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THE LIMITED ANGLE PROBLEM IN COMPUTERIZED TOMOGRAPHY

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Abstract. Fast reconstruction formulae in x-ray computerized tomography demand the directions, in which the measurements are taken, to be equally distributed over the whole circle. In many applications data can only be provided in a restricted range. Here the intrinsic difficulties are studied by giving a singular value decomposition of the Radon transform in a restricted range. Practical limitations are deduced.

1 Introduction

The aim of computerized tomography (CT) is to provide information about the internal structure of the human body for medical diagnosis. This information is presented in form of pictures from slices through the body and it is dependent on the physical system taking the measurements. In x-ray CT the x-ray attenuation coefficient of the scanned tissue is shown, in nuclear magnetic resonance (NMR) CT it is for example the distribution of hydrogen nuclei in the considered region. In the following we want to concentrate on x-ray CT, the mathematical problems in NMR are quite similar.

In x-ray CT x rays are transmitted through the body. The denser the tissue the more photons are attenuated. Finally the photons arriving on the opposite side are counted in a detector. This is repeated for many orientations of the rays. The data measured in this way are to a good approximation proportional to an integral transform of the searched-for density distribution. In x-ray CT and in NMR this integral transform is the Radon transform of the distribution, in x-ray CT in two dimensions, in NMR in two or three dimensions.

In order to provide the pictures on which the medical diagnosis is based the Radon transform has to be inverted. This is possible if an infinite number of projections are available, see e.g. Smith-Solmon-Wagner [18]. In practice this can never be achieved which means that parts of the spectrum of the searched-for distribution are disturbed, see e.g. Louis [10]. As a consequence only details larger than a threshold, depending on the number of data, can reliably be reconstructed.

For the reconstruction discretizations of the inversion formula (2.4) are implemented. Fast algorithms usually demand the rays to be equally distributed. This request can not always be fulfilled in practice. In the following we study the problems arising from distributing the rays only in a restricted range. Although in this case the searched-for density distribution is uniquely determined a singular value decomposition of the integral operator exhibits that some components of the picture can practically not be recovered.

2 The Radon Transform

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function, representing the x-ray attenuation coefficients in the scanned tissue. The measured data lead to the integrals of f along the paths of the x rays. Let

$$\omega \equiv \omega(\varphi) = (\cos\varphi, \sin\varphi)^T, \tag{2.1}$$

then

$$Rf(s,\omega) = \int_{\mathbb{R}} f(s\omega + t\omega^{\perp})\,dt \tag{2.2}$$

where $\omega^{\perp} = \omega(\varphi + \frac{\pi}{2})$. Denoting with S^1 the boundary of the unit ball in $\mathbb{R}^2$, then

$$Rf : \mathbb{R} \times S^1 \to \mathbb{R}$$

is the Radon transform of f. In the applications, that we have in mind, the function f is always compactly supported, hence we assume that, after a possible scaling,

$$\text{supp } f \subset \Omega := V(0,1) \tag{2.3}$$

where V(a,r) denotes the ball with center a and radius r.

The problem of retrieval of f from its transform was first solved by Radon [15] in 1917. This inversion can easily be described with the help of the Fourier transform. Let

$\psi : \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, then its Fourier transform is given as

$$\hat{\psi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \psi(x) e^{-ix\xi} dx.$$

The inverse Radon transform takes on the following form

$$(2.4) \qquad R^{-1} = (2\pi)^{-1} R^{\#} I^{1}$$

where I^1 denotes the Riesz potential acting on the first variable of the function g on $\mathbb{R} \times S^1$,

$$(2.5) \qquad (I^1 g)^{\wedge}(\sigma, \omega) = |\sigma| \hat{g}(\sigma, \omega).$$

Here the one-dimensional Fourier transform has to be taken with respect to the first variable. The operator $R^{\#}$ is the so-called backprojection, it is the adjoint of R in L_2 and is defined as

$$(2.6) \qquad R^{\#} q(x) = \int_0^{\pi} q(x \cdot \omega, \omega) d\varphi.$$

Fast numerical implementations are based on these formulae, see e.g. Herman [4]. In (2.5) the data are filtered to stabilize the inverse transform which is not bounded in L_2, see Natterer [13]. The high frequencies are eliminated by a low-pass filter: they are corrupted with noise and also contain non-neglectable contributions of the ghosts; i.e., the functions in the null space of the transform for finitely many directions, see Louis [10].

