FORSCHUNG - AUSBILDUNG - WEITERBILDUNG

Bericht Nr. 57

ON THE DEPENDENCE OF THE SOLUTION OF GENERALIZED BOLTZMANN EQUATIONS ON THE SCATTERING CROSS SECTION: THE INVERSE PROBLEM

Bernd Wiesen

UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Arbeitsgruppe Technomathematik
Postfach 3049

6750 Kaiserslautern

September 1991

On the Dependence of the Solution of Generalized Boltzmann Equations on the Scattering Cross Section: The Inverse Problem

Bernd Wiesen
Dept. of Mathematics, AGTM
University of Kaiserslautern
Kaiserslautern, GERMANY

Abstract
In this paper we consider the problem of the reconstruction of the scattering cross section from the solution of a generalized Boltzmann equation.

Key words: Generalized Boltzmann Equation, Differential Equations in Banach Spaces, Inverse Problems

1) Introduction: The Boltzmann equation and scattering cross sections

The evolution of the distribution function of a spatially homogeneous gas consisting of molecules with internal energy is given by:

$$\frac{\partial}{\partial t} f(t,v,\varepsilon_1) = J(\sigma,f,f)(t,v,\varepsilon_1) \quad \text{with initial value } f_0 \in L_1(\mathbb{R}^3 x \mathbb{R}_+). \tag{1.1}$$

In (1.1) we have used the following notations:

$$J(\sigma,f,g) = \frac{1}{2} \int_{\Pi'} \sqrt{1-e_1'-e_2'}\ \sigma(E,e_1,e_2,e_1',e_2',\eta\cdot\eta')\,[f'g'_* + f'_*g' - fg_* - f_*g]\,d\mu \tag{1.2}$$

with :

$\Pi' = \mathbb{R}^3 x \mathbb{R}_+ x \Delta_1 x S_2$ with $\Delta_1 = \{(e_1',e_2') : 0 \le e_1',\ 0 \le e_2' \text{ and } e_1' + e_2' \le 1\}$,

$d\mu = d\Omega(\eta')\,de_1'de_2'\ d\varepsilon_2 dw$,

$$E = \frac{1}{2}|v-w|^2 + \varepsilon_1 + \varepsilon_2 \ , \ c' = \sqrt{2E(1 - e_1' - e_2')} \quad \text{and } e_i = \varepsilon_i/E,\ i=1,2. \tag{1.3}$$

$$v' = \frac{1}{2}(v + w + \eta'c'), \qquad \varepsilon_1' = e_1'E,$$
$$w' = \frac{1}{2}(v + w - \eta'c'), \qquad \varepsilon_2' = e_2'E, \tag{1.4}$$

$$f' = f(t,v',\varepsilon_1'),\quad f'_* = f(t,w',\varepsilon_2'),\quad f_* = f(t,w,\varepsilon_2) \tag{1.5}$$

The function σ in (1.2) is considered to be an element of the function space introduced below. As usual we denote the space of continous functions from a metric space X into a metric space Y by $C(X \to Y)$

Definition 1.1: The set S of scattering cross sections is the set of all measurable real valued functions k defined on $\mathbb{R}_+ x \Delta_1 x \Delta_1 x S_2$ which have the properties:

(i) $k \in C(\mathbb{R}_+ x \Delta_1 \to L_1(\Delta_1 x[-1,1]))$

(ii) $k(E,e,e',x) = k(E,e',e,x)$ and $k(E,e,(e_1',e_2'),x) = k(E,e,(e_2',e_1'),-x)$ a.e. (1.6)

(iii) $e_1 + e_2 = 1 \Rightarrow k(E,e,e',x) = 0$ a.e.

(iv) $\|k\| = \sup_{(E,e)} \int_{\Delta_1 x S_2} |k(E,e,e',x)| \sqrt{1 - e_1' - e_2'}\ de'dx < \infty$

The set of all nonnegative functions in S will be denoted by S_+. We denote the closed unit ball of S by B_1, its boundary by ∂B_1 and the open unit ball by B_1'.

Notation: For any $\sigma \in S$ we denote

$$\sigma_\tau(E,e) = \int_{\Delta_1 x S_2} 2\pi\sigma(E,e,e',x) \sqrt{1 - e_1'^2 - e_2'^2}\; de'dx \tag{1.7}$$

and we introduce $L_{1,1} = \{f \in L_1 : \int_{\mathbb{R}^3 x \mathbb{R}_+} (1+|v|^2+\varepsilon_1)\ |f(v,\varepsilon_1)|\ d\varepsilon_1 dv < \infty\}$.

The aim of this paper is to study the possibility to identify the scattering cross section σ from the behavior of the solutions of (1.1) in $C([0,t_o]\to L_1)$. As a first step in this direction we note a scaling property of such solutions. Suppose we have found a solution $f(\cdot,\sigma)$ of (1.1) in $C([0,t_o]\to L_1)$ for some $t_o > 0$. We define for $\lambda,\mu > 0$

$$g(t) = \lambda f(\lambda\mu t,\sigma)$$

and we get

$$\begin{aligned} g(t) = \lambda f(\lambda\mu t,\sigma) &= \lambda f_o + \lambda \int_0^{\lambda\mu t} J(\sigma,f(s),f(s))\ ds \\ &= \lambda f_o + \int_0^t J(\mu\sigma,\lambda f(\lambda\mu s),f(\lambda\mu s))\ ds = g_o + \int_0^t J(\mu\sigma,g(s),g(s))\ ds \end{aligned} \tag{1.8}$$

