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SENSITIVITY IN THE RIGID PUNCH PROBLEM

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Abstract

The rigid punch problem, a certain contact problem, leads to a noncoercive variational inequality. We show that its solution admits a directional derviative with respect to the data.

1 Introduction and problem formulation

In the rigid punch problem one considers a material body deformed by another body (the punch), the latter assumed to be rigid. We discuss here the following specific situation. The linear elastic material body occupies at rest the domain $\Omega \subset \mathbb{R}^N, N = 2$ or 3, and is kept fixed at a part Γ_0 of its boundary. The rigid punch has one degree of freedom, namely the motion in the vertical direction, and presses from above with a fixed force resulting from its own weight into the material body. We assume frictionless contact. See figure 1 for a sketch of the initial and the equilibrium configuration.

(Insert figure 1.)

The variational inequality approach to contact problems is wellknown and subject of several textbooks, see e.g. [2], [4], [7], [8], [10], [11], and in particular [8] for an extended discussion of the rigid punch problem. Here, the unknowns are the components $u_i : \Omega \rightarrow \mathbb{R}^N, 1 \leq i \leq N$, of the displacement of the elastic body and the vertical displacement $y \in \mathbb{R}$ in the downward direction of the rigid punch. For notational simplicity, we write

$$u = (u_1, \ldots, u_N, y) \in V$$

where V is a suitable Hilbert space. The variational inequality then takes on the standard form

$$a(u, v - u) \quad \geq \quad < F, v - u > \quad \forall v \in K \tag{1}$$
$$u \in K$$

Here, a is the usual bilinear form of linear elasticity

$$a(u, v) = \int_\Omega a_{ijkl}\epsilon_{ij}(u)\epsilon_{kl}(v)dx \tag{2}$$

$$\epsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i).$$

The linear form F includes the body forces as well as the contribution from the potential energy of the punch

$$< F, u > = \int_\Omega f_i u_i dx + Py \tag{3}$$

where $P > 0$ corresponds to the weight of the punch. The constraint set K represents a linearized form of the nonpenetration condition (i.e. the condition that the punch is always above the elastic body). We write it in the general form

$$K = \{(v_1, \ldots, v_N, z) : \alpha_i(x)v_i(x) + \beta(x)z \leq \varphi(x), x \in \Gamma_c\} \tag{4}$$

Here, Γ_c is that part of the upper boundary of Ω where contact may possibly occur. The function φ denotes the vertical distance (or some modification of it) of the punch from the elastic body in the initial configuration, and the functions α_i, β are connected to the unit

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normal of the boundary of the punch or of the elastic body. Various forms are possible, for example ($N = 2$)

$$n_i(x)v_i(x) + z \leq \varphi(x)$$
$$n_i(x)v_i(x) + n_2(x)z \leq n_2(x)\varphi(x)$$
$$v_2(x) + z \leq \varphi(x),$$

see e.g. [3], [7], [8]. We also refer to the literature for the formulation of the rigid punch problem as a boundary value problem.

In optimal design, one wants to choose the material coefficients a_{ijkl} or the shape Ω of the elastic body in order to minimize some functional involving the displacement or the stress. Here one encounters the basic mathematical difficulty that the solution u of (1) is a Lipschitz continuous (if the bilinear form is coercive), but not differentiable function of the data. It is therefore of interest to know whether this function is directionally differentiable, and to compute its derivative. Mignot [9] has studied the dependence of u upon F in (1) in the coercive case. Among others, Sokolowski and Zolesio have contributed a lot, partially based on [14]; we just refer to [13].

The specific feature of interest here is the noncoercivity of the bilinear form a due to the presence of the additional unknown y which does not appear in (2). While a lot of work has been done in existence theory, mainly by the Italian school, the question of sensitivity seems to be largely open. Our approach is to extend the methods of [6, 9, 12] to the analysis of the rigid punch problem. This is possible since problem (1)-(4) is not very far from the case of a coercive bilinear form, and since one can obtain local Lipschitz dependence of the solution u upon the data (which is, of course, almost necessary for directional differentiability).

