# Long-time behaviour of Langevin-type dynamics on Riemannian manifolds and scaling limits

On hypocoercivity of fibre lay-down models on smooth spaces

**Inauguraldissertation**

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**Maximilian Constantin Mertin**

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Prof. Dr. Martin Grothaus (TUK) und
Prof. Dr. Fabrice Baudoin (UConn).

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# Contents

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# Introduction

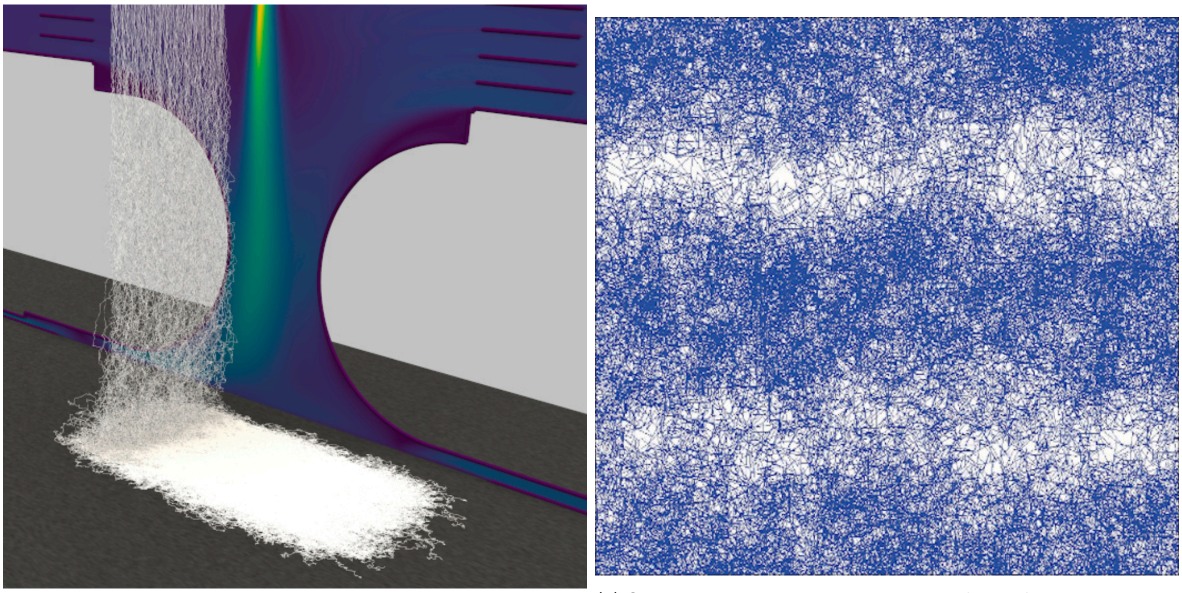
This thesis is motivated by the industrial production process of nonwoven materials consisting of hundreds, even thousands of endless, slender fibre filaments that are entangled due to a stochastic perturbation. These random entanglements on the one hand stabilise the material while on the other hand there are holes of varying size that crucially influence the key properties of the final product: For instance when nonwovens are used as filter material for air, water, or oil, the bigger the holes on average the coarser the resulting filter would be, and vice versa. Besides filter properties, one might be interested in elasticity, tearing strength, sterility, liquid repellence, absorbance capacity, and so on. In daily life, one finds nonwovens in hygiene products such as baby wipes or tissues, in medical products such as wound dressings or one-time scrubs/bedsheets, and clothing. See [KMW09] for an introduction from the point of view of mathematical modelling and [AF12] for a comprehensive compendium addressed to a wide readership of manufacturers, engineers, and users. Famously, during the SARS-Cov-2 pandemic the demand for filter materials used in medical masks skyrocketed, once the immediate danger had been comprehended by the public. Therefore, many companies adjusted their existing machines in order to satisfy the sudden demand.

Even though there are different production methods to consider, our original motivation comes from production of so-called melt-blown nonwovens, but some of the illustrating pictures relate to spunbound nonwovens. Both methods start with thermoplastic polymer and polymer melt is continuously extruded through nozzles or spinnerets. Then, the fibre filaments are stretched, swirled, and entangled by turbulent air flow. To name one of the differences between these aerodynamic methods: For meltblown nonwovens one uses a hot air flow first and cooling flows afterwards, whilst in the spunbound process the fibre filaments are simultaneously swirled and cooled and also might hit at a deflector. The fibre filaments lay down on a moving conveyor belt and are further processed. A general physical model for dynamics of fibre filaments was presented in [MW06; MW07]. See also [HM05; KMW09] for context and further information. In particular, we want to bring a consequence of an inextensibility assumption to the readers attention: The extensibility of the solidified/solidifying fibres is negligible and due to the constant extrusion speed, we can assume that the velocity vectors are normalised to length one, see [KMW09, p. 944] and [KMW12a, Section 2]. This is the model implemented in the software package FIDYST[1] by the department Transport Processes of Fraunhofer Institute for Industrial Mathematics ITWM in Kaiserslautern. FIDYST produces simulations of realistic fibre dynamics, which are used in machinery design and parameter optimisation. See Figure 1 for a few illustrations.

The quality of the simulations comes at the cost of both high computational effort and memory storage, due to the amount of physical details. These difficulties have been circumvented by considering surrogate models that describe fibre filaments just on the conveyor belt instead of the full dynamic. The idea is to adjust parameters of the surrogate model in such a way that the simulated trajectory can be treated as an isolated representative fibre from experimental data. Then, the calibrated surrogate model can be used to simulate hundreds of fibres at once to obtain a whole web and possibly proceed with simulations using FIDYST. The software package SURRO[3], again developed by the Transport Processes department of Fraunhofer ITWM, is a tool to simulate virtual fleeces, see Figure 2. For sake of completeness, we point the reader also to [Lin+17; Str19]. Overall, the aim there is to understand numerical behaviour of the complex model by analytic investigation of the spatially discretised equations, e. g. existence and uniqueness results for solutions, complemented with the numerical experiments.

Surrogate models considered in these applications build upon a system of stochastic differential equations with stochastic forces representing turbulent air flow. The first one was proposed in [Göt+07] modelling a dynamic on a nonmoving conveyor belt. Henceforth, various other models have been suggested that provide additional features: From the basic fibre lay-down model, which could alternatively named 'spherical velocity Langevin model', one obtains trajectories that already share several common features with the physical filament paths. But those trajectories are just continuous, not differentiable, which is not true for these physical paths. To mirror reality more closely, so-called 'smooth' models were proposed in [KMW12b]. We are very careful to call the 'smooth' models in thesis smooth*ed* models instead, for the simple reason that

[1]Fiber Dynamics Simulation Tool
[2]Copyright for all these pictures belongs to Fraunhofer ITWM. I am very grateful for the permission for usage in this thesis.
[3]Software Surrogate Model



(a) Angled view on full spunbound process

(b) Simulated web on conveyor belt moving from left to right

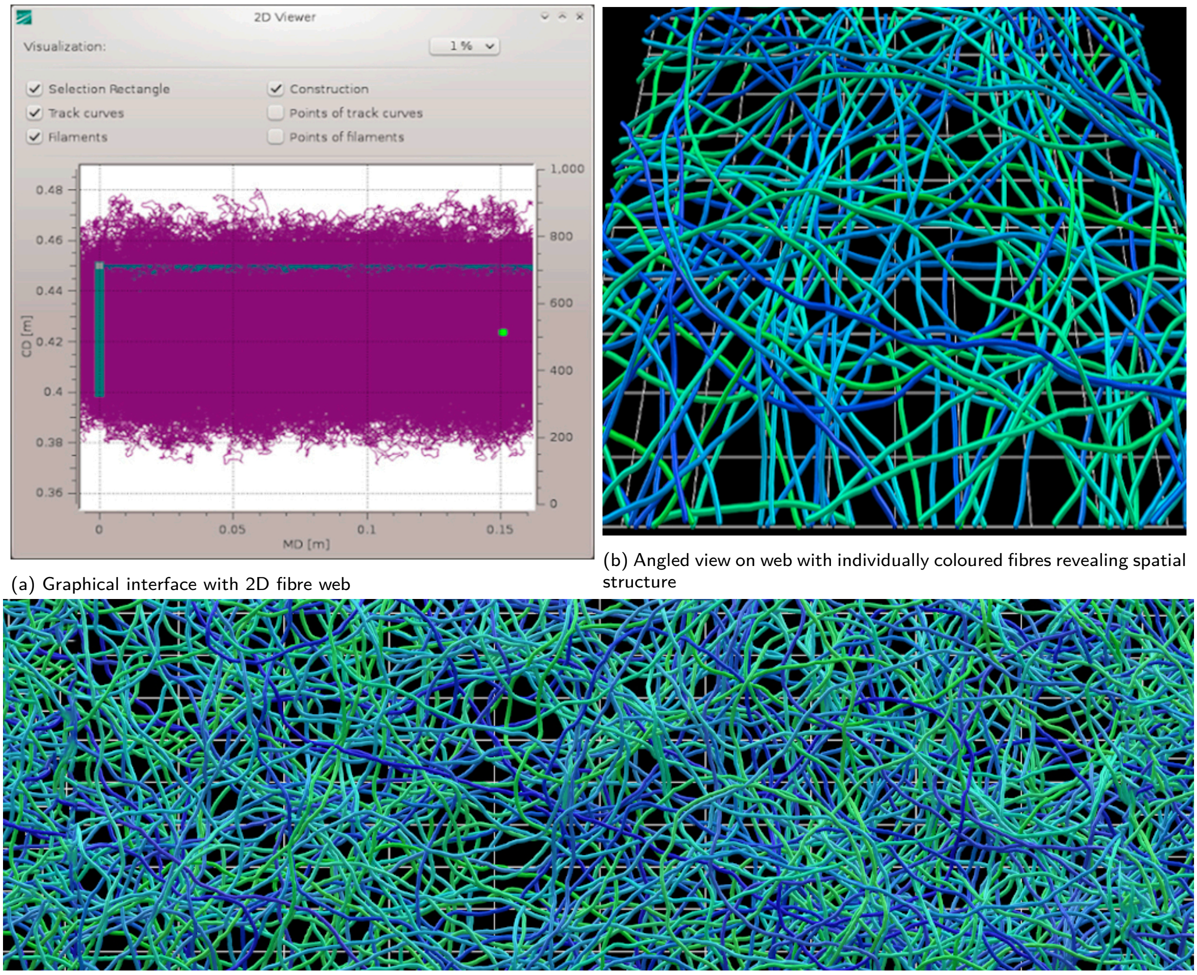
Figure 1: Some simulations done with FIDYST[2]

'smooth' is a quite loaded term and should come with context to make sense. These smoothed models could be viewed as the first truly manifold-valued model with state space being the Cartesian product of an Euclidean space and a corresponding sphere. P. Stilgenbauer discovered that the original version of that new model was not complete and he determined a missing term ensuring that the process really stays on the state manifold. See [Sti14, Section 1.8.1] and for the industrial application see [Mar13, Chapters 4 and 6] as well as [Gro+14]. Moreover, macroscopically the fleece looks as two-dimensional as the conveyor belt, whilst the nonwoven material reveals spatial structure on closer inspection. This structure includes e. g. a certain distribution of angles between fibre filament and it is in turn linked to important properties like permeability. So, a realistic three-dimensional fibre lay-down model is required to incorporate a certain anisotropicity of the velocity directions. For instance, see [KMW12a, Section 4] for the basic fibre lay-down model and [KMW12b, Section 4] for the smoothed model.

Until now, the region of deposition of the fibre filaments was either the plane or three-dimensional space. We contribute the option to consider a large class of Riemannian manifolds instead. One could therefore say that we bring geometric surrogate models to the table. For a simple motivation, one could imagine a sagging conveyor belt or a belt running over a cylindrical roller at the deposition region. In actuality, the macroscopic geometric phenomena arising due to the production process look a bit different. The recently published paper [HMW21] adapts the three-dimensional anisotropic fibre lay-down model in such a way that it captures a geometric feature of nonwovens produced via the airlay method: The material shows sigmoidal contour planes, see [HMW21, Figure 1]. Furthermore, very thick layers of nonwovens are produced to serve as insulation material for the building industry. These layers slope up forwards as material accumulates. In this thesis, we don't restrict ourselves to a specific geometric side condition from applications, we rather look at fibre lay-down models on a finite-dimensional, connected, complete[4] Riemannian manifold. This manifold is abstract in the sense that it is most of the time not embedded into a larger Euclidean space. Compared to the equivalent, but cumbersome point of view that P. Stilgenbauer adopts in his PhD thesis, we use the intrinsic notions of modern differential geometry.

We focus on the long-time analysis of surrogate models. Specifically, we are interested in exponential convergence to equilibrium with known rates. The significance is put in simple words in [GK08]: 'The faster this convergence is, the more uniform the produced textile will be.' Meaning that homogeneity of the real nonwoven material is related to the theoretical topic of hypocoercivity. See [Vil09] for the famous synoptic article on the phenomenon where the name 'hypocoercivity' is firmly established. As hypocoercivity is particularly challenging to show for the degenerate models we are interested in, we employ a powerful hypocoercivity method that is functional analytic by nature. A in-depth discussion on the history

[4] Compare to Remark 1.2.16.




(a) Graphical interface with 2D fibre web

(b) Angled view on web with individually coloured fibres revealing spatial structure

(c) Top-down view on web with individually coloured fibres

Figure 2: Some simulations done with SURRO[2]

of that method with its multiple iterations, extensions, and applications is covered in Section 1.1. Of course, long-time behaviour of stochastic dynamics is a flourishing field and impressive results have been achieved for related equations with wildly different approaches. We cover some particularly noteworthy results in Section 1.1, but that exposition can not be exhaustive due to the sheer amount of material.

In the following, we summarise the results of this thesis:

- Overall, we open geometric modelling perspectives for various fibre lay-down models and Langevin-type models in other applications.
- We slightly shift the focus when thinking about the antisymmetric part of the Kolmogorov backwards generators of fibre lay-down dynamics to semisprays. Even though in the case of Riemannian manifolds the natural choice of such a semispray is the well-known and frequently used geodesic spray, semisprays could be a good starting point for further generalisations e. g. Lagrange spaces.
- We extend preexisting results on hypocoercivity of both Langevin equation and spherical velocity Langevin equations from [GS14; GS16; Sti14] to the abstract manifold situation. We discuss existence of a properly associated Hunt process and infer from hypocoercivity $L^2$-exponential ergodicity. These results given in Chapter 2 have been published before in [GM22].
- We establish the very first hypocoercivity result for a variation on the isotropic smoothed fibre lay-down models. A publication of the results from Chapter 3 is in preparation.
- We propose in Chapter 4 a new modelling technique for anisotropic models which is based on the notion of stratifolds due to Matthias Kreck. With that we can give a rigorous geometric meaning to

reducing the standard fibre of the unit tangent bundle to a sphere of lower dimension. We discuss hypocoercivity of the induced anisotropic models. The main theoretical feature of the new technique is that one can basically restrict oneself to parallelisable manifolds.

- In Appendix A, we give abstract local coordinate expressions and a few explicit examples for nice manifolds.

Although our motivation comes from production nonwoven material, we emphasise that Langevin-type equations appear in many different scientific areas. For a broad historical overview we refer to [CKW12, Chapter 2] and to the subsequent chapters for various applications in physics and more specifically, in chemistry as well as electric engineering. It shouldn't be surprising that geometric Langevin equations have become an intriguing topic before: Coming from the work [Koz89] of S. M. Kozlov on tori with flat metric, M. R. Soloveitchik considered compact connected Riemannian manifolds. Both take the view of Hamiltonian mechanics to geodesic motion subjected to a smooth potential field, start from the corresponding Hamiltonian, and add a Brownian diffusion by adding its generator to the Hamiltonian vector field. Basically, this is the same route pursued by T. Lelièvre, M. Rousset and G. Stoltz e. g. in [LRS12] for multibody dynamics. Our results are compatible with such previous works as we just choose to take the Lagrangian mindset rather than the Hamiltonian. Also we should mention the works by V. N. Kolokoltsov, see the book [Kol00] and references therein, as well as by E. Jørgensen, see [Jør78]. We discuss them in Section 1.3 a bit more.

# 1 Preliminaries

This chapter contains an exposition of the employed hypocoercivity method as well as foundational concepts and vocabulary from differential geometry. One the one hand, we assume that the reader is familiar with the theory of operator semigroups and Dirichlet forms as well as stochastic differential equations (in $\mathbb{R}^{d}$). Knowledge on generalised Dirichlet forms is useful, see [Sta99], but we cover in the appendix at least the basic setup, specifically in Section B.3. One the other hand, we do not assume that the reader has much experience with differential geometry overall.

Most importantly, we report in Section 1.1 on a hypocoercivity strategy, which we are going to refer to as Abstract Hilbert space Hypocoercivity Method (AHHM). It was developed in its original 'algebraic' form by J. Dolbeault, C. Mouhot, and Ch. Schmeiser before it was completed and extended by M. Grothaus and P. Stilgenbauer. The latter study was motivated by the example of fibre lay-down dynamics as previous results in the field of hypocoercivity were not able to cover it. We adopt in this section language from [GS14; GS16; Sti14] that will be used throughout the thesis. In particular, Theorem 1.1.4 is the key theorem we rely upon. Of course, there are remarkable and quite different approaches to convergence to equilibrium for Langevin-type equations like rough path methods or Gamma calculus. We give a faceted overview in the end of Section 1.1.

Afterwards, a quick run-down on topics from differential geometry is given in Section 1.2. Undoubtedly, we just provide patches of the bigger picture discussed in a wide variety of books. Nonetheless, in none of the books that I know of the relation between geometry and second order ordinary differential equations is made as clear as in the semispray formalism that I encountered in the works by I. Bucataru, see Section 1.2.4. Realising these links was important for me personally in order to reconcile preexisting ideas on geometric Langevin dynamics by various authors. As many spaces of interest are fibred in the sense of fibre bundles, we briefly discuss not only integration on manifolds, but also measures on fibre bundles in Section 1.2.2. Finally in Section 1.3, we shade some light on different perspectives how one can understand stochastic differential equations on manifolds.

Before we begin, we shall give some orientation on how notation is used in the thesis over all.

*Notation* 1.0.1 (Some broad paradigmas).

- *upright:* We use upright letters in math mode quite frequently. Of course, special mathematical constants like Euler's number 'e' and the imaginary unity 'i' are set upright, as they should be. Furthermore, we denote Riemannian metrics by small upright letters often corresponding to the symbol for the manifold. When introducing a weight on a Riemannian manifold the symbol of the Riemannian metric becomes boldfaced, compare to Definition 1.2.23.
- *calligraphic:* We use calligraphic letters like $\mathcal{X}$ or $\mathcal{E}$ for vector fields and bilinear forms. In particular, $\mathcal{H}$ always refers to semisprays, see Definition 1.2.27. A second distinct set of calligraphic symbols like $\mathscr{d}$, $\mathscr{b}$ etc. is reserved exclusively for (finite) dimensions.
- *blackboard bold:* As usual the sets of natural numbers, integers, rational numbers, real numbers, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ in this order. Here, the set of natural numbers is the set of nonnegative integers according to DIN 5473:1992-07. If we want to exclude 0, we write $\mathbb{N}_+ := \mathbb{N}\setminus\{0\}$. Other blackboard bold letters like $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{B}$, $\mathbb{E}$ etc. refer to certain topological spaces, oftentimes (smooth) manifolds. In particular, $\mathbb{U}$ refers to open balls, $\mathbb{S}$ to spheres, and $\mathbb{T}$ to tori.
- *Fraktur:* Fraktur typeface in upper case signifies set systems such as power sets, topologies, and $\sigma$-algebras. For instance, $\mathfrak{B}(E)$ denotes the Borel $\sigma$-algebra on a topological space $E$. Fraktur letters in lower case refer to Lie algebras.

## 1.1 ▪ Abstract Hilbert space hypocoercivity method

We assume that the reader is familiar with the basic theory of operator semigroups. See for instance the books [EN00; EN06] by K.-J. Engel and R. Nagel. Before we formulate the hypocoercivity method, we start with a little bit of historical context and further developments.

Following a suggestion by Thierry Gallay, Cédric Villani coined the term 'hypocoercivity' in [Vil09] for describing a phenomenon of convergence to equilibrium at a specified rate when a degenerate dissipation operator and a 'conservative' operator come together in an operator $L$ describing the total dynamic. The name hypocoercivity reflects that the phenomenon is distinct from, but akin to the well-studied hypoellipticity, which focuses on regularity rather than convergence to equilibrium. The survey provided by C. Villani sparked many great works among which we spotlight here a technique introduced by Jean Dolbeault and Clément Mouhot, and Christian Schmeiser in [DMS15] and the preceding paper [DMS09]. They establish exponential decay to equilibrium of the strongly continuous semigroup associated to a given generator $L$, having in mind linear kinetic equations with rather general external potentials see [DMS15, Theorem 2]. The strategy is very appealing, since it is formulated in a general Hilbert space framework and the assumptions seem fairly reasonable. Conceptionally, the great achievement is the introduction of a modified entropy functional: Under so-called hypocoercivity assumptions the time derivative times minus one can be estimated from below in such a way that an application of the Grönwall lemma yields the claim. As mentioned in [DMS15], the approach is inspired by F. Hérau, compare to [Hér06]. Nevertheless, the new hypocoercivity strategy was developed only algebraically: Operator domains were not discussed and domain issues were not taken into account. Martin Grothaus and Patrik Stilgenbauer filled in these gaps in [GS14] and extended the method to the case where a core for the generator $L$ is known. Note that the Lie bracket conditions needed in C. Villani's approach are not satisfied in the situation of spherical velocity Langevin equation, so his result doesn't apply as it stands. To round off the picture, we mention that another motivation for studies by M. Grothaus and P. Stilgenbauer was [LRS12, Proposition 3.2]. That is an ergodicity statement for Langevin dynamics with mechanical constrain, where no explicit rate is given. As we discuss in Section 2.3 ergodicity statements with explicit rates can be inferred from our desired hypocoercivity results.

As a matter of fact, we point out that many articles containing hypoellipticity, hypocoercivity, or results on long-time behaviour by means of other analytic methods consider Fokker-Planck equations. One of the key points in [GS14] is the extension of previous iterations of the hypocoercivity method to Kolmogorov backwards evolution equations reformulated as abstract Cauchy problems. Even though some might call it trivial from the rigorous mathematical point of view, we emphasise that in applications the Fokker-Planck formulation is associated to the Kolmogorov backwards formulation via a unitary transformation of the model Hilbert space, compare e. g. to [Gro+12, Remark 6.2 (i)] or [Ber21, Section 5.3.1]. This might be interesting to know for those applicationists who are mostly familiar with the Fokker-Planck situation.

We deem the papers [GS14; GS16] as well as the PhD thesis [Sti14] our main references for what we are going to call the Abstract Hilbert space Hypocoercivity Method (AHHM) from now on. The main theorem, Theorem 1.1.4, requires that two sets of conditions are fulfilled. One is the aforementioned set (H) of hypocoercivity assumptions, but in a reformulated version compared to [DMS15]. The other one is the set (D) of 'data' assumptions providing the Hilbert space framework, a generator of a strongly continuous semigroup, an invariant measure, and so on. It shall be mentioned that an alternative version of the Theorem 1.1.4 holds with the set of conditions (D) replaced by the generalised data conditions from [Sti14, Section 2.2.3]. By now, a quite abstract generalisation going by the name *weak hypocoercivity* is available. It was developed by M. Grothaus and F.-Y. Wang in [GW19]. For more details we refer to [Ber21, Section 3]; therein A. Bertram contrasts weak hypocoercivity with the 'strong' hypocoercivity that we are presenting here. Furthermore, he proposes rearrangements and reformulations compared to original sources in order to paint the bigger picture.

**Condition 1.1.1** (Data conditions (D)).

(D1) *model Hilbert space:* Let $(E, \mathfrak{E}, \mu)$ be a probability space and choose the Hilbert space $H$ as $L^2(E;\mu) = L^2(\mu)$.

(D2) *strongly continuous semigroup and its infinitesimal generator:* Let $(L, D(L))$ be a linear operator on $H$ and $(T_t)_{t\in[0,\infty)}$ be the strongly continuous semigroup generated by $L$, i. e. $T_0 = \mathrm{Id}_H$ and $T_t f \to f$ as $t \downarrow 0$ for all $f \in H$.

(D3) *core property:* Let $D \subseteq D(L)$ be dense in $H$ and an operator core of $(L, D(L))$, i. e. the closure of $(L, D)$ coincides with $(L, D(L))$. We might also refer to $D$ as core domain.

(D4) *SAD-decomposition of generator $L$ into symmetric and antisymmetric part:* Let $(S, D(S))$ be symmetric and let $(A, D(A))$ be closed and antisymmetric on $H$ such that $D \subseteq D(S) \cap D(A)$ and the restriction of $L$ to the core can be decomposed as $L|_D = S - A$.

(D5) *projection:* Let $P\colon H \to H$ be an orthogonal projection such that $P(H) \subseteq D(S)$ and $SP = 0$ as well as $P(D) \subseteq D(A)$ and $AP(D) \subseteq D(A)$. Define

$$P_S\colon\ H \longrightarrow H,\ f \longmapsto Pf + (f\,,1)_H.$$

(D6) *invariant measure:* Let $\mu$ be invariant for $(L, D)$ in the sense that

$$(Lf\,,1)_H = \int_E Lf\ \mathrm{d}\mu = 0 \qquad \text{for all } f \in D.$$

(D7) *semigroup conservativity:* Let $1 \in D(L)$ and $L1 = 0$.

