sigma dW(t).$$

The solution of this SDE is given by

$$\begin{aligned}
&v_{00}\left(t; (1 - RL^{\bar{\pi}_{(1,1)}}(t))X^{\bar{\pi}_{(1,1)}}(t-)\right)\\
=&v_{00}\left(0; (1 - RL^{\bar{\pi}_{(1,1)}}(0))X^{\bar{\pi}_{(1,1)}}(0)\right)\cdot\exp\Big((1-\gamma)\int_0^t \bar{\pi}_{(1,1)}(s)^T\sigma dW(s)\\
&- \frac{1}{2}(1-\gamma)^2\int_0^t \bar{\pi}_{(1,1)}(s)^T\sigma\sigma^T\bar{\pi}_{(1,1)}(s)ds\Big),
\end{aligned}$$

for $t \in [0, T]$. Together with the integrability assumptions of an admissible portfolio process and Novikov's condition we get that

$$v_{00}\left(t; (1 - RL^{\bar{\pi}_{(1,1)}}(t))X^{\bar{\pi}_{(1,1)}}(t-)\right)$$

is a martingale on $[0, T]$. The boundary condition $RL^{\pi_{(1,1)}}(T) = 0$ extends the martingale property for $M^{\bar{\pi}}$ to $[0, T] \cup \{\infty\}$. By Doob's optional sampling theorem the stopped process is also a martingale, and so we get the minimum constant property on $[0, T] \cup \{\infty\}$.

□

Before we prove the optimality of such a minimum constant portfolio process, we introduce two helpful concepts. First, as analogue to the indifference-optimality we formulate

the minimum constant-optimality principle.

Proposition 6 (Minimum constant-optimality principle)**.** *If $\bar{\pi} \in \mathbb{A}(x)$ is a minimum constant portfolio process and for all $\pi \in \mathbb{A}(x)$ we have*

$$M^{\bar{\pi}}(\tilde{\xi}) \geq M^{\pi}(\tilde{\xi}) \text{ for at least one } \tilde{\xi} \in \Theta,$$

then $\bar{\pi}$ solves the worst-case portfolio problem under stresses (1.5)*.*

Proof. The definition of a minimum constant portfolio process implies:

$$\inf_{\xi \in \Theta} \mathbb{E}\left(M^{\bar{\pi}}(\xi)\right) = \mathbb{E}\left(M^{\bar{\pi}}(\tilde{\xi})\right) \geq \mathbb{E}\left(M^{\pi}(\tilde{\xi})\right) \geq \inf_{\xi \in \Theta} \mathbb{E}\left(M^{\pi}(\xi)\right),$$

and together with Lemma 2 the claim is proven. □

Next, we construct a minimum indifference frontier for the two-stress worst-case problem so that all portfolio processes which exceed this frontier cannot be optimal for the worst-case problem.

Proposition 7 (Minimum indifference frontier)**.** *Let $\pi \in \mathbb{A}(x)$, and let $\bar{\pi} \in \mathbb{A}(x)$ be a minimum constant portfolio process. Set*

$$\tau_1 := \inf\left\{t : RL_{12}^{\pi_{(1,1)}}(t) > RL^{\bar{\pi}_{(1,1)}}(t)\right\},$$
$$\tau_2 := \inf\left\{t : RL_{21}^{\pi_{(1,1)}}(t) > RL^{\bar{\pi}_{(1,1)}}(t)\right\},$$

and define

$${}_1\tilde{\pi}_{(1,1)}(t) := \pi_{(1,1)}(t) \text{ if } t < \tau_1 \text{ and } {}_1\tilde{\pi}_{(1,1)}(t) := \bar{\pi}_{(1,1)}(t) \text{ if } t \geq \tau_1,$$
$${}_2\tilde{\pi}_{(1,1)}(t) := \pi_{(1,1)}(t) \text{ if } t < \tau_2 \text{ and } {}_2\tilde{\pi}_{(1,1)}(t) := \bar{\pi}_{(1,1)}(t) \text{ if } t \geq \tau_2.$$

Then, ${}_1\tilde{\pi}$, ${}_2\tilde{\pi} \in \mathbb{A}(x)$ and the worst-case performances in (1.5) *attained by ${}_1\tilde{\pi}$ and ${}_2\tilde{\pi}$ are at least as big as that achieved by π.*

Proof. We give the proof for ${}_1\tilde{\pi}$ and τ_1. The proof for ${}_2\tilde{\pi}$ follows analogously. First note that ${}_1\tilde{\pi}_{(1,1)}$ is right-continuous by construction. Further, τ_1 is predictable and therefore also ${}_1\tilde{\pi}_{(1,1)}$ is predictable. Thus, we have ${}_1\tilde{\pi} \in \mathbb{A}(x)$.

Now it is enough to look at $\tau_1 < \infty$ as otherwise we directly get $M^{\pi}(t) = M^{{}_1\tilde{\pi}}(t)$ for all $t \in [0, T] \cup \{\infty\}$. By definition, we have ${}_1\tilde{\pi}_{(1,1)}(t) = \pi_{(1,1)}(t)$ if $0 \leq t < \tau_1$, and as $\bar{\pi}$ is minimum constant, we get that $\tilde{\pi}$ is minimum constant for $[\tau_1, T] \cup \{\infty\}$-valued stopping times. Furthermore, the construction of ${}_1\tilde{\pi}$ and τ_1 leads to

$$M^{\pi}(t) = M^{{}_1\tilde{\pi}}(t) \text{ for } t \in [0, \tau_1)$$

and for $t = \tau_1$:

$$\mathbb{E}\left(M^{\pi}(\tau_1)\right) = \mathbb{E}\left(v_{00}\left(\tau_1; (1 - RL^{\pi_{(1,1)}}(\tau_1))X^{\pi_{(1,1)}}(\tau_1-)\right)\right)$$

$$
\begin{aligned}
=&\mathbb{E}\Big(\min\Big\{v_{00}\left(\tau_1;(1-RL_{12}^{\pi_{(1,1)}}(\tau_1))X^{\pi_{(1,1)}}(\tau_1-)\right),\\
&\qquad v_{00}\left(\tau_1;(1-RL_{21}^{\pi_{(1,1)}}(\tau_1))X^{\pi_{(1,1)}}(\tau_1-)\right)\Big\}\Big)\\
\leq&\mathbb{E}\Big(\min\Big\{v_{00}\left(\tau_1;(1-RL^{\bar{\pi}_{(1,1)}}(\tau_1))X^{\pi_{(1,1)}}(\tau_1-)\right),\\
&\qquad v_{00}\left(\tau_1;(1-RL_{21}^{\pi_{(1,1)}}(\tau_1))X^{\pi_{(1,1)}}(\tau_1-)\right)\Big\}\Big)\\
\leq&\mathbb{E}\left(v_{00}\left(\tau_1;(1-RL^{\bar{\pi}_{(1,1)}}(\tau_1))X^{\pi_{(1,1)}}(\tau_1-)\right)\right)=\mathbb{E}\left(M^{1^{\bar{\pi}}}(\tau_1)\right).
\end{aligned}
$$

The second equality follows from the proof of Lemma 2. The first inequality holds, since $v_{00}(t;x)$ is monotonically increasing in x and by the right-continuity properties of admissible portfolio processes. Let Θ^{τ_1} denote the set of $[0,\tau_1]$-valued stopping times. Then, we obtain

$$
\inf_{\xi\in\Theta}\mathbb{E}\left(M^{1^{\bar{\pi}}}(\xi)\right)=\inf_{\xi\in\Theta^{\tau_1}}\mathbb{E}\left(M^{1^{\bar{\pi}}}(\xi)\right)\geq\inf_{\xi\in\Theta^{\tau_1}}\mathbb{E}\left(M^{\pi}(\xi)\geq\inf_{\xi\in\Theta}\mathbb{E}\left(M^{\pi}(\xi)\right),\right.
$$

and the claim follows with Lemma 2.

□

Remark 6 (Minimum indifference frontier)**.** *We briefly discuss the meaning of the two previous Propositions and the concept of the indifference frontier. For a minimum constant portfolio process $\bar{\pi}$ we define the set $ID(\bar{\pi})$ of all admissible portfolio processes that respect the minimum indifference frontier of $\bar{\pi}$ as follows:*

$$
ID(\bar{\pi}):=\{\pi\in\mathbb{A}(x)|RL^{\pi_{(1,1)}}(t)\leq RL^{\bar{\pi}_{(1,1)}}(t)\text{ for all }t\in[0,T]\}.
$$

From Proposition 7 we conclude that we can limit the search for the optimal strategy to $ID(\bar{\pi})$ and from Proposition 6 we know that it is enough to show that the minimum constant portfolio process $\bar{\pi}$ is the best strategy in $ID(\bar{\pi})$ for the no-stress scenario.

In Theorem 2 we state the existence of an optimal minimum constant portfolio process. Its proof is based on the following two lemmata.

Lemma 3 (Left-hand derivative of RL^{π^*})**.** *Let π^* be the optimal post-stress portfolio process. Then the left-hand derivative of the relative worst-case loss RL^{π^*} exists, and we obtain:*

$$
\lim_{\substack{h\to 0\\ h<0}}\frac{RL^{\pi^*}(t+h)-RL^{\pi^*}(t)}{h}<0,
$$

for all $t\in(0,T]$, if $\max\left\{(\pi^*)^T z^{(1)},(\pi^*)^T z^{(2)}\right\}<1.$

Proof. We know that RL^{π^*} is the maximum of the two functions $RL_{12}^{\pi^*}$ and $RL_{21}^{\pi^*}$ with derivatives given by (see Lemma 1)

$$
\begin{aligned}
\left(RL_{12}^{\pi^*}\right)'(t)&=\left(1-(\pi^*)^T z^{(1)}\right)\left(\bar{\pi}_{(0,1)}^T z^{(2)}\right)'(t)<0,\\
\left(RL_{21}^{\pi^*}\right)'(t)&=\left(1-(\pi^*)^T z^{(2)}\right)\left(\bar{\pi}_{(1,0)}^T z^{(1)}\right)'(t)<0,
\end{aligned}
$$

$$\left(RL_{12}^{\pi^*}\right)''(t) = \left(1-(\pi^*)^T z^{(1)}\right)\left(\bar{\pi}_{(0,1)}^T z^{(2)}\right)''(t) < 0,$$
$$\left(RL_{21}^{\pi^*}\right)''(t) = \left(1-(\pi^*)^T z^{(2)}\right)\left(\bar{\pi}_{(1,0)}^T z^{(1)}\right)''(t) < 0,$$

for $t \in [0,T]$. If $RL_{12}^{\pi^*}(t) = RL_{21}^{\pi^*}(t)$ for all $t \in [0,T]$, the claim follows directly by the first derivatives. Otherwise we should note that RL^{π^*} is the maximum of two strictly concave functions. By Lemma 2.11. in [15], the two functions have a maximum of two intersections. Thus RL^{π^*} is differentiable from the left and together with the above first derivatives of $RL_{12}^{\pi^*}$ and $RL_{21}^{\pi^*}$, we obtain for the left-hand limit of the difference quotient:

$$\lim_{\substack{h\to 0\\ h<0}} \frac{RL^{\pi^*}(t+h) - RL^{\pi^*}(t)}{h} < 0,$$

for all $t \in (0,T]$. □

Lemma 4 (Bounded relative worst-case loss)**.** *Let $\bar{\pi} \in \mathbb{A}(x)$ be a minimum constant portfolio process defined as in Proposition 5 and π^* be the optimal post-stress portfolio process. Then, for all $t \in [0,T]$ we get*

$$RL^{\bar{\pi}_{(1,1)}}(t) \le RL^{\pi^*}(t), \text{ if } \max\left\{(\pi^*)^T z^{(1)}, (\pi^*)^T z^{(2)}\right\} < 1.$$

Proof. First we revert time and define $F^{\bar{\pi}_{(1,1)}}(t) = RL^{\bar{\pi}_{(1,1)}}(T-t)$. Then we obtain the forward ODE of $RL^{\bar{\pi}_{(1,1)}}$ for $t \in [0,T]$:

$$\left(F^{\bar{\pi}_{(1,1)}}\right)'(t) = -(1-F^{\bar{\pi}_{(1,1)}}(t))\left(\max_{\substack{\pi\in\mathbb{R}^2\\ RL^{\pi}(T-t)=F^{\bar{\pi}_{(1,1)}}(t)}} \phi_\gamma(\pi) - \phi_\gamma(\pi^*)\right),$$

with $F^{\bar{\pi}_{(1,1)}}(0) = 0$. Further we define $F^{\pi^*}(t) = RL^{\pi^*}(T-t)$ for $t \in [0,T]$ and with Lemma 3 we obtain

$$\lim_{\substack{h\to 0\\ h>0}} \frac{F^{\pi^*}(t+h) - F^{\pi^*}(t)}{h} > 0, \tag{1.10}$$

for all $t \in [0,T)$. Now we have to show that $F^{\pi^*}(t) \ge F^{\bar{\pi}_{(1,1)}}(t)$ for $t \in [0,T]$. Assumption (1.2) leads to

$$F^{\bar{\pi}_{(1,1)}}(0) = 0 < \max\left\{(\pi^*)^T z^{(1)}, (\pi^*)^T z^{(2)}\right\} = F^{\pi^*}(0).$$

Let us assume that there exists a $\bar{t} \in (0,T]$ with $F^{\pi^*}(\bar{t}) < F^{\bar{\pi}_{(1,1)}}(\bar{t})$. Note that $\bar{\pi}_{(1,1)}$ and π^* are continuous and so are $F^{\bar{\pi}_{(1,1)}}$ and F^{π^*}. Therefore, there is at least one point of intersection $t_0 \in (0,\bar{t})$ and an $\epsilon > 0$, such that $F^{\pi^*}(t_0) = F^{\bar{\pi}_{(1,1)}}(t_0)$ and

$$F^{\pi^*}(t_0+h) - F^{\bar{\pi}_{(1,1)}}(t_0+h) < 0 \quad \text{for all } h \in (0,\epsilon]$$

This leads to

$$\lim_{\substack{h\to 0\\ h>0}} \frac{\left(F^{\pi^*}(t_0+h) - F^{\bar{\pi}_{(1,1)}}(t_0+h)\right) - \left(F^{\pi^*}(t_0) - F^{\bar{\pi}_{(1,1)}}(t_0)\right)}{h} \leq 0.$$

Then, the differentiability of $F^{\bar{\pi}_{(1,1)}}$ implies

$$\begin{aligned}\lim_{\substack{h\to 0\\ h>0}} \frac{F^{\pi^*}(t_0+h) - F^{\pi^*}(t_0)}{h} &\leq \lim_{\substack{h\to 0\\ h>0}} \frac{F^{\bar{\pi}_{(1,1)}}(t_0+h) - F^{\bar{\pi}_{(1,1)}}(t_0)}{h}\\ &= \left(F^{\bar{\pi}_{(1,1)}}\right)'(t_0).\end{aligned}$$

However, this is a contradiction, as on one hand inequality (1.10) is valid and on the other hand, π^* is the unconstrained maximizer of ϕ_γ which yields

$$\begin{aligned}\left(F^{\bar{\pi}_{(1,1)}}\right)'(t_0) &= -\left(1 - F^{\bar{\pi}_{(1,1)}}(t_0)\right)\left(\max_{\substack{\pi\in\mathbb{R}^2\\ RL^{\pi}(T-t_0)=F^{\pi^*}(t_0)}} \phi_\gamma(\pi) - \phi_\gamma(\pi^*)\right)\\ &= -\left(1 - F^{\bar{\pi}_{(1,1)}}(t_0)\right)\left(\phi_\gamma(\pi^*) - \phi_\gamma(\pi^*)\right) = 0.\end{aligned}$$

So $F^{\bar{\pi}_{(1,1)}}(t) \leq F^{\pi^*}(t)$ and therefore $RL^{\bar{\pi}_{(1,1)}}(t) \leq RL^{\pi^*}(t)$ for all $t \in [0,T]$. □

Theorem 2 (Solution of the multi-stress worst-case problem)**.** *Let $\bar{\pi} \in \mathbb{A}(x)$ be a minimum constant portfolio process defined as in Proposition 5. Then, $\bar{\pi}$ is a worst-case optimal portfolio process of problem (1.5). For the corresponding value function $v_{11}(t;x)$, we get at time $t = 0$:*

$$v_{11}(0;x) = v_{00}\left(0; (1 - RL^{\bar{\pi}_{(1,1)}}(0))x\right).$$

Proof. Let $\bar{\pi} \in \mathbb{A}(x)$ be a minimum constant portfolio process defined as in Proposition 5. In step (i) of this proof we show that $RL^{\bar{\pi}_{(1,1)}}(t) \leq RL^{\pi^*}(t)$ for all $t \in [0,T]$. Then in step (ii) we prove that

$$\bar{\pi} \in \operatorname*{arg\,max}_{\pi\in ID(\bar{\pi})} \mathbb{E}\left(M^{\pi}(\infty)\right) = \operatorname*{arg\,max}_{\pi\in ID(\bar{\pi})} \mathbb{E}\left(U\left(X^{\pi_{(1,1)}}(T)\right)\right), \tag{1.11}$$

where $ID(\bar{\pi})$ is defined by the minimum indifference frontier as

$$ID(\bar{\pi}) := \{\pi \in \mathbb{A}(x) | RL^{\pi_{(1,1)}}(t) \leq RL^{\bar{\pi}_{(1,1)}}(t) \text{ for all } t \in [0,T]\}.$$

Hence, $\bar{\pi}$ solves the portfolio optimization problem in the stress-free world in the set of all admissible portfolio processes respecting its indifference frontier. Together with Propositions 6 and 7 this directly yields optimality of $\bar{\pi}$.

(i) We first want to show that $RL^{\bar{\pi}_{(1,1)}}(t) \leq RL^{\pi^*}(t)$ for all $t \in [0,T]$. For this, we have to distinguish two cases.

(a) If $\max\left\{(\pi^*)^T z^{(1)}, (\pi^*)^T z^{(2)}\right\} \geq 1$, we get $RL^{\pi^*}(t) \geq 1$ for all $t \in [0,T]$. But by Proposition 5 we know $RL^{\bar{\pi}_{(1,1)}}(t) < 1$ for all $t \in [0,T]$ and therefore $RL^{\bar{\pi}_{(1,1)}}(t) < RL^{\pi^*}(t)$ for all $t \in [0,T]$.

(b) In the case of $\max\left\{(\pi^*)^T z^{(1)}, (\pi^*)^T z^{(2)}\right\} < 1$, Lemma 4 gives us the inequality .

(ii) Before we prove the optimality of $\bar{\pi}$, we show that $\bar{\pi}$ solves the constrained optimization problem

$$\max_{\pi \in ID(\bar{\pi})} \phi_\gamma(\pi_{(1,1)}(t)) \quad \text{for all } t \in [0,T]. \tag{1.12}$$

We maximize the quadratic function ϕ_γ under linear constraints. From part (i) we know that the unique optimizer π^* of ϕ_γ in the unconstrained case does not fulfill the constraints strictly, i.e. $RL^{\pi^*}(t) \not< RL^{\bar{\pi}_{(1,1)}}(t) < 1$ for any $t \in [0,T]$. Thus, the optimizer π^c of the constrained optimization problem (1.12) is a boundary point of $ID(\bar{\pi})$. That means $RL^{\pi^c_{(1,1)}}(t) = RL^{\bar{\pi}_{(1,1)}}(t)$ for all $t \in [0,T]$. The definition of $\bar{\pi}$ in Proposition 5 yields

$$\bar{\pi}_{(1,1)}(t) \in \underset{\substack{\pi \in \mathbb{R}^2 \\ RL^{\pi}(t) = RL^{\bar{\pi}_{(1,1)}}(t)}}{\arg\max} \phi_\gamma(\pi) \quad \text{for all } t \in [0,T],$$

and thus $\bar{\pi}$ is a solution of (1.12).

Now we show (1.11) and hence prove the optimality of $\bar{\pi}$. For this, we apply the terminal utility decomposition of Proposition 1, i.e. for $\pi \in \mathbb{A}(x)$ we have in the no-stress scenario:

$$U(X^{\pi_{(1,1)}}(T)) = U(x) \exp\left((1-\gamma)\int_0^T \phi_\gamma(\pi_{(1,1)}(t))dt\right) Y_T^\gamma(\pi_{(1,1)}),$$

where the process $Y^\gamma(\pi_{(1,1)}) = \left(Y_t^\gamma(\pi_{(1,1)})\right)_{t\in[0,T]}$ is a positive $\mathcal{F}_t$-martingale with $Y_0^\gamma(\pi_{(1,1)}) = 1$. Since $\bar{\pi}$ solves (1.12), we get for $\pi \in ID(\bar{\pi})$

$$\begin{aligned}
&\mathbb{E}\left(M^\pi(\infty)\right) = \mathbb{E}\left(U(X^{\pi_{(1,1)}}(T))\right) \\
&= \mathbb{E}\left(U(x)\exp\left((1-\gamma)\int_0^T \phi_\gamma(\pi_{(1,1)}(t))dt\right) Y_T^\gamma(\pi_{(1,1)})\right) \\
&\leq \mathbb{E}\left(U(x)\exp\left((1-\gamma)\int_0^T \phi_\gamma(\bar{\pi}_{(1,1)}(t))dt\right) Y_T^\gamma(\pi_{(1,1)})\right) \\
&= \mathbb{E}\left(U(x)\exp\left((1-\gamma)\int_0^T \phi_\gamma(\bar{\pi}_{(1,1)}(t))dt\right) Y_T^\gamma(\bar{\pi}_{(1,1)})\right) = \mathbb{E}\left(M^{\bar{\pi}}(\infty)\right).
\end{aligned}$$

Combining this with the arguments given at the beginning of this proof, we have shown that a minimum constant process exists which is a worst-case optimal portfolio process of problem (1.5).

Since $\bar{\pi}$ is a minimum constant portfolio process by construction, we get, together with Lemma 2:

$$\begin{aligned} v_{11}(0;x) &= \sup_{\pi\in\mathbb{A}(x)} \inf_{\xi\in\Theta} \mathbb{E}\left(M^{\pi}(\xi)\right) = \inf_{\xi\in\Theta} \mathbb{E}\left(M^{\bar{\pi}}(\xi)\right) = \mathbb{E}\left(M^{\bar{\pi}}(0)\right) \\ &= \mathbb{E}\left(v_{00}\left(0;(1-RL^{\bar{\pi}_{(1,1)}}(0))x\right)\right) = v_{00}\left(0;(1-RL^{\bar{\pi}_{(1,1)}}(0))x\right). \end{aligned}$$

□

We would like to briefly point out that the previous concepts can be extended to the worst-case portfolio problem with more than two stresses and also to the case of a logarithmic utility function.

Remark 7 (Extension to more than two stresses)**.** *The recursive determination of the optimal strategies, depending on how much stresses can still occur, enables the extension to more than two stresses. The definitions from section 1.5, as well as the minimum indifference frontier and the corresponding minimum indifference-optimality principle, can be adapted for the general n stress case. Since we can define the relative worst-case loss completely analogous to Definition 4, the existence result in Proposition 5 as well as the proof of optimality in Theorem 2 can be used for the n stress case.*

Remark 8 (Logarithmic utility)**.** *The worst-case portfolio problem under stresses can be solved in the case of a logarithmic utility function, i.e. for $U(x) = \ln(x)$, with completely analogous proofs. Note that we obtain, with the help of Itô's formula, a decomposition of the terminal utility as in Proposition 1, which is even additive. Based on this, the classical Merton problem with random initial time can be solved. The one-stress worst-case problem in the multi asset setting is also treated in [29]. The proofs of section 1.5 are quite analogous for the log-utility case, noting that the newly introduced concepts of the relative worst-case loss and minimum constant portfolio processes are defined independently of the concrete form of utility function.*

1.6 Sub-scenarios and a verification theorem

Proposition 2 and Theorem 1 allow us to construct the optimal strategy in the post-stress situation and in the one-stress worst-case setting. In Theorem 2 we have shown that there exists a minimum constant portfolio process in the multi-stress case, which is optimal if both stresses can still occur. However, the proof of existence of this strategy, given in Proposition 5, is not constructive. Therefore, the goal is to derive a verification theorem, on the basis of which we then create an algorithmic framework, that enables us to construct the optimal strategy in a multi-stress worst-case scenario.

Sub-scenarios. We first define and determine indifference strategies for the sub-scenarios, when the order of occurrence of the stresses is known.

Definition 6. *We consider a worst-case portfolio problem with two stress scenarios $(z^{(1)}, \xi_1)$ and $(z^{(2)}, \xi_2)$.*

1. *If there is a $\pi \in \mathbb{A}(x)$, such that*

$$v_{00}\left(t; (1 - RL_{12}^{\pi_{(1,1)}}(t))X^{\pi_{(1,1)}}(t-)\right),\ t \in [0,T],$$

is a martingale on $[0,T]$ with $RL_{12}^{\pi_{(1,1)}}(T) = 0$, we call π a sub-scenario indifference strategy for the sub-scenario $\xi_1 < \xi_2$.

2. *If there is a $\pi \in \mathbb{A}(x)$, such that*

$$v_{00}\left(t; (1 - RL_{21}^{\pi_{(1,1)}}(t))X^{\pi_{(1,1)}}(t-)\right),\ t \in [0,T],$$

is a martingale on $[0,T]$ with $RL_{21}^{\pi_{(1,1)}}(T) = 0$, we call π a sub-scenario indifference strategy for the sub-scenario $\xi_2 < \xi_1$.

The required martingale property enables us to compute such sub-scenario indifference strategies.

Proposition 8 (Sub-scenario indifference strategy)**.** *For $\pi \in \mathbb{A}(x)$, define $f^{\pi_{(1,1)}}$ by*

$$f^{\pi_{(1,1)}}(t) = \pi_{(1,1)}(t)^T(b - r\underline{1}) - \frac{1}{2}\gamma\left(\pi_{(1,1)}(t)^T\sigma\sigma^T\pi_{(1,1)}(t) + (\pi^*)^T\sigma\sigma^T\pi^*\right).$$

1. *Let $\pi \in \mathbb{A}(x)$ and $\pi_{(1,1)}$ be a deterministic solution to the following one-dimensional coupled, non-linear ordinary differential equation*

$$\begin{aligned}&\pi_{(1,1)}{}'(t)^T z^{(1)}\\ =&(1 - \pi_{(1,1)}(t)^T z^{(1)})\left(f^{\pi_{(1,1)}}(t) - \frac{\bar{\pi}'_{(0,1)}(t)^T z^{(2)}}{1 - \bar{\pi}_{(0,1)}(t)^T z^{(2)}}\right),\end{aligned} \tag{1.13}$$

for $t \in [0,T]$ and with $\pi_{(1,1)}(T)^T z^{(1)} = 0$. Then π is a sub-scenario indifference strategy with respect to the sub-scenario $\xi_1 < \xi_2$.

2. *Let $\pi \in \mathbb{A}(x)$ and $\pi_{(1,1)}$ be a deterministic solution to the following one-dimensional coupled, non-linear ordinary differential equation*

$$\begin{aligned}&\pi_{(1,1)}{}'(t)^T z^{(2)}\\ =&(1 - \pi_{(1,1)}(t)^T z^{(2)})\left(f^{\pi_{(1,1)}}(t) - \frac{\bar{\pi}'_{(1,0)}(t)^T z^{(1)}}{1 - \bar{\pi}_{(1,0)}(t)^T z^{(1)}}\right),\end{aligned} \tag{1.14}$$

for $t \in [0,T]$ and with $\pi_{(1,1)}(T)^T z^{(2)} = 0$. Then π is a sub-scenario indifference strategy with respect to the sub-scenario $\xi_2 < \xi_1$.

Proof. We give the proof for $\pi \in \mathbb{A}(x)$, fulfilling equation (1.13). The other case follows analogously. With Definition 4 we get for $t \in [0,T]$ that

$$v_{00}\left(t; (1 - RL_{12}^{\pi_{(1,1)}}(t))X^{\pi_{(1,1)}}(t-)\right)$$

$$=v_{00}\left(t;\left(1-\pi_{(1,1)}(t)^T z^{(1)}\right)\left(1-\bar{\pi}_{(0,1)}(t)^T z^{(2)}\right)X^{\pi_{(1,1)}}(t-)\right).$$

Now we use the same idea as in the proof of Proposition 5. To prove the martingale property we use Itô's rule, the representation of $v_{00}(t;x)$ from Proposition 2, equation (1.13) and get

$$dv_{00}\left(t;x(t)\right)=v_{00}\left(t;x(t)\right)(1-\gamma)\,\pi_{(1,1)}(t)^T\sigma dW(t),$$

for $x(t)=(1-\pi_{(1,1)}(t)^T z^{(1)})(1-\bar{\pi}_{(0,1)}(t)^T z^{(2)})X^{\pi_{(1,1)}}(t-)$. The solution of this SDE is given by

$$\begin{aligned}v_{00}\left(t;x(t)\right)=v_{00}\left(0;x(0)\right)\cdot\exp\Big((1-\gamma)\int_0^t \pi_{(1,1)}(s)^T\sigma dW(s)\\ -\frac{1}{2}(1-\gamma)^2\int_0^t \pi_{(1,1)}(s)^T\sigma\sigma^T\pi_{(1,1)}(s)ds\Big),\end{aligned}$$

where the second factor is a martingale by Novikov's condition and the integrability assumptions of an admissible portfolio process. Together with the condition $\pi(T)^T z^{(1)}=0$, we get the sub-scenario indifference property. □

The following Corollary specifies the conditions under which a portfolio process that fulfills at least one of the two ODEs above is minimum constant.

Corollary 1 (Minimum constant sub-scenario indifference strategy)**.** *Let $\pi\in\mathbb{A}(x)$ be an admissible portfolio process and $t_1,t_2\in[0,T]$ with $t_1\leq t_2$. Further, let $\pi_{(1,1)}$ fulfill one of the following properties:*

(i) $\pi_{(1,1)}$ is a solution of ODE (1.13) on $[t_1,t_2]$ and for all $t\in[t_1,t_2]$ it holds $RL^{\pi_{(1,1)}}(t)=RL_{12}^{\pi_{(1,1)}}(t)$,

(ii) $\pi_{(1,1)}$ is a solution of ODE (1.14) on $[t_1,t_2]$ and for all $t\in[t_1,t_2]$ it holds $RL^{\pi_{(1,1)}}(t)=RL_{21}^{\pi_{(1,1)}}(t)$.

Then we have $\mathbb{E}\left(M^\pi(\tau_1)\right)=\mathbb{E}\left(M^\pi(\tau_2)\right)$ for all $[t_1,t_2]$-valued stopping times τ_1 and τ_2.

Proof. If property (i) is fulfilled, the relative worst-case loss of π is $RL^{\pi_{(1,1)}}(t)=RL_{12}^{\pi_{(1,1)}}(t)$ and with the definition of the minimum process M^π it follows that

$$\begin{aligned}M^\pi(t)&=v_{00}\left(t;(1-RL^{\pi_{(1,1)}}(t))X^{\pi_{(1,1)}}(t-)\right)\\&=v_{00}\left(t;(1-RL_{12}^{\pi_{(1,1)}}(t))X^{\pi_{(1,1)}}(t-)\right)\quad\text{for } t\in[t_1,t_2].\end{aligned}$$

By construction, π is a sub-scenario indifference strategy for the sub-scenario $\xi_1<\xi_2$ on $[t_1,t_2]$, i.e. with the same arguments as in the proof of Proposition 8, we get that the process $v_{00}\left(t;(1-RL_{12}^{\pi}(t))X^{\pi}(t-)\right)$ is a martingale on $[t_1,t_2]$. The result follows with Doob's optional sampling theorem. For property (ii) the statement is proven analogously. □

Verification theorem for the multi-stress worst-case problem. The above considerations on sub-scenario indifference strategies allow us to formulate the verification theorem. Before

we start, however, we would like to look at a short example, which illustrates the general procedure in a simplified way.

Example 1. *We consider the most basic setting of two independent stocks, two possible stresses each affecting only one stock. Therefore we have* $z^{(1)} = (k,0)^T$, $z^{(2)} = (0,k)^T$ *for* $k \geq 0$ *and* $\sigma_{12} = \sigma_{21} = 0$. *With the notation* $\bar{\pi}_{(1,0)} = (\bar{\pi}_{(1,0),1}, \bar{\pi}_{(1,0),2})^T$ *and* $\bar{\pi}_{(0,1)} = (\bar{\pi}_{(0,1),1}, \bar{\pi}_{(0,1),2})^T$ *one can show that*

$$\begin{aligned}
\bar{\pi}'_{(1,0),1}(t)k &= \Big(1 - \bar{\pi}_{(1,0),1}(t)k\Big)\Big(\bar{\pi}_{(1,0),1}(t)\,(b_1 - r) \\
&\quad - \frac{1}{2}\gamma\Big(\bar{\pi}_{(1,0),1}(t)^2\sigma_{11}^2 + (\pi_1^*)^2\sigma_{11}^2\Big)\Big), \\
\bar{\pi}'_{(0,1),2}(t)k &= \Big(1 - \bar{\pi}_{(0,1),2}(t)k\Big)\Big(\bar{\pi}_{(0,1),2}(t)\,(b_2 - r) \\
&\quad - \frac{1}{2}\gamma\Big(\bar{\pi}_{(0,1),2}(t)^2\sigma_{22}^2 + (\pi_2^*)^2\sigma_{22}^2\Big)\Big),
\end{aligned}$$

with $\bar{\pi}_{(1,0)}(T)^T z^{(1)} = \bar{\pi}_{(0,1)}(T)^T z^{(2)} = 0$ *and*

$$\bar{\pi}_{(1,0),2}(t) = \pi_2^* \text{ and } \bar{\pi}_{(0,1),1}(t) = \pi_1^* \text{ for } t \in [0,T].$$

Using Corollary 1, it is easy to verify that $\bar{\pi}$ *with* $\bar{\pi}_{(1,1)} = (\bar{\pi}_{(1,0),1}, \bar{\pi}_{(0,1),2})$ *is a minimum constant strategy as it solves the ODEs* (1.13) *and* (1.14). *Due to the terminal conditions and the martingale properties we get, for all* $t \in [0,T]$,

$$v_{00}\left(t; (1 - RL_{12}^{\bar{\pi}_{(1,1)}}(t))X^{\bar{\pi}_{(1,1)}}(t-)\right) = v_{00}\left(t; (1 - RL_{21}^{\bar{\pi}_{(1,1)}}(t))X^{\bar{\pi}_{(1,1)}}(t-)\right),$$

and we also have $RL^{\bar{\pi}_{(1,1)}}(t) = RL_{12}^{\bar{\pi}_{(1,1)}}(t) = RL_{21}^{\bar{\pi}_{(1,1)}}(t)$, $t \in [0,T]$. *To show that* $\bar{\pi}$ *solves the worst-case portfolio problem under stresses, we know, by the proof of Theorem 2, that it is enough to show that* $\bar{\pi}$ *solves the constrained portfolio optimization problem* (1.12). *We maximize*

$$\phi_\gamma(\pi) = r + \pi^T(b - r\underline{1}) - \frac{1}{2}\gamma\pi^T\sigma\sigma^T\pi \tag{1.15}$$

point wise in π. *The maximizer in the unconstrained case is* $\pi^* = (\pi_1^*, \pi_2^*)$ *and by Lemma* (1) *we have*

$$\pi_{(1,0),1}(t) < \pi_1^* \text{ and } \pi_{(0,1),2}(t) < \pi_2^*. \tag{1.16}$$

Furthermore, the set of all strategies respecting the indifference frontier is

$$\begin{aligned}
ID(\bar{\pi}) &= \{\pi \in \mathbb{A}(x) | RL^{\pi_{(1,1)}}(t) \leq RL^{\bar{\pi}_{(1,1)}}(t) \text{ for all } t \in [0,T]\} \\
&= \{\pi \in \mathbb{A}(x) | \pi_{(1,1)}(t)^T z^{(1)} \leq \bar{\pi}_{(1,1)}(t)^T z^{(1)} \wedge \pi_{(1,1)}(t)^T z^{(2)} \leq \bar{\pi}_{(1,1)}(t)^T z^{(2)} \\
&\quad \text{for all } t \in [0,T]\} \\
&= \{\pi \in \mathbb{A}(x) | \pi_{(1,1),1}(t) \leq \bar{\pi}_{(1,1),1}(t) \wedge \pi_{(1,1),2}(t) \leq \bar{\pi}_{(1,1),2}(t) \text{ for all } t \in [0,T]\},
\end{aligned}$$

where we use again the notation $\pi_{(1,1)} = (\pi_{(1,1),1}, \pi_{(1,1),2})$ *for* $\pi \in \mathbb{A}(x)$. *Together with the*

quadratic form of (1.15) *and the inequalities* (1.16) *we get the optimality of* $\bar{\pi}$ *in the constrained optimization problem* (1.12) *and thus the optimality in the worst-case portfolio problem under stresses.*

We are now able to determine a verification theorem.

Theorem 3 (Verification Theorem)**.** *Consider the worst-case portfolio optimization problem* (1.5) *with stress scenarios* $\left(z^{(1)}, \xi_1\right)$ *and* $\left(z^{(2)}, \xi_2\right)$*. Let* $\bar{\pi} \in \mathbb{A}(x)$ *be an admissible portfolio process and* π^* *be the optimal portfolio process in the post-stress problem, defined in Proposition* 2*. Assume that there exists a* $k \geq 0$ *and* $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = T$*, such that on every interval* $(t_s, t_{s+1}]$*, with* $s = 0, \ldots, k$*, the portfolio process* $\bar{\pi}_{(1,1)}$ *fulfills one of the following properties:*

(i) For a $j \in \{1, 2\}$*, the portfolio process* $\bar{\pi}_{(1,1)}(t)$ *is given by*

$$\bar{\pi}_{(1,1)}(t) = \pi^*(t) - \frac{\lambda_j}{\gamma}\left(\sigma\sigma^T\right)^{-1} z^{(j)}, \tag{1.17}$$

where λ_j *is given by equation* (1.20) *below, with* $\lambda_j \geq 0$*, and* $N_j(t)$ *is the unique solution of ODE* (1.21) *with terminal condition* $N_j(t_{s+1}) = \bar{\pi}_{(1,1)}(t_{s+1})^T z^{(j)}$*. Further, we have for all* $t \in (t_s, t_{s+1}]$*, that* $RL^{\bar{\pi}_{(1,1)}}(t) = RL_{12}^{\bar{\pi}_{(1,1)}}(t)$ *holds if* $j = 1$ *and* $RL^{\bar{\pi}_{(1,1)}}(t) = RL_{21}^{\bar{\pi}_{(1,1)}}(t)$ *if* $j = 2$*.*

(ii) The portfolio process $\bar{\pi}_{(1,1)}(t)$ *is given by*

$$\bar{\pi}_{(1,1)}(t) = \pi^*(t) - \frac{\lambda}{\gamma}\left(\sigma\sigma^T\right)^{-1} z^{(1)} - \frac{\mu}{\gamma}\left(\sigma\sigma^T\right)^{-1} z^{(2)}, \tag{1.18}$$

where λ *and* μ *are given by equation* (1.22) *and* (1.23)*, with* $\lambda \geq 0$ *and* $\mu \geq 0$*, and* $N(t)$ *and* $M(t)$ *are the unique solutions of the ODEs* (1.24) *and* (1.25) *with terminal conditions* $N(t_{s+1}) = \bar{\pi}_{(1,1)}(t_{s+1})^T z^{(1)}$ *and* $M(t_{s+1}) = \bar{\pi}_{(1,1)}(t_{s+1})^T z^{(2)}$*. Furthermore, we have that* $RL_{12}^{\bar{\pi}_{(1,1)}}(t) = RL_{21}^{\bar{\pi}_{(1,1)}}(t)$ *for all* $t \in (t_s, t_{s+1}]$*.*

In case of property (ii), we require that $z^{(1)}$ *and* $z^{(2)}$ *are linearly independent. Finally, we assume that* $RL^{\bar{\pi}_{(1,1)}}(T) = 0$*. Then* $\bar{\pi}$ *solves the worst-case portfolio optimization problem* (1.5) *with stress scenarios* $\left(z^{(1)}, \xi_1\right)$ *and* $\left(z^{(2)}, \xi_2\right)$*.*

Proof. Let $\bar{\pi} \in \mathbb{A}(x)$ be an admissible portfolio process that fulfills the properties of the verification theorem. By Remark 3 it is enough to prove that $\bar{\pi}_{(1,1)}$ is optimal if both stresses can still occur. We use the idea of the proof of Theorem 2, especially part (ii). According to this, we need to show, that

- $\bar{\pi}$ is a minimum constant portfolio process, and
- $\bar{\pi}$ is optimal in the stress-free world under all processes respecting its indifference frontier, i.e. $\bar{\pi}$ solves the constrained optimization problem

$$\max_{\pi \in ID(\bar{\pi})} \phi_\gamma(\pi_{(1,1)}(t)) \text{ for all } t \in [0, T]. \tag{1.19}$$

We look at the interval $(t_s, t_{s+1}]$ for $s \in \{0, \dots, k\}$, and prove that these two conditions are fulfilled.