The backprojection in (2.6) is approximated by numerical integration. For periodic functions the trapezoidal rule with equally distributed integration points is highly accurate but this demands the measured directions also to be equally distributed.

In many applications the directions are distributed only in a restricted range, then (2.4) is no longer applicable. Quite a number of methods have been proposed to overcome this difficulty. Gordon-Herman [3], Lewitt [6] used ART; i.e., point collocation in a suitable subspace. Tuy [20], Lent-Tuy [5], Tam-Perez Mendez-McDonalds [19] gave iterative methods in Fourier space. Rockmore-Macovski [16] used estimation theory. Davison-Grünbaum [2] constructed filters for (2.5) dependent on the direction. In Louis [7,8] and Perez [14] extrapolations of the data in the missing range are computed based on characterizations of the range of the Radon transform. This provides a fast and accurate method to tackle the problem, but the numerical implementations have to be done very carefully as the two different conclusions in these papers show.

3 The Singular Value Decomposition of the Limited Angle Radon Transform

In the following we study the intrinsic difficulties and limitations of the restricted range problem. To this end we start off with idealized assumptions, to wit we assume that complete projections are given in a subset of S^1 with $\omega(\varphi)$ where

(3.1) $$\Phi \le \varphi \le \frac{\pi}{2} \quad , \quad 0 < \Phi < \frac{\pi}{2} \ .$$

In this way we avoid any influence of discretization errors and can study the problem purely stemming from distributing the directions in this restricted range.

For constructing the singular value decomposition we need some special functions. Let $P_n^{(\alpha,\beta)}$ be the Jacobi polynomial of degree m, they are orthogonal on [-1,1] with respect to the weight

(3.2) $$w_{\alpha\beta}(s) = (1-s)^{\alpha}(1+s)^{\beta}.$$

The special case where $\alpha = \beta = \frac{1}{2}$, but with a different normalization, are the Chebyshev polynomials of the second kind

$$U_n(s) = \sin((n+1)\arccos s) \,/\, \sin(\arccos s).$$

Thus they are orthogonal on [-1,1] with respect to the weight

(3.3) $$w(s) = (1-s^2)^{1/2}.$$

For constructing the singular value decomposition of R we first give a complete orthogonal set over the unit ball in $\mathbb{R}^2$ and then compute its Radon transform.

Lemma 3.1

Let $x = r \cdot \omega(\theta)$. Then the functions

(3.4) $$V_{m\ell}(x) = Q_{m\ell}(s) e^{i\ell\varphi}$$

with

$$(3.5) \qquad Q_{m\ell}(s) = s^{\ell} P^{(0,\ell)}_{(m-\ell)/2}(2s^2-1)$$

where $m,\ell \in \mathbb{N}_0$, $0 \leq \ell \leq m$ with $m+\ell$ even, form a complete set of orthogonal functions in $L_2(\Omega)$.

Proof

This is a special case of Theorem 3.1 in [9].

Lemma 3.2

Let $V_{m\ell}$ be as in (3.4). Then its Radon transform is

$$(3.6) \qquad RV_{m\ell}(s,\omega) = (2\pi)^{1/2} \frac{2}{m+1} w(s) U_m(s) e^{i\ell\varphi}.$$

Proof

This is the well-known result of Cormack [1], for different proofs see Marr [12], Louis [9].

It is also shown in [9] that the functions in (3.6) form, for $m,\ell \in \mathbb{N}_0$, $0 \leq \ell \leq m$, $m+\ell$ even, a complete orthogonal set in the range of the Radon transform in $L_2([-1,1] \times S^1, w^{-1})$, thus giving a constructive proof of the Helgason-Ludwig consistency conditions. This is now used for the singular value decomposition for the Radon transform in restricted range.
Let $A(m,\Phi)$ be the $(m+1) \times (m+1)$-matrix with entries

$$(3.7) \qquad a_{jk} = \begin{cases} \dfrac{\sin 2(k-j)\Phi}{(k-j)\pi} & j \neq k \\ \dfrac{2\Phi}{\pi} & j = k. \end{cases}$$

This Toeplitz matrix was studied by Slepian [17] and denoted there by $\rho(m+1,\Phi/\pi)$. He shows that the eigenvalues of this matrix fulfil for $0 < \Phi < \frac{\pi}{2}$ the relation

$$0 < \lambda_0(m,\Phi) < \lambda_1(m,\Phi) < \ldots < \lambda_m(m,\Phi) < 1.$$

With

$$d_\mu(m,\Phi) = (d_\mu(m,\Phi)_0, \ldots, d_\mu(m,\Phi)_m)^T \in \mathbb{R}^{m+1}$$

we denote the corresponding normalized eigenvectors.