**Condition 1.1.2** (Hypocoercivity conditions (H)).

(H1) *algebraic relation:* The composition $PAP$ is trivial on the core, i. e. $PAP|_D = 0$.

(H2) *microscopic coercivity:* There is a constant $\Lambda_m \in (0,\infty)$ such that for all $f \in D$ the following estimate holds:

$$\Lambda_m \|(\mathrm{Id}_H - P_S)f\|_H^2 \le -(Sf\,,f)_H.$$

(H3) *macroscopic coercivity:* There is a constant $\Lambda_M \in (0,\infty)$ such that for all $f \in D((AP)^*(AP))$ the following estimate holds:

$$\Lambda_M \|Pf\|_H^2 \le \|APf\|_H^2.$$

(H4) *boundedness of auxiliary operators:* Define $B := (\mathrm{Id}_H + (AP)^*(AP))^{-1}(AP)^*$ on $D((AP)^*)$. There are constants $c_1, c_2 \in (0,\infty)$ such that for all $f \in D$ the estimates

$$\|BSf\|_H \le c_1 \big\|(\mathrm{Id}_H - P_j)f\big\|_H \quad \text{and} \quad \|BA(\mathrm{Id}_H - P)f\|_H \le c_2 \big\|(\mathrm{Id}_H - P_j)f\big\|_H$$

hold with $P_j \in \{P, P_S\}$, $j \in \{1,2\}$.

*Remark* 1.1.3 ($B$ as bounded operator on $H$). From a purely functional analytic point of view, one can say more about operators $B$ of a form as in (H4): Suppose a closed operator $(Z, D(Z))$, that is either symmetric or antisymmetric, on a Hilbert space $H$. Furthermore, let $P\colon H \to H$ be an orthogonal projection such that for some dense subspace $D$ with $D \subseteq D(Z)$ holds $P(D) \subseteq D(Z)$. Then, $(ZP, D(ZP))$ is densely defined and closed. Applying von Neumann's Theorem, specifically [Ped89, Theorem 5.1.9 (ii)], to the latter operator we obtain that $\mathrm{Id}_H + (ZP)^*(ZP)\colon D((ZP)^*(ZP)) \to H$ with its natural domain

$$D((ZP)^*(ZP)) := \{f \in D(ZP) \mid ZPf \in D((ZP)^*)\}$$

is continuously invertible. Define

$$B := (\mathrm{Id}_H + (ZP)^*(ZP))^{-1}(ZP)^* \qquad \text{on } D((ZP)^*).$$

Using standard results as collected in [Sti14, Lemma 2.2], one can show $B = (ZP)^*(\mathrm{Id}_H + (ZP)(ZP)^*)^{-1}$ as a linear continuous operator on $H$ with norm less or equal than 1, compare to [Ped89, Theorem 5.1.9 (iii)]. This piece of information is just good to keep in mind, but not necessary for the AHHM.

In practice, one needs to impose additional assumptions in order to check the sets of conditions (D) and (H). Typically, those are assumptions on an external force potential $\Psi$, that plays into the antisymmetric operator $A$ and the invariant measure $\mu$ but per se is not part of the abstract framework. We state potential assumptions separately when we want to apply the following theorem.

**Theorem 1.1.4** (Abstract Hilbert space Hypocoercivity Theorem). Assume the conditions (D) and (H). Then, the operator semigroup $(T_t)_{t\in[0,\infty)}$ generated by $(L, D(L))$ is hypocoercive in the sense that there exist constants $\kappa_1, \kappa_2 \in (0,\infty)$ computable in terms of the constants $\Lambda_m$, $\Lambda_M$, $c_1$ and $c_2$ appearing in the assumptions such that

$$\big\|T_t g - (g\,,1)_H\big\|_H \le \kappa_1 \mathrm{e}^{-\kappa_2 t}\, \big\|g - (g\,,1)_H\big\|_H$$

holds for all times $t \in [0,\infty)$ and for all $g \in H$.

See [GS14, Theorem 2.18], [GS16, Theorem 2.2], or [Sti14, Section 2] for the proof. In general, one can choose first some $\delta \in (0,\infty)$ and afterwards $\varepsilon \in (0,\infty)$ such that

$$\frac{\Lambda_M}{\Lambda_M+1} - (1+c_1+c_2)\,\frac{\delta}{2} > 0 \qquad \text{and} \qquad \Lambda_m - \varepsilon(1+c_1+c_2)\Big(1+\frac{1}{2\delta}\Big) > 0 \tag{1.1}$$

hold. Then, choosing $\kappa \in (0,\infty)$ as minimum of the left hand-sides in Equation (1.1) and using that $P$ is an orthogonal projection one finds that

$$\begin{aligned}\kappa\|T_t f\|_H^2 \le \Big(\Lambda_m - \varepsilon(1+c_1+c_2)\Big(1+\frac{1}{2\delta}\Big)\Big)\|(\mathrm{Id}_H - P)T_t f\|_H^2 \\ + \Big(\frac{\Lambda_M}{\Lambda_M+1} - (1+c_1+c_2)\,\frac{\delta}{2}\Big)\|PT_t f\|_H^2\end{aligned}$$

is satisfied for all $f \in D(L)$. Then, one gets $\kappa_1 = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$ and $\kappa_2 = \frac{\kappa}{1+\varepsilon}$. This statement is almost verbatim [Ber21, Remark 3.2.2] and is directly derived from the proof of Theorem 1.1.4. For Langevin-type equations, under the necessary conditions on the potential $\Psi$, one can compute the constants $\kappa_1$ and $\kappa_2$ more explicitly, see [GS16, p. 164-165] or [Sti14, pp. 109-110] respectively, and [Sti14, p. 192]. In fact, one has freedom to choose $\kappa_1 \in (1,\infty)$ and then $\kappa_2$ is determined:

(A) In the case of the (classical) Langevin equation (2.1) on $\mathbb{R}^d$ with friction parameter $\alpha \in (0,\infty)$ one gets

$$\kappa_2 = \frac{\kappa_1 - 1}{\kappa_1}\,\frac{\alpha}{n_1 + n_2\alpha + n_3\alpha^2}$$

where $n_i \in (0,\infty)$ for $i \in \{1,2,3\}$ only depend on the potential $\Phi = \beta\Psi$ and $\beta \in (0,\infty)$. We want to give the constants $n_i$ here as explicit as possible based on the proof, which is more concerned with finding suitable $\kappa$ and $\varepsilon$ for fixed $\delta = \frac{\Lambda_M}{\Lambda_M+1}\frac{1}{(1+c_1+c_2)}$. The constants from the hypocoercivity conditions are $\Lambda_M = {}^{\Lambda}\!/\!_{\beta}$ with $\Lambda$ the Poincaré constant of the measure $\exp(-\Phi)\lambda$, $\Lambda_m = \alpha$, $c_1 = {}^{\alpha}\!/\!_2$, and $c_2 = c_{\Phi;\beta}$ a constant only depending $\Phi$ and $\beta$. During the proof the quantity

$$\bar\varepsilon_{\Phi;\beta}(\alpha) := \frac{\alpha}{(a_1 + a_2\alpha + a_3\alpha^2)}$$

is significant where the constants $a_i$ read as

$$\begin{aligned} a_1 &= (1+c_{\Phi;\beta})\cdot\Big(1 + (1+c_{\Phi;\beta})\frac{\Lambda+\beta}{2\Lambda}\Big) + \frac{1}{2}\frac{\Lambda}{\Lambda+\beta},\\ a_2 &= \frac{1}{2}\Big(1 + (1+c_{\Phi;\beta})\frac{\Lambda+\beta}{2\Lambda}\Big), \quad \text{and} \quad a_3 = \frac{1}{4}\frac{\Lambda+\beta}{2\Lambda}.\end{aligned}$$

Broadly speaking, the quantity $\bar\varepsilon_{\Phi;\beta}(\alpha)$ is derived from the coefficients in the estimate [GS16, Equation (2.9)] or [Sti14, Equation (2.18)] respectively. As a function of $\bar\alpha \in (0,\infty)$ the quantity $\bar\varepsilon_{\Phi;\beta}(\bar\alpha)$ attains a maximum at

$$\begin{aligned}\sqrt{\frac{a_1}{a_3}} &= 2\left((1+c_{\Phi;\beta})\frac{2\Lambda}{\Lambda+\beta} + (1+c_{\Phi;\beta})^2 + \Big(\frac{\Lambda}{\Lambda+\beta}\Big)^2\right)^{\frac{1}{2}}\\ &= 2\Big(1 + c_{\Phi;\beta} + \frac{\Lambda}{\Lambda+\beta}\Big).\end{aligned}$$

Define $\bar\varepsilon_{\Phi;\beta;\mathsf{max}} := \max\Big(1, \bar\varepsilon_{\Phi;\beta}\Big(\sqrt{{}^{a_1}\!/\!_{a_3}}\Big)\Big)$, then the $n_i$ are given as

$$n_i := 4\,\frac{\Lambda+\beta}{\Lambda}\cdot\bar\varepsilon_{\Phi;\beta;\mathsf{max}}\cdot a_i \qquad \text{for all } i \in \{1,2,3\}.$$

(B) Very similar to (A), one gets for the fibre lay-down model or spherical velocity Langevin equation with position space $\mathbb{R}^d$, potential $\Psi$, and diffusion parameter $\sigma \in (0,\infty)$ the form

$$\kappa_2 = \frac{\kappa_1 - 1}{\kappa_1}\,\frac{\sigma^2}{n_1 + n_2\sigma^2 + n_3\sigma^4}$$

where $n_i \in (0,\infty)$ for $i \in \{1,2,3\}$ only depend on the potential $\Psi$. We determine the $n_i$ as in (A), but now the constants from the hypocoercivity conditions are $\Lambda_M = \Lambda/d$ with $\Lambda$ the Poincaré constant of the measure $\exp(-\Phi)\lambda$, $\Lambda_m = \sigma^2/2\,(d-1)$, $c_1 = \sigma^2/4\,(d-1)$, and $c_2 = c_\Psi$ a constant only depending on $\Psi$. Now, the constants in

$$\bar{\varepsilon}_\Psi(\sigma) = \frac{\sigma^2}{(a_1 + a_2\sigma^2 + a_3\sigma^4)} := \frac{\sigma^2(d-1)}{2(\bar{a}_1 + \bar{a}_2\sigma^2 + \bar{a}_3\sigma^4)}$$

are given by $a_i = \frac{2}{d-1}\,\bar{a}_i$ and

$$\bar{a}_1 = (1+c_\Phi)\cdot\left(1 + (1+c_\Psi)\frac{\Lambda + d}{2\Lambda}\right) + \frac{1}{2}\frac{\Lambda}{\Lambda + d},$$
$$\bar{a}_2 = \frac{1}{4}\,(d-1)\left(1 + (1+c_\Psi)\frac{\Lambda+d}{2\Lambda}\right), \quad \text{and} \quad \bar{a}_3 = \left(\frac{1}{4}\,(d-1)\right)^2\frac{\Lambda+d}{2\Lambda}.$$

As before, the constants $n_i$ are defined as

$$n_i := 4\,\frac{\Lambda + d}{\Lambda}\,\bar{\varepsilon}_{\Psi;\max}\cdot a_i \quad \text{for all } i \in \{1,2,3\},$$

where $\bar{\varepsilon}_{\Psi;\max} := \max\left(1, \varepsilon_\Psi\left(\sqrt[4]{a_1/a_3}\right)\right)$.

For good measure, let us finish this section with a very short overview on some relevant long-time investigations that use rather different methods. We start with the discussion of so-called *kinetic Brownian motion* on Riemannian manifolds due to J. Angst, I. Bailleul, and C. Tardif, see [ABT15]. Kinetic Brownian motion is the spherical velocity Langevin equation, that we discuss in Section 2.2, with zero potential. For Euclidean position space the random path is driven by Brownian motion on the sphere as velocity dynamic; in the general Riemannian case the stochastic process takes values in the unit tangent bundle and can be thought as stochastic perturbation of geodesic flow. A long-time analysis for rotationally invariant manifolds can be found in [ABT15, Section 3] employing the so-called *dèvissage method* that determines the Poisson boundary of the process, see [AT16]. Besides, another big topic in [ABT15] is the behaviour when the diffusion parameter approaches infinity, which is tackled with rough path methods. Changing the framework to $\Gamma$-calculus, we refer to the recently published paper [BGH21] devoted to extend hypocoercivity results due to Villani in order to treat Langevin dynamics with even singular potentials as Lennard-Jones potentials. Introducing a weight function $W$ as well as an adapted $H^{1,2}$-norm $\|\cdot\|_{\varsigma,W}$ and requiring some basic structure of the potential as well as the growth condition [BGH21, Assumption 2.7], the authors derive in [BGH21, Theorem 2.23] exponential semigroup convergence with respect to $\|\cdot\|_{\varsigma,W}$. This result heavily relies on appropriate explicit Lyapunov functions and the authors see potential for further fine tuning here as these Lyapunov functions might not be optimal. Note that [Bau17], one of the foundational works for [BGH21], describes the role of generalised Bakry-Émry-type conditions in this $\Gamma$-calculus. To obtain such estimates is a central point in the papers [Bau16] and [BT18]. The former operates in totally geodesic Riemannian foliations and treats Langevin-type equations in [Bau16, Section 7] as an example. In [BT18] F. Baudoin and C. Tardif first cover the case non-totally geodesic foliations and then deduce hypocoercivity results for the spherical velocity Langevin equation. I am grateful to an anonymous reviewer for pointing me at [Bau16; BT18] which I did not know until August 2021.

## 1.2 ▪ Differentiable manifolds and Riemannian geometry

This section provides a gentle, but swift introduction to the foundations of differential and Riemannian geometry. Of course, we do not cram a comprehensive study in here and details might slightly differ from author to author anyways. By and large, the section is based on [Lee13]. Among many other great books we also warmly recommend [Lan95], [Nic96], and [Sak96], as well as [Jän05] for German speaking readers. Sadly, notation is far from being standardised, as the saying goes: 'Differential geometry is the study of properties that are invariant under change of notation.'[1] We try to keep everything accessible even for uninitiates, but take some shortcuts here and there.

The primordial basis for this thesis is the definition of manifolds, of course. Furthermore, fibre bundles (and more general fibrations) appear as the conceptual framework for various important constructions such

[1] At [Lee13, page x] J. M. Lee refers to it as an old joke.The original source seems to be lost to history.

as the tangent bundle or Riemannian metrics. These basic concepts are contained in Section 1.2.1. Latter subsections are devoted mostly to concepts linked to the choice of a Riemannian metric. In particular in Section 1.2.4, we shade light on the relations between semisprays, Ehresmann connections and, (dynamical) covariant derivatives which can be determined by a Riemannian (or Lagrangian) metric. The remaining subsections deal on the one hand with integration theory on manifolds or fibre bundles, see Section 1.2.2, and on the other hand with the most important differential operators, see Section 1.2.3.

### 1.2.1 ▪ Basic definitions

**Definition 1.2.1** (topological and differentiable manifolds).

(i) A topological space $\mathbb{Y}$ is said to be a *$\mathscr{y}$-dimensional topological manifold (over $\mathbb{R}$)* if the following three conditions hold:

  (1) $\mathbb{Y}$ is a Hausdorff space,

  (2) $\mathbb{Y}$ is second countable, i. e. its topology admits a countable base,

  (3) $\mathbb{Y}$ is locally homeomorphic to an open subset of $\mathbb{R}^{\mathscr{y}}$, more formally: For every point $y \in \mathbb{Y}$ exist an open neighbourhood $\mathsf{dom}(h) \subseteq \mathbb{Y}$ of $y$, and a homeomorphism $h\colon \mathsf{dom}(h) \to \mathsf{ran}(h) \subseteq \mathbb{R}^{\mathscr{y}}$ with $\mathsf{ran}(h)$ denoting the range of $h$.

Such a local homeomorphism $h$ is referred to as a *chart (at $y$)*. The inverse of a chart at $y$ can be thought as a local parametrisation of $\mathbb{Y}$ around $y$. For any two charts $h_1$ and $h_2$ the composition $h_2 \circ h_1^{-1}$ (with its natural domain $h_1(\mathsf{dom}(h_1) \cap \mathsf{dom}(h_2))$) is called their *transition map*, see Figure 1.1.

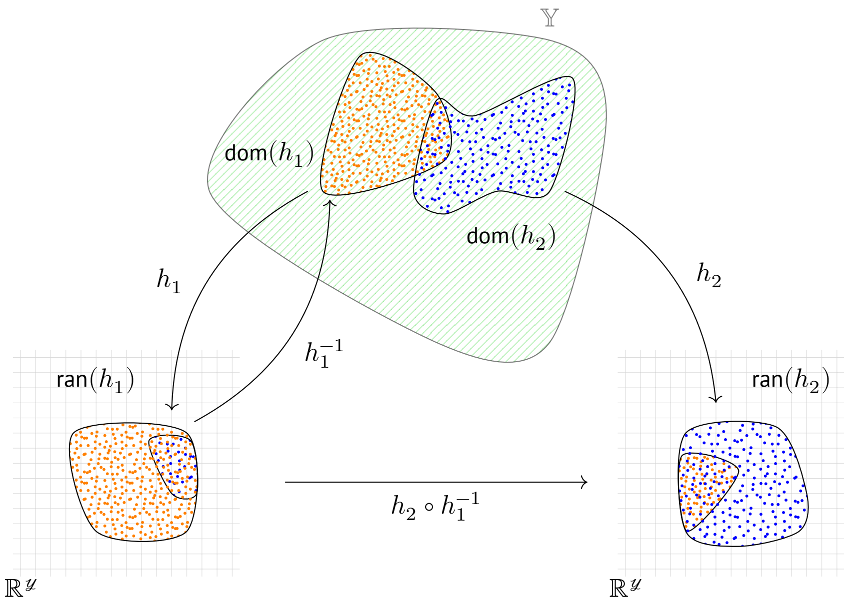



Figure 1.1: Two charts and their transition map

(ii) An *atlas for a topological manifold* $\mathbb{Y}$ is a collection of charts that cover $\mathbb{Y}$. According to part (i) at least one atlas for $\mathbb{Y}$ exists. An atlas $\mathbf{D}$ is called *differentiable* if for two charts in $\mathbf{D}$ the transition map is differentiable in the classical sense. A *differentiable structure on* $\mathbb{Y}$ is a differentiable atlas for $\mathbb{Y}$ which is maximal with respect to inclusion among differentiable atlases.

(iii) If $\mathbb{Y}$ is a $\mathscr{y}$-dimensional topological manifold endowed with a differentiable structure $\mathbf{D}$ for $\mathbb{Y}$, then $\mathbb{Y}$ (or rather the pair $(\mathbb{Y}, \mathbf{D})$) is called a *differentiable manifold*.

(iv) A function $f$ between two differentiable manifolds $(\mathbb{Y}_1, \mathbf{D}_1)$ and $(\mathbb{Y}_2, \mathbf{D}_2)$ is called *differentiable at* $y \in \mathbb{Y}_1$ if for all charts $h_1 \in \mathbf{D}_1$ and $h_2 \in \mathbf{D}_2$ the composition $h_2 \circ f \circ h_1^{-1}$ is differentiable at $h_1(y)$. ⌟

*Notation* 1.2.2 (local coordinates). Once we fix a basis on $\mathbb{R}^{\mathscr{d}}$ a chart on a $\mathscr{d}$-dimensional manifold can be thought as having $\mathscr{d}$ many component functions. Therefore, it's rather common to write the chart as collection of component functions e. g. $(x^j)_{j=1}^{\mathscr{d}}$, where upper indices are chosen to avoid confusing jam of lower indices. Calculations with these component functions, as if they were real numbers, should always be taken at least with a grain of salt. On this matter, Hermann Weyl is often quoted as saying that 'the introduction of numbers as coordinates [...] is an act of violence', see [Wey49]. However, the quote does not stop there and Weyl acknowledges that there is a 'practical vindication' by means of manageable computations. In the same vein, we renunciate local coordinate descriptions for the most part, but it should not be denied that they are needed in some proofs as well as in every fast computer implementation. Therefore, we provide several useful local coordinate expressions in Appendix A. ⌟

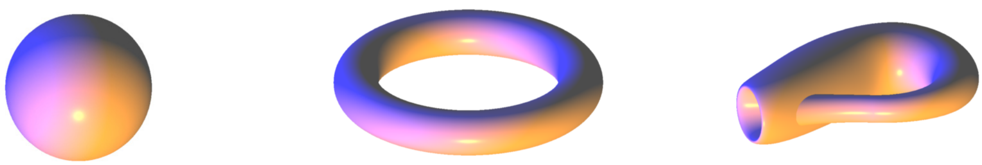
Replacing the notion of differentiability in parts (ii) and (iii) of Definition 1.2.1 by continuous differentiability or any higher order differentiability '$C^k$', $k \in \mathbb{N}_+ \setminus \{1\}$, or even infinitely often differentiability '$C^\infty$' we end up each time with another category[2] of manifolds. In lieu of more regularity, one could also require less regularity such as 'Lipschitz continuity' or 'piecewise linearity' yielding by analogy Lipschitz manifolds and piecewise linear manifolds. It's well-known that every $C^k$-differentiable structure, $k \in \mathbb{N}_+$, admits an up to $C^k$-diffeomorphisms unique $C^\infty$-differentiable structure, see [Whi36][3]. Therefore, we reasonably restrict ourselves to $C^\infty$-differentiable manifolds which we refer to as *(smooth) manifolds* from now on. The set of smooth functions in the sense of part (iv) of Definition 1.2.1 is denoted by $C^\infty(\mathbb{Y}_1; \mathbb{Y}_2)$; as usual, we write $C^\infty(\mathbb{Y}_1)$ for the set of real-valued smooth functions.

When we proceed to recapitulate several definitions more, we impose either continuity or smoothness as regularity, but it goes without saying that constructions like fibre bundles would carry over to other categories.

**Definition 1.2.3** (submanifolds). Consider a $\mathscr{y}$-dimensional manifold $(\mathbb{Y}, \mathbf{D})$, a subset $S \subseteq \mathbb{Y}$, and $j \in \{0, \ldots, \mathscr{y}\}$. Assume that for each $s \in S$ there is a chart $h \in \mathbf{D}$ at $s$ such that $h(\mathrm{dom}(h) \cap S) = \mathrm{ran}(h) \cap (\mathbb{R}^j \times \{0\}^{\mathscr{y}-j})$ and denote by $\mathbf{D}(S)$ the set of those charts $h$ restricted to $S$. Then, $S$ endowed with subspace topology and $\mathbf{D}(S)$ as a smooth structure is a $j$-dimensional manifold. We say that $S$ is a *submanifold of* $\mathbb{Y}$ *(of dimension* $j$*)*. ⌟

We quickly run to a variety of standard examples of manifolds. A more detailed discussion as well as several more examples can be found in [Lee13, Section 1].

**Example 1.2.4.** The most trivial examples are the Euclidean spaces $\mathbb{R}^{\mathscr{d}}$ with the standard smooth structure consisting of the identity as the only (global) chart. The minimal examples are countable discrete spaces $\mathbb{Y}$ as zero-dimensional manifolds; the smooth structure consists just of charts $\{y\} \to \mathbb{R}^0 = \{0\}$ for all $y \in \mathbb{Y}$. More common examples include general ellipsoids, tori, and the Klein bottle, see Figure 1.2.



(a) Sphere $\mathbb{S}^2$ (b) Torus $\mathbb{T}^2$ (c) Klein bottle self-intersecting in $\mathbb{R}^3$

Figure 1.2: Some example manifolds

Furthermore, an open subset $U$ of a manifold $(\mathbb{Y}, \mathbf{D})$ naturally inherits the smooth structure $\mathbf{D}(U) := \{h \mid h \in \mathbf{D}\colon\ \mathrm{dom}(h) \subseteq U\}$ and becomes an open submanifold $(U, \mathbf{D}(U))$. It is straight forward to see that also finite Cartesian products of manifolds are manifolds, which shows that the (topological) $\mathscr{d}$-torus $\mathbb{T}^{\mathscr{d}} = \bigtimes_{j=1}^{\mathscr{d}} \mathbb{S}^1$ becomes a manifold once we made the circle $\mathbb{S}^1$ into a manifold.

An easy method for creating manifolds is provided by the Regular Value Theorem, see [Lee13, Proposition 8.10]: Consider a smooth function $f$ between manifolds $\mathbb{Y}_1$ and $\mathbb{Y}_2$. A point $y_2 \in \mathbb{Y}_2$ is a *regular value* if for all $y \in f^{-1}(\{y_2\})$ the rank[4] of $f$ at $y$ equals the dimension of $\mathbb{Y}_2$. In this situation $f^{-1}(\{y_2\})$ is called *regular level set*. The Regular Value Theorem asserts that regular level sets of smooth

[2] The objects are manifolds with the respective regularity $C$ and the morphisms are $C$-differentiable functions between manifolds.
[3] Also, the modern definition of manifolds in terms of charts and atlases traces back to this paper.
[4] That is the rank of the Jacobian $D_{h_1(y)}(h_2 \circ f \circ h_1^{-1})$ for arbitrary charts $h_1$ at $y$ and $h_2$ at $f(y)$.

maps are submanifolds of $\mathbb{Y}_1$ with dimension $\dim(\mathbb{Y}_2)$. For instance, 1 is regular value of the mapping $f\colon \mathbb{R}^d \to \mathbb{R}$, $x \mapsto \|x\|_{\text{euc}} = \sqrt{(x\,,x)_{\text{euc}}}$ computing the Euclidean norm, thus the sphere $\mathbb{S}^{d-1} = f^{-1}(\{1\})$ can be thought as a smooth submanifold of $\mathbb{R}^d$. $\lrcorner$

Note that manifolds in the sense of Definition 1.2.1 have empty boundary in terms of [Lee13, Section 1.5]. For sake of illustration we sometimes consider manifolds with nonempty boundary; e. g. the Möbius strip is an example which famously has exactly one boundary curve. In such a situation, we might use adequately altered definitions without mentioning it explicitly.