which shows, that $g(\cdot)$ solves (1.1) with data λf_o and $\mu\sigma$ in $C([0,t_o/\mu\lambda]\to L_1)$. Because of this property we assume in the following:

$$\|f_o\| = 1 \quad \text{and} \quad \sigma \in \partial B_1.$$

2) Some known results

In this section we collect some results which are needed in the sequel. For the proofs of the following two propositions see Wiesen (1991,1 and 1991,2). We first note that we can split the collision operator J in (1.2) into a gain and a loss part:

$$J(\sigma,f,g) = G(\sigma,f,g) - V(\sigma,f,g)$$

Proposition 2.1: Let σ be in S. Then both $G(\sigma,\cdot,\cdot)$ and $V(\sigma,\cdot,\cdot)$ are mappings from $L_1 x L_1$ into L_1 and there hold the estimates:

$$\|V(\sigma,f,g)\| \le 2\pi\|\sigma\|\,\|f\|\,\|g\| \quad \text{and} \quad \|G(\sigma,f,g)\| \le 2\pi\|\sigma\|\,\|f\|\,\|g\|. \tag{2.1}$$

Moreover we have for any $\varphi \in C^b(\mathbb{R}^3 x \mathbb{R}_+)$, $f,g \in L_1(\mathbb{R}^3 x \mathbb{R}_+)$:

$$\int_{\mathbb{R}^3\times\mathbb{R}_+} \varphi(v,\varepsilon_1)\, J(\sigma,f,g)(v,\varepsilon_1)\, d\varepsilon_1 dv = \tag{2.2}$$

$$\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}_+\times\Pi'} \sqrt{1-e_1'-e_2'}\; \sigma(E,e_1,e_2,e_1',e_2',\eta\cdot\eta')\, [\varphi'_*-\varphi]\, [fg_*+f_*g]\; d\mu\otimes d\varepsilon_1 dv.$$

For any σ in ∂B_1 and $t_o < [4\pi]^{-1}$ there is a unique function $f(\cdot,\sigma) \in C([0,t_o]\to L_1)$ which solves (1.1) with data f_o and σ. If we introduce the following sequence $\{G_n(\sigma)\}$ of functions

$$G_o(\sigma) = f_o$$

$$G_n(\sigma) = \frac{1}{n}\sum_{\mu=0}^{n-1} J(\sigma,G_{n-1-\mu}(\sigma),G_\mu(\sigma)),\; n \geq 1, \tag{2.3}$$

then we have the estimate: $\|G_n(\sigma)\| \leq [4\pi\|\sigma\|]^n$ and the solution of (1.1) in $[-t_o,t_o]$ may be represented as:

$$f(t,\sigma) = \sum_{n=0}^{\infty} t^n\, G_n(\sigma)\quad ,\quad t \in [-t_o,t_o]. \tag{2.4}$$

If σ and f_o are nonnegative functions then for any $t_o > 0$ there is a unique solution $f(\cdot,\sigma) \in C([0,t_o]\to L_1)$ of (1.1) having the property :

$$\forall t \geq 0: \; \|f(t,\sigma)\| = \|f_o\|. \tag{2.5}$$

<u>Definition 2.1</u>: Let σ and f_o be a nonnegative functions in S and $L_1(\mathbb{R}^3\times\mathbb{R}_+)$ respectively. We define for $h \geq 2\pi\|\sigma\|$ the operator

$$Q_h(\sigma,f,g) = J(\sigma,f,g) + \frac{h}{2}\Big\{ f\int_{\mathbb{R}^3\times\mathbb{R}_+} g(w,\varepsilon_2)\, d\varepsilon_2 dw + g\int_{\mathbb{R}^3\times\mathbb{R}_+} f(w,\varepsilon_2)\, d\varepsilon_2 dw \Big\}, \tag{2.6}$$

and the following sequence $\{H_n(\sigma)\}$ of functions

$$H_o(\sigma) = f_o$$

$$H_n(\sigma) = \frac{1}{n\,h}\sum_{\mu=0}^{n-1} Q_h(\sigma,H_{n-1-\mu}(\sigma),H_\mu(\sigma))\quad ,\; \text{if } n \geq 1 . \tag{2.7}$$

<u>Proposition 2.2:</u> Let $\sigma \in S$ and f_o be nonnegative functions. Then each of the functions $H_n(\sigma)$ in (2.6) is nonnegative and we have $\|H_n(\sigma)\| = \|f_o\| = 1$ for any $n \in \mathbb{N}$. The unique solution of (1.1) can be represented as

$$f(t,\sigma) = \sum_{n=0}^{\infty} e^{-ht}\,(1-e^{-ht})^n\; H_n(\sigma)\,, \tag{2.8}$$

where we can choose any $h \geq 2\pi\|\sigma\|$

<u>Remark:</u> For the rest of this paper we assume f_o to be a nonnegative function.

Remark: For both sequences $\{G_n(\cdot), n \geq 0\}$ and $\{H_n(\cdot), n \geq 0\}$, f_o is called the *start value* of the sequence.

3) Injectivity properties of the solution of (1.1)

Proposition 3.1: Let $f(\cdot,\sigma_1)$ and $f(\cdot,\sigma_2)$ be two solutions of (1.1) with data $\sigma_1 \in B_1'$ and $\sigma_2 \in B_1'$ and suppose: $f(\cdot,\sigma_1) \neq f(\cdot,\sigma_2)$ in $C([0,t_o] \to L_1)$, where t_o is an arbitrary positive time. Then the set $\mathbf{M}$ of all times $t \in \mathbb{R}_+$ with the property: $f(t,\sigma_1) = f(t,\sigma_2)$ in L_1 is closed and there is no finite limit point in $\mathbf{M}$.

