DIPOLYNOMIAL MINIMAL BASES AND LINEAR SYSTEMS IN AR-REPRESENTATION

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April 10, 1995

Abstract

This paper deals with a module theoretic approach to the dipolynomial matrix parametrization of discrete-time behaviour systems as introducted by Willems [1991]. Canonical minimal lag representations for these behaviours are constructed which are tighter than the corresponding polynomial representations.

1 Introduction

Matrices of rational functions play an important role in the description of the input/output behaviour of linear systems via transfer functions (matrices). With the introduction of minimal bases of rational vector spaces Forney [1975] gave elegant algebraic solutions for problems in the context of the input/output behaviour of linear systems; for example, problems of realization and invertibility of linear systems.
For the algebraic description of the state space approach and the system description via polynomial matrices (introduced by Rosenbrock [1970]) modules over the polynomial ring are the relevant structures.
The foundation for the module theoretic treatment of linear systems has been layed in the pioneering chapter 10 of the book of Kalman et al. [1969]. Within the algebraic theory of linear systems, developed in the last twenty years, very important contributions are due to Fuhrmann (cf. Fuhrmann [1976], [1977], [1991]). His approach, originally developed for a better understanding of the concept of strict system equivalence, was extended over the years to a systematic treatment of a variety of problems for state-space-, transfer-function- and polynomial system matrix representations of linear systems.
In the last years Willems developed in a series of papers ([1986a], [1986b], [1987], [1988], [1991]) a general theory of dynamical behaviour systems. In this framework it is shown that every discrete-time, linear, time-invariant complete behaviour system has an autoregressive (AR-) representation, where the representing matrix is dipolynomial (cf. Willems [1991]).

In our paper we consider some aspects of a module theoretic treatment of those sytem representations over the ring of dipolynomials $F[s, s^{-1}]$. In particular, we extend the concept of polynomial minimal bases for modules and rational vector spaces (cf. Forney

[1975], Münzner and Prätzel-Wolters [1979], Kailath [1980]) to the dipolynomial case. We proceed as follows:

Section 2 contains some preliminaries concerning the ring of dipolynomials and dipolynomial matrices.

In Section 3 we introduce dipolynomial minimal bases for rational vector spaces and dipolynomial modules and compare these bases to the existing polynomial concepts. Furthermore, we characterize the dipolynomial minimal basis transformations and derive canonical dipolynomial minimal bases in echelon form.

In Section 4 we apply the results of Section 3 to discrete-time AR-systems. In particular, we identify the dipolynomial minimal module bases as the minimal lag descriptions (cf. Willems [1991]) of the associated behaviour system. This way the canonical form constructed in Section 3 is shown to be a trim canonical form for the dipolynomial matrix parametrization of AR-equations. It coincides with a (modified) canonical representation given in Willems [1991].

2 Preliminaries

Let F denote any field, $F[s]$ the ring of polynomials in the indeterminate $s, F(s)$ the field of rational functions and

$$F[s,s^{-1}] = \{\alpha_L s^L + \ldots + \alpha_\ell s^\ell : L, \ell \in \mathbb{Z}, \ell \leq L,\ \alpha_k \in F \text{ for } k \in \{\ell,\ldots,L\}\} \qquad (2.1)$$

the *ring of dipolynomials* with coefficients in F resp. . Observe that $F[s,s^{-1}]$ is the ring which is obtained by *localization* of $F[s]$ at $S := \{s^k : k \in \mathbb{N}\}$, which is a submonoid of the multiplicative monoid of $F[s]$ (see e.g. Jacobson [1989]); in particular this means that all elements of S are units in $F[s,s^{-1}]$.
It is well known that $F[s]$ is an euclidean ring with respect to the degree-function

$$\deg : \begin{array}{ccc} F[s] & \longrightarrow & \mathbb{N} \\ \alpha_L s^L + \ldots + \alpha_\ell s^\ell & \longrightarrow & L \end{array} \qquad (2.2)$$

The units in $F[s]$ are the nonzero constants $\alpha \neq 0, \alpha \in F$. Modifying (2.2) for $F[s,s^{-1}]$ in the following way:

$$\mathrm{ddeg} : \begin{array}{ccc} F[s,s^{-1}] & \longrightarrow & \mathbb{N} \\ \alpha_L s^L + \ldots + \alpha_\ell s^\ell & \longrightarrow & L-\ell \end{array} \qquad (2.3)$$

one obtains:

Lemma 2.1 $(F[s,s^{-1}], \mathrm{ddeg})$ *is an euclidean ring. The units in* $F[s,s^{-1}]$ *are the elements of the form* $\alpha s^d, d \in \mathbb{Z}, \alpha \in F, \alpha \neq 0$. ■

We skip the easy proof; note that the second statement holds by construction of $F[s,s^{-1}]$. Moreover, the following characterization of *irreducible* elements in $F[s,s^{-1}]$ is straightforward:

Lemma 2.2 *$q(s,s^{-1}) \in F[s,s^{-1}]$ is irreducible if and only if q has a representation of the form $q(s,s^{-1}) = s^k p(s)$, where $k \in \mathbb{Z}$ and $s \neq p(s) \in F[s]$ is irreducible in $F[s]$.* ■

Note that $\deg(s^L) = L$ for $L \in \mathbb{N}$, while $\operatorname{ddeg}(s^L) = 0$ for $L \in \mathbb{Z}$. We denote by $F^{g\times q}[s]$ resp. $F^{g\times q}[s,s^{-1}]$ the set of $g \times q$ polynomial resp. dipolynomial matrices. The units in the *rings* $F^{n\times n}[s]$ resp. $F^{n\times n}[s,s^{-1}]$ are called *unimodular matrices.* A matrix $R \in F^{n\times n}[s](F^{n\times n}[s,s^{-1}])$ is unimodular iff $\det R$ is a unit in $F[s](F[s,s^{-1}])$.

3 Minimal bases

To every rational $k \times n$-matrix $G(s) \in F^{k\times n}(s)$ is associated the $F(s)$-vector space $V_G := F^{1\times k}(s)G(s)$, and there holds $\dim_{F(s)} V_G = \operatorname{rank}_{F(s)} G$.

Conversely there exists for every $F(s)$-vector space $V \subset F^{1\times n}(s)$ with $\dim_{F(s)} V = k$ a matrix $G(s) \in F^{k\times n}(s)$ of $F(s)$ – rank k such that $V = V_G$. Matrices $G(s)$ of full row rank are called *bases* for the vector space V_G. Analogously there are associated free $F[s]$ – ($F[s,s^{-1}]$–) modules of vector polynomials (-dipolynomials) $M_G := F^{1\times k}[s]G(s)\,(\tilde{M}_G := F^{1\times k}[s,s^{-1}]G(s,s^{-1}))$ to every polynomial (-dipolynomial) $k \times n$-matrix $G(s) \in F^{k\times n}[s](G(s,s^{-1}) \in F^{k\times n}[s,s^{-1}])$. Observe that

$$\dim M_G = \operatorname{rank}_{F[s]} G(s),\ \dim \tilde{M}_G = \operatorname{rank}_{F[s,s^{-1}]} G(s,s^{-1}) \tag{3.1}$$

Matrices $G(s)(G(s,s^{-1}))$ with $F[s]$ – ($F[s,s^{-1}]$–) linear independent rows are called *bases* for the modules $M_G(\tilde{M}_G)$. For those basis matrices we have

$$\begin{aligned} M_{G_1} = M_{G_2} &\Leftrightarrow \exists T(s) \in F^{k\times k}[s] \text{ unimodular} \quad s.t. \quad G_1 = TG_2 \\ \tilde{M}_{G_1} = \tilde{M}_{G_2} &\Leftrightarrow \exists T(s) \in F^{k\times k}[s,s^{-1}] \text{ unimodular} \quad s.t. \quad G_1 = TG_2 \end{aligned}$$

The extension of the scalar degree functions deg and ddeg to the vector case associates degree-structure to submodules $M \subset F^{1\times n}[s]$ and $\tilde{M} \subset F^{1\times n}[s,s^{-1}]$:

$$\deg : \begin{array}{ccc} F^{1\times n}[s] & \longrightarrow & \mathbb{N} \\ \alpha_L s^L + \ldots + \alpha_\ell s^\ell & \longrightarrow & L \end{array},\ \operatorname{ddeg} : \begin{array}{ccc} F^{1\times n}[s,s^{-1}] & \longrightarrow & \mathbb{N} \\ \alpha_L s^L + \ldots + \alpha_\ell s^\ell & \longrightarrow & L-\ell \end{array}$$

The pioneering work of Forney [1975] on *minimal bases for rational vector spaces* initiated a series of papers on minimal bases and their relations to controltheoretic constructions. In this section we extend the concept of minimal bases to the dipolynomial case.
For a dipolynomial matrix $G(s,s^{-1}) \in F^{k\times n}[s,s^{-1}]$,

$$G(s,s^{-1}) = \begin{pmatrix} g_1(s,s^{-1}) \\ \vdots \\ g_k(s,s^{-1}) \end{pmatrix} = \begin{pmatrix} \alpha^1_{\nu_1} \cdot s^{n_1} + \ldots + \alpha^1_0 \cdot s^{n_1-\nu_1} \\ \vdots \\ \alpha^k_{\nu_k} \cdot s^{n_k} + \ldots + \alpha^k_0 \cdot s^{n_k-\nu_k} \end{pmatrix} \tag{3.2}$$

the numbers

$$\nu_i := \operatorname{ddeg} g_i(s,s^{-1}),\ i \in \underline{k} \tag{3.3a}$$

$$\nu := \sum_{i=1}^{k} \nu_i \tag{3.3b}$$

are called the *dipolynomial indices* and the *dipolynomial order* of G.