**Definition 1.2.5** (continuous and smooth fibre bundles)**.** A continuous surjection $\pi$ between topological spaces $\mathbb{E}$ and $\mathbb{B}$ is called *continuous fibre bundle with (standard) fibre* $\mathbb{F}$ if $\mathbb{F}$ is a topological space and $\pi$ satisfies the axiom of local trivialisation: For any $b \in \mathbb{B}$ there is an open neighbourhood $U_b \subseteq \mathbb{B}$ as well as a homeomorphism $\varphi$ that sends the preimage $\pi^{-1}(U_b)$ to the Cartesian product $U_b \times \mathbb{F}$. Meaning that $\varphi$ renders the diagram in Figure 1.3 commutative. In this situation one refers to $\mathbb{B}$ as *base space*, to $\mathbb{E}$ as

$$\begin{tikzcd}
\pi^{-1}(U_b) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U_b \times \mathbb{F} \arrow[dl, "\mathrm{pr}_1"] \\
U_b &
\end{tikzcd}$$

Figure 1.3: Local trivialisation of a fibre bundle

*total space*, and to $\varphi$ as *(local) trivialisation chart (at $b$)*. The set $\mathbb{E}_b := \pi^{-1}(\{b\})$ for given $b \in \mathbb{B}$ is called *fibre over* $b$ and is homeomorphic to $\mathbb{F}$, whence the name *standard fibre* for $\mathbb{F}$. One also calls $\pi\colon \mathbb{E} \longrightarrow \mathbb{B}$ the *bundle projection* thinking it as a part rather than the whole object.

If the spaces $\mathbb{B}$, $\mathbb{E}$, $\mathbb{F}$ are manifolds, $\pi$ is smooth, and trivialisation charts are diffeomorphisms, then $\pi\colon \mathbb{E} \longrightarrow \mathbb{B}$ is a *(smooth) fibre bundle over the base manifold* $\mathbb{B}$ *with (standard) fibre* $\mathbb{F}$. $\lrcorner$

**Example 1.2.6.** The simplest examples consist of product spaces $\mathbb{E} = \mathbb{B} \times \mathbb{F}$ and a bundle projection $\pi = \mathrm{pr}_1$ that coincides with projection to the first component. Then, the axiom of local trivialisation is satisfied even globally. If one wants to verify that a surjective submersion $\pi\colon \mathbb{E} \to \mathbb{B}$ is a fibre bundle, then one can check that $\pi$ is proper. This criterion is known as Ehresmann Fibration Theorem, see [Ehr51].

Later on in Definition 1.2.15 we introduce the unit tangent bundle which is an instance of so-called sphere bundles having a sphere as standard fibre. Whilst a torus is a trivial bundle over the circle $\mathbb{B} = \mathbb{S}^1$ with standard fibre $\mathbb{F} = \mathbb{S}^1$, the Klein bottle is a fibre bundle with the same base space and fibre, but by no means a trivial bundle. This phenomenon is easier to visualise via the example of an annulus on the one hand, which is a trivial bundle over the circle $\mathbb{B} = \mathbb{S}^1$ with a certain finite interval as fibre, and on the other hand a Möbius strip which is a nontrivial bundle over the very same base manifold with the same standard fibre but also with a twist in the total manifold.

Enriching the standard fibre with additional structure and making sure that the trivialisation charts respect it, gives us new interesting types of fibre bundles. For instance, *vector bundles (of rank $k$)* are fibre bundles such that the standard fibre is a $k$-dimensional (real) vector space and every trivialisation chart $\varphi$ is linear, hence the restriction of $\varphi$ to the fibre $\mathbb{E}_b$ over $b \in \mathbb{B}$ is a linear isomorphism. $\lrcorner$

**Example 1.2.7** ((co-)tangent bundles)**.** For extensive definitions regarding the tangent bundle $T\mathbb{B}$ over a manifold $\mathbb{B}$ we refer to [Lee13, Section 3, 4]. However, we quickly recall the three equivalent definitions of tangent vectors/spaces at $b \in \mathbb{B}$ as exquisitely explained in [Jän05, Abschnitt 2.2]:

(i) *algebraic definition*: One considers equivalence classes of smooth functions that coincide in a neighbourhood of $b$. Those equivalence classes are called *germs at* $b$. A tangent vector $v$ at $b$ is a so-called *derivation at* $b$ that is a linear functional on the space of germs at $b$ satisfying the product rule, i. e. $v([f]_b \cdot [g]_b) = v([f]_b) \cdot v([g]_b)$ for germs $[f]_b$ and $[g]_b$ at $b$. This is largely considered as the most elegant approach.

(ii) *geometric definition*: Two $\mathbb{B}$-valued smooth curves $\gamma_1$ and $\gamma_2$ such that both their domains are open intervals containing 0 and $\gamma_1(0) = \gamma_2(0) = b$ holds are said to be *tangentially equivalent* if there is a chart $h$ at $b$ such that $\frac{d}{dt}\,(h \circ \gamma_1)\big|_{t=0} = \frac{d}{dt}\,(h \circ \gamma_2)\big|_{t=0}$ holds. If this equality holds for one chart $h$, it holds for every chart at $b$. An equivalence class with respect to tangential equivalence is called tangent vector at $b$.

(iii) *'physical' or chart based definition*: Let $\mathbf{D}$ be the smooth structure of $\mathbb{B}$. Define the tangent space $\mathsf{T}_b\mathbb{B}$ at $b$ as

$$\left\{ v = (v_h)_h \in \bigtimes_{\substack{h \in \mathbf{D} \\ h \text{ chart at } b}} \mathbb{R}^{\dim(\mathbb{B})} \;\middle|\; \forall h_1, h_2 \in \mathbf{D}\colon\; D_{h_1(b)}(h_2 \circ h_1^{-1}) v_{h_1} = v_{h_2} \right\}.$$

To put it in words, a tangent vector $v$ could be understood as a function assigning to a given chart $h$ the representation of $v$ in the chart $h$ and the differential of the transition map tells us how a representation of $v$ transforms under change of charts.

Once the tangent space $\mathsf{T}_b\mathbb{B}$ at $b$ is defined, so is the cotangent space $\mathsf{T}_b^*\mathbb{B}$ at $b$ as dual vector space. The disjoint unions $\mathsf{T}\mathbb{B} = \bigsqcup_{b\in\mathbb{B}} \mathsf{T}_b\mathbb{B}$ and $\mathsf{T}^*\mathbb{B} = \bigsqcup_{b\in\mathbb{B}} \mathsf{T}_b^*\mathbb{B}$ both carry a natural smooth structure and the mappings

$$\pi_0\colon\; \mathsf{T}\mathbb{B} \to \mathbb{B},\; \mathsf{T}_b\mathbb{B} \ni v \longmapsto b, \qquad \text{and} \qquad \mathsf{T}^*\mathbb{B} \to \mathbb{B},\; \mathsf{T}_b^*\mathbb{B} \ni v^* \longmapsto b$$

are vector bundles over $\mathbb{B}$, the so-called *tangent bundle* and *cotangent bundle* respectively, see [Lee13, Lemma 4.1, Proposition 5.3, Proposition 6.5]. The cotangent bundle is an example of the so-called dual bundle of a given vector bundle which is so to say the 'fibrewise dual'. ⌟

As we frequently operate in tangent bundles of higher order, especially in Chapter 3, we might as well introduce some short hand notation at this point.

*Notation* 1.2.8 (Higher order tangent bundles). Define $\mathsf{T}^0\mathbb{B} := \mathbb{B}$. For every $k \in \mathbb{N}_+$ the *kth order tangent bundle* is recursively defined as $\mathsf{T}^k\mathbb{B} := \mathsf{T}(\mathsf{T}^{k-1}\mathbb{B})$. Consistently, this yields $\mathsf{T}^1\mathbb{B} = \mathsf{T}\mathbb{B}$. Denote by $\pi_{k,k-1}$ the bundle projection in the 1st order tangent bundle $\mathsf{T}^k\mathbb{B} \longrightarrow \mathsf{T}^{k-1}\mathbb{B}$. Then consequently, the projection in the $(k-j)$th order bundle $\mathsf{T}^k\mathbb{B} \longrightarrow \mathsf{T}^j\mathbb{B}$ is denoted by $\pi_{k,j}$; explicitly, it is given as the composition of several ordinary tangent bundle projections

$$\pi_{k,j} = \pi_{j+1,j} \circ \pi_{j+2,j+1} \circ \cdots \circ \pi_{k-1,k-2} \circ \pi_{k,k-1}.$$

The $k$th order tangent bundle together with these projections is an instance of a so-called *multi(-fibre-)bundles*, compare to [Sau02, Section 3]. ⌟

Torus and Klein bottle can be viewed as fibre bundles with both same base manifold and same standard fibre, but they are fundamentally different. However, just by looking at the definition of fibre bundles one does not easily spot the difference. The key is to understand the mechanism that, roughly speaking, glues the fibres together. This mechanism is provided in terms of the so-called *structure group* of a fibre bundle. For an extensive discussion see [Nic96, Section 2.3.3], we boil it down to its essence in the following remark. Structure groups will become relevant in the context of local product measures on fibre bundles, see Definition 1.2.19 later on.

*Remark* 1.2.9 (structure groups). Given the (smooth) fibre bundle $\pi\colon\; \mathbb{E} \longrightarrow \mathbb{B}$ we consider two local trivialisation charts $\varphi_i$ and $\varphi_j$ with domains of trivialisation $U_i, U_j \subseteq \mathbb{B}$, and define their *transition map* as

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1}\colon\; U_j \times \mathbb{F} \to U_i \times \mathbb{F}.$$

Due to the fact that $\pi_0 \circ \varphi_k^{-1}(b,v) = b$ for all $(b,v) \in U_k \times \mathbb{F}$, $k \in \{i,j\}$, the transition map can be understood as

$$\varphi_{ij}(b,v) = (b, T_{ij}(b)v) \qquad \text{for all } (b,v) \in U_j \times \mathbb{F},$$

where $T_{ij}(b)\colon\; \mathbb{F} \to \mathbb{F}$ is a diffeomorphism and $T_{ij}(b)$ smoothly depends on the base point $b$. As the diffeomorphisms of $\mathbb{F}$ form a group, the fibre bundle is indeed a so-called *(smooth) $G$-fibre bundle* with the diffeomorphisms of the standard fibre as *structure group* $G$. For a $G$-fibre bundle one says that a diffeomorphism $T\colon\; \mathbb{E} \to \mathbb{E}$ is a *$G$-automorphism* if it maps fibres to fibres, i.e. $\pi \circ T = \pi$ holds, and for any trivialisation chart $\varphi$ with domain $U$ of trivialisation there is a smooth map $g\colon\; U \to G$ such that $\varphi \circ T \circ \varphi^{-1} = (b, g(b)v)$ for all $b \in U$ and $v \in \mathbb{F}$. One can show by means of covering theory[5] that for the total space $\mathbb{E}$ being torus or Klein bottle the set of $G$-automorphisms is isomorphic to the fundamental group of $\mathbb{E}$. As clearly the fundamental groups of torus and Klein bottle are quite different, so are the fibre automorphisms. ⌟

[5] To this end, one considers the universal covering space of $\mathbb{E}$, since its automorphism group is isomorphic to the fundamental group of $\mathbb{E}$. See [Lee11, Proposition 12.6, Example 12.7, Exercise 12.5].

1

**Definition 1.2.10** (differential of smooth functions)**.** Consider a function between manifolds $f \in C^\infty(\mathbb{Y}_1;\mathbb{Y}_2)$, a point $y \in \mathbb{Y}_1$, and a tangent vector $v \in \mathsf{T}_y\mathbb{Y}_1$. We define a tangent vector $d_y f(v) \in \mathsf{T}_{f(y)}\mathbb{Y}_2$ by the assignment

$$\big(d_y f(v)\big)_{h_2} := D_{h_1(b)}(h_2 \circ f \circ h_1^{-1})v_{h_1} \in \mathbb{R}^{\dim(\mathbb{Y}_2)}$$

for all charts $h_1$ and $h_2$ at $y$ and $f(y)$ respectively, compare to (iii) of Example 1.2.7. Note that the assignment does not depend on the choice of the charts and yields a unique tangent vector. In this way $f$ induces a linear mapping $d_y f\colon \mathsf{T}_y\mathbb{Y}_1 \to \mathsf{T}_{f(y)}\mathbb{Y}_2$ called the *differential of $f$ at $y$*. Taking the differential at $y$ on the one hand satisfies $d_y \mathsf{Id}_{\mathbb{Y}_1} = \mathsf{Id}_{\mathsf{T}_y\mathbb{Y}_1}$ and on the other hand obeys the *chain rule* $d_y(f_2 \circ f_1) = d_{f_1(y)} f_2 \circ d_y f_1$ for any two smooth functions $f_1$ from $\mathbb{Y}_1$ to $\mathbb{Y}_2$ and $f_2$ on $\mathbb{Y}_2$. Reminiscent of partial derivatives, we sometimes write $\partial_v f(y) := \langle v \,, d_y f \rangle$ for $v \in \mathsf{T}_y\mathbb{Y}_1$. ⌟

*Notation* 1.2.11 (local coordinate form of tangent vectors and differentials). Suppose a chart $h$ of a $\mathscr{y}$-dimensional manifold $\mathbb{Y}$ and denote the induced local coordinates by $(y^j)_{j=1}^{\mathscr{y}}$. Moreover, consider the tangent vectors $({}^{\partial}/_{\partial x^j}(x))_{j=1}^{\mathscr{y}}$ at $x \in \mathbb{R}^{\mathscr{y}}$ corresponding to the standard coordinates $(x^j)_{j=1}^{\mathscr{y}}$ of $\mathbb{R}^{\mathscr{y}}$ provided by the identity chart. I. e. when the differential of a function $g \in C^\infty(\mathbb{R}^{\mathscr{y}})$ acts on ${}^{\partial}/_{\partial x^j}(x)$, $j \in \{1,\dots,\mathscr{y}\}$, we just obtain the classical partial derivatives $\langle {}^{\partial}/_{\partial x^j}(x) \,, d_x g \rangle = \partial_{x^j} g(x)$ at $x \in \mathbb{R}^{\mathscr{y}}$. For all $y \in \mathbb{Y}$ and $j \in \{1,\dots,\mathscr{y}\}$ as well as functions $f \in C^\infty(\mathbb{Y})$ the expressions ${}^{\partial}/_{\partial y^j}(y)$ and ${}^{\partial f}/_{\partial y^j}(y)$ are defined as

$$\frac{\partial}{\partial y^j}(y) := \left\langle \frac{\partial}{\partial x^j}(h(y)) \,, d_{h(y)}(h^{-1}) \right\rangle \quad \text{and} \quad \frac{\partial f}{\partial y^j}(y) = \frac{\partial}{\partial y^j} f\,(y) := \frac{\partial (f \circ h^{-1})}{\partial x^j}(h(y))$$

See [Lee13, Section 3.3] for more details and formulae that describe a change of coordinates. ⌟

**Definition 1.2.12** (sections and vector fields)**.** Let $\pi\colon\ \mathbb{E} \longrightarrow \mathbb{B}$ be a continuous fibre bundle. A *(global) continuous section* in this bundle is a continuous right inverse of the bundle projection, i. e. a mapping $\mathscr{X}\colon\ \mathbb{B} \to \mathbb{E}$ such that $\pi \circ \mathscr{X} = \mathsf{Id}_{\mathbb{B}}$. The set of continuous sections is denoted by $\Gamma^0(\mathbb{B} \to \mathbb{E})$ or just $\Gamma^0(\mathbb{E})$. A local continuous section is defined in almost the same manner, but rather with an open subset as domain than the whole base space.

Clearly, a *(global) (smooth) section* in a (smooth) fibre bundle is a smooth right inverse of the bundle projection. The set of smooth sections is denoted by $\Gamma^\infty(\mathbb{B} \to \mathbb{E})$ or $\Gamma^\infty(\mathbb{E})$ for sake of brevity. Smooth sections in the tangent bundle are commonly referred to as *vector fields* and can be thought as differential operators in view of Example 1.2.7: Given a smooth function $f \in C^\infty(\mathbb{B})$ and a vector field $\mathscr{X} \in \Gamma^\infty(\mathsf{T}\mathbb{B})$ then the action of $\mathscr{X}$ on $f$ is a smooth function $\mathscr{X} f = \mathscr{X}(f) \in C^\infty(\mathbb{B})$ given as

$$\mathscr{X} f(b) = (\mathscr{X} f)(b) := \partial_{\mathscr{X}} f(b) = \langle \mathscr{X}(b) \,, d_b f \rangle \qquad \text{for all } b \in \mathbb{B}.$$

It's common to denote the $k$-times action of $\mathscr{X}$ on $f$ by $\mathscr{X}^k f = \mathscr{X}^k(f)$ which we usually just need for $k=2$, i. e. $\mathscr{X}^2 f := \mathscr{X}(\mathscr{X}(f))$. ⌟

*Remark* 1.2.13 (parallelisability). Consider a vector bundle $\pi\colon\ \mathbb{E} \to \mathbb{B}$. A *local (smooth) frame* $(E_j)_{j=1}^{\mathscr{e}}$ is an ordered tuple of local sections $E_j \in \Gamma^\infty(U;\mathbb{E})$ with common open domain $U \subseteq \mathbb{B}$ that provides fibrewise a basis of $\mathbb{E}|_U$, i. e. $(E_j(b))_{j=1}^{\mathscr{e}}$ is a basis of $\mathbb{E}_b$ for all $b \in U$. By extension, a global frame is a local frame that is fully supported on $\mathbb{B}$. E. g. given local coordinates $(b^j)_{j=1}^{\mathscr{e}}$ the local vector fields $({}^{\partial}/_{\partial b^j})_{j=1}^{\mathscr{e}}$ form a local frame in $\mathbb{E} = \mathsf{T}\mathbb{B}$. According to [Lee13, Corollary 5.11], a (smooth) vector bundle is trivial if and only if there is a (smooth) global frame.

The manifold $\mathbb{B}$ is said to be *parallelisable* if there is a global frame in $\mathsf{T}\mathbb{B}$. Obviously, $\mathbb{R}^{\mathscr{d}}$ possesses a global frame induced by its global chart of Euclidean coordinates. Many more examples are provided by [Lee13, Proposition 5.15], for instance the $\mathscr{d}$-dimensional torus $\mathbb{T}^{\mathscr{d}}$ is parallelisable. However, most spheres are not parallelisable: Since the works [Ker58] by M. Kervaire and independently [BM58] by R. Bott and J. Milnor it's known that $\mathbb{S}^{\mathscr{d}-1}$ is parallelisable exactly for $\mathscr{d} \in \{1,2,4,8\}$. The case $\mathscr{d}=3$ is well-known as a consequence of the Hairy Ball Theorem, see [Lee13, Exercise 14.22]. This observation motivates our usage of stratifolds in Chapter 4. Moreover, the possible lack of parallelisability causes, but just initially, an issue with the McKean-Gangolli injection scheme for construction of skew Brownian motions on general manifolds. We address this later in Section 1.3 in a bit more detail. ⌟

In terms of sections, one defines the analogues of inner products on vector spaces for vector bundles. These analogues are the Riemannian metrics which are discussed in [Lee13, Section 11.4]. As to be expected, Riemannian metrics provide a fibrewise sense of length of tangent vectors and angles between nonzero tangent vectors, thus also a measure for the length of curves on the base manifold and a fibrewise sense of orthogonality.

**Definition 1.2.14** (Riemannian metrics). Let $\pi\colon \mathbb{E} \to \mathbb{B}$ be a vector bundle and denote by $\mathsf{Symm}(\mathbb{E})$ the space of fibrewise symmetric bilinear forms on $\mathbb{E}$. A *Riemannian metric in* $\mathbb{E}$ is a section in the fibre bundle $\mathsf{Symm}(\mathbb{E}) \to \mathbb{B}$ that is fibrewise an inner product. A Riemannian metric $\mathsf{b}$ yields fibrewise a norm via $|e|_{\mathsf{b}} := \sqrt{\mathsf{b}_b(e\,,e)}$ for all $e \in \mathbb{E}_b$ and $b \in \mathbb{B}$. The pair $(\mathbb{B}, \mathsf{b})$ is called *Riemannian manifold*. Usually, we implicitly consider $\mathbb{E} = \mathsf{T}\mathbb{B}$. Two Riemannian manifolds $(\mathbb{B}_j, \mathsf{b}_j)$, $j \in \{1, 2\}$, are said to be *isometric* if there is a diffeomorphism $i\colon \mathbb{B}_1 \to \mathbb{B}_2$ such that the pushforward $i_*\mathsf{b}_1 := \mathsf{b}_1(di\,\cdot\,, di\cdot)$ coincides with $\mathsf{b}_2$. ⌟

The next definition follows immediately and gives us the most important example of sphere bundles.

**Definition 1.2.15** (unit tangent bundles). Consider a $\mathscr{b}$-dimensional Riemannian manifold $(\mathbb{B}, \mathsf{b})$. The unit tangent space at a point $b \in \mathbb{B}$ is defined as

$$\mathsf{U}_b\mathbb{B} := \left\{ v \in \mathsf{T}_b\mathbb{B} \;\middle|\; |v|_{\mathsf{b}}^2 = \mathsf{b}_b(v\,,v) = 1 \right\}$$

and the disjoint union over all points $b$ yields the *unit tangent bundle* $\pi_{0|\mathsf{U}}\colon \mathsf{U}\mathbb{B} \to \mathbb{B}$ which indeed is a smooth fibre bundle with standard fibre $\mathbb{F} = \mathbb{S}^{\mathscr{b}-1}$. ⌟

Via the length of connecting curves one obtains a notation of the distance between to points in a Riemannian manifold. We make a few comments on the arising metric space.

*Remark* 1.2.16 (Riemannian distance metric and completeness properties). To avoid ambiguities, we refer to a metric in the sense of metric spaces as *distance metric/function*. Given a connected Riemannian manifold $(\mathbb{B}, \mathsf{b})$ the associated length measure for curves in $\mathbb{B}$ induces a distance metric via

$$\mathsf{dist}_{\mathsf{b}}(b_1, b_2) = \inf_{\substack{\gamma\colon [0,1]\to\mathbb{B} \\ \text{(piecewise) smooth with} \\ \gamma(0)=b_1,\,\gamma(1)=b_2}} \mathsf{length}(\gamma) := \inf_{\gamma\cdots} \int_0^1 |\dot\gamma(t)|_{\mathsf{b}}\,\mathrm{d}t$$

for all points $b_1, b_2 \in \mathbb{B}$. By [Lee13, Proposition 11.20], the topology of the metric space $(\mathbb{B}, \mathsf{dist}_{\mathsf{b}})$ coincides with the original topology of the manifold $\mathbb{B}$.

The famous Hopf-Rinow Theorem, see [Nic96, Theorem 4.1.29], states that completeness of connected Riemannian manifold $(\mathbb{B}, \mathsf{b})$ as a metric space is equivalent to several other nice properties, among them are to mention geodesically completeness[6] and that at every point $b \in \mathbb{B}$ the exponential map[7] $\exp_b$ is defined on the entirety of $\mathsf{T}_b\mathbb{B}$. In particular, for two points in a complete manifold $\mathbb{B}$ there is at least one length minimising curve that also is a geodesic with respect to the Levi-Civita connection, which we introduce later in Remark 1.2.28. Spheres with their standard round metric are examples of complete connected Riemannian manifolds with not necessarily unique minimising geodesics: Given two antipodal points there infinitely many length minimising, connecting geodesics. ⌟

A given Riemannian metric not only induces a 'geodesic' structure as outlined in Remark 1.2.16. We foreshadow that it also induces a so-called spectral structure, see Definition 4.1.9, by means of the Laplace-Beltrami operator introduced in Definition 1.2.20. As important as Riemannian metrics may be, one must nevertheless keep in mind that every vector bundle can be endowed with some Riemannian metric[8] and in general there is no canonical way of selecting one among all the possible Riemannian metrics. In this thesis, we usually suppose that a Riemannian metric is given and other structures are compatible with it. But e. g. in the context of semisprays we briefly touch upon the cases where instead another piece of information like a Lagrangian is given.

### 1.2.2 ▪ Integration on manifolds and in fibre bundles

When it comes to integration on manifolds one usually distinguishes the cases of orientable and nonorientable manifold. An *oriented* manifold is a manifold with a maximal[9] oriented atlas by which we mean that

[6] Meaning that the so-called geodesic flow is a global flow, see [Nic96, Definition 4.1.11].
[7] See [Nic96, § 4.1.3] for the definition of the exponential map.
[8] In a domain of local triviality $U \subseteq \mathbb{B}$ one can construct from some smooth map $s_U\colon U \to \mathsf{Symm}(\mathbb{E})|_U$ a Riemannian metric on $U$ as $U \to \mathsf{Symm}(\pi^{-1}(U)) \simeq \mathsf{Symm}(U \times \mathbb{R}^k)$, $b \mapsto (b, s_U(b))$. Now, a global Riemannian metric is built from the local ones by a partition of unity argument.
[9] With respect to inclusion.

the Jacobian determinant of any transition map is everywhere positive. Such a maximal oriented atlas formally represents the concept of an *orientation*. A manifold is orientable if there is an orientation, i. e. a maximal orientable atlas; a nonempty connected orientable manifold admits exactly two different orientations. Famously, the Möbius strip and the Klein bottle are examples of manifolds that are not orientable. Integration on orientable manifolds is defined in terms of volume forms whilst on nonorientable manifolds one uses density fields. A detailed discussion would go far beyond the scope of this thesis, thus we refer to [Lee13, Sections 12-14] or [Gro+01, Section 3.1.2]. The result is integration with respect to the so-called Riemannian volume measure as both constructions yield the same integral on orientable manifolds. Alternatively, one could define the Riemannian volume measure directly as a Radon measure via an integral expression reminiscent of integration with respect to differential forms, compare to [Sak96, Section II.5].