Finally we consider the Radon transform as mapping

(3.8) $$R : L_2(\Omega) \to L_2(Z_\Phi, w^{-1})$$

where

$$Z_\Phi = [-1,1] \times S_\Phi, \quad S_\Phi = \{\omega \in S^1 : |\varphi \pm \tfrac{\pi}{2}| \le \tfrac{\pi}{2} - \Phi\};$$

i.e., the directions are given in S_Φ. The adjoint operator of R is then

(3.9) $$R^*g(x) = \int_{S_\Phi} w^{-1}(x\cdot\omega)\, g(x\cdot\omega,\omega)\, d\varphi.$$

<u>Theorem 3.3</u>

Let λ_μ, d_μ be the eigenvalues and eigenvectors of $A(m,\Phi)$ and

(3.10) $$f_{m\mu}(r\cdot\omega(\theta)) = \sum_{\ell=0}^{m} d_\mu(m,\Phi)_\ell\, Q_{m,|2\ell-m|}(r)\, e^{i(2\ell-m)\theta},$$

(3.11) $$g_{m\mu}(s,\omega) = w(s)U_m(s) \sum_{\ell=0}^{m} d_\mu(m,\Phi)_\ell\, e^{i(2\ell-m)\varphi},$$

(3.12) $$\sigma_{m\mu}(\Phi) = 2\left(\frac{\pi}{m+1}(1-\lambda_\mu(m,\Phi)\right)^{1/2}.$$

Then $(f_{m\mu}, g_{m\mu}; \sigma_{m\mu})$, $m,\mu \in \mathbb{N}_0$, $0 \le \mu \le m$, for a complete singular system of the Radon transform $R : L_2(\Omega) \to L_2(Z_\Phi, w^{-1})$.

<u>Proof</u>

Because of the linear independence of the $d_\mu(m,\Phi)$ the functions $f_{m\mu}$, introduced in (3.10), form a complete orthogonal set in $L_2(\Omega)$. Hence we expand $R^*Rf_{m\mu} \in L_2(\Omega)$ in terms of these functions. The expansion coefficients are then

$$c_{n\nu} = \langle R^*Rf_{m\mu}, f_{n\nu}\rangle_{L_2(\Omega)} \,/\, \langle f_{n\nu}, f_{n\nu}\rangle_{L_2(\Omega)}.$$

Using the adjoint operator and (3.6) we get

$$\langle R^*Rf_{m\mu}, f_{n\nu}\rangle_{L_2(\Omega)} = \langle Rf_{m\mu}, Rf_{n\nu}\rangle_{L_2(Z_\Phi, w^{-1})}$$

$$= \frac{2}{\pi(m+1)(n+1)} \int_{-1}^{1} w(s) U_m(s) U_n(s) ds \cdot$$

$$\sum_{\ell=0}^{m} \sum_{k=0}^{n} d_\mu(m,\Phi)_\ell d_\nu(n,\Phi)_k \int_{S_\Phi} e^{i(2\ell-m+n-2k)\varphi} d\varphi.$$

According to the orthogonality of the Chebyshev polynomials the integral with respect to s has the value $\frac{\pi}{2}\delta_{mn}$. It remains to compute the integral with respect to the directions. Using $m = n$ we get

$$\int_{S_\Phi} e^{2i(k-\ell)\varphi} d\varphi = 2 \int_{\Phi \leq |\varphi| \leq \frac{\pi}{2}} e^{2i(\ell-k)\varphi} d\varphi$$

$$= 2\left(\int_{-\pi/2}^{\pi/2} e^{2i(k-\ell)\varphi} d\varphi - \int_{-\Phi}^{\Phi} e^{2i(k-\ell)\varphi} d\varphi\right)$$

$$= 2\pi(\delta_{k\ell} - a_{k\ell})$$

with $a_{k\ell}$ given in (3.7). With matrix notation the above scalar product can be written as

$$\langle R^*Rf_{m\mu}, f_{n\nu}\rangle_{L_2(\Omega)} = \frac{2}{(m+1)^2}\delta_{mn} d_\mu(m,\Phi)^T (I-A(m,\Phi)) d_\nu(m,\Phi)$$