(i) By using Lagrangian-multiplier methods one can show that $\bar{\pi}$ with

$$\bar{\pi}_{(1,1)}(t) = \pi^*(t) - \frac{\lambda_j}{\gamma} \left(\sigma\sigma^T\right)^{-1} z^{(j)},$$

where

$$\frac{\lambda_j}{\gamma} = \frac{\pi^*(t)^T z^{(j)} - N_j(t)}{(z^{(j)})^T \left(\sigma\sigma^T\right)^{-1} z^{(j)}} \tag{1.20}$$

and N_j is the unique solution of

$$\begin{aligned} N_j'(t) &= (1 - N_j(t)) \left(f^{\bar{\pi}_{(1,1)}}(t) - \frac{\bar{\pi}'_{(0,1)}(t)^T z^{(2)}}{1 - \bar{\pi}_{(0,1)}(t)^T z^{(2)}} \right), \quad \text{if } j = 1, \\ N_j'(t) &= (1 - N_j(t)) \left(f^{\bar{\pi}_{(1,1)}}(t) - \frac{\bar{\pi}'_{(1,0)}(t)^T z^{(1)}}{1 - \bar{\pi}_{(1,0)}(t)^T z^{(1)}} \right), \quad \text{if } j = 2, \end{aligned} \tag{1.21}$$

with $N_j(t_{s+1}) = \bar{\pi}_{(1,1)}(t_{s+1})^T z^{(j)}$ and

$$f^{\bar{\pi}_{(1,1)}}(t) = -\frac{1}{2}\gamma \left((\frac{\lambda_j}{\gamma} z^{(j)})^T \left(\sigma\sigma^T\right)^{-1} (\frac{\lambda_j}{\gamma} z^{(j)}) \right),$$

solves the following constrained optimization problem if $\lambda_j \geq 0$:

$$\max_{\pi \in \widetilde{ID}(\bar{\pi})} \phi_\gamma(\pi_{(1,1)}(t)) \quad \text{for every } t \in (t_s, t_{s+1}].$$

Here, $\widetilde{ID}(\bar{\pi})$ is defined as

$$\widetilde{ID}(\bar{\pi}) = \{\pi \in \mathbb{A}(x) | \pi_{(1,1)}(t)^T z^{(j)} \leq \bar{\pi}_{(1,1)}(t)^T z^{(j)} \text{ for } t \in (t_s, t_{s+1}]\}.$$

Furthermore, $\bar{\pi}_{(1,1)}$ solves ODE (1.13) on $(t_s, t_{s+1}]$ if $j = 1$ and ODE (1.14) if $j = 2$. From $RL^{\bar{\pi}_{(1,1)}}(t) = RL_{12}^{\bar{\pi}_{(1,1)}}(t)$, respectively $RL^{\bar{\pi}_{(1,1)}}(t) = RL_{21}^{\bar{\pi}_{(1,1)}}(t)$, we get, that $RL^{\pi_{(1,1)}}(t) \leq RL^{\bar{\pi}_{(1,1)}}(t)$ implies $\pi_{(1,1)}(t)^T z^{(j)} \leq \bar{\pi}_{(1,1)}(t)^T z^{(j)}$. Therefore, we have $ID(\bar{\pi}) \subseteq \widetilde{ID}(\bar{\pi})$ and thus $\bar{\pi}$ solves the constrained optimization problem (1.19) on $(t_s, t_{s+1}]$. Further, $\bar{\pi}$ fulfills property (i) or (ii) of Corollary 1 on $(t_s, t_{s+1}]$.

(ii) Consider $\bar{\pi}$ with

$$\bar{\pi}_{(1,1)}(t) = \pi^*(t) - \frac{\lambda}{\gamma} \left(\sigma\sigma^T\right)^{-1} z^{(1)} - \frac{\mu}{\gamma} \left(\sigma\sigma^T\right)^{-1} z^{(2)},$$

where

$$\frac{\lambda}{\gamma} l = \frac{\pi^*(t)^T z^{(1)} - N(t)}{(z^{(1)})^T \left(\sigma\sigma^T\right)^{-1} z^{(1)}} - \frac{\left(\pi^*(t)^T z^{(2)} - M(t)\right) (z^{(1)})^T \left(\sigma\sigma^T\right)^{-1} z^{(2)}}{(z^{(1)})^T \left(\sigma\sigma^T\right)^{-1} z^{(1)} (z^{(2)})^T \left(\sigma\sigma^T\right)^{-1} z^{(2)}}, \tag{1.22}$$

$$\frac{\mu}{\gamma} l = \frac{\pi^*(t)^T z^{(2)} - M(t)}{(z^{(2)})^T \left(\sigma\sigma^T\right)^{-1} z^{(2)}} - \frac{\left(\pi^*(t)^T z^{(1)} - N(t)\right) (z^{(1)})^T \left(\sigma\sigma^T\right)^{-1} z^{(2)}}{(z^{(1)})^T \left(\sigma\sigma^T\right)^{-1} z^{(1)} (z^{(2)})^T \left(\sigma\sigma^T\right)^{-1} z^{(2)}}, \tag{1.23}$$

with

$$l = 1 - \frac{\left((z^{(1)})^T \left(\sigma\sigma^T\right)^{-1} z^{(2)}\right)^2}{(z^{(1)})^T \left(\sigma\sigma^T\right)^{-1} z^{(1)} (z^{(2)})^T \left(\sigma\sigma^T\right)^{-1} z^{(2)}}.$$

Using the Cholesky decomposition and the Cauchy-Schwarz inequality we get $l > 0$ by the linear independence assumption. Let N and M denote the unique solutions of

$$N'(t) = (1 - N(t)) \left(f^{\bar{\pi}_{(1,1)}}(t) - \frac{\bar{\pi}'_{(0,1)}(t)^T z^{(2)}}{1 - \bar{\pi}_{(0,1)}(t)^T z^{(2)}} \right), \tag{1.24}$$

$$M'(t) = (1 - M(t)) \left(f^{\bar{\pi}_{(1,1)}}(t) - \frac{\bar{\pi}'_{(1,0)}(t)^T z^{(1)}}{1 - \bar{\pi}_{(1,0)}(t)^T z^{(1)}} \right), \tag{1.25}$$

with $N(t_{s+1}) = \bar{\pi}_{(1,1)}(t_{s+1})^T z^{(1)}$ and $M(t_{s+1}) = \bar{\pi}_{(1,1)}(t_{s+1})^T z^{(2)}$ and

$$f^{\bar{\pi}_{(1,1)}}(t) = -\frac{1}{2}\gamma \left((\frac{\lambda}{\gamma} z^{(1)} + \frac{\mu}{\gamma} z^{(2)})^T \left(\sigma\sigma^T\right)^{-1} (\frac{\lambda}{\gamma} z^{(1)} + \frac{\mu}{\gamma} z^{(2)}) \right).$$

Again, by using Lagrangian-multiplier methods and the Karush-Kuhn-Tucker conditions, one can show that $\bar{\pi}$ solves the following constrained optimization problem if $\lambda \geq 0$ and $\mu \geq 0$:

$$\max_{\pi \in \widetilde{ID}(\bar{\pi})} \phi_\gamma(\pi_{(1,1)}(t)) \quad \text{for every } t \in (t_s, t_{s+1}],$$

with

$$\widetilde{ID}(\bar{\pi}) = \{\pi \in \mathbb{A}(x) | \pi_{(1,1)}(t)^T z^{(1)} \leq \bar{\pi}_{(1,1)}(t)^T z^{(1)} \\ \wedge \pi_{(1,1)}(t)^T z^{(2)} \leq \bar{\pi}_{(1,1)}(t)^T z^{(2)} \text{for } t \in (t_s, t_{s+1}]\}.$$

Furthermore, $\bar{\pi}_{(1,1)}$ solves ODE (1.13) and ODE (1.14). Using the definition of the relative worst-case loss and $RL_{12}^{\bar{\pi}_{(1,1)}}(t) = RL_{21}^{\bar{\pi}_{(1,1)}}(t)$ for all $t \in (t_s, t_{s+1}]$, $RL^{\pi_{(1,1)}}(t) \leq RL^{\bar{\pi}_{(1,1)}}(t)$ for $t \in (t_s, t_{s+1}]$ implies $\pi \in \widetilde{ID}(\bar{\pi})$ and therefore $ID(\bar{\pi}) \subseteq \widetilde{ID}(\bar{\pi})$. So $\bar{\pi}$ solves the constrained optimization problem (1.19) and fulfills property (i) and (ii) of Corollary 1 on $(t_s, t_{s+1}]$.

In summary, this means that $\bar{\pi}$ solves the constrained optimization problem and on each interval $(t_s, t_{s+1}]$ for $s = 0, \ldots, k$ it fulfills at least one of the two properties of Corollary 1. Thus, due to the right-continuity and the boundary condition $RL^{\bar{\pi}_{(1,1)}}(T) = 0$, $\bar{\pi}$ is minimum constant. □

Remark 9 (Notes on the Verification Theorem)**.** *We make a few comments on the verification theorem.*

1. *If $\bar{\pi}_{(1,1)}$, for $\bar{\pi} \in \mathbb{A}(x)$, fulfills property (i) of the verification theorem on the whole interval $[0,T]$, then $\bar{\pi}$ is by construction a sub-scenario indifference strategy. An investor using this portfolio process prefers the stress order $\xi_2 < \xi_1$ if $RL^{\bar{\pi}_{(1,1)}}(t) = RL_{12}^{\bar{\pi}_{(1,1)}}(t)$ and $\xi_1 < \xi_2$ if $RL^{\bar{\pi}_{(1,1)}}(t) = RL_{21}^{\bar{\pi}_{(1,1)}}(t)$.*

2. *An investor with portfolio process $\bar{\pi} \in \mathbb{A}(x)$, where $\bar{\pi}_{(1,1)}$ fulfills property (ii) of the verification theorem on the interval $[0,T]$, with terminal condition $\bar{\pi}_{(1,1)}(T)^T z^{(1)} = 0$ and $\bar{\pi}_{(1,1)}(T)^T z^{(2)} = 0$, is indifferent concerning the time and order of occurrence of the two stresses, since then we have*

$$\begin{aligned}
&\mathbb{E}\left(v_{00}\left(\tau; \left(1 - RL_{12}^{\bar{\pi}_{(1,1)}}(\tau)\right) X^{\bar{\pi}_{(1,1)}}(\tau-)\right)\right)\\
=&\mathbb{E}\left(v_{00}\left(T; \left(1 - RL_{12}^{\bar{\pi}_{(1,1)}}(T)\right) X^{\bar{\pi}_{(1,1)}}(T-)\right)\right)\\
=&\mathbb{E}\left(v_{00}\left(T; \left(1 - RL_{21}^{\bar{\pi}_{(1,1)}}(T)\right) X^{\bar{\pi}_{(1,1)}}(T-)\right)\right)\\
=&\mathbb{E}\left(v_{00}\left(\tau; \left(1 - RL_{21}^{\bar{\pi}_{(1,1)}}(\tau)\right) X^{\bar{\pi}_{(1,1)}}(\tau-)\right)\right) \quad \textit{for all } \tau \in \Theta.
\end{aligned}$$

3. *The verification theorem in the log-utility case is obtained by setting $\gamma = 1$. For this, note that we can compute sub-scenario indifference strategies in the log-utility case by using Itô's rule, analogous to the proof of Proposition 8.*

1.7 Characterization of the solution and numerical illustration.

To apply the verification theorem we are now going to construct, in a heuristic way, general candidates for the optimal portfolio process. These heuristic constructions do not serve as an alternative to a proof of existence, but should make it easier to find a suitable process. As the numerical considerations show, the Lagrange multipliers appearing in the verification theorem depend to a large extent on the correlations of the stocks and the type of stresses. Due to the dependence on the ODEs, it must be checked numerically whether they fulfill the non-negativity conditions of the verification theorem. In the following we describe a procedure that helps to determine the optimal strategy.

In the numerical considerations of this section, the matrix σ is a $d \times d$ lower triangular matrix of full rank.

Linearly dependent stresses. We consider the situation of two linearly dependent stresses $z^{(1)} = az^{(2)}$, with $a \geq 1$. The proof of the verification theorem implies that a portfolio process which satisfies property (ii) solves the two ODEs (1.13) and (1.14). But this is only possible if the stresses are linearly independent. Hence, it is sufficient to consider the case of property (i). The linear dependency and the boundary condition $RL^{\bar{\pi}_{(1,1)}}(T) = 0$ directly yield $\bar{\pi}_{(1,1)}(T)^T z^{(1)} = \bar{\pi}_{(1,1)}(T)^T z^{(2)} = 0$. Here, where $z^{(1)}$ is the worse stress, it now makes sense to compute (1.17) for $j = 1$, check the required properties and if necessary switch to $j = 2$.

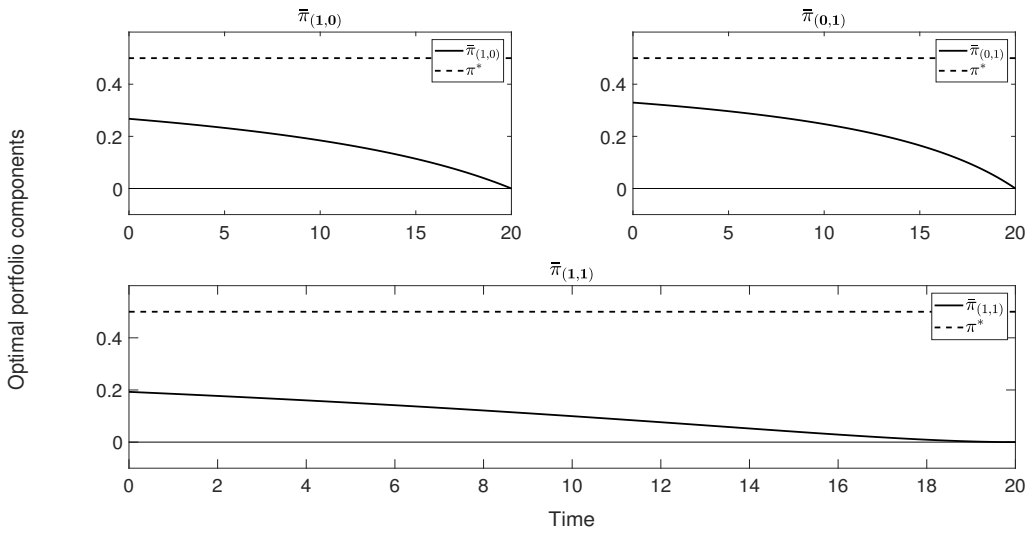

FIGURE 1.1: One stock with $z^{(1)} = 0.25$ and $z^{(2)} = 0.15$.

A simple example is shown in Figure 1.1. Here, we have $r = 0$, $\gamma = 1.5$, $T = 20$, one stock with $b = 0.03$, $\sigma = 0.2$, and two linearly dependent stresses $z^{(1)} = 0.25$ and $z^{(2)} = 0.15$. Solving the ODE (1.13) and verifying the conditions of part (i) of the verification theorem yield that the worst-case optimal strategy is the optimal sub-scenario indifference strategy for the sub-scenario $\xi_1 < \xi_2$, i.e. one should be prepared for the larger stress occurring first.

Linearly independent stresses. By assumption (1.2) and due to $\bar{\pi}_{(1,0)}(T)^T z^{(1)} = 0$ and $\bar{\pi}_{(0,1)}(T)^T z^{(2)} = 0$, we know that

$$\bar{\pi}_{(1,1)}(T) \in \underset{\substack{\pi \in \mathbb{R}^2 \\ \pi^T z^{(1)} \leq 0 \\ \pi^T z^{(2)} \leq 0}}{\arg\max}\, \phi_\gamma(\pi)$$

solves the boundary condition $RL^{\bar{\pi}_{(1,1)}}(T) = 0$, since at least one of the constraints in the above optimization problem is fulfilled with equality for the optimal solution. If both constraints are fulfilled with equality we know that $RL^{\bar{\pi}_{(1,1)}}(T) = RL_{12}^{\bar{\pi}_{(1,1)}}(T) = RL_{21}^{\bar{\pi}_{(1,1)}}(T)$ and therefore we should start to construct $\bar{\pi}_{(1,1)}$ accordingly to (1.18). If, without loss of generality, $\bar{\pi}_{(1,1)}(T)^T z^{(1)} < 0$ we know that $\bar{\pi}_{(1,1)}(T)^T z^{(2)} = 0$ (otherwise π^* would fulfill the constraints). Then, we construct $\bar{\pi}_{(1,1)}$ accordingly to (1.17) with $j = 2$, since $RL^{\bar{\pi}_{(1,1)}}(T) = RL_{21}^{\bar{\pi}_{(1,1)}}(T)$.

Now we look at the interval $(t_s, t_{s+1}]$ for $s \in \{0, \ldots, k\}$ and a not yet defined t_s. If $\bar{\pi}_{(1,1)}$ is given by equation (1.17) with $j = 1$ ($j = 2$ is similar) in t_{s+1}, we define t_s as

$$t_s = \sup\{t \leq t_{s+1} | RL^{\bar{\pi}_{(1,1)}}(t) > RL_{12}^{\bar{\pi}_{(1,1)}}(t)\}.$$

In t_s one should then check whether the transition to (1.18) is possible. On the other hand, if $\bar{\pi}_{(1,1)}$ is given by equation (1.18) in t_{s+1} we define t_s as

$$t_s = \sup\{t \leq t_{s+1} | \lambda(t) < 0 \vee \mu(t) < 0\},$$

where λ and μ are interpreted as functions in t. Let us assume that w.l.o.g. $\lambda(t_s-) \geq 0$ and $\mu(t_s-) < 0$. Then we should switch to (1.17) with $j = 1$, since then, due to continuity, we expect $RL^{\bar{\pi}_{(1,1)}}(t_s-) = RL_{12}^{\bar{\pi}_{(1,1)}}(t_s-)$.

As an example, we look at two uncorrelated stocks with $r = 0$, $\gamma = 1.5$, $b = (0.03, 0.035)$, $\sigma_{11} = 0.2$, $\sigma_{22} = 0.3$ and $T = 20$. In Figure 1.2, we see the optimal portfolio process for $z^{(1)} = (0.15, 0.25)^T$ and $z^{(2)} = (0.25, 0.15)^T$. Here, $\bar{\pi}_{(1,1)}$ fulfills property (ii) of the verification theorem on the whole interval $[0, T]$. Hence, the investor is indifferent with respect to the order of occurrence of the stresses. That is not the case for the optimal portfolio process in Figure 1.3. Here, we have $z^{(1)} = (0.25, 0.25)^T$ and $z^{(2)} = (0.35, 0.15)^T$. On the interval $[0, 16.1]$ the process $\bar{\pi}_{(1,1)}$ fulfills property (ii) of the verification theorem and afterwards property (i) for $j = 1$, i.e. from $t = 16.1$ on, the investor prefers the occurrence of the stress $z^{(2)}$ first.

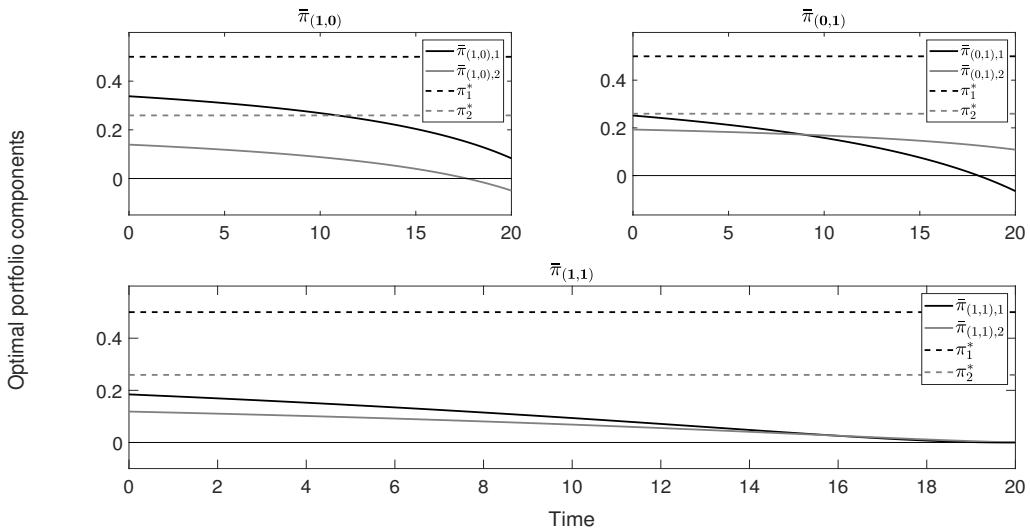

FIGURE 1.2: Two uncorrelated stocks with $z^{(1)} = (0.15, 0.25)^T$ and $z^{(2)} = (0.25, 0.15)^T$.

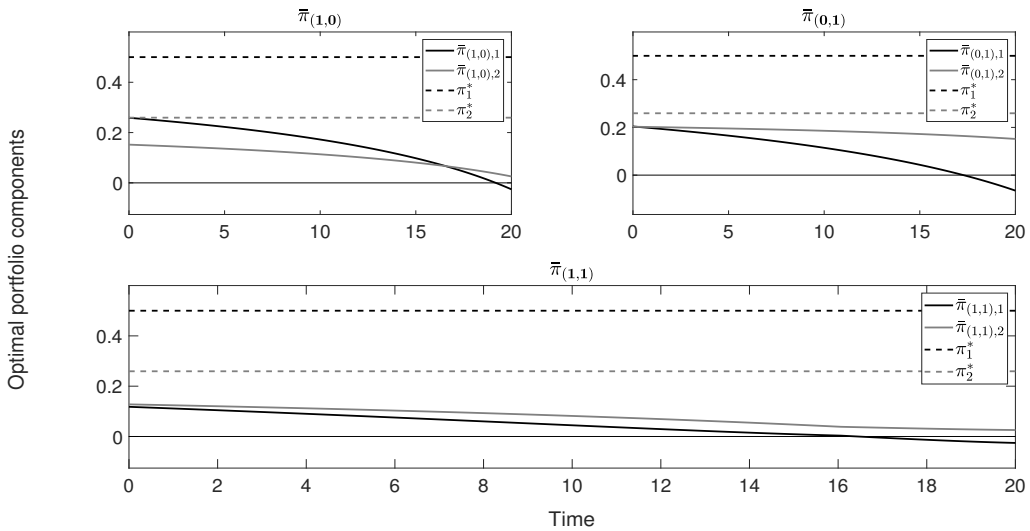

FIGURE 1.3: Two uncorrelated stocks with $z^{(1)} = (0.25, 0.25)^T$ and $z^{(2)} = (0.35, 0.15)^T$.

Further numerical illustrations. In the following, we give four further numerical examples to show the extent to which the choice of the stresses influences the construction of the optimal minimum constant portfolio process. In all examples, we use the same two negatively correlated stocks and step wise vary the stresses. We select the following parameters: $T = 20, \gamma = 1.5, b = (0.03, 0.035)$ and $\sigma_{11} = 0.2, \sigma_{21} = -0.15, \sigma_{22} = 0.2$. It follows that $\ln(S_2)$ has a volatility of 0.25 and the log-prices of the two stocks have a correlation of $\rho = -0.6$. In Figure 1.4 we use the stresses $z^{(1)} = (0.15, 0.25)^T$ and $z^{(2)} = (0.25, 0.15)^T$. The symmetric stresses cause the optimal strategy to satisfy property (ii) of the verification theorem on the interval $[0, T]$ and the investor is therefore indifferent regarding time and order of occurrence of the stresses. Note that in relation to Figure 1.2, the negative correlation leads to significantly higher investments in the risky assets. In Figure 1.5, we now slightly adjust

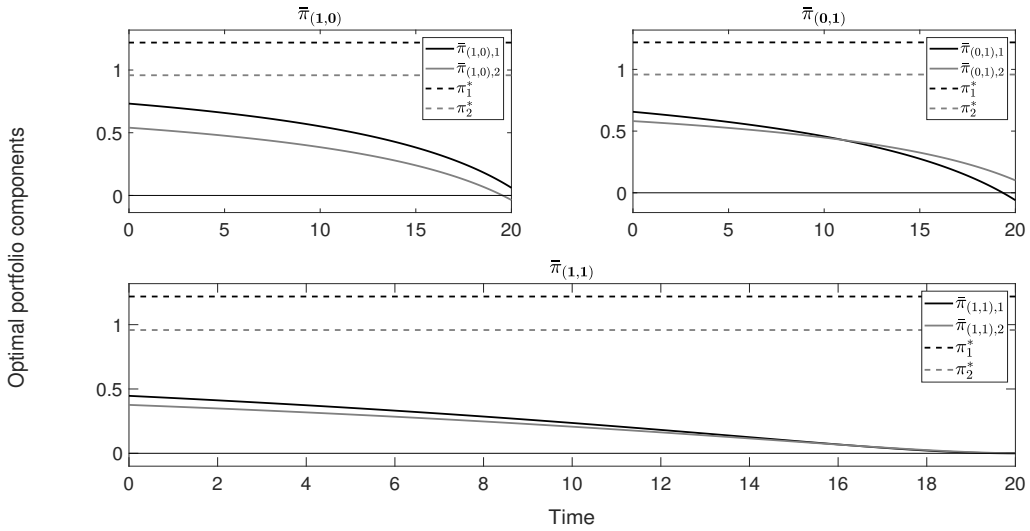

FIGURE 1.4: Two negatively correlated stocks with $z^{(1)} = (0.15, 0.25)^T$ and $z^{(2)} = (0.25, 0.15)^T$.

stress $z^{(2)}$ and consider the stresses $z^{(1)} = (0.15, 0.25)^T$ and $z^{(2)} = (0.15, 0.15)^T$, so that $z^{(1)}$ can be regarded as the worse stress. The optimal strategy $\bar{\pi}_{(1,1)}$ fulfills property (i) for $j = 1$ on the interval $[0, 8.8]$ and afterwards property (ii) of the verification theorem. Hence, the investor first prefers the occurrence of stress $z^{(2)}$, since $RL^{\bar{\pi}_{(1,1)}}(t) = RL_{12}^{\bar{\pi}_{(1,1)}}(t)$, before being indifferent regarding the order of occurrence. The time of transition from property (i) to (ii) is shifted back in time in Figure 1.6 by further reducing stress $z^{(2)}$ and using the stresses $z^{(1)} = (0.15, 0.25)^T$ and $z^{(2)} = (0.1, 0.1)^T$. Here, $\bar{\pi}_{(1,1)}$ fulfills property (i) for $j = 1$ on the interval $[0, 15]$ and afterwards property (ii). Finally, we swap the components of stress $z^{(1)}$ and consider the stresses $z^{(1)} = (0.25, 0.15)^T$ and $z^{(2)} = (0.1, 0.1)^T$ in Figure 1.7. Here we have two transitions between the properties of the verification theorem. On the interval $[0, 14.4]$, the optimal strategy $\bar{\pi}_{(1,1)}$ satisfies property (i) for $j = 1$, on the interval $[14.4, 18.5]$ property (ii) and then again property (i), but this time for $j = 2$. This in particular means that towards the time horizon the investor prefers stress $z^{(1)}$, which is actually perceived as worse. This is because there is a short position in the first stock S_1 if $\bar{\pi}_{(1,1)}$ fulfills property (i) for $j = 2$ close to T.

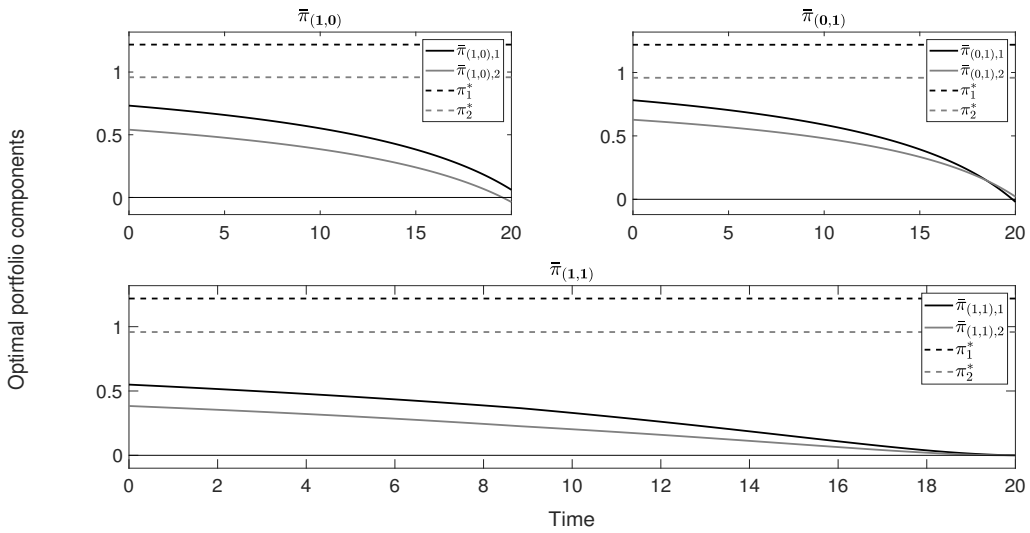

FIGURE 1.5: Two negatively correlated stocks with $z^{(1)} = (0.15, 0.25)^T$ and $z^{(2)} = (0.15, 0.15)^T$.

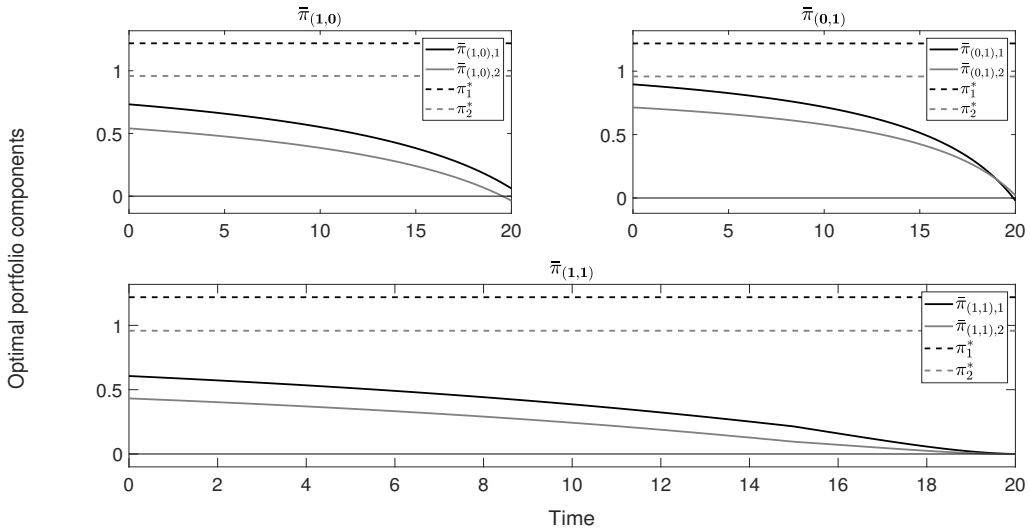

FIGURE 1.6: Two negatively correlated stocks with $z^{(1)} = (0.15, 0.25)^T$ and $z^{(2)} = (0.1, 0.1)^T$.

1.8 HJB-equation framework

In his pioneering work, Merton [47] used classical stochastic control methods to reduce the portfolio problem to a matter of solving a Hamilton-Jacobi-Bellman partial differential equation (HJB-equation). In the previous sections we have solved the worst-case portfolio problem with stress scenarios using indifference arguments. The goal of this section is to embed this portfolio optimization problem in a classical HJB-framework and to establish the connection to the stochastic control theory. For this purpose, the work of Korn and Steffensen [36] on the portfolio optimization problem with a single stress is extended to the case with two stresses. We only consider the power-utility case with relative risk aversion γ, but the log-utility case follows analogously.

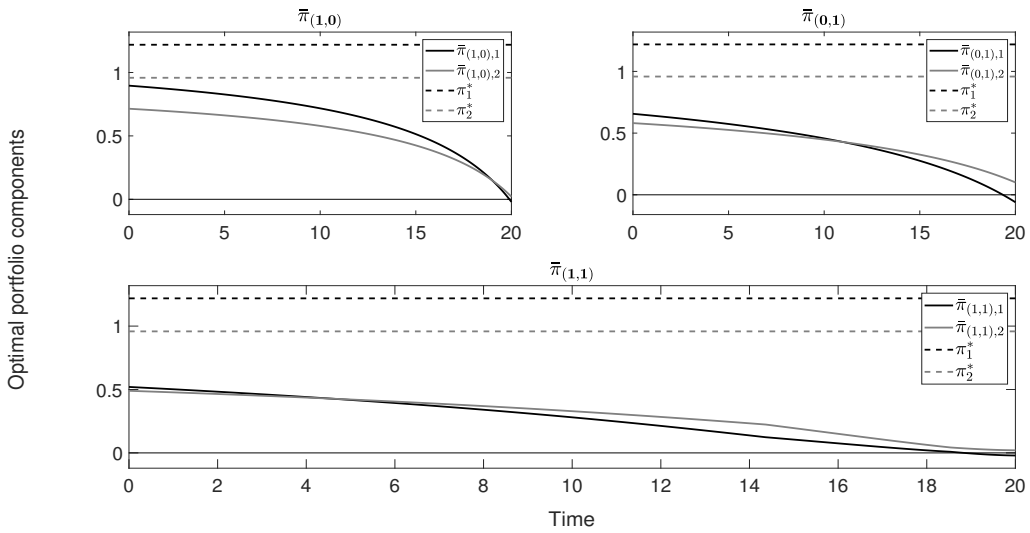

FIGURE 1.7: Two negatively correlated stocks with $z^{(1)} = (0.25, 0.15)^T$ and $z^{(2)} = (0.1, 0.1)^T$.

The Bellman system and the verification theorem. We want to present a Bellman system in a verification theorem. Therefore, for each $V \in C^{1,2}$ we define the differential operator $\mathcal{L}^{\pi} V$ by

$$\mathcal{L}^{\pi} V(t;x) = V_t(t;x) + V_x(t;x)\left(r + \pi^T (b - r\underline{1})\right) x + \frac{1}{2} V_{xx}(t;x) \pi^T \sigma \sigma^T \pi x^2.$$

Since the case of one stress is already solved in [36], we only consider the worst-case portfolio problem with two stresses. For this we denote by $v_{11}(t;x)$ the value function defined in (1.6) and we are interested in determining the optimal portfolio process $\bar{\pi}_{(1,1)}$ if both stresses can still occur. Furthermore, we also adopt the assumption from Remark 3. Now we are able to state the verification theorem which contains an inequality similar to the HJB-equation, but also shows the connection between the value functions before and after a stress.

Theorem 4 (Verification theorem in an HJB-equation framework)**.**
Let $z^{(1)}$, $z^{(2)} \in \mathbb{R}^d$ be two stresses with corresponding stopping times $\xi_1, \xi_2 \in \Theta$. Then we define for every function $V_{11} \in C^{1,2}$ and $x > 0$, the sets $\mathbb{A}'_{11}(t,x)$ and $\mathbb{A}''_{11}(t,x)$ by

$$\begin{aligned}
\mathbb{A}'_{11}(t,x) &= \{\pi : \pi \in \mathbb{A}(t,x), 0 \leq \mathcal{L}^{\pi_{(1,1)}} V_{11}(t,x)\}, \\
\mathbb{A}''_{11}(t,x) &= \{\pi : \pi \in \mathbb{A}(t,x), 0 \leq v_{10}\left(t; \left(1 - \pi_{(1,1)}(t)^T z^{(2)}\right) x\right) - V_{11}(t,x) \\
&\qquad \wedge 0 \leq v_{01}\left(t; \left(1 - \pi_{(1,1)}(t)^T z^{(1)}\right) x\right) - V_{11}(t,x)\},
\end{aligned}$$

respectively. Assume that there exists a polynomially bounded $C^{1,2}$-solution of

$$\begin{aligned}
0 &\leq \sup_{\pi \in \mathbb{A}''_{11}(t,x)} [\mathcal{L}^{\pi_{(1,1)}} V_{11}(t,x)], \\
0 &\leq \sup_{\pi \in \mathbb{A}'_{11}(t,x)} [v_{01}\left(t; \left(1 - \pi_{(1,1)}(t)^T z^{(1)}\right) x\right) - V_{11}(t,x)],
\end{aligned}$$

$$
\begin{aligned}
0 &\leq \sup_{\pi \in \mathbb{A}'_{11}(t,x)} \left[v_{10}\left(t; \left(1 - \pi_{(1,1)}(t)^T z^{(2)}\right) x\right) - V_{11}(t,x)\right], \\
0 &= \Bigg[\sup_{\pi \in \mathbb{A}''_{11}(t,x)} \left[\mathcal{L}^{\pi_{(1,1)}} V_{11}(t,x)\right] \\
&\qquad \cdot \sup_{\pi \in \mathbb{A}'_{11}(t,x)} \left[v_{01}\left(t; \left(1 - \pi_{(1,1)}(t)^T z^{(1)}\right) x\right) - V_{11}(t,x)\right] \\
&\qquad \cdot \sup_{\pi \in \mathbb{A}'_{11}(t,x)} \left[v_{10}\left(t; \left(1 - \pi^T_{(1,1)} z^{(2)}\right) x\right) - V_{11}(t,x)\right] \Bigg], \\
V_{11}(T,x) &= U(x),
\end{aligned}
$$

and that

$$
p(t,x) = \arg \sup_{\pi \in \mathbb{A}''_{11}(t,x)} \left[\mathcal{L}^{\pi_{(1,1)}} V_{11}(t,x)\right].
$$

Then

$$
v_{11}(t,x) = V_{11}(t,x),
$$

and the optimal control function exists and is given by

$$
\bar{\pi}(t) = p(t, X^{\bar{\pi}_{(1,1)}}(t-)).
$$

Proof. We will give the proof by using similar arguments as Korn and Steffensen in [36]. First we fix a point (t,x), choose an arbitrary control (π, ξ), for $\pi \in \mathbb{A}(t,x)$ and $\xi \in \Theta_t$, and assume that $X^\pi(t) = x$. By ξ we denote the control of the first stress, which is why we have to distinguish between the two cases $\xi = \xi_1 < \xi_2$ and $\xi = \xi_2 < \xi_1$. Inserting X^π in V_{11} and using Itô's formula, we obtain

$$
\begin{aligned}
V_{11}(t, X^\pi(t)) =& V_{11}(t,x), \\
dV_{11}(s, X^\pi(s)) =& \mathcal{L}^{\pi_{(1,1)}} V_{11}(s, X^{\pi_{(1,1)}}(s-))ds \\
&+ (V_{11})_x (s, X^{\pi_{(1,1)}}(s-)) X^{\pi_{(1,1)}}(s-) \pi_{(1,1)}(s)^T \sigma dW(s),
\end{aligned}
$$

for $t < s < \xi$, such that

$$
\begin{aligned}
& V_{11}(\xi, X^{\pi_{(1,1)}}(\xi-)) - V_{11}(t,x) \\
=& \int_t^\xi \mathcal{L}^{\pi_{(1,1)}} V_{11}(s, X^{\pi_{(1,1)}}(s-))ds \\
&+ \int_t^\xi (V_{11})_x (s, X^{\pi_{(1,1)}}(s-)) X^{\pi_{(1,1)}}(s-) \pi_{(1,1)}(s)^T \sigma dW(s).
\end{aligned} \tag{1.26}
$$

We fix the portfolio process $\pi(s) = p(s)$, $t \leq s \leq \xi$. From now on the letter p is used for the process $p \in \mathbb{A}(t,x)$ with $p_{(1,1)}(s) = p(s, X^{\pi_{(1,1)}}(s-))$, coming from applying the function $p(s,x)$ on the wealth process X^π. Note that the portfolio process p is admissible by

definition. From the Bellman system it follows that, for $t \leq s \leq \xi$,

$$0 \leq \mathcal{L}^{p_{(1,1)}} V_{11}(s, X^{p_{(1,1)}}(s-))$$

and, since $p(s) \in \mathbb{A}''_{11}(s, X^{p_{(1,1)}}(s-))$, also

$$\begin{aligned} 0 &\leq v_{10}\left(s, \left(1 - p_{(1,1)}(s)^T z^{(2)}\right) X^{p_{(1,1)}}(s-)\right) - V_{11}(s, X^{p_{(1,1)}}(s-)), \\ \text{and } 0 &\leq v_{01}\left(s, \left(1 - p_{(1,1)}(s)^T z^{(1)}\right) X^{p_{(1,1)}}(s-)\right) - V_{11}(s, X^{p_{(1,1)}}(s-)). \end{aligned}$$

If we now insert p in (1.26), we obtain the inequality

$$\begin{aligned} V_{11}(t,x) \leq \min \Big\{ & v_{10}\left(\xi, \left(1 - p_{(1,1)}(\xi)^T z^{(2)}\right) X^{p_{(1,1)}}(\xi-)\right), \\ & v_{01}\left(\xi, \left(1 - p_{(1,1)}(\xi)^T z^{(1)}\right) X^{p_{(1,1)}}(\xi-)\right) \Big\} \\ & - \int_t^{\xi} (V_{11})_x (s, X^{p_{(1,1)}}(s-)) X^{p_{(1,1)}}(s-) p_{(1,1)}(s)^T \sigma dW(s). \end{aligned}$$

We now take expectation on both sides and note that the stochastic integral vanishes due to the requirements on the admissible portfolio processes. Together with the definition of the minimum process of an admissible portfolio process π and equation (1.7), we obtain

$$V_{11}(t,x) \leq \mathbb{E}_{t,x}\left(M^p(\xi)\right).$$

Taking the infimum over ξ on both sides leads to

$$V_{11}(t,x) \leq \inf_{\xi \in \Theta_t} \mathbb{E}_{t,x}\left(M^p(\xi)\right) \tag{1.27}$$

and we can immediately conclude that

$$V_{11}(t,x) \leq \sup_{\pi \in \mathbb{A}(t,x)} \inf_{\xi \in \Theta_t} \mathbb{E}_{t,x}\left(M^{\pi}(\xi)\right). \tag{1.28}$$

Again we fix an arbitrary point (t,x), choose $\pi \in \mathbb{A}(t,x)$ and consider (1.26). Furthermore we fix the stopping time

$$\begin{aligned} \xi^* = \inf \Big\{ & \xi_1,\ \xi_2 \in \Theta_t \Big| \\ & v_{01}\left(\xi_1; \left(1 - \pi_{(1,1)}(\xi_1)^T z^{(1)}\right) X^{\pi_{(1,1)}}(\xi_1-)\right) - V_{11}(\xi_1; X^{\pi_{(1,1)}}(\xi_1-)) \leq 0 \\ & \vee v_{10}\left(\xi_2; \left(1 - \pi_{(1,1)}(\xi_2)^T z^{(2)}\right) X^{\pi_{(1,1)}}(\xi_2-)\right) - V_{11}(\xi_2; X^{\pi_{(1,1)}}(\xi_2-)) \leq 0 \Big\}. \end{aligned}$$

Thus, we get that

$$v_{01}\left(s; \left(1 - \pi_{(1,1)}(s)^T z^{(1)}\right) X^{\pi_{(1,1)}}(s-)\right) - V_{11}(s; X^{\pi_{(1,1)}}(s-)) > 0, \tag{1.29}$$

$$\text{and } v_{10}\left(s; \left(1 - \pi_{(1,1)}(s)^T z^{(2)}\right) X^{\pi_{(1,1)}}(s-)\right) - V_{11}(s; X^{\pi_{(1,1)}}(s-)) > 0, \tag{1.30}$$

for $t \leq s < \xi^*$. Furthermore we obtain with the right-continuity assumption of an admissible portfolio process that

$$v_{01}\left(\xi^*; \left(1 - \pi_{(1,1)}(\xi^*)^T z^{(1)}\right) X^{\pi_{(1,1)}}(\xi^*-)\right) - V_{11}(\xi^*; X^{\pi_{(1,1)}}(\xi^*-)) \leq 0,$$
$$\text{or } v_{10}\left(\xi^*; \left(1 - \pi_{(1,1)}(\xi^*)^T z^{(2)}\right) X^{\pi_{(1,1)}}(\xi^*-)\right) - V_{11}(\xi^*; X^{\pi_{(1,1)}}(\xi^*-)) \leq 0.$$