Proof: The closedness of $\mathbf{M}$ is a direct consequence of the continuity of $f(\cdot,\sigma_1)$ and $f(\cdot,\sigma_2)$, see proposition 2.1.
Assume now, we have a finite limit point $T_1 \in \mathbf{M}$. Then there exists a sequence $\{t_n\}$ in $\mathbf{M}$ with the property:

$$\forall n \in \mathbb{N}: t_n \neq T_1 \qquad \text{and} \qquad \lim_{n\to\infty} t_n = T_1$$

We set $\Psi_o = f(T_1,\sigma_1) = f(T_1,\sigma_2)$ and define the sequences $\{G_n(\sigma_1), n\geq 0\}$ and $\{G_n(\sigma_2), n\geq 0\}$ with start value Ψ_o. In the time intervall $[T_1-[4\pi]^{-1}, T_1+[4\pi]^{-1}]$ the solutions of (1.1) with data σ_1 and σ_2 can be represented as

$$f(t,\sigma_1) = \sum_{n=0}^{\infty} (t-T_1)^n\, G_n(\sigma_1) \quad \text{und} \quad f(t,\sigma_2) = \sum_{n=0}^{\infty} (t-T_1)^n\, G_n(\sigma_2) \tag{3.1}$$

Now the fact: $\forall n \in \mathbb{N}$: $f(t_n,\sigma_1) = f(t_n,\sigma_2)$ yields:

$$\forall k \in \mathbb{N} : \lim_{n\to\infty} \frac{\|f(t_n,\sigma_1)-f(t_n,\sigma_2)\|}{|T_1-t_n|^k} = 0 \tag{3.2}$$

and we get with the help of standard techniques from (3.1) and (3.2):

$$\forall n \in \mathbb{N} : G_n(\sigma_1) = G_n(\sigma_2). \tag{3.3}$$

But (3.3) implies $f(\cdot,\sigma_1) = f(\cdot,\sigma_2)$ in $C([T_1-[4\pi]^{-1}, T_1+[4\pi]^{-1}] \to L_1)$. Now an iteration procedure yields $f(\cdot,\sigma_1) = f(\cdot,\sigma_2)$ in $C([0,t_o] \to L_1)$ which gives a contradiction.

///

An immediate consequence of proposition 3.1 and of the theorem of the continuation of the solution of differential equations in Banach spaces (see e.g. Martin, 1976, chapter 6) is the following

Corollary 3.1: Let σ_1 and σ_2 be two nonnegative functions in S and let $f(\cdot,\sigma_1)$ and $f(\cdot,\sigma_2)$ be the corresponding solutions of (1.1). Then there are equivalent:

(i) There exists a $t_o > 0$ such that $f(\cdot,\sigma_1) = f(\cdot,\sigma_2)$ in $C([0,t_o]\to L_1)$

(ii) $\forall n \in \mathbb{N}: G_n(\sigma_1) = G_n(\sigma_2)$, where the functions $G_n(\cdot)$ are given by (2.3) and for any $t_o > 0$ there holds : $f(\cdot,\sigma_1) = f(\cdot,\sigma_2)$ in $C([0,t_o]\to L_1)$.

Theorem 3.1: Let $\sigma_1, \sigma_2 \in S_+$ be two different scattering cross sections. Let $f(\cdot,\sigma_1)$ and $f(\cdot,\sigma_2)$ be the corresponding solutions of (1.1). We denote $\sigma(\lambda) = \lambda\sigma_1 + (1-\lambda)\sigma_2$, $0\le\lambda\le 1$. Suppose there exists a $t_1 > 0$ such that $f(t_1,\sigma_1) \neq f(t_1,\sigma_2)$. Then, for any $t_o > 0$, the map

$$[0,1] \ni \lambda \to f(\cdot,\sigma(\lambda)) \in C([0,t_o]\to L_1), \text{ with } \sigma(\lambda) = \sigma_1 + \lambda(\sigma_2-\sigma_1),$$

$f(\cdot,\sigma(\lambda))$ being the solution of (1.1) with data f_o and $\sigma(\lambda)$, is injective.

Proof: As a consequence of corollary 3.1 there exists a $N > 0$ such that:

$$G_N(\sigma_1) \neq G_N(\sigma_2),$$

where the functions $G_n(\cdot)$ are given by (2.3). We set $M = \inf\{n : G_n(\sigma_1) \neq G_n(\sigma_2)\}$. Using (2.3) we obtain

$$G_n(\sigma(\lambda)) = \frac{1}{n}\sum_{\mu=0}^{n-1} (1-\lambda)\, J(\sigma_1,G_{n-1-\mu}(\sigma(\lambda)),G_\mu(\sigma(\lambda))) + \lambda\, J(\sigma_2,G_{n-1-\mu}(\sigma(\lambda)),G_\mu(\sigma(\lambda)))$$

which yields

$$G_n(\sigma(\lambda)) = G_n(\sigma_1) = G_n(\sigma_2)\ ,\quad n = 0,1,..,M-1, \quad \text{and}$$

$$G_M(\sigma(\lambda)) = (1-\lambda)\, G_M(\sigma_1) + \lambda\, G_M(\sigma_2). \tag{3.4}$$

Now equation (3.4) implies that we have for any two λ_1 and $\lambda_2 \in [0,1]$:

$$\lambda_1 \neq \lambda_2 \Rightarrow G_M(\sigma(\lambda_1)) \neq G_M(\sigma(\lambda_2))$$

and the assertion follows from corollary 3.1.