$$= \frac{2}{(m+1)^2}(1-\lambda_\mu(m,\Phi))\delta_{mn}\delta_{\mu\nu}.$$

With

$$\langle f_{m\mu}, f_{m\mu}\rangle_{L_2(\Omega)} = \frac{1}{2(m+1)}$$

finally follows

$$R^*Rf_{m\mu} = c_{m\mu} f_{m\mu} = \sigma_{m\mu}^2 f_{m\mu}.$$

<u>Remark</u>

In the case of the full range problem we have $\Phi = 0$ and thus $I-A(m,0) = I$. This gives the singular values

$$\sigma_{m\mu} = 2\left(\frac{\pi}{m+1}\right)^{1/2},\ 0 \leq \mu \leq m.$$

4 Practical Consequences

As a measure for the ill-posedness of a problem usually the ratio between largest and smallest singular value is considered. Clearly this ratio is infinite for the Radon transform as mapping between L_2-spaces. But this only means that this compact operator has an unbounded inverse in an L_2 setting, see also Natterer [13]. Instead we consider the ratio of the largest and smallest of the singular values $\sigma_{m\mu}(\Phi)$ for $m \leq p-1$, and denote it by $\kappa_p(\Phi)$. This corresponds to a stabilization by cutting off the smallest singular values. If we compare $\kappa_p(0)$ for the full range case with $\kappa_p(\pi/6)$, where one third of the range is missing,

$$\kappa_{40}(\pi/6) \sim 10^8 \kappa_{40}(0).$$

This may lead to the conclusion that it is hopeless to reconstruct from data in a limited range.

But on the other hand the above considerations are very pessimistic as they are based only on the worst case. The asymptotic estimates given in Slepian [17] indicate a special behaviour of the singular values. If we consider the eigenvalues of $I-A(m,\Phi)$ then we realize that the spectrum splits into two parts: some eigenvalues are close to 1 and the rest are close to 0, see Figure 4.1. More precisely we can state the following.

Theorem 4.1

For large m the number of singular values $\sigma_{m\mu}(\Phi)$, $0 \leq \mu \leq m$, close to the singular values in the full range case, $\sigma_{m\mu}(0)$, is equal to the integral part of

$$(m+1)\left(1 - \frac{2\Phi}{\pi}\right), \tag{4.1}$$

the rest are close to 0.

This means that it is well possible to reconstruct parts of the picture, namely those singular functions belonging to large singular values. The smaller the missing range, the larger is this information, compare (4.1). Figures 4.2 show the singular functions $f_{20,20}$, $f_{20,16}$, $f_{20,10}$, $f_{20,0}$ for $\Phi = \pi/4$; i.e., half of the range is missing. Here the x rays parallel to the ripples in Figure 4.2 a), showing $f_{20,20}$, and in directions up to $\pm\frac{\pi}{4}$ are given. Contributions belonging to $f_{20,\mu}$, $\mu \geq 11$, can be reconstructed. The singular function $f_{20,10}$ belongs to a mid-sized singular value and is somewhat critical. But it is definitely impossible to reconstruct parts stemming from the singular function $f_{20,0}$ where the ripples are opaque to the x rays.

Finally Figure 4.3 shows reconstructions of the head phantom given in Herman [4], see Figure 4.3 a). In Figure 4.3 b)-d) reconstructions are shown from 144 directions and 165 rays per direction with noisy data according to the photon statistic described in [4]. In Figure 4.3 b) the directions are equally distributed over the full range; i.e., $\Phi = 0$. In Figure 4.3 c), d) they are distributed only over $\frac{2}{3}$ of the circle; i.e., $\Phi = \frac{\pi}{6}$. In 4.3 c) we simply reconstructed with zeros in the missing range, see also Tuy [20] and in Figure 4.3 d) we used the extrapolation procedure given in Louis [8]. The streak artifact between the triangular parts of the bones is obviously stemming from singular functions belonging to small singular values and thus intrinsic to reconstructions from limited angular data. But the effect seems to be larger than it is in reality. The values of the phantom range from 0 to 0.416 whereas the window in the display covers only 2% of this region: values smaller than 0.1945 are represented as black, values larger than 0.22 as white.