With Definition 5, equation (1.7) and by taking expectation, we obtain from the above inequality that

$$\mathbb{E}_{t,x}\left(M^{\pi}(\xi^*)\right) - \mathbb{E}_{t,x}\left(V_{11}(\xi^*; X^{\pi_{(1,1)}}(\xi^*-))\right) \leq 0. \tag{1.31}$$

For $t \leq s < \xi^*$, we then have to note that either $0 > \mathcal{L}^{\pi_{(1,1)}} V_{11}(s, X^{\pi_{(1,1)}}(s-))$ or $0 \leq \mathcal{L}^{\pi_{(1,1)}} V_{11}(s, X^{\pi_{(1,1)}}(s-))$. In the case that $0 \leq \mathcal{L}^{\pi_{(1,1)}} V_{11}(s, X^{\pi_{(1,1)}}(s-))$, we obtain that $\pi(s) \in \mathbb{A}'_{11}(s, X^{\pi_{(1,1)}}(s-))$, and then (1.29) and (1.30) give us that

$$\sup_{\pi \in \mathbb{A}'_{11}(s, X^{\pi_{(1,1)}}(s-))} \left[v_{01}\left(s; \left(1 - \pi_{(1,1)}(s)^T z^{(1)}\right) X^{\pi_{(1,1)}}(s-)\right) - V_{11}(s; X^{\pi_{(1,1)}}(s-))\right] > 0,$$
$$\text{and} \sup_{\pi \in \mathbb{A}'_{11}(s, X^{\pi_{(1,1)}}(s-))} \left[v_{10}\left(s; \left(1 - \pi_{(1,1)}(s)^T z^{(2)}\right) X^{\pi_{(1,1)}}(s-)\right) - V_{11}(s; X^{\pi_{(1,1)}}(s-))\right] > 0.$$

Together with the complementary condition this leads to

$$\sup_{\pi \in \mathbb{A}''_{11}(s, X^{\pi_{(1,1)}}(s-))} \left[\mathcal{L}^{\pi_{(1,1)}} V_{11}(s; X^{\pi_{(1,1)}}(s-))\right] = 0.$$

From (1.29) and (1.30) we obtain that $\pi(s) \in \mathbb{A}''_{11}(s, X^{\pi_{(1,1)}}(s-))$ and therefore

$$\mathcal{L}^{\pi_{(1,1)}} V_{11}(s, X^{\pi_{(1,1)}}(s-)) \leq 0.$$

That means in any case we have $\mathcal{L}^{\pi_{(1,1)}} V_{11}(s, X^{\pi_{(1,1)}}(s-)) \leq 0$, for $t \leq s < \xi^*$. Based on this inequality, we insert ξ^* in (1.26) and get that

$$\begin{aligned} V_{11}(t; x) \geq &- \int_t^{\xi^*} (V_{11})_x (s; X^{\pi_{(1,1)}}(s-)) X^{\pi_{(1,1)}}(s-) \pi_{(1,1)}(s)^T \sigma dW(s) \\ &+ V_{11}(\xi^*; X^{\pi_{(1,1)}}(\xi^*-)). \end{aligned}$$

Now we can again use our requirements on the admissible controls and on the value function, to take the expectation, such that we obtain that

$$V_{11}(t; x) \geq \mathbb{E}_{t,x}\left(V_{11}(\xi^*; X^{\pi_{(1,1)}}(\xi^*-))\right).$$

Next, we can use inequality (1.31) and get

$$V_{11}(t; x) \geq \mathbb{E}_{t,x}\left(M^{\pi}(\xi^*)\right).$$

Then we can conclude that

$$V_{11}(t;x) \geq \inf_{\xi \in \Theta_t} \mathbb{E}_{t,x}\left(M^{\pi}(\xi)\right).$$

Taking the supremum over π on both sides leads to

$$V_{11}(t;x) \geq \sup_{\pi(t) \in \mathbb{A}(t,x)} \inf_{\xi \in \Theta_t} \mathbb{E}_{t,x}\left(M^{\pi}(\xi)\right). \tag{1.32}$$

From (1.28), (1.32) and by Lemma 2 we conclude

$$V_{11}(t;x) = \sup_{\pi(t) \in \mathbb{A}(t,x)} \inf_{\xi \in \Theta_t} \mathbb{E}_{t,x}\left(M^{\pi}(\xi)\right) = v_{11}(t;x).$$

Then the optimality of $p_{11}(t;x)$ follows with (1.27). □

Verification of the optimal minimum constant portfolio process. In the previous sections we have proven that there exists a minimum constant portfolio process that solves the worst-case portfolio optimization problem with two stress scenarios. We now want to show for such a process $\bar{\pi}$, defined in Proposition 5, that the corresponding value function, defined as

$$v_{11}(t,x) := v_{00}\left(t, \left(1 - RL^{\bar{\pi}_{(1,1)}}(t)\right) x\right),$$

for $t \in [0,T]$ and $x > 0$, satisfies the conditions of the above verification theorem. With the definition of v_{00} and $\bar{\pi}_{(1,1)}$ it follows directly that $v_{11} \in C^{1,2}$. We first want to show that $\bar{\pi} \in \mathbb{A}''_{11}(t,x)$. For this let $x > 0$, $t \in [0,T]$ and note that we get with the definitions of the relative worst-case loss, v_{10}, v_{01} and v_{00} that

$$\begin{aligned} v_{11}(t,x) =& v_{00}\left(t, \left(1 - RL^{\bar{\pi}_{(1,1)}}(t)\right) x\right) \\ =& \min\left\{v_{10}\left(t, \left(1 - \bar{\pi}_{(1,1)}(t)^T z^{(2)}\right) x\right), v_{01}\left(t; \left(1 - \bar{\pi}_{(1,1)}(t)^T z^{(1)}\right) x\right)\right\}. \end{aligned} \tag{1.33}$$

This leads to $\bar{\pi} \in \mathbb{A}''_{11}(t,x)$. We now want to show that $\bar{\pi} \in \mathbb{A}'_{11}(t,x)$ and therefore need to compute $\mathcal{L}^{\bar{\pi}_{(1,1)}} v_{11}(t,x)$. Using (1.9) leads to

$$\mathcal{L}^{\bar{\pi}_{(1,1)}} v_{11}(t,x) = 0, \tag{1.34}$$

and thus $\bar{\pi} \in \mathbb{A}'_{11}(t,x)$. Together with (1.33) we get directly that

$$\begin{aligned} 0 &\leq \sup_{\pi \in \mathbb{A}'_{11}(t,x)} [v_{01}\left(t; \left(1 - \pi_{(1,1)}(t)^T z^{(1)}\right) x\right) - v_{11}(t,x)], \\ 0 &\leq \sup_{\pi \in \mathbb{A}'_{11}(t,x)} [v_{10}\left(t; \left(1 - \pi_{(1,1)}(t)^T z^{(2)}\right) x\right) - v_{11}(t,x)]. \end{aligned}$$

Further, equation (1.34) and the fact that $\bar{\pi} \in \mathbb{A}''_{11}$ lead to

$$0 \leq \sup_{\pi \in \mathbb{A}''_{11}(t,x)} [\mathcal{L}^{\pi_{(1,1)}} v_{11}(t,x)].$$

To prove that v_{11} also satisfies the complementary condition of the verification theorem we have to note that we can rewrite the set $\mathbb{A}''_{11}$, using the definition of v_{00} and of the relative worst-case loss, as follows:

$$\begin{aligned}\mathbb{A}''_{11}(t,x) = \{\pi : \pi \in \mathbb{A}(t,x), 0 \leq v_{10}\left(t; \left(1 - \pi_{(1,1)}(t)^T z^{(2)}\right) x\right) - v_{11}(t,x) \\ \wedge 0 \leq v_{01}\left(t; \left(1 - \pi_{(1,1)}(t)^T z^{(1)}\right) x\right) - v_{11}(t,x)\} \\ = \{\pi : \pi \in \mathbb{A}(t,x),\ RL^{\pi_{(1,1)}}(t) \leq RL^{\bar{\pi}_{(1,1)}}(t)\}\end{aligned}$$

for $t \in [0,T]$ and $x > 0$. Thus, we obtain

$$\begin{aligned}&\sup_{\pi \in \mathbb{A}''_{11}(t,x)} [\mathcal{L}^{\pi_{(1,1)}} v_{11}(t,x)] = \sup_{\pi \in \mathbb{A}''_{11}(t,x)} (1-\gamma) v_{11}(t,x) \left(\phi_\gamma(\pi_{(1,1)}) - \phi_\gamma(\bar{\pi}_{(1,1)})\right) \\ =&(1-\gamma) v_{11}(t,x) \left(\sup_{\pi \in \mathbb{A}''_{11}(t,x)} \phi_\gamma(\pi_{(1,1)}) - \phi_\gamma(\bar{\pi}_{(1,1)})\right) \\ =&(1-\gamma) v_{11}(t,x) \left(\phi_\gamma(\bar{\pi}_{(1,1)}) - \phi_\gamma(\bar{\pi}_{(1,1)})\right) = 0,\end{aligned}$$

where we used the reformulation of $\mathbb{A}''_{11}(t,x)$ and the proof of Theorem 2, in particular the optimization problem (1.12). Thus, v_{11} satisfies the conditions of the verification theorem and $\bar{\pi}_{(1,1)}$ is the optimal portfolio process if both stresses can still occur.

1.9 Conclusion

We have modelled a portfolio optimization problem with possible stress scenarios by suitably adapting the worst-case scenario portfolio approach originally introduced in [37]. This included the introduction of different types of stresses and of an uncertain ordering of their occurrence. To deal with such a kind of problem, we generalized the concepts of the indifference frontier and of the indifference-optimality principle as given in [56] via the introduction of the minimum constant process.

We succeed in proving the existence of a minimum constant process which solves the multi-asset multi-stress worst-case problem in case that more than one stress can occur.

With the help of a verification theorem, we developed an algorithmic framework for coming up with candidates for the optimal worst-case portfolios and then verifying the guess by checking the validity of one of the two different sufficient conditions in the theorem.

Numerical examples highlighted various different effects. There can be settings where the investor globally is worst-case hedged against one particular ordering of the stresses. There can also be the situation where the investor is indifferent with respect to the ordering of the occurrence of the stresses, or where he/she is locally indifferent, but also locally is focused

to concentrate on a particular ordering as the worst case.

We also managed to establish a connection between our worst-case portfolio problem and the classical HJB-framework.

In total, the stress scenario setting is more complex than the classical worst-case portfolio problem. It is thus also not surprising that in the verification theorem we can only give sufficient conditions for a worst-case optimal portfolio strategy.

Chapter 2

Optimal portfolio choice with crash and default risk

2.1 Introduction

Investing in the financial market involves facing the risk of crises and the associated uncertain price (down) jumps or asset defaults. It is above all the infinite possibilities of triggers for such a catastrophic event that makes it so difficult in practice to make suitable probabilistic assumptions regarding its occurrence. Especially large economic or political crises, but also natural catastrophes and pandemics, can occur completely unexpectedly and with far-reaching consequences. It is this particular kind of uncertainty that Frank Knight [25] refers to: "There is a fundamental distinction between the reward for taking a known risk and that for assuming a risk whose value itself is not known." According to him "the present and more important task is to follow out the consequences of that higher form of uncertainty not susceptible to measurement". When choosing an optimal portfolio, it is therefore all the more of interest to consider a model that includes the possibility of an asset crash or default, but at the same time does not require the investor to have a comprehensive knowledge of its underlying probabilities.

In this chapter we would like to generalise the worst-case scenario optimal portfolio approach introduced by Korn and Wilmott [37].

The model that will be considered contains the following two assumptions, which clearly distinguish it from the existing literature:

- In our model, a worst-case scenario does not automatically imply an asset crash, instead there can also be assets with a default risk.
- Furthermore, our investor should not know which asset is affected by the worst-case scenario, hence we extend the worst-case approach so that potentially all assets can be exposed to a crash, respectively a default, but at most one scenario occurs, namely the worst.

Based on the idea of the worst-case scenario approach we have neither information on the crash/default times nor on the corresponding jump intensities.

Even though bonds represent a large part of the financial market, only a relatively small part of the research carried out in the area of continuous time portfolio optimization includes securities that are threatened by default.

Under the assumption of constant interest rate and by assuming the default process to be a Poisson process with a constant intensity, Bielecki and Jung [8] derived a closed form solution for an investor, who can allocate his/her wealth among a defaultable bond, a risk-free account, and a stock. Kraft and Steffensen [38] derive the system of Hamilton-Jacobi-Bellman equations for a general Markov chain setting in which various portfolio problems with defaultable bonds can be analysed. Capponi and Figueroa-López [10] consider a portfolio optimization problem in a defaultable market with finitely many economical regimes. The reduced form approach to credit risk is used for the defaultable bond. Korn and Kraft [28], in contrast, avoid the modelling of default intensities and instead assume that credit risk is modelled by a firm value model.

All these models have in common, that an investor who follows an optimal portfolio process in such a setting only hedges a default situation in the mean which does not protect him/her from a loss if a default actually occurs. This will not be the case for the investor in this chapter, who receives full protection by using the optimal portfolio process.

In order to add defaultable assets to the worst-case scenario optimal portfolio approach, the following steps are carried out in this chapter:

- In section 2.3, we set up a suitable portfolio optimization problem based on a model that is introduced in section 2.2.
- In section 2.4 we consider the post-crash/default problem and in section 2.5 the pre-crash/default portfolio problem is solved with the help of the newly defined worst-case crash/default loss.
- Section 2.6 is used to construct an algorithm for the numerical approximation of the optimal pre-crash/default portfolio process and based on this and section 2.5, we analyse some explicit examples in section 2.7.

2.2 Model setting

In this chapter, we would like to consider a portfolio optimization problem where the investor can allocate his/her wealth to a risk-free money market account (MMA), m assets that are exposed to a crash risk and n defaultable assets. We fix a finite time horizon T and all stochastic processes are defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(W(t))_{t\in[0,T]}$ be a d-dimensional Brownian motion, with $d \geq m+n$. For the modelling of the crashes and defaults we further require $m+n$ one-jump processes $N_1, \ldots, N_{m+n}$ defined as

$$N_i(t) := \mathbb{1}_{\{\xi_i \leq t\}},\ t \in [0,T],$$

for $\xi_i \in [0,T] \cup \{\infty\}$, $i \in \{1, \ldots, m+n\}$, where $\xi_i = \infty$ indicates that the i-th asset can not default, respectively the i-th asset does not suffer a crash. We equip our probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ that satisfies the so-called usual conditions and that includes the filtration generated by the Brownian motion $W(t)$ and the one generated by the jump processes $N_1(t), \ldots, N_{m+n}(t)$. This filtration $\mathbb{F}$ is extended to $[0,T] \cup \{\infty\}$ by letting $\mathcal{F}_\infty := \mathcal{F}_T$, so that we can also model the no-default and no-crash scenarios $\xi_i = \infty$ for $i \in \{1, \ldots, m+n\}$. Thus, the ξ_i, $i \in \{1, \ldots, m+n\}$, are stopping times with respect to $\mathbb{F}$, and we denote, for $t \in [0,T]$, the set of all these $[t,T] \cup \{\infty\}$-valued stopping times by Θ_t, with $\Theta := \Theta_0$.

We are now able to introduce the dynamics of the money market account, of the assets with crash risk and of the defaultable assets. For a constant interest rate $r > 0$, the money market account B follows the ODE

$$dB(t) = rB(t)dt,$$

$t \in [0, T]$, with $B(0) = 1$.

We define the drift vector $b = (b_1, \ldots, b_{m+n})^T$, $b_i > r$ for $i \in \{1, \ldots, m+n\}$, and the $(m+n) \times d$ volatility matrix $\sigma = (\sigma_{ij})$ of full rank, for the $m+n$ assets with default or crash risk.

Then, for $i = 1, \ldots, m$, the dynamics of the assets P_i with crash risk are defined as

$$dP_i(t) = P_i(t-) \left(b_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t) - l_i dN_i(t) \right), \quad P_i(0) = p_{0,i},$$

for a crash rate $0 < l_i < 1$ and initial value $p_{0,i} > 0$.

Before we introduce the defaultable assets, we first define the associated risk-free assets $\tilde{P}_i$, $i \in \{m+1, \ldots, m+n\}$, which will not be tradable in our model. These are given, for $i \in \{m+1, \ldots, m+n\}$, by the dynamics

$$d\tilde{P}_i(t) = \tilde{P}_i(t) \left(b_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t) \right), \quad \tilde{P}_i(0) = p_{0,i},$$

for $p_{0,i} > 0$. We then define the n defaultable assets P_i, for $\xi_i \in \Theta$, $i \in \{m+1, \ldots, m+n\}$ and $t \in [0, \min\{T, \xi_i\}]$ as

$$P_i(t) = (1 - l_i N_i(t))\, \tilde{P}_i(t),$$

where $0 < l_i \leq 1$ is the loss rate of the i-th asset. Thus, $(1 - l_i)$, $i \in \{1, \ldots, n\}$, denotes the fraction of the value of the defaultable asset just prior to default that the investor recovers if the i-th asset defaults.

2.3 The worst-case optimization problem

In the further modelling, we assume that at most one of the $m+n$ risky assets can default or suffer a crash. In this context, our understanding of Knightian uncertainty is, that the investor has no probabilistic assumptions regarding the crash, respectively the default. This means that he/she does not know whether there will be a crash or a default, which asset will be affected in the event of a crash/default and at what time such a crash/default will occur.

Admissible portfolio processes. Before we formulate the portfolio optimization problem we need to define the set of admissible portfolio processes. We aim to ensure that the investor only uses self-financing portfolio processes, that he/she observes the possible crash/default but can only react to it after it has occurred and that ruin is avoided.

Since we assume in our model that at most one crash/default can occur, we first have to define, for $t \in [0, T]$, the set $\bar{\Theta}_t$ of possible crash/default scenarios:

$$\bar{\Theta}_t := \{\xi = (\xi_1, \ldots, \xi_{m+n}) \,|\, \xi_i \in \Theta_t \text{ for } i \in \{1, \ldots, m+n\} \text{ and for all}$$

$$\omega \in \Omega \text{ there exists at most one } i \in \{1, \dots, m+n\} \text{ with } \xi_i(\omega) < \infty\}.$$

Note that it is still a possible scenario that none of the assets defaults and no crash occurs. Further we set $\bar{\Theta} := \bar{\Theta}_0$.

This leads to the following technical assumptions.

Definition 7 (Admissible portfolio process)**.** *Consider the financial market model introduced in section 2.2. Then, the set $\mathbb{A}$ of admissible portfolio processes π, corresponding to an initial capital $x > 0$, consists of $\{\mathcal{F}_t, 0 \le t \le T\}$-predictable processes such that*

1. *for $t \in [0, T]$ and $\xi \in \bar{\Theta}$, $\pi(t)$ is of the form*

$$\pi(t) = \pi_{(0)}(t)\mathbb{1}_{\{t \le \min\{\xi_i | i=1,\dots,m+n\}\}} + \sum_{i=1}^{m+n} \pi_{(i)}(t)\mathbb{1}_{\{t > \xi_i\}},$$

for $\pi_{(0)} = \left(\pi_{(0),1}, \dots, \pi_{(0),m+n}\right)^T$, $\pi_{(i)} = \left(\pi_{(i),1}, \dots, \pi_{(i),m+n}\right)^T$ if $i \in \{1, \dots, m\}$, and $\pi_{(i)} = \left(\pi_{(i),1}, \dots, \pi_{(i),m+n-1}\right)^T$ if $i \in \{m+1, \dots m+n\}$, being stochastic processes with right-continuous paths with left limits on $[0, T]$.

2. *the investor's wealth process X^π is given for $t \in [0, T]$ and $\xi \in \bar{\Theta}$ by*

$$X^\pi(t) = X^{\pi_{(0)}}(t)\mathbb{1}_{\{t < \min\{\xi_i | i=1,\dots,m+n\}\}} + \sum_{i=1}^{m+n} X^{\pi_{(i)}}(t)\mathbb{1}_{\{t \ge \xi_i\}},$$

where $X^{\pi_{(0)}}$ needs to be a strong solution of the stochastic differential equation

$$\begin{aligned} X^{\pi_{(0)}} &= x, \\ dX^{\pi_{(0)}}(t) &= X^{\pi_{(0)}}(t)\Big(\left(r + \pi_{(0)}(t)^T (b - r\underline{1})\right) dt \\ &\qquad + \pi_{(0)}(t)^T \sigma dW(t)\Big) \quad \text{for } t \in [0, T], \end{aligned}$$

and for $i \in \{1, \dots, n+m\}$, $X^{\pi_{(i)}}$ has to solve

$$\begin{aligned} X^{\pi_{(i)}}(\xi_i) &= \left(1 - \pi_{(0),i}(\xi_i) l_i\right) X^{\pi_{(0)}}(\xi_i) \\ dX^{\pi_{(i)}}(t) &= X^{\pi_{(i)}}(t)\Big(\left(r + \pi_{(i)}(t)^T \left(b_{(i)} - r\underline{1}\right)\right) dt \\ &\qquad + \pi_{(i)}(t)^T \sigma_{(i)} dW(t)\Big) \quad \text{for } t \in [\xi_i, T], \end{aligned}$$

on $\{\omega \in \Omega | \xi_i(\omega) < \infty\}$. Here, $b_{(i)} = b \in \mathbb{R}^{m+n}$ and $\sigma_{(i)} = \sigma \in \mathbb{R}^{m+n \times d}$ for $i \in \{1, \dots, m\}$, whereas $b_{(i)} \in \mathbb{R}^{m+n-1}$ and $\sigma_{(i)} \in \mathbb{R}^{m+n-1 \times d}$ are obtained from b and σ by removing the i-th row, for $i \in \{m+1, \dots, m+n\}$.

3. *the following no-ruin and integrability conditions are fulfilled for all $\xi \in \bar{\Theta}$:*

$$\begin{aligned} &X^\pi(t) > 0 \; \forall t \in [0, T], \\ &\int_t^T \pi^2_{(i),j}(s) ds < \infty \; P\text{-a.s. for } i \in \{0, \dots, m\},\; j = 1, \dots, m+n, \end{aligned}$$

$$\int_t^T \pi^2_{(i),j}(s)ds < \infty \text{ P-a.s. for } i \in \{m+1,\dots,m+n\},\ j = 1,\dots,m+n-1.$$

Note that from the no-ruin condition of an admissible portfolio process $\pi \in \mathbb{A}$, it follows that π needs to fulfill

$$\max\left\{\pi_{(0),i}(t)l_i | i = 1,\dots,m+n\right\} < 1,$$

for $t \in [0,T]$.

Worst-case optimization problem. Before we formulate the portfolio optimization problem, we would like to take a closer look at the investor in our model.

We assume that a highly risk-averse investor can invest in the money market account and $m+n$ assets. In order to choose the optimal portfolio process, he/she wants to maximize the expected utility of the terminal wealth. At the same time, the investor wants to take into account the possibility that one of the assets may default, respectively suffer a crash, during the investment period. Due to our above described understanding of Knightian uncertainty and the strong risk aversion, we assume a worst-case scenario for the crash/default.

To describe the investor's risk preferences we use a utility function U with constant relative risk aversion $\gamma > 0$, $\gamma \neq 1$:

$$U : (0,\infty) \to \mathbb{R}, \quad U(x) = \frac{1}{1-\gamma}x^{1-\gamma}.$$

The **worst-case crash/default portfolio problem** of the investor is

$$\sup_{\pi\in\mathbb{A}} \inf_{\xi\in\bar{\Theta}} \mathbb{E}_{0,x,1}\left(U\left(X^\pi(T)\right)\right), \tag{2.1}$$

where for $t \in [0,T]$, $\mathbb{E}_{t,x,1}$ denotes the conditional expectation given that $X^\pi(t) = x$ and the crash/default can still occur. The last index of the conditional expectation is zero if we look at the no-crash and no-default scenario or if we consider the optimization problem after the crash/default has occurred.

2.4 The post-crash/default problem

We start by looking at the portfolio problem after the crash/default. This is the well-known Merton problem, which is why we can directly state the following result.

Proposition 9 (Optimal post-crash/default portfolio process)**.** *Let $\xi \in \bar{\Theta}$, $x > 0$ and $i \in \{1,\dots,m+n\}$. On $\{\omega \in \Omega | \xi_i(\omega) < \infty\}$, the post-crash/default value function $v_i(\xi_i, x)$ is given by*

$$v_i(\xi_i, x) := \sup_{\pi\in\mathbb{A}} \mathbb{E}_{\xi_i,x,0}\left(U\left(X^\pi(T)\right)\right) = \mathbb{E}_{\xi_i,x,0}\left(U\left(X^{\bar{\pi}}(T)\right)\right),$$

for every admissible portfolio process $\bar{\pi} \in \mathbb{A}$ with

$$\bar{\pi}_{(i)} := \arg\max_{\pi} \phi_{(i)}(\pi) = \frac{1}{\gamma}\left(\sigma_{(i)}\sigma_{(i)}^T\right)^{-1}\left(b_{(i)} - r\underline{1}\right),$$

for $\phi_{(i)}(\pi) = r + \pi^T(b_{(i)} - r\underline{1}) - \frac{1}{2}\gamma\pi^T\sigma_{(i)}\sigma_{(i)}^T\pi$ with $\pi \in \mathbb{R}^{m+n}$ if $i \leq m$ and $\pi \in \mathbb{R}^{m+n-1}$ else. Furthermore we obtain the representation

$$v_i(\xi_i, x) = \frac{x^{1-\gamma}}{1-\gamma}\exp\left((1-\gamma)\int_{\xi_i}^T \phi_{(i)}(\bar{\pi}_{(i)})ds\right).$$

Proof. We note that this Proposition considers the portfolio optimization problem after the crash/default, which is the classical Merton problem with random initial time ξ_i. For a proof, we refer to [35]. □

For the solution of the pre-crash/default problem we need a more detailed consideration of the post-crash/default value functions. For this purpose, we introduce in the following Lemma the value function v_0 for the related financial market without crashes and defaults and compare it with v_i for $i \in \{1, \ldots, m+n\}$.

Lemma 5 (Reformulation of the post-crash/default value function)**.** *Define $v_0(t,x)$ for $x > 0$ and $t \in [0,T]$:*

$$v_0(t,x) := \frac{x^{1-\gamma}}{1-\gamma}\exp\left((1-\gamma)\int_t^T \phi_{(0)}(\pi^*)ds\right),$$

for $\phi_{(0)}(\pi) = r + \pi^T(b - r\underline{1}) - \frac{1}{2}\gamma\pi^T\sigma\sigma^T\pi$ and

$$\pi^* := \arg\max_{\pi\in\mathbb{R}^{m+n}} \phi_{(0)}(\pi) = \frac{1}{\gamma}\left(\sigma\sigma^T\right)^{-1}(b - r\underline{1}).$$

Then we obtain:

(i) For $i \in \{1, \ldots, m+n\}$, there exists a unique continuous function $k_i(t)$ on $[0,T]$ such that

$$v_i(t,x) = v_0(t, (1-k_i(t))x), \tag{2.2}$$

for $x > 0$ and $t \in [0,T]$.

(ii) For $i \in \{1, \ldots, m\} \cup \{j \in \{m+1, \ldots, m+n\} | \pi_j^* = 0\}$ we have $k_i(t) = 0$ for all $t \in [0,T]$.

(iii) For $i \in \{j \in \{m+1, \ldots, m+n\} | \pi_j^* \neq 0\}$, k_i is strictly decreasing and strictly concave with $0 \leq k_i(t) < 1$ for all $t \in [0,T]$.

Proof. Before we start to prove part (i), (ii) and (iii), let us consider, for $t \in [0,T]$ and $x > 0$, the portfolio optimization problem

$$\sup_{\pi\in\mathbb{A}} \mathbb{E}_{t,x,0}\left(U\left(X^{\pi_{(0)}}(T)\right)\right).$$

This is the classical crash/default-free Merton problem with optimal portfolio process $\pi \in \mathbb{A}$ with $\pi_{(0)} = \pi^*$ and corresponding value function $v_0(t,x)$, as defined above. For a proof of this basic result, we refer to [26].

(i) Using the definition of v_0 and the representation of v_i from Proposition 9, one can prove that k_i, given by

$$k_i(t) = 1 - \exp\left(\int_t^T \left(\phi_{(i)}(\bar{\pi}_{(i)}) - \phi_{(0)}(\pi^*)\right) ds\right), \quad t \in [0,T],$$

is the unique continuous function, for which equation (2.2) is fulfilled.

(ii) For $i \in \{1,\dots,m\} \cup \{j \in \{m+1,\dots,m+n\} | \pi_j^* = 0\}$ we have $\phi_{(i)}(\bar{\pi}_{(i)}) = \phi_{(0)}(\pi^*)$ and therefore $k_i(t) = 0$ for all $t \in [0,T]$.

(iii) Now let $i \in \{j \in \{m+1,\dots,m+n\} | \pi_j^* \neq 0\}$. Then v_i is the value function of the crash/default-free Merton problem in the post-default market with $m+n-1$ assets and v_0 is the value function of the crash/default-free Merton problem in the corresponding pre-crash/default market with $m+n$ assets. As the post-default market is derived from pre-default market by deleting the i-th asset we obtain that

$$v_0(t,x) \geq v_i(t,x), \tag{2.3}$$

for $x > 0$ and $t \in [0,T]$. The definitions of $b_{(i)}$ and $\sigma_{(i)}$ lead to a strict inequality if $\pi_i^* \neq 0$. From inequality (2.3), we then obtain $\phi_{(i)}(\bar{\pi}_{(i)}) < \phi_{(0)}(\pi^*)$ and thus $0 \leq k(t) < 1$ for all $t \in [0,T]$. To prove the monotonicity and concavity, we consider the first two derivatives of k_i and get for $t \in [0,T]$:

$$k_i'(t) = \exp\left(\int_t^T \left(\phi_{(i)}(\bar{\pi}_{(i)}) - \phi_{(0)}(\pi^*)\right) ds\right) \left(\phi_{(i)}(\bar{\pi}_{(i)}) - \phi_{(0)}(\pi^*)\right) < 0,$$
$$k_i''(t) = -\exp\left(\int_t^T \left(\phi_{(i)}(\bar{\pi}_{(i)}) - \phi_{(0)}(\pi^*)\right) ds\right) \left(\phi_{(i)}(\bar{\pi}_{(i)}) - \phi_{(0)}(\pi^*)\right)^2 < 0.$$

□

Remark 10 (Optimal no-crash/default portfolio process)**.** *Note that $\sigma\sigma^T$ is positive definite and $b_i > r$ for $i \in \{1,\dots,m+n\}$ by assumption. This leads to $(b - r\underline{1})^T \pi^* > 0$, and therefore there hast to exist at least one $i \in \{1,\dots,m+n\}$ with $\pi_i^* > 0$.*

Remark 11 (Reduction of the set of admissible portfolio processes)**.** *Proposition 9 gives us the optimal post-default portfolio process $\bar{\pi}_{(i)}$, for $i \in \{1,\dots,m+n\}$. Thus, we assume from now on, without loss of generality, that we have $\pi_{(i)} = \bar{\pi}_{(i)}$ for every $\pi \in \mathbb{A}$ and $i \in \{1,\dots,m+n\}$.*

2.5 The pre-crash/default problem

We now look again at the worst-case crash/default problem (2.1) with corresponding value function v, defined for $x > 0$ at time $t \in [0, T]$ via

$$v(t,x) := \sup_{\pi \in \mathbb{A}} \inf_{\xi \in \bar{\Theta}_t} \mathbb{E}_{t,x,1} \left(U \left(X^{\pi}(T) \right) \right).$$

In Lemma 6 we want to reformulate this value function, which is why we first define the worst-case crash/default loss.

Definition 8 (Worst-case crash/default loss)**.** *For a portfolio process π we define the crash/default losses CDL_i^{π} for $i \in \{1, \ldots, m+n\}$ at time $t \in [0, T]$, by*

$$1 - CDL_i^{\pi}(t) := (1 - \pi_i(t) l_i)(1 - k_i(t)).$$

The worst-case crash/default loss $CDL^{\pi}(t)$ of π at time $t \in [0, T]$ is defined as

$$CDL^{\pi}(t) := \max\{CDL_i^{\pi}(t) | i \in \{1, \ldots, m+n\}\}.$$

Lemma 6 (Reformulation of the pre-crash/default value function)**.** *For $x > 0$, the value function $v(t,x)$ of the worst-case default problem (2.1), at time $t \in [0, T]$, can be rewritten as*

$$v(t,x) = \sup_{\pi \in \mathbb{A}} \inf_{\xi \in \Theta_t} \mathbb{E}_{t,x} \left(v_0 \left(\xi, (1 - CDL^{\pi_{(0)}}(\xi)) X^{\pi_{(0)}}(\xi) \right) \right),$$

with $v_0 \left(\infty, (1 - CDL^{\pi_{(0)}}(\infty)) X^{\pi_{(0)}}(\infty) \right) := U \left(X^{\pi_{(0)}}(T) \right)$.

Proof. Without loss of generality we give the proof for $t = 0$. For $\xi \in \bar{\Theta}$ we define $\tilde{\xi} := \min\{\xi_i | i \in \{1, \ldots, m+n\}\}$. Then we obtain the following list of equalities:

$$\begin{aligned}
&\sup_{\pi \in \mathbb{A}} \inf_{\xi \in \bar{\Theta}} \mathbb{E}_{0,x,1} \left(U \left(X^{\pi}(T) \right) \right) \\
&= \sup_{\pi \in \mathbb{A}} \inf_{\xi \in \bar{\Theta}} \mathbb{E}_{0,x,1} \left(\mathbb{1}_{\tilde{\xi} = \infty} U \left(X^{\pi_{(0)}}(T) \right) + \mathbb{1}_{\tilde{\xi} < \infty} U \left(X^{\pi}(T) \right) \right) \\
&= \sup_{\pi \in \mathbb{A}} \inf_{\xi \in \bar{\Theta}} \mathbb{E}_{0,x,1} \left(\mathbb{1}_{\tilde{\xi} = \infty} U \left(X^{\pi_{(0)}}(T) \right) + \sum_{i=1}^{m+n} \mathbb{1}_{\xi_i < \infty} \mathbb{E}_{\xi_i, X^{\pi}(\xi_i), 0} \left(U \left(X^{\pi}(T) \right) \right) \right) \\
&= \sup_{\pi \in \mathbb{A}} \inf_{\xi \in \bar{\Theta}} \mathbb{E}_{0,x,1} \left(\mathbb{1}_{\tilde{\xi} = \infty} U \left(X^{\pi_{(0)}}(T) \right) + \sum_{i=1}^{m+n} \mathbb{1}_{\xi_i < \infty} v_i \left(\xi_i, X^{\pi}(\xi_i) \right) \right) \\
&= \sup_{\pi \in \mathbb{A}} \inf_{\xi \in \bar{\Theta}} \mathbb{E}_{0,x,1} \left(\mathbb{1}_{\tilde{\xi} = \infty} U \left(X^{\pi_{(0)}}(T) \right) + \sum_{i=1}^{m+n} \mathbb{1}_{\xi_i < \infty} v_0 \left(\xi_i, (1 - k_i(\xi_i)) X^{\pi_{(i)}}(\xi_i) \right) \right) \\
&= \sup_{\pi \in \mathbb{A}} \inf_{\xi \in \bar{\Theta}} \mathbb{E}_{0,x,1} \Bigg(\mathbb{1}_{\tilde{\xi} = \infty} U \left(X^{\pi_{(0)}}(T) \right) \\
&\qquad + \sum_{i=1}^{m+n} \mathbb{1}_{\xi_i < \infty} v_0 \left(\xi_i, \left(1 - CDL_i^{\pi_{(0)}}(\xi_i) \right) X^{\pi_{(0)}}(\xi_i) \right) \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\pi\in\mathbb{A}} \inf_{\xi\in\Theta} \mathbb{E}_{0,x}\Bigg(\mathbb{1}_{\xi=\infty} U\left(X^{\pi_{(0)}}(T)\right) \\
&\qquad + \mathbb{1}_{\xi<\infty} \min\left\{ v_0\left(\xi, \left(1 - CDL_i^{\pi_{(0)}}(\xi)\right) X^{\pi_{(0)}}(\xi)\right) \middle| i \in \{1,\dots,m+n\}\right\}\Bigg) \\
&= \sup_{\pi\in\mathbb{A}} \inf_{\xi\in\Theta} \mathbb{E}_{0,x}\left(\mathbb{1}_{\xi=\infty} U\left(X^{\pi_{(0)}}(T)\right) + \mathbb{1}_{\xi<\infty} v_0\left(\xi, \left(1 - CDL^{\pi_{(0)}}(\xi)\right) X^{\pi_{(0)}}(\xi)\right)\right) \\
&= \sup_{\pi\in\mathbb{A}} \inf_{\xi\in\Theta} \mathbb{E}_{0,x}\left(v_0\left(\xi, \left(1 - CDL^{\pi_{(0)}}(\xi)\right) X^{\pi_{(0)}}(\xi)\right)\right).
\end{aligned}
$$

We used the tower-property, Remark 11 with the definition of the post-crash/default value function, Lemma 5 and the above defined worst-case crash/default loss. For the transition from $\bar{\Theta}$ to Θ in the sixth equality, it should be noted that for all $\omega \in \Omega$ exists at most one $i \in \{1,\dots,m+n\}$ with $\xi_i(\omega) < \infty$ and that one can assume without loss of generality that

$$
\begin{aligned}
&v_0\left(\xi_i, \left(1 - CDL_i^{\pi_{(0)}}(\xi_i)\right) X^{\pi_{(0)}}(\xi_i)\right) \\
&= \min\left\{ v_0\left(\xi_i, \left(1 - CDL_j^{\pi_{(0)}}(\xi_i)\right) X^{\pi_{(0)}}(\xi_i)\right) \middle| j \in \{1,\dots,m+n\}\right\}.
\end{aligned}
$$

on $\{\omega \in \Omega | \xi_i(\omega) < \infty\}$, for $i \in \{1,\dots,m+n\}$.

□

We would like to use the following two Remarks to give an interpretation of the worst-case crash/default loss and to show the parallels between this chapter and the work on worst case portfolio optimization in the presence of stress scenarios by Korn and Müller [34] (see chapter 1).

Remark 12 (Interpretation of the crash/default loss)**.** *We would like to take a closer look at the definition of the crash/default loss $CDL_i^{\pi_{(0)}}$, for $i \in \{1,\dots m+n\}$ and a portfolio process $\pi \in \mathbb{A}$. For $i \in \{1,\dots,m\} \cup \{j \in \{m+1,\dots,m+n\} | \pi_j^* = 0\}$ we get with Lemma 5 (ii) that $k_i(t) = 0$ for $t \in [0,T]$ and therefore $CDL_i^{\pi_{(0)}} = \pi_{(0),i} l_i$. On the other hand, if $i \in \{j \in \{m+1,\dots,m+n\} | \pi_j^* \neq 0\}$, we know by Lemma 5 (iii) that k_i is strictly deceasing with $0 \leq k_i(t) < 1$ for all $t \in [0,T]$. This means that the crash/default loss $CDL_i^{\pi_{(0)}}$ in this case depends on two factors. Firstly, the direct loss $\pi_{(0),i} l_i$ due to the asset default and secondly, the loss incurred by the investor because he/she can only invest in $m+n-1$ assets in the post-crash/default market, instead of $n+m$ assets in the pre-crash/default market. This second part of the loss is given by k_i.*

The maximum of these crash/default losses at time $t \in [0,T]$ is given by $CDL^{\pi_{(0)}}(t)$, so that the reformulated value function v only depends on one stopping time $\xi \in \Theta$, which determines when or if this maximum loss occurs.

Remark 13 (Optimal portfolios in the presence of stress scenarios)**.** *In [34], Korn and Müller use the worst-case scenario approach to model an investment decision problem that includes stress scenarios. A stress can be compared to a crash of one or more assets. Their model contains several different stresses, which can all occur. The occurrence is modelled via the worst-case approach, i.e. the investor does not know which stresses occur and in which order.*

Both in [34] and in this chapter, one minimises over possible scenarios, whereby each of these scenarios also has a time component, that describes the time of occurrence. In [34], the scenarios are the different orders of stresses and in this chapter the scenarios are given by the individual crashes and defaults. In both works, the problem can be reduced to one time component, i.e. to a single stopping time, by a reformulation as in Lemma 6.

Although two fundamentally different problems are addressed, this leads to the fact that the overall proof idea for the optimal pre-crash/default portfolio process, respectively optimal pre-stress portfolio process, follows the same steps.

We now introduce minimum constant portfolio processes, and prove the existence of such a process in Theorem 5.

Definition 9 (Minimum constant portfolio process)**.** *Let $\pi \in \mathbb{A}$ be an admissible portfolio process.*

1. *We define the minimum process $M^{\pi}(t)$ by*

$$M^{\pi}(t) := v_0\left(t, (1 - CDL^{\pi_{(0)}}(t))X^{\pi_{(0)}}(t)\right) \text{ for } t \in [0, T],$$

   with $M^{\pi}(\infty) := U\left(X^{\pi_{(0)}}(T)\right)$.

2. *We call π* minimum constant *if we have*

$$\mathbb{E}_{0,x}\left(M^{\pi}(\tau_1)\right) = \mathbb{E}_{0,x}\left(M^{\pi}(\tau_2)\right) \text{ for all } \tau_1,\ \tau_2 \in \Theta.$$

The wording minimum process is based on the following reformulation.

Remark 14 (Reformulation of the minimum process)**.** *Note that, by using the definition of v_0 from Lemma 5, one can show that*

$$\begin{aligned} M^{\pi}(t) &= v_0\left(t, (1 - CDL^{\pi_{(0)}}(t))X^{\pi_{(0)}}(t)\right) \\ &= \min\{v_0\left(t, (1 - CDL_i^{\pi_{(0)}}(t))X^{\pi_{(0)}}(t)\right) | i \in \{1, \dots, m+n\}\}, \end{aligned}$$

for $t \in [0, T]$.

The following Theorem shows that we can prove the existence of a minimum constant portfolio process with the help of an one-dimensional coupled, non-linear ordinary differential equation.