Definition 3.1 A matrix $G(s,s^{-1}) \in F^{k\times n}[s,s^{-1}]$ is called a *dipolynomial minimal basis* of a $F(s)$-vector space $V \subset F^{1\times n}(s)$ if

(i) G is basis of V

(ii) The dipolynomial order of G is minimal among all dipolynomial bases of V □

Associated with $G(s,s^{-1})$ as in (3.2) are the *highest row coefficient* resp. *lowest row coefficient* matrix

$$[G]_h^r := \begin{pmatrix} \alpha_{\nu_1}^1 \\ \vdots \\ \alpha_{\nu_k}^k \end{pmatrix} \in \mathbb{R}^{k\times n} \ni [G]_\ell^r := \begin{pmatrix} \alpha_0^1 \\ \vdots \\ \alpha_0^k \end{pmatrix} \tag{3.4}$$

A matrix $G(s,s^{-1})$ with $[G]_h^r$ and $[G]_\ell^r$ of full row rank is called *row proper.*

Theorem 3.2 *(dipolynomial minimal vector space bases)*

Let $G(s,s^{-1}) \in F^{k\times n}[s,s^{-1}]$ as in (3.2) be a dipolynomial basis of $V = F^{1\times k}(s)G(s,s^{-1})$. Then the following conditions are equivalent:

(i) G is a dipolynomial minimal basis for V

(ii) (a) The greatest common divisor (gcd) of all $k\times k$-minors of G is a unit in $F[s,s^{-1}]$

(b) $\mathrm{rank}_F[G]_h^r = k$ and $\mathrm{rank}_F[G]_\ell^r = k$

(iii) (a) G is nonsingular modulo $p(s,s^{-1})$ for all irreducible dipolynomials $p(s,s^{-1}) \in F[s,s^{-1}]$

(b) Let $f(s,s^{-1}) = (f_1,\ldots,f_{\binom{n}{k}})$ the vector of all $k\times k$-minors f_i of G. Then

$$\nu := \sum_{i=1}^{k} \nu_i = ddeg(f)$$

(iv) If $y = xG$ is dipolynomial, then

(a) x is dipolynomial

(b) For $x(s,s^{-1}) = (x_1(s,s^{-1}),\ldots,x_k(s,s^{-1}))$ with $x_i(s,s^{-1}) = \beta_{\tau_i}^i s^{k_i} + \ldots + \beta_0^i s^{k_i-\tau_i}$ there holds

$$\mathrm{ddeg}(y) = \max_{i:x_i\neq 0}(k_i+n_i) - \min_{i:x_i\neq 0}(k_i-\tau_i+n_i-\nu_i) \tag{3.5}$$

(predictable degree property)

(v) For all $d \geq 1$

$$\dim_F V_d = \sum_{i:\nu_i\leq d}(d-\nu_i) \tag{3.6}$$

where $V_d := \{y(s) \in V \cap F^{1\times n}[s] : \deg(y(s)) < d\}$.
Furthermore the following conditions are equivalent:
(ii)(a), (iii)(a), (iv)(a) and

(c) $G(s,s^{-1})$ *is a basis of* $\tilde{M}_V := V \cap F^{k\times n}[s,s^{-1}]$

Proof:

The structure of the proof is as follows:

$$\begin{array}{ccccccccc}
 & & (ii)(a) & & (c) & & & & \\
 & & \Updownarrow & & \Updownarrow & & & & \\
 & & (iii)(a) & \Leftrightarrow & (iv)(a) & & & & \\
 & \nearrow & & & & \searrow & & & \\
(i) & & & & & & (v) & \Rightarrow & (i) \\
 & \searrow & & & & \nearrow & & & \\
 & & (ii)(b) & \Leftrightarrow & (iv)(b) & & & & \\
 & & \Updownarrow & & & & & & \\
 & & (iii)(b) & & & & & &
\end{array}$$

The proofs for $(ii)(a) \Leftrightarrow (iii)(a) \Leftrightarrow (iv)(a) \Leftrightarrow (c)$ are easily adapted from the corresponding proofs for the polynomial case given in Forney [1975] using the characterization of irreducible dipolynomials in Lemma 2.2; also the proof of $(iv) \Rightarrow (v)$ and the implication $(v) \Rightarrow (i)$ is straightforward following that reference.

(ii)(b) $\Leftrightarrow$ (iii)(b):

Let N be any $k \times k$-submatrix of G and $[N]_h^*([N]_\ell^*)$ the corresponding submatrix of $[G]_h^r([G]_\ell^r)$. Then

$$\det N = \det[N]_h^* \cdot s^n + \ldots + \det[N]_\ell^* \cdot s^{n-\nu}$$

where

$$n := \sum_{i=1}^{k} n_i \quad \text{and} \quad \nu := \sum_{i=1}^{k} \nu_i$$

Hence (ii)(b) is equivalent to the existence of submatrices N_1 and N_2 of G such that $\det[N_1]_h^* \neq 0 \neq \det[N_2]_\ell^*$, which is equivalent to (iii)(b).

(ii)(b) $\Rightarrow$ (iv)(b):

There holds

$$y = xG = [y]_h s^b + \ldots + [y]_\ell s^e \tag{3.7}$$

with

$$b := \max_{i:x_i\neq 0}(k_i + n_i), \quad e := \min_{i:x_i\neq 0}(k_i - \tau_i + n_i - \nu_i)$$

Let $[G_i]_h([G_i]_\ell)$ denote the i-th row of $[G]_h^r([G]_\ell^r)$; then

$$[y]_h = \sum_{i=1}^{k}[x_i]_h \cdot [G_i]_h, \quad [y]_\ell = \sum_{i=1}^{k}[x_i]_\ell [G_i]_\ell$$

where

$$[x_i]_h = \begin{cases} 0 & , \quad k_i + n_i < b \\ \beta^i_{\tau_i} & , \quad k_i + n_i = b \end{cases}$$

and

$$[x_i]_\ell = \begin{cases} 0 & , \quad k_i - \tau_i + n_i - \nu_i > e \\ \beta^i_0 & , \quad k_i + \tau_i + n_i - \nu_i = e \end{cases}$$

The full rank property of $[G]^r_h$ and $[G]^r_\ell$ implies $[y]_h \neq 0 \neq [y]_\ell$.

(iv)(b) ⇒ (ii)(b):

Assume $\text{rank}[G]^r_h < k$. Then there exist $x_i \in F$, $i \in \{1, \ldots, k\}$, $(x_1, \ldots, x_k) \neq (0, \ldots, 0)$ such that

$$\sum_{i=1}^{k} x_i [G_i]_h = 0 \tag{3.8}$$

Let i_0 an index such that $\nu_{i_0} = \max\{\nu_i : x_i \neq 0\}$; furthermore let

$$x(s, s^{-1}) := (x_1 \cdot s^{n_{i_0} - n_1}, \ldots, x_k \cdot s^{n_{i_0} - n_k})$$

and

$$\hat{g}(s, s^{-1}) = x(s, s^{-1}) G(s, s^{-1}) = \sum_{i=1}^{k} x_i g_i(s, s^{-1}) \cdot s^{n_{i_0} - n_i}$$

Then because of (3.8) $\text{ddeg}\, \hat{g}(s, s^{-1}) < \nu_{i_0}$; however

$$\max_{i: x_i \neq 0} (k_i + n_i) = \max_{i: x_i \neq 0} (n_{i_0} - n_i + n_i) = n_{i_0}$$

and

$$\min_{i: x_i \neq 0} (k_i - \tau_i + n_i - \nu_i) = \min_{i: x_i \neq 0} ((n_{i_0} - n_i) + (n_i - \nu_i)) = \min_{i: x_i \neq 0} (n_{i_0} - \nu_i) = n_{i_0} - \nu_{i_0}$$

Thus

$$\max_{i: x_i \neq 0} (k_i + n_i) - \min_{i: x_i \neq 0} (k_i - \tau_i + n_i - \nu_i) = \nu_{i_0}$$

contradicting (iv)(b). The proof for $[G]^r_\ell$ is completely analogous.