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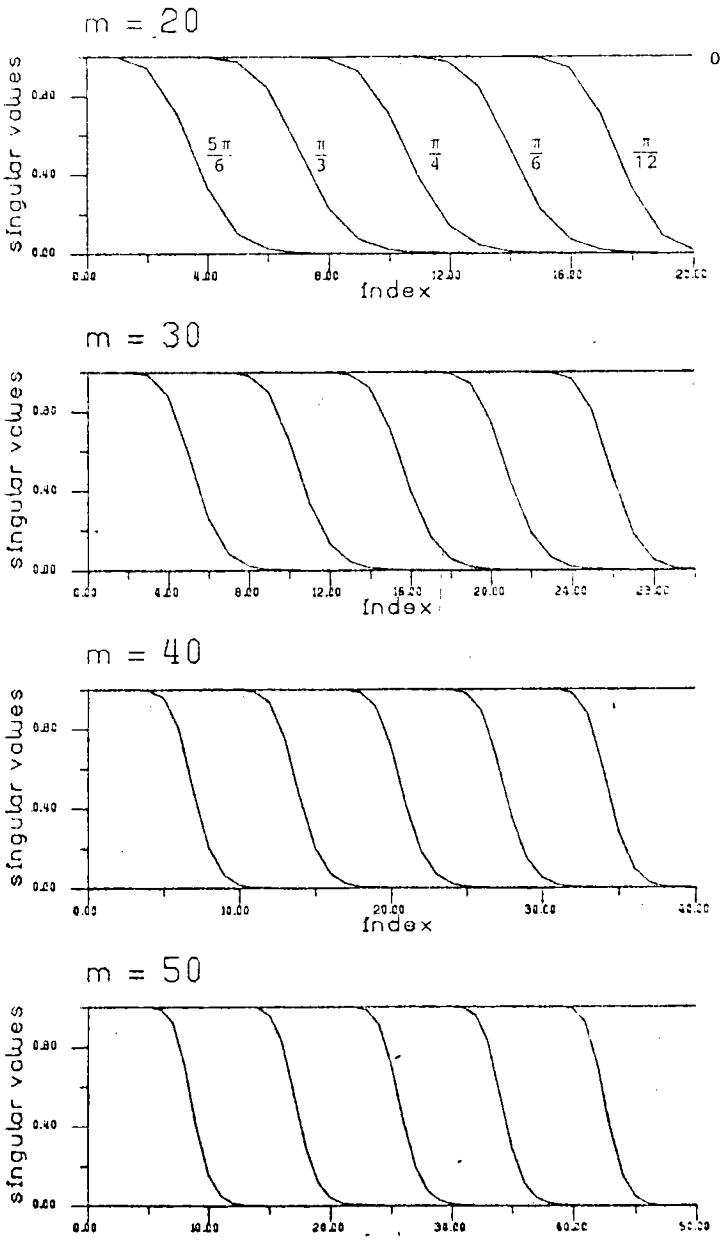