///

4) Separation properties of the collision operator $J(\cdot,\cdot,\cdot)$

In this section we use the injectivity criterion of corollary 3.1 to see that there exists for any two different scattering cross sections σ_1 and σ_2 an initial condition f_o such that the corresponding solutions of (1.1) are different, regardless of the observation time t_o. To this end we note that the function G_1 of the sequence (2.3) is given by $G_1(\sigma) = J(\sigma,f,f)$ where f is the start function of the sequence. So all we have to show is that, for any two different σ_1 and σ_2, there is a nonnegative function f such that $J(\sigma_1,f,f) \neq J(\sigma_2,f,f)$.

Lemma 4.1: The family of mappings

$\{S \ni \sigma \;\to\; J(\sigma,f,f)$, f nonnegative function in $L_{1,1}\}$

separates on S.

Proof: Let σ_1 and σ_2 be two different scattering cross sections. Due to property (i) of definition 1.1 there exist *rational* numbers $E_O \in \mathbb{R}_+$ and $e_O=(e_{1O},e_{2O}) \in \Delta_1$ such that

$$k(E_O,e_O,\ .\ ,\ .\ ) : (e',x) \;\to\; \sigma_1(E_O,e_O,e',x) - \sigma_2(E_O,e_O,e',x) \tag{4.1}$$

is not the null function. We have to show : $\exists f \in L_{1,1} : J(k,f,f) \neq 0$ in L_1. Suppose this is not the case. Then we have:

$$\forall\varphi \in C^b(\mathbb{R}^3 x\mathbb{R}_+),\ f \in L_{1,1} \quad : \quad \int_{\mathbb{R}^3 x\mathbb{R}_+} \varphi(v,\varepsilon_1)\ J(k,f,f)(v,\varepsilon_1)\ d\varepsilon_1 dv \;=\; 0 \tag{4.2}$$

Due to the properties of the function k, recall (i), (ii) and (iii) in definition 1.1, the map

$$\mathbb{R}^3 x\mathbb{R}_+ x\mathbb{R}^3 x\mathbb{R}_+ \backslash\{(v,\varepsilon_1,v,\varepsilon_2),\ v \in \mathbb{R}^3,\ \varepsilon_1,\varepsilon_2 \in \mathbb{R}_+\} \ni (v,\varepsilon_1,w,\varepsilon_2) \to$$
$$\int_{\Delta_1 x S_2} k(E,e,e',\eta\cdot\eta')\ [\varphi(v',\varepsilon_1') - \varphi(v,\varepsilon_1)]\ \sqrt{1-e_1'-e_2'}\ \ d\Omega(\eta')de', \tag{4.3}$$

is a continous function in its variables $(v,\varepsilon_1,w,\varepsilon_2)$. Recall that we used in (4.3) the notations (1.3) and (1.4). We set

$$\varepsilon_{1O} = e_{1O}E_O,\quad \varepsilon_{2O} = e_{2O}E_O,\quad v_1 = (r,0,0),\quad v_2 = (-r,0,0)\ ,\qquad r \;=\frac{E_O(1-e_{1O}-e_{2O})}{2}$$

and define for $n \in \mathbb{N}$:

$$g_n(v,\varepsilon) \;= \frac{n}{2}\sqrt{\frac{n}{\pi}}^{\,3}\ [H(\varepsilon-\varepsilon_{1O})\ \exp[-n(|v-v_1|^2+\varepsilon)] + H(\varepsilon-\varepsilon_{2O})\ \exp[-n(|v-v_2|^2+\varepsilon)]], \tag{4.4}$$

where $H(\cdot)$ is the Heaviside function. It can be seen easily that for $n \in \mathbb{N}$ g_n is in $L_{1,1}$. In addition we have

$$\forall n \in \mathbb{N} : \int_{\mathbb{R}^3 x\mathbb{R}_+} g_n(v,\varepsilon_1)\ d\varepsilon_1 dv = 1$$

Because of (4.2) there holds:

$$\forall n \in \mathbb{N}\ ,\ \varphi \in C^b(\mathbb{R}^3 x\mathbb{R}_+): \int_{\mathbb{R}^3 x\mathbb{R}_+} \varphi(v,\varepsilon_1)\ J(k,g_n,g_n)(v.\varepsilon_1)\ d\varepsilon_1 dv = 0. \tag{4.4}$$

Due to the continuity of the mapping (4.3) we get (see e.g. Folland, 1976, Thm. 0.13):

$$\lim_{n\to\infty} \int_{\mathbb{R}^3 x \mathbb{R}_+} \varphi(v,\varepsilon_1)\, J(k,g_n,g_n)(v,\varepsilon_1)\, d\varepsilon_1 dv =$$

$$\frac{1}{4} \int_{\Delta_1 x S_2} k(E_0,e_0,e',\eta_0\cdot\eta')\, \sqrt{1-e_1'-e_2'} \cdot [\varphi(\tfrac{\eta'}{2}\sqrt{2E_0(1-e_1'-e_2')},\, e_1'E_0) - \varphi(v_1,\varepsilon_{10})]\, d\Omega(\eta')de'$$
$$+ \quad \frac{1}{4} \int_{\Delta_1 x S_2} k(E_0,e_0,e',-\eta_0\cdot\eta')\, \sqrt{1-e_1'-e_2'} \cdot [\varphi(\tfrac{\eta'}{2}\sqrt{2E_0(1-e_1'-e_2')},\, e_1'E_0) - \varphi(v_2,\varepsilon_{20})]\, d\Omega(\eta')de' , \qquad (4.5)$$