Theorem 5 (Existence of a minimum constant portfolio process)**.** *There exists a minimum constant portfolio process $\bar{\pi} \in \mathbb{A}$, such that*

$$\bar{\pi}_{(0)}(t) \in \underset{\substack{\pi \in \mathbb{R}^{m+n} \\ CDL^{\pi}(t) = N(t)}}{\arg\max} \ \phi_{(0)}(\pi) \quad \text{for } t \in [0, T],$$

where N is defined for $t \in [0,T]$ by the differential equation

$$N'(t) = (1 - N(t)) \left(\max_{\substack{\pi \in \mathbb{R}^{m+n} \\ CDL^{\pi}(t) = N(t)}} \phi_{(0)}(\pi) - \phi_{(0)}(\pi^*) \right),$$

with $N(T) = 0$. Further, for all $t \in [0,T]$, we have $CDL^{\bar{\pi}_{(0)}}(t) < 1$.

Proof. With the same arguments as in [34] one can prove the existence of an admissible portfolio process $\bar{\pi}$ with $CDL^{\bar{\pi}_{(0)}}(t) < 1$, as claimed in the assertion. We will now use similar arguments as in [34] , to show the minimum constant property of the defined portfolio process. For this note, that we get from the definition of $\bar{\pi}$ that

$$\left(CDL^{\bar{\pi}_{(0)}}\right)'(t) = (1 - CDL^{\bar{\pi}_{(0)}}(t)) \left(\phi_{(0)}(\bar{\pi}_{(0)}(t)) - \phi_{(0)}(\pi^*) \right).$$

We use this equation together with Itô's rule and obtain

$$dv_0\left(t, x(t)\right) = v_0\left(t, x(t)\right)(1-\gamma)\, \bar{\pi}_{(0)}(t)^T \sigma dW(t),$$

for $x(t) = (1 - CDL^{\bar{\pi}_{(0)}}(t)) X^{\bar{\pi}_{(0)}}(t)$. Solving this SDE leads to

$$\begin{aligned} v_0\left(t, x(t)\right) = v_0\left(0, x(0)\right) \cdot \exp\Big(&(1-\gamma) \int_0^t \bar{\pi}_{(0)}(s)^T \sigma dW(s) \\ &- \frac{1}{2}(1-\gamma)^2 \int_0^t \bar{\pi}_{(0)}(s)^T \sigma \sigma^T \bar{\pi}_{(0)}(s) ds \Big), \end{aligned}$$

where the second factor is a martingale by Novikov's condition and the integrability assumptions of an admissible portfolio process. With the terminal condition $CDL^{\bar{\pi}_{(0)}}(T) = 0$ and Doob's optional sampling theorem we obtain the minimum constant property. □

Remark 15 ($CDL^{\bar{\pi}_{(0)}}$ of the minimum constant process $\bar{\pi}$)**.** *Let $\bar{\pi} \in \mathbb{A}$ be a minimum constant portfolio process, defined as in Theorem 5. Note that by construction of $\bar{\pi}$, the corresponding crash/default loss $CDL^{\bar{\pi}_{(0)}}$ fulfills:*

$$\begin{aligned} \left(CDL^{\bar{\pi}_{(0)}}\right)'(t) &= (1 - CDL^{\bar{\pi}_{(0)}}(t)) \left(\max_{\substack{\pi \in \mathbb{R}^{m+n} \\ CDL^{\pi}(t) = CDL^{\bar{\pi}_{(0)}}(t)}} \phi_{(0)}(\pi) - \phi_{(0)}(\pi^*) \right) \\ &= (1 - CDL^{\bar{\pi}_{(0)}}(t)) \left(\phi_{(0)}(\bar{\pi}_{(0)}(t)) - \phi_{(0)}(\pi^*) \right). \end{aligned}$$

Before we prove the minimum constant optimality condition in Proposition 10 we need to cite the following helpful Lemma.

Lemma 7 (Terminal Utility Decomposition)**.** *Let $\pi \in \mathbb{A}$. Then we obtain the following decomposition for $t \in [0,T]$:*

$$U(X^{\pi_{(0)}}(T)) = U(X^{\pi_{(0)}}(t)) \exp\left((1-\gamma) \int_t^T \phi_{(0)}(\pi_{(0)}(s)) ds \right) Y_t^T(\pi_{(0)}),$$

where $Y(\pi_{(0)}) = \left(Y_t^s(\pi_{(0)})\right)_{s\in[t,T]}$ *is a positive* $\{\mathcal{F}_s\}$*-martingale with* $Y_t^t(\pi_{(0)}) = 1$.

Proof. See [35]. □

We now use this result to state a condition for a minimum constant portfolio process to be optimal.

Proposition 10 (Minimum constant optimality condition)**.** *Let* $\bar{\pi} \in \mathbb{A}$ *be a minimum constant portfolio process with*

$$\bar{\pi}_{(0)}(t) \in \underset{\substack{\pi\in\mathbb{R}^{m+n} \\ CDL^{\pi}(t)\leq CDL^{\bar{\pi}_{(0)}}(t)}}{\arg\max} \phi_{(0)}(\pi) \quad \textit{for all } t \in [0,T].$$

Then $\bar{\pi}$ *solves the worst-case crash/default portfolio problem* (2.1).

Proof. Let $\bar{\pi} \in \mathbb{A}$ be a minimum constant portfolio process. For $\pi \in \mathbb{A}$ we define η^{π} by

$$\eta^{\pi} := \inf\{t \in [0,T] | CDL^{\pi_{(0)}}(t) > CDL^{\bar{\pi}_{(0)}}(t)\},$$

with the standard convention $\inf\emptyset = \infty$. We now give the proof in two steps. In step (i) we show that for every portfolio process $\pi \in \mathbb{A}$ there exists another portfolio process $\tilde{\pi} \in \mathbb{A}$ with $\eta^{\tilde{\pi}}(\omega) = \infty$ for all $\omega \in \Omega$ and whose worst-case performance is at least as good as the one of π. In step (ii) we prove that $\bar{\pi}$ has the best worst-case performance in the set of all portfolio processes $\pi \in \mathbb{A}$ with $\eta^{\pi}(\omega) = \infty$ for all $\omega \in \Omega$.

(i) For $\pi \in \mathbb{A}$, $t \in [0,T]$, we define

$$\tilde{\pi}_{(0)}(t) = \begin{cases} \pi_{(0)}(t) & \text{if } t < \eta^{\pi}, \\ \bar{\pi}_{(0)}(t) & \text{if } t \geq \eta^{\pi}. \end{cases}$$

It follows directly that $\tilde{\pi} \in \mathbb{A}$ and $\eta^{\tilde{\pi}}(\omega) = \infty$ for all $\omega \in \Omega$. Further we have $M^{\pi}(t) = M^{\tilde{\pi}}(t)$ for $t \in [0, \eta^{\pi})$ and together with the right-continuous paths condition of admissible portfolio processes and the definition of η^{π} we obtain

$$\begin{aligned}
&\mathbb{E}_{0,x}\left(M^{\pi}(\eta^{\pi})\right) = \mathbb{E}_{0,x}\left(v_0\left(\eta^{\pi}, (1 - CDL^{\pi_{(0)}}(\eta^{\pi}))X^{\pi_{(0)}}(\eta^{\pi})\right)\right) \\
=&\mathbb{E}_{0,x}\Big(\mathbb{1}_{\eta^{\pi}=\infty}U\left(X^{\pi_{(0)}}(T)\right) \\
&+ \mathbb{1}_{\eta^{\pi}<\infty}v_0\left(\eta^{\pi}, (1 - CDL^{\pi_{(0)}}(\eta^{\pi}))X^{\pi_{(0)}}(\eta^{\pi})\right)\Big) \\
\leq&\mathbb{E}_{0,x}\Big(\mathbb{1}_{\eta^{\pi}=\infty}U\left(X^{\tilde{\pi}_{(0)}}(T)\right) \\
&+ \mathbb{1}_{\eta^{\pi}<\infty}v_0\left(\eta^{\pi}, (1 - CDL^{\bar{\pi}_{(0)}}(\eta^{\pi}))X^{\tilde{\pi}_{(0)}}(\eta^{\pi})\right)\Big) = \mathbb{E}_{0,x}\left(M^{\tilde{\pi}}(\eta^{\pi})\right).
\end{aligned}$$

In addition note that $\tilde{\pi}$ is minimum constant for all $[\eta^{\pi}, T] \cup \{\infty\}$-valued stopping times. If we now define by $\Theta^{\eta^{\pi}}$ the set of $[0, \eta^{\pi}]$-valued stopping times, we obtain

$$\inf_{\xi\in\Theta} \mathbb{E}_{0,x}\left(M^{\tilde{\pi}}(\xi)\right) = \inf_{\xi\in\Theta^{\eta^{\pi}}} \mathbb{E}_{0,x}\left(M^{\tilde{\pi}}(\xi)\right) \geq \inf_{\xi\in\Theta^{\eta^{\pi}}} \mathbb{E}_{0,x}\left(M^{\pi}(\xi)\right)$$

$$\geq \inf_{\xi \in \Theta} \mathbb{E}_{0,x}\left(M^{\pi}(\xi)\right).$$

Together with Lemma 9 we get, that we can reduce the search for the optimal pre-crash/default portfolio process to the set of portfolio processes $\pi \in \mathbb{A}$ with $\eta^{\pi}(\omega) = \infty$ for all $\omega \in \Omega$.

(ii) Now let $\pi \in \mathbb{A}$ with $\eta^{\pi}(\omega) = \infty$ for all $\omega \in \Omega$. From the definition of η^{π} we obtain $CDL^{\pi_{(0)}}(t) \leq CDL^{\bar{\pi}_{(0)}}(t)$ for all $t \in [0,T]$. Using the terminal utility decomposition from Lemma 7 and the assumption of this Theorem leads to

$$\begin{aligned}
&\inf_{\xi \in \Theta} \mathbb{E}_{0,x}\left(M^{\pi}(\xi)\right) \leq \mathbb{E}_{0,x}\left(M^{\pi}(\infty)\right) = \mathbb{E}_{0,x}\left(U(X^{\pi_{(0)}}(T))\right) \\
=&\mathbb{E}_{0,x}\left(U(x)\exp\left((1-\gamma)\int_0^T \phi_{(0)}(\pi_{(0)}(t))dt\right)Y_0^T(\pi_{(0)})\right) \\
\leq&\mathbb{E}_{0,x}\left(U(x)\exp\left((1-\gamma)\int_0^T \phi_{(0)}(\bar{\pi}_{(0)}(t))dt\right)Y_0^T(\pi_{(0)})\right) \\
=&\mathbb{E}_{0,x}\left(U(x)\exp\left((1-\gamma)\int_0^T \phi_{(0)}(\bar{\pi}_{(0)}(t))dt\right)Y_0^T(\bar{\pi}_{(0)})\right) \\
=&\mathbb{E}_{0,x}\left(M^{\bar{\pi}}(\infty)\right) = \inf_{\xi \in \Theta} \mathbb{E}_{0,x}\left(M^{\bar{\pi}}(\xi)\right).
\end{aligned}$$

With the definition of a minimum constant portfolio process and Lemma 6 we get directly the optimality of $\bar{\pi}$.

□

Before we can prove the optimality of the minimum constant portfolio process defined in Theorem 5, we need to take a look at the following Lemmata.

Lemma 8 (Properties of CDL^{π^*})**.** *Let π^* be defined as in Proposition 5 and for*

$$I_{k=0} := \{1,\dots,m\} \cup \{j \in \{m+1,\dots,m+n\} | \pi_j^* = 0\},$$

we define

$$t_0 := \inf\{t \in [0,T] | CDL^{\pi^*}(t) = CDL_i^{\pi^*}(t) \text{ for an } i \in I_{k=0}\}.$$

Further assume that $\max\{\pi_i^* l_i | i \in \{1,\dots,m+n\}\} < 1$.

Then CDL^{π^*} is constant on $[t_0,T]$, if $t_0 \leq T$, and for $t \in (0, \min\{t_0,T\}]$ the left-hand derivative $\frac{\partial^-}{\partial t}CDL^{\pi^*}(t)$ exists with $\frac{\partial^-}{\partial t}CDL^{\pi^*}(t) < 0$, if $t_0 > 0$.

Proof. Since π^* is constant, we get together with the Definition of $CDL_i^{\pi^*}$, $i \in \{1,\dots,m+n\}$, and Lemma 5 (ii), that $CDL_i^{\pi^*}$ is constant if $i \in I_{k=0}$. Further we obtain with Lemma 5 (iii), together with $\max\{\pi_i^* l_i | i \in \{1,\dots,m+n\}\} < 1$, that $CDL_i^{\pi^*}$ is strictly decreasing and strictly concave if $i \in \{1,\dots,m+n\} \setminus I_{k=0}$.

If $t_0 \leq T$, we obtain from the continuity of CDL^{π^*} and the definition of t_0 that CDL^{π^*} is constant on $[t_0,T]$.

On the interval $(0, \min\{t_0, T\}]$, for $t_0 > 0$, we now that CDL^{π^*} is the maximum of finitely many strictly decreasing and strictly concave functions. By Lemma 2.11. in [15], these functions can only have finitely many intersections. Thus the left-hand derivative $\frac{\partial^-}{\partial t} CDL^{\pi^*}(t)$ exists for $t \in (0, \min\{t_0, T\}]$ and is strictly negative. □

Lemma 9 (Properties of $CDL^{\bar{\pi}_{(0)}}$)**.** *Let $\bar{\pi} \in \mathbb{A}$ be a minimum constant portfolio process as defined in Theorem 5 and π^* be as defined in Proposition 5. Then we obtain for $CDL^{\bar{\pi}_{(0)}}$:*

(i) If there exists an $s \in [0, T]$ with $CDL^{\bar{\pi}_{(0)}}(s) = CDL^{\pi^*}(s)$, then the derivative of $CDL^{\bar{\pi}_{(0)}}$ is zero in s, i.e. $\left(CDL^{\bar{\pi}_{(0)}}\right)'(s) = 0$.

(ii) If there exists a subinterval $[s_1, s_2] \subset [0, T]$, such that $CDL^{\bar{\pi}_{(0)}}(s_2) = CDL^{\pi^*}(s_2)$ and CDL^{π^*} is constant on $[s_1, s_2]$, then $CDL^{\bar{\pi}_{(0)}}(t) = CDL^{\pi^*}(t)$ for $t \in [s_1, s_2]$.

Proof. For the proof we use the representation of the crash/default loss as a differential equation.

(i) We get with Remark 15 that

$$\left(CDL^{\bar{\pi}_{(0)}}\right)'(s) = (1 - CDL^{\bar{\pi}_{(0)}}(s)) \left(\max_{\substack{\pi \in \mathbb{R}^{m+n} \\ CDL^{\pi}(s) = CDL^{\bar{\pi}_{(0)}}(s)}} \phi_{(0)}(\pi) - \phi_{(0)}(\pi^*) \right).$$

Further we know from the definition of s that $CDL^{\bar{\pi}_{(0)}}(s) = CDL^{\pi^*}(s)$. Using the definition of π^*, given in Proposition 5, we obtain that

$$\left(CDL^{\bar{\pi}_{(0)}}\right)'(s) = (1 - CDL^{\bar{\pi}_{(0)}}(s)) \left(\phi_{(0)}(\pi^*) - \phi_{(0)}(\pi^*) \right) = 0.$$

(ii) Consider N, defined in Theorem 5 as

$$N'(t) = (1 - N(t)) \left(\max_{\substack{\pi \in \mathbb{R}^{m+n} \\ CDL^{\pi}(t) = N(t)}} \phi_{(0)}(\pi) - \phi_{(0)}(\pi^*) \right). \tag{2.4}$$

By assumption we have

$$N(s_2) = CDL^{\bar{\pi}_{(0)}}(s_2) = CDL^{\pi^*}(s_2),$$

and thus we get that $N(t) = CDL^{\pi^*}(t)$ solves the differential equation (2.4) for $t \in [s_1, s_2]$, since CDL^{π^*} is constant on $[s_1, s_2]$ and

$$\begin{aligned} \left(CDL^{\pi^*}\right)'(t) &= (1 - CDL^{\pi^*}(t)) \left(\max_{\substack{\pi \in \mathbb{R}^{m+n} \\ CDL^{\pi}(t) = CDL^{\pi^*}(t)}} \phi_{(0)}(\pi) - \phi_{(0)}(\pi^*) \right) \\ &= 0, \end{aligned}$$

with the same arguments as in part (i). Together with the construction of $\bar{\pi}_{(0)}$ in Theorem 5 we get $CDL^{\bar{\pi}_{(0)}}(t) = CDL^{\pi^*}(t)$ for $t \in [s_1, s_2]$.

□

Now we want to use Lemma 8 and Lemma 9 to give an upper bound for the worst-case crash/default loss of the minimum constant portfolio process $\bar{\pi}$, derived in Theorem 5.

Lemma 10 (Crash/Default loss of a minimum process)**.** *Let $\bar{\pi} \in \mathbb{A}$ be a minimum constant portfolio process as defined in Theorem 5. Then we get for π^*, defined in Proposition 5, that $CDL^{\bar{\pi}_{(0)}}(t) \leq CDL^{\pi^*}(t)$ for all $t \in [0, T]$.*

Proof. Depending on $\max\{\pi_i^* l_i | i \in \{1, \dots, m+n\}\}$, we divide the proof into part (i) and (ii).

(i) First, we consider the case that $\max\{\pi_i^* l_i | i \in \{1, \dots, m+n\}\} > 1$. Note that $k_i(t) < 1$ for all $t \in [0, T]$ and $i \in \{1, \dots, m+n\}$. Together with the definition of CDL^{π^*} we obtain that $CDL^{\pi^*} \geq 1$ and thus $CDL^{\bar{\pi}_{(0)}}(t) \leq CDL^{\pi^*}(t)$ for all $t \in [0, T]$, since $CDL^{\bar{\pi}_{(0)}}(t) < 1$ for all $t \in [0, T]$ by Theorem 5.

(ii) For $\max\{\pi_i^* l_i | i \in \{1, \dots, m+n\}\} < 1$ we want to proof this Lemma by contradiction. For this we assume that there exists a $\bar{t} \in [0, T]$ with $CDL^{\bar{\pi}_{(0)}}(\bar{t}) > CDL^{\pi^*}(\bar{t})$.

Further let t_0 be defined as in Lemma 8. Then the proof consists of three steps. Under the above assumption we prove in step (a) the existence of an intersection point s of CDL^{π^*} and $CDL^{\bar{\pi}_{(0)}}$. In step (b) and (c) we consider the cases $s \leq t_0$ and $s > t_0$ and lead the assumption to a contradiction.

(a) With Remark 10 we obtain that

$$CDL^{\bar{\pi}_{(0)}}(T) = 0 < CDL^{\pi^*}(T).$$

Then the assumption, as well as the continuity of $CDL^{\bar{\pi}_{(0)}}$ and CDL^{π^*}, lead to the existence of a point of intersection $s \in (\bar{t}, T)$ and of an $\epsilon > 0$, such that $CDL^{\pi^*}(s) = CDL^{\bar{\pi}_{(0)}}(s)$ and

$$CDL^{\pi^*}(s-h) - CDL^{\bar{\pi}_{(0)}}(s-h) < 0 \text{ for all } h \in (0, \epsilon), \tag{2.5}$$

and thus

$$\lim_{\substack{h \to 0 \\ h > 0}} \frac{\left(CDL^{\pi^*}(s-h) - CDL^{\bar{\pi}_{(0)}}(s-h)\right) - \left(CDL^{\pi^*}(s) - CDL^{\bar{\pi}_{(0)}}(s)\right)}{h} \leq 0. \tag{2.6}$$

(b) If $s \leq t_0$, we know from Lemma 8 that $\frac{\partial^-}{\partial t} CDL^{\pi^*}(s) < 0$. Together with inequality (2.6), we obtain

$$\begin{aligned} 0 > \frac{\partial^-}{\partial t} CDL^{\pi^*}(s) &= \lim_{\substack{h \to 0 \\ h > 0}} \frac{CDL^{\pi^*}(s) - CDL^{\pi^*}(s-h)}{h} \\ &\geq \lim_{\substack{h \to 0 \\ h > 0}} \frac{CDL^{\bar{\pi}_{(0)}}(s) - CDL^{\bar{\pi}_{(0)}}(s-h)}{h} = \left(CDL^{\bar{\pi}_{(0)}}\right)'(s). \end{aligned}$$

This is a contradiction, since we get by the construction of s and Lemma 9 (i) that $\left(CDL^{\bar{\pi}_{(0)}}\right)'(s) = 0$.

(c) If $s > t_0$, we get from Lemma 8 that CDL^{π^*} is constant on $[t_0, s]$. By the definition of s and with part (ii) of Lemma 9 we obtain that $CDL^{\pi^*}(t) = CDL^{\bar{\pi}_{(0)}}(t)$ for all $t \in [t_0, s]$, which is a contradiction to (2.5).

□

Using this property helps us to prove the optimality of the minimum constant portfolio process defined in Theorem 5.

Theorem 6 (Worst-case optimal pre-crash/default portfolio process)**.** *A minimum constant portfolio process $\bar{\pi} \in \mathbb{A}$, as defined in Theorem 5, solves the worst-case crash/default portfolio problem* (2.1)*.*

Proof. Let $\bar{\pi} \in \mathbb{A}$ be a minimum constant portfolio process as defined in Theorem 5. To prove the optimality it is enough to show that $\bar{\pi}$ fulfills the assumptions of Proposition 10, i.e. we need to prove that

$$\bar{\pi}_{(0)}(t) \in \underset{\substack{\pi \in \mathbb{R}^{m+n} \\ CDL^{\pi}(t) \leq CDL^{\bar{\pi}_{(0)}}(t)}}{\arg\max} \phi_{(0)}(\pi) \quad \text{for all } t \in [0, T].$$

For this note that $\phi_{(0)}(\pi)$ is a strictly concave quadratic function in π, which is maximized under linear constraints. For the unconstrained optimizer π^* we get by Lemma 10 that $CDL^{\bar{\pi}_{(0)}}(t) \leq CDL^{\pi^*}(t)$ for all $t \in [0, T]$. It follows that the optimal solution π_c of the above optimization problem is a boundary point of the side constraint, i.e. $CDL^{\pi_c}(t) = CDL^{\bar{\pi}_{(0)}}(t)$ for all $t \in [0, T]$. By construction of $\bar{\pi}$ we know that

$$\bar{\pi}_{(0)}(t) \in \underset{\substack{\pi \in \mathbb{R}^{m+n} \\ CDL^{\pi}(t) = CDL^{\bar{\pi}_{(0)}}(t)}}{\arg\max} \phi_{(0)}(\pi) \quad \text{for } t \in [0, T],$$

and thus $\bar{\pi}$ solves the worst-case crash/default portfolio problem (2.1).

□

Remark 16 (No-short selling constraint and maximum crash/loss rate)**.** *In the majority of the existing worst-case scenario portfolio optimization literature, including the initial work by Korn and Wilmott [26], a short position in the risky assets is ruled out by the set of admissible portfolio processes. Furthermore, no deterministic jump height is assumed for the crash, but a maximum height. We would now like to briefly point out that it is also possible to include these two assumptions in our extension of the model. We consider the following modification of the portfolio optimization problem* (2.1)*:*

$$\sup_{\pi \in \mathbb{A}_0} \inf_{\substack{\xi \in \bar{\Theta} \\ l \in (0, l^*]}} \mathbb{E}_{0,x,1}\left(U\left(X^{\pi}(T)\right)\right),$$

where the set of admissible portfolio processes is defined as

$$\mathbb{A}_0 := \{\pi \in \mathbb{A} | \pi(t) \text{ is non-negative for all } t \in [0,T]\},$$

and where $l^* \in (0,1)$ *denotes a maximum crash/loss rate, such that*

$$l_1 = \ldots = l_{m+n} = l \leq l^*.$$

This means, that if there is a crash or default of the i-th asset at time $\xi_i \in \Theta$*, the investor suffers a direct non-negative loss* $\pi_{(0),i}(\xi_i)l \geq 0$.

This extension of the portfolio problem can be solved quite analogously. The necessary adjustment for the optimal post-crash/default portfolio process is the restriction of the optimization problem in Proposition 9 to non-negative portfolio processes.

The pre-crash/default value function can be reformulated analogously to Lemma 6 so that the following representation is obtained

$$v(t,x) = \sup_{\pi \in \mathbb{A}_0} \inf_{\substack{\xi \in \Theta_t \\ l \in (0,l^*]}} \mathbb{E}_{t,x} \left(v_0 \left(\xi, (1 - CDL^{\pi_{(0)}}(\xi)) X^{\pi_{(0)}}(\xi) \right) \right),$$

for $t \in [0,T]$ *and* $x > 0$*. With the definition of the worst-case crash/default loss* $CDL^{\pi_{(0)}}$ *and considering the no-short selling constraint, we can set* $l = l^*$ *without loss of generality. To prove the existence of an optimal minimum constant portfolio process, we only need to restrict the admissible sets in the optimization problems of section 2.4 to non-negative solutions. In particular, we should note that the differential equation in Theorem 5 is still well-defined. This can be proven with Lemma 4.2. in [56].*

2.6 Construction of the optimal portfolio process

In most of the examples considered in section 2.7, it will be necessary to compute numerically the optimal portfolio process defined in Theorem 5. To do this, a differential equation must be solved together with a quadratic optimization problem. In order to make this process more efficient, we first consider the optimal portfolio process using the Lagrangian-multiplier methods and then set up an algorithmic framework.

Analysis of the optimal portfolio process $\bar{\pi}$. We look at a minimum constant portfolio process $\bar{\pi} \in \mathbb{A}$ as defined in Theorem 5. Let us define the set $I(t)$ of worst-case scenarios at time $t \in [0,T]$ by

$$I(t) := \{i \in \{1,\ldots,m+n\} | CDL_i^{\bar{\pi}_{(0)}}(t) = CDL^{\bar{\pi}_{(0)}}(t)\}. \tag{2.7}$$

In Theorem 5, $\bar{\pi}$ is defined as

$$\bar{\pi}_{(0)}(t) \in \underset{\substack{\pi \in \mathbb{R}^{m+n} \\ CDL^{\pi}(t) = N(t)}}{\arg\max} \quad \phi_{(0)}(\pi) \quad \text{for } t \in [0,T],$$

where N is given for $t \in [0, T]$ by the differential equation

$$N'(t) = (1 - N(t)) \left(\max_{\substack{\pi \in \mathbb{R}^{m+n} \\ CDL^{\pi}(t) = N(t)}} \phi_{(0)}(\pi) - \phi_{(0)}(\pi^*) \right),$$

with $N(T) = 0$.

Using Lagrangian-multiplier methods we obtain that $\bar{\pi}$ solves the first order conditions of the constrained optimization problem, i.e. the following equations hold true for $t \in [0, T]$:

$$\begin{aligned} 0 &= -\gamma \sigma \sigma^T \bar{\pi}_{(0)}(t) + (b - r\underline{1}) - \sum_{i \in I(t)} \lambda_i(t) l_i (1 - k_i(t)) \underline{1}_i, \\ 0 &= N(t) - CDL_i^{\bar{\pi}_{(0)}}(t) \quad \text{for } i \in I(t), \end{aligned}$$

with $\lambda_i(t) \in \mathbb{R}$ for $i \in I(t)$. Here, $\underline{1}_i \in \mathbb{R}^{m+n}$ denotes a vector where the i-th entry is one and all others are zero. The first equation leads to

$$\bar{\pi}_{(0)}(t) = \pi^* - \frac{1}{\gamma} \left(\sigma \sigma^T \right)^{-1} \sum_{i \in I(t)} \lambda_i(t) l_i (1 - k_i(t)) \underline{1}_i. \tag{2.8}$$

From the remaining equations we get

$$\bar{\pi}_{(0),i}(t) = \frac{1}{l_i} - \frac{1 - N(t)}{l_i(1 - k_i(t))} \quad \text{for } i \in I(t).$$

Inserting (2.8) leads to $|I(t)|$ equations of the form

$$\left(\pi^* - \frac{1}{\gamma} \left(\sigma \sigma^T \right)^{-1} \sum_{i \in I(t)} \lambda_i(t) l_i (1 - k_i(t)) \underline{1}_i \right)_i = \frac{1}{l_i} - \frac{1 - N(t)}{l_i(1 - k_i(t))}, \tag{2.9}$$

for $i \in I(t)$, where $N(t)$ solves

$$N'(t) = (1 - N(t)) \left(\phi_{(0)}(\bar{\pi}_{(0)}(t)) - \phi_{(0)}(\pi^*) \right),$$

with $N(T) = 0$.

Algorithmic framework for the construction of $\bar{\pi}_{(0)}$. We would like to use the above observations to specify an algorithmic framework for computing a numerical approximation of the optimal pre-crash/default portfolio process. For this purpose, we use the first-order Euler method and reduce the solving of the optimization problem to the linear equation system (2.9), belonging to the Lagrangian-multipliers. This leads us to the framework stated in Algorithm 1.

This algorithm essentially consists of two parts. In the first part, the variables are initialised and the optimal pre-crash/default portfolio process $\bar{\pi}_{(0)}$ at time T is determined. For this, we use the boundary condition $N(T) = 0$ from Theorem 5. In the second part, we go backwards in time and compute $\bar{\pi}_{(0)}$. To do this, we use use the first-order Euler method

to approximate N. For the further procedure note, that we know from the above considerations that there has to exist a set I, defined by (2.7), such that the solution of the system of equations (2.9) leads together with (2.8) to the optimal portfolio process $\bar{\pi}_{(0)}$. At each point in time $t_k \in [0, T]$, we iterate over the finitely many candidates J for I. For every J we use the system of equations (2.9) and equation (2.8) to determine a candidate π_J. From Theorem 5 and the proof of Theorem 6 we know that the optimal pre-crash/default portfolio process $\bar{\pi}_{(0)}$ at time t_k is given by the process π_{J^*}, that maximizes the function $\phi_{(0)}$ over all processes π_J, with $J = \{i \in \{1, \dots, m+n\} | CDL_i^{\pi_J}(t_k) = CDL^{\pi_J}(t_k)\}$.

Algorithm 1: Numerical construction of $\bar{\pi}_{(0)}$

Input: Number of steps M

Output: Approximation of the optimal pre-crash/default portfolio process $\bar{\pi}_{(0)}$ and the set of worst-case scenarios I

`/* Initialise the grid. */`

Set $t_k = Tk/M$ for $k = 0, \dots, M$.

Based on the boundary condition set $N(t_M) = 0$.

`/* The next step can be done by using Lagrangian-multiplier methods. */`

Set $\bar{\pi}_{(0)}(t_M) = \arg\max_{\substack{\pi \in \mathbb{R}^{m+n} \\ CDL^{\pi}(t_M)=0}} \phi_{(0)}(\pi)$.

Set $I(t_M) = \{i \in \{1, \dots, m+n\} | CDL_i^{\bar{\pi}_{(0)}}(t_M) = CDL^{\bar{\pi}_{(0)}}(t_M)\}$.

`/* Based on the previous considerations, we can go backwards in time. */`

for *k=M-1:0* **do**

  Set $N(t_k) = N(t_{k+1}) - \frac{T}{M}(1 - N(t_{k+1}))\left(\phi_{(0)}(\bar{\pi}_{(0)}(t_{k+1})) - \phi_{(0)}(\pi^*)\right)$.

  for $J \subset \{1, \dots, m+n\}$, $J \neq \emptyset$ **do**

    Solve the following system of linear equations for λ_i^J, $i \in J$:

$$\left(\pi^* - \frac{1}{\gamma}\left(\sigma\sigma^T\right)^{-1} \sum_{i \in J} \lambda_i(t_k) l_i (1 - k_i(t)) \underline{1}_i\right)_i = \frac{1}{l_i} - \frac{1 - N(t_k)}{l_i(1 - k_i(t_k))},$$

    for $i \in J$.

    Compute $\pi_J = \pi^* - \left(\sigma\sigma^T\right)^{-1} \sum_{i \in J} \lambda_i^J l_i (1 - k_i(t)) \underline{1}_i$.

    Set $I_J = \{i \in \{1, \dots, m+n\} | CDL_i^{\pi_J}(t_k) = CDL^{\pi_J}(t_k)\}$.

  Choose

$$J^* \in \arg\max_{\substack{J \subset \{1, \dots, m+n\}, J \neq \emptyset \\ J = I_J}} \phi_{(0)}(\pi_J).$$

  Set $I(t_k) = J^*$ and $\bar{\pi}_{(0)}(t_k) = \pi_{J^*}$.

2.7 Analysis of selected examples

We now use the results from section 2.5 and section 2.6 to examine a few selected examples.

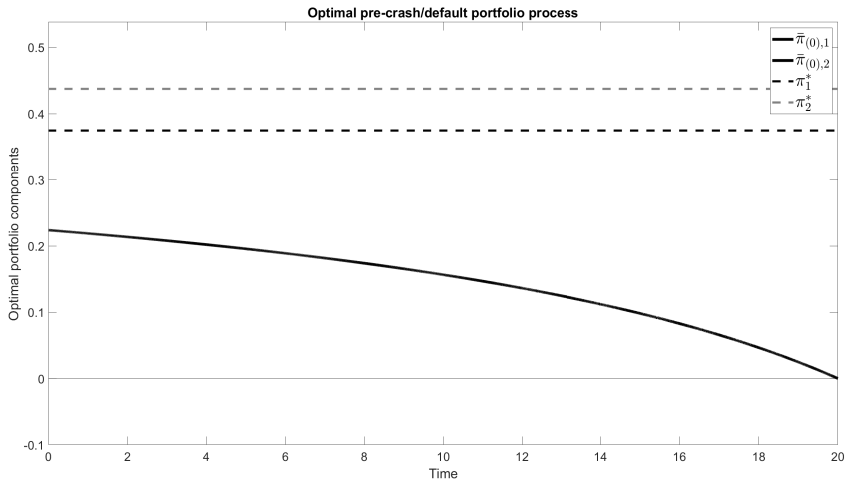

FIGURE 2.1: Set II with $m = 2$, $n = 0$ and $l_1 = l_2 = 0.5$.

For the numerical considerations in this section, we set $T = 20$, $\gamma = 2$, $r = 0$ and consider two assets with $b = (0.03, 0.035)^T$ and three different volatility matrices:

$$\begin{aligned}
\text{Set I: } \sigma_{11} &= \sigma_{22} = 0.2,\ \sigma_{12} = 0,\ \text{and } \sigma_{21} = 0.15,\\
\text{Set II: } \sigma_{11} &= \sigma_{22} = 0.2 \text{ and } \sigma_{12} = \sigma_{21} = 0,\\
\text{Set III: } \sigma_{11} &= \sigma_{22} = 0.2,\ \sigma_{12} = 0,\ \text{and } \sigma_{21} = -0.15.
\end{aligned}$$

In particular, the two assets are uncorrelated for the parameters from Set II. If we take the volatility matrix from Set I, the logarithms of the two assets have a correlation of $\rho = 0.6$ and in Set III these are negatively correlated with $\rho = -0.6$.

One asset with crash risk. Korn and Willmot [37] introduced the worst-case scenario optimal portfolio approach in the case of one stock with crash risk. This situation can also be represented in our model for the power utility function. For this we set $m = 1$ and $n = 0$. Using Theorem 5 and Theorem 6 we can show that the optimal pre-crash portfolio process $\bar{\pi}_{(0)}$ is given by

$$\bar{\pi}'_{(0)}(t) = \frac{(1 - \bar{\pi}_{(0)}(t) l_1)}{l_1} \left(\phi_{(0)}(\bar{\pi}_{(0)}) - \phi_{(0)}(\pi^*) \right),$$

with $\bar{\pi}_{(0)}(T) = 0$.

Multiple assets with crash risk. We take a look at the case of two assets with crash risk, i.e. we set $m = 2$ and $n = 0$. In Figure 2.1, we see the results for $l_1 = l_2 = 0.5$ and Set II. Although the two assets have a different market price of risk, we obtain $\bar{\pi}_{(0),1} = \bar{\pi}_{(0),2}$. This is because the worst-case approach causes the asset in which more capital is invested to suffer a crash if $l_1 = l_2$.

If we now take a look at the negative correlation from Set III in Figure 2.2, more wealth is invested in the two assets, but we still have $\bar{\pi}_{(0),1} = \bar{\pi}_{(0),2}$.

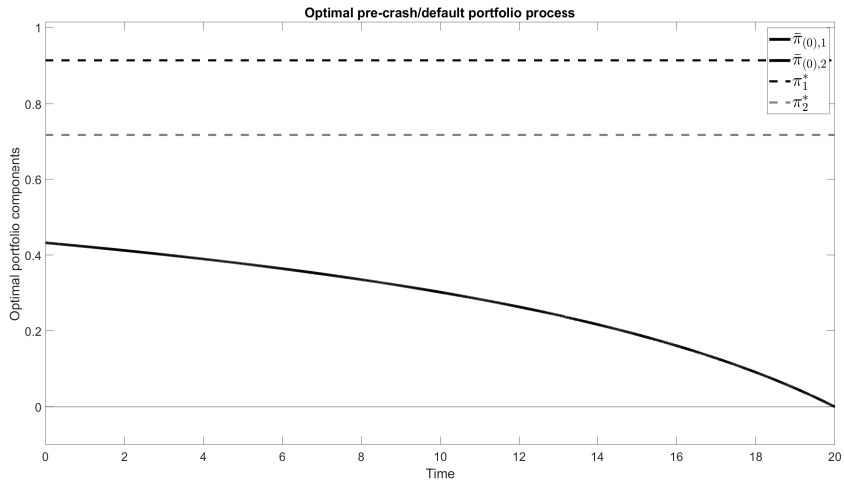

FIGURE 2.2: Set III with $m = 2$, $n = 0$ and $l_1 = l_2 = 0.5$.

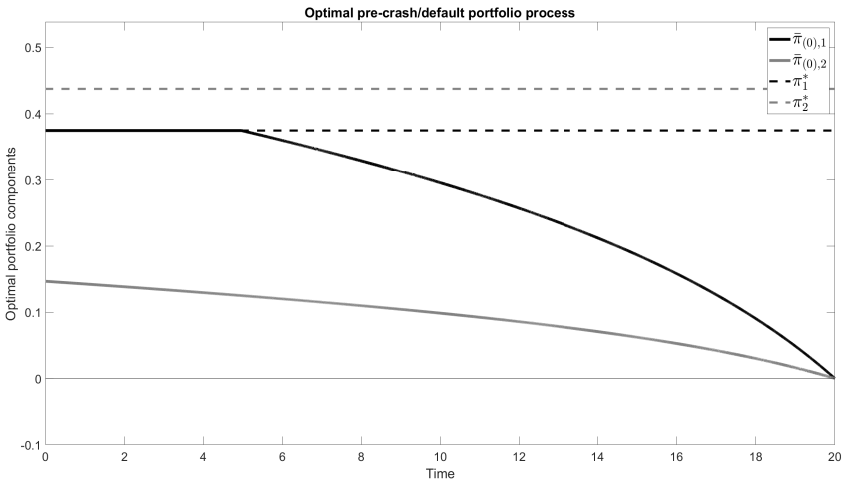

FIGURE 2.3: Set II with $m = 2$, $n = 0$, $l_1 = 0.25$ and $l_2 = 0.75$.

The situation is different in Figure 2.3, where we consider Set II with $l_1 = 0.25$ and $l_2 = 0.75$. Here, the optimal investor puts much more capital into the asset with the smaller crash rate. Since the two assets are uncorrelated, we get $\bar{\pi}_{(0),1} \leq \pi_1^*$ and therefore $\bar{\pi}_{(0),1}$ is constant on $[0, 4.97]$. Further, we obtain on this interval that $I(t) = \{2\}$ for I defined by (2.7), i.e. on $[0, 4.97]$ the crash of asset P_2 is the worst-case scenario.

One defaultable asset. For the first time in the worst-case scenario optimal portfolio literature, we take into account the default of assets. The key difference between an asset crash and an asset default is that the market is reduced by one investment opportunity after the default. In the case of $m = 0$ and $n = 1$, i.e. the market only consists of the money market account and one defaultable asset, we therefore get that $\bar{\pi}_{(0)} = 0$ is the optimal pre-default portfolio process. Before we give an interpretation of this result, we would like to prove it with the help of Proposition 10.

For $\bar{\pi}_{(0)} = 0$, using the definition of the wealth process and v_0 leads to:

$$X^{\bar{\pi}_{(0)}}(t) = x\exp(rt) \text{ and } v_0(t,x) = U(x)\exp\left((1-\gamma)\int_t^T \left(\phi_{(0)}(\pi^*)\right) ds\right),$$

for $t \in [0,T]$. Since the post-default market only consists of the money market account, we obtain for $t \in [0,T]$ that

$$\begin{aligned} k_1(t) &= 1 - \exp\left(\int_t^T \left(\phi_{(1)}(\bar{\pi}_{(1)}) - \phi_{(0)}(\pi^*)\right) ds\right) \\ &= 1 - \exp\left(r(T-t) - \int_t^T \left(\phi_{(0)}(\pi^*)\right) ds\right) \end{aligned}$$

and thus

$$\begin{aligned} \left(1 - CDL^{\bar{\pi}_{(0)}}(t)\right) X^{\bar{\pi}_{(0)}}(t) &= (1-k_1(t))X^{\bar{\pi}_{(0)}}(t) \\ &= x\exp\left(rT - \int_t^T \left(\phi_{(0)}(\pi^*)\right) ds\right), \end{aligned}$$

for $t \in [0,T]$. Putting everything together results in the following minimum process:

$$\begin{aligned} M^{\bar{\pi}_{(0)}}(t) &= v_0\left(t; (1 - CDL^{\pi_{(0)}}(t))X^{\pi_{(0)}}(t)\right) = U\left(x\exp\left(rT\right)\right) \text{ for } t \in [0,T], \\ M^{\bar{\pi}_{(0)}}(\infty) &= U\left(X^{\pi_{(0)}}(T)\right) = U\left(x\exp\left(rT\right)\right). \end{aligned}$$

In particular, this means that $\bar{\pi}$ with $\bar{\pi}_{(0)} = 0$ is a minimum constant portfolio process. Note that by assumption $b_1 > r$. Thus, we get $\pi^* > \bar{\pi}_{(0)} = 0$ and thus $CDL^{\pi_{(0)}}(t) < CDL^{\pi^*}(t)$ for $t \in [0,T]$. With the same arguments as in the proof of Theorem 6 we get that $\bar{\pi}_{(0)}$ also solves the constrained optimization problem of Proposition 10 and therefore it is the optimal pre-default portfolio process.