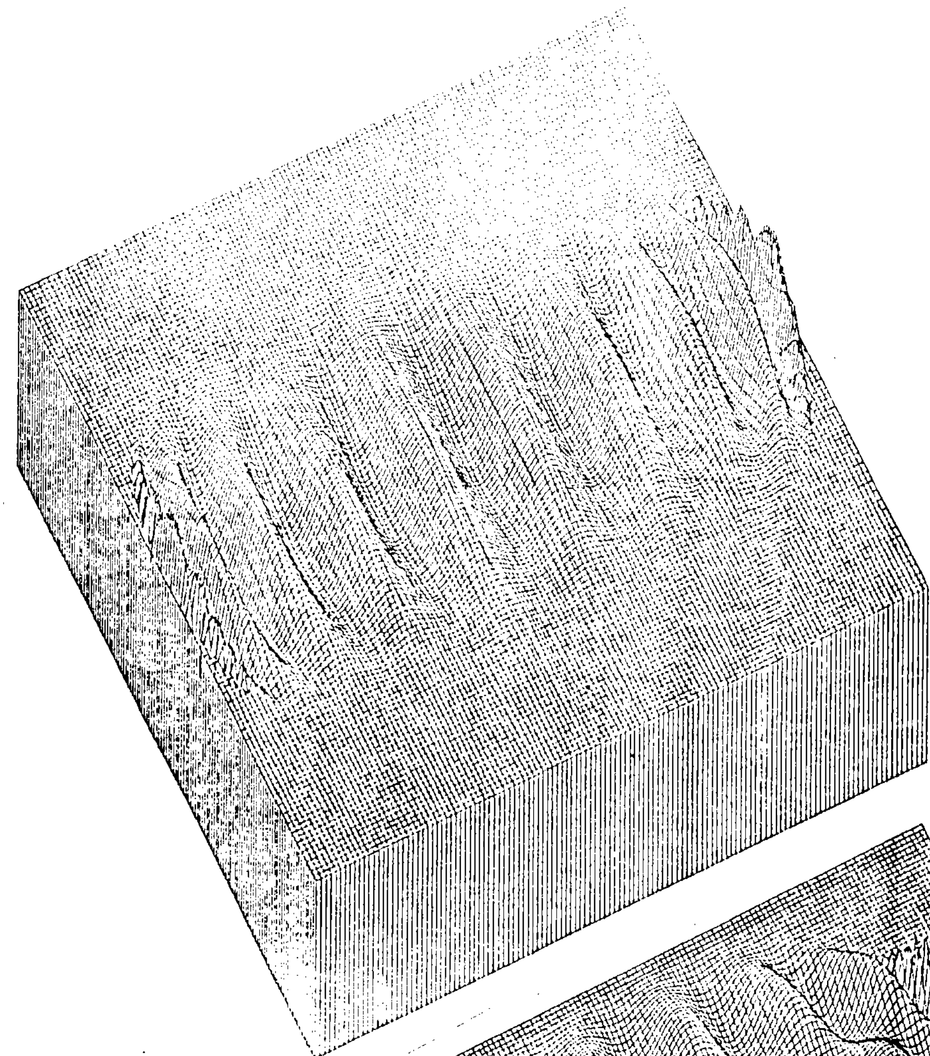
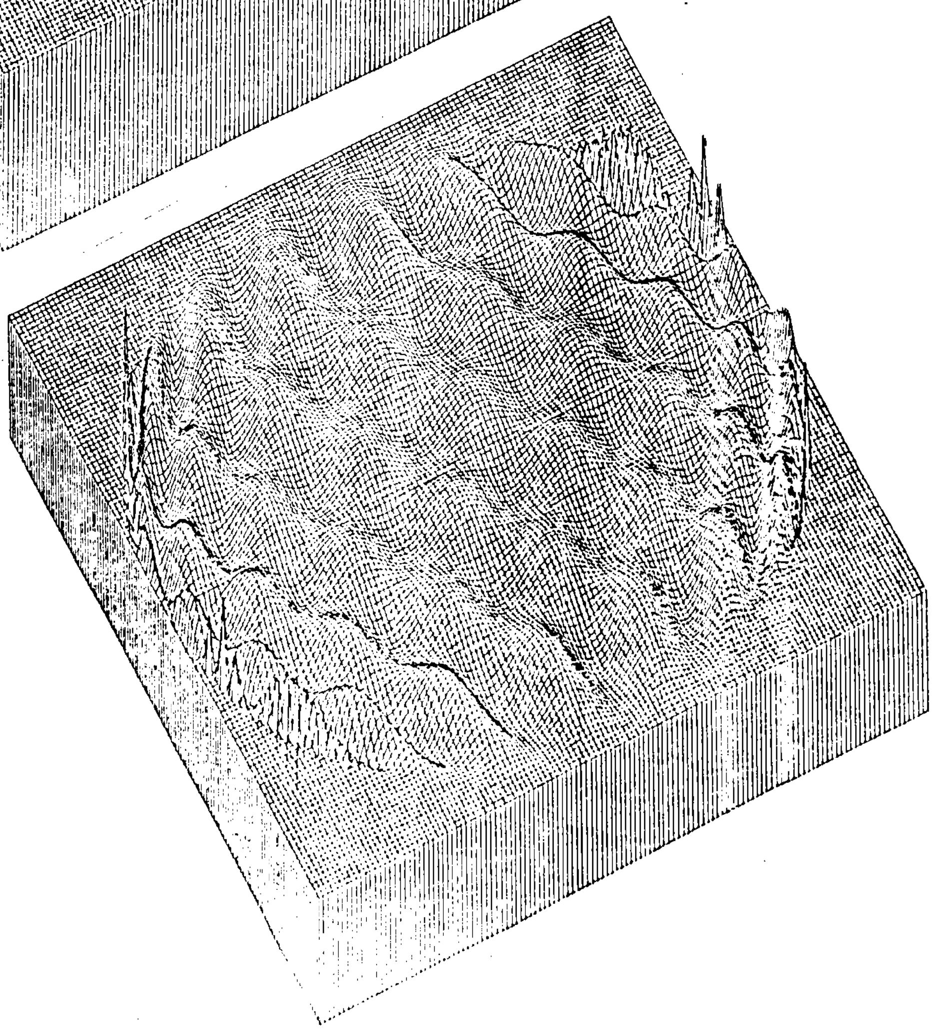
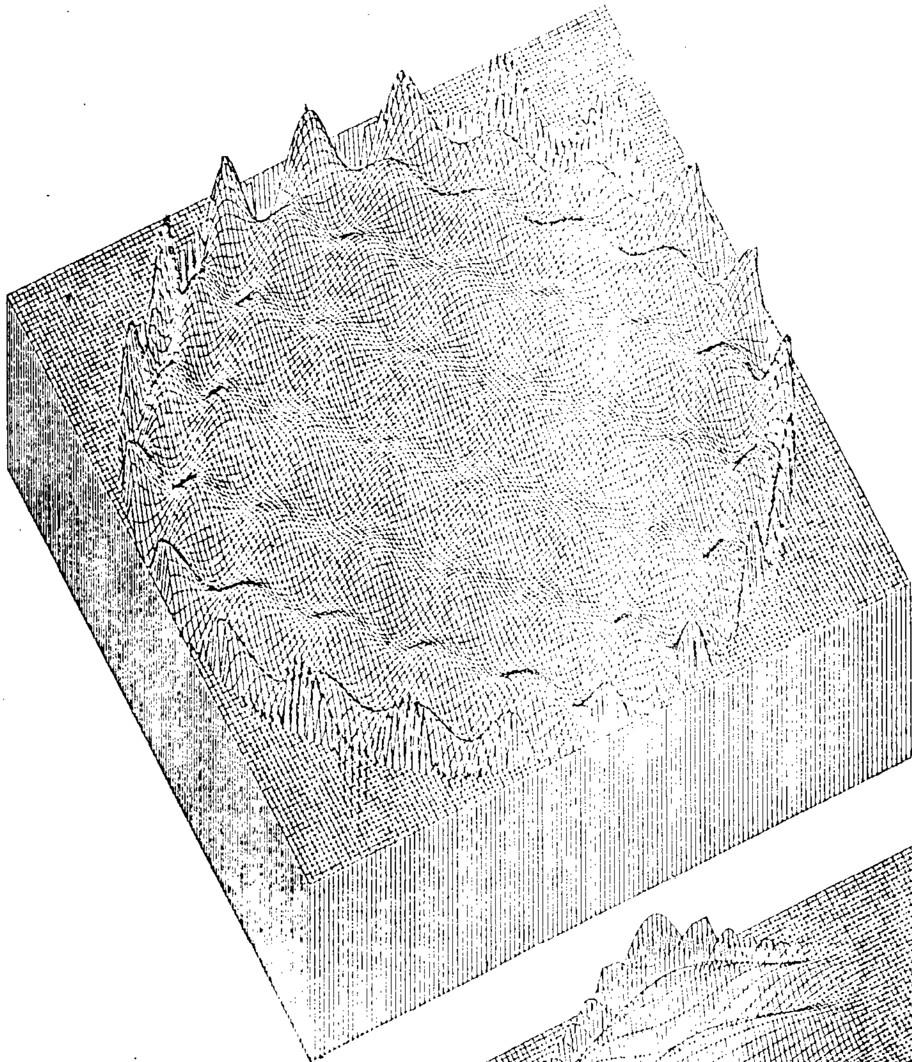