In (4.5) η_0 denots the unit vector in the direction of v_1. If we use now the symmetry properties of k, see (4.1) and (1.6), we get from (4.4) and (4.5):

$\forall\varphi \in C^b(\mathbb{R}^3 x \mathbb{R}_+)$:

$$0 = \int_{\Delta_1 x S_2} k(E_0,e_0,e',\eta_0\cdot\eta')\, \sqrt{1-e_1'-e_2'} \quad [2\varphi(\tfrac{\eta'}{2}\sqrt{2E_0(1-e_1'-e_2')},\, e_1'E_0) - \varphi(v_1,\varepsilon_{10}) - \varphi(v_2,\varepsilon_{20})]\, d\Omega(\eta')\, de'. \qquad (4.6)$$

To discuss the integral on the right hand side of (4.6) we perform the following changes of integration variables:

$(e_1',e_2') \to (z = e_1'+e_2',\ y = e_1'-e_2')$

$y \to y' = \frac{y}{z}$, $\quad z \to z' = 1-z$, $\quad z \to r = \sqrt{\frac{E_0 z'}{2}}$,

$y' \to y'' = -(E_0-2r^2)(1+y')$

and we obtain:

$$0 = \int_{G(E_0)} k(E_0,(e_{10},e_{20}),(e_1'(x,z),e_2'(x,z)),\eta_0\cdot\eta'(x))\, \sqrt{\frac{8}{E_0^5}} \cdot [\varphi(x,z) - \tfrac{1}{2}(\varphi(v_1,\varepsilon_{10})-\varphi(v_2,\varepsilon_{20}))]\ dz \otimes d^3x , \qquad (4.7)$$

where we have: $G(E_0) = \{(x,z) \in \mathbb{R}^3 x \mathbb{R}_+ : |x|^2 \le E_0/2 \text{ and } 0 \le z \le E_0-2|x|^2\}$,

$e_1'(x,z) = \frac{z}{E_0}$, $\quad e_2'(x,z) = \frac{E_0-2|x|^2-z}{E_0}$, $\quad \eta'(x) = \frac{x}{|x|}$.

We consider the following measures on $G(E_0)$ and its induced Borel σ algebra.

$$d\mu_1(x,z) = \sqrt{\frac{8}{E_0^5}}\ k(E_0, e_{10},e_{20},e_1'(x,z),e_2'(x,z),\eta_0\cdot\eta'(x))\ dz \otimes d^3x \qquad (4.8)$$

and

$$d\mu_2(x,z) = \frac{\pi}{2}\left[\int_{\Delta_1 x[-1,1]} k(E_o,e_o,e',x)\sqrt{1-e_1'-e_2'}\; dxde'\right] \qquad (4.9)$$

$$\cdot\ [\delta(x-v_1)\otimes\delta(z-\varepsilon_{1o}) + \delta(x-v_2)\otimes\delta(z-\varepsilon_{2o})]$$

and note that (4.7) implies:

$$\forall\varphi \in C^b(\mathbb{R}^3 x \mathbb{R}_+): \int_{G(E_o)} \varphi(x,z)\, d\mu_1(x,z) - \int_{G(E_o)} \varphi(x,z)\, d\mu_2(x,z) = 0 \qquad (4.10)$$

Because of (4.1), $d\mu_1(x,z)$ is not the zero measure. A comparison of (4.8) and (4.9) shows that $d\mu_1$ is absolutely continous with respect to the 4 dimensional Lebesque measure whereas $d\mu_2$ is singular to this measure. So both measures are different.

Now, using that the dual space of the real measures on $G(E_o)$ is isomorphic to $C^b(G(E_o))$, we get:

$$\exists\varphi' \in C^b(G(E_o)) : \int_{G(E_o)} \varphi'(x,z)\, d\mu_1(x,z) - \int_{G(E_o)} \varphi'(x,z)\, d\mu_2(x,z) \neq 0 \qquad (4.11)$$

A comparison of (4.10) and (4.11) shows that we get a contradiction, provided we can show that the restriction map

$$R(E_o) : C^b(\mathbb{R}^3 x \mathbb{R}_+) \to C^b(G(E_o)) \quad : \quad \varphi \to \varphi\big|_{G(E_o)}$$

is surjective.

Due to the compactness of $G(E_o)$ there exists a $r > 0$ such that

$$G(E_o) \subset B_4(0,r) = \{(x,z) \in \mathbb{R}^3 x \mathbb{R}_+ : |x|^2+z \le r\}.$$

Now Tietze's extension theorem (see Reed-Simon, 1980, Thm. IV.11) yields the surjectivity of the mapping

$$R_r : C^b(B_4(0,r)) \to C^b(G(E_o)) \quad : \quad \varphi \to \varphi\big|_{G(E_o)} .$$

The surjectivity of the map $R(E_o)$ is now a direct consequence of Urysohn's lemma (see Rudin, 1970, Thm. 2.12).

///

Remark: The set $\sqrt{\mathbb{Q}_+} = \{x \in \mathbb{R} : x^2 \in \mathbb{Q}\}$ is countable.