2 Sensitivity analysis of a noncoercive variational inequality.

We consider the variational inequality

$$a(u, v - u) \geq\ <F, v - u>\quad \forall v \in K \tag{5}$$
$$u \in K$$

where $a : V \times V \to \mathbb{R}$ is a continuous bilinear form on a Hilbert space V, $F : V \to \mathbb{R}$ is a continuous linear functional, and K is a closed convex subset of V. We assume that

$$a(u, u) \geq 0 \quad \forall u \in V \tag{6}$$

but neither that a is coercive nor that it is symmetric. We consider the case where the variational inequality (5) has a unique solution

$$u = \pi(F) \tag{7}$$

and look for the directional derivative of the mapping $\pi : V' \to V$ in some direction $H \in V'$, i.e. we want to find $g \in V$ with

$$\pi(F + tH) = \pi(F) + tg + o(t). \tag{8}$$

From the paper of Mignot [9] it is clear that g should solve an associated variational inequality where the convex set K is replaced by a certain subset of its tangent cone at $u = \pi(F)$. More precisely, let us denote the set of admissible directions at $u \in K$ by

$$C(u) = \{v \in V : u + tv \in K \text{ for some } t \geq 0\} \tag{9}$$

and the tangent cone by

$$S(u) = cl(C(u)) \tag{10}$$

If necessary, we will write $C_K(u), S_K(u)$ for clarity.

We also define for $u \in K, F \in V'$

$$S(u, F) = S(u) \cap H(u, F), C(u, F) = C(u) \cap H(u, F) \tag{11}$$

where $H(u, F)$ denotes the closed hyperplane

$$H(u, F) = \{v \in V : a(u, v) = <F, v>\}. \tag{12}$$

The associated variational inequality for the directional derivative g of π at F in the direction H is given by

$$a(g, v - g) \quad \geq \quad <H, v - g> \quad \forall v \in S(u, F) \tag{13}$$
$$g \in S(u, F).$$

Mignot [9] and Haraux [6] have shown that the following additional property of the convex set K is crucial for the analysis of the coercive case.

Definition 2.1
A closed convex set $K \subset V$ is called polyhedric if

$$S(u) \cap H = cl(C(u) \cap H)$$

for any $u \in K$ and any hyperplane H in V.

$\square$

The main result of this section is now the following.

Theorem 2.2
Assume that

(i) the variational inequality (5) has a unique solution $u = \pi(F)$ for an $F \in V'$, and the map $t \to \pi(F + tH)$ is locally Lipschitz continuous for any $F, H, \in V'$,

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(ii) the variational inequality (13) has at most one solution,

(iii) the set K is polyhedric.

Then the difference quotient

$$\Delta_t u = \frac{u(t) - u(0)}{t}, u(t) = \pi(F + tH),$$

satisfies

$$\Delta_t u \to g \quad \text{weakly in } V,$$
$$a(\Delta_t u - g, \Delta_t u - g) \to 0$$

for $t \downarrow 0$, where g is the unique solution of (13).

Proof: This is similar to that of proposition 1 in [6]. By (i), $\{\Delta_t u\}$ is bounded in V for small t, and $\Delta_{t_n} u \to g$ weakly for some $g \in V$ and some sequence $t_n \downarrow 0$. Let now $v \in C(u(0), F)$. Then for $t \geq 0$ small enough we have $u(0) + tv \in K$, and

$$
\begin{aligned}
t^2 [a(\Delta_t u, v - \Delta_t u) - &< H, v - \Delta_t u >] = \\
= \ & a(u(t), u(0) + tv - u(t)) - a(u(0), u(0) + tv - u(t)) - < tH, u(0) + tv - u(t) > \\
\geq \ & < F + tH, u(0) + tv - u(t) > + a(u(0), u(t) - u(0)) - ta(u(0), v) - \\
& - < tH, u(0) + tv - u(t) > \\
= \ & a(u(0), u(t) - u(0)) - < F, u(t) - u(0) > \geq 0,
\end{aligned}
$$

so in particular

$$t[a(\Delta_t u, v - \Delta_t u) - < H, v - \Delta_t u >] \geq a(u(0), \Delta_t u) - < F, \Delta_t u > \geq 0. \tag{14}$$

From (14) we immediately obtain $a(u(0), g) = < F, g >$ and therefore $g \in S(u(0), F)$, since $\Delta_t u \in C(u(0), F)$ for small t. We now shown that g solves (13). By (14) and since K is polyhedric,

$$a(\Delta_t u, v - \Delta_t u) \geq < H, v - \Delta_t u > \quad \forall v \in S(u(0), F) \tag{15}$$

and by the standard argument involving Minty's lemma, g solves (13). By assumption (ii), this solution is unique, so it does not depend on the sequence $\{t_n\}$. Using $v = g$ in (15), we obtain

$$0 \leq a(g - \Delta_t u, g - \Delta_t u) \leq a(g, g - \Delta_t u) - < H, g - \Delta_t u >,$$

so

$$a(g - \Delta_t u, g - \Delta_t u) \to 0.$$

$\square$

In the rigid punch problem, a is coercive except on a finite dimensional subspace, so the following corollary applies.