(i) ⇒ (ii):

See the reduction algorithms below. ■

Starting with a basis $G(s) \in F^{k \times n}(s)$, one obtains a dipolynomial minimal basis by the following 3 steps:

(a) Multiply each row of G by the least common multiple of the denominators to make G dipolynomial

(b) Reduce the resulting matrix to a basis of $\tilde{M}_V = V \cap F^{k\times n}[s, s^{-1}]$

(c) Reduce further to obtain full rank highest and lowest coefficient matrices

Algorithm performing step (b):

This algorithm proves the implication (i) $\Rightarrow$ (iii)(a) of Theorem 3.2.
Let $G(s, s^{-1})$ be given in the form (3.2). Define:

$$\bar{G}(s, s^{-1}) = \begin{pmatrix} \bar{g}_1(s, s^{-1}) \\ \vdots \\ \bar{g}_k(s, s^{-1}) \end{pmatrix} = \mathrm{diag}(s^{-(n_1-\nu_1)}, \ldots, s^{-(n_k-\nu_k)}) G(s, s^{-1})$$

Let $\epsilon(s)$ resp. $\delta(s)$ denote the gcd of the $k \times k$-minors of G resp. $\bar{G}$. Then

$$\delta(s) = s^{-\rho}\epsilon(s) \quad \text{where} \quad \rho := \sum_{i=1}^{k}(n_i - \nu_i) \tag{3.9}$$

If $\epsilon(s)$ is a unit in $F[s, s^{-1}]$ then nothing has to be proved because of Theorem 3.2 (iii)(a) $\Leftrightarrow$ (c).
Assume $\epsilon(s)$ is not a unit in $F[s, s^{-1}]$. Then by Lemma 2.2 $\epsilon(s)$ has an irreducible polynomial factor $p(s) \neq s$, which in view of (3.9) is also a factor of $\delta(s)$. Hence, modulo p, $\bar{G}$ does not have full rank. By applying Algorithm 2 in Forney [1975] we can replace a row $\bar{g}_{i_0}$ of $\bar{G}$ by a row $\bar{g}$ of polynomial degree strictly less than that of $\bar{g}_{i_0}$. Since $\bar{g}$ is a linear combination of the rows of $\bar{G}$ and the coefficient of $\bar{g}_{i_0}$ in this linear combination is not equal to zero, the resulting matrix $\tilde{G}(s, s^{-1})$ is also a basis and $G'(s, s^{-1}) := \mathrm{diag}(s^{(n_1-\nu_1)}, \ldots, s^{(n_k-\nu_k)})\tilde{G}(s, s^{-1})$ is a dipolynomial basis of lower dipolynomial order than $G(s, s^{-1})$, since

$$\mathrm{ddeg}(\bar{g}) \leq \deg(\bar{g}) < \deg(\bar{g}_{i_0}) = \mathrm{ddeg}(\bar{g}_{i_0})$$

■

Algorithm performing step (c):

This algorithm provides a proof for the implication (i) $\Rightarrow$ (ii)(b) of Theorem 3.2.
Assume $\mathrm{rank}[G]_h^r < k$ and let $\hat{g}(s, s^{-1})$ be constructed as in the proof of Theorem 3.2, (iv)(b) $\Rightarrow$ (ii) (b). Replace row g_{i_0} by $\hat{g}(s, s^{-1})$ to obtain a basis $\hat{G}(s, s^{-1})$ of V_G of lower dipolynomial order. Iterate until $\mathrm{rank}_F[G]_h^r = k$. If $\mathrm{rank}[G]_\ell^r < k$ proceed analogously to the case $\mathrm{rank}[G]_h^r < k$. ■

In view of condition (v) of Theorem 3.2 it is clear that the dipolynomial indices do not depend - up to ordering - on the specific minimal basis but only on the vector space V. Hence one can define:

Definition 3.3 The *dipolynomial indices* $(\nu)_V = (\nu_1, \ldots, \nu_k)$ and the *dipolynomial order* $\nu_V = \sum_{i=1}^{k} \nu_i$ of a k-dimensional $F(s)$-vector space $V \subset F^{1\times n}(s)$ are the dipolynomial indices resp. the dipolynomial order of any dipolynomial minimal basis of V (in descending order).
□

Replacing the $F(s)$-vector space $V \subset F^{1\times n}(s)$ by free $F[s,s^{-1}]$- *submodules* $\tilde{M} \subset F^{1\times n}[s,s^{-1}]$ the notions of bases, dipolynomial indices and dipolynomial order are defined completely analogous to the vector space setting. Without proof we state:

Theorem 3.4 *Let $G(s,s^{-1})$ in the form (3.2) be a dipolynomial basis of the $F[s,s^{-1}]$-module $\tilde{M}_G = F^{1\times k}[s,s^{-1}]\cdot G(s,s^{-1})$. Then the following conditions are equivalent:*

(i) G is a minimal basis for $\tilde{M}_G$

(ii) $\operatorname{rank}_F[G]_h^r$ *and* $\operatorname{rank}_F[G]_\ell^r = k$

(iii) Let $f(s,s^{-1}) = (f_1,\ldots,f_{\binom{n}{k}})$, where the f_i are the $k\times k$-minors of G, then

$$\nu := \sum_{i=1}^{k}\nu_i = \operatorname{ddeg}(f)$$

(iv) For $y = xG$ with $x(s,s^{-1}) = (x_1(s,s^{-1}),\ldots,x_k(s,s^{-1}))$ and $x_i(s,s^{-1}) = \beta^i_{\tau_i}s^{k_i} + \ldots + \beta^i_0 s^{k_i-\tau_i}$ there holds

$$\operatorname{ddeg}(y) = \max_{i:x_i\neq 0}(k_i+n_i) - \min_{i:x_i\neq 0}(k_i-\tau_i+n_i-\nu_i)$$

(predictable degree property)

(v) For all $d\geq 1$ we have

$$\dim_F \tilde{V}_d = \dim_F\{y\in\tilde{M}_G\cap F^{1\times n}[s] : \deg(y) < d\} = \sum_{i:\nu_i\leq d}(d-\nu_i)$$

■

It should be mentioned that the equivalence of (i), (ii) and (iii) is stated in Willems [1991], however, derived there in a quite different manner.
Again condition (v) in Theorem 3.4 shows that the dipolynomial indices only depend on the module $\tilde{M}$. Hence we can also introduce the *dipolynomial indices*

$$(\tilde{\nu})_{\tilde{M}} = (\tilde{\nu}_1,\ldots,\tilde{\nu}_k),\quad \tilde{\nu}_1\geq\ldots\geq\tilde{\nu}_k$$

and the dipolynomial order

$$\tilde{\nu}_{\tilde{M}} = \sum_{i=1}^{k}\tilde{\nu}_i$$

of a k-dimensional submodule $\tilde{M}\subset F^{1\times n}[s,s^{-1}]$ as the indices resp. order of any dipolynomial minimal basis of $\tilde{M}$.

For polynomial modules $M \subset F^{1\times n}[s]$ polynomial minimal bases, indices $(\nu)_M = (\nu_1,\ldots,\nu_k), \nu_1\geq\ldots\geq\nu_k$ and order $\nu_M = \sum_{i=1}^{k}\nu_i$ are defined and analyzed in Münzner and Prätzel-Wolters [1979].

The following results summarize some properties of the different polynomial and dipolynomial concepts. Their proofs are straightforward consequences of the characterizations of minimal bases in Forney [1975], Münzner and Prätzel-Wolters [1979] and this paper.

Theorem 3.5 *Let $V \subseteq F^{1\times n}(s)$ a $F(s)$-vector space. Then there holds:*

(i) Every Forney-minimal basis of V is also a dipolynomial minimal basis.