<u>Theorem</u> <u>4.1:</u> There exists a sequence $\{f_n\}$ of nonnegative functions in $L_{1,1}$ with the properties

(i) $\forall n \in \mathbb{N} : \int_{\mathbb{R}^3 \times \mathbb{R}_+} f_n(v,\varepsilon_1)\, d\varepsilon_1 dv \leq 1$ (4.12)

(ii) $\forall n \in \mathbb{N} : \int_{\mathbb{R}^3 \times \mathbb{R}_+} v\, f_n(v,\varepsilon_1)\, d\varepsilon_1 dv = 0$ (4.13)

(iii) $\exists C \in \mathbb{R}_+ : \forall n \in \mathbb{N} : \int_{\mathbb{R}^3 \times \mathbb{R}_+} (1 + |v|^2 + \varepsilon_1)\, f_n(v,\varepsilon_1)\, d\varepsilon_1 dv \leq C$, (4.14)

such that the set of mappings $\{S \ni \sigma \to J(\sigma,f_n,f_n) \in L_1\}$ separates on S.

Proof: We start with the sequence $\{g_n\}$ of functions introduced in (4.4). These functions are parametrized by the the values r, ε_{10}, ε_{20} . Therefore we write here $g_n(r,\varepsilon_{10},\varepsilon_{20})(v,\varepsilon)$. We have the following properties:

(a) $\int_{\mathbb{R}^3 \times \mathbb{R}_+} g_n(r,\varepsilon_{10},\varepsilon_{20})(v,\varepsilon)\, d\varepsilon dv = 1$ for arbitrary $n,r,\varepsilon_{10},\varepsilon_{20}$

(b) $\int_{\mathbb{R}^3 \times \mathbb{R}_+} v\, g_n(r,\varepsilon_{10},\varepsilon_{20})(v,\varepsilon)\, d\varepsilon dv = 0$ for arbitrary $n,r,\varepsilon_{10},\varepsilon_{20}$

(c) $\int_{\mathbb{R}^3 \times \mathbb{R}_+} (1 + |v|^2 + \varepsilon)\, g_n(r,\varepsilon_{10},\varepsilon_{20})(v,\varepsilon) d\varepsilon dv = 1 + r^2 + \frac{1}{2}(\varepsilon_{10}+\varepsilon_{20}) + \frac{5}{2n}$,

because we have $v_1=(r,0,0)$ und $v_2=(-r,0,0)$.

We define for arbitrary but fixed $C > 0$ the functions:

$$f_n(r,\varepsilon_{10},\varepsilon_{20}, C) = \lambda(r,\varepsilon_{10},\varepsilon_{20},C)\, g_n(r,\varepsilon_{10},\varepsilon_{20}) \quad \text{with}$$

$$\lambda(r,\varepsilon_{10},\varepsilon_{20},C) = \min\left[1 , \frac{C}{r^2 + \frac{1}{2}(\varepsilon_{10}+\varepsilon_{20}) + \frac{7}{2}}\right]$$

Obviously the set $M_S = \{ f_n(r,\varepsilon_{10},\varepsilon_{20},C),\ n \in \mathbb{N},\ r \in \sqrt{\mathbb{Q}_+},\ \varepsilon_{10},\varepsilon_{20} \in \mathbb{Q}_+\}$ is countable and the functions f_n have the properties (4.12), (4.13) and (4.14). Moreover we can see easily from the proof of lemma 4.1 that the set of mappings

$\{S \ni \sigma \to J(\sigma,f,f) \in L_1,\ \ f \in M_S\}$

separates on S.

///

<u>Defintion 4.1:</u> We call a sequence $\{h_n\}$ of nonnegative functions in $L_{1,1}$ a *separating sequence*, if it has the properties mentioned in theorem 4.1. The constant C occuring in (4.14) is called *energy bound* of the sequence $\{h_n\}$.

Notation: As usual we denote the space of the bounded sequences over some Banach space X by $l_\infty(X)$. It is equipped with the norm

$$l_\infty \ni y \to \|y\|_\infty = \sup_{n \in \mathbb{N}} \|y_n\|_X .$$

Suppose now we have a separating sequence $\{h_n\}$. Then theorem 4.1 motivates the study of the following mapping

$$S_+ \ni \sigma \to F(\sigma) = (f_1(\cdot,\sigma), f_2(\cdot,\sigma), \cdot , \cdot) \in l_\infty(C([0,t_o] \to L_1)) . \tag{4.15}$$

In (4.15) t_o is any positive time and the functions $f_i(\cdot,\sigma)$ are the solutions of (1.1) with data σ and h_i. It is a direct consequence of theorem 4.1 that this mapping is injective. So we study the inverse of the mapping (4.15) on the image of S_+ which will be denoted by $F(S_+)$.

5) On the inverse problem

Lemma 5.1: Let f_1 and f_2 be two nonnegative functions in L_1 with the following property: there are constants C_1, $C_2 > 0$ such that:

$$\forall R > 0 : \int_{B_4{}^c(R)} f_i(v,\varepsilon)\, d\varepsilon dv \le \frac{C_i}{1 + R^2} , \ i = 1,2 , \tag{5.1}$$

where we have $B_4{}^c(R) = \{(v,\varepsilon) : |v|^2 + \varepsilon > R^2\}$. Let $k \in S_+$ a function with the property:

$$\exists D > 0 : \forall E \le 2D^2, \ e \in \Delta_1 : k(E,e,\cdot,\cdot) = 0 . \tag{5.2}$$

Then we have :

$$\|J(k,f_1,f_2\| \le \frac{4\pi\|k\|}{1 + D^2} (C_1\|f_2\| + C_2\|f_1\|) . \tag{5.3}$$