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Corollary 2.3

In the situation of theorem 2.2, assume that $V = V_1 \oplus V_2$ with $a|V_1$ coercive and $\dim(V_2) < \infty$. Then

$$\Delta_t u \to g \quad strongly \ in \ V,$$

so g is the directional derivative of π at F in the direction H.

$\square$

In optimization problems involving an elliptic variational inequality (5), one often wants to optimize with respect to the coefficients of the bilinear form, so one is led to consider the variational inequality parametrized by some $t \geq 0$

$$a^t(u^t, v - u^t) \quad \geq \quad < F^t, v - u^t > \quad \forall v \in K \qquad (16)$$
$$u^t \in K$$

Again, one looks for the derivative

$$g = \lim_{t \downarrow 0} \frac{1}{t}(u^t - u^0) \qquad (17)$$

under the assumption that (16) has a unique solution $u^t = \pi^t(F^t)$. Naturally, one has to assume differentiability of a^t and F^t with respect to t in some sense. Sokolowski and Zolesio [12] have shown how to reduce (16), (17) to the situation (5), (8) in the coercive case. Concerning the differentiability of a^t, they assume that there exists a linear continuous map $A' : V \to V'$ such that

$$\lim_{t \downarrow 0} \sup_{u,v \in B} |\frac{1}{t}[a^t(u,v) - a^0(u,v)] - < A'u, v > | = 0, \qquad (18)$$

where B denotes the unit ball in V, and for F^t they assume that there exists $F' \in V'$ such that

$$\limsup_{t \downarrow 0} \sup_{u \in B} |\frac{1}{t} < F^t - F^0, u > - < F', u > | = 0, \qquad (19)$$

We remark that (18) implies

$$|a^t(u,v)| \leq M ||u|| \, ||v|| \qquad (20)$$

with M independent from t.
We now present the result, a generalization of theorem 1 in [12].

Theorem 2.4

Assume that

(i) *the variational inequality (16) has a unique solution $u^t = \pi^t(F^t)$ for any $F^t \in V'$, and the map π^0 is locally Lipschitz continuous around u^0,*

(ii) *the map π^0 has a directional derivative $g(H)$ at F^0 in any direction $H \in V'$,*

(iii) *conditions (6) and (18) hold for the bilinear forms a^t, and (19) holds for F^t.*

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Then

$$u^t = u^0 + tg(F' - A'u^0) + o(t).$$

Proof: We modify the proof of theorem 1 in [12] in a suitable way. One checks with elementary computations that the element $u^t = \pi^t(F^t)$ satisfies

$$u^t = \pi^0(F^0 + t(F' - A'u^0) + \sum_{i=1}^{3} r_i(t)) \tag{21}$$

where

$$
\begin{aligned}
r_1(t) &= F^t - F^0 - tF' \\
< r_2(t), v > &= a^0(u^0, v) - a^t(u^0, v) + t < A'u^0, v > \quad \forall v \in V \\
< r_3(t), v > &= a^0(u^t - u^0, v) - a^t(u^t - u^0, v) \quad \forall v \in V
\end{aligned}
$$

We may interpret u^t as a fixed point of

$$z = P(t, z)$$

where $P : \mathbb{R}_+ \times V \to V$ is defined by

$$
\begin{aligned}
P(t, z) &= \pi^0(F^0 + t(F' - A'u^0) + r_1(t) + r_2(t) + R(t, z)), \\
< R(t, z), v > &= a^0(z - u^0, v) - a^t(z - u^0, v).
\end{aligned}
$$

Because of (18), (19) and the local Lipschitz continuity of π^0, for any ϵ-ball B_ϵ around u^0, the mapping $z \to P(t, z)$ is a contraction on B_ϵ for t small enough, so in particular $u^t \in B_\epsilon$ and therefore $u^t \to u^0$ in V. Since

$$| < r_3(t), v > | \le Ct||u^t - u^0|| \, ||v|| \quad \forall v \in V$$

for some constant C, this implies that $||r_i(t)|| = o(t)$ for $1 \le i \le 3$ and the assertion now follows from assumption (ii) and the local Lipschitz continuity of π^0.
$\square$

