Algebraic Aspects of Controllability for AR-Systems

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This paper examines some aspects of controllability for behaviour systems in AR-representation. The approach is based on the theory of dipolynomial modules and dipolynomial matrices. Explicit criteria for controllability in terms of the representing matrices are derived and an effective test algorithm is given. Furthermore, controllability indices are introduced for AR-systems. The new theory is compared to existing index concepts in the literature on linear systems.

1 Introduction

In the recent years J.C. Willems developed in a series of papers a general theory of dynamical behaviour systems (see e.g. Willems (1986a, 1986b, 1987, 1988, 1991)). In this framework controllability is defined as an intrinsic system property which does neither depend on special dynamical properties like linearity, finite dimensionality etc. nor on the model representation. In particular, the considered models are not necessarily state-space models. However, in Willems (1991) it is shown that the general concept coincides with the classical controllability concept if the systems are time-invariant, linear and finite dimensional state-space systems.
In this paper our starting point are linear time-invariant behaviour systems which in the case of instantaneously specified behaviour (in continuous time) resp. complete behaviour (in discrete time) can always be represented as autoregressive (AR)-models. Controllability of those systems can be characterized by the relative right primeness (RRP) of the representing polynomial resp. dipolynomial matrices (cf. Willems (1991)). In the polynomial case explicit equivalent criteria for RRP in terms of the coefficients of the representing matrices are given in the paper of Emre and Silverman (1977). In particular, the structure algorithm of Silverman (1969) can be used as an efficient procedure to test controllability of AR-systems with time domain $T = \mathbb{Z}_+, \mathbb{R}_+$ resp. $\mathbb{R}$. In this paper we extend these results to dipolynomial AR-representations associated to the time domain $T = \mathbb{Z}$ and derive explicit controllability test algorithms for AR-representations.
Furthermore we apply the module theoretic concepts introduced for behaviour systems in Hoffmann and Prätzel-Wolters (1992) to define a list of algebraic controllability indices for linear dynamical systems in AR-representation. Our approach is a straightforward extension of the characterization of controllability indices as minimal indices of the F[s]-modules $\ker(sI - A, B)$ in the state-space setting. It covers Fagnani's general

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geometric concept of controllability indices (cf. Fagnani (1991)) if the system class is restricted to linear, time-invariant complete behaviour systems with time axis $T = \mathbb{Z}$.
Section 2 contains some preliminary remarks concerning the ring of dipolynomials, dipolynomial modules and dipolynomial matrices.
In section 3 we shortly summarize some basic results from the theory of behaviour systems.
Section 4 deals with the derivation of controllability criteria for behaviour systems in AR-representation. After a summary of criteria for RRP of polynomial matrices we derive controllability tests for dipolynomial representations.
Finally, in section 5 controllability indices are defined for linear systems in AR-representation. A characterization of controllability via the controllability indices and an effective algorithm for their computation is given. Moreover, in the discrete-time case we establish a tight connection between degree-preserving isomorphisms of the associated modules of return to zero of two AR-systems and bicausal isomorphisms of their behaviours; in particular, this indicates that our index list is equal to the Fagnani list for $T = \mathbb{Z}$.

2 Preliminaries: Dipolynomials

Let F denote any field, $F[s]$ the ring of polynomials in the indeterminate s with coefficients in F, $F(s)$ the field of rational functions with coefficients in F and

$$F[s, s^{-1}] = \left\{ \alpha_L s^L + \ldots + \alpha_\ell s^\ell,\ L, \ell \in \mathbb{Z}, \ell \leq L, \alpha_k \in F \text{ for } k \in \{\ell, \ldots, L\} \right\} \tag{2.1}$$

the ring of dipolynomials with coefficients in F. It is well known that $F[s]$ is an euclidian ring with respect to the degree function

$$deg : \begin{array}{ccc} F[s] & \longrightarrow & \mathbb{N} \\ \alpha_L s^L + \ldots + \alpha_\ell s^\ell & \longrightarrow & L \end{array} \tag{2.2}$$

The units in $F[s]$ are the nonzero constants $\alpha \neq 0$, $\alpha \in F$. Modifying (2.2) for $F[s, s^{-1}]$ in the following way:

$$ddeg : \begin{array}{ccc} F[s, s^{-1}] & \longrightarrow & \mathbb{N} \\ \alpha_L s^L + \ldots + \alpha_\ell s^\ell & \longrightarrow & L - \ell \end{array} \tag{2.3}$$

we obtain:

2.1 Lemma:

$(F[s, s^{-1}], \mathrm{ddeg})$ is an euclidean ring. The units in $F[s, s^{-1}]$ are the elements of the form αs^d, $d \in \mathbb{Z}$, $a \in F$, $\alpha \neq 0$. □

Note that $\deg(s^L) = L$ for $L \in \mathbb{N}$, while $\mathrm{ddeg}(s^L) = 0$ for $L \in \mathbb{Z}$. Let further $F^{g\times q}[s]$ resp. $F^{g\times q}[s,s^{-1}]$ denote the set of $g \times q$ polynomial resp. dipolynomial matrices. The units in the rings $F^{n\times n}[s]$, $F^{n\times n}[s,s^{-1}]$ are called unimodular matrices. An easy calculation shows:

$$\begin{array}{rcl} R(s) \in F^{n\times n}[s] \text{ unimodular} & \Leftrightarrow & \det(R) = \alpha,\ \alpha \in F,\ \alpha \neq 0 \\ R(s) \in F^{n\times n}[s,s^{-1}] \text{ unimodular} & \Leftrightarrow & \det(R) = \alpha s^d,\ \alpha \neq 0,\ \alpha \in F,\ d \in \mathbb{Z} \end{array}$$

For dipolynomial matrices the notions of common right divisors, greatest common right divisors (gcrd) and relative right primeness (RRP) are defined analogous to the corresponding polynomial notions. Also the proof of the following result is completely analogous to the polynomial case:

2.2 Proposition:

Two dipolynomial matrices $P(s,s^{-1}) \in F^{k\times m}[s,s^{-1}]$, $Q(s,s^{-1}) \in F^{\ell\times m}[s,s^{-1}]$, $k+\ell > m$ are RRP if and only if

$$Sm_R = \begin{pmatrix} I_m \\ 0 \end{pmatrix} \text{ for } R := \begin{pmatrix} P(s,s^{-1}) \\ Q(s,s^{-1}) \end{pmatrix}$$

where Sm_R denotes the Smith-Form of R over $F[s,s^{-1}]$ (cf. Newman (1972)). □

To every dipolynomial matrix $G(s,s^{-1}) \in F^{k\times n}[s,s^{-1}]$,

$$G(s,s^{-1}) = \begin{pmatrix} g_1(s,s^{-1}) \\ \vdots \\ g_k(s,s^{-1}) \end{pmatrix} = \begin{pmatrix} \alpha^1_{v_1} \cdot s^{n_1} + \ldots + \alpha^1_0 \cdot s^{n_1 - v_1} \\ \vdots \\ \alpha^k_{v_k} \cdot s^{n_k} + \ldots + \alpha^k_0 \cdot s^{n_k - v_k} \end{pmatrix} \tag{2.4}$$

there is associated the free $F[s,s^{-1}]$-module of vector dipolynomials M_G spanned by the rows of G:

$$\begin{array}{rcl} M_G & := & F^{1\times k}[s,s^{-1}]G(s,s^{-1}) \\ \dim_{F[s,s^{-1}]} M_G & = & \mathrm{rank}_{F[s,s^{-1}]} G(s,s^{-1}) \end{array} \tag{2.5}$$

