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Discrete positive real systems and high gain stability

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1 Introduction

The original definition of positive real (PR) rational functions $g(s) \in \mathbb{C}(s)$ goes back to the thirties. Brune (1931) introduced PR-functions to characterize time-invariant, rational one port passive networks. Gewertz (1933) extended the PR-concept to symmetric, rational PR-matrices $G(s) \in \mathbb{C}(s)^{p \times p}$ for which the quadratic from $x^* G(s) x \in \mathbb{C}(s)$ is a PR-function. For $p = 2$ he showed that rational PR-matrices coincide with the impedance matrices of certain 2-port passive networks. This result was extended to arbitrary $p \in \mathbb{N}$ by Oono (1950), Mc Millan (1952) and Bayard (1949). In the fiftieth passive networks were embedded into the general theory of dissipative dynamical systems. The equivalence of the positive realness of a $p \times p$-transfer-function $G(s)$ and the passivity of a minimal realization (A, B, C, D) of $G(s)$ was shown nearly simultaneously by Meixner (1954,1958,1964) and Youla, Castriota and Carlin(1959).
In the following PR-functions gained increasing interest, in particular in control theory. There are several papers extending the theory in different aspects like:

- Generalizations to non rational PR-matrices $G = (g_{ij})_{i,j=1,\dots,p}$, $g_{ij} : \mathbb{C} \to \mathbb{C}$.
- Modification of PR-matrices to strict positive real (SPR) and almost strict positive real (ASPR) matrices.
- Relations between high gain resp. hyper stable systems and PR-transfer functions.

The purpose of this paper is to complete the discrete version of the theory of PR-systems. To the best of our knowledge discrete positive real (DPR) systems are defined and algebraically characterized the first time in Hitz and Anderson (1969). The algebraic characterization in the DPR-lemma is extended in Anderson (1986) to discrete strict positive real (DSPR) systems, however only for the scalar case.
In section 2 of this paper the extension of the DSPR-lemma to the multivariable case and an alternative characterization of DSPR-systems which can be interpreted as the discrete versions of theorem 2.1 in Tao and Iannou (1988) and lemma 10 in Narendra and Taylor (1973) is given.
In section 3 discrete almost strict positive real (DASPR) systems and their relations to high gain stable systems are analyzed. The presented results are partly contained in Bar-Kana (1986), however, stated there with incomplete or weakly formalized proofs.

2 Discrete strict positive real functions

The following definition is due to Hitz and Anderson (1969).

2.1 Definition:

A square rational matrix $G(z) \in \mathbb{R}(z)^{p \times p}$ is called discrete positive real (DPR) if:

(i) The entries $g_{ij}(z), i, j \in \underline{p}$ of $G(z)$ are analytic in $\Gamma = \{z \in \mathbb{C}; |z| > 1\}$.

(ii) $G(z) + \overline{G(z)}^T$ is positive semidefinite hermitian in Γ.

An algebraic characterization of DPR matrices is given by the discrete positive real lemma:

2.2 Lemma (Hitz and Anderson (1969)):

Let $G(z) \in \mathbb{R}(z)^{p\times p}$, where $G(z)$ has no poles outside the unit disc and simple poles only on the unit circle. Further let $\sum = (A, B, C, D)$ be a minimal realization of $G(z)$. $G(z) = D + C[zI - A]^{-1}B$. Then $G(z)$ is DPR if and only if there exist real matrices P, L and $W, P > 0, P$ symmetric such that:

$$\begin{array}{lll} (i) & A^TPA - P & = -LL^T \\ (ii) & A^TPB & = C^T - LW \\ (iii) & W^TW & = D + D^T - B^TPB. \end{array} \tag{2.1}$$

□

2.3 Remark:

a) In particular this lemma implies that DPR matrices $G(z)$ are always proper rational with:

$$\lim_{z\to\infty} G(z) \neq 0 \tag{2.2}$$

b) Furthermore it is shown in Hitz and Anderson (1969) that the poles of a DPR-matrix lie in $\Gamma^C = \{z \in \mathbb{C} | |z| \leq 1\}$ and are simple on the unit circle $\{z \in \mathbb{C} |\ |z| = 1\}$ and that $G(e^{i\omega}) + G^T(e^{-i\omega})$ is positive semidefinite hermitian if $e^{i\omega}$ is not a pole of $G(z)$.