We remark that in the situation of corollary 2.3, assumption 2.4 (ii) holds, and the directional derivative $g(F' - A'u^0)$ is characterized by the variational inequality

$$
\begin{aligned}
a^0(g, v - g) &\ge \quad < F' - A'u^0, v - g > \quad \forall v \in S(u^0, F^0) \\
g &\in S(u^0, F^0).
\end{aligned}
$$

3   Application to the rigid punch problem

We repeat from the introduction the variational inequality formulation of the rigid punch problem, assuming that the rigid punch has one vertical of freedom. It reads

$$a(u, v - u) \quad \geq \quad <F, v - u> \quad \forall v \in K \tag{22}$$
$$u \in K \subset V.$$

The underlying Hilbert space V is given by

$$V = V(\Omega)^N \times \mathbb{R}, \quad N = 2 \text{ or } 3, \tag{23}$$
$$V(\Omega) = \{v : v \in H^1(\Omega), v|\Gamma_0 = 0\}.$$

so $u \in V$ has the form $u = (u_1, u_2, y)$ resp. $u = (u_1, u_2, u_3, y)$. The bilinear form a is the standard one from linear elasticity. For $N = 2$ it is written as

$$a(u, v) = \int_\Omega a_{ijkl}(x) \epsilon_{ij}(u) \epsilon_{kl}(v) dx \tag{24}$$
$$\epsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

We assume that a is coercive on $V(\Omega)^N$:

$$a(u, u) \geq c \, ||(u_1, \ldots, u_N)||^2_{H_1(\Omega)^N}. \tag{25}$$

The right hand side F is given by

$$< F, u > = \int_\Omega f_i(x) u_i(x) dx + Py, \tag{26}$$

where $P > 0$ is the total force exerted on the elastic body by the rigid punch. The convex set K is defined by

$$K = \{u \in V : \alpha_i(x) u_i(x) + \beta(x) y \leq \varphi(x) \text{ on } \Gamma_c\}, \tag{27}$$

where the zone of possible contact Γ_c, is compactly embedded into the upper boundary Γ_1 of Ω. We assume that φ, α_i, β are smooth functions satisfying

$$0 < c_0 \leq ||\alpha(x)|| \leq c_1, \quad 0 < c_0 \leq \beta(x) \leq c_1 \tag{28}$$

on Γ_c for some constants c_0, c_1.

The existence of a solution of (22) can be derived from [5], compare also [8]. For convenience, we use [1].

Theorem 3.1
In the situation (23) - (28), the variational inequality (22) has a solution.

Proof: We apply theorem 3.1 in [1]. One easily checks that (2.17i), (2.17ii) on p. 625 are satisfied, since $Y \cap rcK = \{(0, v) : y \leq 0\}$, and that (1.18) on p. 619 holds with $\Pi_0 = 0, \Pi_1 = $ projection on the last component.
□

Uniqueness and local Lipschitz stability can be proved directly.

Proposition 3.2

In the situation (23)-(28), the solution of (22) is unique and the map $\pi : V' \to V, u = \pi(F)$, is locally Lipschitz.

Proof: Let us write $u = (\tilde{u}, y), \tilde{u} = (u_1, \ldots, u_N), F = (\tilde{F}, P), < F, u > \; = \; < \tilde{F}, \tilde{u} > + Py$. Let $F_i = (\tilde{F}_i, P_i)$ be given with $P_i > 0$ and denote the corresponding solution of (22) by $u^i = (\tilde{u}_i, y_i)$. Using (22) twice and adding, we obtain

$$0 \leq a(u^1 - u^2, u^1 - u^2) \leq \; < \tilde{F}_1 - \tilde{F}_2, \tilde{u}_1 - \tilde{u}_2 > + (P_1 - P_2)(y_1 - y_2) \qquad (29)$$