This example shows us that an investor who can only invest in one asset, and this asset defaults at the worst possible time for him/her, transfers all his/her wealth to the bank account. This is because of using the optimal post-default process prior to default, already gives him/her a minimum constant portfolio process. This is no longer possible if he/she can invest in more than one asset with risk to default, which leads us to consider the following examples.

Multiple defaultable assets. We now consider the worst-case crash/default problem with two assets, both of which are at risk of default, i.e. we set $m = 0$ and $n = 2$. To do so, we look at the case $l_1 = l_2 = 0.5$ in Figure 2.4 with parameters from Set II. Here we directly see a central difference to the analogous case in Figure 2.1, where we considered two uncorrelated stocks with crash risk. In the example we are looking at now, it is no longer the case that $\bar{\pi}_{(0),1}(t) = \bar{\pi}_{(0),2}(t)$ for all $t \in [0,T]$. This is because we no longer have $k_1(t) = k_2(t)$ for all $t \in [0,T]$, i.e. it is important for the investor in the post-default market which asset has suffered the default and can no longer be traded. This is also why more wealth is invested in the asset with the lower market price of risk.

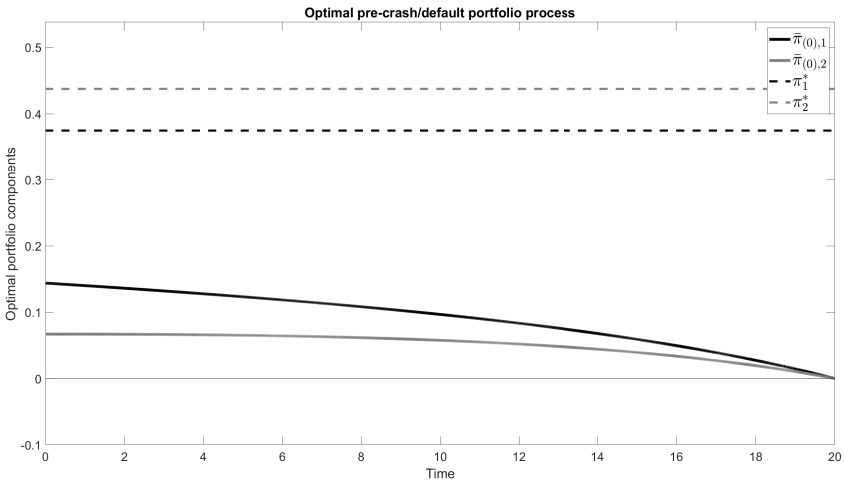

FIGURE 2.4: Set II with $m = 0$, $n = 2$ and $l_1 = l_2 = 0.5$.

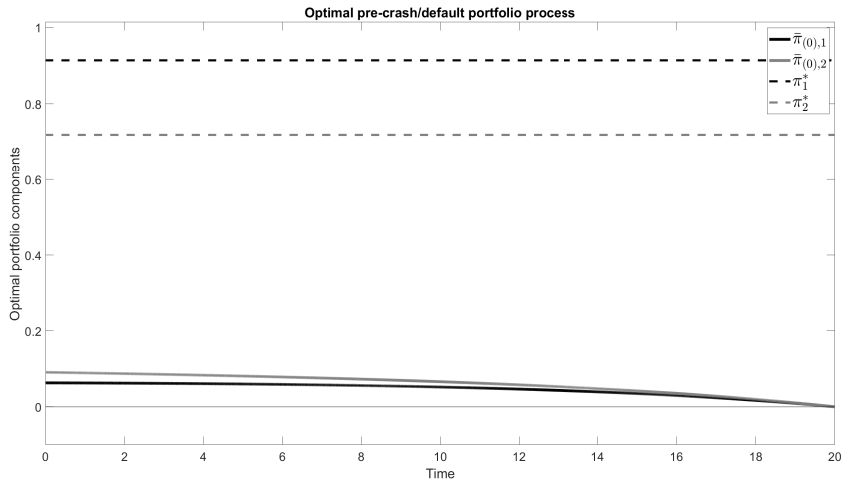

FIGURE 2.5: Set III with $m = 0$, $n = 2$ and $l_1 = l_2 = 0.5$.

In contrast to Figure 2.2, we can see in Figure 2.5 that the negative correlation in Set III does not lead to a significantly higher investment in the two risky assets. This is because the negative correlation leads to a higher k_1 and k_2 and therefore to a higher worst-case crash/default loss $CDL^{\bar{\pi}_{(0)}}$. This can be explained by the fact that the investor now makes a higher loss in the event of a default, due to the market being reduced by one negatively correlated asset.

Stock price crash or bond default? Next, we consider the worst-case portfolio problem with $m = 1$ and $n = 1$, meaning that the investor can invest in two assets, one of which has a crash risk and the other one can suffer a default. In both of the following examples we set $l_1 = l_2 = 0.5$. In Figure 2.6 we see the results for Set I. The investor is willing to invest more in the asset with crash risk as in the asset with default risk. It should be noted that in the event of a default, in addition to the direct loss $\bar{\pi}_{(0),2} l_2$, he/she must also consider that the market will be reduced by one asset. In this example, the two assets are positively

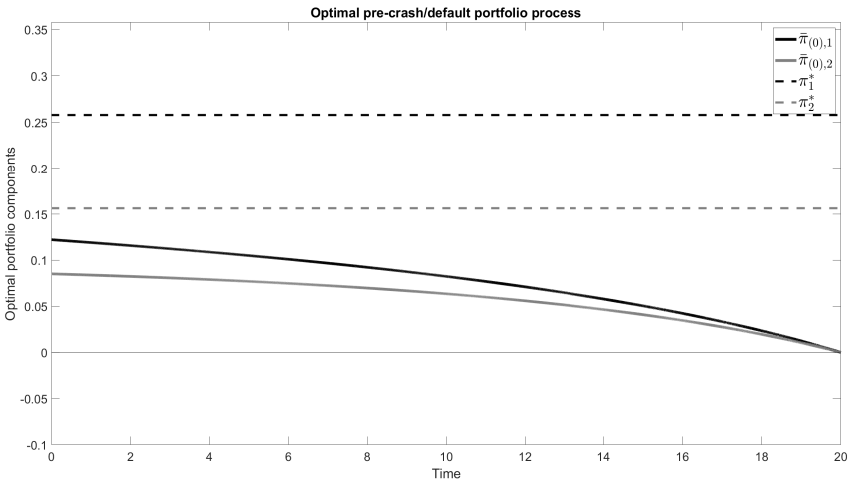

FIGURE 2.6: Set I with $m = 1$, $n = 1$ and $l_1 = l_2 = 0.5$.

correlated, so that a default has significantly less severe consequences in the post-default market than in Figure 2.7, where Set III, with the negative asset correlation, is used.

In relation to Figure 2.6, the negative correlation leads to a larger k_2 and thus a significantly lower investment in the defaultable asset. In this context, one should also look at the set I, defined in (2.7) , to see for which asset the worst-case crash/default loss $CDL^{\bar{\pi}_{(0)}}$ is attained. In the previous example, we have $I(t) = \{1, 2\}$ for all $t \in [0, T]$. In this case now, however, $I(t) = \{2\}$ on the interval $[0, 10]$ and $I(t) = \{1, 2\}$ on $[10, 20]$. To get a better understanding of this result, let us consider the optimal investor in $t = 0$. Due to the small position in the defaultable asset, he/she initially receives a loss $\bar{\pi}_{(0),2}(0)l_2$ that is close to zero, in case of a default. However, at this point in time the default of the asset P_2 and not the possible crash of the asset P_1 is the worst-case scenario, i.e. $CDL_1^{\bar{\pi}_{(0)}}(0) < CDL_2^{\bar{\pi}_{(0)}}(0)$. This is due to the high value $k_2(0) = 0.35 >> k_1(0) = 0$, which results from the fact that in the event of a default at time $t = 0$, the investor can only invest in one risky asset for the next 20 years instead of in two negatively correlated assets.

Efficiency loss. Finally, we would like to consider the loss suffered by an investor who chooses the Merton strategy π^* instead of the optimal portfolio process $\bar{\pi}$, i.e. the efficiency loss will be defined as a loss that can be avoided by adapting the portfolio process. Given that x_M is the wealth of the Merton investor, we define x_0 such that

$$\left(1 - CDL^{\pi^*}(t)\right) x_M = \left(1 - CDL^{\bar{\pi}_{(0)}}(t)\right) x_0,$$

for $t \in [0, T]$. Due to the constant portfolio process π^* and the decreasing k_i for $i \in \{m + 1, \ldots, m + n\}$, the worst-case scenario for the Merton investor is a crash or default at initial time. On the other hand, the minimum constant property makes the optimal investor indifferent to the time of occurrence of the worst-case scenario. Therefore, the above definition leads to an initial wealth x_0, so that we achieve exactly the same worst-case expected utility with the optimal portfolio process as the Merton investor with initial wealth

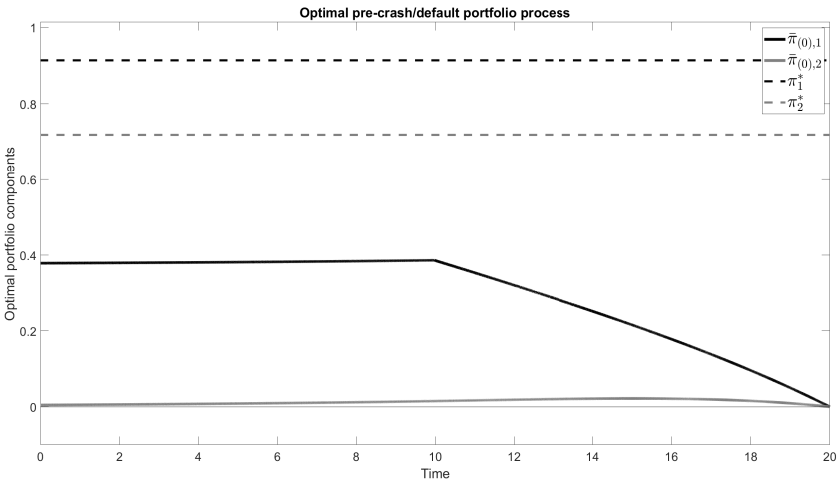

FIGURE 2.7: Set III with $m = 1$, $n = 1$ and $l_1 = l_2 = 0.5$.

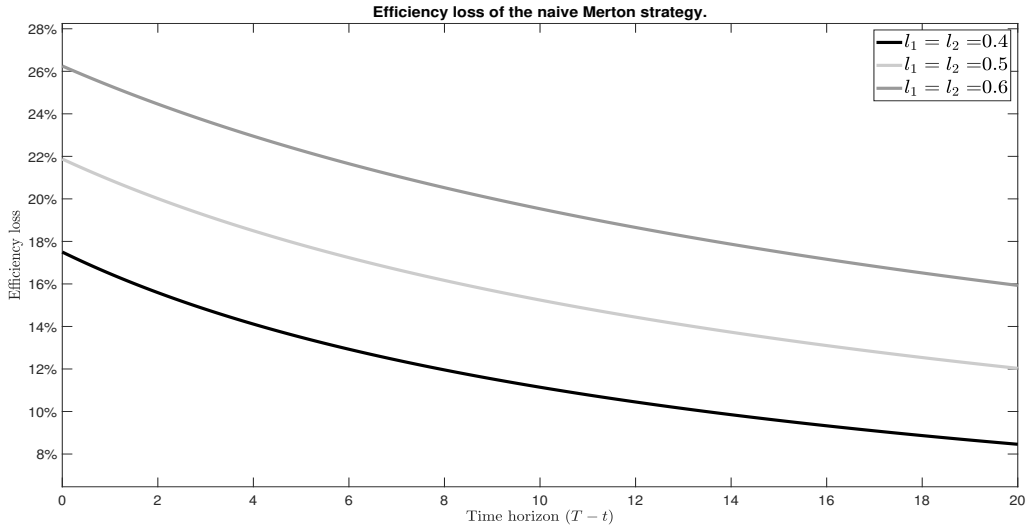

FIGURE 2.8: Efficiency loss for Set II with $m = 2$ and $n = 0$.

x_M, for the time horizon $T - t$. Based on this, we define the efficiency loss as

$$\text{efficiency loss} = \frac{x_M - x_0}{x_M} = \frac{CDL^{\pi^*}(t) - CDL^{\bar{\pi}_{(0)}}(t)}{1 - CDL^{\bar{\pi}_{(0)}}(t)},$$

for $t \in [0, T]$. In Figure 2.8 we consider the case of two assets with crash risk, i.e. $m = 2$ and $n = 0$. We see the results for Set II and different crash rates $l_1 = l_2$. The efficiency loss decreases with increasing time horizons. However, even for a time horizon of 20 years, the efficiency loss for $l_1 = l_2 = 0.5$ is still around 12%. Figure 2.9 shows the analogous results for two defaultable assets. Here, the efficiency loss decreases less significantly with increasing time horizon. This is because there is an additional loss due to the reduced post-default market, expressed by the k_i for $i \in \{1, 2\}$, which is independent of the selected portfolio processes.

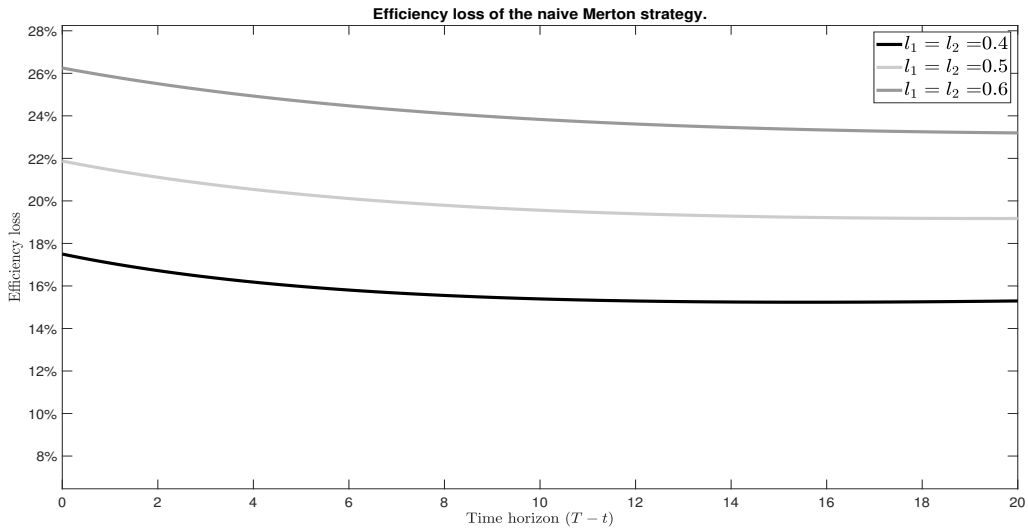

FIGURE 2.9: Efficiency loss for Set II with $m = 0$ and $n = 2$.

2.8 Conclusion

In this chapter, we have solved the worst-case crash/default portfolio problem. For the first time in the worst-case scenario optimal portfolio literature, it is assumed that there can be an asset crash as well as an asset default and that the investor does not know which worst-case scenario will occur. This extension requires the definition of the worst-case crash/default loss, which takes into account, that in the event of a default, in addition to the direct loss, there is also a loss due to the post-default market being reduced by one investment opportunity. We have derived a non-linear ordinary differential equation for the optimal worst-case crash/default loss that enables us to prove the existence of a so called minimum constant portfolio process which solves the newly introduced worst-case problem.

The analysis of the numerical examples shows that the addition of defaultable assets, as well as the assumption that at most one of the worst-case scenarios occurs, has a significant impact on the optimal portfolio processes. In the case of two assets with crash risks, the optimal pre-crash portfolio process can lead to the same investment in the two assets even though the market prices of risk differ. For two defaultable assets, there can even be a higher investment in the asset with the lower market price of risk, since the default of this asset has a lower impact on the post-default market. It is also possible that, for the optimal investor at a given time $t \in [0, T]$, not every possible crash or default represents a worst-case scenario. In fact, the set of worst-case scenarios may even change over time. In the final analysis of the efficiency loss, we have highlighted the importance of adjusting the portfolio process when adding the possibility of a crash or default at the worst possible time to the portfolio optimization problem.

Chapter 3

Optimal portfolio choice with crash risk and model ambiguity

3.1 Introduction

Risk, uncertainty and motivation. The investor in this chapter has the opportunity to invest in a risk-free asset, the money market account, and d risky stocks. For the modeling of the stock price dynamics, we want to add additional uncertainty to the already risky price movements. For this purpose, we refer to Frank Knight's definition of risk and uncertainty from 1921. In [25], Knight distinguishes between "measurable uncertainty" or "risk", and "unmeasurable uncertainty". In his opinion, "the present and more important task is to follow out the consequences of that higher form of uncertainty not susceptible to measurement". In 1961, Ellsberg [16] specifies the fundamental distinction between risk and uncertainty. According to him, if the probabilities of occurrence of a random event are known, it is risky. If this is not the case, it is an uncertain event. Furthermore, he notes, that people are tending to give preference to a risky event rather than to an uncertain event.
We will integrate uncertainty into our portfolio optimization problem using two different approaches. Instead of defining a unique underlying probability measure and associated unique drift and volatility parameters for our stock price processes, we extend our model to sets of possible stock price parameters, but do not assume a probability distribution on this parameter set. This concept is part of the model ambiguity approach, which is described in the subsequent paragraph. Furthermore, we will use the worst-case scenario approach, introduced by Korn and Wilmott [37], to model a possible crash scenario. The combination of these two approaches is of particular interest because, on the one hand, a crash scenario is an event that rarely occurs and thus it is almost impossible to precisely estimate its small probabilities of occurrence and, on the other hand, a high degree of uncertainty about the market's behaviour can be observed in the context of a crash scenario. In combining these two approaches, we also take up some ideas of Frank Seifried as stated in the extensions section of [56].

Model ambiguity. The classical portfolio optimization problems require a precise knowledge of the probability measure underlying the financial market model and of the associated asset price parameters, such as stock price drift and volatility. The aim of robust portfolio optimization, respectively portfolio optimization with ambiguity, is to directly integrate uncertainty about the asset price parameters into the optimization problem. A set of possible probability measures or stock price parameters is considered, which ideally is constructed in such a way that the true measure – respectively the true parameters – are included. In determining the set of possible probability measures, a distinction is basically made between two cases, the dominated and the undominated case. Examples of a dominated framework can be found in [11], [20], [54], [42], among others. Here, dominated means that there exists a particular probability measure with respect to which all other possible probability measures are absolutely continuous. This is not the case in an undominated framework, as it can be found in [21], [52], [51], among others. Models with uncertainty related to jump processes are also discussed in the literature. Lim et al. [40] set up a robust portfolio problem by introducing a fictitious benchmark investor and their model allows to include assets

with uncertain jump distributions. Neufeld and Nutz [50] consider a portfolio optimization problem with uncertain instantaneous drift, volatility, and jump characteristics of the price process. Of particular interest for this chapter is the publication of Lin and Riedel [41] and the work of Biagini and Pınar [7]. In [41], the authors introduce a financial market model, where the risky asset is driven by a G-Brownian motion with volatility bounds $[\underline{\sigma}^2, \bar{\sigma}^2]$ and also with box-ambiguity with regard to the drift parameter. A robust Merton problem is solved in [7], where ellipsoidal uncertainty sets are used for the modeling of the drift.

Summary and outline. In this chapter we set up and solve a portfolio optimization problem that contains ambiguity about the stock price parameters and that allows for the occurrence of a crash scenario at the worst possible time for the investor. Here, we combine the two concepts of portfolio optimization under uncertainty and the worst-case scenario approach. For, this we can build on some ideas and claims already stated in [56]. Further, we would like to take into account, that the set of possible stock price parameters can change after a crash. This leads to the following outline of our chapter:

- In section 3.2 we introduce our financial market model. In particular, we introduce the sets of possible stock price parameters and state the definition of a crash scenario. Based on this, we define the set of admissible portfolio processes and formulate the worst-case crash portfolio problem with ambiguity in section 3.3.
- We solve the optimization problem by going backwards in time. Therefore, in section 3.4 we look at the post-crash portfolio optimization problem. This can be seen as a portfolio optimization problem with ambiguity and random initial time.
- In section 3.5 we solve the pre-crash portfolio problem by using indifference arguments and thus also derive the optimal portfolio process for our optimization problem introduced in section 3.3.
- The sets of possible stock price parameters are defined in section 3.2 in very general terms and are then restricted by assumptions made in section 3.4. In section 3.6 and section 3.7 we give two specific examples of possible parameter sets that fulfill these assumptions and are thus admissible for our portfolio problem. In section 3.8 we analyse some numerical examples, based on the parameter sets in section 3.7.

3.2 Financial market and crash scenarios

Mathematical setting. We fix a time horizon $T > 0$. All processes below are defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Our financial market consists of one riskless money market account B and d risky stocks S_i, $i \in \{1, \dots d\}$. For the modeling of the stock price dynamics we use an n-dimensional Brownian motion $(W(t))_{t\in[0,T]}$, with $n \geq d$. We assume that our probability space is equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ that satisfies the so called usual conditions and that includes the filtration generated by the Brownian motion W. For the modeling of the crash scenarios in the further course of this chapter, we need to

extend $\mathbb{F}$ to $[0,T] \cup \{\infty\}$ by letting $\mathcal{F}_\infty := \mathcal{F}_T$. Based on this we define Θ as the set of all $[0,T] \cup \{\infty\}$-valued stopping times. Further, we define for $\tau \in \Theta$ the set Θ_τ, respectively Θ^τ, as the set of all $\xi \in \Theta$ with $\xi(\omega) \geq \tau(\omega)$ for all $\omega \in \Omega$, respectively all $\xi \in \Theta$ with $\xi(\omega) \leq \tau(\omega)$ for all $\omega \in \Omega$. By $\|\cdot\|$ we denote the L^2-norm on $\mathbb{R}^d$.

Financial market. We now look at the exact definition of the underlying financial market model and introduce the dynamics of the money market account and the d risky stocks.

For a constant interest rate $r \in \mathbb{R}$, the dynamic evolution of the money market account B is given by

$$dB(t) = B(t)rdt, \qquad B(0) = 1.$$

Since we want to consider the portfolio optimization problem of an ambiguity averse investor, we introduce a robust framework for the stock price parameters. For this, we define two compact sets $D_1 \subset \mathbb{R}^d$ and $D_0 \subset \mathbb{R}^d$ for possible values of the drift parameters of the stock price dynamics in the pre- and post-crash market. Analogously, for the volatility matrices, define the compact sets $V_1 \subset \mathbb{R}^{d\times n}$ and $V_0 \subset \mathbb{R}^{d\times n}$ of matrices with full rank. The set P_1 of possible stock price parameters in the pre-crash market and the corresponding set P_0 for the post-crash market are defined for $i \in \{0,1\}$ as

$$P_i := \{p_i = (b_i, \sigma_i) \text{ progressively measurable} | b_i(t) \in D_i \text{ and } \sigma_i(t) \in V_i \text{ for } t \in [0,T]\}.$$

Before defining the stock price dynamics we need to introduce the definition of a crash scenario.

Definition 10 (Crash scenario). *A vector $z \in \mathbb{R}^d$ with components $z_i \in [0,1)$ is called a crash. A tuple (z,ξ), consisting of a crash z and a stopping time $\xi \in \Theta$, is called crash scenario. We will refer to ξ as the time of occurrence of the crash z, where we formally set $\{\xi = \infty\}$ if the crash does not occur. For a crash scenario (z,ξ), we further introduce the (indicator) jump process*

$$J_z(t) := \mathbb{1}_{\{\xi \leq t\}},\ t \in [0,T].$$

Based on the previous considerations, we can now introduce the dynamic evolution of the stock price vector $S = (S_1, \ldots, S_d)$, which is defined for a crash scenario (z,ξ) by

$$dS(t) = Diag(S(t-))\left(b(t)dt + \sigma(t)dW(t) - zdJ_z(t)\right),$$

for $t \in [0,T]$, where $Diag(S(t-))$ is a diagonal $d \times d$ matrix with i-th diagonal element equal to $S_i(t-)$, and $S(0) = s > 0$. The drift vector b and the volatility matrix σ are defined at time $t \in [0,T]$ as

$$b(t) := (1 - J_z(t))b_1(t) + J_z(t)b_0(t),$$
$$\sigma(t) := (1 - J_z(t))\sigma_1(t) + J_z(t)\sigma_0(t),$$

for $p_1 = (b_1, \sigma_1) \in P_1$ and $p_0 = (b_0, \sigma_0) \in P_0$.

Remark 17 (Undominated framework)**.** *In the introduction we subdivided the model ambiguity approach into the dominated case and the undominated case, depending on whether there exists a probability measure with respect to which all other possible probability measures are absolutely continuous. However, we have now introduced our model using a unique probability measure P. It can be shown directly that we can also introduce stock price dynamics in a very analogous way, using a plausible set of probability measures, as in Definition 1 in [7]. It then follows that we have defined an undominated framework. For a clearer presentation of the portfolio problem in our work, we prefer the above definition of the stock price based on a single probability measure P and sets of possible stock price parameters.*

3.3 The portfolio optimization problem with crash risk and model ambiguity

We now introduce the portfolio optimization problem of an extremely risk-averse investor, who considers a financial market with crash risk and parameter uncertainty. To do this, we first define the wealth process, depending on the admissible portfolio process chosen by the investor.

Admissible portfolio process and wealth dynamics. When defining the set of admissible portfolio processes, particular attention should be paid to the fact that the investor can observe the possible crash but can only react to it after it has occurred. Furthermore, the admissible processes should be self-financing.

Definition 11 (Admissible portfolio process)**.** *Consider the financial market model introduced in section 3.2.*

1. *Let $x > 0$, $(p_1, p_0) \in P_1 \times P_0$ and (z, ξ) be a crash scenario. The set $\mathbb{A}^{p_1,\xi,p_0}$ of admissible portfolio processes π given (p_1, ξ, p_0) consists of $\{\mathcal{F}_t, 0 \leq t \leq T\}$-predictable, real-valued processes, such that*

   (a) for $t \in [0, T]$, $\pi(t)$ is of the form

   $$\pi(t) = \mathbb{1}_{\{\xi \geq t\}} \pi_1(t) + \mathbb{1}_{\{\xi < t\}} \pi_0(t),$$

   for $\pi_1(t), \pi_0(t) \in \mathbb{R}^d$ being $\{\mathcal{F}_t, 0 \leq t \leq T\}$-predictable stochastic processes on $[0, T]$, where $\pi_{(1)}(t)^T z$ has right-continuous paths with left limits.

   (b) the investor's wealth process X^{π,p_1,ξ,p_0} is the unique solution of the stochastic differential equation

   $$X^{\pi,p_1,\xi,p_0}(0) = x,$$
   $$dX^{\pi,p_1,\xi,p_0}(t) = X^{\pi,p_1,\xi,p_0}(t-)\Bigg(\Big(r + \pi(t)^T \left(b(t) - r\underline{1}_d\right)\Big) dt$$

$$+ \pi(t)^T \sigma(t) dW(t) - \pi(t)^T z dJ_z(t) \Bigg),$$

for $t \in [0, T]$.

(c) *the following no-ruin and integrability conditions are fulfilled:*

$$X^{\pi, p_1, \xi, p_0}(t) \geq 0 \; \forall t \in [0, T],$$
$$\int_0^T \pi_i^2(t) dt < \infty \; P\text{-a.s. for } i = 1, \ldots, d.$$

2. *For* $x > 0$ *and a crash scenario* (z, ξ)*, the set* $\mathbb{A}$ *of admissible portfolio processes* π *is defined as*

$$\mathbb{A} := \bigcap_{(p_1, \xi, p_0) \in P_1 \times \Theta \times P_0} \mathbb{A}^{p_1, \xi, p_0}.$$

Of particular interest is the change in the wealth process at the time of occurrence of a possible crash.

Remark 18 (Wealth at crash-time ξ)**.** *Let* $x > 0$, $(p_1, p_0) \in P_1 \times P_0$ *and* (z, ξ) *be a crash scenario. Note that for the wealth process of an admissible portfolio process* $\pi \in \mathbb{A}$ *at the time of crash occurrence* ξ *we have:*

$$X^{\pi, p_1, \xi, p_0}(\xi) = \left(1 - \pi_1(\xi)^T z\right) X^{\pi, p_1, \xi, p_0}(\xi -),$$

on $\{\omega \in \Omega | \xi(\omega) < \infty\}$.

The optimization problem. The investor in our model can be seen as highly risk averse. No probabilistic assumptions are made regarding the choice of the stock price parameters and the occurrence of the crash, but instead the Knightian uncertainty approach [25] is used. This means that the crash occurs at the worst possible time for the investor, and the stock price parameters - both before and after the crash - are taken into account using a worst-case approach.

We use a utility function U with constant relative risk aversion $\gamma > 0$, $\gamma \neq 1$, for the description of the investor's risk preferences:

$$U : (0, \infty) \to \mathbb{R}, \quad U(x) = \frac{1}{1 - \gamma} x^{1 - \gamma},$$

with $U(0) := \lim_{x \searrow 0} U(x)$.

For a crash scenario (z, ξ), the **worst-case crash portfolio problem with ambiguity** can be defined as

$$\sup_{\pi \in \mathbb{A}} \inf_{\substack{(p_1, \xi, p_0) \\ \in P_1 \times \Theta \times P_0}} \mathbb{E}\left(U\left(X^{\pi, p_1, \xi, p_0}(T)\right)\right). \tag{3.1}$$

3.4 Post-crash problem and no-crash scenario

We go backwards in time and start by looking at the post-crash portfolio problem, i.e. for an admissible portfolio process $\pi \in \mathbb{A}$ we want to determine the optimal π_0. Further, as this can be solved with similar arguments, we also consider the no-crash portfolio problem, i.e. $\xi(\omega) = \infty$ for all $\omega \in \Omega$. This will be of particular interest when we derive the optimal pre-crash portfolio process π_1, for $\pi \in \mathbb{A}$, in the next section.

The post-crash portfolio problem can be seen as a portfolio optimization problem with ambiguity and random initial time ξ. This problem is similar to the robust Merton problem discussed in [7]. In contrast to this work, however, we avoid the use of a Hamilton–Jacobi–Bellman–Isaac equation. Instead, our proofs are based on a decomposition of the terminal utility. Such a decomposition is already used in [56], among others.

Terminal utility decomposition. We start by decomposing the utility of the wealth process and based on this we can already determine the worst-case stock price parameters in the post-crash market, respectively the worst-case pre-crash parameters in the no-crash scenario, for an investor with deterministic portfolio process π_0, respectively π_1.

Proposition 11 (Utility decomposition)**.** *Let (z, ξ) be a crash scenario. We define the* local utility drift *as the mapping ϕ for $\pi \in \mathbb{R}^d$ and $p = (b_i, \sigma_i) \in D_i \times V_i$, with $i \in \{0, 1\}$, by*

$$\phi(p, \pi) := r + \pi^T (b_i - r\underline{1}_d) - \frac{1}{2}\gamma \pi^T \sigma_i \sigma_i^T \pi.$$

Further, for an admissible portfolio process $\pi \in \mathbb{A}$ and $p_i \in P_i$, $i \in \{0, 1\}$, we define the stochastic exponential $Y_s^t(\pi_i, p_i)$ for $s, t \in [0, T]$, $s \leq t$ as

$$\begin{aligned} Y_s^t(\pi_i, p_i) = \exp\Bigg(&(1-\gamma) \int_s^t \pi_i(u)^T \sigma(u) dW(u) \\ &- \frac{1}{2}(1-\gamma)^2 \int_s^t \pi_i(u)^T \sigma(u) \sigma(u)^T \pi_i(u) du \Bigg). \end{aligned}$$

Now let $(p_1, \xi, p_0) \in P_1 \times \Theta \times P_0$ and $\pi \in \mathbb{A}$.

1. *On $\{\omega \in \Omega | \xi(\omega) < \infty\}$, we obtain the following decomposition:*

$$\begin{aligned} &U(X^{\pi, p_1, \xi, p_0}(T)) \\ =&U(X^{\pi, p_1, \xi, p_0}(\xi)) \exp\left((1-\gamma) \int_\xi^T \phi(p_0(t), \pi_0(t)) dt \right) Y_\xi^T(\pi_0, p_0). \end{aligned}$$

2. *If $\xi(\omega) = \infty$ for all $\omega \in \Omega$, the following decomposition holds:*

$$\begin{aligned} &U(X^{\pi, p_1, \xi, p_0}(T)) \\ =&U(x) \exp\left((1-\gamma) \int_0^T \phi(p_1(t), \pi_1(t)) dt \right) Y_0^T(\pi_1, p_1). \end{aligned}$$

3. *On $\{\omega \in \Omega | \xi(\omega) < \infty\}$, we obtain the following decomposition:*

$$\begin{aligned} &U(X^{\pi,p_1,\xi,p_0}(\xi)) \\ =&U\left(\left(1-\pi_1(\xi)^T z\right)x\right)\exp\left((1-\gamma)\int_0^{\xi}\phi(p_1(t),\pi_1(t))dt\right)Y_0^{\xi}(\pi_1,p_1). \end{aligned}$$

Proof. All three parts are direct consequences of Itô's formula. □

So far, we have defined the sets of possible stock price parameters in very general terms. We now want to change this by specifying requirements for the solvability of optimization problems based on the newly defined function ϕ. Later we will see that these are the only assumptions that lead to possible restrictions of the parameter sets P_1 and P_0. Furthermore, the examples in section 3.6 and section 3.7 show that these assumptions still allow the consideration of relevant applications.

Assumption 1. *For every crash scenario (z,ξ), every $\pi \in \mathbb{A}$ and $i \in \{0,1\}$, we consider the minimization of the local utility drift:*

$$\inf_{p_i \in P_i} \phi(p_i(t),\pi_i(t)),$$

defined for $t \in [0,T]$. The infimum is attained since we minimize point-wise over the compact sets D_i and V_i and the function $\phi(\cdot,\pi_i(t))$ is point-wise continuous. We assume the existence of a $p_i^{\pi} \in P_i$ such that

$$p_i^{\pi} \in \arg\inf_{p_i \in P_i} \phi(p_i(t),\pi_i(t)), \text{ for all } t \in [0,T],$$

and denote the non-empty set of all these p_i^{π} by $P_i^{\pi} \subset P_i$. Note that all $p_i^{\pi} \in P_i^{\pi}$ are deterministic if the corresponding π_i is deterministic.

Further, for $i \in \{0,1\}$ and $t \in [0,T]$ we assume that

$$\min_{p_i \in P_i} \phi(p_i(t),\pi_i(t)),$$

seen as a function in $\pi_i(t)$ for $\pi_i(t) \in \mathbb{R}^d$, is a strictly concave function that converges to $-\infty$ for $||\pi_i(t)|| \to \infty$. In particular, we require the existence of a unique admissible portfolio process $\pi^* \in \mathbb{A}$, such that

$$\sup_{\pi \in \mathbb{A}} \inf_{p_i \in P_i} \phi(p_i(t),\pi_i(t)) = \phi(p_i^{\pi^*}(t),\pi_i^*(t)) < \infty,$$

for all $t \in [0,T]$, every crash scenario (z,ξ), $p_i^{\pi^*} \in P_i^{\pi^*}$, $i \in \{0,1\}$.

Remark 19 (Existence of an optimal constant portfolio process π^* in the crash-free ambiguity setting)**.** *We want to point out how strong Assumption 1 actually is. Indeed, the assumption that the solution of the minimization problem of $\phi(.,\pi_i(t))$ is a strictly concave function that converges to $-\infty$ for $||\pi_i(t)|| \to \infty$, implies that maximizing it over all admissible values $\pi_i(t)$ (i.e. those that*

satisfy the no-ruin constraint) leads to a finite maximum. As the sets D_i and V_i, as well as the set of admissible values for $\pi_i(t)$, are time-independent, we obtain the same maximum for all $t \in [0,T]$ and every crash scenario (z,ξ), i.e. we get for $i \in \{0,1\}$ that

$$\sup_{\pi\in\mathbb{A}} \inf_{p_i\in P_i} \phi(p_i(t),\pi_i(t)) = \sup_{\substack{\pi\in\mathbb{R}^d \\ \pi^T z\leq 1 \text{ iff } i=1}} \inf_{p\in D_i\times V_i} \phi(p,\pi),$$

and this leads to

$$\pi_i^*(t) = \pi_i^*$$

for all $t \in [0,T]$. This is in particular true for the choice of $i = 0$ which corresponds to the portfolio problem under ambiguity after the crash.

For admissible portfolio processes $\pi \in \mathbb{A}$, with deterministic π_0 or π_1, we can already proof claims concerning the worst-case stock price parameters.

Proposition 12 (Deterministic portfolio processes)**.** *Let $\pi \in \mathbb{A}$ be an admissible portfolio process, (z,ξ) be a crash scenario and $x > 0$.*

1. *If π_0 is deterministic, then for every $p_0^\pi \in P_0^\pi$ and every $p_1 \in P_1$ we obtain*

$$\begin{aligned}\inf_{p_0\in P_0} \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0}(T)\right)|\mathcal{F}_\xi\right) &= \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0^\pi}(T)\right)\Big|\mathcal{F}_\xi\right)\\ =&U\left(X^{\pi,p_1,\xi,p_0}(\xi)\right)\exp\left((1-\gamma)\int_\xi^T \phi(p_0^\pi(t),\pi_0(t))dt\right),\end{aligned}$$

   on $\{\omega \in \Omega | \xi(\omega) < \infty\}$.

2. *If π_1 is deterministic and $\xi(\omega) = \infty$ for all $\omega \in \Omega$, then for every $p_0 \in P_0$ and every $p_1^\pi \in P_1^\pi$ we obtain*

$$\begin{aligned}\inf_{p_1\in P_1} \mathbb{E}\left(U\left(X^{\pi,p_1,\infty,p_0}(T)\right)\right) &= \mathbb{E}\left(U\left(X^{\pi,p_1^\pi,\infty,p_0}(T)\right)\right)\\ =&U(x)\exp\left((1-\gamma)\int_0^T \phi(p_1^\pi(t),\pi_1(t))dt\right).\end{aligned}$$

Proof. We give the proof for part 1 in detail and then specify what needs to be adapted for the proof of part 2.

1. Let (z,ξ) be a crash scenario, $(p_1,p_0) \in P_1 \times P_0$, and $\pi \in \mathbb{A}$ be an admissible portfolio process with deterministic π_0. Without loss of generality we can assume that, given $\mathcal{F}_\xi$, $X^{\pi,p_1,\xi,p_0}(\xi) = x > 0$ for all $p_0 \in P_0$. For this, note that by admissibility of the portfolio process we must have $x \geq 0$ anyway. However, the case of $x = 0$ would lead to the uninteresting case of $X^{\pi,p_1,\xi,p_0}(t) = 0$ for all $t \geq \xi$.

   For every $p_0 \in P_0$, we note that $Y_\xi(\pi_0,p_0) = \{Y_\xi^t(\pi_0,p_0), \xi \leq t \leq T\}$, defined in Proposition 11, is a positive $\{\mathcal{F}_t, \xi \leq t \leq T\}$-martingale with $Y_\xi^\xi(\pi_0,p_0) = 1$. Then, with the help of the decomposition in Proposition 11 we obtain on $\{\omega \in \Omega | \xi(\omega) <$

$\infty\}$:

$$
\begin{aligned}
&\mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0}(T)\right)|\mathcal{F}_\xi\right)\\
=&\mathbb{E}\left(U(x)\exp\left((1-\gamma)\int_\xi^T \phi(p_0(t),\pi_0(t))dt\right)Y_\xi^T(\pi_0,p_0)\Big|\mathcal{F}_\xi\right)\\
\geq&\mathbb{E}\left(U(x)\exp\left((1-\gamma)\int_\xi^T \phi(p_0^\pi(t),\pi_0(t))dt\right)Y_\xi^T(\pi_0,p_0)\Big|\mathcal{F}_\xi\right)\\
=&U(x)\exp\left((1-\gamma)\int_\xi^T \phi(p_0^\pi(t),\pi_0(t))dt\right)\mathbb{E}\left(Y_\xi^T(\pi_0,p_0)\Big|\mathcal{F}_\xi\right)\\
=&U(x)\exp\left((1-\gamma)\int_\xi^T \phi(p_0^\pi(t),\pi_0(t))dt\right)\\
=&U(x)\exp\left((1-\gamma)\int_\xi^T \phi(p_0^\pi(t),\pi_0(t))dt\right)\mathbb{E}\left(Y_\xi^T(\pi_0,p_0^\pi)\Big|\mathcal{F}_\xi\right)\\
=&\mathbb{E}\left(U(x)\exp\left((1-\gamma)\int_\xi^T \phi(p_0^\pi(t),\pi_0(t))dt\right)Y_\xi^T(\pi_0,p_0^\pi)\Big|\mathcal{F}_\xi\right)\\
=&\mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0^\pi}(T)\right)\Big|\mathcal{F}_\xi\right),
\end{aligned}
$$

for every $p_0^\pi \in P_0^\pi$.

2. Let us point out, that the problem we are considering here is in fact identical to the post crash setting of part 1 with $\xi = 0$, however with P_1 instead of P_0. Note that also $Y_0(\pi_1, p_1) = \{Y_0^t(\pi_1, p_1), 0 \leq t \leq T\}$ is a positive $\{\mathcal{F}_t, 0 \leq t \leq T\}$-martingale with $Y_0^0(\pi_1, p_1) = 1$, for every $p_1 \in P_1$. Using the decomposition from Proposition 11 part 2, the proof follows in a completely analogous way to the proof of part 1.

□

Post-crash problem and no-crash scenario. Using the previous results, we are now able to state the optimal post-crash and no-crash portfolio processes. The determination of the optimal post-crash portfolio process can be seen as the consideration of a portfolio problem with ambiguous stock price parameters and random initial time.