2.4 Definition (Anderson (1986)):

A square rational matrix $G(z) \in \mathbb{R}^{p\times p}(z)$ is called discrete strict positive real (DSPR) if:

$$\exists \rho \in (0,1) \text{ such that } G(\rho z) \text{ is DPR.} \tag{2.3}$$

□

The following lemmata are characterizations of DSPR-matrices $G(z)$. They are obtained as applications of analagous results for DPR-systems (Lemma 2.2, Remark 2.3) and can be interpreted as discrete versions of theorem 2.1 in Tao and Iannou (1988) and lemma 10 in Narendra and Taylor (1973).

2.5 Lemma

Let $\sum = (A, B, C, D)$ be a minimal realization of the proper rational transfer function $G(z) = D + C[zI - A]^{-1}B$. Then $G(z)$ is DSPR if and only if there exist a real positive definite symmetric matrix P, real matrices L and W and a real number γ such that

$$(i) \quad A^TPA - P = -LL^T - \gamma^2 P \tag{2.4a}$$

$$(ii) \quad A^TPB = C^T - LW \tag{2.4b}$$

$$(iii) \quad W^TW = D + D^T - B^TPB \tag{2.4c}$$

Proof: Let $G(\rho z)$ DPR for some $\rho, 0 < \rho < 1$. Then $\sum = (\frac{1}{\rho}A, B, \frac{1}{\rho}C, D)$ is a minimal realization of $G(\rho z) = \frac{1}{\rho}C(zI - \frac{1}{\rho}A)^{-1}B + D$. By lemma 2.2 $G(\rho z)$ is DPR if there exist real matrices $\tilde{L}, W$ and P, P positive definite symmetric such that

$$\begin{aligned}
\frac{1}{\rho^2}A^TPA - P &= -\tilde{L}\tilde{L}^T \\
\frac{1}{\rho}A^TPB &= \frac{1}{\rho}C^T - \tilde{L}W \\
W^TW &= D + D^T - B^TPB.
\end{aligned}$$

Equivalent is:

$$\begin{aligned}
A^TPA - \rho^2 P &= -\rho^2\tilde{L}\tilde{L}^T \\
A^TPB &= C^T - \rho\tilde{L}W \\
W^TW &= D + D^T - B^TPB.
\end{aligned}$$

$0 < \rho < 1$ implies $0 < 1 - \rho^2 < 1$. With $L := \rho\tilde{L}, \gamma^2 := 1 - \rho^2$ we obtain:

$$\begin{aligned}
A^TPA - P &= -LL^T - \gamma^2 P \\
A^TPB &= C^T - LW \\
W^TW &= D + D^T - B^TPB
\end{aligned}$$

□

2.6 Proposition:

Let $G(z) \in \mathbb{R}(z)^{p\times p}$ be DSPR and let $\sum = (A, B, C, D)$ be a minimal realisation of $G(z)$.

1. Then:

   (i) $g_{ij}(z), i, j \in \underline{p}$, analytic in $\overline{\Gamma} = \{z \in \mathbb{C} | \ |z| \geq 1\}$

   (ii) $G(e^{i\omega}) + G^T(e^{-i\omega})$ positive semidefinite hermitian for all $w \in \mathbb{R}$

2. If additional

   a) $rk_{\mathbb{R}(z)}(G(z) + G^T(\frac{1}{z})) = p$ or

   b) $\sigma_{min}(B) > 0$

   then

   (ii') $G(e^{i\omega}) + G^T(e^{-i\omega})$ positive definite hermitian for all $w \in \mathbb{R}$

3. $G(z)$ DSPR if (i) and (ii').

Proof:

1. (i) $G(z)$ DSPR $\Rightarrow G(\rho z)$ DPR for all $\rho \in [\rho^*, 1]$ and some $\rho^*, 0 < \rho^* < 1$. By definition 2.1 all entries $g_{ij}(z)$ of $G(z)$ are analytic in $|z| > \rho^*$ in particular in $\overline{\Gamma}$.