**Definition 1.2.17** (Riemannian volume form/density field)**.** Consider a $\mathscr{b}$-dimensional Riemannian manifold $(\mathbb{B}, \mathsf{b})$. Suppose that $\mathbb{B}$ is oriented. Then, there is a unique nonvanishing smooth differential form $\mathrm{d}\lambda_{\mathsf{b}}$ of degree $\mathscr{b}$ such that $\mathrm{d}\lambda_{\mathsf{b}}(E_1, \dots, E_{\mathscr{b}}) = 1$ holds for every oriented local orthonormal frame $(E_j)_{j=1}^{\mathscr{b}}$, see [Lee13, Proposition 13.22]. This differential form is called *Riemannian volume form* or Riemannian volume element. If $\mathbb{B}$ is not necessarily orientable, there still is a unique positive, smooth density field $|\mathrm{d}\lambda_{\mathsf{b}}|$ such that $|\mathrm{d}\lambda_{\mathsf{b}}|(E_1, \dots, E_{\mathscr{b}}) = 1$ holds for every local orthonormal frame $(E_j)_{j=1}^{\mathscr{b}}$, see [Lee13, Proposition 14.33]. One calls it the *Riemannian density field* by analogy. $\lrcorner$

*Remark* 1.2.18 ($\mathrm{d}\lambda_{\mathsf{b}} = |\mathrm{d}\lambda_{\mathsf{b}}|$). Riemannian volume form and Riemannian density field coincide for oriented Riemannian manifolds. We adopt the common practice to write integrals of real-valued functions $f$ on $\mathbb{B}$ always as $\int_{\mathbb{B}} f \,\mathrm{d}\lambda_{\mathsf{b}}$ no matter if there might be an orientation or not. The Radon measure $\lambda_{\mathsf{b}}$ defined by $\left\langle f\,, \lambda_{\mathsf{b}} \right\rangle := \int_{\mathbb{B}} f \,\mathrm{d}\lambda_{\mathsf{b}}$ for all $f \in C^0_{\mathsf{c}}(\mathbb{B})$ is the *Riemannian volume measure*. For local coordinate expressions see Section A.1. $\lrcorner$

Once integration on manifolds is established, one proceeds to introduce $L^p$-spaces $L^p(\mathbb{B}; \lambda_{\mathsf{b}})$ and Sobolev spaces $H^{m,p}(\mathbb{B}) = H^{m,p}(\mathbb{B}; \lambda_{\mathsf{b}})$, see [Heb00]. This might be a good point to mention that for locally Lipschitzian functions $f_0$ on $\mathbb{B}$ holds $f_0 \in H^{1,p}_{\mathsf{loc}}(\mathbb{B})$ for all $p \in [1, \infty)$, see [Heb00, Proposition 2.4] and references therein for the proof.[10] Furthermore, if $\mathbb{B}$ is complete, then the set of smooth functions with compact support is dense in $H^{1,p}(\mathbb{B})$, see [Heb00, Theorem 3.1]; the case of $m > 1$ is discussed subsequently there.

Moving on, we also desire 'product-like' measures in the total space of a fibre bundle corresponding to given measures on the base space and the standard fibre. The general situation was investigated in [Goe59] by A. Götz. We choose to adapt his wording slightly and speak either of 'bundle measures' or 'local product measures'.

**Definition 1.2.19** (bundle measures)**.** Consider a continuous fibre bundle $\pi\colon\ \mathbb{E} \to \mathbb{B}$ with standard fibre $\mathbb{F}$ such that both $\mathbb{B}$ and $\mathbb{F}$ are locally compact.[11] Let $\mu_{\mathbb{B}}$ and $\nu_{\mathbb{F}}$ be a Baire measures on $\mathbb{B}$ and $\mathbb{F}$ respectively. A Baire measure $\mu_{\mathbb{E}}$ on $\mathbb{E}$ is a *bundle measure* or *local product measure with base measure* $\mu_{\mathbb{B}}$ *and fibre measure* $\nu_{\mathbb{F}}$ if for every local trivialisation chart $\varphi$ with domain of trivialisation $U$ and all Baire sets $A \subseteq \pi^{-1}(U) \times \mathbb{F}$ holds $\mu_{\mathbb{E}}(\varphi^{-1}(A)) = \mu_{\mathbb{B}} \otimes \nu_{\mathbb{F}}(A)$. In this case, we write $\mu_{\mathbb{B}} \otimes_{\mathsf{loc}} \nu_{\mathbb{F}} := \mu_{\mathbb{E}}$. Recall the notion of structure groups from Remark 1.2.9. According to [Goe59, Theorem 1], the local product measure $\mu_{\mathbb{B}} \otimes_{\mathsf{loc}} \nu_{\mathbb{F}}$ exists if and only if $\nu_{\mathbb{F}}$ is invariant under transformation by the structure group.

Notably, a local product measure $\mu_{\mathbb{B}} \otimes_{\mathsf{loc}} \nu_{\mathbb{F}}$ disintegrates in the following way: For all Baire sets $Z$ in $\mathbb{E}$ and integrable functions $f\colon\ Z \to \mathbb{R}$ hold

$$\mu_{\mathbb{B}} \otimes_{\mathsf{loc}} \nu_{\mathbb{F}}(Z) = \int_{\mathbb{B}} \nu_{\mathbb{F};b}(Z)\ \mu_{\mathbb{B}}(\mathrm{d}b)$$

$$\text{and} \qquad \int_Z f \,\mathrm{d}\mu_{\mathbb{B}} \otimes_{\mathsf{loc}} \nu_{\mathbb{F}} = \int_{\pi(Z)} \int_{\mathbb{E}_b} f \,\mathrm{d}\nu_{F;b}\ \mu_{\mathbb{B}}(\mathrm{d}b),$$

where $\nu_{\mathbb{F};b}$ can roughly be thought as a copy of $\nu_{\mathbb{F}}$ on the fibre $\mathbb{E}_b$. For the construction of $\nu_{\mathbb{F};b}$ see part b) of the proof of [Goe59, Theorem 1]. $\lrcorner$

[10] On a related note, a variant of Rademacher's Theorem asserts that $f_0$ is differentiable almost everywhere and the set of points where $f_0$ is differentiable is dense in $\mathbb{B}$. This variant is given in [AFL05, Theorem 5.7].

[11] Then, also $\mathbb{E}$ is locally compact. Clearly, the assumption is satisfied for smooth fibre bundles.

### 1.2.3 ▪ Divergence Theorem and Laplace-Beltrami operator

**Definition 1.2.20** (the 'big three' differential operators on Riemannian manifolds). Consider a Riemannian manifold $(\mathbb{B}, \mathsf{b})$.

(i) Given a function $f \in C^\infty(\mathbb{B})$ its *gradient* $\nabla_\mathsf{b} f$ is the unique vector field that fulfils

$$\mathsf{b}\big(\nabla_\mathsf{b} f\,, \mathcal{X}\big) = \partial_\mathcal{X} f \quad \text{for all } \mathcal{X} \in \Gamma^\infty(\mathrm{T}\mathbb{B}). \tag{1.2}$$

(ii) The divergence operator is defined in terms of the Lie derivative[12] $\mathscr{L}$, since this definition reminds us that the divergence models infinitesimal change of volume under the flow of a vector field. Suppose that $\mathbb{B}$ is orientable with volume form $\mathrm{d}\lambda_\mathsf{b}$. Given a vector field $\mathcal{X} \in \Gamma^\infty(\mathrm{T}\mathbb{B})$ the *divergence* $\mathrm{div}_\mathsf{b} \mathcal{X}$ is a smooth function that satisfies

$$\mathscr{L}_\mathcal{X}\, \mathrm{d}\lambda_\mathsf{b} = \mathrm{div}_\mathsf{b} \mathcal{X} \cdot \mathrm{d}\lambda_\mathsf{b}.$$

This definition actually doesn't depend on the orientation at all, see [Lee13, Problem.14-20], so the volume form can be replaced by the corresponding volume density field.

(iii) For a smooth function $f \in C^\infty(\mathbb{B})$ the so-called *Laplace-Beltrami operator* $\Delta_\mathsf{b} f$ is the divergence of the gradient of $f$: $\Delta_\mathsf{b} f = \mathrm{div}_\mathsf{b}\big(\nabla_\mathsf{b} f\big)$. Some authors choose a different sign convention and define the Laplace-Beltrami as minus divergence of the gradient. ⌟

*Remark* 1.2.21 (Divergence Theorem and formal adjoint of vector fields). The Divergence Theorem states that for every smooth vector field $\mathcal{X}$ one has

$$\int_\mathbb{B} f \cdot \mathrm{div}_\mathsf{b} \mathcal{X} \;\mathrm{d}\lambda_\mathsf{b} = -\int_\mathbb{B} \mathsf{b}\big(\nabla_\mathsf{b} f\,, \mathcal{X}\big)\;\mathrm{d}\lambda_\mathsf{b} = -\int_\mathbb{B} \mathcal{X} f \;\mathrm{d}\lambda_\mathsf{b} \quad \text{for all } f \in C^\infty_\mathrm{c}(\mathbb{B}). \tag{1.3}$$

Compare to [Lee13, Theorems 14.23 and 14.34]. If one of two functions $f, g \in C^\infty(\mathbb{B})$ has compact support, then it follows the analogue of integration by parts:

$$\int_\mathbb{B} \Delta_\mathsf{b} f \cdot g \;\mathrm{d}\lambda_\mathsf{b} = -\int_\mathbb{B} \mathsf{b}\big(\nabla_\mathsf{b} f\,, \nabla_\mathsf{b} g\big)\;\mathrm{d}\lambda_\mathsf{b} = \int_\mathbb{B} f \cdot \Delta_\mathsf{b} g \;\mathrm{d}\lambda_\mathsf{b}. \tag{1.4}$$

Furthermore, we can use Equation (1.3) to determine the formal adjoint of a vector field $\mathcal{X}$: Observing that

$$\begin{aligned}\int_\mathbb{B} \mathcal{X} f \cdot g \;\mathrm{d}\lambda_\mathsf{b} &= \int_\mathbb{B} \mathsf{b}\big(\nabla_\mathsf{b} f\,, g\mathcal{X}\big)\;\mathrm{d}\lambda_\mathsf{b} \overset{(1.3)}{=} -\int_\mathbb{B} f \cdot \mathrm{div}_\mathsf{b}(g\mathcal{X}) \;\mathrm{d}\lambda_\mathsf{b} \\ &= -\int_\mathbb{B} fg \cdot \mathrm{div}_\mathsf{b} \mathcal{X} + f \cdot \mathcal{X} g \;\mathrm{d}\lambda_\mathsf{b}\end{aligned}$$

we infer that $\mathcal{X}^* = -\mathcal{X} - \mathrm{div}_\mathsf{b} \mathcal{X} \cdot \mathrm{Id}$. ⌟

It can not be overstated how important the question of essential self-adjointness of the Laplace-Beltrami operator is. Unsurprisingly, that question has been tackled with a large variety of different tools.

**Theorem 1.2.22** (essential self-adjointness of Laplace-Beltrami). Suppose that $\mathbb{B}$ is complete as in Remark 1.2.16. The Laplace-Beltrami operator $\big(\Delta_\mathsf{b}, C^\infty_\mathrm{c}(\mathbb{B})\big)$ is essentially self-adjoint on $L^2(\mathbb{B}; \lambda_\mathsf{b})$. ⌟

For a proof of the theorem as it stands we refer to [Str83, Theorem 2.4] which relies on an eigenvalue discussion. We mention a few different proofs without going into details, since the result is that central. P. R. Chernoff formulates in [Che73, Section 2] a criterion for all powers of operators $T = -\mathrm{i}\, L$ being essentially self-adjoint, where $L$ is a formally antisymmetric differential operator on the Hilbert space of square integrable sections in a vector bundle with Riemannian metric over a complete manifold. The criterion is then applied to the Hodge-de Rham operator[13] $(\mathrm{d} + \delta)^2 = (\mathrm{d}\delta + \delta\mathrm{d})$ which differs from the Laplace-Beltrami operator by a sign. In contrast, in [Roe60, Satz 8] the author shows that the deficiency indices are zero, see the corollary on [RS75, p. 141] or see [Wer11, Lemma VII.2.7 (a), Satz VII.2.8]. Other classical references on the matter are [Gaf51; Gaf54a; Gaf54b], [LM89, Section II.5] and [Wol73]. If we would assume nontrivial boundary, then essentially self-adjointness of the Laplace-Beltrami operator is rather delicate. This challenging problem has been tackled in [Pér13] and the noteworthy paper [ILP15].

12See [Lee13, Section 18].
13See [Sak96, p. 302].

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**Definition 1.2.23** (weighted Riemannian manifolds). Let $(\mathbb{B}, \mathsf{b})$ be a Riemannian manifold. Consider a nonconstant, smooth[14] function $\rho^2_{\mathbb{B}}\colon \mathbb{B} \to (0,\infty)$ and define $\rho_{\mathbb{B}} := \sqrt{\rho^2_{\mathbb{B}}}$. The *weighted Riemannian metric* $\mathbf{b}$ (with weight function $\rho_{\mathbb{B}}$) is defined by

$$\mathbf{b}(v\,,w) = (\rho_{\mathbb{B}}\mathsf{b})(v,w) := \mathsf{b}(\rho_{\mathbb{B}}v\,,\rho_{\mathbb{B}}w)$$

for all $v,w \in \mathsf{T}\mathbb{B}$ with $\pi_0(v) = \pi_0(w)$. The main differential operators associated to the Riemann metric change in a consistent way as follows: For the weighted gradient $\nabla_{\mathbf{b}} := {}^1\!/\!_{\rho_{\mathbb{B}}} \cdot \nabla_{\mathsf{b}}$ the relation (1.2) transforms into $\mathbf{b}\big(\nabla_{\mathbf{b}} f\,, \mathcal{X}\big) = \rho_{\mathbb{B}} \cdot \partial_{\mathcal{X}} f$ for $f \in C^\infty(\mathbb{B})$ and $\mathcal{X} \in \Gamma^\infty(\mathsf{T}\mathbb{B})$. We define the weighted divergence as $\mathrm{div}_{\mathbf{b}}\mathcal{X} := {}^1\!/\!_{\rho_{\mathbb{B}}} \cdot \mathrm{div}_{\mathsf{b}}(\rho^2_{\mathbb{B}}\mathcal{X})$ for $\mathcal{X} \in \Gamma^\infty(\mathsf{T}\mathbb{B})$. Consequently, following the logic of part (iii) of Definition 1.2.20, we define the weighted Laplace-Beltrami as $\Delta_{\mathbf{b}} := \mathrm{div}_{\mathbf{b}}\nabla_{\mathbf{b}} = {}^1\!/\!_{\rho_{\mathbb{B}}} \cdot \mathrm{div}_{\mathsf{b}}\big(\rho_{\mathbb{B}}\nabla_{\mathsf{b}}\big)$. The definitions are chosen such that the Divergence Theorem and its consequences hold: We have that

$$\int_{\mathbb{B}} f \cdot \mathrm{div}_{\mathbf{b}}\mathcal{X}\ \mathrm{d}\lambda_{\mathbf{b}} = -\int_{\mathbb{B}} \mathbf{b}\big(\nabla_{\mathbf{b}} f\,, \mathcal{X}\big)\ \mathrm{d}\lambda_{\mathbf{b}}$$

and

$$\int_{\mathbb{B}} f \cdot \Delta_{\mathbf{b}} g\ \mathrm{d}\lambda_{\mathbf{b}} = -\int_{\mathbb{B}} \mathsf{b}\big(\nabla_{\mathsf{b}} f\,, \nabla_{\mathsf{b}} g\big)\ \mathrm{d}\lambda_{\mathbf{b}} = \int_{\mathbb{B}} f \cdot \Delta_{\mathbf{b}} g\ \mathrm{d}\lambda_{\mathbf{b}} \tag{1.5}$$

hold for all $f \in C^\infty_{\mathsf{c}}(\mathbb{B})$ and $g \in C^\infty(\mathbb{B})$. For sake of readability, we write $L^p(\mathbb{B};\mathbf{b})$ instead of $L^p(\mathbb{B};\lambda_{\mathbf{b}})$.

Suppose a fibre bundle $\mathbb{E} \to \mathbb{B}$ over $\mathbb{B}$ with some bundle measure $\mu_{\mathbb{B}} \otimes_{\mathsf{loc}} \nu_{\mathbb{F}}$. If $\mu_{\mathbb{B}}$ is weighted by $\rho_{\mathbb{B}}$ and $\nu_{\mathbb{F}}$ is weighted by a function $\rho_{\mathbb{F}}$ that is invariant with respect to transformations of the structure group, we denote the Radon-Nikodým derivative of the weighted loc-product measure by $\rho_{\mathbb{B}} \otimes_{\mathsf{loc}} \rho_{\mathbb{F}}$. ⌟

## 1.2.4 ▪ Ehresmann connections and semisprays

A definition of Ehresmann connections for general fibre bundles can be found in [KMS93, Section III.9]. Here, we restrict ourselves to the case of vector bundles.

**Definition 1.2.24** (vertical bundle). Consider a fibre bundle $\pi\colon\ \mathbb{E} \longrightarrow \mathbb{B}$. The null space of the differential $d\pi\colon\ \mathsf{T}\mathbb{E} \to \mathsf{T}\mathbb{B}$. of the bundle projection is called the *space of vertical vectors* and denoted by $\mathsf{V}\mathbb{E} := \mathsf{null}(d\pi)$. We think vertical vectors as being tangent to the fibres of $\pi$. Then, $\pi\colon\ \mathsf{V}\mathbb{E} \longrightarrow \mathbb{E}$ is referred to as the *vertical bundle*. Indeed: As $d_v\pi$ is surjective for all $v \in \mathbb{E}$, the vertical bundle is a smooth subbundle. The fibrewise projection to the space of vertical vectors is denoted by vpr. If $\pi\colon\ \mathbb{E} \longrightarrow \mathbb{B}$ happens to be a vector bundle, than we additionally can define the *vertical lift at* $v \in \mathbb{E}$ fixed as the mapping $\mathsf{vl}_v\colon\ \mathbb{E}_{\pi(v)} \to \mathsf{T}_v\mathbb{E}$ that acts on test functions $f \in C^\infty(\mathbb{E}_{\pi(v)})$ as

$$\langle \mathsf{vl}_v(w)\,, d_v f\rangle = \frac{d}{dt} f(v + tw)\,\big|_{t=0}.$$

If $\mathbb{E} = \mathsf{T}\mathbb{B}$, then $\langle \mathsf{vl}_v(w)\,, d_v df\rangle = \langle w\,, df\rangle$ for all $f \in C^\infty(\mathbb{B})$ determines the lift uniquely. The smooth section $\mathcal{V} \in \Gamma^\infty(\mathbb{E};\mathsf{V}\mathbb{E})$ given as $v \mapsto \mathcal{V}(v) := \mathsf{vl}_v(v)$ is called *canonical vector field*. ⌟

**Definition 1.2.25** (Ehresmann connections in vector bundles). Consider a vector bundle $\pi\colon\ \mathbb{E} \longrightarrow \mathbb{B}$. A (smooth) subbundle $\pi\colon\ \mathsf{H}\mathbb{E} \to \mathbb{B}$ is called *Ehresmann connection* or *horizontal bundle* if the total space decomposes into the *Whitney sum* $\mathsf{T}\mathbb{E} = \mathsf{H}\mathbb{E} \oplus \mathsf{V}\mathbb{E}$ meaning that

$$\mathsf{T}_v\mathbb{E} = \mathsf{H}_v\mathbb{E} \oplus \mathsf{V}_v\mathbb{E} \qquad \text{for all } v \in \mathbb{E}.$$

In this case, we denote the projection of vectors in $\mathbb{E}$ to their horizontal parts by hpr. Furthermore, the *horizontal lift at* $v$ of some vector $w \in \mathbb{E}$ is the unique vector $\mathsf{hl}_v(w) \in \mathsf{H}_v\mathbb{E}$ such that $w = \langle \mathsf{hl}_v(w)\,, d\pi\rangle$. ⌟

In general there is no canonical choice of a Ehresmann connection, rather such a choice adds additional geometric information. We discuss this phenomenon after the following definition a bit more. Since for this thesis our main concern are Riemannian manifolds, we specifically look for an Ehresmann connection associated to the Riemannian structure.

[14] As mentioned earlier, local-Lipschitz regularity would suffice, since then is locally integrable and possesses a weak gradient.

**Definition 1.2.26** (connector map and Riemannian horizontal bundle). Consider a Riemannian manifold $(\mathbb{B}, \mathsf{b})$ and a generic point $o \in \mathbb{B}$. Let $U \subseteq \mathbb{B}$ be a neighbourhood of $o$ and $V := \pi_0^{-1}(U) \subseteq \mathsf{T}\mathbb{B}$ such that the exponential[15] $\exp_o\colon \mathsf{T}_o\mathbb{B} \to \mathbb{B}$ maps a 0-neighbourhood to $U$ diffeomorphically. Denote by $\tau\colon V \to \mathsf{T}_o\mathbb{B}$ parallel transport of $v \in V$ along the unique geodesic arc connecting $\pi(v)$ and $o$. Let $r_{-u}\colon \mathsf{T}_o\mathbb{B} \to \mathsf{T}_o\mathbb{B}$ be the translation $w \mapsto w - u$ by the vector $u \in \mathsf{T}_o\mathbb{B}$. Now, consider the mapping

$$\kappa\colon\; V \longrightarrow \mathbb{B},\; v \longmapsto (\exp_o \circ r_{-u} \circ \tau)(v).$$

The dependency on the chart vanishes when passing to the differential

$$d\kappa_u\colon\; \mathsf{T}^2_u\mathbb{B} \longrightarrow \mathsf{T}_{\pi_0(u)}\mathbb{B},\; a \longmapsto \Big\langle a\,, d\Big(\exp_{\pi_0(u)} \circ r_{-u} \circ \tau\Big)\Big\rangle$$

which is called the *connector map*, compare to [Dom62, Section 2]. Via the assignment $\mathsf{HT}\mathbb{B} := \mathsf{null}(d\kappa)$ we gain an Ehresmann connection which we call *Riemannian* since it depends by means of the exponential map on the Riemannian metric, see e. g [Dom62, Appendix (ii)]. ⌟

Some authors view the mappings $d\pi_0$ and $d\kappa$ as horizontal and vertical projections respectively, but this could lead to confusion as they don't map fibrewise into the same vector space. One observes that the dependency of the connector map on the Riemannian metric in Definition 1.2.26 essentially lies in the notion of a geodesic on the base manifold.

The geometric information of an Ehresmann connection is closely linked to the concept of semisprays, which we introduce next. Semisprays naturally arise in the study of second order (ordinary) differential equations on manifolds or more generally Lagrange spaces. One motivation comes from Newton's (second) law see [Gli97, Section 4] or from Euler-Lagrange equations [MR99, Chapter 7]. The semispray considered in these classical situations appears as Lagrangian vector field and is again referred to as geodesic spray due to the fact that induces the geodesic flow. In the neat book by M. do Carmo one finds in [Car92, Lemma III.2.3] a construction of that semispray just in terms of geodesics corresponding to the Riemannian metric[16] which also justifies the name geodesic spray. As explained in [Buc06], starting from a Lagrange space with regular Lagrangian one obtains from the Euler-Lagrange equations a system of second order differential equations that yield local coefficients for a semispray. There is a unique associated Ehresmann connection that is compatible with both the metric tensor[17] and the symplectic structure on the tangent space – just compatibility with respect to the metric tensor would not suffice. Thence, a possibility for future studies would be not to choose a Riemannian metric, but rather consider Lagrange spaces.

**Definition 1.2.27** (semisprays). A section $\mathcal{H} \in \Gamma^\infty(\mathsf{T}\mathbb{B}; \mathsf{T}^2\mathbb{B})$ is a called a *semispray* or *second order vector field* if it satisfies $\langle \mathcal{H}\,, d\pi_0\rangle = \mathsf{Id}_{\mathsf{T}\mathbb{B}}$ or equivalently if any integral curve[18] $s\colon I \to \mathsf{T}\mathbb{B}$ takes the form $s = (\pi_0 \circ s)'$. A curve $c\colon I \to \mathbb{B}$ is called *geodesic of the semispray* $\mathcal{H}$ if there is an integral curve $s\colon I \to \mathsf{T}\mathbb{B}$ such that $c = \pi_0 \circ s$. Equivalently, $c$ is geodesic if $\mathcal{H} \circ \dot{c} = \ddot{c}$. A semispray is called *(full) spray* if it additionally fulfils $[\mathcal{V}\,, \mathcal{H}] = \mathcal{H}$. ⌟

For a description of semisprays in local coordinates we refer to Remark A.1.2. Note that, as every semispray comes with its own set of geodesics, it introduces a whole 'geodesic structure'. If one is starting from a semispray rather than having it already as generator of the geodesic flow, a natural assumption would be that the associated geodesic structure induces a nice metric space structure via minimising geodesics.