Conversely, there exists for every free $F[s,s^{-1}]$-module $M \subset F^{1\times n}[s,s^{-1}]$ with $\dim_{F[s,s^{-1}]} M = k$ a matrix $G(s,s^{-1}) \in F^{k\times n}[s,s^{-1}]$ of $F[s,s^{-1}]$-rank k such that $M = M_G$.
Matrices $G(s,s^{-1})$ with $F[s,s^{-1}]$-linear independent rows are called bases for the module M_G. For those basis matrices we have:

$$M_{G_1} = M_{G_2} \quad \Leftrightarrow \quad \exists T(s) \in F^{k\times k}[s,s^{-1}] \text{ unimodular } s.t. \quad G_1 = TG_2$$

The extension of the scalar degree function ddeg to the vector case associates degree-structures to submodules $M \subset F^{1\times n}[s, s^{-1}]$:

$$ddeg: \begin{array}{ccc} F^{1\times n}[s,s^{-1}] & \longrightarrow & \mathbb{N} \\ \alpha_L s^l + \ldots + \alpha_\ell s^\ell & \longrightarrow & L-\ell \end{array}$$

For a matrix $G(s, s^{-1})$ as in (2.4) the numbers

$$\begin{array}{rcl} v_i & := & \text{ddeg } g_i(s,s^{-1}),\ i \in \underline{k} \\ v & := & \sum_{i=1}^k v_i \end{array} \tag{2.6}$$

are called the dipolynomial indices and the dipolynomial order of G.

2.3 Definition:

A matrix $G(s, s^{-1}) \in F^{k\times n}[s, s^{-1}]$ is called a dipolynomial minimal basis of a $F[s, s^{-1}]$-module $M \subset F^{1\times n}[s, s^{-1}]$ if

(i) G is a basis of M.

(ii) The dipolynomial order of G is minimal among all dipolynomial bases of M. □

Associated with $G(s, s^{-1})$ as in (2.4) are the highest row coefficient resp. lowest row coefficient matrix

$$[G]_h^r := \begin{pmatrix} \alpha_{v_1}^1 \\ \vdots \\ \alpha_{v_k}^k \end{pmatrix} \in F^{k\times n} \ni [G]_\ell^r := \begin{pmatrix} \alpha_0^1 \\ \vdots \\ \alpha_0^k \end{pmatrix} \tag{2.7}$$

Hoffmann and Prätzel-Wolters (1992) contains several equivalent conditions for the minimality of a dipolynomial basis. We only quote two conditions here which will be applied in the sequel; it should be mentioned that these conditions are also derived in Willems (1991) in a matrix theoretical framework.

2.4 Theorem:

Let $G(s, s^{-1})$ of the form (2.4) be a dipolynomial basis of M_G. Equivalent are:

(i) G is a minimal basis for M_G.

(ii) $\text{rank}_F[G]_h^r = k$ and $\text{rank}_F[G]_\ell^r = k$.

(iii) Let $f(s, s^{-1}) = (f_1, \ldots, f_{\binom{n}{k}})$, where the f_i are the $k \times k$-minors of G, then

$$v := \sum_{i=1}^k v_i = \text{ddeg}(f) \qquad \square$$

Moreover, the dipolynomial indices do not depend - up to ordering - on the specific minimal basis but only on the module M. Hence one can define the dipolynomial indices and the dipolynomial order of a k-dimensional submodule $M \subset F^{1\times n}[s, s^{-1}]$ as the (decreasingly ordered) indices resp. order of any dipolynomial minimal basis of M.

3 Preliminaries: Behaviour systems

In the recent years J.C. Willems developed in a series of papers a general theory of dynamical behaviour systems $\Sigma = (T, W, \mathcal{B})$ with time axis $T \subseteq \mathbb{R}$, signal alphabet W and behaviour $\mathcal{B} \subseteq W^T$ (see e.g. Willems (1986a, 1986b, 1987, 1988, 1991)).
Σ is called time invariant if T is an additive subgroup of $\mathbb{R}$ and $\mathcal{B}$ is invariant with respect to all t-shifts

$$\sigma^t : W^T \to W^T,\ w(\hat{t}) \ \to\ w(\hat{t}+t),\ t \in T$$

A time invariant system Σ with time axis $T = \mathbb{Z}$ or $T = \mathbb{R}$ is called controllable if, for every w_1 and w_2 in $\mathcal{B}$, there exists $0 \leq t \in T$ and $w \in \mathcal{B}$ such that $w^- = w_1^-$ and $(\sigma^t w)^+ = w_2^+$, where $w^- := w\mid_{(-\infty,0)\cap T}$, $w^+ := w\mid_{[0,\infty)\cap T}$.
Σ is said to be complete if

$$\{w \in \mathcal{B}\} \ \Leftrightarrow\ \left\{w\mid_{[t_1,t_2]} \in \mathcal{B}\mid_{[t_1,t_2]},\ \forall t_1, t_2 \in T,\ -\infty < t_1 \leq t_2 < \infty\right\}.$$

In Willems (1991) it is shown that every linear time-invariant complete system
$\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B})$ has an autoregressive (AR)-representation:

$$\begin{array}{rcl} \mathcal{B} & = & \mathcal{B}(P) = \ker P(\sigma, \sigma^{-1}) \\ P(s, s^{-1}) & = & P_L s^L + \ldots + P_\ell s^\ell \in \mathbb{R}^{p\times q}[s, s^{-1}] \end{array} \tag{3.1}$$

The operator

$$P(\sigma, \sigma^{-1}) : \begin{array}{ccc} (\mathbb{R}^q)^{\mathbb{Z}} & \longrightarrow & (\mathbb{R}^p)^{\mathbb{Z}} \\ w(t) & \longrightarrow & P_L w(t+L) + \ldots + P_\ell w(t+\ell) \end{array} \qquad , t \in \mathbb{Z}$$

is called a dipolynomial shift operator. If $\ell \geq 0$ then $P(\sigma, \sigma^{-1})$ is polynomial and denoted by $P(\sigma)$. q denotes the dimension of the signal alphabet space $W = \mathbb{R}^q$, whereas p, the number of equations representing $\mathcal{B}$, is flexible. However, among all dipolynomial matrices $P(s, s^{-1})$ satisfying (3.1) there exist those with full row rank. They are unique up to multiplication from the left by unimodular matrices $U(s, s^{-1})$.
Willems (1991) calls a full row rank matrix P a minimal lag description, if among all full row rank AR-representations its total lag, i.e. the sum of the row degrees of P, is as small as possible.
Moreover, in the case of time axis $T = \mathbb{Z}_+$, $\mathbb{R}_+$ or $\mathbb{R}$ we consider analogous polynomial AR-representations with:

$$\mathcal{B} = \ker P(\sigma) \text{ resp. } \mathcal{B} = \ker P(\frac{d}{dt}) \tag{3.2}$$

where $P(s) \in \mathbb{R}^{p\times q}[s]$.
In the literature different signal spaces for continuous time systems are discussed (see e.g. Willems (1991), Blomberg and Ylinen (1983)). If we consider e.g. $\mathcal{B} \subseteq C^\infty(T, \mathbb{R}^q)$ for $T = \mathbb{R}, \mathbb{R}_+ = (0, \infty)$ it can be shown that the dipolynomial parametrization results completely carry over to the cases $T = \mathbb{Z}_+$, $\mathbb{R}_+$ or $\mathbb{R}$; the dipolynomial concepts only have to be replaced by the corresponding polynomial notions.