(ii) Every dipolynomial minimal basis $G(s, s^{-1})$ of V in the form (3.2) with $n_i = \nu_i$ for $i = 1, \ldots, k$ is also a Forney-minimal basis.

(iii) Forney indices (order) and dipolynomial indices (order) of V coincide. ■

Remark 3.6 Contrary to Forney-minimal bases the maximal degree of the $k \times k$- minors of a dipolynomial minimal basis can be smaller than the order ν. Equality holds only in the case when there exist columns $j_1, \ldots, j_k$ of $[G]_h^r$ and $[G]_\ell^r$ such that the matrices formed by them are nonsingular, i.e.

$$\det\,[G]_h^{j_1,\ldots,j_k} \neq 0 \neq \det[G]_\ell^{j_1,\ldots,j_k}$$

Consider the example

$$W(s) = \begin{pmatrix} s & 1 & s+2 \\ s^2+2s & s+1 & 2s^2+1 \end{pmatrix}$$

Then

$$\text{rank}[W]_h^r = \text{rank}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} = 2$$

The 2×2-minors of W are $-s, s^2-3s-1$ and $s(s^2-4s-3)$ which are $\mathbb{R}[s]$-coprime, hence W is a Forney-minimal basis with indices $\nu_1 = 2, \nu_2 = 1$ and order 3. However, there does not exist a 2×2-minor of dipolynomial degree 3 although there exists one of polynomial degree 3. The reason is that for $j = (j_1, j_2) = (1, 3)$ $\det[W]_h^j \neq 0$ while $\det[W]_\ell^j = 0$. □

Let $G(s) \in F^{k\times n}[s]$ with $\text{rank}_{F[s]}G = k$. As before we associate with G the $F(s)$-vector space $V_G := F^{1\times k}(s)\cdot G(s)$. Furthermore, we define $\tilde{M}_{V_G} := V_G \cap F^{1\times n}[s, s^{-1}]$ and $M_{V_G} := V_G \cap F^{1\times n}[s]$. Thus

$$F^{1\times n}(s) \supset V_G \supset \tilde{M}_{V_G} \supset M_{V_G}$$

On the other hand, let $M_G := F^{1\times k}[s]\cdot G(s)$ resp. $\tilde{M}_G := F^{1\times k}[s, s^{-1}]\cdot G(s)$ the polynomial resp. dipolynomial $F[s]$- resp. $F[s, s^{-1}]$-modules associated with G. Then

$$F^{1\times n}[s] \supset M_G \subset \tilde{M}_G \subset V_G$$

Now one can easily derive conditions for the equality of the polynomial resp. dipolynomial structures in the above inclusions.

Theorem 3.7 *Let $G(s) \in F^{k\times n}[s]$ with rank$_{F[s]}G = k$.*

(i) G is a Forney-minimal basis for V_G iff G is a polynomial minimal basis for M_G and there holds

$$M_G = V_G \cap F^{1\times n}[s] = M_{V_G} \tag{3.10}$$

Furthermore, the following conditions are equivalent to (3.10):

(a) The gcd of the $k \times k$-minors of G is a unit in $F[s]$

(b) G is nonsingular modulo $p(s)$ for all irreducible polynomials $p(s) \in F[s]$

(c) G is basis for M_{V_G}

(d) All left divisors of G are unimodular elements of $F^{k\times k}[s]$

(ii) G is a dipolynomial minimal vector space basis for V_G iff G is a dipolynomial minimal basis for $\tilde{M}_G$ and

$$\tilde{M}_G = V_G \cap F^{1\times n}[s, s^{-1}] = \tilde{M}_{V_G} \tag{3.11}$$

Conditions (ii)(a) , (iii)(a), (iv)(a) and (c) of Theorem 3.2 are equivalent conditions for (3.11).

(iii) Assume G is in the form (3.2) with $n_i = \nu_i$ for $i = 1, \ldots, k$.
Then G is a dipolynomial minimal basis of $\tilde{M}_G$ iff G is a polynomial minimal basis of M_G and

$$M_G = \tilde{M}_G \cap F^{1\times n}[s] \tag{3.12}$$

(iv) G is a dipolynomial minimal vector space basis for V_G iff G is a dipolynomial minimal basis for $\tilde{M}_G$ and there holds:

$$M_{V_G} = \tilde{M}_G \cap F^{1\times n}[s]$$

Proof:

(i) Münzner and Prätzel-Wolters [1979]

(ii) Obvious

(iii) "$\Leftarrow$" Since G is a polynomial minimal basis, we obtain $\text{rank}[G]_h^r = k$. Assume $\text{rank}[G]_\ell^r < k$. Then there exists $0 \neq x_0 \in F^{1\times k}$ such that $x_0[G]_\ell^r = 0$. Moreover, $x := x_0 \cdot s^{-1} \in F^{1\times k}[s, s^{-1}] \backslash F^{1\times k}[s]$ and $xG \in F^{1\times n}[s]$. Hence $xG \in \tilde{M}_G \cap F^{1\times n}[s]$, but $xG \notin M_G$. Thus G is a dipolynomial minimal basis (Theorem 3.4 (ii)).

"$\Rightarrow$" Let G a dipolynomial minimal basis. Then G is a polynomial minimal basis. Obviously, $M_G \subseteq \tilde{M}_G \cap F^{1\times n}[s]$. Let $x \in \tilde{M}_G \cap F^{1\times n}[s]$. Then there exists $y \in F^{1\times k}[s, s^{-1}]$ with $x = yG \in F^{1\times n}[s]$ and $[x]_\ell = [y]_\ell \cdot [G]_\ell^r \neq 0$. Since $[G]_\ell^r$ is of order 0, x polynomial implies y polynomial, and thus (3.12) holds.

(iv) "$\Rightarrow$" If G is a dipolynomial minimal basis of V_G, we obtain by (ii) $\tilde{M}_G = V_G \cap F^{1\times n}[s, s^{-1}]$. Then

$$\tilde{M}_G \cap F^{1\times n}[s] = (V_G \cap F^{1\times n}[s, s^{-1}]) \cap F^{1\times n}[s] = V_G \cap F^{1\times n}[s] = M_{V_G}$$

"$\Leftarrow$" We show: $\tilde{M}_G = V_G \cap F^{1\times n}[s, s^{-1}]$; the statement then follows with (ii). Obviously, $\tilde{M}_G \subseteq V_G \cap F^{1\times n}[s, s^{-1}]$. Let $x \in V_G \cap F^{1\times n}[s, s^{-1}]$. Then there exists $y \in F^{1\times k}(s)$ such that $x = yG \in F^{1\times n}[s, s^{-1}]$. Let $k \in \mathbb{N}$ such that $s^k x \in F^{1\times n}[s]$. Then $s^k x = (s^k y)G \in V_G \cap F^{1\times n}[s]$ and, by assumption, there exists $z \in F^{1\times k}[s, s^{-1}]$ such that $s^k \cdot x = z \cdot G$. Thus, $x = (s^{-k}z)G \in F^{1\times k}[s, s^{-1}]G = \tilde{M}_G$. ∎

The proof of the following corollary is an immediate consequence of Theorem 3.7.

Corollary 3.8 *Let* $G(s) \in F^{k\times n}[s]$, $\mathrm{rank}_{F[s]}G(s) = k$. *Then:*

(i) $v_i(M_G) \geq \tilde{\nu}_i(\tilde{M}_G) \geq v_i(V_G)$ *for* $i = 1, \ldots, k$

(ii) $v_i(M_G) = \tilde{\nu}_i(\tilde{M}_G)$ *for* $i = 1, \ldots k \Leftrightarrow M_G = \tilde{M}_G \cap F^{1\times n}[s]$

(iii) $\tilde{v}_i(\tilde{M}_G) = v_i(V_G)$ *for* $i = 1, \ldots, k \Leftrightarrow \tilde{M}_G = V_G \cap F^{1\times n}[s, s^{-1}]$ ■

Examples:

(i) Let $G_1(s) = \begin{pmatrix} s & 1 & s+2 \\ s^2+2s & s^2+s & s^2 \end{pmatrix}$, $\mathrm{rank}[G_1]_h^r = 2 = \mathrm{rank}[G_1]_\ell^r$ and the gcd of the 2×2-minors is s. Hence $G_1(s)$ is a polynomial minimal basis of M_{G_1}, a dipolynomial minimal basis of $\tilde{M}_{G_1}$ and a dipolynomial minimal basis of V_{G_1}, however not a Forney-minimal basis, because the gcd of the 2×2-minors is not a unit in $F[s]$.