Assume that an Ehresmann connection $\mathsf{T}^2\mathbb{B} = \mathsf{H}\mathbb{B} \oplus \mathsf{V}\mathbb{B}$ is given. We already know that for a given vector $v \in \mathsf{T}\mathbb{B}$ there is a unique horizontal vector $a \in \mathsf{T}^2_v\mathbb{B}$ such that $v = \langle a\,, d\pi_0\rangle$, namely the horizontal lift of $a = \mathsf{hl}_v(v)$. Furthermore, $a$ depends on $v$ smoothly. Hence, the assignment $\mathcal{H}(v) := \mathsf{hl}_v(v)$ yields a semispray *associated to the Ehresmann connection*. If the Ehresmann connection is the one corresponding to a Riemannian metric $\mathsf{b}$ and its Levi-Civita connection, then we denote the associated semispray by $\mathcal{H}_\mathsf{b}$ and call it *Riemannian semispray*.

*Remark* 1.2.28 ((dynamical) covariant derivatives and Levi-Civita connection). In general, an Ehresmann connection and a semispray give rise to a so-called *dynamical covariant derivative*. see [Buc06, Section 2] for details. Here, we stick to the classical case of covariant derivatives in Riemannian geometry as one uses it e. g. to define geodesics.

[15] Recall [Nic96, § 4.1.3].
[16] Via the Levi-Civita connection, see Remark 1.2.28 later.
[17] Rather its induced dynamical covariant derivative is compatible with the metric, compare to Remark 1.2.28.
[18] An integral curve of a given smooth vector field $\mathcal{X} \in \Gamma^\infty(\mathsf{T}\mathbb{Y})$ is a smooth curve $\gamma\colon I \to \mathbb{Y}$ with $\dot{\gamma}(t) = \mathcal{X}(\gamma(t))$ for all times $t \in I$. See [Lee13, Chapter 17].

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By a *covariant derivative* or *Koszul connection* we mean a mapping

$$\nabla\colon\ \mathrm{T}\mathbb{B}\times\Gamma^\infty(\mathrm{T}\mathbb{B})\to\mathrm{T}\mathbb{B},\ (w,\mathcal{X})\mapsto\nabla_w\mathcal{X}$$

that is fibrewise linear in the first argument, additive in the second argument, satisfies the Leibniz rule $\nabla_{\mathcal{W}}(f\mathcal{X})=\mathcal{W}f\cdot\mathcal{X}+f\cdot\nabla_{\mathcal{W}}\mathcal{X}$ for all $\mathcal{X},\mathcal{W}\in\Gamma^\infty(\mathrm{T}\mathbb{B})$ and $f\in C^\infty(\mathbb{B})$, and yields smooth vector fields $b\mapsto\nabla_{\mathcal{W}(b)}\mathcal{X}(b)$ for any $\mathcal{X},\mathcal{W}\in\Gamma^\infty(\mathrm{T}\mathbb{B})$. A *geodesic with respect to* $\nabla$ is a smooth curve $\gamma$ such that $\nabla_{\dot\gamma}\dot\gamma=0$ holds.

The Fundamental Theorem of Riemann Geometry states that there is a unique Koszul connection $\nabla^{\mathsf{b}}$ that is both torsionfree and compatible with the given Riemannian metric $\mathsf{b}$. That unique connection is called *Levi-Civita connection*. 'Torsionfree' means that[19] $[\mathcal{X}\,,\mathcal{Y}]=\nabla^{\mathsf{b}}_{\mathcal{X}}\mathcal{Y}-\nabla^{\mathsf{b}}_{\mathcal{Y}}\mathcal{X}$ holds for all $\mathcal{X},\mathcal{Y}\in\Gamma^\infty(\mathrm{T}\mathbb{B})$. The term 'compatible with the Riemannian metric' means that

$$\mathcal{Z}(\mathsf{b}(\mathcal{X}\,,\mathcal{Y}))=\mathsf{b}\big(\nabla^{\mathsf{b}}_{\mathcal{Z}}\mathcal{X}\,,\mathcal{Y}\big)+\mathsf{b}\big(\mathcal{X}\,,\nabla^{\mathsf{b}}_{\mathcal{Z}}\mathcal{Y}\big)\qquad\text{for all }\mathcal{X},\mathcal{Y},\mathcal{Z}\in\Gamma^\infty(\mathrm{T}\mathbb{B}).$$

Compare to [Nic96, Section 3.3] and [Nic96, § 4.1.2]. ⌟

Later on in Section 2.1, we make use of a few other related facts, specifically the so-called Koszul formula and the constructive definition of covariant derivatives from parallel transport. The Koszul formula appears in the proof of existence of the Levi-Civita connection, see [Nic96, Proposition 4.1.8]. The fact that parallel transport determines a covariant derivative is well-known, an explicit proof can be found e. g. in [HT94, Bemerkung 7.85].

As a key assertion for our purposes, we need that the Riemannian semispray is an antisymmetric differential operator. The precise statement is the version of Liouville's Theorem which we present below in Theorem 1.2.30 – it is a well-known theorem, but comes in quite different flavours. Before we can give Theorem 1.2.30, we have to introduce the so-called Sasaki metric first, originally defined in [Sas58].

**Definition 1.2.29** (Sasakian metric). Let $(\mathbb{B},\mathsf{b})$ be a Riemannian manifold and some given Ehresmann connection of the tangent bundle. The *Sasaki metric* $\mathsf{t}$ is a Riemannian metric on $\mathrm{T}\mathbb{B}$ uniquely characterised as respecting inner products for either both vertically or both horizontally lifted vector fields, and turning the Ehresmann connection into an fibrewise orthogonal Whitney sum. In more detail, the Sasaki metric $\mathsf{t}$ on the one hand is *natural* in the sense of [GK02] meaning that

$$\mathsf{t}(\mathsf{hl}(\mathcal{X})\,,\mathsf{hl}(\mathcal{Y}))=\mathsf{b}(\mathcal{X}\,,\mathcal{Y})\circ\pi_0\qquad\text{and}\qquad\mathsf{t}(\mathsf{vl}(\mathcal{X})\,,\mathsf{hl}(\mathcal{Y}))=0$$

for all $\mathcal{X},\mathcal{Y}\in\Gamma^\infty(\mathrm{T}\mathbb{B})$. On the other hand, the metric $\mathsf{t}$ additionally satisfies

$$\mathsf{t}(\mathsf{vl}(\mathcal{X})\,,\mathsf{vl}(\mathcal{Y}))=\mathsf{b}(\mathcal{X}\,,\mathcal{Y})\circ\pi_0$$

for all $\mathcal{X},\mathcal{Y}\in\Gamma^\infty(\mathrm{T}\mathbb{B})$. Therefore, one could say that the Sasaki metric respects the Ehresmann connection. The metric $\mathsf{t}$ is explicitly given as the sum of pullbacks of the metric tensor of the base manifold:

$$\begin{aligned}\mathsf{t}(\bar v\,,\bar w)&:=\kappa^*\mathsf{b}(\bar v\,,\bar w)+\pi_0^*\mathsf{b}(\bar v\,,\bar w)\\&=\mathsf{b}\big(\,\langle\bar v\,,d\kappa\rangle\,,\,\langle\bar w\,,d\kappa\rangle\big)+\mathsf{b}\big(\,\langle\bar v\,,d\pi_0\rangle\,,\,\langle\bar w\,,d\pi_0\rangle\big)\end{aligned}$$

for all $\bar v,\bar w\in\mathrm{T}^2\mathbb{B}$. We call the component $\mathsf{h}:=\pi_0^*\mathsf{b}$ the *horizontal (Sasaki) metric*. Mutatis mutandis, the *vertical (Sasaki) metric* $\mathsf{v}$ is defined as $\mathsf{v}:=\kappa^*\mathsf{b}$. More condensed, we write $\mathsf{t}=\mathsf{v}+\mathsf{h}$. Compare to [GK02, Definition 7.1].

Similarly, one gets a Riemannian metric $\mathsf{u}$ on the unit tangent bundle $\mathrm{U}\mathbb{B}$ by requiring that

$$\begin{aligned}&\mathsf{u}(\mathsf{hl}\,\mathcal{X}\,,\mathsf{hl}\,\mathcal{Y})=\mathsf{t}(\mathsf{hl}\,\mathcal{X}\,,\mathsf{hl}\,\mathcal{Y}),\qquad\mathsf{u}(\mathsf{tl}\,\mathcal{X}\,,\mathsf{hl}\,\mathcal{Y})=0\qquad\text{and}\\&\mathsf{u}(\mathsf{tl}\,\mathcal{X}\,,\mathsf{tl}\,\mathcal{Y})=\mathsf{t}\big(\mathcal{X}\circ\pi_{1,0}-\mathsf{b}\big(\mathcal{X}\circ\pi_{1,0}\,,\mathsf{Id}\big)\mathsf{Id}\,,\mathcal{Y}\circ\pi_{1,0}-\mathsf{b}\big(\mathcal{Y}\circ\pi_{1,0}\,,\mathsf{Id}\big)\mathsf{Id}\big)\end{aligned}$$

for all $\mathcal{X},\mathcal{Y}\in\Gamma^\infty(\mathrm{T}\mathbb{B})$. Here 'tl' denotes the adaptation of the vertical lift that ensures that the result lives in the unit tangent bundle – we go into this later on in Definition 2.2.1, but also refer here to [BV01, p. 525]. We call the metric $\mathsf{u}$ *unit Sasaki metric*. This Riemannian metric splits into two parts as $\mathsf{u}=\mathsf{v}|_{\mathrm{U}}+\mathsf{h}$ just as the Sasaki metric does. ⌟

[19] Of course, $[\mathcal{X}\,,\mathcal{Y}]:=\mathcal{X}\mathcal{Y}-\mathcal{Y}\mathcal{X}$ is the usual Lie bracket, see [Lee13, Section 4.3].

Now, we can formulate Liouville's Theorem. Afterwards, we turn to differential operators associated to the Sasaki metric.

**Theorem 1.2.30** (Liouville's Theorem). The Riemannian semispray $\mathcal{H}_{\mathsf{b}}$ is solenoidal with respect to Sasaki metric $\mathsf{t}$.

One proof could be done via explicitly calculating the Sasakian volume form in so-called normal coordinates, compare to [Car92, Exercise 3.14] and [Nic96, § 4.1.3] for the notion of normal coordinates. Another way would be to observe that the unit tangent bundle is invariant under the geodesic flow, see [Nic96, p. 122].

As every Riemannian metric does, the Sasaki metric naturally induces gradient, divergence and Laplace-Beltrami operator, but furthermore each one of those splits along $\mathsf{t} = \mathsf{v} + \mathsf{h}$. into a vertical and a horizontal part: $\nabla_{\mathsf{t}} = \nabla_{\mathsf{v}} + \nabla_{\mathsf{h}}$, $\operatorname{div}_{\mathsf{t}} = \operatorname{div}_{\mathsf{v}} + \operatorname{div}_{\mathsf{h}}$, and $\Delta_{\mathsf{t}} = \Delta_{\mathsf{v}} + \Delta_{\mathsf{h}}$. Similar notation is used for the unit Sasaki metric by analogy. We want to identify two classes of functions on which vertical and horizontal gradient or Laplace-Beltrami operator act characteristically.

*Remark* 1.2.31 (vertical and horizontal lift of functions). Let $f_0$ be a real-valued function with domain in $\mathbb{B}$. We call the pullback of $f_0$ with respect to the tangent bundle projection $\pi_0$ the *vertical lift of* $f_0$: We write $f_0^{\mathsf{v}} = \pi_0^* f_0 := f_0 \circ \pi_0$, whenever it's defined. A horizontal lift of $f_0$ as direct analogy to the vertical lift might be expected to be the pullback with respect $\kappa$ declaring $f_0^{\mathsf{h}} = \kappa^* f_0 := f_0 \circ \kappa$, whenever it's defined. However, this is not straight forward since $\kappa$ depends on the choice of a certain neighbourhood $U$ of some base point $o$ and a translation vector $u$, compare to Definition 1.2.26. Splitting a double tangent vector in vertical and horizontal part and projecting these components into the tangent bundle by means of $d\kappa$ and $d\pi_0$ we find that $df_0(d\kappa + d\pi_0) = df_0 \circ d\kappa + df_0^{\mathsf{v}}$ independent of the framework used to define $\kappa$. We define the *horizontal lift* $f_0^{\mathsf{h}}$ of sufficiently regular $f_0$ algebraically as solution of the problem $\langle a\,, df_0^{\mathsf{h}}\rangle = \langle a\,, df_0 \circ d\kappa\rangle$ for $\lambda_{\mathsf{t}}$-almost all $a \in \mathsf{T}^2\mathbb{B}$ up to constant offsets. An analytic intuition would be $f_0^{\mathsf{h}}(v) = \int_{\mathsf{T}^2_v\mathbb{B}} df_0 \circ d\kappa \,\mathrm{d}\lambda$, where $\lambda$ is the Lebesgue measure on the standard fibre $\mathbb{R}^{2d}$ of the double tangent bundle. The simplest instance consists of $f_0$ being smooth and having compact support in the neighbourhood $U$ as in Definition 1.2.26 as then $f_0^{\mathsf{h}} = f_0 \circ \kappa$ as originally expected. If $f_0 \in C_{\mathsf{c}}^\infty(\mathbb{B})$, then we cover the support by open sets as in Definition 1.2.26, localise by means of a partition of unity subordinate to this covering, lift the elements of the partition of unity horizontically, and glue the lifted peaces together.

**Example 1.2.32.** The Riemannian semispray $\mathcal{H}_{\mathsf{b}}$ acts on vertically lifted functions $f = f_0^{\mathsf{v}}$ with $f_0 \in C^\infty(\mathbb{B})$ as

$$\mathcal{H}_{\mathsf{b}} f = \mathsf{h}\big(\mathcal{H}_{\mathsf{b}}\,, \nabla_{\mathsf{h}} f_0^{\mathsf{v}}\big) = \mathsf{b}_{\pi_0}\big(\mathsf{Id}_{\mathsf{T}\mathbb{B}}\,, \nabla_{\mathsf{b}} f_0 \circ \pi_0\big).$$

In case of $\mathbb{B} = \mathbb{R}^d$ with standard Riemannian metric this action reads as $\mathcal{H}_{\mathsf{euc}} f(x,v) = (v\,, \nabla_x f_0(x))_{\mathsf{euc}}$ for the smooth function $f\colon\, \mathbb{R}^d_x \times \mathbb{R}^d_v \to \mathbb{R}$, $(\bar{x},\bar{v}) \mapsto f_0(\bar{x})$ and all $x, v \in \mathbb{R}^d$.

By analogy, the canonical vector field $\mathcal{V}$ acts on horizontally lifted functions $f = f_0^{\mathsf{h}}$ as

$$\mathcal{V} f = \mathsf{v}\big(\mathcal{V}\,, \nabla_{\mathsf{v}} f_0^{\mathsf{h}}\big) = \mathsf{b}_{\pi_0}\big(\mathsf{Id}_{\mathsf{T}\mathbb{B}}\,, \nabla_{\mathsf{b}} f_0 \circ \pi_0\big).$$

In the Euclidean case, this action reads as $\mathcal{V} f(x,v) = (v\,, \nabla_v f_0(v))_{\mathsf{euc}}$ with the horizontal lift being of the form $f\colon\, \mathbb{R}^d_x \times \mathbb{R}^d_v \to \mathbb{R}$, $(\bar{x},\bar{v}) \mapsto f_0(\bar{v})$ and $x, v \in \mathbb{R}^d$.

*Remark* 1.2.33. Clearly, the vertical lift of a function $f_0 \in C_{\mathsf{c}}^\infty(\mathbb{B})$ is smooth but not compactly supported on $\mathsf{T}\mathbb{B}$, as the standard fibre $\mathbb{F} = \mathbb{R}^{d}$ is not compact. But for $\mathbb{B}$ satisfying the assumptions (M), that we impose for the position manifold starting from Chapter 2, we can approximate $f_0^{\mathsf{v}}$ nicely. We need a cut-off function $\varphi \in C_{\mathsf{c}}^\infty(\mathbb{B}; [0,1])$ with $\varphi = 1$ on $\mathbb{U}(o;1)$ and $\varphi = 0$ outside of $\mathbb{U}(o;2)$ for some base point $o \in \mathbb{B}$. Given $\varphi$, we define $\varphi_n := \varphi \circ \gamma_{\mathsf{Id}}(1/n)$ for all $n \in \mathbb{N}_+$, where $\gamma_b\colon\, [0,1] \to \mathbb{B}$ denotes the minimising geodesic with $\gamma_b(0) = o$ and $\gamma_b(1) = b$ for $b \in \mathbb{B}$. Then, $(\varphi_n)_{n\in\mathbb{N}_+}$ converges pointwise to the constant function 1. We define $f_n = f_0^{\mathsf{v}} \otimes \varphi_n^{\mathsf{h}} := f_0^{\mathsf{v}} \cdot \varphi_n^{\mathsf{h}}$ for $n \in \mathbb{N}_+$ and use $(f_n)_{n\in\mathbb{N}_+}$ as approximating sequence for $f_0^{\mathsf{v}}$ later on. An explicit choice of $\varphi$ is given as follows: Let $h$ be a chart at $o$. Without loss of generality the ball $\mathbb{U}(o;2)$ is contained in the chart domain. Define the auxiliary 'mountain' function

$$m\colon\;\; \mathbb{R} \to \mathbb{R},\;\; t \longmapsto \exp\left(-\frac{1}{t(1-t)}\right) \cdot \mathbb{1}_{(0,1)}(t).$$

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We transform the 'mountain' $m$ to a 'table mountain' $\tau$ by the assignment

$$\tau\colon\ \mathbb{R}^d \to [0,1],\ y \longmapsto \frac{\int_0^{2-\|y\|_2} m(t)\ \mathrm{d}t}{\int_0^1 m(t)\ \mathrm{d}t}.$$

Then, the choice $\varphi := \tau \circ h \cdot \mathbb{1}_{\mathbb{U}(o;2)}$ yields a smooth function with the desired properties. ⌟

**Lemma 1.2.34** (Tensor product of lifted test functions)**.** Define the space

$$D_0 := \pi_0^* C_c^\infty(\mathbb{B}) \otimes \kappa^* C_c^\infty(\mathbb{B}) := \mathrm{span}\{f_0^\mathsf{v} \otimes g_0^\mathsf{h} := f_0^\mathsf{v} \cdot g_0^\mathsf{h} \mid f_0, g_0 \in C_c^\infty(\mathbb{B})\}.$$

Then, $D_0$ is dense in $C_c^\infty(\mathrm{T}\mathbb{B})$. By this we mean dense with respect to the usual locally convex topology induced by appropriate seminorms, which implies uniform convergence of all derivatives on compacts; for the well-known Euclidean case see [Hor66, Example 2.4.10] and for the generalised geometric case see [Gro+01, Section 3.1.3].

*Proof.* Consider a local trivialisation with domain $U$ and trivialisation chart $\varphi\colon V \to U \times \mathbb{R}^d$. Then, we immediately conclude the relations

$$\pi_0^* C_c^\infty(U) = \varphi^*(C_c^\infty(U) \otimes \{1\}) \qquad \text{and} \qquad \kappa^* C_c^\infty(U) = \varphi^*(\{1\} \otimes C_c^\infty(\mathbb{R}^d)).$$

Thus, by [Hor66, Proposition 4.8.1] the tensor product $\pi_0^* C_c^\infty(U) \otimes \kappa^* C_c^\infty(U)$ is dense in $C_c^\infty(V) = \varphi^* C_c^\infty(U \times \mathbb{R}^d)$ and the proof is finished via a partition of unity argument. □

For the following characterisation lemma I don't know a reference, hence a short proof is given.

**Lemma 1.2.35** (differential operators induced by the Sasaki metric)**.** The gradient, divergence and Laplace-Beltrami operators corresponding to the vertical Sasaki metric fulfil

$$\nabla_\mathsf{v} f_0^\mathsf{h} = \mathsf{vl}\big(\nabla_\mathsf{b} f_0 \circ \pi_0\big),\ \ \mathsf{div}_\mathsf{v}(\mathsf{vl}\,\mathcal{X}) = \big(\mathsf{div}_\mathsf{b}\mathcal{X}\big) \circ \pi_0 \ \text{ and }\ \Delta_\mathsf{v} f_0^\mathsf{h} = \big(\Delta_\mathsf{b} f_0\big) \circ \pi_0$$

for all $f_0 \in C^\infty(\mathbb{B})$ and $\mathcal{X} \in \Gamma^\infty(\mathrm{T}\mathbb{B})$. Similarly, for the case of the horizontal Sasaki metric we have that

$$\nabla_\mathsf{h} f_0^\mathsf{v} = \mathsf{hl}\big(\nabla_\mathsf{b} f_0 \circ \pi_0\big),\ \ \mathsf{div}_\mathsf{h}(\mathsf{hl}\,\mathcal{X}) = \big(\mathsf{div}_\mathsf{b}\mathcal{X}\big) \circ \pi_0 \ \text{ and }\ \Delta_\mathsf{h} f_0^\mathsf{v} = \big(\Delta_\mathsf{b} f_0\big) \circ \pi_0$$

for all $f_0 \in C^\infty(\mathbb{B})$ and $\mathcal{X} \in \Gamma^\infty(\mathrm{T}\mathbb{B})$.

*Proof.* We restrict ourselves to the vertical case, since the other statements follow analogously. Let $f_0 \in C^\infty(\mathbb{B})$ and $\mathcal{X} \in \Gamma^\infty(\mathrm{T}\mathbb{B})$ be fixed. Then, we know that the following two equations hold simultaneously which characterises the vertical gradient:

$$\begin{aligned} &\mathsf{v}\big(\mathsf{vl}\,\mathcal{X}\,,\nabla_\mathsf{v} f_0^\mathsf{h}\big) = (\mathcal{X} f_0) \circ \pi_0 \\ \text{and}\quad &\mathsf{v}\Big(\mathsf{vl}\,\mathcal{X}\,,\mathsf{vl}\big(\nabla_\mathsf{b} f_0\big)\Big) = \mathsf{b}\big(\mathcal{X}\,,\nabla_\mathsf{b} f_0\big) = (\mathcal{X} f_0) \circ \pi_0. \end{aligned}$$

Recall the defining equations of divergence for the Riemannian metrics b and v from Definition 1.2.20:

$$\mathscr{L}_\mathcal{X}\,\lambda_\mathsf{b} = \mathsf{div}_\mathsf{b}(\mathcal{X})\cdot\lambda_\mathsf{b} \quad \text{and} \quad \mathscr{L}_{\mathsf{vl}\,\mathcal{X}}\,\lambda_\mathsf{v} = \mathsf{div}_\mathsf{v}(\mathsf{vl}\,\mathcal{X})\cdot\lambda_\mathsf{v}.$$

Cartan's magical formula, see [Lee13, Proposition 18.13], applied to $\lambda_\mathsf{b}$ as volume form yields the equation[20]

$$\begin{aligned} \mathscr{L}_{\mathsf{vl}\,\mathcal{X}}\,\lambda_\mathsf{v} &= d\big(\mathsf{vl}(\mathcal{X}) \lrcorner\, \kappa^*\lambda_\mathsf{b}\big) + \mathsf{vl}(\mathcal{X}) \lrcorner\, d\big(\kappa^*\lambda_\mathsf{b}\big) = \big(d\mathcal{X} \lrcorner\, \lambda_\mathsf{b}\big) \circ \pi_0 + \mathsf{vl}(\mathcal{X}) \lrcorner\, d\kappa^*\lambda_\mathsf{b} \\ &= \big(d\mathcal{X} \lrcorner\, \lambda_\mathsf{b}\big) \circ \pi_0 + \big(\mathcal{X} \lrcorner\, d\lambda_\mathsf{b}\big) \circ \pi_0 = \big(\mathscr{L}_\mathcal{X}\,\lambda_\mathsf{b}\big) \circ \pi_0. \end{aligned}$$

Thus, the statement follows by reduction of the nonorientable to the orientable case as mentioned in Definition 1.2.20 before. By combining the results for gradient and divergence we get the equation for the Laplace-Beltrami operator. □

[20] Here, the symbol '$\lrcorner$' refers to the partial evaluation map that usually goes by the confusing name *interior product*: If $\omega$ is an alternating $k$-multilinear map, $k \in \mathbb{N}_+$, on a finite-dimensional vector space $V$ and $v \in V$ is a vector, then the partial evaluation at $v$ is defined as $v \lrcorner\, \omega := \omega(v,\cdot)$. Since $v \lrcorner\, \omega$ is an alternating $(k-1)$-multilinear map, J. M. Lee also calls the mapping $\lrcorner$ 'contraction'. The partial evaluation of a constant mapping $\omega$ at $v$ is zero by definition. Now, the entire construction extends to differential forms which are partially evaluated at a given vector field.

Let us finish this subsection with an almost unduly short note on curvatures. For the following definitions we refer to [Nic96, Definitions 4.2.1 and 4.2.4].