   (ii) Let $\sum = (A, B, C, D)$ be a minimal realization of $G(z), G(z)$ DSPR. Then there exist real matrices $P > 0, L, W$ and a real number γ, such that (2.4) holds. From (2.4a):

$$\begin{aligned} & (\overline{z}I - A^T)P(zI - A) + A^T P(zI - A) + (\overline{z}I - A^T)PA \\ = \; & |z|^2 P - A^T PA = (|z|^2 - 1 + \gamma^2)P + LL^T. \end{aligned} \tag{2.5}$$

Applying (2.4b) and (2.4c) we obtain

$$\begin{aligned} G(z) + G^T(\overline{z}) &= D + D^T + B^T(\overline{z}I - A^T)^{-1}C^T + C(zI - A)^{-1}B \\ &= W^T W + B^T PB + B^T(\overline{z}I - A^T)^{-1}(A^T PB + LW) \\ &+ (B^T PA + W^T L^T)(zI - A)^{-1}B \\ &= W^T W + B^T PB + B^T\left((\overline{z}I - A^T)^{-1}A^T P + PA(zI - A)^{-1}\right)B \\ &+ B^T(\overline{z}I - A^T)^{-1}LW + W^T L^T(zI - A)^{-1}B \\ &= W^T W + B^T PB + B^T(\overline{z}I - A^T)^{-1}\left(A^T P(zI - A)\right. \\ &+ \left.(\overline{z}I - A^T)PA\right)(zI - A)^{-1}B \\ &+ B^T(\overline{z}I - A^T)^{-1}LW + W^T L^T(zI - A)^{-1}B, \end{aligned}$$

Using (2.5)

$$\begin{aligned} &= W^T W + B^T PB \\ &+ B^T(\overline{z}I - A^T)^{-1}\left((|z|^2 + \gamma^2 - 1)P + LL^T - (\overline{z}I - A^T)P(zI - A)\right)(zI - A)^{-1}B \\ &+ B^T(\overline{z}I - A^T)^{-1}LW + W^T L^T(zI - A)^{-1}B \\ &= W^T W + B^T PB + (|z|^2 + \gamma^2 - 1)B^T(\overline{z}I - A^T)^{-1}P(zI - A)^{-1}B \\ &+ B^T(\overline{z}I - A^T)^{-1}LL^T(zI - A)^{-1}B - B^T PB \\ &+ B^T(\overline{z}I - A^T)^{-1}LW + W^T L^T(zI - A)^{-1}B \\ &= (|z|^2 + \gamma^2 - 1)B^T(\overline{z}I - A^T)^{-1}P(zI - A)^{-1}B \\ &+ (W^T + B^T(\overline{z}I - A^T)^{-1}L)(W + L^T(zI - A)^{-1}B). \end{aligned}$$

In particular from:

$$\begin{aligned} G(e^{i\omega}) + G^T(e^{-i\omega}) &= \gamma^2 B^T(e^{-i\omega}I - A^T)^{-1}P(e^{i\omega}I - A)^{-1}B \\ &\quad +(W^T + B^T(e^{-i\omega}I - A^T)^{-1}L)(W + L^T(e^{i\omega}I - A)^{-1}B), \end{aligned}$$

we have for w $\neq 0$:

$$\begin{aligned}
\overline{\mathrm{w}}^T(G^T(e^{i\omega})+G^T(e^{-i\omega}))\mathrm{w} &\geq \gamma^2\sigma_{min}(P)\sigma_{min}^2(B)\sigma_{min}^2(e^{i\omega}I-A)^{-1}||\mathrm{w}||_2^2 \\
\geq \gamma^2\frac{\sigma_{min}(P)\sigma_{min}^2(B)}{||e^{i\omega}I-A||_2^2}||\mathrm{w}||_2^2 &\geq \gamma^2\frac{\sigma_{min}(P)\sigma_{min}^2(B)}{(1+||A||_2)^2}||\mathrm{w}||_2^2 \\
\begin{cases} \geq 0 & \text{if } \sigma_{min}(B)=0 \\ > 0 & \text{if } \sigma_{min}(B)>0 \end{cases} &
\end{aligned} \tag{2.6}$$

hence

$$G(e^{i\omega})+G^T(e^{-i\omega}) \geq 0.$$

2. $rk_{\mathbb{R}(z)}(G(z)+G^T(\frac{1}{z}))=p$, together with $G(z)$ analytic on $\{z \in \mathbb{C} |\, |z|=1\}$ implies

$$rk(G(e^{i\omega})+G^T(e^{-i\omega}))=p \text{ for all } \omega \in \mathbb{R}. \tag{2.7}$$

$\sigma_{min}(B)>0$, implies (2.6). (2.6) as well as (2.7) imply:

$$G(e^{i\omega})+G^T(e^{-i\omega}) > 0.$$

3. Let $\sum=(A,B,C,D)$ be a minimal realization of $G(z)$. There exists $\rho_0, 0<\rho_0<1$, such that $G(\rho z)$ is analytic in $|z|>1$ for all $\rho \in [\rho_0, 1]$. In the following let $\rho > \rho_0$. Because $G(e^{i\omega})+G^T(e^{-i\omega})>0$ for all $\omega \in \mathbb{R}$ there exists a $\eta>0$, such that

$$G(e^{i\omega})+G^T(e^{-i\omega}) > \eta I \text{ for all } \omega \in \mathbb{R}.$$