Fix $\tilde{F}_1 = \tilde{F}_2 = \tilde{F}$ and $P_1 > 0$. For any $P_2 \geq P_1$ we have by (29) and (22), applied to $u = u^1, v = u^2$,

$$\begin{aligned}
0 \; \leq \; y_2 - y_1 &\leq \frac{1}{P_1}(a(u^1, u^2 - u^1) - \; < \tilde{F}, \tilde{u}_2 - \tilde{u}_1 >) \\
&\leq \frac{1}{P_1}\|\tilde{u}_2 - \tilde{u}_1\|(M\|\tilde{u}_1\| + \|\tilde{F}\|)
\end{aligned}$$

From the partial coercivity (25) and (29) we get

$$\|\tilde{u}_2 - \tilde{u}_1\| \leq \frac{1}{cP_1}(P_2 - P_1)(M\|\tilde{u}_1\| + \|\tilde{F}\|)$$

This proves that on any interval $\{(\tilde{F}, P) : P_1 \leq P \leq P_2\}$, the solution map π is well defined and Lipschitz continuous. But if we fix $P_1 = P_2$ in (29), we immediately obtain

$$\|\tilde{u}_2 - \tilde{u}_1\| \leq \frac{1}{c}\|\tilde{F}_1 - \tilde{F}_2\|,$$

so the assertion is proved.

$\square$

We now turn our attention to the convex set K. We have to check that K is polyhedric, and we want to compute the set $S(u, F)$, defined in (11) and needed in the associated variational inequality (13). To this end, we introduce the map

$$\begin{aligned}
T : V \; &= \; V(\Omega)^N \times \mathbb{R} \to H^{\frac{1}{2}}(\Gamma_c) \qquad (30) \\
(Tv)(x) \; &= \; \alpha_i(x)v_i(x) + \beta(x)z, \quad x \in \Gamma_c.
\end{aligned}$$

Due to the trace theorem and to (28), T is well defined and surjective. We set

$$K_c = \{w \in H^{\frac{1}{2}}(\Gamma_c) : w \leq \varphi \text{ a.e. on } \Gamma_c\}, \qquad (31)$$

so K_c is closed and convex in $H^{\frac{1}{2}}(\Gamma_c)$ and $K = T^{-1}(K_c)$. The following result is due to [9], compare also the exposition in [6].

Proposition 3.3

The set K_c defined in (31) is polyhedric, and its tangent cone

$$S(w_c) = \{w \in H^{\frac{1}{2}}(\Gamma_c) : w \leq 0 \quad q.e. \text{ on } Z(w_c - \varphi)\}.$$

Here q.e. (quasi everywhere) means except on a set of capacity zero, and $Z(w)$ denotes the set of zeroes of any quasicontinuous representative of w.

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Proof: Apply lemma 3.2' and theorem 3.2 in [9] to $H^{\frac{1}{2}}(\Gamma_c)$.

$\square$

Due to the following proposition, we will be able to transfer the result of proposition 3.3 to the set K.

Proposition 3.4

Let V, W be Hilbert space, $T : V \to W$ be linear, continuous, and surjective. Let $L \subset W$ be closed and convex, set $K = T^{-1}(L)$. Let U be a closed subspace of V such that $T(U)$ is closed in W.
Then

$$S_L(Tu) \cap T(U) = cl(C_L(Tu) \cap T(U))$$

implies

$$T^{-1}(S_L(Tu)) \cap U = cl(C_K(u) \cap U) \tag{32}$$

and in particular, setting $U = V$, we always have

$$S_K(u) = T^{-1}(S_L(Tu))$$

Proof: On easily checks from the definition that

$$C_K(u) = T^{-1}(C_L(Tu)). \tag{33}$$

From this, we conclude

$$cl(C_K(u) \cap U) \subset U \cap cl(T^{-1}(C_L(Tu))) \subset U \cap T^{-1}(S_L(Tu))$$

so it remains to prove "$\subset$" in (32). Let $v \in T^{-1}(S_L(Tu)) \cap U$, so $Tv \in S_L(Tu) \cap T(U)$. By assumption, we may choose $w_n \in C_L(Tu) \cap T(U)$ with $w_n \to Tv$. We also choose $z_n \in U$ with

$$Tz_n = w_n - Tv, \quad \|z_n\| \leq \delta(\|w_n - Tv\|),$$

where $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ is a certain function with $\delta(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$. This is possible due to the open mapping theorem, applied to $T : U \to T(U)$. Now set $v_n = v + z_n$, then

$$v_n \in U, \; Tv_n = w_n, v_n \to v$$

and $v_n \in C_K(u)$ by (33), so (32) is proved.