**Definition 1.2.36** (Riemannian and Ricci curvature). The *Riemannian curvature* is the curvature of the Levi-Civita connection. That is the tensor $R_{\mathsf{b}}$ uniquely determined by

$$R_{\mathsf{b}}(\mathcal{X},\mathcal{Y})\mathcal{Z} = \left[\nabla^{\mathsf{b}}_{\mathcal{X}}\,,\nabla^{\mathsf{b}}_{\mathcal{Y}}\right]\mathcal{Z} - \nabla^{\mathsf{b}}_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z} = \nabla^{\mathsf{b}}_{\mathcal{X}}\nabla^{\mathsf{b}}_{\mathcal{Y}}\mathcal{Z} - \nabla^{\mathsf{b}}_{\mathcal{Y}}\nabla^{\mathsf{b}}_{\mathcal{X}}\mathcal{Z} - \nabla^{\mathsf{b}}_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z}$$

for all $\mathcal{X},\mathcal{Y},\mathcal{Z} \in \Gamma^\infty(\mathrm{T}\mathbb{B})$. Let $\mathcal{X},\mathcal{Y} \in \Gamma^\infty(\mathrm{T}\mathbb{B})$, then the mapping $v \mapsto R_{\mathsf{b}}(v,\mathcal{X})\mathcal{Y}$ is an endomorphism of $\mathrm{T}\mathbb{B}$. Taking the trace of this endomorphism as

$$\mathsf{Ric}_{\mathsf{b}}(\mathcal{X},\mathcal{Y}) := \mathsf{tr}_{\mathsf{b}}(v \mapsto R_{\mathsf{b}}(v,\mathcal{X})\mathcal{Y})$$

defines a symmetric (0,2)-tensor field $\mathsf{Ric}_{\mathsf{b}}$. This tensor field is called *Ricci curvature*. ⌟

*Remark* 1.2.37 (significance of the Ricci curvature). It is well-known that Poincaré inequalities as well as isoperimetric or Harnack inequalities and many more analytic estimates often are linked to a lower bound $K \in \mathbb{R}$ on the Ricci curvature. In formulae, lower boundedness of the Ricci curvature means that $\mathsf{Ric}_{\mathsf{b}} \geq (\mathscr{b}-1)\,K \cdot \mathsf{b}$. Just to give one reference here, we refer to [Sal02, Section 5.6.3] which is based on the works by S. T. Yau. In Section 4.1 we briefly touch upon the generalisations by K. T. Sturm as well as J. Lott and C. Villani for metric measure spaces. We point out there that lower Ricci curvature bounds are stable under a Gromov-Hausdorff-type convergence of certain metric measure spaces. ⌟

## 1.3 ▪ SDEs on manifolds

We assume that the reader is to a comfortable extend familiar with stochastic integration and stochastic differential equations (SDE) on $\mathbb{R}^d$. It is largely considered as most convenient to consider Stratonovich SDEs on manifolds, since Stratonovich integration leads under transformations to a chain rule compared to Itô integration, which yields the famous Itô formula. See the discussions e. g. in [Elw82, Section VII.1] and [IW81, Section V]. Some people disagree like Y. Gliklikh in [Gli97, Section 4.13]: 'In the theory of SDEs on smooth manifolds, as well as in the Euclidean space, the Itô equations are more fundamental objects than the Stratonovich equations.' He advocates to use Itô bundles as introduced by Y. I. Belopol'skaya and Y. L. Daletskij in [BD82]. We think the majority opinion is justified by the fundamental significance of the chain rule in differential geometry and choose to only formulate Stratonovich SDEs in this thesis. Apart from the decision whether Itô or Stratonovich, different branches in the study of SDEs on manifolds can be recognised. They are very briefly sketched here as the choice among all those options matters at the modelling stage.

We start with the formulation that is the easiest one to grasp: Consider some smooth vector fields $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_k \in \Gamma^\infty(\mathrm{T}\mathbb{Y})$ on the smooth manifold $\mathbb{Y}$. If $\mathbb{Y}$ is compact, then $\hat{\mathbb{Y}} := \mathbb{Y}$, otherwise $\hat{\mathbb{Y}}$ denotes the one-point compactification of $\mathbb{Y}$. Furthermore, let $W = (W_t)_{t\in[0,\infty)} = \left((W^\ell_t)^k_{\ell=1}\right)_t$ be a $k$-dimensional real Wiener process adapted to a given filtration $\mathfrak{F}$ and starting at 0 almost surely. Suppose that there is a $\hat{\mathbb{Y}}$-valued continuous process $Y = (Y_t)_{t\in[0,\infty)}$ starting on $\mathbb{Y}$ almost surely that can not escape the eventual 'point at infinity'. We say that $Y$ is a *solution to the Stratonovich SDE*

$$\mathrm{d}Y_t = \mathcal{A}_0(Y_t)\,\mathrm{d}t + \sum_{\ell=1}^k \mathcal{A}_\ell(Y_t) \circ \mathrm{d}W^\ell_t \tag{1.6}$$

if for all $f \in C^\infty_{\mathrm{c}}(\mathbb{Y})$ extended trivially to $\hat{\mathbb{Y}}$ the Stratonovich integral equation

$$f(Y_t) - f(Y_0) = \int_0^t (\mathcal{A}_0 f)(Y_s)\,\mathrm{d}s + \sum_{\ell=1}^k \int_0^t (\mathcal{A}_\ell f)(Y_s) \circ \mathrm{d}W^\ell_s$$

on $\mathbb{R}$ holds for all times $t$. The Kolmogorov backwards generator $\mathcal{L}$ associated to Equation (1.6) attains the Hörmander-type form

$$\mathcal{L} f = \mathcal{A}_0 f + \frac{1}{2}\sum_{\ell=1}^k \mathcal{A}^2_\ell f \qquad \text{for all } f \in C^\infty_{\mathrm{c}}(\mathbb{Y}). \tag{1.7}$$

See [IW81, Section V.1] specifically [IW81, Theorem V.1.2]. As mentioned in [IW81, Remark V.1.1] this intrinsic formulation is straight up compatible with the extrinsic one, where the manifold is embedded into a Euclidean ambient space. Unfortunately, the Laplace-Beltrami operator can not be written globally in the Hörmander form (1.7) unless $\mathbb{Y}$ is parallelisable. Therefore, in order to gain more general nondegenerate diffusions – as e. g. Brownian motion on $\mathbb{Y}$– one constructs a 'parent' dynamic in the bundle of orthonormal frames over $\mathbb{Y}$ that projects to the desired diffusion. This is the most widespread approach and the background is explained in the introductory paragraphs of [IW81, Section V.4]: Bochner proposed the idea that given an inky Wiener process in the plane one could role a sphere along the trajectory and the ink trace on the sphere should describe a spherical Brownian motion. By means of an (affine) connection the idea has been realised in [EE76] and is elaborated in [IW81, Section V.4] subsequently. In the end, if one has a second order differential operator that is given in local coordinates $(y^j)_{j=1}^{\mathscr{y}}$, $\mathscr{y} := \dim(\mathbb{Y})$, as

$$\mathscr{L}f(y) = \sum_{i=1}^{\mathscr{y}} b^i(y) \cdot \frac{\partial f}{\partial y^i} + \frac{1}{2}\sum_{j=1}^{\mathscr{y}} a^{ij}(y) \cdot \frac{\partial f}{\partial y^i \partial y^j} \qquad \text{for } f \in C_c^\infty(\mathbb{Y}), y \in \mathbb{Y} \tag{1.8}$$

with $(a^{ij})$ being symmetric and strictly positive definite, then [IW81, Theorem V.4.2] together with [IW81, Proposition V.4.3] yields exactly the desired nondegenerate diffusion $Y$ with Kolmogorov backwards generator $\mathscr{L}$ as in (1.8). The corresponding 'parent' dynamic in the orthonormal frame bundle is referred to as horizontal lift of $Y$. In context, this nomenclature absolutely makes sense, however we can't help but notice that a 'horizontal' stochastic process might just as well be a process in a horizontal tangent bundle. Now, one is in the position to develop more martingale theory, analysis on path spaces etc. on manifolds; we recommend the two books [Hsu02; HT94].

The SDEs in this thesis are to be read in the sense of [Jør78, Proposition 2]. E. Jørgensen used the *McKean-Gangolli injection scheme* to construct what he called an 'Ornstein-Uhlenbeck process'. The appeal of the injection scheme its rather intuitive founding idea: Basically, one injects increments of skew Brownian motion in tangent spaces via the exponential map and repeats the process from the new points. Compare to [Gan64, Equation (2.3)]. We think that it might be worth considering the injection scheme for simulation purposes. In Figure 1.4 one finds realisations of a Wiener process on a sphere for diffusion parameter 0.4, which are generated using the injection scheme. The theoretical groundwork of that method was done by H. P. McKean Jr. for Lie groups, see [McK60], and the general case was addressed by R. Gangolli in [Gan64]. As briefly mentioned before in the margins of Remark 1.2.13 though, the procedure in [Gan64] is flawed in regards to implicitly requiring parallelisability. The problem arises from the compatibility conditions and was pointed out by E. Jørgensen in the remark at the end of [Jør78, Section 1]. So, stochastic processes in local coordinate patches still need to be stitched together to a global process which is an issue of topological nature. E. g. K. Itô in [Itô50] imposes a suitable transformation rule for the local coefficients of the considered SDE under change of coordinates. The first part of [Jør78] consists of fixing the injection scheme by employing it in the bundle of frames and projecting the resulting process via the bundle projection. A central point is an extension of a theorem by E. B. Dynkin on measurable transformations of Markov processes, see [Dyn65, Theorem 10.13, p. 325], to ensure that the projection of the process constructed in the holonomy bundle to the base manifold again is a Markov process, see [Jør78, p. 76].

Miscellaneously, we list a few other formulations that occur in the literature. In [Kol00, Chapter 4] V. N. Kolokoltsov studies 'curvilinear Ornstein-Uhlenbeck processes' in terms of stochastic Hamiltonian systems which are heavily based on local coordinate expressions. One notes that everything is formulated in terms of smooth tensor fields such that the local coordinate forms transform correctly. Going off on a small tangent here, more ideas from classical Hamiltonian mechanics have been enriched with stochastic aspects: E. g. [BO09] caught our attention wherein the authors construct variational integrators from stochastic variational principles and discuss physically interesting features of these schemes. In contrast to painstaking local coordinate formulations, there also is the class of approaches using second order objects: E. g. in [Elw82, Chapter VI] the second order objects are second order vectors, in [AB18] the authors promote the idea to use 2-jets. While the former work is quite famous and well developed, the latter using 2-jets is not widespread at all. It might get more attention in the future though, as some mathematicians and physicists deem the jet formalism most appropriate for ordinary or partial differential equations on manifolds in invariant form.

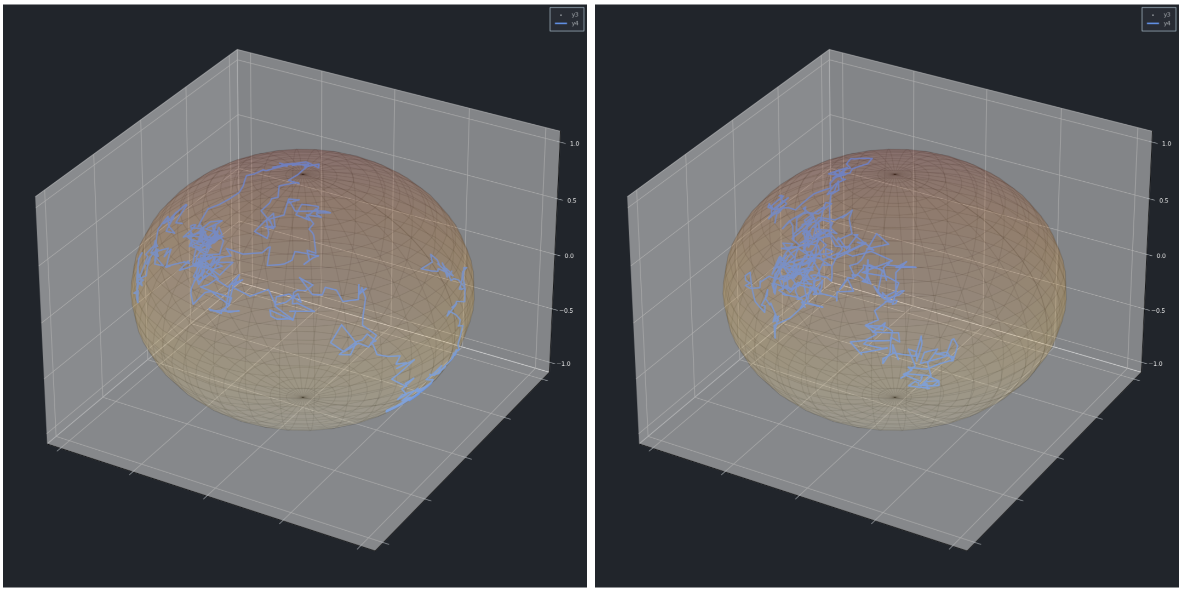



Figure 1.4: Different realisations of a Wiener process on a sphere via the McKean-Gangolli injection scheme

# 2 Hypocoercivity for Langevin-type equations on manifolds

In this chapter, we are going to apply the Abstract Hilbert space Hypocoercivity Method (AHHM) as introduced in Section 1.1 to both the Langevin equation and the spherical velocity Langevin equation on an abstract, connected, finite dimensional smooth Riemannian manifold $\mathbb{X}$. The results of Section 2.1 and Section 2.2 have been published before in [GM22]. The main theorems are the hypocoercivity results Theorem 2.1.4 and Theorem 2.2.4. We learned the techniques for many of our proofs concerning hypocoercivity from [GS14; GS16]. We extend them and justify for several steps that are obvious just in the Euclidean case. Afterwards in Section 2.3, we construct on the one hand an in a reasonable way associated Hunt processes by means of generalised Dirichlet form theory, and on the other hand show ergodicity for these processes at an explicit rate that even is optimal in regards to the time dependence. Furthermore, that section contains some remarks on what we can say when the diffusion parameter approaches infinity. In the end of the chapter in Section 2.4, we allude to various perspectives of new research topics, possibly interesting thoughts for optimisation in industrial application, and more related models.

## 2.1 ▪ Geometric Langevin equation

As already mentioned in the introduction, we think the Langevin equation in the classical Euclidean setting as the SDE system

$$\begin{aligned} \mathrm{d}x_t &= v_t \,\mathrm{d}t \\ \mathrm{d}v_t &= -\alpha \cdot v_t \,\mathrm{d}t - \nabla\Psi(x_t)\,\mathrm{d}t + \sigma \circ \mathrm{d}W_t, \end{aligned} \tag{2.1}$$

where $x_t \in \mathbb{R}^d_x$ are positions in a space $\mathbb{R}^d_x$ and $v_t \in \mathbb{R}^d_v$ are velocities for all times $t \in [0,\infty)$ respectively – the spaces $\mathbb{R}^d_x$ and $\mathbb{R}^d_v$ are independent copies of $\mathbb{R}^d$. The model parameters $\alpha$ and $\sigma$ both are positive real numbers. The potential $\Psi\colon \mathbb{R}^d_x \to \mathbb{R}$ has to satisfy certain regularity properties that are relatively weak – we give the details below in Condition 2.1.3 and the subsequent discussion. By $W = (W_t)_{t\in[0,\infty)}$ we denote a $d$-dimensional Wiener process and the symbol '$\circ$' signifies Stratonovich integration. In classical statistic physics, Equation (2.1) describes a particle at time $t$ with position $x_t$ and velocity $v_t$ moving in the presence of a force field $-\nabla\Psi$ (up to a multiple) and experiencing random collisions caused by smaller particles. One interprets $\alpha$ as a friction parameter, while the diffusion parameter $\sigma$ regulates the impact of the collisions. Note that later on, we introduce the parameter $\beta \in (0,\infty)$ rescaling $\Phi = \beta\Psi$ and impose the relation $\sigma = \sqrt{2\alpha/\beta}$ linking the parameters.

As we usually choose the mind set of Lagrangian dynamics we adopt its nomenclature and clarify a few other manners of speaking in the classical framework: We refer to the space $\mathbb{R}^d_x$ as the *position space*; the fundamental issue of this thesis is the choice of an abstract manifold as the position space, the *position manifold* $\mathbb{X}$, incorporating some smooth side condition for the position variables. Position space and the 'space of velocities' together form the *configuration manifold* $\mathsf{Q} = \mathbb{R}^d_x \times \mathbb{R}^d_v$; in general the configuration manifold is the tangent space of the position manifold, in formulae $\mathsf{Q} = \mathrm{T}\mathbb{X}$. Recall Remark 1.2.13 where we mentioned that a tangent space is not a trivial bundle unless the base manifold is parallelisable. Oftentimes, for instance in [MR99], people rather use the term 'configuration space', what we purposefully don't do even though it causes slight inconvenience in Chapter 4. We reserve this term for the configuration space as a model of multiparticle systems, see [Röc98] for an overview on this concept. The configuration space formalism might become handy when studying ensembles of multiple, possibly interacting filaments at once, however this is not part of this thesis. Whilst $\mathbb{X}$ encodes a smooth side condition on the position, a side condition on the velocity is implemented in Section 2.2 by choosing $\mathsf{Q}$ as a subbundle of the tangent bundle. Now, we specify which properties (M) of $\mathbb{X}$ are required henceforth and throughout the thesis.

**Condition 2.1.1** (Position manifold (M)).

(M1) *General geometry:* Let $(\mathbb{X}, \mathsf{x})$ be a real, finite dimensional, connected Riemannian manifold with $d := \dim(\mathbb{X}) \geq 2$.

(M2) *Completeness:* Let $\mathbb{X}$ endowed with the intrinsic metric be a complete metric space.



In view of Section 1.3, we consider as the natural generalisation of Equation (2.1) the following Stratonovich SDE on Q:

$$\mathrm{d}\eta = \mathcal{H}_{\mathsf{x}}\ \mathrm{d}t - \alpha \cdot \mathcal{V}\ \mathrm{d}t + \mathsf{vl}_\eta(-\nabla_{\mathsf{x}}\Psi)\ \mathrm{d}t + \sigma \sum_{j=1}^{d} \mathsf{vl}_\eta\left(\frac{\partial}{\partial x_\eta^j}\right) \circ \mathrm{d}W_t^j, \tag{2.2}$$

where $\eta\colon I \to \mathsf{T}\mathbb{X}$ is a curve with time interval $I$ and $\left(x_\eta^1, x_\eta^2, \ldots, x_\eta^d\right)$ is a chart at $\pi_0(\eta)$ providing normal coordinates.[1] Recall that $\mathcal{V}$ denotes the canonical vector field. Below in Condition 2.1.3 we specify certain assumptions on the potential $\Psi\colon \mathbb{X} \to \mathbb{R}$. The nonnegative model parameters $\alpha$ and $\sigma$ are related by $\sigma = \sqrt{2\alpha/\beta}$, where $\beta$ is a nonnegative rescaling of the potential as $\Phi = \beta\Psi$. We call Equation (2.2) the *Langevin equation on* $\mathbb{X}$ or *geometric Langevin equation* in the vein of [GS16; Sti14]. The Kolomogorov backwards generator formally reads as

$$L = \mathcal{H}_{\mathsf{x}} - \nabla_{\mathsf{v}}\Psi^{\mathsf{h}} + \frac{\sigma^2}{2}\,\Delta_{\mathsf{v}} - \alpha\mathcal{V} = \mathcal{H}_{\mathsf{x}} - \frac{1}{\beta}\nabla_{\mathsf{v}}\Phi^{\mathsf{h}} + \frac{\alpha}{\beta}\,\Delta_{\mathsf{v}} - \alpha\mathcal{V}, \tag{2.3}$$

compare to Equation (1.8). For short, we refer to $L$ as the *Langevin generator*. Evidently, this operator is defined for all smooth functions on Q. Once the model Hilbert space $H$ is introduced in the next definition, the space of smooth functions with compact support offers itself as candidate for the core $D$.

It should be noted that we can very easily write down both Equation (2.2) and the operator $L$ in local coordinate form. For sake of convenience, we give local coordinate forms in Appendix A for abstract coordinates and a few examples of position manifolds $\mathbb{X}$.

**Definition 2.1.2** (model Hilbert space (D1)). Consider the probability space

$$(E, \mathfrak{E}, \mu) = \left(\mathsf{T}\mathbb{X}, \mathfrak{B}(\mathsf{T}\mathbb{X}), \lambda_{\mathbf{t}}\right),$$

where $\lambda_{\mathbf{t}} = \lambda_{\mathbf{x}} \otimes_{\mathsf{loc}} \nu$ is the weighted Sasaki volume measure, specifically $\mathbb{X}$ is weighted by $\rho_{\mathbb{X}} := \exp(-\Phi) = \exp(-\beta\Psi)$ with $\beta \in (0,\infty)$ such that $\lambda_{\mathbf{x}}$ is a probability measure on $(\mathbb{X}, \mathfrak{B}(\mathbb{X}))$ and $\nu = \mathsf{N}(0; \beta^{-1}\mathsf{Id}_d)$ is the zero-mean normal distribution on the fibre $\mathbb{F} = \mathbb{R}^d$ with covariance matrix $\beta^{-1}\mathsf{Id}_d$. In other words, $\lambda_{\mathbf{t}}$ has the loc-density $\rho = \rho_{\mathbb{X}} \otimes_{\mathsf{loc}} \rho_{\mathbb{F}} = \exp(-\Phi) \otimes_{\mathsf{loc}} \prod_{j=1}^{d} \varphi_{0,\beta^{-1}} \circ \mathsf{pr}_j$, where $\varphi_{0,\beta^{-1}}$ denotes the density of a one-dimensional normal distribution with variance $\beta^{-1}$. The model Hilbert space is $H := L^2(E;\mu) = L^2(\mathsf{T}\mathbb{X}; \mathbf{t})$.

Note that the weighted Sasaki volume measure $\mu = \lambda_{\mathbf{t}}$ really is well-defined in terms of Definition 1.2.19. Indeed, this follows from the facts that on the one hand the standard fibre $\mathbb{F} = \mathbb{R}^d$ of $\mathsf{T}\mathbb{X}$ is considered as a linear metric space and on the other hand that the Gaußian measure $\nu$ is invariant with respect to rotations.

As of yet, we didn't impose assumptions on the potential $\Psi$, but of course such assumptions are crucial in order to verify conditions (D) and (H) in applications. First, we give typical requirements for the potential and afterwards discuss a few already contemplated variations.

**Condition 2.1.3** (Potential conditions (P)).

(P1) *General regularity and boundedness:* Let $\Phi = \beta\Psi$ a loc-Lipschitz potential which is bounded from below and such that $\lambda_{\mathbf{x}} = \rho_{\mathbb{X}}\,\lambda_{\mathsf{x}} = \exp(-\Phi)\,\lambda_{\mathsf{x}}$ is a probability measure on $(\mathbb{X}, \mathfrak{B}(\mathbb{X}))$.

(P2) *Poincaré inequality:* The weighted Riemannian measure $\lambda_{\mathbf{x}}$ satisfies the Poincaré inequality

$$\left\|\nabla_{\mathsf{x}} f_0\right\|^2_{L^2(\mathbb{X}\to\mathsf{T}\mathbb{X};\mathbf{x})} \geq \Lambda \cdot \left\|f_0 - (f_0\,,1)_{L^2(\mathbb{X};\mathbf{x})}\right\|^2_{L^2(\mathbb{X};\mathbf{x})} \tag{2.4}$$

for all $f_0 \in C_{\mathsf{c}}^\infty(\mathbb{X})$ and some $\Lambda \in (0,\infty)$.

[1] We choose normal coordinates here for sake of sheer convenience, since in such coordinates a Laplace-Beltrami operator attains a sum of squares form. Compare to Section A.1.

(P3) *Hessian dominated by gradient:* Assume $\Phi \in C^2(\mathbb{X})$. There is a constant $c \in (0,\infty)$ such that

$$\left|\mathsf{Hess}_{\mathsf{x}}(\Phi)(x)\right| \leq c\left(1+\left|\nabla_{\mathsf{x}}\Phi(x)\right|_{\mathsf{x}}\right) \qquad \text{holds for all } x \in \mathbb{X}.$$

Here, '$\mathsf{Hess}_{\mathsf{x}}(\cdot)$' denotes the Hessian with respect to the given Riemannian metric and the norm '$|\mathsf{Hess}_{\mathsf{x}}(\cdot)|$' of the Hessian is the Frobenius tensor norm induced by the Riemannian metric. ⌟



Clearly, the Poincaré inequality (2.4) is the most restrictive condition on the geometry of the weighted position manifold $(\mathbb{X}, \mathbf{x})$. Recall Remark 1.2.37. It is explained in [GS14, Remark 3.16] the condition (P3) as above can be weakened to the assumptions (A3), (A4), (A5) as in [GS14, Appendix A].

Next, we step by step verify the data and hypocoercivity assumptions relying at large on the methods of proving form [GS14] and [GS16]. Though, I have to elaborate on several computations that didn't need further justification in case of Euclidean position spaces. This yields the main theorem of this section:

**Theorem 2.1.4** (Hypocoercivity of the geometric Langevin dynamic)**.** Consider parameters $\alpha, \beta \in (0,\infty)$ and a Riemannian manifold $(\mathbb{X}, \mathsf{x})$ satisfying (M). Assume that the potential $\Phi\colon\ \mathbb{X} \to \mathbb{R}$ fulfils the conditions (P). Denote by $\nu$ the zero-mean Gaußian measure with covariance matrix $\beta^{-1}\mathsf{Id}$ and define $\mu := \lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu$. Then, the Langevin operator

$$(L, C_{\mathsf{c}}^{\infty}(\mathsf{T}\mathbb{X})) = \left(\frac{\alpha}{\beta}\,\Delta_{\mathsf{v}} - \alpha\mathcal{V} + \mathcal{H}_{\mathsf{x}} - \frac{1}{\beta}\,\nabla_{\mathsf{v}}\Phi^{\mathsf{h}}, C_{\mathsf{c}}^{\infty}(\mathsf{T}\mathbb{X})\right)$$

is closable in $H = L^2(\mathsf{T}\mathbb{X};\mu)$. Moreover, its closure $(L, D(L))$ generates a SCCS $(T_t)_{t\in[0,\infty)}$. Finally, for all $\kappa_1 \in (1,\infty)$ there is a constant $\kappa_2 \in (0,\infty)$ such that for all $g \in H$ and times $t \in [0,\infty)$ holds

$$\left\|T_t g - (g\,,1)_H\right\|_H \leq \kappa_1 \mathsf{e}^{-\kappa_2 t}\left\|g - (g\,,1)_H\right\|_H,$$

and $\kappa_2$ is given as

$$\kappa_2 = \frac{\kappa_1 - 1}{\kappa_1}\,\frac{\alpha}{n_1 + n_2\alpha + n_3\alpha^2} \tag{2.5}$$

where $n_i \in (0,\infty)$ for $i \in \{1,2,3\}$ only depend on $\Phi$ and $\beta$, compare to the calculation (A) in Section 1.1. ⌟

## Data conditions

One of the first incidents where I had to expand on the usual computations for the Euclidean position space concerns the SAD-decomposition. Relying just on integration by parts, one ostensibly has to impose the in view of Weyl's theorem rather restrictive assumption of a weakly harmonic potential. In second view, we realise that one can get rid of this assumption by means of a Poincaré bracket.