Proof: Using proposition 2.1 and the nonnegativity of f_1, f_2 and k we get

$$\begin{aligned} \|J(k,f,g)\| &\le 2 \int_{\mathbb{R}^3 x \mathbb{R}_+} V(k,f,g)(v,\varepsilon_1)\, d\varepsilon_1 dv \\ &= \int_{\mathbb{R}^3 x \mathbb{R}_+ x \mathbb{R}^3 x \mathbb{R}_+ x \Delta_1 x S_2} k(E,e,e',\eta\cdot\eta') \sqrt{1-e_1'-e_2'}\ [f_1(v,\varepsilon_1) f_2(w,\varepsilon_2) \\ &\qquad + f_2(v,\varepsilon_1) f_1(w,\varepsilon_2)] d\Omega(\eta') de' d\varepsilon_2 dw d\varepsilon_1 dv. \end{aligned} \tag{5.4}$$

Recall that we have in (5.4):

$$E = \frac{1}{2}|v-w|^2 + \varepsilon_1 + \varepsilon_2 \le |v|^2 + \varepsilon_1 + |w|^2 + \varepsilon_2 \tag{5.5}$$

The set $\Gamma = \mathbb{R}^3 x \mathbb{R}_+ x \mathbb{R}^3 x \mathbb{R}_+$ can be decomposed into four disjoint sets:

$$\Gamma = B_4(D)xB_4(D) \cup B_4{}^c(D)xB_4(D) \cup B_4(D)xB_4{}^c(D) \cup B_4{}^c(D)xB_4{}^c(D) .$$

Now (S.3), (S.4) and (S.5) yield:

$$\int_{B_4(D)xB_4(D)} k_\tau(E,e) \, [f_1(v,\varepsilon_1)f_2(w,\varepsilon_2) + f_2(v,\varepsilon_1)f_1(w,\varepsilon_2)] \; d\varepsilon_2 dw d\varepsilon_1 dv = 0 , \qquad (5.6)$$

where the function k_τ is given by (1.7). As a consequence of (5.6) we get:

$$2 \int_{\mathbb{R}^3 x \mathbb{R}_+} V(k,f_1,f_2) \; d\varepsilon_1 dv \le 4\pi \|k\| \left[\int_{B_4{}^c(D)} f_1(v,\varepsilon_1) \; d\varepsilon_1 dv \int_{\mathbb{R}^3 x \mathbb{R}_+} f_2(w,\varepsilon_2) \; d\varepsilon_2 dw \right.$$

$$\left. + \int_{B_4{}^c(D)} f_2(v,\varepsilon_1) \; d\varepsilon_1 dv \int_{\mathbb{R}^3 x \mathbb{R}_+} f_1(w,\varepsilon_2) \; d\varepsilon_2 dw \right].$$

Now (5.3) is a direct consequence of (5.1).

///

Motivated by this lemma we introduce a shift operrator on a subset of S.

<u>Definition 5.1:</u> We denote by S' the set of all scattering cross sections k with the property:

$\forall\ e,e' \in \Delta_1,\ x \in [0,1] : \quad k(0,e,e',x) = 0.$

$S'_{+,1}$ denotes the set of all nonnegative $k \in S'$ with $\|k\| = 1$. We define the s-hift operator T_D:

T_D: $S' \to S : \sigma \to \sigma_D$ with

$$\sigma_D(E,e,e',x) = \begin{cases} 0 & , \quad 0 \le E \le 2D^2 \\ \sigma(E-2D^2,e,e',x) & , \quad \text{else} . \end{cases}$$

<u>Theorem 5.1:</u> Let t_0 be an arbitrary, but fixed positive time and let ε and C be positive real numbers. There exists a positive constant $D(t_0,\varepsilon,C)$ such that we have for all separating sequences $\{h_n\}$ with energy bound C and all $k \in S'_{+,1}$ and $\sigma \in B_1$:

$$\|F(\sigma+k_D) - F(\sigma)\|_\infty < \varepsilon. \qquad (5.7)$$

<u>Remark:</u> Recall that we have in (5.7):

$$\|F(\sigma+k_D) - F(\sigma)\|_\infty = \sup_{n\in\mathbb{N}} \sup_{t\in[0,t_o]} \|f_n(t,\sigma+k_D)-f(t,\sigma)\|$$

To prove theorem 5.1 we need a technical lemma:

Lemma 5.1: Let c and d be nonnegative real numbers with $c \le 2$. Suppose we have a sequence $\{a_n\}$ of nonnegative numbers satisfying

(i) $a_o \ge 0$

(ii) $a_n \le \frac{c}{n} \sum_{\mu=0}^{n-1} a_\mu + d\ ,\ n \ge 1.$

Then there holds the estimate: $a_n \le (n+1)a_o + nd$ for $n \ge 0$.