4 Controllability Criteria for Behaviour Systems in AR-Representation

Whether or not a behaviour system in AR-representation is controllable can be read off from the behavioural equations:

4.1 Theorem [Willems (1991)]

Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(P))$ a dynamical system in AR-representation with $P(s, s^{-1}) \in \mathbb{R}^{p\times q}[s, s^{-1}]$ of full row rank. Then the following conditions are equivalent:

(i) Σ is controllable.

(ii) $\text{rank}_{\mathbb{C}} P(\lambda, \lambda^{-1}) = p$ for all $0 \neq \lambda \in \mathbb{C}$

(iii) $Sm_{P(s,s^{-1})} = [I_p \quad 0_{p\times(q-p)}]$ □

4.2 Remark:

For $\underline{T = \mathbb{Z}_+, \mathbb{R}_+ \text{ resp. } \mathbb{R}}$ Theorem 4.1 remains true if we replace $P(s, s^{-1})$ by a polynomial matrix $P(s) \in \mathbb{R}^{p\times q}[s]$ and require condition (ii) for all $\lambda \in \mathbb{C}$, i.e.

$$\text{rank}_{\mathbb{C}} P(\lambda) = p \quad \underline{\forall \lambda \in \mathbb{C}}$$

In particular, a necessary condition for controllability of a polynomial AR-representation is:

$$\text{rank}_{\mathbb{C}} P(0) = \text{rank}_{\mathbb{C}} P_0 = p \tag{4.1}$$

□

Conditions (ii) and (iii) in the above Theorem are not very explicit criteria to test controllability. One likes to have an analogous criterion to the state-space rank condition:

$$\text{rank}(B \quad AB \quad \ldots \quad A^{n-1}B) = n \tag{4.2}$$

for systems $x(t+1) = Ax(t) + Bu(t)$, $x(t) \in \mathbb{R}^n$.
For polynomial AR-representations ($T = \mathbb{Z}_+, \mathbb{R}_+, \mathbb{R}$) we suggest explicit criteria for controllability via the RRP of the representing matrices $P(s)$. In the literature there can be found several results which generalize the classical Sylvester Resultant matrix test for coprimeness of two scalar polynomials to the matrix case (see e.g. Kung et al. (1976), Barnett (1983) and the references therein). We take here an equivalent characterization of RRP which is available from the paper of Emre and Silverman (1977).
Combining Theorem 4.1 and Proposition 2.2 we have for

$$P(s) = \sum_{j=0}^{n} P_j s^j \in \mathbb{R}^{p\times q}[s],\ P_n \neq 0,\ \text{rank}_{\mathbb{R}[s]} P(s) = p \tag{4.3}$$

the following equivalence:

$$\mathcal{B}(P(s)) \quad \text{controllable} \quad \Leftrightarrow \quad P^T(s) \;\; RRP \tag{4.4}$$

4.3 Theorem [Emre and Silverman (1977)]

Let $P(s)$ as in (4.3) with $\operatorname{rank} P_0 = p$. Then the following conditions are equivalent:

(i) $P^T(s)$ is RRP.

(ii) Let

$$M_0 := \begin{pmatrix} P_0^T & \dots & \dots & P_{n-1}^T \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & P_0^T \end{pmatrix} \quad M_1 := \begin{pmatrix} P_n^T & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ P_1^T & \dots & \dots & P_n^T \end{pmatrix}$$

Then

$$\operatorname{rank} \begin{pmatrix} M_0 & M_1 & 0 & \dots & \dots & 0 \\ 0 & M_0 & M_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & M_0 & M_1 \end{pmatrix}$$

$$= \operatorname{rank} \begin{pmatrix} M_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \operatorname{rank} \begin{pmatrix} M_1 & 0 & \dots & \dots & \dots & 0 \\ M_0 & M_1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & M_0 & M_1 \end{pmatrix}$$

(the above matrices have p block rows) □

Under the assumption $\operatorname{rank} P_0 = p$, which is necessary for controllability of polynomial AR-representations condition (ii) of Theorem 4.3 generalizes the state-space condition (4.2). Indeed, given the special AR-system $\Sigma_{A,B} = (T, \mathbb{R}^n \times \mathbb{R}^m, \mathcal{B}(P))$, where $P(s) := (sI - A, -B) \in \mathbb{R}^{n \times (n+m)}[s]$ an application of (ii) yields - after some algebra - equation

(4.2) as a criterion for the controllability of $\Sigma_{A,B}$; for the details of the proof we refer to Hoffmann and Prätzel-Wolters (1991). Moreover the structure algorithm of Silverman (1969) gives an effective recursive procedure for the rank test in (ii).
In the following we extend these results to AR-systems with time domain $T = \mathbb{Z}$, i.e. dipolynomial AR-representations. Let:

$$\begin{aligned} \mathcal{B} &= \ker \tilde{P}(\sigma, \sigma^{-1}) \\ \tilde{P}(s, s^{-1}) &\in \mathbb{R}^{p \times q}[s, s^{-1}] \\ \operatorname{rank}_{\mathbb{R}[s,s^{-1}]} \tilde{P}[s, s^{-1}] &= p \end{aligned} \tag{4.5}$$

4.4 Theorem:

Let $\tilde{P}(s, s^{-1})$ as in (4.5). Then there exists $T(s, s^{-1}) \in \mathbb{R}^{p \times p}[s, s^{-1}]$ unimodular such that

$$P(s, s^{-1}) := T(s, s^{-1})\tilde{P}(s, s^{-1}) \tag{4.6}$$

is polynomial and

$$\operatorname{rank}_{\mathbb{R}}[P(s, s^{-1})]_0 = p \tag{4.7}$$

Proof:
Let $W(s, s^{-1}) := \operatorname{diag}(s^{k_j}) \in \mathbb{R}^{p \times p}[s, s^{-1}]$ such that

$$Q^{(0)}(s, s^{-1}) := W(s, s^{-1})\tilde{P}(s, s^{-1}) = \sum_{k=0}^{p_0} Q_k^{(0)} s^k, \quad p_0 \geq 1.$$

Assume $g_1 := \operatorname{rank} Q_0^{(0)} < p$ (otherwise (4.7) is satisfied). Then there exists $U^{(1)} \in GL_p(\mathbb{R})$ such that

$$U^{(1)} Q_0^{(0)} = \begin{pmatrix} \tilde{Q} \\ 0_{(p-g_1) \times q} \end{pmatrix} \tag{4.8}$$

with $\operatorname{rank}_{\mathbb{R}} \tilde{Q} = g_1$. Define

$$Q^{(1)}(s, s^{-1}) := V^{(1)}(s, s^{-1}) U^{(1)} Q^{(0)}(s, s^{-1}),$$

where

$$V^{(1)}(s, s^{-1}) := \begin{pmatrix} I_{g_1} & 0 \\ 0 & s^{-1} I_{p-g_1} \end{pmatrix}$$

is unimodular. Then $Q^{(1)}(s, s^{-1}) = \sum_{k=0}^{p_1} Q_k^{(1)} s^k$ is polynomial because of (4.8). If $p_1 = 0$, we have:

$$\operatorname{rank}_{\mathbb{R}[s,s^{-1}]} Q^{(1)}(s, s^{-1}) = \operatorname{rank}_{\mathbb{R}} Q_0^{(1)} = \operatorname{rank}_{\mathbb{R}[s,s^{-1}]} \tilde{P}(s, s^{-1}) = p$$