Proposition 13 (Optimal post-crash and no-crash portfolio processes)**.** *Let (z, ξ) be a crash scenario and $x > 0$.*

1. *There exists an admissible portfolio process $\bar{\pi} \in \mathbb{A}$ with $\bar{\pi}_0$ being constant such that for every $p_0^{\bar{\pi}} \in P_0^{\bar{\pi}}$ we obtain*

$$\phi(p_0^{\bar{\pi}}(t), \bar{\pi}_0) = \max_{\pi\in\mathbb{A}} \min_{p_0\in P_0} \phi(p_0(t), \pi_0(t)) \text{ for all } t \in [0, T].$$

We call such a portfolio process $\bar{\pi}$ post-crash optimal. Now we fix an arbitrary post-crash optimal portfolio process $\bar{\pi}$ and define $\tilde{\mathbb{A}} \subset \mathbb{A}$ as

$$\tilde{\mathbb{A}} := \{\pi \in \mathbb{A} |\ \pi_0(t) = \bar{\pi}_0(t) \text{ for all } t \in [0, T]\}.$$

Then, for every $p_1 \in P_1$ and every $p_0^{\bar{\pi}} \in P_0^{\bar{\pi}}$ we obtain

$$\sup_{\pi \in \mathbb{A}} \inf_{p_0 \in P_0} \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0}(T)\right)|\mathcal{F}_\xi\right) = \sup_{\pi \in \tilde{\mathbb{A}}} \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0^{\bar{\pi}}}(T)\right)|\mathcal{F}_\xi\right).$$

2. *There exists an admissible portfolio process $\pi^* \in \mathbb{A}$ with π_1^* being constant such that for every $p_1^{\pi^*} \in P_1^{\pi^*}$ we obtain*

$$\phi(p_1^{\pi^*}(t), \pi_1^*) = \max_{\pi \in \mathbb{A}} \min_{p_1 \in P_1} \phi(p_1(t), \pi_1(t)) \text{ for all } t \in [0,T].$$

If $\xi(\omega) = \infty$ for all $\omega \in \Omega$, then for every $p_1^{\pi^*} \in P_1^{\pi^*}$ and every $p_0 \in P_0$ we get

$$\begin{aligned}\sup_{\pi \in \mathbb{A}} \inf_{p_1 \in P_1} \mathbb{E}\left(U\left(X^{\pi,p_1,\infty,p_0}(T)\right)\right) &= \mathbb{E}\left(U\left(X^{\pi^*,p_1^{\pi^*},\infty,p_0}(T)\right)\right)\\ &= U(x)\exp\left((1-\gamma)\int_0^T \phi(p_1^{\pi^*}(t), \pi_1^*)dt\right).\end{aligned}$$

Such a portfolio process π^* is called no-crash optimal portfolio process.

Proof. We give the proof for part 1 in detail and then specify what needs to be adapted for part 2.

1. The existence of a portfolio process $\bar{\pi}$ as defined above is a direct consequence of Assumption 1 and Remark 19. We now prove its optimality. For this let $p_1 \in P_1$ and $\pi \in \mathbb{A}$ be arbitrary and (z, ξ) be a crash scenario. Then we define $\tilde{\pi} \in \tilde{\mathbb{A}}$ with $\tilde{\pi}_1(t) = \pi_1(t)$ and $\tilde{\pi}_0(t) = \bar{\pi}_0(t)$ for all $t \in [0,T]$. Without loss of generality, we assume that $X^{\pi,p_1,\xi,p_0^\pi}(\xi) = X^{\tilde{\pi},p_1,\xi,p_0^{\bar{\pi}}}(\xi) = x > 0$ for every $p_0^\pi \in P_0^\pi$ and $p_0^{\bar{\pi}} \in P_0^{\bar{\pi}}$, given $\mathcal{F}_\xi$. Then we obtain with Proposition 11 and Proposition 12 for every $p_0^\pi \in P_0^\pi$ and every $p_0^{\bar{\pi}} \in P_0^{\bar{\pi}}$:

$$\begin{aligned}&\inf_{p_0 \in P_0} \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0}(T)\right)|\mathcal{F}_\xi\right) \leq \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0^\pi}(T)\right)|\mathcal{F}_\xi\right)\\ =&\mathbb{E}\left(U(x)\exp\left((1-\gamma)\int_\xi^T \phi(p_0^\pi(t), \pi_0(t))dt\right) Y_\xi^T(\pi_0, p_0^\pi)|\mathcal{F}_\xi\right)\\ \leq&\mathbb{E}\left(U(x)\exp\left((1-\gamma)\int_\xi^T \phi(p_0^{\bar{\pi}}(t), \bar{\pi}_0)dt\right) Y_\xi^T(\pi_0, p_0^\pi)|\mathcal{F}_\xi\right)\\ =&U(x)\exp\left((1-\gamma)\int_\xi^T \phi(p_0^{\bar{\pi}}(t), \bar{\pi}_0)dt\right)\mathbb{E}\left(Y_\xi^T(\pi_0, p_0^\pi)|\mathcal{F}_\xi\right)\\ =&U(x)\exp\left((1-\gamma)\int_\xi^T \phi(p_0^{\bar{\pi}}(t), \bar{\pi}_0)dt\right)\mathbb{E}\left(Y_\xi^T(\bar{\pi}_0, p_0^{\bar{\pi}})|\mathcal{F}_\xi\right)\\ =&\mathbb{E}\left(U\left(X^{\tilde{\pi},p_1,\xi,p_0^{\bar{\pi}}}(T)\right)|\mathcal{F}_\xi\right) = \inf_{p_0 \in P_0} \mathbb{E}\left(U\left(X^{\tilde{\pi},p_1,\xi,p_0}(T)\right)|\mathcal{F}_\xi\right),\end{aligned}$$

on $\{\omega \in \Omega | \xi(\omega) < \infty\}$. We have used similar arguments as in the proof of Proposition 12, in particular that $Y_\xi(\pi_0, p_0) = \{Y_\xi^t(\pi_0, p_0), \xi \leq t \leq T\}$ is a positive $\{\mathcal{F}_t, \xi \leq t \leq T\}$-martingale with $Y_\xi^\xi(\pi_0, p_0) = 1$ for every $\pi \in \mathbb{A}$ and every $p_0 \in P_0$. For the second inequality we used the definition of $\bar{\pi}$ and the fact that $U(x) > 0$ iff $(1-\gamma) > 0$. The

last equality follows with Proposition 12 part 1. Thus, we obtain

$$\sup_{\pi\in\mathbb{A}} \inf_{p_0\in P_0} \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0}(T)\right)|\mathcal{F}_\xi\right) \leq \sup_{\pi\in\tilde{\mathbb{A}}} \inf_{p_0\in P_0} \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0}(T)\right)|\mathcal{F}_\xi\right).$$

The reverse inequality follows directly from the construction of $\tilde{\mathbb{A}}$ and thus the assertion is proven with part 1 of Proposition 12.

2. Using the definition of π^* together with the second parts of Proposition 11 and Proposition 12 leads to the proof of the no-crash optimality with the same arguments as in part 1.

□

Based on this result we want to introduce the value function of the post-crash portfolio problem, which becomes important for the reformulation of the value function of problem (3.1) in the next section.

Remark 20 (Post-crash value function)**.** *Note that under our assumptions there exists a constant $\phi_0 \in \mathbb{R}$ such that we have*

$$\phi_0 := \phi(p_0^{\bar{\pi}}(t), \bar{\pi}_0),$$

for all $t \in [0,T]$ and every post-crash optimal portfolio process $\bar{\pi} \in \mathbb{A}$ with corresponding $p_0^{\bar{\pi}} \in P_0^{\bar{\pi}}$. Now let (z,ξ) be a crash scenario, $x > 0$ and $p_1 \in P_1$. We define the ***post-crash worst-case portfolio problem with ambiguity*** *on $\{\omega \in \Omega | \xi(\omega) < \infty\}$ as*

$$\sup_{\pi\in\mathbb{A}} \inf_{p_0\in P_0} \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0}(T)\right)|\mathcal{F}_\xi\right).$$

From Proposition 13 we get that the problem is solved by every post-crash optimal portfolio process $\bar{\pi} \in \mathbb{A}$ and therefore we define the post-crash value function v_0, given that $X^{\pi,p_1,\xi,p_0}(\xi) = x$, as

$$\begin{aligned} v_0(\xi,x) :&= \sup_{\pi\in\mathbb{A}} \inf_{p_0\in P_0} \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0}(T)\right)|\mathcal{F}_\xi\right) \\ &= \mathbb{E}\left(U\left(X^{\bar{\pi},p_1,\xi,p_0^{\bar{\pi}}}(T)\right)|\mathcal{F}_\xi\right) \\ &= U(x)\exp\left((1-\gamma)(T-\xi)\phi_0\right), \end{aligned}$$

on $\{\omega \in \Omega | \xi(\omega) < \infty\}$. The equations follow for every post-crash optimal portfolio process $\bar{\pi} \in \mathbb{A}$ with Proposition 12 and Proposition 13.

In order to make the possible crash scenario truly a worst-case scenario, we make the following assumption with respect to a no-crash optimal portfolio process.

Assumption 2 (No-crash optimal portfolio process)**.** *We assume that for every crash $z \in \mathbb{R}^d$ and every no-crash optimal portfolio process $\pi^* \in \mathbb{A}$, defined in Proposition 13, we have*

$$(\pi_1^*)^T z > 0,$$

i.e. every investor who chooses a no-crash optimal portfolio process suffers a loss due to the crash .

3.5 The pre-crash problem

Knowing the optimal π_0 for an admissible portfolio process $\pi \in \mathbb{A}$, we can now start determining the optimal corresponding π_1 and thus solve the worst-case crash portfolio problem with ambiguity (3.1). As usual for the worst-case scenario approach introduced by Korn and Wilmott [37], the solution to the portfolio problem will be based on indifference arguments.

Reformulation of the optimization problem. First, we would like to reformulate the value function corresponding to the optimization problem (3.1). To do this, we use the results from the previous section, i.e. we make use of the fact that we already know the post-crash optimal π_0 for every portfolio process $\pi \in \mathbb{A}$.

Proposition 14 (Reformulation of the value function.)**.** *For $t \in [0, T]$ and $x > 0$ we define the value function v_1, of the worst-case crash portfolio problem with ambiguity* (3.1)*, given that $X^{\pi,p_1,\xi,p_0}(t) = x$, as*

$$v_1(t,x) := \sup_{\pi\in\mathbb{A}} \inf_{\substack{(p_1,\xi,p_0)\\ \in P_1\times\Theta_t\times P_0}} \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0}(T)\right)|X^{\pi,p_1,\xi,p_0}(t) = x\right).$$

Let $\bar{\pi} \in \mathbb{A}$ be an arbitrary but fixed post-crash optimal portfolio process as introduced in Proposition 13. We define the sub-set of admissible portfolio processes $\tilde{\mathbb{A}} \subset \mathbb{A}$ as

$$\tilde{\mathbb{A}} := \{\pi \in \mathbb{A}|\ \pi_0(t) = \bar{\pi}_0(t) \text{ for all } t \in [0,T]\}.$$

With $v_0\left(\infty, X^{\pi,p_1,\infty,p_0}(\infty)\right) := U\left(X^{\pi,p_1,\infty,p_0}(T)\right)$, we then obtain the representation

$$v_1(t,x) = \sup_{\pi\in\tilde{\mathbb{A}}} \inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta_t}} \mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1,\xi,p_0^{\bar{\pi}}}(\xi)\right)|X^{\pi,p_1,\xi,p_0^{\bar{\pi}}}(t) = x\right),$$

for every $p_0^{\bar{\pi}} \in P_0^{\bar{\pi}}$.

Proof. Let $\bar{\pi} \in \mathbb{A}$ be an arbitrary post-crash optimal portfolio process as introduced in Proposition 13 and $\tilde{\mathbb{A}} \subset \mathbb{A}$ be as defined above. Without loss of generality we give the proof for $t = 0$ and $x > 0$. Using the results of section 3.4, especially part 1 of Proposition 13, and the definition of $\tilde{\mathbb{A}} \subset \mathbb{A}$ we get

$$\begin{aligned}
v_1(0,x) &= \sup_{\pi\in\mathbb{A}} \inf_{\substack{(p_1,\xi,p_0)\\ \in P_1\times\Theta\times P_0}} \mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0}(T)\right)\right)\\
&= \sup_{\pi\in\mathbb{A}} \inf_{\substack{(p_1,\xi,p_0)\\ \in P_1\times\Theta\times P_0}} \mathbb{E}\left(\mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0}(T)\right)|\mathcal{F}_\xi\right)\right)\\
&= \sup_{\pi\in\tilde{\mathbb{A}}} \inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta}} \mathbb{E}\left(\mathbb{E}\left(U\left(X^{\pi,p_1,\xi,p_0^{\bar{\pi}}}(T)\right)|\mathcal{F}_\xi\right)\right)
\end{aligned}$$

$$= \sup_{\pi \in \tilde{\mathbb{A}}} \inf_{\substack{(p_1,\xi) \\ \in P_1 \times \Theta}} \mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1,\xi,p_0^{\bar{\pi}}}(\xi)\right)\right),$$

for every $p_0^{\bar{\pi}} \in P_0^{\bar{\pi}}$. □

The previous result shows that we can reduce the set of admissible portfolio processes.

Assumption 3 (Reduction of $\mathbb{A}$)**.** *Without loss of generality we assume from now on that every admissible portfolio process is post-crash optimal. In particular, we choose an arbitrary but fixed post-crash optimal portfolio process $\bar{\pi}$ and reduce the set of admissible portfolio processes $\mathbb{A}$ to $\tilde{\mathbb{A}}$, defined in Proposition 14. Further, from now on, the post-crash parameter p_0 is arbitrary but fixed with $p_0 \in P_0^{\bar{\pi}}$.*
Note that for $\xi \in \Theta^T$ and $\pi \in \mathbb{A}$ we obtain

$$v_0\left(\xi, X^{\pi,p_1,\xi,p_0}(\xi)\right) = v_0(\xi, 0) = 0,$$

for every $p_1 \in P_1$ on $\{\omega \in \Omega | \pi_1(\xi)^T z(\omega) = 1\}$. On the other hand a pure investment in the money market account always leads to a strictly positive wealth process. Thus, we can assume from now on, that $\pi_1(\xi)^T z < 1$ for all admissible portfolio processes $\pi \in \mathbb{A}$ and every $\xi \in \Theta^T$.

Indifference optimality. As in [56] and [35], we first show the existence of a so-called crash-indifference process based on the following definition.

Definition 12 (Crash-indifference process)**.** *Let $z \in \mathbb{R}^d$ be a crash. An admissible portfolio process $\pi \in \mathbb{A}$ is called crash-indifference process if we have*

$$\inf_{p_1 \in P_1} \mathbb{E}\left(v_0(\xi_1, X^{\pi,p_1,\xi_1,p_0}(\xi_1))\right) = \inf_{p_1 \in P_1} \mathbb{E}\left(v_0(\xi_2, X^{\pi,p_1,\xi_2,p_0}(\xi_2))\right),$$

for all possible times of occurrences $\xi_1, \xi_2 \in \Theta$ of z.

For the proof of existence of a crash-indifference process, we need the following rather technical lemma. There, we consider the worst-case stock price parameters for a portfolio process that is at least piecewise given by a deterministic portfolio process whose post-crash value function fulfills the martingale property.

Lemma 11 (Piecewise martingale property)**.** *Let $z \in \mathbb{R}^d$ be a crash and $\hat{\pi} \in \mathbb{A}$ with $\hat{\pi}_1$ being deterministic such that $v_0\left(t, X^{\hat{\pi},p_1^{\hat{\pi}},t,p_0}(t)\right)$ is a martingale on $[0,T]$ for every $p_1^{\hat{\pi}} \in P_1^{\hat{\pi}}$, with $\hat{\pi}_1(T)^T z = 0$. Further, let $\pi \in \mathbb{A}$ and $\tau \in \Theta$ such that*

$$\pi_1(t) = \hat{\pi}_1(t) \text{ for } t \geq \tau.$$

Next, we fix an arbitrary $p_1^{\hat{\pi}} \in P_1^{\hat{\pi}}$ and for every $p_1 \in P_1$ we define $p_1^{\tau} \in P_1$ by

$$p_1^{\tau}(t) = p_1^{\hat{\pi}}(t) \quad t \geq \tau,$$
$$p_1^{\tau}(t) = p_1(t) \quad t < \tau.$$

Then we get for every $\xi \in \Theta_\tau$:

$$\begin{aligned}\inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1,\xi,p_0}(\xi)\right)\right) &= \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1^\tau,\xi,p_0}(\xi)\right)\right) \\ &= \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\tau, X^{\pi,p_1,\tau,p_0}(\tau)\right)\right).\end{aligned}$$

Proof. See Appendix 3.10. □

Based on the previous consideration we are now able to proof the existence of a crash-indifference process.

Theorem 7 (Existence of a crash-indifference process)**.** *Let $z \in \mathbb{R}^d$ be a crash. There exists an admissible crash-indifference process $\hat{\pi} \in \mathbb{A}$ with $\hat{\pi}_1$ being deterministic, such that*

$$\hat{\pi}(t) \in \operatorname*{arg\,max}_{\substack{\pi \in \mathbb{A} \\ \pi_1(t)^T z = N(t)}} \min_{p_1 \in P_1} \phi(p_1(t), \pi_1(t)) \quad \textit{for all } t \in [0,T], \tag{3.2}$$

where N is defined for $t \in [0,T]$ by the differential equation

$$N'(t) = (1 - N(t)) \left(\max_{\substack{\pi \in \mathbb{R}^d \\ \pi^T z = N(t)}} \min_{p \in D_1 \times V_1} \phi(p, \pi) - \phi_0 \right),$$

with $N(T) = 0$.

Proof. We want to give the proof in three steps. In the first part we consider the differential equation for N. Then we proof the existence of an admissible portfolio process $\hat{\pi}$ as defined in (3.2). In the last step we look at the crash-indifference property of such a portfolio process.

1. We define the function $F : (-\infty, 1] \to \mathbb{R}$ with

   $$F(N) = \max_{\substack{\pi \in \mathbb{R}^d \\ \pi^T z = N}} \min_{p \in D_1 \times V_1} \phi(p, \pi) - \phi_0.$$

   Then F is locally Lipschitz continuous and well-defined by Assumption 1 and the definition of D_1 and V_1. Further we get from Lemma 4.2. in [56] that N, given by

   $$N'(t) = (1 - N(t))F(N(t)), \quad N(T) = 0,$$

   is also well-defined and uniquely determined with $N(t) < 1$ for all $t \in [0,T]$.

2. Now we look at the optimization problem (3.2). First we should note that

   $$\{\pi \in \mathbb{A} | \pi_1(t)^T z = N(t) \text{ for all } t \in [0,T]\} \neq \emptyset,$$

   since the condition $\pi_1(t)^T z = N(t)$ for all $t \in [0,T]$ leads to the right-continuity of $\pi_1(t)^T z$ (N is differentiable) and to a bounded worst-case loss, i.e. $\pi_1(t)^T z = N(t) < 1$

for all $t \in [0,T]$. The corresponding pre-crash process $\hat{\pi}_1$ is deterministic as by the definition of $\mathbb{A}$ and P_1 we obtain

$$\hat{\pi}_1(t) \in \underset{\substack{\pi_1(t),\, \pi \in \mathbb{A} \\ \pi_1(t)^T z = N(t)}}{\arg\max} \min_{p_1 \in P_1} \phi(p_1(t), \pi_1(t)) = \underset{\substack{\pi \in \mathbb{R}^d \\ \pi^T z = N(t)}}{\arg\max} \min_{p \in D_1 \times V_1} \phi(p, \pi), \tag{3.3}$$

for all $t \in [0,T]$, where N, D_1 and V_1 are deterministic. Due to the strict concavity, required in Assumption 1, we can use a maximum theorem as in [57], to show that the solution of (3.2) is unique and continuous. Therefore, the integrability conditions for admissible portfolio processes are also fulfilled.

3. Now let $\hat{\pi} \in \mathbb{A}$ be the admissible portfolio process solving (3.2) with $\hat{\pi}_1$ being deterministic. In particular note that by construction we have $N(t) = \hat{\pi}_1(t)^T z$ for all $t \in [0,T]$. Thus, (3.3) and the definition of N imply

$$(\hat{\pi}_1(t)^T z)'(t) = (1 - \hat{\pi}_1(t)^T z) \left(\phi(p_1^{\hat{\pi}}(t), \hat{\pi}_1(t)) - \phi_0 \right), \; \hat{\pi}_1(T)^T z = 0, \tag{3.4}$$

for all $p_1^{\hat{\pi}} \in P_1^{\hat{\pi}}$.

Now let $p_1^{\hat{\pi}} = (b_1^{\hat{\pi}}, \sigma_1^{\hat{\pi}}) \in P_1^{\hat{\pi}}$ be arbitrary but fixed. Then, Itô's rule, Remark 18, the definition of the post-crash value function v_0 and the local utility drift ϕ imply

$$\begin{aligned} & dv_0\left(t, X^{\hat{\pi}, p_1^{\hat{\pi}}, t, p_0}(t)\right) = \\ & v_0\left(t, X^{\hat{\pi}, p_1^{\hat{\pi}}, t, p_0}(t)\right)(1-\gamma)\left(-\frac{(\hat{\pi}_1(t)^T z)'(t)}{1 - \hat{\pi}_1(t)^T z} + \phi(p_1^{\hat{\pi}}(t), \hat{\pi}_1(t)) - \phi_0\right) dt \\ & + v_0\left(t, X^{\hat{\pi}, p_1^{\hat{\pi}}, t, p_0}(t)\right)(1-\gamma)\, \hat{\pi}_1(t)^T \sigma_1^{\hat{\pi}}(t) dW(t), \end{aligned}$$

for $t \in [0,T]$. Using the ODE (3.4) leads to

$$dv_0\left(t, X^{\hat{\pi}, p_1^{\hat{\pi}}, t, p_0}(t)\right) = v_0\left(t, X^{\hat{\pi}, p_1^{\hat{\pi}}, t, p_0}(t)\right)(1-\gamma)\, \hat{\pi}_1(t)^T \sigma_1^{\hat{\pi}}(t) dW(t),$$

for $t \in [0,T]$.

Thus, due to the integrability condition for admissible portfolio processes and the bounded stock price parameters, $v_0\left(t, X^{\hat{\pi}, p_1^{\hat{\pi}}, t, p_0}(t)\right)$ is a martingale on $[0,T]$ for every $p_1^{\hat{\pi}} \in P_1^{\hat{\pi}}$. With $\hat{\pi}_1(T)^T z = 0$ and Remark 18, it is in particular a martingale on $[0,T] \cup \{\infty\}$. Now we use Lemma 11 with $\tau = 0$ and obtain for two stopping times $\xi_1, \xi_2 \in \Theta$ that we have

$$\begin{aligned} \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi_1, X^{\hat{\pi}, p_1, \xi_1, p_0}(\xi_1)\right)\right) &= \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(0, X^{\hat{\pi}, p_1, 0, p_0}(0)\right)\right) \\ &= \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi_2, X^{\hat{\pi}, p_1, \xi_2, p_0}(\xi_2)\right)\right), \end{aligned}$$

i.e. $\hat{\pi}$ is a crash-indifference process.

□

The existence of a crash-indifference process allows us to narrow down the set of possible pre-crash optimal portfolio processes with the help of the following Proposition.

Proposition 15 (Indifference frontier). *Let $z \in \mathbb{R}^d$ be a crash and $\hat{\pi} \in \mathbb{A}$ be a crash-indifference process as defined in Theorem 7. Fix an arbitrary $\pi \in \mathbb{A}$ and define the stopping time τ as*

$$\tau := \inf\{t \in [0,T] | \pi_1(t)^T z > \hat{\pi}_1(t)^T z\},$$

with the standard convention $\inf \emptyset = \infty$. *Based on this we define $\tilde{\pi}$ for $t \in [0,T]$ by*

$$\tilde{\pi}_1(t) := \mathbb{1}_{t<\tau}\pi_1(t) + \mathbb{1}_{t\geq\tau}\hat{\pi}_1(t).$$

Then, $\tilde{\pi} \in \mathbb{A}$ and we obtain

$$\inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta}} \mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1,\xi,p_0}(\xi)\right)\right) \leq \inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta}} \mathbb{E}\left(v_0\left(\xi, X^{\tilde{\pi},p_1,\xi,p_0}(\xi)\right)\right).$$

Proof. Let z, $\hat{\pi}$, τ and $\tilde{\pi}$ be defined as above for an arbitrary $\pi \in \mathbb{A}$. First note that $\tilde{\pi}_1$ is piece wise given by parts of admissible portfolio processes. Further, $\tilde{\pi}_1(t)^T z$ has right-continuous paths with left limits by definition. Thus $\tilde{\pi}$ is an admissible portfolio process.

For $\xi \in \Theta$, we define $\xi_{max} := \max\{\xi, \tau\}$. By Lemma 11 and $\xi_{max} \in \Theta_\tau$, we get

$$\begin{aligned}
&\inf_{p_1\in P_1} \mathbb{E}\left(\mathbb{1}_{\xi>\tau} v_0\left(\xi, X^{\tilde{\pi},p_1,\xi,p_0}(\xi)\right)\right) + \inf_{p_1\in P_1} \mathbb{E}\left(\mathbb{1}_{\xi_{max}=\tau} v_0\left(\tau, X^{\tilde{\pi},p_1,\tau,p_0}(\tau)\right)\right)\\
&= \inf_{p_1\in P_1} \mathbb{E}\left(v_0\left(\xi_{max}, X^{\tilde{\pi},p_1,\xi_{max},p_0}(\xi_{max})\right)\right)\\
&= \inf_{p_1\in P_1} \mathbb{E}\left(v_0\left(\tau, X^{\tilde{\pi},p_1,\tau,p_0}(\tau)\right)\right)\\
&= \inf_{p_1\in P_1} \mathbb{E}\left(\mathbb{1}_{\xi>\tau} v_0\left(\tau, X^{\tilde{\pi},p_1,\tau,p_0}(\tau)\right)\right) + \inf_{p_1\in P_1} \mathbb{E}\left(\mathbb{1}_{\xi_{max}=\tau} v_0\left(\tau, X^{\tilde{\pi},p_1,\tau,p_0}(\tau)\right)\right).
\end{aligned}$$

Here, the first and last equality are valid, since we minimize path-wise over the post crash parameters $p_1 \in P_1$ and the indicator functions divide Ω into two disjoint subsets. From this list of equations we in particular obtain

$$\inf_{p_1\in P_1} \mathbb{E}\left(\mathbb{1}_{\xi>\tau} v_0\left(\xi, X^{\tilde{\pi},p_1,\xi,p_0}(\xi)\right)\right) = \inf_{p_1\in P_1} \mathbb{E}\left(\mathbb{1}_{\xi>\tau} v_0\left(\tau, X^{\tilde{\pi},p_1,\tau,p_0}(\tau)\right)\right)$$

and therefore

$$\begin{aligned}
&\inf_{p_1\in P_1} \mathbb{E}\left(v_0\left(\xi, X^{\tilde{\pi},p_1,\xi,p_0}(\xi)\right)\right)\\
&= \inf_{p_1\in P_1} \mathbb{E}\left(\mathbb{1}_{\xi>\tau} v_0\left(\xi, X^{\tilde{\pi},p_1,\xi,p_0}(\xi)\right)\right) + \inf_{p_1\in P_1} \mathbb{E}\left(\mathbb{1}_{\xi\leq\tau} v_0\left(\xi, X^{\tilde{\pi},p_1,\xi,p_0}(\xi)\right)\right)\\
&= \inf_{p_1\in P_1} \mathbb{E}\left(\mathbb{1}_{\xi>\tau} v_0\left(\tau, X^{\tilde{\pi},p_1,\tau,p_0}(\tau)\right)\right) + \inf_{p_1\in P_1} \mathbb{E}\left(\mathbb{1}_{\xi\leq\tau} v_0\left(\xi, X^{\tilde{\pi},p_1,\xi,p_0}(\xi)\right)\right)\\
&= \inf_{p_1\in P_1} \mathbb{E}\left(v_0\left(\xi_{min}, X^{\tilde{\pi},p_1,\xi_{min},p_0}(\xi_{min})\right)\right),
\end{aligned}$$

where we define $\xi_{min} := \min\{\xi, \tau\}$. Thus, it is enough to consider all $\xi \in \Theta^\tau$ while looking at the worst-case scenario for $\tilde{\pi}$, i.e. we have

$$\inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta}} \mathbb{E}\left(v_0\left(\xi, X^{\tilde{\pi},p_1,\xi,p_0}(\xi)\right)\right) = \inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta^\tau}} \mathbb{E}\left(v_0\left(\xi, X^{\tilde{\pi},p_1,\xi,p_0}(\xi)\right)\right). \tag{3.5}$$

Note that $\tilde{\pi}_1(t) = \pi_1(t)$ for all $t < \tau$ and thus

$$v_0\left(t, X^{\tilde{\pi},p_1,t,p_0}(t)\right) = v_0\left(t, X^{\pi,p_1,t,p_0}(t)\right), \quad \text{for } t \in [0,\tau), \tag{3.6}$$

and every $p_1 \in P_1$. Further, the definition of τ and the right-continuity assumption of an admissible portfolio process yield

$$\begin{aligned}
&v_0\left(\tau, X^{\tilde{\pi},p_1,\tau,p_0}(\tau)\right) = v_0\left(\tau, \left(1-\tilde{\pi}_1(\tau)^T z\right) X^{\tilde{\pi},p_1,\tau,p_0}(\tau-)\right)\\
=&v_0\left(\tau, \left(1-\hat{\pi}_1(\tau)^T z\right) X^{\pi,p_1,\tau,p_0}(\tau-)\right) \geq v_0\left(\tau, \left(1-\pi_1(\tau)^T z\right) X^{\pi,p_1,\tau,p_0}(\tau-)\right)\\
=&v_0\left(\tau, X^{\pi,p_1,\tau,p_0}(\tau)\right),
\end{aligned}$$

on $\{\omega \in \Omega | \tau(\omega) < \infty\}$ for every $p_1 \in P_1$. Together with (3.5) and (3.6), this leads to

$$\begin{aligned}
&\inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta}} \mathbb{E}\left(v_0\left(\xi, X^{\tilde{\pi},p_1,\xi,p_0}(\xi)\right)\right) = \inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta^\tau}} \mathbb{E}\left(v_0\left(\xi, X^{\tilde{\pi},p_1,\xi,p_0}(\xi)\right)\right)\\
\geq &\inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta^\tau}} \mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1,\xi,p_0}(\xi)\right)\right) \geq \inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta}} \mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1,\xi,p_0}(\xi)\right)\right).
\end{aligned}$$

□

We now have to introduce, in the two following lemmata, the supermartingale property for an admissible portfolio process and have to consider the relationship between a crash-indifference process and a no-crash optimal portfolio process.

Lemma 12 (Supermartingale property)**.** *Let $z \in \mathbb{R}^d$ be a crash and $\hat{\pi} \in \mathbb{A}$ with $\hat{\pi}_1$ being deterministic such that $v_0\left(t, X^{\hat{\pi},p_1^{\hat{\pi}},t,p_0}(t)\right)$ is a supermartingale on $[0,T] \cup \{\infty\}$ for every $p_1^{\hat{\pi}} \in P_1^{\hat{\pi}}$. Further, let $\pi \in \mathbb{A}$ and $\tau \in \Theta$ such that*

$$\pi_1(t) = \hat{\pi}_1(t) \text{ for } t \leq \min\{\tau, T\}.$$

Then, for $\xi \in \Theta^\tau$ we obtain

$$\inf_{p_1\in P_1} \mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1,\xi,p_0}(\xi)\right)\right) \geq \inf_{p_1\in P_1} \mathbb{E}\left(v_0\left(\tau, X^{\pi,p_1,\tau,p_0}(\tau)\right)\right).$$

Proof. See Appendix 3.10. □

Lemma 13 (Crash-indifference and no-crash optimal process)**.** *Let $z \in \mathbb{R}^d$ be a crash and $\hat{\pi} \in \mathbb{A}$ be a crash-indifference process as defined in Theorem 7. Further, let $\pi^* \in \mathbb{A}$ be a no-crash optimal portfolio process as defined in Proposition 13. For an arbitrary $p_1^{\pi^*} \in P_1^{\pi^*}$ we then define ϕ_1*

as

$$\phi_1 := \phi(p_1^{\pi^*}(t), \pi_1^*) \text{ for all } t \in [0, T].$$

Using ϕ_0, defined in Remark 20, we obtain:

1. *If $\phi_1 \geq \phi_0$, then $\hat{\pi}_1(t)^T z \leq (\pi_1^*)^T z$ for all $t \in [0, T]$.*
2. *If $\phi_1 < \phi_0$, then $\hat{\pi}_1(t)^T z \leq (\pi_1^*)^T z$ for all $t \in [0, T]$ or there exists a $T_0 \in (0, T]$ such that $\hat{\pi}_1(t)^T z \geq (\pi_1^*)^T z$ for $t \in [0, T_0]$ and $\hat{\pi}_1(t)^T z \leq (\pi_1^*)^T z$ for $t \in [T_0, T]$.*

Proof. See Appendix 3.10. □

Now we define an admissible portfolio process $\bar{\pi} \in \mathbb{A}$ for which the no-crash scenario is the worst-case scenario, and for which we then prove its optimality in Theorem 8.

Proposition 16 (No-crash worst-case scenario)**.** *Let $z \in \mathbb{R}^d$ be a crash and $\hat{\pi} \in \mathbb{A}$ be a crash-indifference process as defined in Theorem 7. Further, let $\pi^* \in \mathbb{A}$ be a no-crash optimal portfolio process as defined in Proposition 13. Based on Proposition 15, we define the set ID as*

$$ID := \{\pi \in \mathbb{A} | \pi_1(t)^T z \leq \hat{\pi}_1(t)^T z \text{ for all } t \in [0, T]\},$$

and the portfolio process $\bar{\pi} \in \mathbb{A}$ such that

$$\bar{\pi} \in \arg\max_{\pi \in ID} \min_{p_1 \in P_1} \phi(p_1(t), \pi_1(t)) \quad \text{for all } t \in [0, T].$$

Then we obtain

$$\begin{aligned}\inf_{\substack{(p_1,\xi)\\ \in P_1 \times \Theta}} \mathbb{E}\left(v_0\left(\xi, X^{\bar{\pi}, p_1, \xi, p_0}(\xi)\right)\right) &= \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\infty, X^{\bar{\pi}, p_1, \infty, p_0}(\infty)\right)\right) \\ &= \mathbb{E}\left(U\left(X^{\bar{\pi}, p_1^{\bar{\pi}}, \infty, p_0}(T)\right)\right),\end{aligned}$$

for every $p_1^{\bar{\pi}} \in P_1^{\bar{\pi}}$.

Proof. Let $z \in \mathbb{R}^d$ be a crash and $\bar{\pi}$ be as defined in the assertion. First, we look at the definition of $\bar{\pi}$. From Assumption 1 we know that, for $t \in [0, T]$,

$$\min_{p_1 \in P_1} \phi(p_1(t), \pi_1(t))$$

is point-wise a strictly concave function in $\pi_1(t) \in \mathbb{R}^d$ with unique maximum. Due to the linear constraint given by ID; the Karush-Kuhn-Trucker conditions and the definition of π^* imply that for every $t \in [0, T]$ the point-wise maximizer $\bar{\pi}_1(t)$ is given by

$$\bar{\pi} \in \arg\min_{\pi \in \{\pi^*, \hat{\pi}\}} \{\pi_1(t)^T z\}, \text{ for } t \in [0, T]. \tag{3.7}$$

From Lemma 13 we know that $\bar{\pi}$ is indeed admissible with $\bar{\pi}_1$ being deterministic. Further it is sufficient to distinguish between two cases:

1. If $\hat{\pi}_1(t)^T z \leq (\pi_1^*)^T z$ for all $t \in [0,T]$, then $\bar{\pi}_1(t) = \hat{\pi}_1(t)$ for all $t \in [0,T]$ and and the assertion follows with the crash-indifference property of $\hat{\pi}$ and Proposition 12 part 2.

2. Now we look at the case that $\phi_1 < \phi_0$ and there exists a $T_0 \in (0,T]$, as defined in part 2 of Lemma 13, such that $\bar{\pi}_1(t) = \pi_1^*$ for $t \in [0,T_0]$ and $\bar{\pi}_1(t) = \hat{\pi}_1(t)$ for $t \in [T_0,T]$. Note that for every $p_1^{\bar{\pi}} \in P_1^{\bar{\pi}}$ there exists a $p_1^{\pi*} \in P_1^{\pi*}$, such that $p_1^{\bar{\pi}}(t) = p_1^{\pi*}(t)$ for all $t \in [0,T_0]$. Thus, applying Itô's formula for $t \in [0,T_0]$ and an arbitrary $p_1^{\bar{\pi}} = (b_1^{\bar{\pi}}, \sigma_1^{\bar{\pi}}) \in P_1^{\bar{\pi}}$ yields

$$\begin{aligned} &dv_0\left(t, X^{\bar{\pi},p_1^{\bar{\pi}},t,p_0}(t)\right) = dv_0\left(t, X^{\pi^*,p_1^{\pi^*},t,p_0}(t)\right) \\ =&v_0\left(t, X^{\bar{\pi},p_1^{\bar{\pi}},t,p_0}(t)\right)\left((1-\gamma)(\phi_1-\phi_0)dt + (1-\gamma)\sigma_1^{\bar{\pi}}(t)^T\bar{\pi}_1(t)dW(t)\right). \end{aligned}$$

Here we used the definition of ϕ_1 from Lemma 13. With $\phi_1 < \phi_0$, we obtain that $v_0(t, X^{\bar{\pi},p_1^{\bar{\pi}},t,p_0}(t))$ is a supermartingale on $[0,T_0]$. From the construction of $\hat{\pi}$ and the proof of Theorem 7 we further get that $v_0(t, X^{\bar{\pi},p_1^{\bar{\pi}},t,p_0}(t))$ is a martingale on $[T_0,T] \cup \{\infty\}$. Thus, $v_0(t, X^{\bar{\pi},p_1^{\bar{\pi}},t,p_0}(t))$ is a supermartingale on $[0,T] \cup \{\infty\}$ and the proof follows with Lemma 12 and Proposition 12 part 2.

□

The final step is to proof that the portfolio process $\bar{\pi} \in \mathbb{A}$, defined in Proposition 16, is optimal for our worst-case problem.

Theorem 8 (Optimal pre-crash portfolio process)**.** *Let $z \in \mathbb{R}^d$ be a crash and $\bar{\pi} \in \mathbb{A}$ be as defined in Proposition 16. Then $\bar{\pi}$ solves the worst-case crash portfolio problem with ambiguity* (3.1)*.*

Proof. Let $z \in \mathbb{R}^d$ be a crash and $\bar{\pi} \in \mathbb{A}$ be defined as in the assertion. Consider the stopping time τ, defined in Proposition 15 for an admissible portfolio process $\pi \in \mathbb{A}$. We know from Proposition 15 that we can reduce the search for the optimal portfolio processes to all processes $\pi \in \mathbb{A}$ with corresponding $\tau = \infty$. Thus, we fix a $\pi \in \mathbb{A}$ with $\pi_1(t)^T z(t) \leq \hat{\pi}_1(t)^T z$ for all $t \in [0,T]$.