$\square$

We now combine the results of (24) and (25) to compute the tangent cone for the constraint set K of the variational inequality.

Proposition 3.5

The set K defined in (27) is closed, convex and polyhedric in V defined in (23), and its tangent cone in $u = (u_1, \ldots, u_N, y) \in K$ is given by

$$S(u) = \{v \in V : \alpha_i(x)v_i(x) + \beta(x)z \leq 0 \text{ on } Z \subset \Gamma_c\}$$

where

$$Z = \{x \in \Gamma_c : \alpha_i(x)u_i(x) + \beta(x)y = \varphi(x)\}$$

interpreted as coincidence set of quasicontinuous representatives.

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Proof: We use T and K_c from (30) and (31), then $K = T^{-1}(K_c)$ is closed and convex in V. We apply proposition 3.4 with $W = H^{\frac{1}{2}}(\Gamma_c)$ and $L = K_c$. The form of $S(u)$ now follows from proposition 3.3 and 3.4. Since, for any closed hyperplane H in V, the codimension of $T(H)$ is at most 1, the polyhedricity of K follows from that of K_c by (32), if we know that $T(H)$ is closed. But this is the case, since through the canonical isomorphism between W and $(\ker T)^{\perp}$ we have the representation

$$T(H) \cong (\ker T)^{\perp} \cap P^{-1}(P(\ker T))$$

where $P : V \to V$ denotes the orthogonal projection on the one dimensional orthogonal complement $H^{\perp}$ of H in V.

$\square$

We summarize the results.

Theorem 3.6
Consider the rigid punch problem (22) - (28). Then the solution map $\pi : V' - V, u = \pi(F)$, has a directional derivative $g \in V$ in any direction $H \in V'$, and g is characterized as the unique solution of the variational inequality

$$a(g, v - g) \qquad \geq \qquad <H, v - g> \quad \forall v \in S(u, F) \qquad (34)$$
$$g \in S(u, F)$$

where $S(u, F) = S(u) \cap \{v : a(u, v) = <F, v>\}$ and $S(u)$ is given in proposition 3.5.

Proof: Due to theorem 3.1 and propositions 3.2 to 3.5, the assertion follows from theorem 2.2 and corollary 2.3 provided we check that (34) has at most one solution. Assume that g_1 and g_2 solve (34). Then by the standard argument,

$$a(g_2 - g_1, g_2 - g_1) \geq 0$$

but since a is coercive on $V(\Omega)^N$ this implies that

$$g_2 - g_1 = (0, \ldots, 0, \gamma), \gamma \in \mathbb{R}.$$

Since $g_i \in S(u, F)$,
$$0 = a(u, g_1 - g_2) - <F, g_1 - g_2> = -P\gamma$$

so $\gamma = 0$.

4 Remarks on coefficient optimization and shape optimization.

Assume we want to compute the directional derivative of the solution u as we vary the coefficients of the bilinar form or the shape of, say, the lower boundary of Ω.
This leads to the parametrized variational inequality (16), repeated for convenience:

$$a^t(u^t, v - u^t) \quad \geq \quad <F^t, v - u^t> \quad \forall v \in K \qquad (35)$$
$$u^t \in K$$

In the case of coefficient variation, we have $F^t = F$ and

$$a^t = a + tb$$

where $(N = 2)$

$$b(u, v) = \int_\Omega b_{ijkl} \epsilon_{ij}(u) \epsilon_{kl}(v) dx.$$

For t small enough, the assumptions of theorem 2.4 are satisfied because of theorem 3.6, and because theorem 3.1 and proposition 3.2 remain true if we replace a by a^t in 22. It follows from theorem 2.4 that the directional derivative of u^t at $t = 0$ is equal to the unique solution g of the variational inequality

$$a(g, v - g) \quad \geq \quad - b(u^0, v - g) \quad \forall v \in S(u^0, F) \qquad (36)$$
$$g \in S(u^0, F).$$

The situation is more complicated if we vary the shape of the boundary. One may use the method of [14] to describe a one parameter family of domains Ω_t. One then transforms to the fixed domain formulation 35. In order to apply theorem 2.4, one additionally has to check (18) - (20) and to compute the mappings A' and F'. This procedure has been carried out in section 4 of [12]; in particular, the form of A' and F' is given in (4.56) and (4.55) of [12]. Therefore, the assertions of theorem 2.4 remain true for the case of shape variation of the rigid punch problem.

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