If $p_1 > 0$ and $\text{rank}_{\mathbb{R}} Q_0^{(1)} = p$, then (4.7) is proved.
If $p_1 > 0$ and $\text{rank}_{\mathbb{R}} Q_0^{(1)} < p$, then repeat (4.8).
After a finite number d of these steps either $\text{rank} Q_0^{(d)} = p$ or $Q^{(d)}(s, s^{-1})$ is constant because in every step the column degree of at least one column of the $Q^{(k)}$-matrices is reduced by 1. Hence:

$$\text{rank}_{\mathbb{R}[s,s^{-1}]} Q^{(d)}(s, s^{-1}) = \text{rank}_{\mathbb{R}} Q_0^{(d)} = \text{rank}_{\mathbb{R}[s,s^{-1}]} \tilde{P}(s, s^{-1}) = p$$

□

In view of equation (4.6) there holds $\mathcal{B}(\tilde{P}) = \mathcal{B}(P)$. Since controllability is a property of the behaviour, this implies that $\mathcal{B}(\tilde{P})$ is controllable iff $\mathcal{B}(P)$ is controllable. Thus by (4.7) and Theorem 4.1 (ii) there holds:

$$\Sigma = (\mathbb{Z}, \mathbb{R}^q, B(\tilde{P})) \text{ controllable } \Leftrightarrow \quad \text{rank}_{\mathbb{C}} P(\lambda, \lambda^{-1}) = p \qquad \forall \lambda \in \mathbb{C} \tag{4.9}$$

Combining (4.9) and Theorem 4.3 we obtain that Silverman's structure algorithm also gives a controllability test for dipolynomial AR-representations if we preassume $\tilde{P}$ polynomial and

$$\text{rank}\, \tilde{P}_0 = p \tag{4.10}$$

for $\tilde{P}(s)$ as defined in (4.5). However, in contrary to the polynomial case condition (4.10) is <u>not necessary</u> for controllability for $T = \mathbb{Z}$. Hence, in order to be able to test controllability one has to perform the reduction step in Theorem 4.4 in advance.
The proof of this Theorem leads to a construction algorithm for the transformation matrix T. However, the calculation of the matrix $U^{(k)}$ is quite involved and the following algorithm simplifies this step.

4.5 Lemma:

Let $U(s, s^{-1}) := I + s^{-1}B \in \mathbb{R}^{p \times p}[s, s^{-1}]$, $B \in \mathbb{R}^{p \times p}$. Then $U(s, s^{-1})$ is unimodular iff B is nilpotent.

<u>Proof:</u>
$(I + s^{-1}B)$ unimodular $\Leftrightarrow 0 \notin \sigma(I + \lambda^{-1}B)$ for all $0 \neq \lambda \in \mathbb{C}$.
For $\lambda \neq 0$:

$$I + \lambda^{-1}B \text{ singular } \Leftrightarrow \quad \lambda^{-1}(\lambda I + B) \quad \text{singular}$$

$$\Leftrightarrow \quad \lambda I + B \quad \text{singular} \quad \Leftrightarrow \quad -\lambda \in \sigma(B)$$

Hence $U(s, s^{-1})$ unimodular iff $\sigma(B) = \{0\}$. □

Let $\tilde{P}(s, s^{-1}) := \sum_{k=0}^{n} \tilde{P}_k s^k \in \mathbb{R}^{p \times q}[s, s^{-1}]$ of full row rank and $U(s, s^{-1})$ as in Lemma 4.5. Then it is easy to prove that $U(s, s^{-1})\tilde{P}(s, s^{-1})$ is polynomial if and only if $B\tilde{P}_0 = 0$.

4.6 Lemma:

Let $\tilde{P}(s, s^{-1})$ as above with $\text{rank}_{\mathbb{R}} \tilde{P}_0 < p$. Let

$$v\tilde{P}_0 = 0, \quad 0 \neq v = \begin{pmatrix} v_1 & v_2 & \dots & v_p \end{pmatrix} \in \mathbb{R}^{1\times p}$$

and

$$\sum_{i=1}^{p} v_i \lambda_i = 0 \tag{4.11}$$

with $0 \neq (\lambda_1, \lambda_2, \dots, \lambda_p) \in \mathbb{R}^{1\times p}$. Then $B := (\lambda_1 v^T, \dots, \lambda_p v^T)^T \neq 0$ and

(i) $B\tilde{P}_0 = 0$.

(ii) B is nilpotent.

Proof:

(i)

$$B\tilde{P}_0 = \begin{pmatrix} \lambda_1 v\tilde{P}_0 \\ \vdots \\ \lambda_p v\tilde{P}_0 \end{pmatrix} = 0$$

(ii) Assume $\mu \neq 0$ is an eigenvalue of B and hence of B^T. Then there exists $x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} \neq 0$, such that $B^T x = \mu x$, i.e. $v_j \sum_{i=1}^{p} \lambda_i x_i = \mu x_j$ for $1 \leq j \leq p$.

Define $\eta := \sum_{i=1}^{p} \lambda_i x_i$. Then:

$$v_j \eta = \mu x_j, \quad j = 1, \dots, p \tag{4.12}$$

This implies $\lambda_j v_j \eta = \mu \lambda_j x_j, \quad j = 1, \dots, p$ and

$$\sum_{j=1}^{p} \lambda_j v_j \eta = \sum_{j=1}^{p} \mu \lambda_j x_j = \mu\eta$$

From (4.11) we conclude $\mu\eta = 0$ and $\eta = 0$ because $\mu \neq 0$. From (4.12) it follows: $0 = \mu x_j$, $1 \leq j \leq p$, hence: $x_j = 0$, $1 \leq j \leq p$, contradicting the fact that x is an eigenvector. □

Finally we need:

4.7 Lemma:

Let $A \in \mathbb{R}^{p \times q}$, $p \le q$, $\text{rank}_{\mathbb{R}} A < p$ and $0 \ne v \in \mathbb{R}^{1 \times p}$ such that $vA = 0$. Let further $(\lambda_1, \ldots, \lambda_p)$ and B as in Lemma 4.6. Finally let $C \in \mathbb{R}^{p \times q}$, $vC \ne 0$. Then $\text{rank}_{\mathbb{R}}(A + BC) \ge \text{rank}_{\mathbb{R}} A$. □

Summarizing we obtain the following algorithm:

Given: $\tilde{P}(s, s^{-1}) = \sum_{k=0}^{n} \tilde{P}_k s^k \in \mathbb{R}^{p \times q}[s, s^{-1}],\ p \le q,\ \tilde{P}_0 \ne 0,$
$\text{rank}\, \tilde{P}_0 < p,\ \text{rank}_{\mathbb{R}[s,s^{-1}]} \tilde{P}(s, s^{-1}) = p$

Select: $0 \ne v \in \mathbb{R}^{1 \times p}$ such that $v\tilde{P}_0 = 0$, $\lambda_1, \ldots, \lambda_p$ and B as in Lemma 4.6.