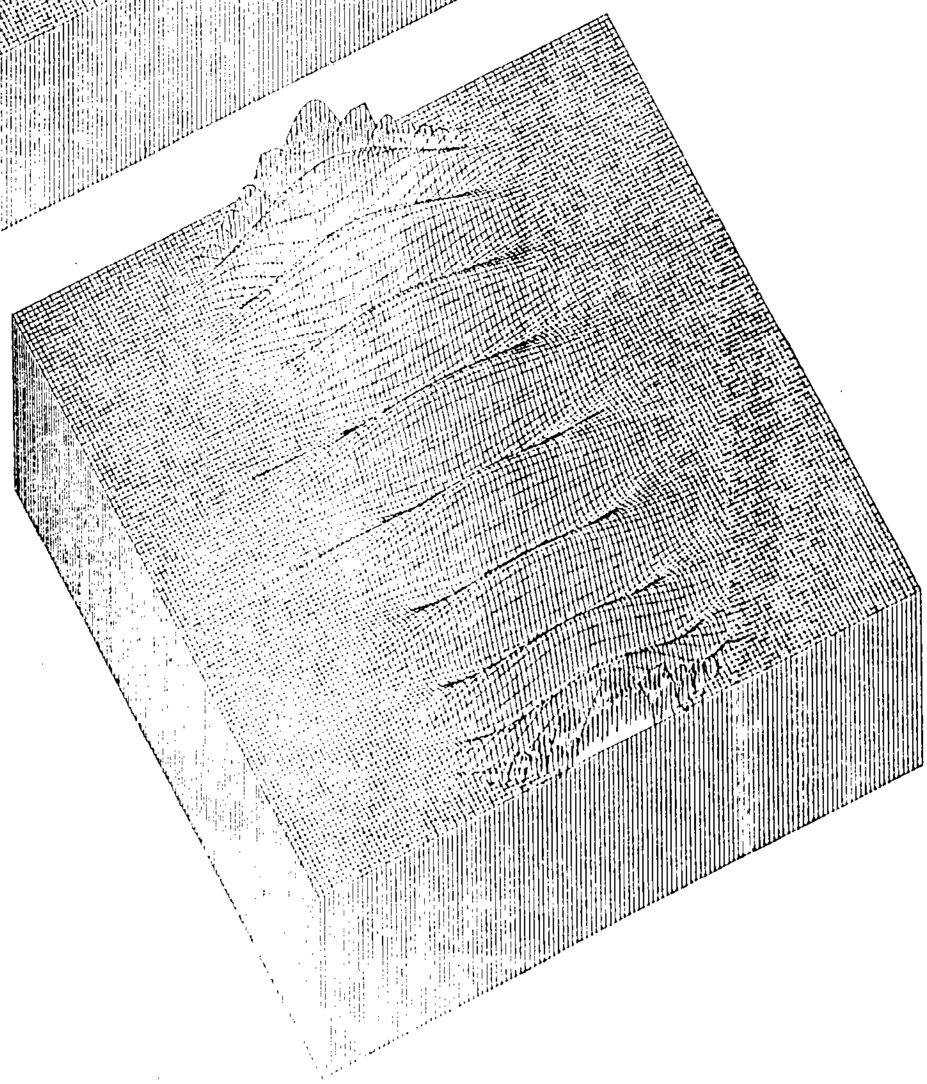
Figure 4.1: Eigenvalues of I-A(m,Φ), see (3.12), for $m = 10,20,30,40$ and $\Phi = 0, \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{6}$.