**Lemma 2.1.5** (SAD-decomposition (D3), (D4), (D6))**.** Assume that condition (P1) holds and let $\Phi$ be weakly harmonic. Consider the SAD-decomposition $L = S - A$ on $D$ with

$$Sf := \frac{\alpha}{\beta}\,\Delta_{\mathbf{v}} f = \frac{\alpha}{\beta}\,\Delta_{\mathsf{v}} f - \alpha\cdot\mathcal{V}f$$

$$\text{and} \quad Af = -\mathcal{H}_{\mathbf{x}} f := -\mathcal{H}_{\mathsf{x}} f + \frac{1}{\beta}\left(\nabla_{\mathsf{v}}\Phi^{\mathsf{h}}\right)(f) \quad \text{for all } f \in D.$$

Then, the following assertions hold:

(i) $(S, D)$ is symmetric and nonpositive definite.

(ii) $(A, D)$ is antisymmetric.

(iii) For all $f \in D$ we have that $Lf \in L^1(\mathsf{T}\mathbb{X};\mu)$ and $\int_{\mathsf{T}\mathbb{X}} Lf \,\mathsf{d}\mu = (Lf\,,1)_H = 0$.

*Proof.*

(i) In order to avoid tedious formulae, we condense notation a bit calculating the logarithmic derivative as

$$\frac{1}{\rho_{\mathbb{F};x}}\nabla_{\mathsf{v}}\rho_{\mathbb{F};x}(f) = \nabla_{\mathsf{euc}}\left(-\frac{\beta}{2}\,|\mathsf{Id}_{\mathbb{F}}|^2_{\mathsf{euc}}\right)(f) = -\beta\left\langle \mathsf{Id}_{\mathsf{T}\mathbb{X}}\,, df_0\right\rangle = -\beta\cdot\mathcal{V}f$$

2

on $\mathsf{T}_x\mathbb{X}$ holds for all $f = f_0^{\mathsf{h}} \in \kappa^* C_{\mathsf{c}}^\infty(\mathbb{X})$. Hence, it follows for all $f, g \in D_0$ that

$$(Sf\,,g)_H = -\int_{\mathsf{T}\mathbb{X}} \mathsf{v}\big(\nabla_{\mathsf{v}} f\,, \nabla_{\mathsf{v}} g\big)\;\mathsf{d}\mu = -\int_{\mathsf{T}\mathbb{X}} \mathbf{v}\big(\nabla_{\mathbf{v}} f\,, \nabla_{\mathbf{v}} g\big)\;\mathsf{d}\mu,$$

and therefore $(S, D_0)$ is symmetric and nonpositive definite. Since $S$ is well defined on $D$ and $D_0$ is dense in $D$ with respect to graph norm, the statement follows.

(ii) By Liouville's Theorem we know the adjoint of the Riemannian semispray with respect to $L^2(\mathsf{T}\mathbb{X};\mathbf{t})$-scalar product:

$$\begin{aligned}\mathcal{H}_{\mathsf{x}}{}^* &= -\mathcal{H}_{\mathsf{x}} - 0 - \frac{1}{\rho}\,\mathcal{H}_{\mathsf{x}}\rho = -\mathcal{H}_{\mathsf{x}} + \frac{1}{\rho_{\mathbb{F}}}\,\beta\rho_{\mathbb{F}}\cdot\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\\ &= -\mathcal{H}_{\mathsf{x}} + \beta\cdot\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}.\end{aligned}$$

Furthermore, we compute the adjoint (with respect to $L^2(\mathsf{T}\mathbb{X};\mathbf{t})$-scalar product) of $\nabla_{\mathsf{v}}\Psi^{\mathsf{h}} = \frac{1}{\beta}\nabla_{\mathsf{v}}\Phi^{\mathsf{h}}$ using that $\Psi$ is weakly harmonic:

$$\begin{aligned}\big(\nabla_{\mathsf{v}}\Psi^{\mathsf{h}}\big)^* &= -\nabla_{\mathsf{v}}\Psi^{\mathsf{h}} - \underbrace{\mathsf{div}_{\mathsf{t}}\big(\nabla_{\mathsf{v}}\Phi^{\mathsf{h}}\big)}_{=0} - \frac{1}{\rho}\big(\nabla_{\mathsf{v}}\Psi^{\mathsf{h}}\big)(\rho)\\ &= -\nabla_{\mathsf{v}}\Psi^{\mathsf{h}} - \underbrace{\mathsf{v}\big(\nabla_{\mathsf{v}}\Psi^{\mathsf{h}}\,, -\beta\mathcal{V}\big)}_{=\mathsf{h}\left(\nabla_{\mathsf{h}}\Psi^{\mathsf{v}}, -\beta\mathcal{H}_{\mathsf{x}}\right)} = -\nabla_{\mathsf{v}}\Psi^{\mathsf{h}} + \beta\cdot\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}.\end{aligned}$$

This shows that $(A, D)$ is an antisymmetric operator. Since it is densely defined on $H$, it is closable.

(iii) Integrability is rather clear. Moreover, consider some test function $h \in D$, then by part (i) we have $\int_{\mathsf{T}\mathbb{X}} Sh\,\mathsf{d}\mu = 0$, and by part (ii) $\int_{\mathsf{T}\mathbb{X}} Ah\,\mathsf{d}\mu = 0$. Finally, the densely defined operator $(L, D)$ is dissipative, thus closable.

□

*Notation* 2.1.6 (closures of $S$, $A$ and $L$). Form Lemma 2.1.5 we know that $(S, D)$, $(A, D)$ and $(L, D)$ are closable. Their respective closures are denoted by $(S, D(S))$, $(A, D(A))$ and $(L, D(L))$.

Of course, we are not content with assuming a weakly harmonic potential. Hoping to replace the assumption, we revisit the easiest example of Euclidean space as in [GS16] the authors don't need assumption more restrictive than 2.1.3.

**Example 2.1.7.** Let's look at $\mathbb{X} = \mathbb{R}^d_x$. The inverse $\begin{pmatrix} 0 & -\mathsf{Id}_d \\ \mathsf{Id}_d & 0 \end{pmatrix}$ of 'the' symplectic matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ yields an almost complex structure $\mathcal{J}$ on $\mathsf{T}\mathbb{R}^d_x \simeq \mathbb{R}^d_x \times \mathbb{R}^d_v$, that is a mapping $\mathcal{J}\colon \mathsf{T}\mathbb{X} \to \mathsf{T}\mathbb{X}$ fulfilling $\mathcal{J}^2 = -\mathsf{Id}$. Taking our convention of listing the vertical component first and the horizontal one second into account, it's the same as the one constructed in [Dom62, Paragraph 5] or later in [TO62]. Moreover, it is *compatible* with the Euclidean metric and the canonical symplectic form $\Omega$ on $\mathsf{T}^*\mathbb{R}^d_x \simeq \mathbb{R}^d_x \times (\mathbb{R}^d)^*$, see [Lee13, Section 12.6] for the basic definition, in the sense that

$$(v\,,w)_{\mathsf{euc}} = \Omega(v, \mathcal{J}w) \quad \text{and} \quad \Omega(v,w) = (\mathcal{J}v\,,w)_{\mathsf{euc}} \qquad \text{for all } v,w \in \mathbb{R}^{2d},$$

compare to [MR99, Exercise 2.2.1]. The symplectic form gives rise to a Poisson bracket $\{\cdot\,,\cdot\}$ such that the corresponding Poisson tensor reads as

$$\{f\,,g\}(x,v) = -(\nabla_{\mathsf{euc}} f(x,v)\,, \mathcal{J}\nabla_{\mathsf{euc}} g(x,v))_{\mathsf{euc}} = \Omega_{(x,v)}(df, dg)$$

for all $f, g \in C^\infty(\mathbb{R}^{2d})$ and $(x,v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v$. In [MR99] one finds a broad discussion of Poisson brackets in physics and some historical background.

Using Poisson bracket terminology, the proof of [GS16, Lemma 3.4 part ii)] relies on linking the operator $(A, D)$ to the antisymmetric bilinear form $(\mathcal{A}, D)$ of integrating minus the Poisson bracket with respect to $\mu$, in formulae:

$$\mathcal{A}(f,g) := \int_{\mathbb{R}^{2d}} -\{f\,,g\}\;\mathsf{d}\mu \quad \text{for all } f,g \in D = C_{\mathsf{c}}^\infty(\mathbb{R}^{2d}).$$

Via integration by parts one can show that $(\beta\, Af\,,g)_{L^2(\mu)} = \mathcal{A}(f,g)$ holds for all $f,g \in C_c^\infty(\mathbb{R}^{2d})$. This can be done *without the assumption* of a weakly harmonic potential, since the vector field action can be represented in terms of the Hamiltonian vector fields. Indeed: Denote by $\mathsf{H}_f$ the *Hamiltonian vector field* of $f \in C^\infty(\mathbb{R}^{2d})$, i.e. it fulfils $\mathsf{H}_f(g) = \{f\,,g\}$ for all $g \in C^\infty(\mathbb{R}^{2d})$. We get the explicit formula $-\mathsf{H}_f = (\nabla_v f\,, \nabla_x)_{\mathsf{euc}} - (\nabla_x f\,, \nabla_v)_{\mathsf{euc}}$ and conclude that $\mathsf{H}_f$ is solenoidal by Schwarz's theorem. Hence, the following equality is true:

$$\mathcal{A}(f,g) = \int_{\mathbb{R}^{2d}} \mathsf{H}_f(\rho)\cdot g \,\mathsf{d}\lambda = (\beta\, Af\,,g)_{L^2(\mu)} \qquad \text{for all } f,g \in D,$$

where $\rho := {}^{d\mu}/_{d\lambda}$ is the (loc-)density of $\mu$. ⌟

**Proposition 2.1.8** (SAD-decomposition (2nd version))**.** The assertions of Lemma 2.1.5 are true without the assumption of $\Psi$ being weakly harmonic.

*Proof.* We only have to look at part (ii) of Lemma 2.1.5 and mimic the technique discussed in the previous example. Denote by $\mathcal{J}$ minus the almost complex structure on $\mathsf{T}\mathbb{X}$ constructed in [Dom62], and let $\Omega$ be the canonical symplectic form on $\mathsf{T}(\mathbb{X})$. By construction, the Sasaki metric, $\mathcal{J}$ and $\Omega$ are compatible, see [MR99, page 341]. Hence, they define the same Poisson bracket $\{\cdot\,,\cdot\}$ on $\mathsf{T}\mathbb{X}$ via both assignments

$$\Omega_v(df,dg) =: \{f\,,g\}(v) := -\mathsf{t}\big(\nabla_{\mathsf{t}} f(v)\,, \mathcal{J}\nabla_{\mathsf{t}} g(v)\big)v$$

for all $f,g \in C^\infty(\mathsf{T}\mathbb{X})$ and $v \in \mathsf{T}\mathbb{X}$. Thus, for any fixed $f \in C^\infty(\mathsf{T}\mathbb{X})$ there is a unique Hamilton vector field $\mathsf{H}_f$ by [MR99, Proposition 10.2.1]. Let a chart $(v^j)_{j=1}^{2d}$ that gives normal coordinates and respects the Ehresmann connection such that $(\partial v^i)_{i=1}^d$ provides a local basis for vertical vector fields and $(\partial v^{k+d})_{k=1}^d$ provides a basis for the horizontal vector fields. In these coordinates we can write the Hamiltonian vector fields in the form

$$-\mathsf{H}_f = \sum_{k=1}^d \partial_{v^k} f \cdot \partial_{v^{k+d}} - \sum_{i=1}^d \partial_{v^{i+d}} f \cdot \partial_{v^i}.$$

Then, we easily compute that $\mathsf{divt}(\mathsf{H}_f) = 0$ using Schwarz's Theorem. In other words, all $\mathsf{H}_f$ are solenoidal.

Define the antisymmetric bilinear form $(\mathcal{A}, D)$ by $\mathcal{A}(f,g) := \int_{\mathsf{T}\mathbb{X}} -\{f\,,g\}\,\mathsf{d}\mu$ for all $f,g \in D = C_c^\infty(\mathsf{T}\mathbb{X})$. From the Divergence Theorem it follows that

$$\mathcal{A}(f,g) = \int_{\mathsf{T}\mathbb{X}} -\mathsf{H}_f(g)\,\mathsf{d}\lambda_{\mathsf{t}} = \int_{\mathsf{T}\mathbb{X}} \mathsf{H}_f(\rho)\cdot g\,\mathsf{d}\lambda_{\mathsf{t}}.$$

We infer that $\mathsf{H}_f(\rho) = -\rho\cdot\mathcal{H}_{\mathsf{x}} f$. In a nutshell, we localise in the support of $f$ via a partition of unity argument where the corresponding open cover is formed by charts $(v^j)_{j=1}^{2d}$ that are respecting the Ehresmann connection and also are restricted to domains of local trivialisation; therein, we can use the local coordinate form of $\mathsf{H}_f$ and that the loc-density $\rho$ trivialises to a product of exponential-type densities. Hence, we gain that $(\beta\, Af\,,g)_H = \mathcal{A}(f,g)$ for all $f,g \in D$ which finishes the proof. ☐

Turning to the projections $P$ and $P_S$ mentioned in (D5) they always are built from some kind of fibrewise average. Given any $f \in H$ the *fibrewise average of $f$* is the following mapping

$$\mathsf{E}_\nu[f]\colon\ \mathbb{X} \to \mathbb{R},\ x \longmapsto \int_{\mathsf{T}_x\mathbb{X}} f\,\mathsf{d}\nu_x.$$

The fibrewise average acts trivially on vertically lifted functions as they are fibrewise constant: $\mathsf{E}_\nu[f_0^{\mathsf{v}}] = f_0$ for all $f_0 \in L^2(\mathbb{X};\mathbf{x})$. The projection $P_S$ is chosen as the vertical lift of the fibrewise average assigning to an element $v$ of the tangent bundle the average of $f$ over the fibre corresponding over $\pi_0(v)$: $P_S f := \mathsf{E}_\nu[f]\circ\pi_0$. Note that $\mathsf{ran}(P_S)$ coincides with $\{f_0^{\mathsf{v}} \in H \mid f_0 \in L^2(\mathbb{X};\mathbf{x})\}$, which is closed, and $P_S$ is a projection indeed. Furthermore, for all $f \in H$ we compute that

$$\begin{aligned}\int_{\mathsf{T}\mathbb{X}} (P_S f)^2\,\mathsf{d}\mu &= \int_{\mathsf{T}\mathbb{X}} (\mathsf{E}_\nu[f]\circ\pi_0)^2\,\mathsf{d}\lambda_{\mathbf{x}}\otimes_{\mathsf{loc}}\nu = \int_{\mathsf{T}\mathbb{X}} (\mathsf{E}_\nu[f])^2\circ\pi_0\,\mathsf{d}\lambda_{\mathbf{x}}\otimes_{\mathsf{loc}}\nu\\ &= \int_{\mathbb{X}} \mathsf{E}_\nu[f]^2\,\mathsf{d}\lambda_{\mathbf{x}}.\end{aligned}$$

This implies $\|P_S f\|_H = \|\mathsf{E}_\nu[f]\|_{L^2(\mathbb{X};\mathsf{x})} \leq \|f\|_H$ meaning that $P_S$ is continuous with norm 1. Thus, $P_S$ even is an orthogonal projection by [Con90, Proposition 3.3]. Eventually, we define the projection $P$ from (D5) via the assignment $P := P_S - (\cdot\,, 1)_H$. Then, we are in the position to verify not only the part of the data conditions involving $P$, but also semigroup conservativity.

2

**Lemma 2.1.9** (Properties of projection $P$ and semigroup conservativity (D5), (D7)). Assume that condition (P1) holds. Then, we have $P(H) \subseteq D(S)$, $SP = 0$, as well as $P(D) \subseteq D(A)$, and $AP(D) \subseteq D(A)$. Furthermore, $1 \in D(L)$ and $L1 = 0$.

*Proof.* The proof is based on [GS16, Lemma 3.4 part (iv)] for the Euclidean case. First of all, we deal with the statements related to $S$. In order to deal with the noncompact standard fibre, we consider the sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{X};[0,1])$ as constructed in Remark 1.2.33 which serves as an approximation of the constant function 1 on $\mathbb{X}$. In particular, recall that there is some constant $c \in (0,\infty)$ such that

$$|\nabla_{\mathsf{x}}\varphi_n(x)|_{\mathsf{x}} \leq \frac{c}{n} \quad \text{and} \quad |\mathsf{Hess}_{\mathsf{x}}(\varphi_n)(x)|_\infty \leq \frac{c}{n^2}$$

for all $x \in \mathbb{X}$ and $n \in \mathbb{N}_+$, compare to [GS16, Definition 3.3]. Let $f_0 \in C_{\mathsf{c}}^\infty(\mathbb{X})$ and define

$$f_n := f_0^{\mathsf{v}} \otimes \varphi_n^{\mathsf{h}} = f_0^{\mathsf{v}} \cdot \varphi_n^{\mathsf{h}} \qquad \text{for all } n \in \mathbb{N}_+. \tag{2.6}$$

The sequence $(f_n)_{n\in\mathbb{N}_+}$ lives in $D$ and approximates $f_0^{\mathsf{v}}$ in $H$. By dominated convergence it follows that

$$Sf_n = \frac{\alpha}{\beta}\,\Delta_{\mathsf{v}} f_n = \frac{\alpha}{\beta}\, f_0^{\mathsf{v}} \cdot \Delta_{\mathsf{v}}\varphi_n^{\mathsf{h}} - \alpha f_0^{\mathsf{v}} \cdot \mathcal{V}\varphi_n^{\mathsf{h}} \longrightarrow 0 \qquad \text{in } H \text{ as } n \to \infty.$$

Note that we used here that the function $\mathsf{T}\mathbb{X} \to \mathbb{R}$, $v \mapsto |\mathcal{V}(v)|_{\mathsf{t}} = |v|_{\mathsf{x}}$ is in $H$, since the mapping $\left|\mathsf{Id}_{\mathsf{T}_x\mathbb{X}}\right|_{\mathsf{x}}$ is in $L^2(\mathsf{T}_x\mathbb{X};\nu_x)$ for all $x \in \mathbb{X}$. Using closedness of $(S, D(S))$ now, we infer that $f_0^{\mathsf{v}} \in D(S)$, and even more $Sf_0^{\mathsf{v}} = 0$. This already gives as an idea for the next assertion namely that $P$ maps into the null space of $S$. As we know that the range of $P$ is contained in $\pi_0^* L^2(\mathbb{X};\mathsf{x})$, we just take one $g_0 \in L^2(\mathbb{X};\mathsf{x})$ and show that $g_0^{\mathsf{v}} \in \mathsf{null}(S)$. Since the space $C_{\mathsf{c}}^\infty(\mathbb{X})$ is dense in $L^2(\mathbb{X};\mathbf{x})$, we consider for given $g_0 \in L^2(\mathbb{X};\mathsf{x})$ a sequence $(g_n)_{n\in\mathbb{N}_+}$ in $C_{\mathsf{c}}^\infty(\mathbb{X})$ approximating $g_0$ in $L^2(\mathbb{X};\mathbf{x})$. We have seen above that the vertically lifted sequence $(g_n^{\mathsf{v}})_{n\in\mathbb{N}_+}$ is a sequence in $\mathsf{null}(S)$. Again, from closedness of $(S, D(S))$ it follows that $g_0^{\mathsf{v}} \in D(S)$ and $Sg_0^{\mathsf{v}} = 0$.

Concerning the statements involving $A$ we start by fixing a function $f \in C_{\mathsf{c}}^\infty(\mathbb{X})$ and define the sequence $(f_n)_{n\in\mathbb{N}_+} := (f_0^{\mathsf{v}} \otimes \varphi_n^{\mathsf{h}})_n$ as in Equation (2.6). First, we observe that

$$\begin{aligned} Af_n &= -\varphi_n^{\mathsf{h}} \cdot \mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}} + \frac{1}{\beta} f_0^{\mathsf{v}} \cdot \left(\nabla_{\mathsf{v}}\Phi^{\mathsf{h}}\right)\varphi_n^{\mathsf{h}} = -\varphi_n^{\mathsf{h}} \cdot \mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}} + \frac{1}{\beta} f_0^{\mathsf{v}} \cdot \mathsf{x}\left(\nabla_{\mathsf{x}}\Phi\,, \nabla_{\mathsf{x}}\varphi_n\right) \\ &\longrightarrow -\mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}} + \frac{1}{\beta} f_0^{\mathsf{v}} \cdot 0 = -\mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}} \qquad \text{in } H \text{ as } n \to \infty. \end{aligned}$$

To see this we note that $|\mathcal{H}_{\mathsf{x}}|_{\mathsf{t}} = \left|\mathsf{Id}_{\mathsf{T}\mathbb{X}}\right|_{\mathsf{x}} \in H$, furthermore $\nabla_{\mathsf{x}}\Phi \in L^2(\mathbb{X} \to \mathsf{T}\mathbb{X};\mathbf{x})$, and then use dominated convergence. As $(A, D(A))$ is closed we get $f_0^{\mathsf{v}} \in D(A)$ and $Af_0^{\mathsf{v}} = -\mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}}$. It's enough to show that $1 \in D(A)$ and $A1 = 0$ for the inclusion $P(D) \subseteq D(A)$, therefore we take $(\varphi_n)_{n\in\mathbb{N}_+}$ as before and compute that

$$A\varphi_n^{\mathsf{v}} = -\mathcal{H}_{\mathsf{x}}\varphi_n^{\mathsf{v}} \longrightarrow 0 \qquad \text{in } H \text{ as } n \to \infty$$

using dominated convergence again. In order to prove now the inclusion $AP(D) \subseteq D(A)$ we adhere to our approximation strategy and define $h_n := \mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}} \cdot \varphi_n^{\mathsf{h}}$ for all $n \in \mathbb{N}_+$. By construction, the sequence $(h_n)_{n\in\mathbb{N}}$ converges to $\mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}}$ both pointwisely and in $H$. We note that the function $\mathcal{H}_{\mathsf{x}}^2 f_0^{\mathsf{v}} = \mathcal{H}_{\mathsf{x}}(\mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}})$ is dominated by

$$\|\mathcal{H}_{\mathsf{x}}\|_{L^2(\mathsf{T}\mathbb{X}\to\mathsf{T}^2\mathbb{X};\mu)} \cdot \left\|\mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}}\right\|_H \leq \|\mathcal{H}_{\mathsf{x}}\|^2_{L^2(\mathsf{T}\mathbb{X}\to\mathsf{T}^2\mathbb{X};\mu)} \cdot \|f_0^{\mathsf{v}}\|_H = \left\||\mathsf{Id}_{\mathsf{T}\mathbb{X}}|_{\mathsf{x}}\right\|_H^2 \cdot \|f_0^{\mathsf{v}}\|_H$$

due to the Cauchy-Bunyakovsky-Schwarz inequality (CSBI) applied twice. The left hand side is finite due to the choice of the Gaußian fibre measure: On each fibre $\mathsf{T}_x\mathbb{X}$ the function $\left|\mathsf{Id}_{\mathsf{T}_x\mathbb{X}}\right|^2$ is in $L^2(\mathsf{T}_x\mathbb{X};\nu_x)$. Together with aforementioned facts that $\left|\mathsf{Id}_{\mathsf{T}\mathbb{X}}\right|_{\mathsf{x}} \in H$ and $\nabla_{\mathsf{x}}\Phi \in L^2(\mathbb{X} \to \mathsf{T}\mathbb{X};\mathbf{x})$ we conclude that

$$Ah_n = -\varphi_n^{\mathsf{h}} \cdot \mathcal{H}_{\mathsf{x}}^2 f_0^{\mathsf{v}} + \frac{1}{\beta} \cdot \left(\nabla_{\mathsf{v}}\Phi^{\mathsf{h}}\right) h_n$$

$$
\begin{aligned}
&= -\varphi_n^{\mathsf{h}} \cdot \mathcal{H}_{\mathsf{x}}^2 f_0^{\mathsf{v}} + \frac{1}{\beta} \cdot \Big( \big( \mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}} \big) \cdot \big( \nabla_{\mathsf{v}} \Phi^{\mathsf{h}} \big) \varphi_n^{\mathsf{h}} + \varphi_n^{\mathsf{h}} \cdot \big( \nabla_{\mathsf{v}} \Phi^{\mathsf{h}} \big) \big( \mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}} \big) \Big) \\
&\longrightarrow -\mathcal{H}_{\mathsf{x}}^2 f_0^{\mathsf{v}} + 0 + \frac{1}{\beta} \cdot \big( \nabla_{\mathsf{v}} \Phi^{\mathsf{h}} \big) \big( \mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}} \big) \qquad \text{in } H \text{ as } n \to \infty
\end{aligned}
$$

by dominated convergence. Since $(A, D(A))$ is closed, the function $\mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}}$ is an element of $D(A)$.