Form: $\bar{P}(s, s^{-1}) = (I + s^{-1}B)\tilde{P}(s, s^{-1})$ and iterate

Then we obtain: $\bar{P}(s, s^{-1}) = \sum_{k=0}^{q} \bar{P}_k s^k \in \mathbb{R}^{p \times q}[s]$ with $\text{rank}\, \bar{P}_0 = p$

5 Controllability Indices for AR-Systems

In the literature there are several approaches for the investigation of controllability indices (c.i.) for different representations of linear systems (cf. e.g. Münzner and Prätzel-Wolters (1979)). Here we define a list of algebraic controllability indices for linear dynamical systems in AR-representation. Our approach is a straightforward extension of the characterization of controllability indices as minimal indices of the $F[s]$-modules $\ker[sI - A, B]$ in the state-space setting.
Assume that $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ is a dynamical system in AR-representation:

$$\begin{aligned} \mathcal{B} &= \ker R(\sigma, \sigma^{-1}) \\ R(s, s^{-1}) &= R_L s^L + \ldots + R_\ell s^\ell \in \mathbb{R}^{p \times q}[s, s^{-1}] \\ &\text{rank}_{\mathbb{R}[s,s^{-1}]} R(s, s^{-1}) = p \end{aligned} \tag{5.1}$$

Here we implicitly assume that $p < q$; otherwise the following construction does not lead to a reasonable definition of c.i.'s; observe that $p = q$ corresponds to the autonomous case (compare Willems (1991)).
Interpreting $R(s, s^{-1})$ as the $\mathbb{R}[s, s^{-1}]$-linear mapping:

$$R(s, s^{-1}) : \begin{array}{ccc} \mathbb{R}^q[s, s^{-1}] & \longrightarrow & \mathbb{R}^p[s, s^{-1}] \\ x(s, s^{-1}) & \longrightarrow & R(s, s^{-1})x(s, s^{-1}) \end{array}$$

we obtain that $M(R) := \ker R(s, s^{-1})$ is a free $\mathbb{R}[s, s^{-1}]$-submodule of $\mathbb{R}^q[s, s^{-1}]$, satisfying:

$$M(R) = M(UR) \text{ for } U(s, s^{-1}) \in \mathbb{R}^{p \times p}[s, s^{-1}] \text{ unimodular}$$

Following the notation in Münzner and Prätzel-Wolters (1979) we call $M_\Sigma := M(R(s,s^{-1}))$ the module of return to zero. The list of dipolynomial indices

$$v(\Sigma) := (v_1(\Sigma),\dots,v_m(\Sigma)),\ m := q-p$$

of the module M_Σ is called the list of (algebraic) controllability indices.

5.1 Remarks:

(i) Another possible way to introduce c.i.'s is to define the index list as the polynomial indices of $M_\Sigma \cap \mathbb{R}^q[s]$ (cf. Münzner and Prätzel-Wolters (1979)). However, these two sets of integers coincide.

(ii) Observe that $M(R) = M(FR)$ for $F(s,s^{-1}) \in \mathbb{R}^{p\times p}[s,s^{-1}]$, $\det F \not\equiv 0$ holds true. Expressed in system theoretic terms this means that two different AR-systems which have identical controllable parts (see Willems (1991)) give rise to the same module of return to zero. □

The above definition leads to a natural equivalence concept for behaviour systems. We call two AR-systems Σ_1 and Σ_2 index equivalent ($\Sigma_1 \overset{ind}{\approx} \Sigma_2$) if there exists a degree-preserving $\mathbb{R}[s,s^{-1}]$-isomorphism between M_{Σ_1} and M_{Σ_2}; observe that this condition is equivalent to the equality of the associated index lists. In other words:

$$\Sigma_1 \overset{ind}{\approx} \Sigma_2 \ \Leftrightarrow\ v(\Sigma_1) = v(\Sigma_2) \tag{5.2}$$

If $\Sigma = (T, \mathbb{R}^q, \mathcal{B})$ is a linear time-invariant system in AR-representation with time axis $T = \mathbb{Z}_+, \mathbb{R}_+$ or $\mathbb{R}$, i.e. $\mathcal{B} = \mathcal{B}(R)$ where $R(s)$ is a polynomial $p\times q$-matrix satisfying $\mathrm{rank}_{\mathbb{R}[s]} R(s) = p$, then the developed algebraic construction carries over completely to the $\mathbb{R}[s]$-linear mapping:

$$R(s):\ \begin{array}{ccc} \mathbb{R}^q[s] & \longrightarrow & \mathbb{R}^p[s] \\ x(s) & \longrightarrow & R(s)x(s) \end{array}$$

and the associated module

$$M(R) = \ker R(s) \subset \mathbb{R}^q[s]$$

(see also Kuijper (1992)). However, we will only elaborate the dipolynomial case in this paper.

5.2 Remark:

For arbitrary Rosenbrock-type polynomial system matrices (and hence especially for state-space forms):

$$R(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \in \mathbb{R}^{(\ell+p)\times(\ell+m)}[s] \tag{5.3}$$

$$\det T(s) \not\equiv 0,\ (VT^{-1}U + W) \text{ strict proper rational}$$

as well as for <u>singular state-space systems</u>

$$E\dot{x} = Ax + Bu \tag{5.4}$$

$$E, A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ \det[sE - A] \not\equiv 0$$

the lists of controllability indices defined in the literature (cf. Münzner and Prätzel-Wolters (1979) and Glüsing-Lüerßen (1991)) coincide with the list $v(\Sigma)$ of $\Sigma = (\mathbb{Z}, \mathbb{R}^{\ell+m}, \mathcal{B}(T(s), U(s)))$ with T, U as in (5.3) respectively the list $v(\tilde{\Sigma})$ with $\tilde{\Sigma} = (\mathbb{Z}, \mathbb{R}^{n+m}, \mathcal{B}(sE - A, B))$ and E, A, B from (5.4). □

Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ be again a dynamical system in AR-representation with R satisfying (5.1). Let further

$$f(s, s^{-1}) = \left(f_1, \ldots, f_{\binom{q}{p}}\right)$$

be the vector of all $p \times p$-minors f_i of R. Willems (1991) defines the <u>Mc Millan degree</u> of Σ, $Mm(\Sigma)$, as:

$$Mm(\Sigma) = Mm(R) = \text{ddeg} f(s, s^{-1}) \tag{5.5}$$

$Mm(\Sigma)$ is well defined because $Mm(R) = Mm(UR)$ for any unimodular U. Even $Mm(RQ) = Mm(R)$ is true for nonsingular constant matrices Q.

5.3 Theorem:

Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ where R satisfies (5.1). Let further $v(\Sigma) = (v_1, \ldots, v_m)$ be the list of controllability indices of Σ. Then:

$$\Sigma \text{ controllable} \quad \Leftrightarrow \quad Mm(\Sigma) = \sum_{i=1}^{m} v_i$$

<u>**Proof:**</u>
An easy argument shows that there always exists a permutation matrix $Q \in \mathbb{R}^{q\times q}$ such that for $RQ := (R_1, R_2)$, $R_1 \in \mathbb{R}^{p\times p}[s, s^{-1}]$, $R_2 \in \mathbb{R}^{p\times(q-p)}[s, s^{-1}]$ there holds: $R_1^{-1}R_2$ is proper rational.
Moreover, let $U(s, s^{-1}) \in \mathbb{R}^{p\times p}[s, s^{-1}]$ such that $P := URQ \in \mathbb{R}^{p\times q}[s]$ is a dipolynomial minimal basis for the $\mathbb{R}[s, s^{-1}]$-module $\mathbb{R}^{1\times p}[s, s^{-1}] \cdot P$ satisfying rank $P_0 = p$.
Observe that for $P = (P_1, P_2)$ partitioned as above $P_1^{-1}P_2 = (UR_1)^{-1}UR_2 = R_1^{-1}R_2$ is proper rational. Furthermore, the polynomial and dipolynomial row degrees of P coincide; we denote them by $(\mu_1, \ldots, \mu_p)$. The transformations U and Q leave controllability invariant, i.e. $\mathcal{B}(R)$ controllable iff $\mathcal{B}(P)$ controllable; moreover $Mm(R) = Mm(P)$.