For $G(\rho e^{i\omega})$ we obtain:

$$\begin{aligned}
G(\rho e^{i\omega}) &= D+C(\rho e^{i\omega}I-A)^{-1}B = G(e^{i\omega})+C((\rho e^{i\omega}I-A)^{-1}-(e^{i\omega}I-A)^{-1}))B \\
&= G(e^{i\omega})+C\left((e^{i\omega}I-A)(\rho e^{i\omega}I-A)^{-1}(e^{i\omega}I-A)^{-1}\right. \\
&- \left.(\rho e^{i\omega}I-A)(\rho e^{i\omega}I-A)^{-1}(e^{i\omega}I-A)^{-1}\right)B \\
&= G(e^{i\omega})+(1-\rho)e^{i\omega}C(\rho e^{i\omega}I-A)^{-1}(e^{i\omega}I-A)^{-1}B.
\end{aligned}$$

Hence:

$$\begin{aligned}
G(\rho e^{i\omega})+G^T(\rho e^{-i\omega}) \geq\ & \eta I+(1-\rho)e^{i\omega}\left(C(\rho e^{i\omega}I-A)^{-1}(e^{i\omega}I-A)^{-1}B\right. \\
& \left.+B^T(e^{-i\omega}I-A^T)^{-1}(\rho e^{-i\omega}I-A^T)^{-1}C^T\right),
\end{aligned}$$

or

$$\overline{\mathrm{w}}^T\left(G(\rho e^{i\omega})+G^T(\rho e^{-i\omega})\right)\mathrm{w} \geq \eta||\mathrm{w}||^2+\delta,$$

with

$$\begin{aligned}\delta &= (1-\rho)e^{i\omega}\overline{\mathrm{w}}^T\left(C(\rho e^{i\omega}I-A)^{-1}(e^{i\omega}I-A)^{-1}B\right.\\ &+ \left.B^T(e^{-i\omega}I-A^T)^{-1}(\rho e^{-i\omega}I-A^T)^{-1}C^T\right)\mathrm{w}.\end{aligned}$$

Then:

$$|\delta| \le 2(1-\rho)||C||\,||B||\,||(\rho e^{i\omega}I-A)^{-1}||\,||(e^{i\omega}I-A)^{-1}||\,||\mathrm{w}||^2.$$

From $\rho e^{i\omega} \notin \sigma(A)$ ($G(z)$ analytic in $|z| \ge 1$ implies $||A||_2 \le \rho_0 < 1$) we obtain:

$$||(\rho e^{i\omega}I-A)^{-1}|| \ge \frac{1}{|\rho - ||A||\,|}.$$

Hence

$$|\delta| \le \frac{2(1-\rho)||C||_2||B||_2}{(1-||A||_2)(\rho-||A||_2)}||\mathrm{w}||_2^2.$$

The right hand side converges to zero monotonically for $\rho \to 1$. Hence there exists a ρ_1 with $1 < \rho_1 < \rho_o$ such that

$$\frac{2||C||_2||B||_2}{1-||A||_2}\frac{1-\rho}{\rho-||A||_2} < \eta \quad \text{for} \quad \rho \in (\rho_1, 1].$$

Therefore $|\delta| \le \eta||\mathrm{w}||^2$ for $\rho > \rho_1$ and $G(\rho e^{i\omega}) + G^T(\rho e^{-i\omega}) \ge 0$ for all $\omega \in \mathbb{R}$.

Lemma 2 from Hitz and Anderson (1969) (cf. remark 2.3) implies that $G(\rho z)$ is DPR for $\rho > \rho_1$.

2.7 Remark:

The condition a) in proposition 2.6 excludes the singularity of $G(e^{i\omega}) + G^T(e^{-i\omega})$ for all $\omega \in \mathbb{R}$. For example $G(z) = \frac{z-0.5}{z+0.6}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ satisfies definition 2.4, but not 2.(ii') in proposition 2.6.