Proof of the lemma: Obviously the above inequality is true for $n = 0$. Now suppose it is true for all $k \in \{0,1,..,n-1\}$, $n\ge1$. Then we have

$$a_n \le \frac{c}{n} \sum_{\mu=0}^{n-1} [(\mu+1)a_o + \mu d] + d$$

$$= \frac{c}{n}\frac{n+1}{2} n\, a_o + \frac{c}{n}\frac{n}{2}(n-1)\, d + d$$

$$\le (n+1)a_o + (n-1)d + d \quad , \quad \text{because we assumed : } c \le 2,$$

which implies that the above estimate is true for all $k \in \{0,1,...,n\}$

///

Proof of theorem 5.1: We first note that we have for any $k \in S'_{+,1}$:

$\|\sigma + k_D\| = \|\sigma\| + 1$ and $\|\sigma + k_D - \sigma\| = 1.$

Without any restriction we may assume that the null function is not an element of our separating sequence so that we have:

$$\int_{\mathbb{R}^3\times\mathbb{R}_+} h_k(v,\varepsilon_1)\, d\varepsilon_1 dv = a_k > 0, \quad k \in \mathbb{N}.$$

For $h = 8\pi(\|\sigma\|+1)$ we can use the series representation (2.8) of the solution of (1.1):

$$f_k(t,\sigma') = \sum_{n=0}^{\infty} a_k \exp[-ha_k t](1-\exp[-ha_k t])^n H_{nk}(\sigma') \quad , \quad \sigma' \in \{\sigma,\sigma+k_D\}\ . \tag{5.8}$$

Note that we have used the scaling property (1.8). The functions $\{H_{nk}(\sigma'),\ n \in \mathbb{N}\}$ in (5.8) have to be calcluated as shown in (2.7) and use h_k/a_k as start function. Using this formula we can calculate recursively the difference $\Delta H_{nk} = H_{nk}(\sigma+k_D) - H_{nk}(\sigma)$:

$$\begin{aligned}\Delta H_{nk} &= H_{nk}(\sigma+k_D) - H_{nk}(\sigma)\\ &= \frac{1}{nh}\sum_{\mu=0}^{n-1} J(k_D,H_{n-1-\mu,k}(\sigma),H_{\mu,k}(\sigma))\\ &= \frac{1}{nh}\sum_{\mu=0}^{n-1} J(\sigma+k_D,H_{n-1-\mu,k}(\sigma)+H_{n-1-\mu,k}(\sigma+k_D),\Delta H_{\mu,k}) + h\Delta H_{\mu,k}\end{aligned} \tag{5.9}$$

If we use now (2.2), proposition 2.2 and the nonnegativity of $H_{nk}(\sigma')$, $\sigma' \in \{\sigma, \sigma+k_D\}$, we obtain:

$$\forall\ k \in \mathbb{N},\ n \in \mathbb{N} : \int_{\mathbb{R}^3\times\mathbb{R}_+} (1+|v|^2+\varepsilon_1)\ H_{nk}(\sigma')\ d\varepsilon_1 dv \ \le\ C/a_k \tag{5.10}$$

$$\text{which implies} \quad \int_{B_4{}^c(D)} H_{nk}(\sigma')(v,\varepsilon_1)\ d\varepsilon_1 dv \le \frac{C}{a_k(1+D^2)} .$$

The constant C in (5.10) is the energy bound of the separating sequence. From (5.10) we can see that there holds for any $k \in S'_{+,1}$:

$$\|J(k_D,H_{n-1-\mu}(\sigma),H_\mu(\sigma)\| \ \le\ \frac{8\pi C}{a_k(1+D^2)} .$$

If we use this estimate in (5.9) we get:

$$\|\Delta H_{n,k}\| \ \le\ [\ \frac{1}{n}\sum_{\mu=0}^{n-1} 2\|\Delta H_{\mu,k}\|_o\ ] \ +\ \frac{C}{a_k(1+D^2(1+\|\sigma\|)} \tag{5.11}$$

If use now (5.11) and (5.8) and lemma 5.1 we get with the help of the simple formula

$$\sum_{n=0}^{\infty} nq^n \ =\ \frac{q}{(1-q)^2} \ :$$

$$\begin{aligned}\|f_k(t,\sigma+k_D)-f_k(t,\sigma)\| &\le \frac{C}{(1+\|\sigma\|)(1+D^2)}\ [\exp[8\pi a_k(1+\|\sigma\|)\,t] - 1]\\ &\le \frac{C}{(1+\|\sigma\|)(1+D^2)}\ [\exp[8\pi(1+\|\sigma\|)t] - 1]\end{aligned} \tag{5.12}$$

If we choose now

$$D(\varepsilon,t_o,C) \ >\ \sqrt{\max\left[0, \frac{C\,[\exp[16\pi t] - 1]}{2} - 1\right]}$$

(5.7) is a direct consequence of (5.12).

///

6) Conclusions

In this paper we have considered the problem of the reconstruction of the scattering cross from the knowledge of the solution of a generalized Boltzmann equation. To this end we first introduced a suitable function space for the scattering cross sections and we discussed some injectivity properties of the solution of the Boltzmann equation.
In the next step we have shown that there is a countable family of initial conditions such that the collision operator separates on our function space. As a consequence of this result we have proposed the function space $l_\infty(C([0,t_0]\to L_1))$ as solution space for the reconstruction of σ. In the last section we have shown the inverse of the mapping, which assigns to a given σ a sequence of solutions of the generalized Boltzmann equation, is discontinous. This shows that our inverse problem is *ill posed.*

Acknowledgement: The author wishes to thank Dr. Hans Babovsky for fruitful discussions.

References

Folland, G.B. 1976 *Introduction to Partial Differential Equations*, Princeton University Press

Martin, R.H. 1976 *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley & Sons

Reed, M., Simon B. 1980 *Functional Analysis*, Academic Press

Rudin, W. 1970 *Real and Complex Analysis*, McGraw-Hill

Wiesen, B. 1991,1 *Zur Abhängigkeit verallgemeinerter Boltzmann-Gleichungen von Streuquerschnitt*, PhD thesis, FB Mathematik, Universität Kaiserslautern

Wiesen, B. 1991,2 On a Phenomenological Generalized Boltzmann Equation, preprint Nr. 203, FB Mathematik, Universität Kaiserslautern