In view of (4.9) controllability of $\mathcal{B}(P)$ is equivalent to controllability of P viewed as a polynomial Rosenbrock-type system matrix. For these matrices we have:

$$P \text{ controllable} \iff \deg(\det P_1) = \sum_{i=1}^{m} \tilde{v}_i \tag{5.6}$$

where $\tilde{v}_i$, $i = 1, \ldots, m$ are the polynomial indices of $M(P) \cap \mathbb{R}^q[s]$; by Remark 5.1 (i) we have $\tilde{v}_i = v_i$, $i = 1, \ldots, m$. Moreover,

$$\deg(\det P_1) = \sum_{i=1}^{p} \mu_i \tag{5.7}$$

since by Theorem 2.4 (ii) there holds $\operatorname{rank}[P]_h^r = p$.
Finally, an application of Theorem 2.4 (iii) yields $Mm(P) = \sum_{i=1}^{p} \mu_i$. This together with (5.6) and (5.7) proves the result. □

For the derivation of an effective algorithm for the calculation of controllability indices we need the following lemma:

5.4 Lemma:

Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ where R satisfies (5.1). Furthermore, let $U \in \mathbb{R}^{p\times p}[s, s^{-1}]$ unimodular and $Q \in \mathbb{R}^{q\times q}$ nonsingular such that

$$P := URQ = \sum_{k=0}^{\deg} P_k s^k =: \sum_{k=0}^{\deg} (\bar{P}_k, \tilde{P}_k) s^k$$

with $P_0 \neq 0$, $\operatorname{rank} \bar{P}_{\deg} = p$ and $\tilde{P}_{\deg} = 0_{p\times(q-p)}$.
Then URQ is strict system equivalent to the state space form $(sI_{\deg\cdot p} - A_\Sigma, B_\Sigma)$ where:

$$A_\Sigma := \begin{pmatrix} 0 & 0 & \ldots & \ldots & 0 & -\bar{P}_0 \cdot \bar{P}_{deg}^{-1} \\ I_p & 0 & 0 & & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & 0 & -\bar{P}_{deg-2} \cdot \bar{P}_{deg}^{-1} \\ 0 & \ldots & \ldots & 0 & I_p & -\bar{P}_{deg-1} \cdot \bar{P}_{deg}^{-1} \end{pmatrix}, \quad B_\Sigma = \begin{pmatrix} \tilde{P}_0 \\ \tilde{P}_1 \\ \vdots \\ \tilde{P}_{deg-1} \end{pmatrix}$$

<u>Proof:</u>
We will show that there exist matrices $M_{1e}, M_{2e} \in \mathbb{R}^{\deg\cdot p\times \deg\cdot p}[s]$ polynomial unimodular and $K \in \mathbb{R}^{\deg\cdot p\times(q-p)}[s]$ such that

$$M_{1e}(sI_{\deg\cdot p} - A_\Sigma, B_\Sigma) \begin{pmatrix} M_{2e} & K \\ 0 & I_{q-p} \end{pmatrix} = \begin{pmatrix} I_{(\deg-1)\cdot p} & 0 \\ 0 & P \end{pmatrix} \tag{5.8}$$

Now

$$(sI_{\deg\cdot p}-A_\Sigma, B_\Sigma) = \begin{pmatrix} sI_p & 0 & \dots & \dots & 0 & \bar{P}_0\cdot\bar{P}_{deg}^{-1} & \tilde{P}_0 \\ -I_p & sI_p & 0 & & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & & \ddots & \ddots & sI_p & \bar{P}_{deg-2}\cdot\bar{P}_{deg}^{-1} & \tilde{P}_{deg-2} \\ 0 & \dots & \dots & 0 & -I_p & sI_p+\bar{P}_{deg-1}\cdot\bar{P}_{deg}^{-1} & \tilde{P}_{deg-1} \end{pmatrix}$$

Successive multiplication from the left by the unimodular matrices

$$\begin{pmatrix} I_p & & & \\ & \ddots & & \\ & & I_p & sI_p \\ & & & I_p \end{pmatrix}, \begin{pmatrix} I_p & & & & \\ & \ddots & & & \\ & & I_p & sI_p & \\ & & & I_p & \\ & & & & I_p \end{pmatrix}, \dots, \begin{pmatrix} I_p & sI_p & & \\ & I_p & & \\ & & \ddots & \\ & & & I_p \end{pmatrix}$$

yields the matrix

$$\begin{pmatrix} 0 & \dots & \dots & \dots & 0 & X_1 & Y_1 \\ -I_p & \ddots & & & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & X_{deg-1} & Y_{deg-1} \\ 0 & \dots & \dots & 0 & -I_p & X_{deg} & Y_{deg} \end{pmatrix}$$

where

$$X_i := s^{\deg-i+1}I_p + s^{\deg-i}\bar{P}_{\deg-1}\cdot\bar{P}_{\deg}^{-1} + \dots + \bar{P}_{i-1}\cdot\bar{P}_{\deg}^{-1} \in \mathbb{R}^{p\times p}[s]$$

and

$$Y_i := s^{\deg-i}\tilde{P}_{\deg-1} + \dots + \tilde{P}_{i-1} \in \mathbb{R}^{p\times(q-p)}[s]$$

for $i = 1, \dots, \deg$.
Multiplying successively from the right by the unimodular matrices

$$\begin{pmatrix} I_p & & & & \\ & \ddots & & & \\ & & I_p & X_{\deg} & Y_{\deg} \\ & & & I_p & 0 \\ & & & & I_{q-p} \end{pmatrix}, \dots, \begin{pmatrix} I_p & & & X_2 & Y_2 \\ & \ddots & & & \\ & & I_p & & \\ & & & I_p & \\ & & & & I_{q-p} \end{pmatrix}$$

and $\begin{pmatrix} I_p & & & & \\ & \ddots & & & \\ & & I_p & & \\ & & & \bar{P}_{\deg} & \\ & & & & I_{q-p} \end{pmatrix}$ one obtains $\begin{pmatrix} 0 & \dots & \dots & \dots & 0 & P \\ -I_p & \ddots & & & \vdots & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & \dots & 0 & -I_p & 0 \end{pmatrix}$

which gets transformed by elementary row transformations into

$$\begin{pmatrix} I_{(\deg -1)\cdot p} & 0 \\ 0 & P \end{pmatrix}$$

In total all the transformations are of the form (5.8). □

5.5 Remark:

(i) Let $\Sigma_1 = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$, where R satisfies (5.1). Furthermore, let

$\Sigma_2 = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(RT))$ where $T \in \mathbb{R}^{q\times q}$ is nonsingular. Then $\Sigma_1 \overset{ind}{\approx} \Sigma_2$, since the mapping

$$\begin{array}{ccc} M_{\Sigma_1} & \longrightarrow & M_{\Sigma_2} \\ x(s) & \longrightarrow & T^{-1}x(s) \end{array}$$

is a degree-preserving $\mathbb{R}[s, s^{-1}]$-isomorphism.

(ii) Let $\Sigma_i = (\mathbb{Z}, \mathbb{R}^{q_i}, \mathcal{B}(R_i))$, $R_i = (T_i, U_i)$, $q_i = \ell_i + m$, $T_i \in \mathbb{R}^{\ell_i\times\ell_i}[s]$, $\det T_i \not\equiv 0$, $U_i \in \mathbb{R}^{\ell_i\times m}[s]$, $i = 1, 2$. Moreover assume that $T_i^{-1}U_i$ is strictly proper for $i = 1, 2$ and that Σ_1 and Σ_2 are strict system equivalent. Then there holds $\Sigma_1 \overset{ind}{\approx} \Sigma_2$. For the proof we refer to Theorem 3.4 in Münzner and Prätzel-Wolters (1979). □

Based on Lemma 5.4 and Remark 5.5 we obtain an effective algorithm for the determination of the controllability index list.
Starting with a system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ satisfying (5.1) we first construct a strict system equivalent state-space system $(A_\Sigma, B_\Sigma) \in \mathbb{R}^{(\deg\cdot p)\times(\deg -1)\cdot p+q}$ according to Lemma 5.4. For the explicit construction of the transformation matrix Uwe use the reduction algorithm in section 4; an elementary step is now of the form $I + sB$, B nilpotent and $B \cdot P_{\deg} = 0$. Note that (A_Σ, B_Σ) is not uniquely determined; however, all possible state-space systems generate the same index list. Having obtained (A_Σ, B_Σ) we determine $v(\Sigma)$ by the Kalman-Rosenbrock deleting procedure.