□

3 Discrete positive realness and high gain stability

A scalar system ($p = 1$) with transfer function $g(z) \in \mathbb{R}(z)$ is called high gain stable if there exists some real number $K^* \ge 0$ such that the closed loop system:

$$y = G(z)u, \quad u = -Ky \tag{3.1}$$

is asymptotically stable for all $K > K^*$. To generalize this concept to multivariable systems, $G(z) \in \mathbb{R}(z)^{p\times p}, K \in \mathbb{R}^{p\times p}$, the problem is to define what is meant by "$K \to \infty$"

(cf. Brockett and Byrnes (1981)). Here we restrict ourselves to the following class of feedback-gains:

3.1 Definition:

A linear time-invariant discrete system $\sum$ with transfer function $G(z) \in \mathbb{R}(z)^{p\times p}$ is called high gain stable if there exists some $\lambda^* \in \mathbb{R}_+$ such that the poles of the closed loop transfer function $(I+\lambda G(z))^{-1}G(z)$ lie inside the unit disc $\{z \in \mathbb{C} |\ |z| \leq 1\}$ for all $\lambda > \lambda^*$.

□

The system $(I+\lambda G(z))^{-1}G(z)$ is a special case ($K = I_p$) of the closed loop system:

$$y = G(z)u, \quad u = -\lambda K y, \ \det K \neq 0 \tag{3.2}$$

whose high gain pole behavior ($\lambda \to \infty$) is characterized as follows:

3.2 Lemma:

Let $G(z) \in \mathbb{R}(z)^{p\times p}, K \in \mathbb{R}^{p\times p}, \det K \neq 0$.

(i) If $\det G(z) \not\equiv 0$ then in (3.2) as many poles as there are transmission zeros converge to these zeros. The remaining poles go to infinity.

(ii) If $\det G(z) \equiv 0$, $rk_{\mathbb{R}(z)}G(z) = r < p$, $s = rk_{\mathbb{R}}G(\infty), n(z)$ the zero polynomial of $G(z)$, and $\chi_0(z)$ the pole polynomial of $G(z)$, and

$$t(z) = \frac{\chi_0(z)}{n(z)} \lim_{\lambda\to\infty} \frac{\det[I+\lambda G(z)K]}{\lambda^r} \in \mathbb{R}[z] \tag{3.3}$$

then for $\lambda \to \infty$ as many poles as there are transmission zeros ($\deg n(z)$) converge to these zeros, some poles converge to the zeros of $t(z)$ and the remaining poles go to infinity.

Proof:

(i) is proved in McFarlane and Postlethwaite (1977) for $G(\infty) = 0$. This proof carries over to $G(\infty) \neq 0$. For (ii) let $G(z) = (g_{ij})_{1\leq i,j\leq p}$, $g_{ij}(z) = \frac{x_{ij}(z)}{y_{ij}(z)}, x_{ij}, y_{ij}$ coprime and y_{ij} normalized. Let further $d(z) = lcm\{y_{ij}, 1 \leq i,j \leq p\}$,
$G(z) = \frac{1}{d(z)}\, N(z), N(z) \in \mathbb{R}^{p\times p}[z], r = rk_{\mathbb{R}(z)}G(z)$. The Smith form $S(z)$ of $N(z)$ is of the form

$$S(z) = \begin{pmatrix} S^*(z)_{r\times r} & 0_{r\times(p-r)} \\ 0_{(p-r)\times r} & 0_{(p-r)\times(p-r)} \end{pmatrix}, S(z) = L(z)N(z)R(z),\ L(z) \text{ and } R(z) \text{ unimodular}$$

with

$$S^*(z) = diag(i_j(z), 1 \leq j \leq r), \quad i_j(z) = \frac{d_j(z)}{d_{j-1}(z)}$$

and

$$
\begin{aligned}
d_o(z) &\equiv 1 \\
d_j(z) &= gcd\left\{det\left(N\left(\begin{array}{c} k_1,\ldots,k_j \\ l_1,\ldots,l_j \end{array}\right)(z)\right), 1 \le k_1 < \cdots < k_j \le p, 1 \le l_1 < \cdots < l_j \le p\right\},
\end{aligned}
$$

i.e. $d_j(z)$ is the gcd of all minors of order j of $N(z)$.
Then

$$\det S^*(z) = \prod_{k=1}^{r} i_k(z) = d_r(z).$$

The McMillan-Form $M(z)$ of $G(z)$ and $G(z)K$ is:

$$
\begin{aligned}
M(z) &= \frac{S(z)}{d(z)} = \left(\begin{array}{cc} M^*(z)_{r\times r} & 0_{r\times(p-r)} \\ 0_{(p-r)\times r} & 0_{(p-r)\times(p-r)} \end{array}\right) \\
M^*(z) &= diag\left(\frac{\epsilon_i(z)}{\psi_i(z)}, 1 \le i \le r\right).
\end{aligned}
$$