(ii) Let $G_2(s) = \begin{pmatrix} s & 1 & s+2 \\ s^2+2s & s^2+s & s^2+1 \end{pmatrix}$, $\mathrm{rank}[G_2]_h^r = 2 = rank[G_2]_\ell^r$ and the gcd of the 2×2-minors is 1. Hence $G_2(s)$ is Forney-, dipolynomial and polynomial minimal basis for $V_{G_2}, \tilde{M}_{G_2}$ and M_{G_2}, resp. . □

We close this section with the characterization of *dipolynomial minimal basis transformations* and some results concerning canonical dipolynomial minimal bases in echelon form.

Theorem 3.9 *Let* $G(s, s^{-1}) \in F^{k\times n}[s, s^{-1}]$ *be a dipolynomial minimal basis for* $V_G(\tilde{M}_G)$ *of the form (3.2) with ordered dipolynomial indices* $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_k$ *and row-degrees** $\deg g_i(s, s^{-1}) = n_i$, $i \in \underline{k}$. *Let* $\bar{G} = TG$, $T(s, s^{-1}) \in F^{k\times k}[s, s^{-1}]$ *unimodular. Then* $\bar{G}$ *is a dipolynomial minimal basis with ordered indices for* $V_G(\tilde{M}_G)$ *if and only if:*

$$t_{ij}(s, s^{-1}) = 0 \quad \textit{for} \quad \nu_j > \nu_i \tag{3.13}$$

$$\rho_{ij} + \mathrm{ddeg}\, t_{ij}(s, s^{-1}) \leq \nu_i - \nu_j \quad \textit{for} \quad \nu_j \leq \nu_i \tag{3.14}$$

where

$$\rho_{ij} = \deg\, t_{ij} - \mathrm{ddeg}\, t_{ij} + n_j - \nu_j - \min_{j\in\underline{n}}(\deg\, t_{ij} - \mathrm{ddeg}\, t_{ij} + n_j - \nu_j) \tag{3.15}$$

Proof:

Let $\tau_i := \min_{j\in\underline{n}}(\deg\, t_{ij} - \mathrm{ddeg}\, t_{ij} + n_j - \nu_j)$, $i \in \underline{k}$. Let further $\tilde{T} = \mathrm{diag}(s^{-\tau_1}, \ldots, s^{-\tau_k})T$. Then the elements $\tilde{t}_{ij}$ of $\tilde{T}$ are of the form $\tilde{t}_{ij}(s, s^{-1}) = s^{-(n_j-\nu_j)+\rho_{ij}} \cdot p_{ij}(s)$, where $p_{ij}(s) \in$

*Observe that we now also use the polynomial degree function deg for dipolynomials:

$\deg : F[s, s^{-1}] \to \mathbb{Z}, \quad \alpha_L s^L + \ldots + \alpha_\ell s^\ell \to L$

$F[s]$ and $p_{ij}(0) \neq 0$ for all $(i,j) \in \underline{k} \times \underline{n}$ for which $t_{ij} \neq 0$. Note that $\deg p_{ij} = \operatorname{ddeg} t_{ij}$. Furthermore the numbers ρ_{ij} are nonnegative and for every $i \in \underline{k}$ there exists a $j_0 \in \{1, \ldots, k\}$ such that $\rho_{ij_0} = 0$.
Assume now that $\overline{G} = TG$ is a dipolynomial minimal basis with the same ordered indices $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_k$. Then for $i \in \underline{k}$

$$\operatorname{ddeg}(\bar{g}_i(s,s^{-1})) = \operatorname{ddeg}(\sum_{j=1}^{k} t_{ij}(s,s^{-1}) g_j(s,s^{-1}))$$

$$= \operatorname{ddeg}\ (s^{\tau_i}(\sum_{j=1}^{k} \tilde{t}_{ij}(s,s^{-1}) \cdot g_j(s,s^{-1})))$$

$$= \operatorname{ddeg}(\sum_{j=1}^{k} \tilde{t}_{ij}(s,s^{-1}) \cdot g_j(s,s^{-1})) = v_i$$

Condition (iv) (b) in Theorem 3.2 gives

$$\max_{j:t_{ij}\neq 0}(-(n_j-\nu_j)+\rho_{ij}+\deg\ p_{ij}+n_j) - \min_{j:t_{ij}\neq 0}(-(n_j-\nu_j)+\rho_{ij}+(n_j-\nu_j)) = \nu_i$$

or, equivalently

$$\max_{j:t_{ij}\neq 0}(\nu_j+\rho_{ij}+\deg\ p_{ij}) - \min_{j:t_{ij}\neq 0}(\rho_{ij}) = \nu_i$$

which implies (3.13) and (3.14).
Let conversely T satisfy (3.13) and (3.14). Then

$$\begin{aligned}
\bar{g}_i(s,s^{-1}) &= s^{\tau_i}(\sum_{j=1}^{k} \tilde{t}_{ij}(s,s^{-1}) \cdot g_j(s,s^{-1})) \\
&= s^{\tau_i}(\sum_{j:\nu_j\leq\nu_i} s^{-(n_j-\nu_j)+\rho_{ij}} \cdot p_{ij}(s)(\alpha^j_{\nu_j} \cdot s^{n_j} + \ldots + \alpha^j_0 \cdot s^{n_j-\nu_j})) \\
&= s^{\tau_i}(\sum_{j:\nu_j\leq\nu_i} s^{\nu_j+\rho_{ij}} \cdot p_{ij}(s)(\alpha^j_{\nu_j} + \ldots + \alpha^j_0 \cdot s^{-\nu_j}))
\end{aligned}$$

(3.14) shows that

$$\begin{aligned}
&\rho_{ij} + \deg\ p_{ij} = 0 \quad \text{for} \quad \nu_i = \nu_j \\
&\nu_j + \rho_{ij} + \deg\ p_{ij} \leq \nu_j + \nu_i - \nu_j = \nu_i \quad \text{if} \quad \nu_i \neq \nu_j
\end{aligned} \tag{3.16}$$

Hence

$$\bar{g}_i(s,s^{-1}) = [\bar{g}_i]_h \cdot s^{\tau_i+\nu_i} + \ldots + [\bar{g}_i]_\ell \cdot s^{\tau_i}$$

Now

$$[\bar{g}_i]_\ell = \sum_{j:\substack{\nu_j\leq\nu_i \\ \rho_{ij}=0}} p_{ij}(0) \cdot \alpha^j_0 =: q_i[G]^r_\ell \tag{3.17}$$

where $q_i \in F^{1\times k}$ and $q_{ij} = 0$ if j does not appear as summation index in (3.17).
Since T is unimodular and upper block triangular, in every block on the diagonal there is

at least one nonzero element; hence because $p_{ij}(0) \neq 0$ we have $q_i \neq 0$ and $[\bar{q}_i]_\ell \neq 0$ with Theorem 3.2, (ii)(b) (Theorem 3.4, (ii)). Thus

$$[\bar{G}]_\ell^r = \begin{pmatrix} q_1 \\ \vdots \\ q_k \end{pmatrix} [G]_\ell^r =: Q[G]_\ell^r \tag{3.18}$$

However, Q is of the same block structure as T - indeed, the elements in the diagonal blocks coincide. Hence Q is of full rank and thus $\text{rank}[\bar{G}]_\ell^r = k$.
On the other hand,

$$[\bar{g}_i]_h = \sum_{j: \substack{\nu_j \leq \nu_i \\ \rho_{ij} + \deg p_{ij} = \nu_i - \nu_j}} [p_{ij}]_h \cdot \alpha_{\nu_j}^j =: w_i[G]_h^r \tag{3.19}$$

with $w_i \in F^{1\times k}$ and $w_{ij} = 0$ if j is no summation index. Again (3.16) shows that the above sum is not empty, and the unimodularity of T gives $w_i \neq 0$ and hence $[\bar{g}_i]_h \neq 0$ (again because of the row properness of G). Then

$$[\bar{G}]_h^r = \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} [G]_h^r =: W[G]_h^r \tag{3.20}$$

Analogously as above one concludes $\text{rank}[\bar{G}]_h^r = k$. Thus $\bar{G}$ is a dipolynomial minimal basis as a consequence of Theorem 3.2, (ii)(b) (Theorem 3.4, (ii)). ■

Remark 3.10 Contrary to *polynomial* minimal basis transformations for the modules $M_V \subset F^{1\times n}[s]$ as characterized in Münzner and Prätzel-Wolters [1979], the *dipolynomial* minimal basis transformations depend on the degree structure of the particular basis $G(s, s^{-1})$ which has to be transformed. The numbers $n_i = \deg g_i(s, s^{-1})$ are neither vector space nor module invariants. □