5.6 Example:

Consider the nonsingular system of difference equations:

$$w_1(t+2) + 3w_3(t+2) + 6w_4(t+2) + 3w_5(t+2)+$$

$$+2w_1(t+1) + w_2(t+1) - w_3(t+1) + w_5(t+1) +$$
$$+w_1(t) + 2w_2(t) + 2w_4(t) + 3w_5(t) = 0$$
$$2w_1(t+2) + w_4(t) = 0$$

for $t \in \mathbb{Z}$ with the associated dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^5, \mathcal{B}(R))$, where:

$$R(s, s^{-1}) := \begin{pmatrix} s^2+2s+1 & s+2 & 3s^2-s & 6s^2+2 & 3s^2+s+3 \\ & & & & \\ 2s^2 & 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 5}[s, s^{-1}]$$

Hence $p = 2$, $q = 5$ and $\deg = 2$. Obviously $\mathrm{rank}_{\mathbb{R}[s,s^{-1}]} R(s, s^{-1}) = 2$. Furthermore, simple calculations show that the gcd of the 2×2-minors of R is a dipolynomial unit, which yields the controllability of Σ (cf. Willems (1991)).
Observe that R is polynomial; moreover, R is a (dipolynomial) minimal lag description with $Mm(\Sigma) = 4$. Define $Q \in \mathbb{R}^{5\times 5}$ by

$$Q := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$P(s) := R(s) \cdot Q = \begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix} s^2 + \begin{pmatrix} 2 & -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} s +$$
$$\begin{pmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Now P is in the form as in Lemma 5.4. For the matrices A_Σ and B_Σ we obtain

$$A_\Sigma = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{3} & -\frac{7}{6} \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_\Sigma = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Moreover,

$$(B_\Sigma, A_\Sigma B_\Sigma, A_\Sigma^2 B_\Sigma, A_\Sigma^3 B_\Sigma) = \begin{pmatrix} 2 & 2 & 3 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & \frac{7}{3} & \frac{8}{3} & \frac{11}{3} & \frac{7}{9} & -\frac{5}{18} & \frac{11}{9} & \frac{7}{27} & -\frac{16}{27} & \frac{11}{27} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the controllability indices of Σ are $v_1 = 2 > v_2 = v_3 = 1$.
Finally $\sum_{i=1}^{3} v_i = 4 = \deg \cdot p = Mm(\Sigma)$. □

Recently, Fagnani (1991) has introduced a general concept of c.i.'s for linear time-invariant behaviour systems $\Sigma = (T, \mathbb{R}^q, \mathcal{B})$ with time domain $T = \mathbb{Z}$ exclusively in terms of the behaviour $\mathcal{B}$, i.e. independent of a certain system representation. We suggest to call this approach the geometric description of controllability indices. In the sequel we will elaborate the connections to our algebraic approach.
The geometric index list is determined by the dimensions of quotient spaces of truncated behaviour spaces of $\mathcal{B}$. Fagnani showed that the c.i.'s are invariants with respect to a "controllability equivalence relation" on the set of all linear time-invariant behaviour systems defined as follows:
Two linear time-invariant systems $\Sigma_i = (\mathbb{Z}, \mathbb{R}^{q_i}, \mathcal{B}_i)$, $i = 1, 2$, are said to be controllably equivalent ($\Sigma_1 \overset{co}{\approx} \Sigma_2$) if there exists a linear bijection $\psi : \mathcal{B}_1 \to \mathcal{B}_2$ such that:

(i) For all $t \in \mathbb{Z}$ there holds

$$\psi \circ \sigma^t = \sigma^t \circ \psi \tag{5.9}$$

(ii) Let $w \in \mathcal{B}_1$; then

$$\begin{array}{lcl} w^- = 0 & \Leftrightarrow & (\psi(w))^- = 0 \\ w^+ = 0 & \Leftrightarrow & (\psi(w))^+ = 0 \end{array} \tag{5.10}$$

Observe that controllability is preserved under controllable equivalence. Moreover the geometric c.i.'s constitute a complete set of invariants under this equivalence relation if the considered behaviours are complete and controllable.
The geometric and algebraic controllability indices coincide for complete discrete-time behaviour systems:

5.7 Theorem:

Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ where R satisfies (5.1). Denote by $c(\Sigma)$ the list of geometric controllability indices of Σ. Then $c(\Sigma) = v(\Sigma)$. □

For the proof of the theorem we need the following lemma:

5.8 Lemma:

(i) Let $\Sigma_1 = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ where R satisfies (5.1). Let $\Sigma_2 = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(RT))$ where $T \in \mathbb{R}^{q \times q}$ is nonsingular. Then $c(\Sigma_1) = c(\Sigma_2)$.

(ii) Let $\Sigma_i = (\mathbb{Z}, \mathbb{R}^{q_i}, \mathcal{B}(R_i))$, $R_i = (T_i, U_i)$, $q_i = \ell_i + m$, $T_i \in \mathbb{R}^{\ell_i \times \ell_i}[s]$, $\det T_i \not\equiv 0$, $U_i \in \mathbb{R}^{\ell_i \times m}[s]$, $i = 1, 2$. Furthermore, assume that $T_i^{-1} U_i$ is strictly proper rational for $i = 1, 2$ and that Σ_1 and Σ_2 are strictly system equivalent.

Then $c(\Sigma_1) = c(\Sigma_2)$.

(iii) If Σ is in state space form, i.e.

$$\Sigma = (\mathbb{Z}, \mathbb{R}^{n+m}, \mathcal{B}((sI_n - A, B))), \quad (A, B) \in \mathbb{R}^{n\times(n+m)},$$

then the list $c(\Sigma)$ coincides with the list of the ordinary c.i.'s for state space systems.

Proof:

(i) Define $\psi : \mathcal{B}(T) \to \mathcal{B}(RT)$, $w \to T^{-1}w$. Then ψ is an isomorphism and clearly satisfies conditions (5.9) and (5.10). Hence $\Sigma_1 \overset{co}{\approx} \Sigma_2$.

(ii) By the definition of strict system equivalence there exists $q \geq \max(\ell_1, \ell_2)$ and polynomial matrices M_{1e}, M_{2e} and Y with M_{1e}, M_{2e} polynomial unimodular such that

$$M_{1e} \begin{pmatrix} I_{q-\ell_1} & 0 & 0 \\ 0 & T_1 & U_1 \end{pmatrix} = \begin{pmatrix} I_{q-\ell_2} & 0 & 0 \\ 0 & T_2 & U_2 \end{pmatrix} \begin{pmatrix} M_{2e} & -Y \\ 0 & I_m \end{pmatrix} \tag{5.11}$$

Let $\tilde{\Sigma}_i = (\mathbb{Z}, \mathbb{R}^{q+m}, \mathcal{B}(\tilde{R}_i))$ where $\tilde{R}_i = \begin{pmatrix} I_{q-\ell_i} & 0 & 0 \\ 0 & T_i & U_i \end{pmatrix} \in \mathbb{R}^{q\times(q+m)}[s]$,

$i = 1, 2$. Then $c(\Sigma_i) = c(\tilde{\Sigma}_i)$ for $i = 1, 2$. Define:

$$\psi : \begin{array}{ccc} \mathcal{B}(\tilde{R}_1) & \longrightarrow & \mathcal{B}(\tilde{R}_2) \\ w & \longrightarrow & \begin{pmatrix} M_{2e}(\sigma) & -Y(\sigma) \\ 0 & I_m \end{pmatrix} w \end{array} \tag{5.12}$$

Then ψ is an isomorphism (cf. (5.11)) which commutes with the shift σ. It remains to show that (5.10) is satisfied.