With $n(z) = \prod_{i=1}^{r} \epsilon_i(z)$ zero polynomial and $\chi_0(z) = \prod_{i=1}^{r} \psi_i(z)$ pole polynomial we have

$$\det M^*(z) = \frac{n(z)}{\chi_0(z)} = \det\left(\frac{S^*(z)}{d(z)}\right) = \frac{d_r(z)}{d(z)^r} \tag{3.4}$$

For a $p \times p$-Matrix A we have:

$$\det(I + \lambda A) = 1 + \lambda tr(A) + \sum_{i=2}^{p-1} \lambda^i \sum_{1 \le k_1 < \cdots < k_i \le p} \det A\left(\begin{array}{c} k_1,\ldots,k_i \\ k_1,\ldots,k_i \end{array}\right) + \lambda^p \det(A), \tag{3.5}$$

(cf. Markus (1973)), hence for $G(z)$:

$$
\begin{aligned}
\det(I + \lambda G(z)K) &= 1 + \lambda tr(G(z)K) + \sum_{i=2}^{r-1} \lambda^i \sum_{1 \le k_1 < \ldots < k_i \le p} \det\left((G(z)K)\left(\begin{array}{c} k_1,\ldots,k_i \\ k_1,\ldots,k_i \end{array}\right)\right) \\
&\quad \underbrace{+\lambda^r \sum_{1 \le k_1 < \ldots < k_r \le p} \det\left((G(z)K)\left(\begin{array}{c} k_1,\ldots,k_r \\ k_1,\ldots,k_r \end{array}\right)\right)}_{:= \lambda^r g(z)}. \qquad (3.6)
\end{aligned}
$$

$$\lambda^r g(z) = \frac{\lambda^r}{d(z)^r} \sum_{1 \le k_1 < \ldots < k_r \le p} \det\left(N(z)K\left(\begin{array}{c} k_1,\ldots,k_r \\ k_1,\ldots,k_r \end{array}\right)\right).$$

Because d_r is the gcd of all minors of order r of $N(z)$, d_r divides every principle minor of order r of $N(z)K$. Let $t_{k_1,\ldots,k_r}(z)$ be the associated quotient polynomial, i.e.

$$\lambda^r g(z) = \frac{\lambda^r}{d(z)^r} d_r(z) \sum_{1 \le k_1 < \ldots < k_r \le p} t_{k_1,\ldots,k_r}(z).$$

With $t(z) = \sum_{1 \le k_1 < k_r \le p} t_{k_1,\ldots,k_r}(z) \in \mathbb{R}[z]$, (3.4) gives:

$$\lambda^r g(z) = \lambda^r \det M^*(z) t(z) = \lambda^r \frac{n(z)t(z)}{\chi_0(z)}. \tag{3.7}$$

With $s = rk(D) \le rk_{\mathbb{R}(z)} G(z)$ we obtain with (3.6) and (3.7)

$$\begin{aligned} \frac{\chi_{\lambda K(z)}}{\chi_0(z)} &= \frac{\det(I + \lambda G(z)K)}{\det(I + \lambda DK)} = \frac{1 + \lambda tr(G(z)K) + \cdots + \lambda^r \frac{n(z)t(z)}{\chi_0(z)}}{1 + a_1\lambda + \cdots + a_s\lambda^s}, & (3.8) \\ a_i &= \sum_{1 \le k_1 < \cdots < k_i \le p} \det\left((DK)\begin{pmatrix} k_1, \ldots, k_i \\ k_1, \ldots, k_i \end{pmatrix}\right) & (3.9) \end{aligned}$$

and therefore

$$\lim_{\lambda \to \infty} \frac{\chi_{\lambda K(z)}}{\lambda^{r-s}} = \frac{n(z)t(z)}{a_s}$$

Putting (3.6) and (3.7) together the claimed equation for $t(z)$ follows.

□

3.3 Corollary

Let $G_{\sum}(z) \in \mathbb{R}^{p \times p}(z)$ be the transfer function of a discrete linear system $\sum$. Then:

(i) If $\det G_{\sum}(z) \not\equiv 0$ then $\sum$ is high gain stable if and only if $G_{\sum}(z)$ has relative degree 0 and all transmission zeros of $G_{\sum}(z)$ are asymptotically stable

(ii) If $\det G_{\sum}(z) \equiv 0$ then the condition of (i) is sufficient for the high gain stability of $\sum$.