Theorem 3.9 is a useful tool in determining "canonical" dipolynomial minimal bases.
Given a dipolynomial minimal bases $G(s, s^{-1}) \in F^{k\times n}[s, s^{-1}]$ with ordered dipolynomial indices $\nu_1 \leq \nu_2 \leq \ldots \leq \nu_k$ we denote by γ_i the smallest integer, such that the matrix G_i^h which consists of the intersection of columns $\gamma_1, \ldots, \gamma_i$ of $[G]_h^r$ and rows of $[G]_h^r$ which correspond to indices $\leq v_i$, is of full rank. The γ_i are called *pivot indices* (cf. Hinrichsen and Prätzel-Wolters (1983)).
It is easy to show that the pivot indices are as well vector space invariants as module invariants; the proof essentially follows the line of the proof given in Forney [1975] and is thus omitted.
Refining the notation (3.2) by

$$\begin{aligned} g_i(s, s^{-1}) &= \alpha_{\nu_i}^i \cdot s^{n_i} + \ldots + \alpha_0^i \cdot s^{n_i - \nu_i} \\ &= (\alpha_{\nu_i}^{i,1}, \ldots, \alpha_{\nu_i}^{i,n}) \cdot s^{n_i} + \ldots + (\alpha_0^{i,1}, \ldots, \alpha_0^{i,n}) \cdot s^{n_i - \nu_i} \end{aligned} \tag{3.21}$$

for $i \in \{1, \ldots, k\}$, well call a row proper matrix $G \in F^{k\times n}[s, s^{-1}]$ with row indices $\nu_1, \ldots, \nu_k$ and pivot indices $\gamma_1, \ldots, \gamma_k$ in *echelon form*, if the following conditions hold:

(i) $\nu_1 \leq \nu_2 \leq \ldots \leq \nu_k$ (3.22)

(ii) $n_i = \nu_i$ for $i \in \{1, \ldots, k\}$ and $\alpha_{\nu_i}^{i,\gamma_i} = 1$ (3.23)

(iii) For $i, j \in \{1, \ldots, k\}, i \neq j$ and $\nu_i \leq \nu_j$ there holds

$$\alpha_{\nu_j}^{j,\gamma_i} = \alpha_{\nu_j - 1}^{j,\gamma_i} = \ldots = \alpha_{\nu_i}^{j,\gamma_i} = 0 \quad (\text{i.e. ddeg}\,(g_{j,\gamma_i}(s, s^{-1})) < \nu_i) \tag{3.24}$$

Note that this definition coincides with Forneys definition of minimal echelon bases for rational vector spaces (with a modified notion of row properness) and, up to condtion (ii), also with the definition of polynomial echelon bases for full polynomial submodules $M \subset F^{1\times n}[s]$ in Hinrichsen and Prätzel-Wolters [1983]. The proof of the following explicit characterization of dipolynomial minimal bases in echelon form is completely analogous to the proof of the corresponding result (Proposition 5.7) in the last mentioned paper. The only modification which has to be considered is the generalization to rectangular basis matrices instead of square ones.

Proposition 3.11 *A row proper dipolynomial matrix $G(s, s^{-1}) \in F^{k\times n}[s, s^{-1}]$, $k \leq n$ with rows $g_i(s, s^{-1})$, $i = 1, \ldots, k$, of the form (3.21) and ordered row indices $\nu_1 \leq \ldots \leq \nu_k$ is in echelon form iff:*

(i) $G(s, s^{-1})$ is polynomial with $n_i = \nu_i$ for $i \in \{1, \ldots, k\}$ and

$$\operatorname{rank} \begin{pmatrix} \alpha_0^{1,1} & \ldots & \alpha_0^{1,n} \\ \vdots & & \vdots \\ \alpha_0^{k,1} & \ldots & \alpha_0^{k,n} \end{pmatrix} = k \tag{3.25}$$

(ii) There exists a $k \times k$-permutation matrix P with entries 1 at (β_j, j) such that for $\ell, j \in \underline{k}$

$$\ell < j \quad \textit{and} \quad \nu_j = \nu_\ell \Rightarrow \beta_\ell < \beta_j \tag{3.26}$$

and

$$(g^{\beta_1}, \ldots, g^{\beta_k})_h^c = I_k{}^\dagger \tag{3.27}$$

(iii) The i-th row of $[G]_h^r$ is of the form

$$(0, \ldots, 0, \underset{\substack{\uparrow \\ \beta_i}}{1}, *, \ldots, *), \; i \in \underline{k}$$

■

$^\dagger(g^{\beta_1} \ldots, g^{\beta_k})_h^c$ denotes the submatrix of the highest *column* coefficient matrix $[G]_h^c$ formed by the columns $\beta_1, \ldots, \beta_k$ of $[G]_h^c$

Remark 3.12 The indices β_i introduced above coincide with the pivot indices γ_i. □

Theorem 3.13 *Every $F[s,s^{-1}]$-module $\tilde{M} \subseteq F^{1\times n}[s,s^{-1}]$ has a unique dipolynomial minimal basis in echelon form.*

Proof:
Every dipolynomial minimal basis G of $\tilde{M}$ in echelon form is polynomial because $n_i = v_i$ for $i \in \{1,\ldots,k\}$. Furthermore, by Theorem 3.7 (iii) G is also a polynomial minimal basis for $M = \tilde{M} \cap F^{1\times n}[s]$ and in echelon form as defined by Hinrichsen and Prätzel-Wolters ([1983], Def. 5.5). The uniqueness part of Theorem 3.13 then follows with the corresponding uniqueness result therein (Corollary 5.9).
To show existence, let $G(s)$ be a dipolynomial minimal basis of $\tilde{M}$ with

$$[G]_\ell^r = G(0) = \begin{pmatrix} \alpha_0^1 \\ \vdots \\ \alpha_0^k \end{pmatrix} \quad \text{of full rank } k$$

In particular, $G(s)$ is polynomial. Let $\bar{G}(s)$ the minimal basis in echelon form of the $F[s]$-module $F^{1\times k}[s]G(s)$. Then $\bar{G}(s)$ satisfies the conditions (ii) and (iii) of Proposition 3.11. Furthermore, $\bar{G}(s) = T(s)G(s)$ where $T(s)$ is a polynomial unimodular minimal basis transformation. These transformations leave the rank of $G(0)$ invariant: $\bar{G}(0) = T(0)G(0)$, $\operatorname{rank}\bar{G}(0) = k$, i.e. also (i) of Proposition 3.11 is satisfied and $\bar{G}(s)$ indeed is a dipolynomial minimal basis in echelon form. ■

Remark 3.14 The corresponding result for dipolynomial bases of $F(s)$-vector spaces follows directly from the characterization of the echelon form in Forney [1975], since a dipolynomial basis in echelon form is a Forney-basis by Theorem 3.5(ii). □

4 Trim canconical forms for systems in AR-representation

In the recent years J.C. Willems developed in a series of papers a general theory of dynamical systems $\Sigma = (T, W, B)$ with time axis $T \subseteq \mathbb{R}$, signal alphabet W and behaviour $\mathcal{B} \subset W^T$ (see e.g. Willems [1986a, 1986b, 1987, 1988, 1991]). In this framework it is shown that every linear time-invariant complete system Σ with time axis $T = \mathbb{Z}$ has an autoregressive *AR-representation*

$$\mathcal{B} = \ker R(\sigma,\sigma^{-1}) \tag{4.1a}$$

$$R(s,s^{-1}) = R_L s^L + \ldots, + R_\ell s^\ell \in \mathbb{R}^{p\times q}[s,s^{-1}] \tag{4.1b}$$

Here $\sigma^t : W^T \to W^T$, $w(\hat{t}) \to w(\hat{t}+t), t \in T$ is called the *t-shift*. The operator

$$R(\sigma,\sigma^{-1}) : \begin{array}{c} (\mathbb{R}^q)^{\mathbb{Z}} \longrightarrow (\mathbb{R}^p)^{\mathbb{Z}} \\ \\ w(t) \longrightarrow R_L w(t+L) + \ldots + R_\ell w(t+\ell) \end{array} \quad , \ t \in \mathbb{Z}$$

is called a *dipolynomial shift operator*. If $\ell \geq 0$ then $R(\sigma,\sigma^{-1})$ is polynomial and denoted by $R(\sigma)$. q denotes the dimension of the signal alphabet space $W = \mathbb{R}^q$, whereas p, the

number of equations representing $\mathcal{B}$, is flexible.
For $T = \mathbb{Z}_+, \mathbb{R}, \mathbb{R}_+$ there are analogous polynomial AR-representations with

$$\mathcal{B} = \ker\, R(\sigma) \quad \text{resp.} \quad \mathcal{B} = \ker\, R\left(\frac{d}{dt}\right) \quad \text{where} \quad R(s) \in \mathbb{R}^{p\times q}[s] \tag{4.2}$$