From the construction of $\bar{\pi}$ we obtain for every $p_1^{\bar{\pi}} \in P_1^{\bar{\pi}}$ and $p_1^{\pi} \in P_1^{\pi}$ that

$$\phi(p_1^{\bar{\pi}}(t), \bar{\pi}_1(t)) \geq \phi(p_1^{\pi}(t), \pi_1(t)) \quad \text{for all } t \in [0,T].$$

For every $p_1^{\bar{\pi}} \in P_1^{\bar{\pi}}$ and every $p_1^{\pi} \in P_1^{\pi}$ Proposition 16 leads to

$$\begin{aligned} &\inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta}} \mathbb{E}\left(v_0\left(\xi, X^{\bar{\pi},p_1,\xi,p_0}(\xi)\right)\right) = \mathbb{E}\left(U\left(X^{\bar{\pi},p_1^{\bar{\pi}},\infty,p_0}(T)\right)\right) \\ =&\mathbb{E}\left(U(x)\exp\left((1-\gamma)\int_0^T \phi(p_1^{\bar{\pi}}(t), \bar{\pi}_1(t))dt\right) Y_0^T(\bar{\pi}_1, p_1^{\bar{\pi}})\right) \\ =&U(x)\exp\left((1-\gamma)\int_0^T \phi(p_1^{\bar{\pi}}(t), \bar{\pi}_1(t))dt\right)\mathbb{E}\left(Y_0^T(\bar{\pi}_1, p_1^{\bar{\pi}})\right) \\ =&U(x)\exp\left((1-\gamma)\int_0^T \phi(p_1^{\bar{\pi}}(t), \bar{\pi}_1(t))dt\right)\mathbb{E}\left(Y_0^T(\pi_1, p_1^{\pi})\right) \end{aligned}$$

$$
\begin{aligned}
=&\mathbb{E}\left(U(x)\exp\left((1-\gamma)\int_0^T \phi(p_1^{\bar{\pi}}(t),\bar{\pi}_1(t))dt\right)Y_0^T(\pi_1,p_1^{\pi})\right)\\
\geq&\mathbb{E}\left(U(x)\exp\left((1-\gamma)\int_0^T \phi(p_1^{\pi}(t),\pi_1(t))dt\right)Y_0^T(\pi_1,p_1^{\pi})\right)\\
=&\mathbb{E}\left(v_0\left(\infty,X^{\pi,p_1^{\pi},\infty,p_0}(\infty)\right)\right)\geq \inf_{\substack{(p_1,\xi)\\ \in P_1\times\Theta}}\mathbb{E}\left(v_0\left(\xi,X^{\pi,p_1,\xi,p_0}(\xi)\right)\right).
\end{aligned}
$$

In addition, we used Proposition 12 part 2, Proposition 11 part 2 and that $Y_0(\pi_1,p_1)=\{Y_0^t(\pi_1,p_1),0\leq t\leq T\}$ is a positive $\{\mathcal{F}_t,0\leq t\leq T\}$-martingale with $Y_0^0(\pi_1,p_1)=1$ for every $\pi\in\mathbb{A}$ and $p_1\in P_1$. Then the optimality follows with Proposition 14. □

3.6 Box ambiguity

We consider a first simple example with one risky asset and box ambiguity. The parameter choice is based on the work of Lin and Riedel [41]. We have a model as introduced in section 3.2, consisting of one money market account with constant interest rate r and one risky asset S with $n=d=1$. Further, we consider a crash scenario (z,ξ), with $z>0$, and unknown pre- and post-crash parameters. The sets of possible values of the drift parameters $D_1\subset\mathbb{R}$ and $D_0\subset\mathbb{R}$ in the pre-crash and post-crash market are defined as

$$D_1:=[\underline{b}_1,\bar{b}_1]\text{ and }D_0:=[\underline{b}_0,\bar{b}_0],\text{ with }0\leq r\leq\underline{b}_1,\underline{b}_0.$$

Similar, we define the volatility sets $V_1\subset\mathbb{R}$ and $V_0\subset\mathbb{R}$ as

$$V_1:=[\underline{\sigma}_1,\bar{\sigma}_1]\text{ and }V_0:=[\underline{\sigma}_0,\bar{\sigma}_0],\text{ with }0<\underline{\sigma}_1,\underline{\sigma}_0.$$

Based on the results of section 3.4 and section 3.5 we are now able to solve the corresponding worst-case crash portfolio problem with ambiguity (3.1). Assumption 1 is fulfilled, as for every $\pi\in\mathbb{A}$ and $i\in\{0,1\}$ we directly get

$$
\begin{aligned}
\inf_{p_i\in P_i}\phi(p_i(t),\pi_i(t))=&\phi(p_i^{\pi}(t),\pi_i(t))\\
=&r+\pi_i(t)(\underline{b}_i\mathbb{1}_{\pi_i(t)\geq 0}+\bar{b}_i\mathbb{1}_{\pi_i(t)<0}-r\underline{1})-\frac{1}{2}\gamma\pi_i(t)^2\bar{\sigma}_i^2,
\end{aligned}
$$

for all $t\in[0,T]$. In particular this is point-wise a strictly concave function in $\pi_i(t)\in\mathbb{R}$, for $i\in\{0,1\}$, that converges to $-\infty$ for $||\pi_i(t)||\to\infty$. Proposition 13 yields the post-crash optimal portfolio process $\bar{\pi}$ as

$$\bar{\pi}_0=\frac{1}{\gamma}\frac{\underline{b}_0-r}{\bar{\sigma}_0^2}.$$

Using Theorem 7, we can construct a crash-indifference process $\hat{\pi}$ with

$$\hat{\pi}_1(t)'=\frac{(1-\hat{\pi}_1(t)z)}{z}\left(\phi(p_1^{\hat{\pi}}(t),\hat{\pi}_1(t))-\phi_0\right),\quad\hat{\pi}_1(T)=0,$$

for $t \in [0, T]$ and $p_1^{\hat{\pi}} \in P_1^{\hat{\pi}}$, with

$$\phi_0 = r + \frac{1}{2\gamma}\frac{(\underline{b}_0 - r)^2}{\bar{\sigma}_0^2},$$
$$\phi(p_1^{\hat{\pi}}(t), \hat{\pi}_1(t)) = r + \hat{\pi}(t)(\underline{b}_1 - r\underline{1}) - \frac{1}{2}\gamma\hat{\pi}(t)^2\bar{\sigma}_1^2,$$

since $\hat{\pi}_1 \geq 0$ by Lemma 4.1 in [56]. Further we know from Proposition 13, and the no-ruin condition of admissible portfolio processes, that $\pi^* \in \mathbb{A}$ with

$$\pi_1^* = \min\left\{\frac{1}{\gamma}\frac{\underline{b}_1 - r}{\bar{\sigma}_1^2}, \frac{1}{z}\right\},$$

is a no-crash optimal portfolio process. Theorem 8, together with (3.7) in the proof of Proposition 16, lead to the optimality of the post-crash optimal portfolio process $\bar{\pi}$ with

$$\bar{\pi}_1(t) = \min\{\pi_1^*, \hat{\pi}_1(t)\}, \text{ for } t \in [0, T].$$

3.7 Ellipsoidal drift ambiguity with bounded volatility matrix

Now we examine a multi-stock setting consisting of a money-market account, d risky assets with ellipsoid shaped drift ambiguity and bounded volatility matrix and a crash scenario (z, ξ), with $z \neq 0$. The definition of the parameter sets goes back to the robust Merton framework of Biagini and Pınar [7]. In particular, we take up the setting of section 3.2 and define ellipsoid shaped parameter sets for the pre- and post-crash drift values and give upper bounds for the volatility matrices by using the Loewener ordering. For this, we fix $\epsilon_1, \epsilon_0 \in \mathbb{R}_{>0}$ and let $\Sigma_1, \Sigma_0 \in \mathbb{R}^{d\times d}$ be symmetric and positive definite. Based on this, we define the sets of possible values of the drift parameters in the pre- and post-crash market as

$$D_i := \{b \in \mathbb{R}^d | (b - \hat{b})^T \Sigma_i^{-1}(b - \hat{b}) \leq \epsilon_i^2\},$$

for a $\hat{b} \in \mathbb{R}^d$ and $i \in \{0, 1\}$. We further assume that $r\underline{1} \notin D_0$. The sets of possible values of the volatility matrices are defined for $i \in \{0, 1\}$ by

$$V_i := \{\sigma \in \mathbb{R}^{d\times n} \text{ with full rank} \mid x^T\sigma\sigma^T x \leq x^T\Sigma_i x \text{ for all } x \in \mathbb{R}^d\}.$$

Before we solve the corresponding worst-case crash portfolio problem with ambiguity (3.1), we need to show that Assumption 1 is fulfilled.

Lemma 14 (Verification of Assumption 1)**.** *Consider the worst-case crash portfolio problem with ellipsoidal drift ambiguity and bounded volatility matrix. Then, Assumption 1 is satisfied. In particular, for $\pi \in \mathbb{A}$ we have*

$$\inf_{p_i \in P_i} \phi(p_i(t), \pi_i(t))$$

$$=r + \pi_i(t)^T(\hat{b} - r\underline{1}_d) - \epsilon_i\sqrt{\pi_i(t)^T\Sigma_i\pi_i(t)} - \frac{1}{2}\gamma\pi_i(t)^T\Sigma_i\pi_i(t),$$

for all $t \in [0,T]$ *and* $i \in \{0,1\}$.

Proof. For $\pi \in \mathbb{A}$ and $i \in \{0,1\}$ we first have to show that the minimization problem

$$\inf_{p_i \in P_i} \phi(p_i(t), \pi_i(t)) = \inf_{p_i \in P_i} r + \pi_i(t)^T(b_i(t) - r\underline{1}_d) - \frac{1}{2}\gamma\pi_i(t)^T\sigma_i(t)\sigma_i(t)^T\pi_i(t)$$

contains a finite solution for $t \in [0,T]$. We fix an arbitrary $\pi \in \mathbb{A}$, $t \in [0,T]$ and consider the optimization problem of the drift parameter:

$$\inf_{b_i(t) \in D_i} \pi_i(t)^T b_i(t).$$

Using the Karush-Kuhn-trucker conditions one can directly show that the minimizer b_i^π for $i \in \{0,1\}$ at time $t \in [0,T]$ is

$$b_i^\pi(t) = \hat{b} - \epsilon_i \frac{\Sigma_i\pi_i(t)}{\sqrt{\pi_i(t)^T\Sigma_i\pi_i(t)}},$$

if $\pi_i(t) \neq 0$, and $b_i^\pi(t) = \hat{b}$ otherwise. The minimization problem over the set of possible volatility matrices at time $t \in [0,T]$ is given by

$$\inf_{\sigma_i(t) \in V_i} -\frac{1}{2}\gamma\pi_i(t)^T\sigma_i(t)\sigma_i(t)^T\pi_i(t)$$

for $i \in \{0,1\}$. The definition of the sets V_0 and V_1 implies that the minimizer σ_i^π for $i \in \{0,1\}$ and all $t \in [0,T]$ is the matrix of full rank $\sigma_i^\pi \in \mathbb{R}^{d\times n}$ with

$$\sigma_i^\pi(\sigma_i^\pi)^T = \Sigma_i.$$

Such matrices exist since Σ_0 and Σ_1 are symmetric and positive definite. This leads to

$$\begin{aligned}&\inf_{p_i \in P_i} \phi(p_i(t), \pi_i(t))\\ =&r + \pi_i(t)^T(\hat{b} - r\underline{1}_d) - \epsilon_i\sqrt{\pi_i(t)^T\Sigma_i\pi_i(t)} - \frac{1}{2}\gamma\pi_i(t)^T\Sigma_i\pi_i(t),\end{aligned}$$

for $i \in \{0,1\}$ and $t \in [0,T]$. With the definition of σ_i^π, we get for $\pi \in \mathbb{A}$ and $i \in \{0,1\}$ that

$$\|\sigma_i^\pi\pi_i(t)\| = \sqrt{\pi_i(t)^T\Sigma_i\pi_i(t)}.$$

Together with the convexity of the norm we then obtain that the above infimum is point-wise a strictly concave function in $\pi_i(t) \in \mathbb{R}^d$ for $i \in \{0,1\}$ that converges to $-\infty$ for $||\pi_i(t)|| \to \infty$. □

Now we can use Proposition 13 and Theorem 8 to derive the optimal portfolio process $\bar{\pi} \in \mathbb{A}$ for our worst-case problem with model ambiguity (3.1). Applying Proposition 13 for

the post-crash portfolio problem gives us $\bar{\pi}_0$, defined as

$$\bar{\pi}_0 = \frac{H - \epsilon_0}{\gamma H} \Sigma_0^{-1} \left(\hat{b} - r\underline{1} \right), \text{ with } H := \sqrt{\left(\hat{b} - r\underline{1} \right)^T \Sigma_0^{-1} \left(\hat{b} - r\underline{1} \right)}. \tag{3.8}$$

Also with Proposition 13, or with similar arguments as in [7], we can determine a no-crash optimal portfolio process $\pi^* \in \mathbb{A}$. Based on Theorem 7, we need to construct numerically the crash-indifference process $\hat{\pi}$, such that the optimal $\bar{\pi}_1$ can be computed with Proposition 16 and Theorem 8.

3.8 Numerical illustration of the ellipsoidal drift ambiguity case

We now want to take a closer look at the worst-case portfolio optimization problem with ellipsoid shaped drift ambiguity and bounded volatility matrix, as introduced in section 3.7. To this end, we will first analyse the indifference strategy in this model, and then look at some numerical examples. We use the same notation as in section 3.7 and in addition we assume that $r\underline{1} \notin D_1$.

Analysis of the indifference strategy $\hat{\pi}$. For the computation of the optimal pre-crash portfolio process $\bar{\pi}_1$, we need to construct numerically the crash-indifference process $\hat{\pi}$, defined in Theorem 7. For this we want to take a closer look at the optimization problem (3.2). Using the notation of section 3.7, together with the results of Lemma 14, we obtain that $\hat{\pi}$ solves

$$\begin{aligned}
\hat{\pi}(t) \in & \underset{\substack{\pi \in \mathbb{A} \\ \pi_1(t)^T z = N(t)}}{\arg\max} \min_{p_1 \in P_1} \phi(p_1(t), \pi_1(t)) \\
= & \underset{\substack{\pi \in \mathbb{A} \\ \pi_1(t)^T z = N(t)}}{\arg\max} \; r + \pi_1(t)^T (\hat{b} - r\underline{1}_d) - \epsilon_1 \sqrt{\pi_1(t)^T \Sigma_1 \pi_1(t)} - \frac{1}{2} \gamma \pi_1(t)^T \Sigma_1 \pi_1(t),
\end{aligned}$$

for all $t \in [0, T]$, with N defined for $t \in [0, T]$ by the differential equation

$$N'(t) = (1 - N(t)) \left(\phi(p_1^{\hat{\pi}}(t), \hat{\pi}_1(t)) - \phi_0 \right),$$

with $N(T) = 0$ and

$$\phi(p_1^{\hat{\pi}}(t), \hat{\pi}_1(t)) = r + \hat{\pi}_1(t)^T (\hat{b} - r\underline{1}_d) - \epsilon_1 \sqrt{\hat{\pi}_1(t)^T \Sigma_1 \hat{\pi}_1(t)} - \frac{1}{2} \gamma \hat{\pi}_1(t)^T \Sigma_1 \hat{\pi}_1(t),$$

for $t \in [0, T]$. The definition of ϕ_0 in Remark 20 leads to

$$\phi_0 = r + \bar{\pi}_0^T (\hat{b} - r\underline{1}_d) - \epsilon_0 \sqrt{\bar{\pi}_0^T \Sigma_0 \bar{\pi}_0} - \frac{1}{2} \gamma \bar{\pi}_0^T \Sigma_0 \bar{\pi}_0,$$

where the optimal post-crash portfolio $\bar{\pi}_0$ is given by (3.8). We look at the constrained optimization problem which determines $\hat{\pi}$ and apply Lagrangian-multiplier methods. For

this we should note that

$$r + \pi_1(t)^T(\hat{b} - r\underline{1}_d) - \epsilon_1\sqrt{\pi_1(t)^T\Sigma_1\pi_1(t)} - \frac{1}{2}\gamma\pi_1(t)^T\Sigma_1\pi_1(t)$$

is smooth for $\pi_1(t) \in \mathbb{R}\backslash\{0\}$. Thus, $\hat{\pi}_1(t)$ fulfills the following first-order conditions at time $t \in [0, T]$, if $\hat{\pi}_1(t) \neq 0$:

$$\begin{aligned} 0 &= (\hat{b} - r\underline{1}_d) - \left(\frac{\epsilon_1}{\sqrt{\hat{\pi}_1(t)^T\Sigma_1\hat{\pi}_1(t)}} + \gamma\right)\Sigma_1\hat{\pi}_1(t) - \lambda z, \\ 0 &= N(t) - \hat{\pi}_1(t)^T z, \end{aligned}$$

for $\lambda \in \mathbb{R}$. For solving the first equation we define s by

$$s := \sqrt{\hat{\pi}_1(t)^T\Sigma_1\hat{\pi}_1(t)},$$

which leads to

$$0 = (\hat{b} - r\underline{1}_d) - \left(\frac{\epsilon_1}{s} + \gamma\right)\Sigma_1\hat{\pi}_1(t) - \lambda z,$$

and therefore

$$\hat{\pi}_1(t) = \frac{s}{\epsilon_1 + s\gamma}\Sigma_1^{-1}\left(\hat{b} - r\underline{1}_d - \lambda z\right), \tag{3.9}$$

if $\hat{\pi}_1(t) \neq 0$. Now we use the definition of s, plug in our representation of $\hat{\pi}_1$ and obtain:

$$s^2 = \frac{s^2}{(\epsilon_1 + s\gamma)^2}\left(\hat{b} - r\underline{1}_d - \lambda z\right)\Sigma_1^{-1}\left(\hat{b} - r\underline{1}_d - \lambda z\right). \tag{3.10}$$

Next, we define R as

$$R := \sqrt{\left(\hat{b} - r\underline{1}_d - \lambda z\right)\Sigma_1^{-1}\left(\hat{b} - r\underline{1}_d - \lambda z\right)},$$

which leads to

$$0 < \epsilon_1 + s\gamma = R,$$

for $\hat{\pi}_1(t) \neq 0$ and thus we get the following positive root for equation (3.10):

$$s = \frac{R - \epsilon_1}{\gamma}.$$

Plugging this in (3.9), gives us for $t \in [0, T]$ that

$$\hat{\pi}_1(t) = \frac{R - \epsilon_1}{\gamma R}\Sigma_1^{-1}\left(\hat{b} - r\underline{1}_d - \lambda z\right),$$

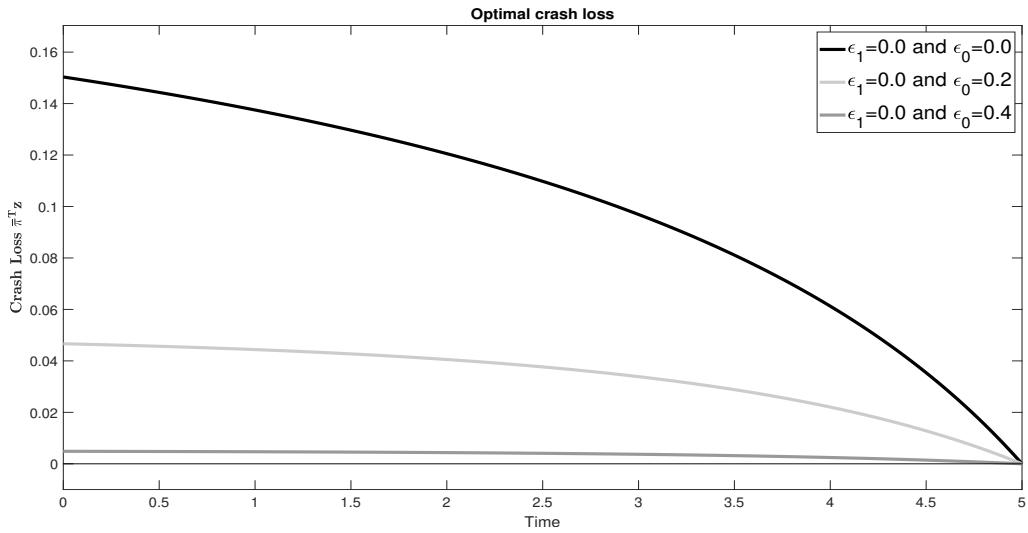

FIGURE 3.1: One stock with post-crash ambiguity.

if $\hat{\pi}_1(t) \neq 0$, for a $\lambda \in \mathbb{R}$ that solves the second equation of the first-order conditions, i.e. a $\lambda \in \mathbb{R}$ such that

$$0 = N(t) - \frac{R - \epsilon_1}{\gamma R} z^T \Sigma_1^{-1} \left(\hat{b} - r\underline{1}_d - \lambda z \right).$$

Since R also depends on λ, this equation can be solved numerically for the optimal λ. Note that because of $\hat{\pi}_1(t)^T z = N(t)$, the case $\hat{\pi}_1(t) = 0$ only needs to be considered if $N(t) = 0$ for $t \in [0, T]$. We are now able to numerically determine the crash-indifference strategy $\hat{\pi}$ with possible loss $\hat{\pi}(t)^T z = N(t)$ for $t \in [0, T]$.

Numerical examples for $d = 1$. We begin our considerations with the simple case of one stock, i.e. we have $d = n = 1$. Using the notation of section 3.7, we choose $\hat{b} = 0.1$, $\Sigma_1 = \Sigma_0 = 0.04$, $r = 0$, $T = 5$, $\gamma = 1.5$ and $z = 0.25$. In the numerical examples we then use different ϵ_1 and ϵ_0 to describe the uncertainty in the pre- and post-crash market. In the one stock case we thus obtain the following sets of possible values of the drift and volatility parameters in the pre- and post-crash market:

$$D_i = \{b \in \mathbb{R} | \, |b - 0.1| \leq 0.2\epsilon_i\} \text{ and } V_i = \{\sigma \in \mathbb{R}_{\neq 0} | \sigma^2 \leq 0.04\},$$

for $i \in \{0, 1\}$. In the numerical analysis we consider the optimal crash loss. This means, for the pre-crash optimal portfolio process $\bar{\pi}$ we look at $\bar{\pi}_1^T z$ for different degrees of uncertainty in the pre- and post-crash market. For comparison, we see in each figure the no-ambiguity case with $\epsilon_1 = \epsilon_0 = 0$. In Figure 3.1, we see the optimal crash loss with post-crash ambiguity. For this we look at $\epsilon_0 = 0.2$ and $\epsilon_0 = 0.4$ and note that the optimal crash loss is chosen significantly lower if there is a higher degree of uncertainty regarding the parameters in the post-crash market. Other effects can be seen in Figure 3.2, where we consider the analogous examples with pre-crash parameter uncertainty. Here, the higher parameter uncertainty, in combination with the worst-case approach, leads to a pre-crash market that is less attractive

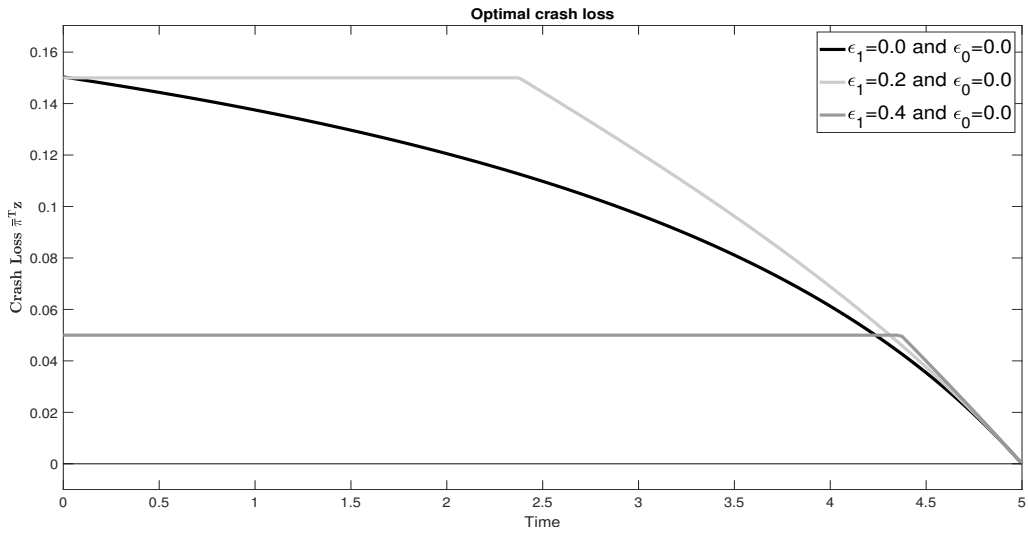

FIGURE 3.2: One stock with pre-crash ambiguity.

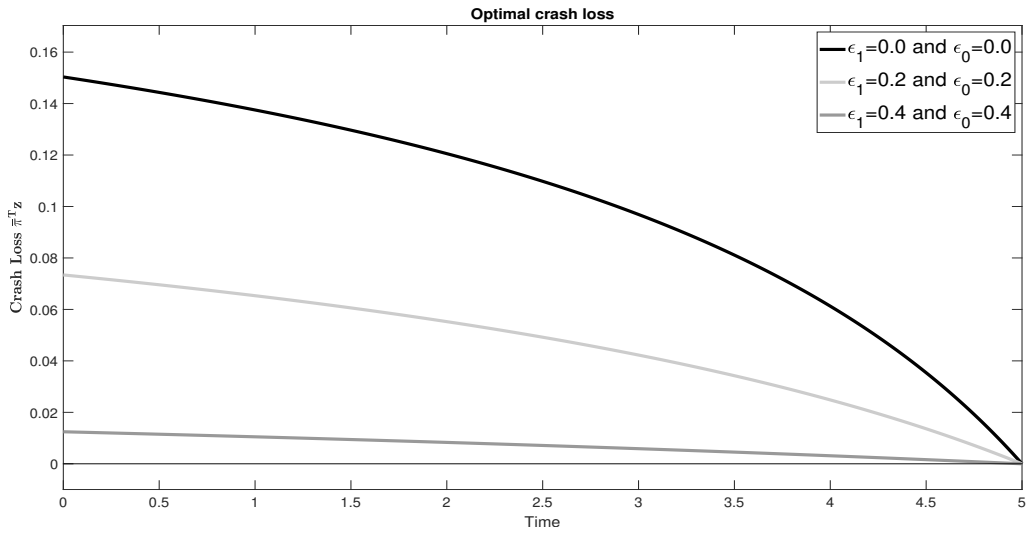

FIGURE 3.3: One stock with pre- and post-crash ambiguity.

for the investor. This can result in a pre-crash optimal portfolio process $\bar{\pi}_1$ that is in large parts identical to the constant no-crash optimal portfolio process π_1^*. In Figure 3.3, there is parameter uncertainty in the pre- and post-crash market. If we now compare the choice $\epsilon_1 = \epsilon_0 = 0.2$ in Figure 3.3 with the post-crash uncertainty $\epsilon_1 = 0$ and $\epsilon = 0.2$ in Figure 3.1, we see that the optimal crash loss in Figure 3.3 is higher despite the additional pre-crash uncertainty. A possible interpretation of this result is, that an investor who follows a crash-indifference process must be indifferent with respect to the loss due to the crash and to the increasing market uncertainty after the crash, in the case of Figure 3.1. In contrast, the indifferent investor in Figure 3.3 does not have to take into account any change in the parameter uncertainty when the crash occurs, since the ambiguity in the post-crash market is the same as in the pre-crash market.

Numerical examples for $d = 2$. We extend the numerical considerations to $d = n = 2$. Again, we set $r = 0$, $T = 5$, $\gamma = 1.5$, $z = (0.25, 0.25)^T$ and consider different values for ϵ_1

and ϵ_0. Further we choose

$$\hat{b} = \begin{pmatrix} 0.08 \\ 0.1 \end{pmatrix} \text{ and } \Sigma_1 = \Sigma_0 = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.03 \end{pmatrix}.$$

In Figure 3.4 we see that a high uncertainty in the post-crash market can lead to an optimal crash loss $\bar{\pi}^T z$ that is even negative. In Figure 3.5, we consider the reverse case with pre-

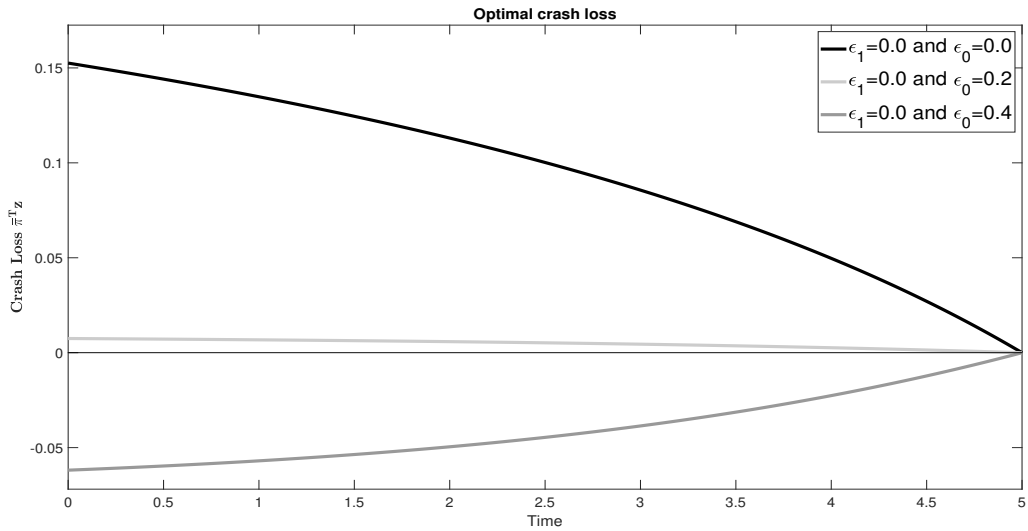

FIGURE 3.4: Two stocks with post-crash ambiguity.

crash uncertainty and without post-crash uncertainty. We see that the optimal crash loss is significantly higher than in the results shown in Figure 3.4. It is even possible that the optimal crash loss in a market with pre-crash uncertainty is higher than in a market without parameter ambiguity. In a market with pre- and post-crash uncertainty, as in Figure 3.6, we

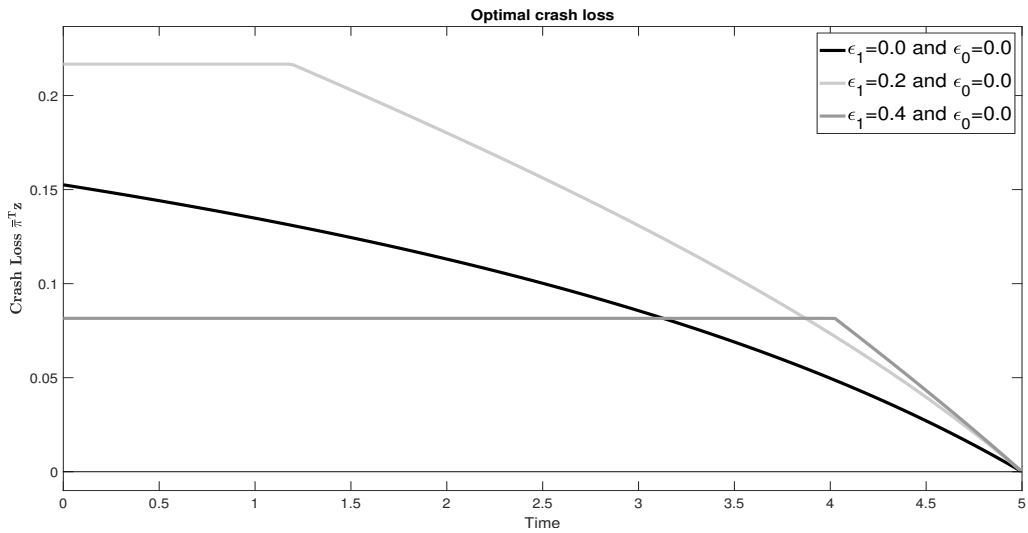

FIGURE 3.5: Two stocks with pre-crash ambiguity.

see a similar effect as in the one stock examples. Despite the high degree of ambiguity, the optimal crash loss is higher than in a market where we only have post-crash uncertainty, as in Figure 3.4.

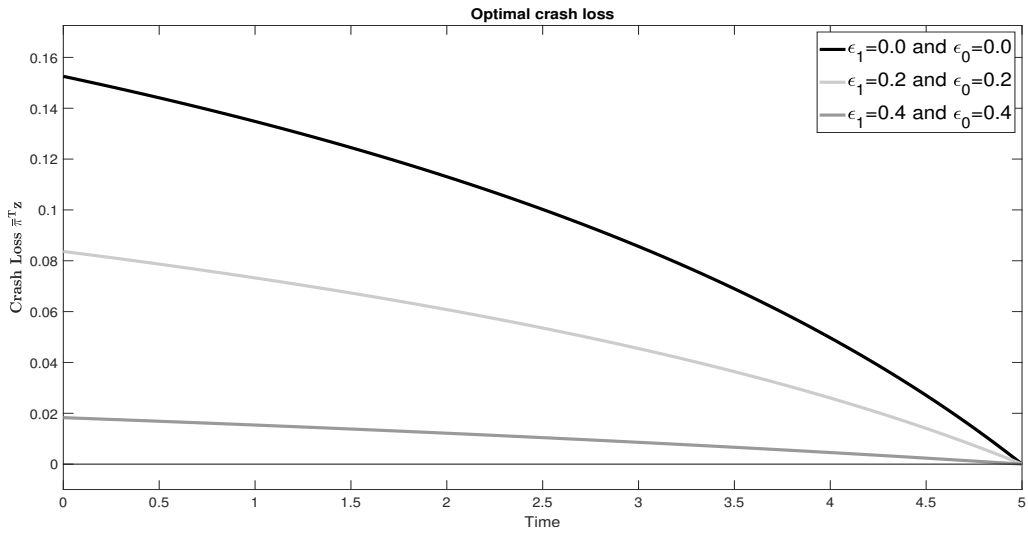

FIGURE 3.6: Two stocks with pre- and post-crash ambiguity.

3.9 Conclusion

We have considered a portfolio optimization problem that includes both, parameter ambiguity and the possibility of a crash scenario. The solution to this problem is divided into the post-crash problem, which can be considered as a portfolio optimization problem with parameter uncertainty and random initial time, and the pre-crash optimization problem. For the post-crash problem we used a decomposition of the utility of the wealth process similar to the change-of-measure device in [56]. The solution of the pre-crash problem combines classical indifference arguments of the worst-case scenario approach with martingale arguments that initially require a discrete consideration of the investment period in order to determine the worst-case parameters of a so-called crash-indifference process. In the final examples, it is shown that the general definition of possible stock price parameters, given in the description of the financial market, includes some concrete examples known from the literature.

3.10 Appendix

Before we add the missing proofs in this appendix, we use the next Lemma to introduce a new notation. Building on this, we will look at properties of the value function, which will enable us to create a more comprehensible proof structure in the further course. Throughout the Appendix, the assumptions of section 3.4 and 3.5 are assumed to be satisfied.

Lemma 15. *Let $z \in \mathbb{R}^d$ be a crash, $p_1 \in P_1$ and Y be defined as in Proposition 11. For $s, t \in [0, T]$ with $s \leq t$ and $\pi \in \mathbb{A}$, we define*

$$\Gamma_s^{\pi,p_1}(t) := \frac{U\left(1-\pi_1(t)^T z\right)}{U\left(1-\pi_1(s)^T z\right)} \exp\left((1-\gamma)\int_s^t \left(\phi(p_1(u),\pi_1(u)) - \phi_0\right) du\right) Y_s^t(\pi_1, p_1).$$

Further we set

$$\Gamma_s^{\pi,p_1}(\infty) := \frac{1}{(1-\pi_1(T)^T z)^{1-\gamma}}\,\Gamma_s^{\pi,p_1}(T).$$

We then obtain:

1. *For $\xi_1, \xi_2 \in \Theta$ with $\xi_1(\omega) \leq \xi_2(\omega)$ for all $\omega \in \Omega$ and $\pi \in \mathbb{A}$ we have*

$$v_0\left(\xi_2, X^{\pi,p_1,\xi_2,p_0}(\xi_2)\right) = v_0\left(\xi_1, X^{\pi,p_1,\xi_1,p_0}(\xi_1)\right)\Gamma_{\xi_1}^{\pi,p_1}(\xi_2).$$

2. *If $s,t \in [0,T] \cup \{\infty\}$ with $s \leq t$ and $\pi \in \mathbb{A}$ with π_1 being deterministic on $[s,t]$, then for every $p_1^\pi \in P_1^\pi$ we get*

$$\mathbb{E}\left(v_0\left(s, X^{\pi,p_1,s,p_0}(s)\right)\Gamma_s^{\pi,p_1}(t)|\mathcal{F}_s\right) \geq \mathbb{E}\left(v_0\left(s, X^{\pi,p_1,s,p_0}(s)\right)\Gamma_s^{\pi,p_1^\pi}(t)|\mathcal{F}_s\right).$$

3. *If $\xi_1, \xi_2 \in \Theta$ with $\xi_1(\omega) \leq \xi_2(\omega)$ for all $\omega \in \Omega$ and $\pi \in \mathbb{A}$ such that $v_0\left(t, X^{\pi,p_1,t,p_0}(t)\right)$ is a martingale on $[0,T]$ with $\pi_1(T)^T z = 0$, then*

$$\mathbb{E}\left(\Gamma_{\xi_1}^{\pi,p_1}(\xi_2)|\mathcal{F}_{\xi_1}\right) = 1.$$

4. *If $\xi_1, \xi_2 \in \Theta$ with $\xi_1(\omega) \leq \xi_2(\omega)$ for all $\omega \in \Omega$ and $\pi \in \mathbb{A}$ such that $v_0\left(t, X^{\pi,p_1,t,p_0}(t)\right)$ is a supermartingale on $[0,T] \cup \{\infty\}$, then*

$$\mathbb{E}\left(\Gamma_{\xi_1}^{\pi,p_1}(\xi_2)|\mathcal{F}_{\xi_1}\right) \leq 1 \text{ if } (1-\gamma) > 0 \text{ and } \mathbb{E}\left(\Gamma_{\xi_1}^{\pi,p_1}(\xi_2)|\mathcal{F}_{\xi_1}\right) \geq 1 \text{ else.}$$

Proof. Let ξ_1, ξ_2, s, t, z and p_1 be defined as above.

1. For $\pi \in \mathbb{A}$, the definition of the post-crash value function v_0 yields

$$v_0(\xi_2, X^{\pi,p_2,\xi_2,p_0}(\xi_2)) = U\left(X^{\pi,p_2,\xi_2,p_0}(\xi_2)\right)\exp\left((1-\gamma)(T-\xi_2)\phi_0\right),$$

on $\{\omega \in \Omega | \xi_2(\omega) < \infty\}$. By the utility decomposition from Proposition 11 part 3 we get a representation for $U\left(X^{\pi,p_1,\xi_2,p_0}(\xi_2)\right)$. Doing the same for $v_0(\xi_1, X^{\pi,p_1,\xi_1,p_0}(\xi_1))$ and using the definition of $\Gamma_{\xi_1}^{\pi,p_1}(\xi_2)$ yields the claim. The case $\xi_2(\omega) = \infty$ follows analogously.

2. We give the proof for $t \leq T$. For $t = \infty$ we can use the same arguments. Now assume that $\pi \in \mathbb{A}$, with π_1 being deterministic on $[s,t]$. Note that

$$c_s(t) := v_0(s, X^{\pi,p_1,s,p_0}(s))\frac{U\left(1-\pi_1(t)^T z\right)}{U\left(1-\pi_1(s)^T z\right)} > 0 \text{ iff } (1-\gamma) > 0.$$

Further, if π_1 is deterministic on $[s,t]$, also every $p_1^\pi \in P_1^\pi$ is deterministic on $[s,t]$. Now consider an arbitrary $p_1^\pi \in P_1^\pi$. This leads, together with the definition of Γ_s^{π,p_1}

and Γ_s^{π,p_1^π}, to

$$\begin{aligned}
&\mathbb{E}\left(v_0(s, X^{\pi,p_1,s,p_0}(s))\Gamma_s^{\pi,p_1}(t)|\mathcal{F}_s\right)\\
=&c_s(t)\mathbb{E}\left(\exp\left((1-\gamma)\int_s^t (\phi(p_1(u),\pi_1(u))-\phi_0)\,du\right)Y_s^t(\pi_1,p_1)|\mathcal{F}_s\right)\\
\geq&c_s(t)\mathbb{E}\left(\exp\left((1-\gamma)\int_s^t (\phi(p_1^\pi(u),\pi_1(u))-\phi_0)\,du\right)Y_s^t(\pi_1,p_1)|\mathcal{F}_s\right)\\
=&c_s(t)\exp\left((1-\gamma)\int_s^t (\phi(p_1^\pi(u),\pi_1(u))-\phi_0)\,du\right)\mathbb{E}\left(Y_s^t(\pi_1,p_1)|\mathcal{F}_s\right)\\
=&c_s(t)\exp\left((1-\gamma)\int_s^t (\phi(p_1^\pi(u),\pi_1(u))-\phi_0)\,du\right)\mathbb{E}\left(Y_s^t(\pi_1,p_1^\pi)|\mathcal{F}_s\right)\\
=&\mathbb{E}\left(v_0(s, X^{\pi,p_1,s,p_0}(s))\Gamma_s^{\pi,p_1^\pi}(t)|\mathcal{F}_s\right),
\end{aligned}$$

where we used again that $Y_s(\pi_1,p_1) = \{Y_s^t(\pi_1,p_1), s \leq t \leq T\}$ is a positive $\{\mathcal{F}_t, s \leq t \leq T\}$-martingale with $Y_s^s(\pi_1,p_1) = 1$ for every $p_1 \in P_1$.

3. Note that

$$\Gamma_s^{\pi,p_1}(\infty) = \frac{1}{(1-\pi_1(T)^T z)^{1-\gamma}}\,\Gamma_s^{\pi,p_1}(T) = \Gamma_s^{\pi,p_1}(T)$$

by definition. Then the claim is a direct consequence of part 1, the martingale property and Doob's optional sampling Theorem.

4. The claim is a direct consequence of part 1, the supermartingale property and Doob's optional sampling Theorem.

□

Before we give the proof of Lemma 11, we have to proof a similar result for discrete stopping times.

Lemma 16. *Let $z \in \mathbb{R}^d$ be a crash and $\hat{\pi} \in \mathbb{A}$ with $\hat{\pi}_1$ being deterministic such that the process $v_0\left(t, X^{\hat{\pi},p_1^{\hat{\pi}},t,p_0}(t)\right)$ is a martingale on $[0,T]$ for every $p_1^{\hat{\pi}} \in P_1^{\hat{\pi}}$, with $\hat{\pi}_1(T)^T z = 0$. Further, let $n \geq 1$ and $t_0, \ldots, t_{n+1} \in [0,T] \cup \{\infty\}$ with $0 = t_0 < \ldots < t_n = T < t_{n+1} = \infty$. Then we denote by ξ and τ two $\{t_0, \ldots, t_{n+1}\}$-valued stopping times with $\tau \leq \xi$.*

Now we consider an admissible portfolio process $\pi \in \mathbb{A}$ with $\pi_1(t) = \hat{\pi}_1(t)$ if $t \geq \tau$. Next, we fix an arbitrary $p_1^{\hat{\pi}} \in P_1^{\hat{\pi}}$ and for every $k \in \{1, \ldots, n+1\}$ and $p_1 \in P_1$ we define $p_1^{\tau,k} \in P_1$ by

$$\begin{aligned}
p_1^{\tau,k}(t) &= p^{\hat{\pi}}(t) && \text{if } t \geq \tau \text{ and } t \geq t_{k-1},\\
p_1^{\tau,k}(t) &= p_1(t) && \text{else.}
\end{aligned}$$

Then we obtain

$$\mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1,\xi,p_0}(\xi)\right)\right) \geq \mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1^{\tau,k},\xi,p_0}(\xi)\right)\right)$$

for all $k \in \{1, \ldots, n+1\}$ and every $p_1 \in P_1$.

Proof. Let z, τ, ξ, π, $p_1 \in P_1$, $\hat{\pi}$ and $p_1^{\hat{\pi}} \in P_1^{\hat{\pi}}$ be defined as above. We give the proof by backwards induction over $k \in \{1, \dots, n+1\}$. For $k = n+1$ we obtain directly with the definition of $p_1^{\tau,n+1}$ that we have

$$\mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1,\xi,p_0}(\xi)\right)\right) = \mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1^{\tau,n+1},\xi,p_0}(\xi)\right)\right),$$

for every $p_1 \in P_1$.