□

Consider now (3.1) with $G(z) \in \mathbb{R}(z)^{p \times p}, K \in \mathbb{R}^{p \times p}$.

3.4 Lemma:

Let $G(z)$ DSPR and $K + K^T$ positive semidefinite then the closed loop system (3.1) is asymptotically stable for all $\lambda \ge 0$. In particular DSPR systems are high gain stable.

Proof:
By Landau (1979) we have that (3.1) is asymptotically hyperstable because the Popov-inequality:

$$\begin{aligned} \eta(k_0, k_1) &= \sum_{k=k_0}^{k_1} u^T(k)y(k) = \lambda \sum_{k=k_0}^{k_1} y^T(k)K^T y(k) \\ &= \frac{1}{2}\lambda \sum_{k=k_0}^{k_1} y^T(k)(K + K^T)y(k) \ge 0 \end{aligned}$$

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is satisfied by the associated feedback-block. Then (3.1) is asymptotically stable for $\lambda \geq 0$.

□

Combining proposition 2.6 and corollary 3.3(i) we have:

3.5 Corollary:

A discrete linear system $\sum$ with invertible DSPR transfer function $G_{\sum}(z)$ has relative degree 0 and asymptotically stable zeros and poles.

In Bar-Kana (1989) a class of systems is considered which are high gain stable, however not DSPR. The following results complete the theory developed in Bar-Kana (1986) and provide complete proofs for the results therein.

3.6 Definition: (Bar-Kana)

$G(z) \in \mathbb{R}(z)^{p \times p}$ is called discrete almost strict positive real (DASPR) if

$$\exists K \in \mathbb{R}^{p \times p} \quad \text{such that} \quad H(z) = (I + G(z)K)^{-1}G(z) \text{ is DSPR}$$

□

In Pugh and Ratcliffe (1981) it is shown that the zeros, the infinite zeros and the number of (finite and infinite) poles of a rational transfer function are invariant with respect to constant output feedback. This together with corollory 3.5 implies:

3.7 Corollary

Let $G(z)$ invertible and DASPR then the zeros of $G(z)$ are asymptotically stable and the number of zeros coincides with the number of finite and infinite poles of $G(z)$.

□

Discrete almost strict positive systems are high gain stable:

3.8 Lemma:

Let $G(z)$ DASPR and $K \in \mathbb{R}^{p \times p}$ such that $H(z) = (I + G(z)K)^{-1}G(z)$ is DSPR. Then the closed loop system:

$$y = G(z)u \ , \quad u = -Fy \tag{3.10}$$

is asymptotically stable if

$$(F + F^T) - (K + K^T) \geq 0. \tag{3.11}$$

In particular $G(z)$ is high gain stable.

Proof:
Let $S(A,B,C,D)$ a minimal realization of $G(z)$. Then:

$$\begin{aligned}
\hat{A} &= A - BK(I+DK)^{-1}C \\
\hat{B} &= B - BK(I+DK)^{-1}D \\
\hat{C} &= (I+DK)^{-1}C \\
\hat{D} &= (I+DK)^{-1}D
\end{aligned}$$

is a minimal realization of $H(z)$. Let $E := F - K$. Consider the system matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ of $S(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, which results from $S(A,B,C,D)$ by output feedback with F. Then

$$\begin{aligned}
\tilde{A} &= A - B(K+E)(I+D(K+E))^{-1}C \\
&= A - BK(I+DK)^{-1}C - BK\left((I+DK+DE)^{-1} - (I+DK)^{-1}\right)C \\
&\quad -BE(I+DK+DE)^{-1}C.
\end{aligned}$$

Because $(I+DK+DE)^{-1} - (I+DK)^{-1} = -(I+DK)^{-1}DE(I+DK+DE)^{-1}$ we have:

$$\begin{aligned}
\tilde{A} &= \hat{A} + BK(I+DK)^{-1}DE(I+DK+DE)^{-1}C - BE(I+DK+DE)^{-1}C \\
&= \hat{A} - (-BK(I+DK)^{-1}D + B)E(I+DK+DE)^{-1}C \\
&= \hat{A} - \hat{B}E(I-DK+DE)^{-1}(I+DK)\hat{C} \\
&= \hat{A} - \hat{B}E\left((I+DK)^{-1}(I+DK+DE)\right)^{-1}\hat{C} \\
&= \hat{A} - \hat{B}E(I+\hat{D}E)^{-1}\hat{C}.
\end{aligned}$$