Among all dipolynomial matrices $R(s, s^{-1})$ satisfying (4.1a) there exist those with full row rank. They are unique up to multiplication from the left by unimodular matrices $U(s, s^{-1})$. Let

$$R(s,s^{-1}) = \begin{pmatrix} r_1(s,s^{-1}) \\ \vdots \\ r_p(s,s^{-1}) \end{pmatrix} = \begin{pmatrix} \alpha^1_{\nu_1} \cdot s^{n_1} + \ldots + \alpha^1_0 \cdot s^{n_1-\nu_1} \\ \vdots \\ \alpha^p_{\nu_p} \cdot s^{n_p} + \ldots + \alpha^p_0 \cdot s^{n_p-\nu_p} \end{pmatrix} \tag{4.3}$$

$n_i, \nu_i \in \mathbb{Z}, \nu_i \geq 0$ such a full row rank dipolynomial matrix. Then $R(s, s^{-1})$ is called a *minimal lag description* of Σ (c.f. Willems [1991]), if among all full row rank *AR*-representations of Σ it has the *total lag* $L_{tot} = \sum_{i=1}^{p} \nu_i$ as small as possible. Because all the representations $R(s, s^{-1})$ of Σ are obtained from a particular one by multiplication form the left by unimodular matrices $U(s, s^{-1})$ the minimal lag descriptions are exactly the dipolynomial minimal bases of the module $\tilde{M}_R := \mathbb{R}^{1\times p}[s, s^{-1}]R(s, s^{-1})$ characterized in Theorem 3.4 by the conditions

$$\text{rank}[R]^r_h = \text{rank}[R]^r_\ell = p \tag{4.4}$$

Furthermore, the row degrees $(\nu_1, \ldots, \nu_p)$ are the same for every minimal lag *AR*-description of a given *AR*-system Σ. They coincide with the minimal indices of the module $\tilde{M}_R$. Associated to these indices in a one to one relation are the *structure indices* $(\rho_t)_{t\in\mathbb{N}}$ of an *AR*-system Σ defined by

$$\rho_t = \text{ the number of } \nu_i' s \text{ equal to } t \tag{4.5a}$$

The characterization (4.4) is also derived in Willems [(1991), Proposition X.5 ("bilaterally row proper")]. Furthermore, in that paper there is described a *uniquely determined minimal lag description* which yields a *trim canonical form* for the dipolynomial matrix parametrization of *AR*-equations. However, the definition given there is intricate and the existence and uniqueness result is not completely proved. In the following we show that this trim canonical form coincides with the dipolynomial bases in echelon form derived in section 3. This way we obtain a characterization which is easier to verify and to handle and a complete proof of Willems result.

Theorem 4.1 *(Willems [1991]) For every full row rank dipolynomial matrix* $R(s, s^{-1}) \in \mathbb{R}^{p\times q}[s, s^{-1}]$ *with associated minimal indices* $(\nu_i)_{i\in\underline{p}}$ *and structure indices* $(\nu_t)_{t\in\mathbb{N}}$ *there exists a uniquely determined integer list*

$$(q^k_t)_{t\in\underline{\rho_k}},\ k \in \mathbb{N}_0,\ 0 \leq q^k_1 < q^k_2 < \ldots < q^k_{\rho_k} \leq q \tag{4.5b}$$

and exactly one polynomial matrix $\bar{R}(s, s^{-1})$ *which is unimodularly left equivalent to* $R(s, s^{-1})$ *and satisfies:*

(i) *If* $j = q_{\tilde{t}}^t$
then if $i = \rho_0 + \rho_1 + \ldots + \rho_{t-1} + \tilde{t}$
then $\deg \bar{r}_{ij} = t$ *and* $\bar{r}_{ij}(s)$ *is monic*
and if instead $i \neq \rho_0 + \rho_1 + \ldots + \rho_{t-1} + \tilde{t}$
then $\deg \bar{r}_{ij} < t$

(ii) *If* $i = \rho_0 + \rho_1 + \ldots + \rho_{t-1} + \tilde{t},\ 1 \leq \tilde{t} \leq \rho_t$, *and* $j \neq q_{\tilde{t}}^t$
then if $j \leq q_{\tilde{t}}^t$
then $\deg \bar{r}_{ij} < t$
and if instead $j > q_{\tilde{t}}^t$
then $\deg \bar{r}_{ij} \leq t$

(iii) *The matrix*

$$\begin{pmatrix} \bar{r}_{11}(0) & \bar{r}_{12}(0) & \ldots & \bar{r}_{1q}(0) \\ \bar{r}_{21}(0) & \bar{r}_{22}(0) & \ldots & \bar{r}_{2q}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{r}_{p1}(0) & \bar{r}_{p2}(0) & \ldots & \bar{r}_{pq}(0) \end{pmatrix}$$

has full row rank

■

Remark 4.2 In Willems [1991] condition (ii) is formulated with the if-conditions "if $j \leq q_{\rho_t}^t$" and "if $j > q_{\rho_t}^t$", i.e. $q_{\rho_t}^t$ is taken instead of $q_{\tilde{t}}^t$. Morever, in view of Theorem 3.9 it can be checked that the class of admissable transformations taken into consideration in the above reference is too restrictive. Consider for example:

$$R(s, s^{-1}) = \begin{pmatrix} 0 & 0 & s+1 & 0 \\ s^2+1 & p(s) & 0 & 0 \\ 0 & 0 & 0 & s^2+1 \end{pmatrix}$$

with $p(s) = \sum_{k=0}^{2} p_k s^k$ a polynomial of degree 2. Then

$$[R]_h^r = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & p_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad [R]_\ell^r = R(0) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & p_0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence $R(s, s^{-1})$ is a dipolynomial minimal lag AR-representation with structure index list $\rho = (0, 1, 2, 0, 0, \ldots)$.
By Theorem 3.9 every unimodular transformation $U(s, s^{-1})$ which transforms $R(s, s^{-1})$ into another dipolynomial minimal lag matrix is of the form

$$U(s, s^{-1}) = \begin{pmatrix} u_{11} & 0 & 0 \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}$$

with $u_{11}, u_{22}, u_{23}, u_{32}, u_{33} \in \mathbb{R}$, $u_{11} \neq 0$, $\det \begin{pmatrix} u_{22} & u_{23} \\ u_{32} & u_{33} \end{pmatrix} \neq 0$ and $u_{21} = as + b$, $u_{31} = c \cdot s + d$, $a, b, c, d \in \mathbb{R}$. Then

$$\tilde{R}(s, s^{-1}) = U(s, s^{-1}) R(s, s^{-1})$$

$$= \begin{pmatrix} 0 & 0 & u_{11}(s+1) & 0 \\ u_{22}(s^2+1) & u_{22}\cdot p(s) & u_{21}(s+1) & u_{23}(s^2+1) \\ u_{32}(s^2+1) & u_{32}\cdot p(s) & u_{31}(s+1) & u_{33}(s^2+1) \end{pmatrix} \tag{4.6}$$

To obtain $\tilde{R}(s,s^{-1})$ in canonical minimal lag description we have to fix the integer lists $(q_t^k)_{t\in\underline{\rho_k}}$, $k \in \mathbb{N}_0$. Because $\rho = (0,1,2,0,0,\dots)$ and $\tilde{R}(s,s^{-1})$ is of the form (4.6) one obtains

$$q_t^k = 0 \quad \text{for} \quad k \notin \{1,2\},\ q_1^1 = 3 \quad \text{and}\ (q_1^2, q_2^2) \quad \text{is either} \quad (1,2),\ (1,4)\ \text{or}\ (2,4)$$

Now assume that $\tilde{R}(s,s^{-1})$ is in canconical form with $q_1^2 = 1$. Then by condition (i) in the original version of Theorem 4.1, $u_{22} = 1$. However, condition (ii) requires $u_{22} = 0$ for both cases $q_2^2 = 2$ and $q_2^2 = 4$.
If $q_1^2 = 2$, again by condition (i) $u_{22} \neq 0$ but also $u_{22} = 0$ by condition (ii).
Summarizing this example shows that there are dipolynomial matrices for which there does not exist a minimal lag representation satisfying the original condition (ii) in Willems Theorem. □