Now let $k \leq n$ and assume that the claim holds for $i \in \{k+1, \dots, n+1\}$. We choose an arbitrary $p_1 \in P_1$ and note that the discrete stopping times, as well as part 1 of Lemma 15, lead to

$$\begin{aligned}
&v_0\left(\xi, X^{\pi,p_1^{\tau,k+1},\xi,p_0}(\xi)\right) = v_0\left(\tau, X^{\pi,p_1^{\tau,k+1},\tau,p_0}(\tau)\right) \\
&\quad + \sum_{i=1}^{n+1} \mathbb{1}_{\tau<t_i\leq\xi}\left(v_0\left(t_i, X^{\pi,p_1^{\tau,k+1},t_i,p_0}(t_i)\right) - v_0\left(t_{i-1}, X^{\pi,p_1^{\tau,k+1},t_{i-1},p_0}(t_{i-1})\right)\right) \\
=&v_0\left(\tau, X^{\pi,p_1^{\tau,k+1},\tau,p_0}(\tau)\right) \\
&\quad + \sum_{i=1}^{n} \mathbb{1}_{\tau<t_i\leq\xi}\left(v_0\left(t_{i-1}, X^{\pi,p_1^{\tau,k+1},t_{i-1},p_0}(t_{i-1})\right)\left(\Gamma_{t_{i-1}}^{\hat{\pi},p_1^{\tau,k+1}}(t_i) - 1\right)\right).
\end{aligned}$$

For the second equality we used in addition that $\pi_1(t) = \hat{\pi}_1(t)$ if $t \geq \tau$, that $t_{i-1} \geq \tau$ if $t_i > \tau$ for $i \in \{1, \dots, n\}$ and that $\Gamma_T^{\hat{\pi},p_1^{\tau,k+1}}(\infty) - 1 = 0$ since $\hat{\pi}_1(T)^T z = 0$. Further we have for $k+1 \leq i \leq n$ with the definition of $p_1^{\tau,k+1}$ that

$$\begin{aligned}
&\mathbb{E}\left(\mathbb{1}_{\tau<t_i\leq\xi}\left(v_0(t_{i-1}, X^{\pi,p_1^{\tau,k+1},t_{i-1},p_0}(t_{i-1}))\left(\Gamma_{t_{i-1}}^{\hat{\pi},p_1^{\tau,k+1}}(t_i) - 1\right)\right)\right) \\
=&\mathbb{E}\left(\mathbb{1}_{\tau<t_i\leq\xi}\left(v_0(t_{i-1}, X^{\pi,p_1^{\tau,k+1},t_{i-1},p_0}(t_{i-1}))\left(\Gamma_{t_{i-1}}^{\hat{\pi},p_1^{\hat{\pi}}}(t_i) - 1\right)\right)\right) \\
=&\mathbb{E}\left(\mathbb{1}_{\tau<t_i\leq\xi}\left(v_0(t_{i-1}, X^{\pi,p_1^{\tau,k+1},t_{i-1},p_0}(t_{i-1}))\mathbb{E}\left(\Gamma_{t_{i-1}}^{\hat{\pi},p_1^{\hat{\pi}}}(t_i) - 1|\mathcal{F}_{t_{i-1}}\right)\right)\right) \\
=&0,
\end{aligned}$$

by Lemma 15 part 3 and the definition of $\hat{\pi}$. Together with the induction hypothesis this leads to

$$\begin{aligned}
&\mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1,\xi,p_0}(\xi)\right)\right) \geq \mathbb{E}\left(v_0\left(\xi, X^{\pi,p_1^{\tau,k+1},\xi,p_0}(\xi)\right)\right) \\
=&\mathbb{E}\left(v_0\left(\tau, X^{\pi,p_1^{\tau,k+1},\tau,p_0}(\tau)\right)\right) \\
&\quad + \sum_{i=1}^{k-1} \mathbb{E}\left(\mathbb{1}_{\tau<t_i\leq\xi}\left(v_0\left(t_{i-1}, X^{\pi,p_1^{\tau,k+1},t_{i-1},p_0}(t_{i-1})\right)\left(\Gamma_{t_{i-1}}^{\hat{\pi},p_1^{\tau,k+1}}(t_i) - 1\right)\right)\right) \\
&\quad + \mathbb{E}\left(\mathbb{1}_{\tau<t_k\leq\xi}\mathbb{E}\left(v_0\left(t_{k-1}, X^{\pi,p_1^{\tau,k+1},t_{k-1},p_0}(t_{k-1})\right)\left(\Gamma_{t_{k-1}}^{\hat{\pi},p_1^{\tau,k+1}}(t_k) - 1\right)|\mathcal{F}_{t_{k-1}}\right)\right) \\
\geq&\mathbb{E}\left(v_0\left(\tau, X^{\pi,p_1^{\tau,k+1},\tau,p_0}(\tau)\right)\right) \\
&\quad + \sum_{i=1}^{k-1} \mathbb{E}\left(\mathbb{1}_{\tau<t_i\leq\xi}\left(v_0\left(t_{i-1}, X^{\pi,p_1^{\tau,k+1},t_{i-1},p_0}(t_{i-1})\right)\left(\Gamma_{t_{i-1}}^{\hat{\pi},p_1^{\tau,k+1}}(t_i) - 1\right)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\left(\mathbb{1}_{\tau<t_k\leq\xi}\left(v_0\left(t_{k-1},X^{\pi,p_1^{\tau,k+1},t_{k-1},p_0}(t_{k-1})\right)\mathbb{E}\left(\Gamma_{t_{k-1}}^{\hat{\pi},p_1^{\hat{\pi}}}(t_k)-1|\mathcal{F}_{t_{k-1}}\right)\right)\right)\\
=&\mathbb{E}\left(v_0\left(\tau,X^{\pi,p_1^{\tau,k},\tau,p_0}(\tau)\right)\right)\\
&+\sum_{i=1}^{k-1}\mathbb{E}\left(\mathbb{1}_{\tau<t_i\leq\xi}\left(v_0\left(t_{i-1},X^{\pi,p_1^{\tau,k},t_{i-1},p_0}(t_{i-1})\right)\left(\Gamma_{t_{i-1}}^{\hat{\pi},p_1^{\tau,k}}(t_i)-1\right)\right)\right)\\
=&\mathbb{E}\left(v_0\left(\xi,X^{\pi,p_1^{\tau,k},\xi,p_0}(\xi)\right)\right).
\end{aligned}
$$

Here, the first inequality follows by assumption and the first equality follows with the above considerations. The second inequality is a direct consequence of Lemma 15 part 2 and part 3 of Lemma 15 gives us the second to last equality, together with $p_1^{\tau,k}(t) = p_1^{\tau,k+1}(t)$ for $t < t_{k-1}$. The last equality is valid, since we get with the definition of $p_1^{\tau,k}$ and the same arguments as above that

$$
\mathbb{E}\left(\mathbb{1}_{\tau<t_i\leq\xi}\left(v_0\left(t_{i-1},X^{\pi,p_1^{\tau,k},t_{i-1},p_0}(t_{i-1})\right)\left(\Gamma_{t_{i-1}}^{\hat{\pi},p_1^{\tau,k}}(t_i)-1\right)\right)\right)=0,
$$

for $k \leq i \leq n$. □

Now we are in a position to prove Lemma 11.

Proof of Lemma 11. Let $z,\ \tau,\ \xi,\ \pi,\ p_1 \in P_1,\ \hat{\pi}$ and $p_1^{\hat{\pi}} \in P_1^{\hat{\pi}}$ be defined as in the assertion. We now prove in two steps that we have

$$
\inf_{p_1\in P_1}\mathbb{E}\left(v_0\left(\xi,X^{\pi,p_1,\xi,p_0}(\xi)\right)\right)=\inf_{p_1\in P_1}\mathbb{E}\left(v_0\left(\xi,X^{\pi,p_1^{\tau},\xi,p_0}(\xi)\right)\right). \tag{3.11}
$$

Together with the tower-property, Lemma 15 part 3, the definition of π, $\hat{\pi}$ and p_1^{τ}, we directly obtain

$$
\begin{aligned}
&\inf_{p_1\in P_1}\mathbb{E}\left(v_0\left(\xi,X^{\pi,p_1^{\tau},\xi,p_0}(\xi)\right)\right)\\
=&\inf_{p_1\in P_1}\mathbb{E}\left(v_0\left(\tau,X^{\pi,p_1^{\tau},\tau,p_0}(\tau)\right)\mathbb{E}\left(\Gamma_{\tau}^{\pi,p_1^{\tau}}(\xi)|\mathcal{F}_{\tau}\right)\right)\\
=&\inf_{p_1\in P_1}\mathbb{E}\left(v_0\left(\tau,X^{\pi,p_1,\tau,p_0}(\tau)\right)\mathbb{E}\left(\Gamma_{\tau}^{\hat{\pi},p_1^{\hat{\pi}}}(\xi)|\mathcal{F}_{\tau}\right)\right)\\
=&\inf_{p_1\in P_1}\mathbb{E}\left(v_0\left(\tau,X^{\pi,p_1,\tau,p_0}(\tau)\right)\right).
\end{aligned}
$$

1. First, we define, for $n \in \mathbb{N}$, the two discrete stopping times $\tau^{(n)}$ and $\xi^{(n)}$ by

$$
\begin{aligned}
\tau^{(n)}&=\sum_{k=1}^{2^n}\frac{kT}{2^n}\mathbb{1}_{\frac{(k-1)T}{2^n}\leq\tau<\frac{kT}{2^n}}+T\mathbb{1}_{\tau=T}+\infty\mathbb{1}_{\tau=\infty},\\
\xi^{(n)}&=\sum_{k=1}^{2^n}\frac{kT}{2^n}\mathbb{1}_{\frac{(k-1)T}{2^n}\leq\xi<\frac{kT}{2^n}}+T\mathbb{1}_{\xi=T}+\infty\mathbb{1}_{\xi=\infty}.
\end{aligned}
$$

Then we have $\tau \leq \tau^{(n)}$, $\xi \leq \xi^{(n)}$ and $\tau^{(n)} \leq \xi^{(n)}$. Now we define $p_1^{\tau^{(n)}}$ in the same way as p_1^{τ}, i.e. $p_1^{\tau^{(n)}}(t) = p_1^{\hat{\pi}}(t)$ if $t \geq \tau^{(n)}$ and $p_1^{\tau^{(n)}}(t) = p_1(t)$ otherwise. Applying Lemma

16 with $k = 1$ gives us

$$\begin{aligned}\mathbb{E}\left(v_0\left(\xi^{(n)}, X^{\pi,p_1,\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right)\right) \geq & \mathbb{E}\left(v_0\left(\xi^{(n)}, X^{\pi,p_1^{\tau^{(n)},1},\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right)\right)\\ = & \mathbb{E}\left(v_0\left(\xi^{(n)}, X^{\pi,p_1^{\tau^{(n)}},\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right)\right),\end{aligned}$$

for every $p_1 \in P_1$. Thus we get

$$\inf_{p_1\in P_1} \mathbb{E}\left(v_0\left(\xi^{(n)}, X^{\pi,p_1,\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right)\right) \geq \inf_{p_1\in P_1} \mathbb{E}\left(v_0\left(\xi^{(n)}, X^{\pi,p_1^{\tau^{(n)}},\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right)\right).$$

On the other hand we know that

$$\inf_{p_1\in P_1} \mathbb{E}\left(v_0\left(\xi^{(n)}, X^{\pi,p_1,\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right)\right) \leq \mathbb{E}\left(v_0\left(\xi^{(n)}, X^{\pi,p_1^{\tau^{(n)}},\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right)\right),$$

for every $p_1 \in P_1$ and therefore we have proven equality (3.11) for discrete stopping times $\tau \leq \tau^{(n)}$ and $\xi \leq \xi^{(n)}$, i.e. we have

$$\begin{aligned}&\inf_{p_1\in P_1} \mathbb{E}\left(v_0\left(\xi^{(n)}, X^{\pi,p_1,\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right)\right) \\ &= \inf_{p_1\in P_1} \mathbb{E}\left(v_0\left(\xi^{(n)}, X^{\pi,p_1^{\tau^{(n)}},\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right)\right).\end{aligned} \tag{3.12}$$

2. From part 1 we know that equation (3.12) holds. Note that we get from Lemma 15 part 1, for every $p_1 \in P_1$, that

$$v_0\left(\xi^{(n)}, X^{\pi,p_1,\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right) = v_0\left(\xi, X^{\pi,p_1,\xi,p_0}(\xi)\right)\Gamma_{\xi}^{\pi,p_1}\left(\xi^{(n)}\right), \tag{3.13}$$

and with the definition of $p_1^{\tau^{(n)}}$ also

$$v_0\left(\xi^{(n)}, X^{\pi,p_1^{\tau^{(n)}},\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right) = v_0\left(\tau, X^{\pi,p_1,\tau,p_0}(\tau)\right)\Gamma_{\tau}^{\pi,p_1^{\tau^{(n)}}}\left(\xi^{(n)}\right). \tag{3.14}$$

With the right-continuity assumptions on an admissible portfolio process, we obtain for every $p_1 \in P_1$ the following almost sure convergences:

$$\begin{aligned}\Gamma_{\xi}^{\pi,p_1}\left(\xi^{(n)}\right) &\xrightarrow{n\to\infty} 1,\\ \Gamma_{\tau}^{\pi,p_1^{\tau^{(n)}}}\left(\xi^{(n)}\right) = \Gamma_{\tau}^{\pi,p_1}\left(\tau^{(n)}\right)\Gamma_{\tau^{(n)}}^{\pi,p_1^{\hat{\tau}}}\left(\xi^{(n)}\right) &\xrightarrow{n\to\infty} \Gamma_{\tau}^{\pi,p_1^{\hat{\tau}}}(\xi).\end{aligned}$$

Note that both, the elements of the convergent sequences and the limits, are positive and that the limits are independent of the choice of $p_1 \in P_1$. Thus we can even extend this with (3.13) and (3.14) to the almost sure convergences

$$\inf_{p_1\in P_1} v_0\left(\xi^{(n)}, X^{\pi,p_1,\xi^{(n)},p_0}\left(\xi^{(n)}\right)\right) \xrightarrow{n\to\infty} \inf_{p_1\in P_1} v_0\left(\xi, X^{\pi,p_1,\xi,p_0}(\xi)\right),$$

$$\inf_{p_1 \in P_1} v_0\left(\xi^{(n)}, X^{\pi, p_1^{\tau^{(n)}}, \xi^{(n)}, p_0}\left(\xi^{(n)}\right)\right) \xrightarrow{n\to\infty} \inf_{p_1 \in P_1} v_0\left(\xi, X^{\pi, p_1^{\tau}, \xi, p_0}(\xi)\right).$$

Putting everything together leads to

$$\begin{aligned}
&\inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi, X^{\pi, p_1, \xi, p_0}(\xi)\right)\right) \\
&= \lim_{n\to\infty} \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi^{(n)}, X^{\pi, p_1, \xi^{(n)}, p_0}(\xi^{(n)})\right)\right) \\
&= \lim_{n\to\infty} \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi^{(n)}, X^{\pi, p_1^{\tau^{(n)}}, \xi^{(n)}, p_0}(\xi^{(n)})\right)\right) \\
&= \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi, X^{\pi, p_1^{\tau}, \xi, p_0}(\xi)\right)\right).
\end{aligned}$$

Here one should note in addition, that we take the infimum over the possible pre-crash parameters path-wise for each $\omega \in \Omega$. Together with the considerations at the beginning, this completes the proof. □

Proof of Lemma 12. Let $z,\ \tau,\ \pi,\ p_1 \in P_1,\ \hat{\pi}$ and $p_1^{\hat{\pi}} \in P_1^{\hat{\pi}}$ be defined as above. Further fix an arbitrary $\xi \in \Theta^{\tau}$. With part 4 of Lemma 15 we get that

$$\mathbb{E}\left(\Gamma_{\xi}^{\hat{\pi}, p_1^{\hat{\pi}}}(\tau) | \mathcal{F}_{\xi}\right) \leq 1 \text{ if } (1-\gamma) > 0 \text{ and } \mathbb{E}\left(\Gamma_{\xi}^{\hat{\pi}, p_1^{\hat{\pi}}}(\tau) | \mathcal{F}_{\xi}\right) \geq 1 \text{ else.}$$

Further note that for every $p_1 \in P_1$, there exists a $\tilde{p}_1 \in P_1$ with $\tilde{p}_1(t) = p_1(t)$ if $t \leq \tau$ and $\tilde{p}_1(t) = p_1^{\hat{\pi}}(t)$ otherwise. Then we obtain that

$$\begin{aligned}
&\inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\tau, X^{\pi, p_1, \tau, p_0}(\tau)\right)\right) = \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi, X^{\pi, p_1, \xi, p_0}(\xi)\right) \mathbb{E}\left(\Gamma_{\xi}^{\hat{\pi}, p_1}(\tau) | \mathcal{F}_{\xi}\right)\right) \\
&\leq \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi, X^{\pi, p_1, \xi, p_0}(\xi)\right) \mathbb{E}\left(\Gamma_{\xi}^{\hat{\pi}, p_1^{\hat{\pi}}}(\tau) | \mathcal{F}_{\xi}\right)\right) \leq \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi, X^{\pi, p_1, \xi, p_0}(\xi)\right)\right)
\end{aligned}$$

and therefore we get

$$\inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\xi, X^{\pi, p_1, \xi, p_0}(\xi)\right)\right) \geq \inf_{p_1 \in P_1} \mathbb{E}\left(v_0\left(\tau, X^{\pi, p_1, \tau, p_0}(\tau)\right)\right),$$

for all $\xi \in \Theta^{\tau}$. □

Proof of Lemma 13. Let $z \in \mathbb{R}^d$ be a crash, $\hat{\pi} \in \mathbb{A}$ be a crash-indifference process as defined in the assertion and $\pi^* \in \mathbb{A}$ be a no-crash optimal portfolio process. Note that $\pi_1^*(t)$ is constant for all $t \in [0, T]$ (see Remark 19) and $\hat{\pi}_1(t)^T z$ is continuous on $[0, T]$. At time T we have by construction of $\hat{\pi}$ and Assumption 2 that $0 = \hat{\pi}_1(T)^T z < (\pi_1^*)^T z$.

1. Assume that there exists a $\bar{t} \in [0, T)$ with $\hat{\pi}_1(\bar{t})^T z > (\pi_1^*)^T z$. Then there has to exist an $s \in (\bar{t}, T)$ with $\hat{\pi}_1(s)^T z = (\pi_1^*)^T z$ and $\left(\hat{\pi}_1(s)^T z\right)' \leq 0$.

(a) If $\phi_1 = \phi_0$, we get that $\hat{\pi}_1(t)^T z = (\pi_1^*)^T z$ for all $t \in [0, s]$, since $(\pi_1^*)^T z$ is constant. From the construction of $\hat{\pi}$ in Theorem 7 we then get

$$\begin{aligned}
\left(\hat{\pi}_1(t)^T z\right)'(t) &= (1 - \hat{\pi}_1(t)^T z) \left(\max_{\substack{\pi \in \mathbb{A} \\ \pi_1(t)^T z = \hat{\pi}_1(t)^T z}} \min_{p_1 \in P_1} \phi(p_1(t), \pi_1(t)) - \phi_0 \right) \\
&= (1 - \hat{\pi}_1(t)^T z) \left(\max_{\substack{\pi \in \mathbb{A} \\ \pi_1(t)^T z = (\pi_1^*)^T z}} \min_{p_1 \in P_1} \phi(p_1(t), \pi_1(t)) - \phi_0 \right) \\
&= (1 - \hat{\pi}_1(t)^T z)(\phi_1 - \phi_0) = 0,
\end{aligned}$$

if $\hat{\pi}_1(t)^T z = (\pi_1^*)^T z$. That is a contradiction to the definition of $\bar{t}$ and therefore $\hat{\pi}(t)^T z \leq (\pi_1^*)^T z$ for all $t \in [0, T]$.

(b) On the other hand, if $\phi_1 > \phi_0$, the same arguments as above yield

$$\begin{aligned}
\left(\hat{\pi}_1(s)^T z\right)'(s) &= (1 - \hat{\pi}_1(s)^T z) \left(\max_{\substack{\pi \in \mathbb{A} \\ \pi_1(s)^T z = \hat{\pi}_1(s)^T z}} \min_{p_1 \in P_1} \phi(p_1(s), \pi_1(s)) - \phi_0 \right) \\
&= (1 - \hat{\pi}_1(s)^T z)(\phi_1 - \phi_0) > 0.
\end{aligned}$$

That is a contradiction to the definition of s and therefore we get $\hat{\pi}(t)^T z \leq (\pi_1^*)^T z$ for all $t \in [0, T]$.

2. We can use the same arguments as in part 1 by defining T_0 as

$$T_0 := \sup\{t \in [0, T] |\ \hat{\pi}(t)^T z = (\pi_1^*)^T z\}.$$

Since $0 = \hat{\pi}_1(T)^T z < (\pi_1^*)^T z$ by Assumption 2, we know that we have $\hat{\pi}_1(t)^T z < (\pi_1^*)^T z$ for $t > T_0$. If $T_0 > 0$, we assume that there exists a point $\bar{t} \in [0, T_0)$ with $\hat{\pi}_1(\bar{t})^T z < (\pi_1^*)^T z$. Then there has to exist an $s \in (\bar{t}, T_0]$ with $\hat{\pi}_1(s)^T z = (\pi_1^*)^T z$ and $\left(\hat{\pi}_1(s)^T z\right)' \geq 0$. Again, we get a contradiction since

$$\left(\hat{\pi}_1(s)^T z\right)'(s) = (1 - \hat{\pi}_1(s)^T z)(\phi_1 - \phi_0) < 0.$$

Thus we obtain that $\hat{\pi}_1(t)^T z \geq \pi_1^*(t)^T z$ for $t \in [0, T_0]$.

□

Chapter 4

Extensions and further research

In the previous chapters, we have analysed and solved various extensions of the worst-case scenario approach. Further problems and research questions can now build on this.

One possibility is to combine the different existing portfolio problems. As an example of such an extension, we combine the definition of the stress scenarios in chapter 1 with the problem formulation in chapter 2, to ensure that at most one of the $n > 1$ possible stress scenarios occurs, namely the worst one for the investor. We solve such a problem in section 4.1.

Another research approach is to integrate further stress scenarios into the worst-case portfolio problem. We take a first step in section 4.2 by looking at an optimal dynamic reinsurance problem with worst-case default of the reinsurer. The problem formulation is based on the Swiss Solvency Test.

4.1 Optimal portfolios and worst-case stress scenario

Introduction

In chapter 1 we have introduced a portfolio optimization problem with d assets and n stress scenarios. A stress is defined as a vector $z \in \mathbb{R}^d \backslash \{0\}$ and potentially all these n stresses can occur. In contrast, we look in chapter 2 at a portfolio problem where all assets are exposed to a crash or default risk, but at most one asset is affected by such a scenario. Here we would like to combine these two approaches by defining n stress scenarios as in chapter 1, of which, however, at most one can occur, which is analogous to the problem formulation in chapter 2.

Financial market and optimization problem

We consider a complete probability space $(\Omega, \mathcal{F}, P)$, equipped by a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$, for a finite time horizon $T > 0$. The filtration $\mathbb{F}$ is extended to $[0, T] \cup \{\infty\}$ by letting $\mathcal{F}_\infty := \mathcal{F}_T$ and Θ is the corresponding set of all $[0, T] \cup \{\infty\}$-valued stopping times. We adopt the definition of a stress scenario from chapter 1 and extend it by the set of possible times of occurrence $\bar{\Theta}$, defined in chapter 2.

Definition 13 (Stress Scenarios). *A vector $z \in \mathbb{R}^d \backslash \{0\}$ with components $z_i \in [0, 1)$ is called a stress. A pair (z, ξ), consisting of a stress z and a stopping time $\xi \in \Theta$, is called a stress scenario. For $n > 1$ stress scenarios $(z_1, \xi_1), \ldots, (z_n, \xi_n)$ the vector of stopping times $\xi = (\xi_1, \ldots, \xi_n)$ is in the set $\bar{\Theta}$, defined as*

$$\begin{aligned}\bar{\Theta} := \{\xi = (\xi_1, \ldots, \xi_n) \,|\, &\xi_i \in \Theta \text{ for } i \in \{1, \ldots, n\} \text{ and for all} \\ &\omega \in \Omega \text{ there exists at most one } i \in \{1, \ldots, n\} \text{ with } \xi_i(\omega) < \infty\}.\end{aligned}$$

Now we fix $n > 1$ stress scenarios $(z_1, \xi_1), \ldots, (z_n, \xi_n)$, and define the dynamics of the money market account B and the d stocks $S_1, \ldots, S_d$. For this let $(W(t))_{t \in [0,T]}$ be a d-dimensional Brownian motion with respect to $\mathbb{F}$. We fix a constant interest rate r, a drift

vector $b = (b_1, ..., b_d)^T \in \mathbb{R}^d$ and a volatility matrix $\sigma \in \mathbb{R}^{d\times d}$ of full rank. Now we define B and the stock price vector $S = (S_1, \dots, S_d)$ by

$$\begin{aligned} dB(t) &= B(t)rdt, \qquad B(0) = 1, \\ dS(t) &= Diag(S(t-))\left(bdt + \sigma dW(t) - \sum_{i=1}^{n} z_i d\mathbb{1}_{\{\xi_i \le t\}}\right), \; S(0) = s > 0, \end{aligned}$$

for $t \in [0, T]$, where $Diag(S(t-))$ is a diagonal $d \times d$ matrix with i-th diagonal element equal to $S_i(t-)$. Here we should note, that by definition of $\bar{\Theta}$, path-wise at most one of the n stresses affects the stock price dynamics.

The investor's portfolio process $\pi(t)$, with values in $\mathbb{R}^d$ for $t \in [0, T]$, is decomposed in the pre-stress portfolio process π_1 and the post-stress portfolio process π_0, i.e. $\pi(t) = \pi_1(t)$ if $t \le \min\{\xi_1, \dots \xi_n\}$ and $\pi(t) = \pi_0(t)$ otherwise. The set $\mathbb{A}$ of admissible portfolio processes $\pi = (\pi_1, \pi_0)$ consists of all processes π, where π_1 and π_0 are bounded, predicable and for which $\pi_1(t)^T z_i$ has right-continuous path with $\pi_1(t)^T z_i < 1$ for all $i \in \{1, \dots, n\}$ and $t \in [0, T]$. Further, the investor's wealth process X^π has to be the unique solution of the stochastic differential equation

$$\begin{aligned} X^\pi(0) &= x > 0, \\ dX^\pi(t) &= X^\pi(t-)\Bigg(\left(r + \pi(t)^T\left(b - r\underline{1}_d\right)\right)dt + \pi(t)^T\sigma dW(t) \\ &\qquad - \sum_{i=1}^{n} \pi(t)^T z_i d\mathbb{1}_{\{\xi_i \le t\}}\Bigg), \end{aligned}$$

for $t \in [0, T]$.

Now the **worst-case portfolio problem with worst-case stress scenario** is defined as

$$\sup_{\pi\in\mathbb{A}} \inf_{\xi\in\bar{\Theta}} \mathbb{E}\left(U\left(X^\pi(T)\right)\right), \tag{4.1}$$

for the power-utility function $U(x) = x^{1-\gamma}/(1-\gamma)$, $\gamma > 0$, $\gamma \neq 1$. As proven in [35], the optimal portfolio process $\bar{\pi}_0$, if no more stresses can occur, is given by

$$\bar{\pi}_0 = \frac{1}{\gamma}\left(\sigma\sigma^T\right)^{-1}\left(b - r\underline{1}\right). \tag{4.2}$$

For a stress scenario to be a true risk, we must require that the no-stress optimal portfolio process results in a loss, i.e. we need to demand that

$$\max\{\bar{\pi}_0^T z_j | j \in \{1, \dots, n\}\} > 0.$$

Main result

We now introduce the worst-case loss of an investor with portfolio process $\pi \in \mathbb{A}$ at time t, for $t \in [0, T]$.

Definition 14 (Worst-case loss)**.** *For a portfolio process $\pi \in \mathbb{A}$ we define the worst-case loss WCL^{π} at time $t \in [0, T]$, by*

$$WCL^{\pi}(t) := \max\{\pi_1(t)^T z_j | j \in \{1, \dots, n\}\}.$$

As in Lemma 6, we can reformulate our optimization problem (4.1) and obtain the following representation:

$$\sup_{\pi \in \mathbb{A}} \inf_{\xi \in \Theta} \mathbb{E}\left(U\left(X^{\pi}(T)\right)\right) = \sup_{\pi \in \mathbb{A}} \inf_{\xi \in \Theta} \mathbb{E}\left(v_0\left(\xi, \left(1 - WCL^{\pi}(\xi)\right) X^{\pi}(\xi-)\right)\right),$$

if we define $v_0\left(\infty, \left(1 - WCL^{\pi}(\infty)\right) X^{\pi}(\infty-)\right) := X^{\pi}(T)$ in the no-stress scenario. Now we have everything together to formulate the final result and to prove it analogous to Theorem 6.

Theorem 9 (Optimal portfolio process)**.** *There exists a portfolio process $\bar{\pi} \in \mathbb{A}$ with $\bar{\pi}_0$ defined in equation (4.2), such that*

$$\bar{\pi}_1(t) \in \underset{\substack{\pi \in \mathbb{R}^d \\ WCL^{\pi}(t) = N(t)}}{\arg\max} \quad \phi(\pi) \quad \textit{for } t \in [0, T],$$

where N is defined for $t \in [0, T]$ by the differential equation

$$N'(t) = (1 - N(t)) \left(\max_{\substack{\pi \in \mathbb{R}^d \\ WCL^{\pi}(t) = N(t)}} \phi(\pi) - \phi(\bar{\pi}_0) \right),$$

with $N(T) = 0$ and ϕ defined as

$$\phi(\pi) = r + \pi^T (b - r\underline{1}) - \frac{1}{2} \gamma \pi^T \sigma \sigma^T \pi.$$

The process $\bar{\pi}$ solves the worst-case portfolio problem with worst-case stress scenario (4.1).

Proof. We use the same proof idea as in chapter 2.

1. Analogous to Definition 9, we can define a minimum constant portfolio process for our new portfolio optimization problem (4.1), using the definition of the worst-case loss. As in Theorem 5, we then prove that the above process $\bar{\pi}$ is well-defined and satisfies the minimum constant property.

2. Thus, we can also apply a minimum constant optimality principle as in Proposition 10.

3. We note that π^* with $\pi_1^* = \bar{\pi}_0$ is optimal for the no-stress scenario and that the corresponding worst-case loss WCL^{π^*} is constant. Then we can use similar arguments as in Lemma 10, to show that $WCL^{\bar{\pi}}(t) \leq WCL^{\pi^*}(t)$ for all $t \in [0, T]$.

4. The last step of the proof can now be done in exactly the same way as the proof of Theorem 6.

□

4.2 Optimal dynamic reinsurance with worst-case default of the reinsurer

Introduction

We model the surplus process of an insurance company with proportional reinsurance, using a diffusion model as in [55], [59] and [44]. In contrast to these works, where the primary goal is to minimize the ruin probability, we consider an expected utility maximization problem. This is also done in [3], [39], among others. Furthermore, our optimization problem includes the default of the reinsurer. In [2] and [9], the authors incorporate the reinsurer counterparty default risk in a one-period model. We are adapting the worst-case scenario approach as introduced to portfolio optimization by Korn and Willmott [37].

To model the default of a reinsurer, we look at a translated quotation from the technical description of the scenarios in the Swiss Solvency Test [58]:

> This scenario deals with the risk of a reinsurer default. It assumes a situation where the insurer is faced with a large insurance loss. In addition, the economic environment for reinsurers is adverse, leading to their ratings being downgraded. A number of reinsurers default, so that they can no longer (fully) meet their obligations. The primary insurer suffers a loss as a result, which is composed as follows
>
> - The reinsurers can no longer cover the reinsured portion of the occurred major loss.
> - Since a number of reinsurers have defaulted, the primary insurer has to buy new cover and pay another premium for it.
> - Reinsurers can only partially settle the primary insurer's outstanding debts from old claims.

Mathematical setting and model description

For a finite time horizon $T > 0$, we consider a complete probability space $(\Omega, \mathcal{F}, P)$, that is equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$. The filtration $\mathbb{F}$ is extended to $[0, T] \cup \{\infty\}$ by letting $\mathcal{F}_\infty := \mathcal{F}_T$ and Θ is the corresponding set of all $[0, T] \cup \{\infty\}$-valued stopping times.

To model the surplus process of the primary insurer we choose a diffusion model and a proportional reinsurance. We include the possibility of a default of the reinsurer at time $\tau \in \Theta$, where $\{\tau = \infty\}$ denotes the no-default scenario.

The set $\mathbb{A}$ of admissible risk exposures of the insurer contains all predictable processes a defined on $[0, T]$ with values in $[0, 1]$. For $a \in \mathbb{A}$ we denote by a_1 the risk exposure up to the default and by a_0 the corresponding risk exposure after the default. In addition, we require that a_1 has right-continuous paths.

To model the default we use the above quoted technical description of the Swiss Solvency Test. We note that the primary insurer's loss can be divided into two main components. First, there is a loss C that is independent of the risk exposure and results from the major damage, which is the trigger for the reinsurer's default. In addition, the insurer suffers a loss $(1 - a_1(\tau))F$ that depends on the risk exposure due to the new covers to be purchased and the only partially settled former claims.

By $\mu \in \mathbb{R}_{>0}$ and $\sigma \in \mathbb{R}_{>0}$ we denote the drift and volatility of the surplus process without reinsurance. Further, $\lambda,\ C,\ F \in \mathbb{R}_{>0}$ are used to include premiums to the reinsurer and the above decomposition of the loss. For $a \in \mathbb{A}$, $\tau \in \Theta$ and a one-dimensional Brownian motion $(W(t))_{t\in[0,T]}$ with respect to $\mathbb{F}$, the (approximated) dynamics for the surplus process $R^{a,\tau}$ are defined as

$$\begin{aligned}
R^{a,\tau}(0) &= x \in \mathbb{R}, \\
dR^{a,\tau}(t) &= (\mu - (1 - a(t))\lambda)\, dt + a(t)\sigma dW(t), \quad t \in [0, \tau) \cup (\tau, T], \\
R^{a,\tau}(\tau) &= R^{a,\tau}(\tau-) - L^{a,\tau} := R^{a,\tau}(\tau-) - C - (1 - a_1(\tau))F, \quad \text{if } \tau < \infty.
\end{aligned}$$

Worst-case optimization problem

Using an exponential utility function with constant absolute risk aversion $\gamma > 0$, we define the **optimal dynamic reinsurance problem with worst-case default of the reinsurer** as

$$\sup_{a\in\mathbb{A}} \inf_{\tau\in\Theta} \mathbb{E}\left(-\exp\left(-\gamma R^{a,\tau}(T)\right)\right). \tag{4.3}$$

Standard results from portfolio optimization and stochastic control, as e.g. [53], imply that the optimal risk exposure $\bar{a}_0$ after the default is

$$\bar{a}_0 = \min\left\{\frac{\lambda}{\gamma\sigma^2}, 1\right\}. \tag{4.4}$$

Let v_0 denote the value function after the default has occurred. Then we obtain for $x \in \mathbb{R}$ and $\tau \in \Theta$ that we have

$$\begin{aligned}
v_0(\tau, x) :&= \sup_{a\in\mathbb{A}} \mathbb{E}\left(-\exp\left(-\gamma R^{a,\tau}(T)\right) | R^{a,\tau}(\tau) = x\right) \\
&= \sup_{\substack{a\in\mathbb{A} \\ a_0=\bar{a}_0}} \mathbb{E}\left(-\exp\left(-\gamma R^{a,\tau}(T)\right) | R^{a,\tau}(\tau) = x\right)
\end{aligned}$$

$$= -\exp\left(-\gamma x - \gamma\left(\mu - (1-\bar{a}_0)\lambda - \frac{1}{2}\gamma\bar{a}_0^2\sigma^2\right)(T-\tau)\right),$$

on $\{\omega \in \Omega | \tau(\omega) < \infty\}$. Thus, we reduce the set $\mathbb{A}$ to all risk exposures a with corresponding $a_0 = \bar{a}_0$. Further, we note that for every risk exposure $a \in \mathbb{A}$, due to the constant loss $C > 0$ and the non-negativity of $(1 - a_1(T))F$, a default at time T is worse than no default. Therefore we also reduce the set Θ to all $[0, T]$-valued stopping times and obtain the following reformulation of our optimization problem (4.3):

$$\begin{aligned}
&\sup_{a\in\mathbb{A}}\inf_{\tau\in\Theta}\mathbb{E}\left(-\exp\left(-\gamma R^{a,\tau}(T)\right)\right) = \sup_{a\in\mathbb{A}}\inf_{\tau\in\Theta}\mathbb{E}\left(\mathbb{E}\left(-\exp\left(-\gamma R^{a,\tau}(T)\right)|\mathcal{F}_\tau\right)\right)\\
=&\sup_{a\in\mathbb{A}}\inf_{\tau\in\Theta}\mathbb{E}\left(v_0\left(\tau, R^{a,\tau}(\tau)\right)\right) = \sup_{a\in\mathbb{A}}\inf_{\tau\in\Theta}\mathbb{E}\left(v_0\left(\tau, R^{a,\tau}(\tau-) - L^{a,\tau}\right)\right).
\end{aligned}$$

Main result.

To determine the optimal risk exposure $\bar{a}_1$, used up to the time of default τ, we first look at the indifference optimality principle, which is well-known in the worst-case scenario literature. Further, we consider the special case of a default at terminal time T.

Lemma 17 (Indifference optimality principle)**.** *Let $\bar{a} \in \mathbb{A}$ be a risk exposure such that*

$$\mathbb{E}\left(v_0\left(\tau, R^{\bar{a},\tau}(\tau-) - L^{\bar{a},\tau}\right)\right) = \mathbb{E}\left(v_0\left(\xi, R^{\bar{a},\xi}(\xi-) - L^{\bar{a},\xi}\right)\right),$$

for all $\tau, \xi \in \Theta$. If for every $a \in \mathbb{A}$ there exists a $\bar{\tau} \in \Theta$ with

$$\mathbb{E}\left(v_0\left(\bar{\tau}, R^{\bar{a},\bar{\tau}}(\bar{\tau}-) - L^{\bar{a},\bar{\tau}}\right)\right) \geq \mathbb{E}\left(v_0\left(\bar{\tau}, R^{a,\bar{\tau}}(\bar{\tau}-) - L^{a,\bar{\tau}}\right)\right),$$

then $\bar{a}$ is optimal for the optimization problem (4.3)*.*

Proof. Analogous to the proof of Proposition 4.1 in [56]. □

Lemma 18 (Default at terminal time)**.** *Let $\bar{a} \in \mathbb{A}$ be a risk exposure with $\bar{a}(T) = 1$. Then, at the terminal time T we have*

$$\mathbb{E}\left(v_0\left(T, R^{\bar{a},T}(T-) - L^{\bar{a},T}\right)\right) = \mathbb{E}\left(-\exp\left(-\gamma(x-C) - \gamma\int_0^T \phi(\bar{a}_1(t))dt\right)\right),$$

for $\phi : [0,1] \mapsto \mathbb{R}$ defined by $\phi(a) := \mu - (1-a)\lambda - \frac{1}{2}\gamma a^2\sigma^2$.

Proof. Note that $L^{\bar{a},T} = C$. Then the proof follows from the definition of the surplus process and $\mathbb{A}$. □

We can use these results to compute the optimal pre-default risk exposure $\bar{a}_1$.

Theorem 10 (Optimal risk exposure)**.** *Let $\bar{a} \in \mathbb{A}$ with $\bar{a}_0$ defined in* (4.4)*.*

1. *If $\lambda < \gamma\sigma^2$, we define $\bar{a}_1(t)$, for $t \in [0,T]$, by*

$$\bar{a}_1(t) = \bar{a}_0 + \frac{2F}{\gamma\sigma^2(T-t) + (2F/(1-\bar{a}_0))}.$$

2. *If $\lambda \geq \gamma\sigma^2$, we set $\bar{a}_1 = 1$.*

Then $\bar{a}$ solves the portfolio optimization problem (4.3).

Proof. 1. We give the proof in three steps. But first we show that $\bar{a}$ is well-defined, i.e. that $\bar{a}_1(t) \in [0,1]$ for all $t \in [0,T]$. For this note that $\bar{a}_1 \geq \bar{a}_0 \geq 0$. Further, $\bar{a}_1(t)$ is increasing in t with $\bar{a}_1(T) = 1$ and thus $\bar{a}_1 \leq 1$.

(a) Applying Itô's formula and using the definition of $\bar{a}_1$ leads to

$$dv_0\left(t, R^{\bar{a},t}(t-) - L^{\bar{a},t}\right) = -v_0\left(t, R^{\bar{a},t}(t-) - L^{\bar{a},t}\right)\gamma\bar{a}_1(t)\sigma dW(t),$$

for $t \in [0,T]$. Thus, $v_0\left(t, R^{\bar{a},t}(t-) - L^{\bar{a},t}\right)$ is a martingale on $[0,T]$. Due to Doob's optional sampling theorem we can apply Lemma 17 from now on.

(b) Now let $a \in \mathbb{A}$ and define τ_a as

$$\tau_a = \inf\{t \in [0,T] | a_1(t) < \bar{a}_1(t)\}, \text{with } \inf\emptyset := \infty.$$

We construct $\tilde{a} \in \mathbb{A}$ such that $\tilde{a}_1(t) = a_1(t)$ on $[0,\tau_a)$ and $\tilde{a}_1(t) = \bar{a}_1(t)$ else. Due to the right-continuity of a_1 we have $L^{\tilde{a},\tau_a} = L^{\bar{a},\tau_a} \leq L^{a,\tau_a}$ if $\tau_a < \infty$. Together with the above martingale property we obtain

$$\begin{aligned}\inf_{\tau\in\Theta} \mathbb{E}\left(v_0\left(\tau, R^{\tilde{a},\tau}(\tau-) - L^{\tilde{a},\tau}\right)\right) &= \inf_{\substack{\tau\in\Theta\\ \tau\leq\tau_a}} \mathbb{E}\left(v_0\left(\tau, R^{\tilde{a},\tau}(\tau-) - L^{\tilde{a},\tau}\right)\right)\\ \geq \inf_{\substack{\tau\in\Theta\\ \tau\leq\tau_a}} \mathbb{E}\left(v_0\left(\tau, R^{a,\tau}(\tau-) - L^{a,\tau}\right)\right) &\geq \inf_{\tau\in\Theta} \mathbb{E}\left(v_0\left(\tau, R^{a,\tau}(\tau-) - L^{a,\tau}\right)\right).\end{aligned}$$

Thus, we can reduce the search for the optimal risk exposure to all $a \in \mathbb{A}$ with $a_1(t) \geq \bar{a}_1(t)$ for all $t \in [0,T]$.

(c) Next we fix such an arbitrary $a \in \mathbb{A}$. In particular, we have that $a_1(t) \geq \bar{a}_1(t) \geq \bar{a}_0$ with $a(T) = \bar{a}(T) = 1$. Now we look at the quadratic, strictly concave function ϕ as defined in Lemma 18. This function is maximized on $[0,1]$ by $\bar{a}_0$, and we thus obtain for $t \in [0,T]$ that

$$\max\{\phi(a_1(t)), \phi(\bar{a}_1(t))\} = \phi(\bar{a}_1(t)).$$

Now we apply Lemma 18 and obtain at terminal time T:

$$\mathbb{E}\left(v_0\left(T, R^{\bar{a},T}(T-) - L^{\bar{a},T}\right)\right) \geq \mathbb{E}\left(v_0\left(T, R^{a,T}(T-) - L^{a,T}\right)\right).$$

Then the claim follows with Lemma 17.

2. As in this case we have $\bar{a}_1 = \bar{a}_0 = 1$, the assumptions of Lemma 17 and Lemma 18 are valid and the proof follows as in part 1.

$\square$

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