Figure 4.2: Singular functions $f_{20,20}$ (a), $f_{20,16}$ (b), $f_{20,10}$ (c), $f_{20,0}$ (d) for $\Phi = \frac{\pi}{4}$.









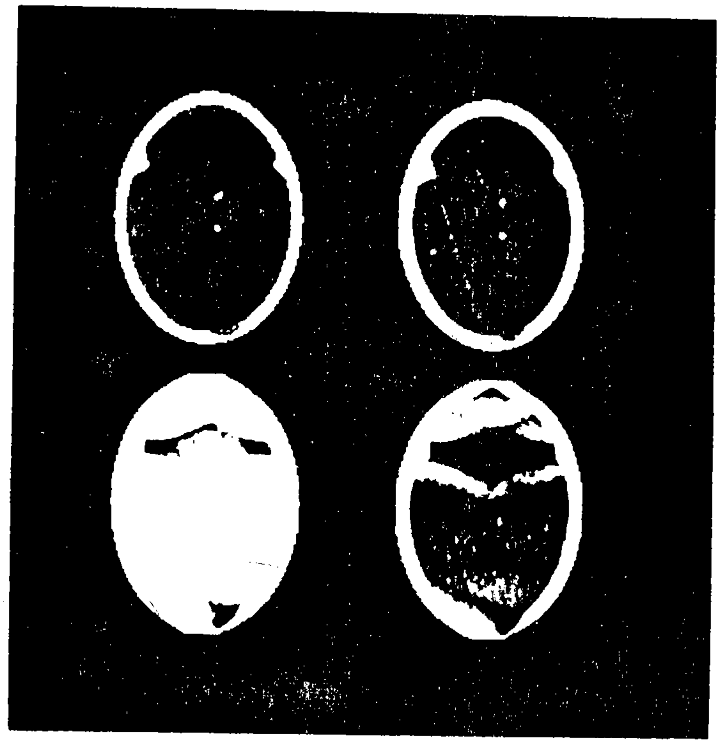


Figure 4.3 a): Test phantom for slice through the human head as described in Herman [4], p. 60.

b)-d): Reconstructions from noisy data simulating the effects of photon statistics, see [4]. Here 144 directions and 165 rays per direction are used in

b) the full range; i.e., $\Phi = 0$

c),d) the restricted range with $\Phi = \frac{\pi}{6}$ with

c) zeros in the missing range,

d) data extrapolation according to the algorithm given in [8].

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