Similarily we obtain for $\tilde{B}, \tilde{C}$ and $\tilde{D}$:

$$\begin{aligned}
\tilde{B} &= B - B(K+E)(I+D(K+E))^{-1}D \\
&= \hat{B} - \hat{B}E(I+\hat{D}E)^{-1}\hat{D} \\
\tilde{C} &= (I+D(K+E))^{-1}C = (I+\hat{D}E)^{-1}\hat{C} \\
\tilde{D} &= (I+D(K+E))^{-1}D = (I+\hat{D}E)^{-1}\hat{D}.
\end{aligned}$$

From these formulas it is evident, that $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ describe also the system, which is obtained by output feedback with E around $S(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. By lemma 3.4 $H(z)$ DSPR and $E + E^T \geq 0$ imply that $S(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is asymptotically stable.
If $F = \lambda I$ then there always exist a $\lambda^* > 0$ s.t. $F - K + (F-K)^T = 2\lambda I - (K + K^T) > 0$ for $\lambda > \lambda^*$, hence $S(A,B,C,D)$ is high gain stable.

□

The following theorem of Bar-Kana describes a "principal" possibility to obtain DASPR systems by augmentation from strict causal linear systems which are stabilizable by constant output feedback.

3.9 Theorem:
Let $G(z) \in \mathbb{R}(z)^{p\times p}$, be strict proper rational and $K \in \mathbb{R}^{p\times p}, \det K \neq 0$, a matrix such that $H(z) = (I + G(z)K)^{-1}G(z)$ is asymptotically stable, then

$$F(z) = G(z) + K^{-1} \tag{3.12}$$

is DASPR.

Proof:
Let $S(A, B, C, D), D = K^{-1}$ a minimal realization of $F(z)$. Let $k_1 := \max\{\lambda | \lambda \in \mathbb{R},\ \lambda \in \sigma(D^{-1}) \cup \{0\}\}$, then $\det(I + kD) \neq 0$ for $k > k_1$. For the closed loop system $S(A_k, B_k, C_k, D_k)$ with:

$$\begin{aligned}
A_k &= A - kB(I + kD)^{-1}C \\
B_k &= B - kB(I + kD)^{-1}D = B(I + kD)^{-1} \\
C_k &= (I + kD)^{-1}C \\
D_k &= (I + kD)^{-1}D
\end{aligned}$$

and transfer function

$$G_k(z) = C_k(zI - A_k)^{-1}B_k + D_k$$

we have:

(i) $\lim_{k\to\infty} A_k = A - BD^{-1}C = A - BKC$ is asymptotically stable by assumption. Then there exists $k_2 > 0$, such that for $k > k_2, A_k$ is asymptotically stable. Hence

$$G_k(z) \text{ analytic in } |z| \geq 1 \text{ for } k > k_2 \tag{3.13}$$

(ii) Define

$$\begin{aligned}
H_k(z) &:= C_k(zI - A_k)^{-1}B_k \\
&= (I + kD)^{-1}C\left(zI - (A - kB(I + kD)^{-1}C)\right)^{-1} B(I + kD)^{-1}.
\end{aligned}$$

Then $\det D \neq 0$ implies

$$\lim_{k\to\infty} k^2 H_k(z) = D^{-1}C(zI - (A - BD^{-1}C))^{-1}BD^{-1}.$$

Hence:

$$\lim_{k\to\infty} kH_k(z) = 0, \quad \text{for } z \notin \sigma(A - BD^{-1}C).$$

Therefore:

$$\lim_{k\to\infty} kG_k(z) = \lim_{k\to\infty}(kH_k(z) + kD_k) = \lim_{k\to\infty} kD_k = \lim_{k\to\infty} k(D^{-1} + kI)^{-1} = I$$

for $z \notin \sigma(A - BD^{-1}C)$ and so:

$$\lim_{k \to \infty} k(G_k(e^{i\omega}) + G_k^T(e^{-i\omega})) = 2I > 0, \quad \omega \in \mathbb{R}.$$

However then there exists $k_3 > 0$ such that for $k > k_3$

$$G_k(e^{i\omega}) + G_k^T(e^{-i\omega}) > 0, \quad \text{for all } \omega \in \mathbb{R}. \tag{3.14}$$

(3.13) and (3.14) together with prop. 2.6 part 3 imply DSPR for $k > \max(k_1, k_2, k_3)$. Hence $F(z)$ is DASPR by definition (3.6).

□

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