Let $\pi : (\mathbb{R}^{q+m})^{\mathbb{Z}} \to (\mathbb{R}^m)^{\mathbb{Z}}$ denote the projection $\begin{pmatrix} x \\ y \end{pmatrix} \to y$ and let $w \in \mathcal{B}(\tilde{R}_1)$. Assume $w^- = 0$. Then by (5.12) $\pi(\psi(w))^- = \pi w^- = 0$; now

$$(1 - \pi)(\psi(w))^- = \begin{pmatrix} 0_{q-\ell_2} \\ T_2^{-1} U_2(\sigma) \pi(\psi(w))^- \end{pmatrix}$$

since $T_2^{-1}U_2$ is strictly proper rational, and hence $(1 - \pi)(\psi(w))^- = 0$. The converse implication $(\psi(w))^- = 0 \Rightarrow w^- = 0$ is proven analogously because ψ^{-1} is of the form

$$\psi^{-1} : \begin{array}{ccc} \mathcal{B}(\tilde{R}_2) & \longrightarrow & \mathcal{B}(\tilde{R}_1) \\ w & \longrightarrow & \begin{pmatrix} M_{2e}^{-1}(\sigma) & M_{2e}^{-1} Y(\sigma) \\ 0 & I_m \end{pmatrix} w \end{array} \tag{5.13}$$

with M_{2e}^{-1} polynomial.

Assume $w^+ = 0$. Since M_{2e} and Y are polynomial we have $(\psi(w))^+ = \psi(w^+)$, which gives $(\psi(w))^+ = 0$. Furthermore, the implication $(\psi(w))^+ = 0 \Rightarrow w^+ = 0$ is an immediate consequence of the unimodularity of M_{2e} and (5.13). Summarizing, there holds $\Sigma_1 \overset{co}{\approx} \Sigma_2$.

(iii) See Fagnani (1991). □

Proof of Theorem 5.7:

Let $P := URQ$, A_Σ and B_Σ as defined in Lemma 5.4 and let $\Sigma_1 := (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(P))$ and $\Sigma_2 := (\mathbb{Z}, \mathbb{R}^{(\deg -1)\cdot p+q}, \mathcal{B}(sI_{\deg \cdot p} - A_\Sigma, B_\Sigma))$. Since left multiplication of R by a unimodular matrix U does not change the behaviour we obtain $c(\Sigma_1) = c(\Sigma)$ and $v(\Sigma_1) = v(\Sigma)$ by Lemma 5.8 (i) resp. Remark 5.5 (i). By Lemma 5.4 Σ_1 and Σ_2 are strict system equivalent and satisfy the assumptions of Remark 5.5 (ii) resp. Lemma 5.8 (ii). Finally, by Lemma 5.8 (iii) the list $c(\Sigma_2)$ coincides with the list of ordinary c.i.'s for state-space systems, which is identical to $v(\Sigma_2)$ (cf. Remark 5.2). □

5.9 Corollary:

Let $\Sigma_i = (\mathbb{Z}, \mathbb{R}^{q_i}, \mathcal{B}(R_i))$ where R_i satisfies (5.1), $i = 1, 2$.
Then:

$$\Sigma_1 \overset{co}{\approx} \Sigma_2 \quad \Rightarrow \quad \Sigma_1 \overset{ind}{\approx} \Sigma_2$$

If Σ_1 and Σ_2 are controllable, the converse is also true. □

5.10 Remark:

The converse implication in Corollary 5.9 is in general not true. As mentioned above, systems with the same controllable part give rise to the same module of return to zero. Hence in one equivalence class under index equivalence there are as well controllable as non-controllable systems. However, this is not the case for controllable equivalence. □

In Corollary 5.9 there is established a connection between bicausal isomorphisms of behaviours and degree-preserving isomorphisms of modules of return to zero. Starting from a given bicausal isomorphism one can explicitly construct the induced isomorphism on the level of the modules of return to zero.

For this purpose we identify sequences and <u>doubly infinite Laurent series</u> via the extended $\mathcal{Z}$-transform

$$\mathcal{Z} : (w(k))_{k\in\mathbb{Z}} \longrightarrow \sum_{v=-\infty}^{\infty} w(v) s^{-v}$$

where $(w(k))_{k\in\mathbb{Z}} \in (\mathbb{R}^q)^{\mathbb{Z}}$.

Now let Σ_i, $i = 1, 2$ as in Corollary 5.9 be controllable equivalent and let

$\psi : \mathcal{B}(R_1) \to \mathcal{B}(R_2)$ the associated isomorphism satisfying (5.9) and (5.10). Define

$$\varphi_\psi : \begin{array}{ccc} \mathbb{R}^{q_1}[s,s^{-1}] & \longrightarrow & \mathbb{R}^{q_2}[s,s^{-1}] \\ x(s,s^{-1}) & \longrightarrow & \mathcal{Z}\psi\mathcal{Z}^{-1}x(s,s^{-1}) \end{array}$$

Then $\varphi_{\psi|M_{\Sigma_1}}$ is a degree-preserving $\mathbb{R}[s,s^{-1}]$-isomorphism from M_{Σ_1} onto M_{Σ_2}:
An easy calculation shows that $x(s,s^{-1}) \in M_{\Sigma_1}$ implies $R_2(s,s^{-1})\varphi_\psi x = 0$. Using property (5.10) one can show that φ_ψ is degree-preserving. That φ_ψ is an isomorphism follows from the fact that $s^\ell \mathcal{Z} = \mathcal{Z}\sigma^\ell$, $\ell \in \mathbb{Z}$. □

6 Conclusions

The purpose of this paper was the development of explicit controllability tests for linear time-invariant behaviour systems (cf. Willems (1991)) in AR-representation and the definition of controllability indices for these systems in an algebraic framework.
Our starting point was the characterization of controllability via the relative right primeness of the representing matrices. In the polynomial case, this can be achieved by applying a result of Emre and Silverman (1977) which provides a test for RRP in terms of the coefficients of the representing matrices. As has been shown in Hoffmann and Prätzel-Wolters (1991) this test leads to the classical rank test for controllability if state-space models are considered.
Furthermore it is proven that the result of Emre and Silverman remains valid in the dipolynomial case. However, to obtain a characterization of controllability in this case, the constant coefficient of the representing matrix has to have full row rank. A reduction algorithm which always achieves this by unimodular left multiplication is given.
It should be mentioned that the obtained results can directly be applied to test controllability of ARMA-systems. Furthermore, observability (as defined by Willems (1991)) for MA- and ARMA-systems can be included. The corresponding results as well as a detailed description of the different test algorithms can be found in Hoffmann (1991), which also contains the realization of Silverman's structure algorithm (cf. Silverman (1969)) and the reduction algorithm as Matlab programs.
The investigation of the index structure of the module of return to zero associated to a complete behaviour system leads to the definition of a set of algebraic controllability indices; this new concept incorporates several existing concepts of controllability indices for different representations of linear systems as special cases. The tight connection of our approach to the recently developed theory in Fagnani (1991) is worked out. Because of this connections, the effective algorithm for the calculation of the algebraic index list given in this paper is also useful in the geometric framework.

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