We call an AR-respresentation statisfying the conditions (i)-(ii) of Theorem 4.1 a (modified) *canonical minimal lag representation.*

Remark 4.3 Observe that $R(s,s^{-1})$ in Remark 4.2 is already a (modified) canonical minimal lag representation with $(q_1^2, q_2^2) = (1,4)$.
Furthermore, $R(s,s^{-1})$ is also in echelon form with high pivot indices $\gamma_1 = 3,\ \gamma_2 = 1,\ \gamma_3 = 4$, which is immediate from Proposition 3.11. □

The above remark indicates that there is a tight connection between the concept of canonical minimal lag description and the echelon forms. Indeed a further examination yields:

Theorem 4.4 *A full row rank dipolynomial matrix $R(s,s^{-1}) \in \mathbb{R}^{p\times q}[s,s^{-1}]$ is a canonical minimal lag representation if and only if $R(s,s^{-1})$ is a minimal dipolynomial basis in echelon form of the module $\mathbb{R}^{1\times p}[s,s^{-1}]\cdot R(s,s^{-1})$.*

Proof:

Assume $R(s,s^{-1})$ is in echelon form with ordered indices $\nu_1 \leq \ldots \leq \nu_p$ and high pivot indices $\{\gamma_1, \ldots \gamma_p\}$. Let $(\rho_t)_{t\in\mathbb{N}}$ denote the integer sequence defined by (4.5a) and let $(q_t^k)_{t\in\mathbb{N}_0}$ be such that

$$q_t^k = \begin{cases} 0 & \text{if} \quad k \notin \{\nu_1, \ldots, \nu_p\} \\ \gamma_{\rho_0+\ldots+\rho_{k-1}+t} & \text{if} \quad k \in \{\nu_1, \ldots, \nu_p\} \wedge t \leq \rho_k \end{cases}$$

The ordering (4.5b) of the q_t^k is a consequence of the ordering of the high pivot indices

$$\gamma_{\rho_0+\ldots+\rho_{k-1}+1} < \gamma_{\rho_0+\ldots+\rho_{k-1}+2} < \cdots < \gamma_{\rho_0+\ldots+\rho_{k-1}+\rho_k}$$

for $k \in \{\nu_1, \ldots, \nu_p\}$. Then it is easily seen that conditions (i), (ii) and (iii) of Theorem 4.1 are equivalent to conditions (ii), (iii) and (i) in Proposition 3.11. Hence $R(s,s^{-1})$ is a canonical minimal lag representation.

If conversely $R(s,s^{-1})$ satisfies the conditions of Theorem 4.1, then it suffices to show that the indices $(q_1^{\nu_1},\ldots,q_{\rho_{\nu_1}}^{\nu_1},\ldots,q_1^{\nu_p},\ldots,q_{\rho_{\nu_p}}^{\nu_p})$ coincide with the pivot indices of $R(s,s^{-1})$. However, by conditions (i) and (ii) of Theorem 4.1 the i-th row of $[R]_h^r$ is of the form $(0,\ldots,0,1,*,\ldots,*)$ with 1 at position $q_{\tilde{t}}^{\nu_i}$, if $i=\rho_0+\ldots+\rho_{\nu_i-1}+\tilde{t}$, hence $q_{\tilde{t}}^{\nu_i}=\gamma_{\rho_0+\ldots+\rho_{\nu_i-1}+\tilde{t}}$ by the definition of the high pivot indices. ■

Remark 4.5 (i) In the cases $T=\mathbb{Z}_+,\mathbb{R}_+,\mathbb{R}$ one obtains all representations of $\mathcal{B}=\mathcal{B}(R)$, R polynomial of full row rank as in (4.2), by polynomial unimodular left multiplications. Now the minimal lag descriptions as defined in Willems [1991] coincide with the polynomial minimal bases of the module $M_R=\mathbb{R}^{1\times p}[s]\cdot R(s)$, and the associated (rectangular) bases in echelon form (cf. Hinrichsen and Prätzel-Wolters [1983]) yield a trim canonical form for the polynomial matrix-parametrization of AR-equations.

(ii) Given a discrete-time linear time-invariant complete behaviour, one can always find a polynomial AR-representation for this set of trajectories. Hence the question arises whether it is sufficient to restrict the representation of such behaviours to polynomial matrices and to regard only polynomial unimodular transformations. However, in view of Corollary 3.8 (i) this leads to *longer* minimal lag descriptions.

(iii) So far we have given a system theoretic interpretation to the polynomial resp. dipolynomial minimal *module bases* by identifying them as the minimal lag descriptions of the associated behaviours. However, the corresponding *vector space* constructions are also neatly connected with system theoretic concepts.

For $T=\mathbb{Z}(\mathbb{Z}_+,\mathbb{R}_+,\mathbb{R})$ a system Σ for the form 4.1 (4.2) with $R(s,s^{-1})\in F^{k\times n}[s,s^{-1}](R(s)\in F^{k\times n}[s])$ of full row rank is controllable if and only if all left divisors of $R(s,s^{-1})(R(s))$ are unimodular elements of $F^{k\times k}[s,s^{-1}](F^{k\times k}[s])$. By Theorem 3.7 (ii) (Theorem 3.7 (i)) this is equivalent to the module identity $\tilde{M}_R=V_R\cap F^{1\times n}[s,s^{-1}](M_R=V_R\cap F^{1\times n}[s])$, hence the minimal basis in echelon form for modules $\tilde{M}\subset F^{1\times n}[s,s^{-1}](M\subset F^{1\times n}[s])$ which are saturated in their rational extensions give rise to a parametrization of all controllable AR-systems over $T=\mathbb{Z}(\mathbb{Z}_+,\mathbb{R}_+,\mathbb{R})$. Furthermore Willems [1991] defines the *controllable part* of a given system Σ with behaviour $\mathcal{B}(R)$ as the largest controllable linear time-invariant complete subsystem of Σ and shows that this subsystem has the behaviour $\mathcal{B}(\tilde{R})$ where $\tilde{R}$ is left prime and related with R by $R=F\cdot\tilde{R}$, $\det F\neq 0$, F dipolynomial resp. polynomial.
Now the reduction algorithm for step (b) in the proof of Theorem 3.2 constructs such a factorization of R. Hence, starting with $\mathcal{B}=\mathcal{B}(R)$ and interpreting R as a basis of the vector space V_R, we obtain:

- Reduction of R to a basis of $\tilde{M}_{V_R}\,(M_{V_R})$ corresponds to the construction of the controllable part of $\mathcal{B}(R)$
- Reduction of R to a module basis of $\tilde{M}_R\,(M_R)$ corresponds to the construction of a minimal lag description of Σ

(iv) The reduction algorithm given in Hoffmann [1993] realizes the transformation of a given AR-representation of $\mathcal{B}$ into a minimal lag description. This algorithm performs a unimodular transformation of the lowest coefficient matrix to full rank. An algorithm for the highest coefficient matrix can be obtained analogously (note that an elementary step of this algorithm is $I + s \cdot B$ with B nilpotent; this transformation is polynomial unimodular and hence can be also be used for the reduction to minimal lag descriptions in the case $T = \mathbb{Z}_+, \mathbb{R}_+, \mathbb{R}$). Hence the associated Matlab program in Hoffmann [1991] can be applied for the reduction.

(v) The McMillan degree $\mathrm{Mm}(\Sigma)$ of a system $\Sigma = (\mathbb{Z}, \mathbb{R}^n, \mathcal{B}(R))$ of the form 4.1 is defined (c.f. Willems [1991]) as the dipolynomial degree of the vector $(f_1, \ldots, f_{\binom{n}{k}})$ formed by all $k \times k$-minors f_i of R. By Theorem 3.4 this McMillan degree coincides with the dipolynomial order of the module $\tilde{M}_R$. □

5 Conclusion

The purpose of this paper was to provide a module theoretic framework for the investigation of linear time-invariant complete behaviour systems as introduced by Willems [1991].

Our starting point was a generalization of the concept of polynomial minimal bases to the case of dipolynomial matrices and the investigation of minimal basis transformations and canonical forms.
A control theoretic interpretation for the minimal dipolynomial bases and a trim canonical form for the dipolynomial matrix parametrization of AR-equations was obtained. Moreover, we showed that the incorporation of dipolynomial concepts for the system representation in discrete time results in tighter minimal lag descriptions.

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