Closing the proof, the statements on $L$ follow in the same vein: Let $f_0 \in C_{\mathsf{c}}^\infty(\mathbb{X})$ and define the sequence $(f_n)_{n \in \mathbb{N}_+}$ as in Equation (2.6). Then, repeat the previous steps and conclude from closedness of $(L, D(L))$ that $f = f_0^{\mathsf{v}} \in D(L)$ as well as $L f_0^{\mathsf{v}} = -A f_0^{\mathsf{v}}$. In particular, the sequence $(L\varphi_n^{\mathsf{v}})_{n \in \mathbb{N}}$ converges in $H$ to 0 as $n \to \infty$, and by closedness it follows $1 \in D(L)$ with $L1 = 0$. □

As quite useful consequence, we have explicit formulae for $AP$ on $D$. First, for $f \in D$ we know by the previous lemma that $Pf \in D(A)$, thus we can calculate that

$$
APf = -\mathcal{H}_{\mathsf{x}}(\mathsf{E}_\nu[f]^{\mathsf{v}}) = -\left\langle \mathcal{H}_{\mathsf{x}} \,,\, d\mathsf{E}_\nu[f] \circ d\pi_0 \right\rangle = -\left\langle \mathsf{Id}_{\mathsf{T}\mathbb{X}} \,,\, d\mathsf{E}_\nu[f] \right\rangle = -d\mathsf{E}_\nu[f]. \tag{2.7}
$$

This attains a more intuitive form

$$
\begin{aligned}
APf &= -\mathcal{H}_{\mathsf{x}}(Pf) = -\mathcal{H}_{\mathsf{x}}(\mathsf{E}_\nu[f]^{\mathsf{v}}) = -\mathsf{h}\big(\mathcal{H}_{\mathsf{x}} \,,\, \nabla_{\mathsf{h}} P_S f\big) \\
&= -\mathsf{h}\big(\mathcal{H}_{\mathsf{x}} \,,\, \mathsf{hl}(\nabla \mathsf{x} \mathsf{E}[f])\big) = -\mathsf{x}_{\pi_0}\big(\mathsf{Id}_{\mathsf{T}\mathbb{X}} \,,\, \nabla_{\mathsf{x}} \mathsf{E}_\nu[f] \circ \pi_0\big).
\end{aligned} \tag{2.8}
$$

Unlike Equation (2.7), we immediately recognise Equation (2.8) as the appropriate generalisation of [GS16, Equation (3.12)].

The most involved one among all the data condition certainly is (D2) requiring that $(L, D(L))$ generates a SCCS. In view of the Lumer-Philips Theorem, this is equivalent to m-dissipativity of $(L, D(L))$. In order to show that $(L, D)$ is essentially m-dissipative, we adopt a threefold strategy used in [GS14, Section 4] wherein step by step less regular potentials are allowed: At first, we look at smooth potentials and use hypoellipticity methods from [HN05]; afterwards, we employ the Kato Perturbation Theorem for m-dissipative operators to cover the case of globally Lipschitzian potentials, before we repeat the localisation argument of M. Grothaus and P. Stilgenbauer that yields the case of locally Lipschitzian potentials.

To start of, we briefly quote a consequence of the Hörmander Theorem, namely [GS14, Proposition A.1]. Assume a second order differential operator $T$ on a $\mathscr{b}$-dimensional Riemannian manifold $(\mathbb{B}, \mathsf{b})$ of the form $T = c + \mathcal{X}_0 + \sum_{k=1}^{\ell} \mathcal{X}_k^2$ for some $c \in C^\infty(\mathbb{B})$ and $\mathcal{X}_k \in \Gamma^\infty(\mathsf{T}\mathbb{B})$ for all $k \in \{0, \dots, \ell\}$. This operator $T$ is said to satisfy the *Hörmander condition* if at any point $b \in \mathbb{B}$ it holds

$$
\dim(\mathsf{Lie}_b(\mathcal{X}_0, \dots, \mathcal{X}_\ell)) = \dim(\mathsf{T}_b\mathbb{B}) = \mathscr{b},
$$

where $\mathsf{Lie}_b(\mathcal{X}_0, \dots, \mathcal{X}_\ell)$ denotes the generated Lie algebra at $b$. See [Lee13, Sections 4.4 and 20] for definitions and discussion on Lie algebras and related topics.

**Proposition 2.1.10.** Assume that $T$ satisfies the Hörmander condition and let $f \in L^1_{\mathsf{loc}}(\mathbb{B}; \mathsf{b})$ such that $\int_{\mathbb{B}} f \cdot T\psi \, \mathrm{d}\lambda_{\mathsf{b}} = 0$ for all $\psi \in C_{\mathsf{c}}^\infty(\mathbb{B})$. Then, $f$ has a smooth representative.

*Proof.* The proof is virtually the same as for [GS14, Proposition A.1]: It is done in chart domains, and within these domains it's perfectly fine to consider $\mathcal{X}_j = \partial x^j = \partial/\partial x^j$, where $(x^j)_{j=1}^{\mathscr{b}}$ are local coordinates provided by the given chart. Then, we have smooth representatives in chart domains serving as starting point for a partition of unity argument. Thus, even if $T$ is just available in local coordinate form, the proposition applies as soon as the respective chart domains form an open cover. □

**Lemma 2.1.11** (Hörmander condition for the Langevin generator)**.** Consider a smooth potential $\Psi \in C^\infty(\mathbb{X})$. Then, the Langevin generator satisfies the Hörmander condition. More precisely, let $v \in \mathsf{T}\mathbb{X}$ and $(x^j)_{j=1}^{d}$ be local coordinates corresponding to a chart at $\pi_0(v)$. Then, we have that

$$
\dim\Big(\mathsf{Lie}_v\Big(\mathcal{H}_{\mathsf{x}}, \mathsf{vl}\big(\nabla_{\mathsf{x}} \Psi\big), \mathsf{vl}(\partial x^1), \dots, \mathsf{vl}(\partial x^d), \mathcal{V}\Big)\Big) = \dim(\mathsf{T}_v^2\mathbb{X}) = \dim(\mathsf{T}\mathbb{X}) = 2d.
$$

*Proof.* From [GK02, Proposition 5.1] we know explicit forms for Lie brackets of vertically and horizontally lifted vector fields in any combination. With that said, we have the equations

$$
[\mathsf{vl}(\mathcal{X}) \,,\, \mathsf{vl}(\mathcal{Y})] = 0 \qquad \text{and} \qquad [\mathsf{hl}(\mathcal{Y}) \,,\, \mathsf{vl}(\mathcal{X})] = \mathsf{vl}\big(\nabla_{\mathcal{Y}}^{\mathsf{x}} \mathcal{X}\big)
$$

for all $\mathcal{X},\mathcal{Y}\in\Gamma^\infty(\mathrm{T}\mathbb{X})$. Neither $\mathsf{vl}\big(\nabla_{\mathsf{x}}\Psi\big)$ nor $\mathcal{V}$ contribute anything to the generated Lie algebra. Hence, the only nontrivial pairing of the seeding vector fields $\mathcal{X},\mathcal{Y}\in\{\partial x^1,\dots,\partial x^d\}$ is $[\mathsf{hl}(\mathcal{Y})\,,\mathsf{vl}(\mathcal{X})]=\mathsf{vl}\big(\nabla^{\mathsf{x}}_{\mathcal{Y}}\mathcal{X}\big)$. All vector fields of this form are linear dependent of the vertical vector fields $\{\mathsf{vl}(\partial x^1),\dots,\mathsf{vl}(\partial x^d)\}$ generating the vertical elements in the Lie algebra. This just shows that the collection $\{\partial x^1,\dots,\partial x^d\}$ even if we would build our generator with both kinds of liftings is not enough to generate a $2d$-dimensional Lie algebra at $v$.

The previous statement regarding Lie brackets of a vertically and a horizontally lifted vector field does not apply to $\mathcal{H}_{\mathsf{x}}$, since a semispray can not arise as horizontal lift of a vector field. Therefore, we compute in Lemma 2.1.12 that $\big[\mathcal{H}_{\mathsf{x}}\,,\mathsf{vl}(\partial x^j)\big]=\mathsf{hl}(\partial x^j)-\sum_{i=1}^d N^i_j\cdot\mathsf{vl}(\partial x^i)$ for certain functions $N^i_j$. This yields $d$ many linear independent horizontal vector fields in the generated Lie algebra which finishes the proof. □

I derived the auxiliary lemma used in the preceding proof directly from the local coordinate form of semisprays as I couldn't find any reference for this result. After this general calculation is presented, we are in the position to show essential m-dissipativity of smooth potentials in Proposition 2.1.13. That proof is based on [HN05, Proposition 5.5], compare to [GS14, Theorem 4.2].

**Lemma 2.1.12.** Compare Remark A.1.2 for abstract local coordinate forms of a semispray $\mathcal{H}$ and vertical as well as horizontal lifts. Then, it holds

$$[\mathcal{H}\,,\mathsf{vl}(\partial x^k)]=\mathsf{hl}(\partial x^k)-\sum_{j=1}^d N^j_k\cdot\mathsf{vl}(\partial x^j)\qquad\text{for all }k\in\{1,\dots,d\}.$$

*Proof.* Let $k\in\{1,\dots,d\}$. Straightforward calculation shows that indeed

$$\begin{aligned}
[\mathcal{H}\,,\mathsf{vl}(\partial x^k)]=[\mathcal{H}\,,\partial v^{d+k}]&=\sum_{j=1}^d\big[v^{d+j}\partial v^j\,,\partial v^{d+k}\big]-2\big[G^j\partial v^{d+j}\,,\partial v^{d+k}\big]\\
&=\sum_{j=1}^d\underbrace{\frac{\partial v^{d+j}}{\partial v^{d+k}}}_{=\delta_{jk}}\cdot\partial v^j+v^{d+j}\underbrace{\big[\partial v^j\,,\partial v^{d+k}\big]}_{=0}\\
&\qquad-2\bigg(\frac{\partial G^j}{\partial v^{d+k}}\cdot\partial v^{d+j}+G_j\underbrace{\big[\partial v^{d+j}\,,\partial v^{d+k}\big]}_{=0}\bigg)\\
&=\sum_{j=1}^d\delta_{jk}\cdot\partial v^j-2N^j_k\cdot\partial v^{d+j}=\partial v^k-2\sum_{j=1}^d N^j_k\cdot\partial v^{d+j}\\
&=\mathsf{hl}(\partial x^k)-\sum_{j=1}^d N^j_k\cdot\partial v^{d+j}=\mathsf{hl}(\partial x^k)-\sum_{j=1}^d N^j_k\cdot\mathsf{vl}(\partial x^j).
\end{aligned}$$

□

**Proposition 2.1.13** ((D2) for smooth potentials using a hypoellipticity startegy)**.** Let $\Psi\in C^\infty(\mathbb{X})$ be a smooth potential. Then, $(L,D)$ is essentially m-dissipative. Thus, its closure $(L,D(L))$ generates a SCCS.

*Proof.* From Lemma 2.1.5 we know that $(L,D)$ is dissipative. We show that the range $(\mathsf{Id}_H-L)(D)$ is dense in $H$. To this end, let $f\in H$ fixed such that

$$((\mathsf{Id}_H-L)u\,,f)_H=0\qquad\text{for all }u\in D.\tag{2.9}$$

We claim that $f=0$.

Due to the choice of $f$ we have that $\exp(-\Phi^{\mathsf{v}})f\in L^1_{\mathsf{loc}}\big(\mathrm{T}\mathbb{X};\lambda_{\mathsf{x}}\otimes_{\mathsf{loc}}\nu\big)$, thus we can assume that $\exp(-\Phi^{\mathsf{v}})f$ is smooth by Lemma 2.1.11. Consider the sequence $(\varphi_n)_{n\in\mathbb{N}}$ of cut-off functions as in Remark 1.2.33. Now, define $u_n:=(\varphi^{\mathsf{v}}_n)^2f$ for all $n\in\mathbb{N}$. It's clear that

$$(u_n\,,f)_H\overset{(2.9)}{=}(Lu_n\,,f)_H=(Su_n\,,f)_H-(Au_n\,,f)_H.$$

Due to the fact that $(S,D)$ is nonpositive definite and multiplying with $\varphi$ commutes with the action of $S$, we see that $(Su_n\,,f)_H=(S(\varphi^{\mathsf{v}}_nf)\,,\varphi^{\mathsf{v}}_nf)_H\le 0$. Similarly, using antisymmetry we get that

$$(Au_n\,,f)_H=(A(\varphi^{\mathsf{v}}_n)\cdot\varphi^{\mathsf{v}}_nf\,,f)_H+(A(\varphi^{\mathsf{v}}_nf)\,,\varphi^{\mathsf{v}}_nf)_H=(A(\varphi^{\mathsf{v}}_n)\,\varphi^{\mathsf{v}}_nf\,,f)_H.$$

Altogether, this yields the estimate

$$(u_n\,,f)_H = \int_{\mathsf{T}\mathbb{X}} (\varphi_n^{\mathsf{v}})^2 f^2 \,\mathsf{d}\mu \le \frac{1}{n} C \int_{\mathsf{T}\mathbb{X}} \varphi_n^{\mathsf{v}} f^2 \,\mathsf{d}\mu \le \frac{1}{n} C \|f\|_H$$

using (2.8). This implies that $\|f\|_H^2 \le 0$ by dominated convergence, thus $f = 0$. $\square$

For the case of a Lipschitzian potential, we view the logarithmic derivative caused by the base weight as a perturbation, thus we consider the nonweighted position manifold, but keep the fibre weight. In other words, we could think of $\mathsf{T}\mathbb{X}$ endowed with bundle weight $1 \otimes_{\mathsf{loc}} \rho_F$. Let $\Psi \in L^1_{\mathsf{loc}}(\mathbb{X};\mathsf{x})$ and define $L_0 := \frac{\alpha}{\beta}\,\Delta_{\mathsf{v}} + \mathscr{H}_{\mathsf{x}}$ on $D_0$.

**Lemma 2.1.14.** The operator $(L_0, D_0)$ on $L^2(\mathsf{T}\mathbb{X};\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)$ is essentially m-dissipative.

*Proof.* Applying Proposition 2.1.13 to the case of the zero potential, we know that $(L_0, D)$ on the Hilbert space $L^2(\mathsf{T}\mathbb{X};\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)$ is essentially m-dissipative. One has to show that $(L_0, D)$ is contained in the closure of $(L_0, D_0)$.

For any $f \in D$ there is an approximating sequence $(f_n)_{n\in\mathbb{N}}$ in $D_0$ with respect to usual locally convex topology implying uniform convergence of all derivatives on compacts, compare to Lemma 1.2.34. Furthermore, there is a common compact set in $\mathsf{T}\mathbb{X}$ large enough containing $\mathsf{supp}(f)$ and $\mathsf{supp}(f_n)$ for all $n \in \mathbb{N}$. Therefore, we have that $\sup_{v\in\mathsf{T}\mathbb{X}}|L_0 f_n(v) - L_0 f(v)| \to 0$ as $n \to \infty$. Thus, $f_n \to f$ and $L_0 f_n \to L_0 f$ in $L^2(\mathsf{T}\mathbb{X};\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)$ as $n \to \infty$. $\square$

Denote by $H_0^{1,\infty}(\mathbb{X})$ the closure of $C_{\mathsf{c}}^\infty(\mathbb{X})$ with respect to $H^{1,\infty}$-norm and define the set

$$D_1 := \pi_0^* H_0^{1,\infty}(\mathbb{X}) \otimes \kappa^* C_{\mathsf{c}}^\infty(\mathbb{X}).$$

We realise that Lemma 2.1.5 still does apply to the operator $(L, D_1)$ as the proof has to be adapted just slightly. Furthermore, we consider the unitary isomorphism

$$U\colon\; H \to L^2(\mathsf{T}\mathbb{X};\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu),\;\; f \longmapsto \exp\Big(-\frac{1}{2}\Phi\Big)^{\mathsf{v}} \cdot f,$$

note that $U(D_1) = D_1$, and thus define the operator $\widetilde{L} := ULU^{-1}$ on $D_1$. Observe that

$$\widetilde{L} = \widetilde{S} - \widetilde{A} := U\,\Delta_{\mathsf{v}}\,U^{-1} - UAU^{-1} \qquad \text{holds on } D_1$$

with $(\widetilde{S}, D_1)$ being symmetric, nonpositive definite and $(\widetilde{A}, D_1)$ being antisymmetric. Moreover, we find that

$$\begin{aligned}
\widetilde{A} f_0^{\mathsf{v}} &= -U\Big(f_0^{\mathsf{v}} \cdot \mathsf{x}_{\pi_0}\big(\mathsf{Id}_{\mathsf{T}\mathbb{X}}\,,\nabla_{\mathsf{x}} \exp(\Phi/2) \circ \pi_0\big) + \exp(\Phi/2)^{\mathsf{v}} \cdot \mathsf{x}_{\pi_0}\big(\mathsf{Id}_{\mathsf{T}\mathbb{X}}\,,\nabla_{\mathsf{x}} f_0 \circ \pi_0\big)\Big) \\
&= -\frac{1}{2} f_0^{\mathsf{v}} \cdot \mathsf{x}_{\pi_0}\big(\mathsf{Id}_{\mathsf{T}\mathbb{X}}\,,\nabla_{\mathsf{x}} \Phi \circ \pi_0\big) - \mathsf{x}_{\pi_0}\big(\mathsf{Id}_{\mathsf{T}\mathbb{X}}\,,\nabla_{\mathsf{x}} f_0 \circ \pi_0\big) \\
&= -\frac{1}{2} f_0^{\mathsf{v}} \cdot \mathscr{H}_{\mathsf{x}} \Phi^{\mathsf{v}} - \mathscr{H}_{\mathsf{x}} f_0^{\mathsf{v}} \qquad \text{for all } f_0 \in H_0^{1,\infty}(\mathbb{X}).
\end{aligned}$$

For a given function $f = f_0^{\mathsf{v}} \otimes g_0^{\mathsf{h}} \in D_1$ we get that

$$\begin{aligned}
\widetilde{A} f &= -\frac{1}{2} f_0^{\mathsf{v}} g_0^{\mathsf{h}} \cdot \mathscr{H}_{\mathsf{x}} \Phi^{\mathsf{v}} - g_0^{\mathsf{h}} \cdot \mathscr{H}_{\mathsf{x}} f_0^{\mathsf{v}} + f_0^{\mathsf{v}} \cdot \frac{1}{\beta} \big(\nabla_{\mathsf{v}} \Phi^{\mathsf{h}}\big)(g_0^{\mathsf{h}}) \\
&= -\frac{1}{2} f \cdot \mathscr{H}_{\mathsf{x}} \Phi^{\mathsf{v}} - \mathscr{H}_{\mathsf{x}} f + \frac{1}{\beta} \big(\nabla_{\mathsf{v}} \Phi^{\mathsf{h}}\big)(f).
\end{aligned}$$

For sake of completeness, we recall the well-known perturbation theorem on which the argument in the next lemma is based. See [Dav80, Corollary 3.8, Lemma 3.9 and Problem 3.10] for the proof.

**Theorem 2.1.15** (Kato perturbation of an essentially m-dissipative operator)**.** Let an essentially m-dissipative operator $Z$ and a dissipative operator Pert have common domain in some given Hilbert space with norm $\|\cdot\| := \sqrt{(\cdot\,,\cdot)}$. Assume that there are constants $c_1 \in \mathbb{R}$ and $c_2 \in (0,\infty)$ such that

$$\|\mathsf{Pert} f\|^2 \le c_1 (Zf\,,f) + c_2 \|f\|^2$$

holds for all $f$ from the common domain. Then, the perturbated operator $Z + \mathsf{Pert}$ defined on the common domain of $Z$ and Pert is essentially m-dissipative. ⌟

2

**Lemma 2.1.16** (Essential m-dissipativity in case of globally Lipschitzian potentials)**.** Assume that $\Psi$ is globally Lipschitzian. Then, $(\widetilde{L}, D_1)$ is essentially m-dissipative on $L^2(\mathrm{T}\mathbb{X}; \lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)$. Hence, $(L, D_1)$ is essentially m-dissipative on the space $H = L^2(\mathrm{T}\mathbb{X}; \lambda_{\mathbf{x}} \otimes_{\mathsf{loc}} \nu)$.

*Proof.* Define $Z := L_0$ on $D_1$. Then, $(Z, D_1)$ is a dissipative extension of $(Z, D_0)$. Thus, $(Z, D_1)$ is essentially m-dissipative on $L^2(\mathrm{T}\mathbb{X}; \lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)$ by Lemma 2.1.14. Define the perturbation

$$\mathsf{Pert} f := -\frac{1}{\beta}\big(\nabla_{\mathsf{v}} \Phi^{\mathsf{h}}\big)(f) + \frac{1}{2} f \cdot \mathcal{H}_{\mathsf{x}} \Phi^{\mathsf{v}}$$

for all $f = f_0^{\mathsf{v}} \otimes g_0^{\mathsf{h}} \in D_1$. Since by Liouville's Theorem $(\mathcal{H}_{\mathsf{x}}, D_1)$ is antisymmetric and also $(\widetilde{A}, D_1)$ is antisymmetric in $L^2(\mathrm{T}\mathbb{X}; \lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)$, $(\mathsf{Pert}, D_1)$ is antisymmetric as well. Thus, $(\mathsf{Pert}, D_1)$ is dissipative.

Choose $g$ as the preimage of $f$ under the transformation $U$, i. e. $f = Ug$. Using the CSBI we get that

$$\begin{aligned}
&\int_{\mathrm{T}\mathbb{X}} \big(\nabla_{\mathsf{v}} \Phi^{\mathsf{h}}(f)\big)^2 \,\mathrm{d}\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu \\
&= \int_{\mathrm{T}\mathbb{X}} |\mathsf{v}\big(\nabla_{\mathsf{v}} \Phi^{\mathsf{h}}\,, \nabla_{\mathsf{v}} f\big)|^2 \,\mathrm{d}\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu = \int_{\mathrm{T}\mathbb{X}} |\mathsf{v}\big(\nabla_{\mathsf{v}} \Phi^{\mathsf{h}}\,, \nabla_{\mathsf{v}} g\big)|^2 \,\mathrm{d}\lambda_{\mathbf{x}} \otimes_{\mathsf{loc}} \nu \\
&\leq \|\nabla_{\mathsf{v}} \Phi^{\mathsf{h}}\|^2_{L^2(\mathrm{T}\mathbb{X} \to \mathrm{T}^2\mathbb{X}; \lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)} \cdot \int_{\mathrm{T}\mathbb{X}} |\nabla_{\mathsf{v}} g|^2_{\mathsf{v}} \,\mathrm{d}\lambda_{\mathbf{x}} \otimes_{\mathsf{loc}} \nu \\
&= \|\nabla_{\mathsf{x}} \Phi\|^2_{L^2(\mathbb{X} \to \mathrm{T}\mathbb{X}; \mathsf{x})} \cdot \big(-\Delta_{\mathbf{v}} g\,, g\big)_{L^2(\mathrm{T}\mathbb{X}; \lambda_{\mathbf{x}} \otimes_{\mathsf{loc}} \nu)} \\
&= \|\nabla_{\mathsf{x}} \Phi\|^2_{L^2(\mathbb{X} \to \mathrm{T}\mathbb{X}; \mathsf{x})} \cdot \big(-\widetilde{S} f\,, f\big)_{L^2(\mathrm{T}\mathbb{X}; \lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)}.
\end{aligned}$$

We abbreviate $C_\Phi := \|\nabla_{\mathsf{x}} \Phi\|^2_{L^\infty(\mathbb{X} \to \mathrm{T}\mathbb{X}; \mathsf{x})}$. Then, we conclude

$$\begin{aligned}
\|\mathsf{Pert} f\|^2_{L^2(\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)} &\leq \frac{1}{\beta^2} C_\Phi \cdot \big(-\widetilde{S} f\,, f\big)_{L^2(\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)} + \frac{1}{4} C_\Phi \cdot \|f\|^2_{L^2(\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)} \\
&= c_1(-Zf\,, f)_{L^2(\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)} + c_2 \|f\|^2_{L^2(\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)}
\end{aligned}$$

with $c_1 := \frac{2}{\alpha\beta} C_\Phi$ and $c_2 := \frac{1}{4} C_\Phi$, since we know $\big(\mathcal{H}_{\mathsf{x}} f\,, f\big)_{L^2(\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu)} = 0$. Finally, the claim follows by applying Theorem 2.1.15 to $(Z + \mathsf{Pert}, D_1)$. □

**Corollary 2.1.17** ((D2) for globally Lipschitzian potentials)**.** Assume that $\Psi$ is a globally Lipschitzian potential. Then, $(L, D)$ is essentially m-dissipative on $H$.

*Proof.* Note that $(L, D)$ is a dissipative extension of $(L, D_0)$. Thus, we show that $(L, D_1)$ is contained in the closure of $(L, D_0)$ and then apply Lemma 2.1.16.

Let $f = f_0^{\mathsf{v}} \in \pi_0^* H_0^{1,\infty}(\mathbb{X})$, $g \in \kappa^* C_{\mathsf{c}}^\infty(\mathbb{X})$, and a sequence $(f_n)_{n \in \mathbb{N}_+}$ in $C_{\mathsf{c}}^\infty(\mathbb{X})$ such that its vertical lifting approximates $f$ in $H^{1,2}$-sense, i. e.

(i) $f_n \longrightarrow f_0$ as $n \to \infty$ in $L^2(\mathbb{X}; \mathsf{x})$-sense and

(ii) ${}^{\partial f_n}/_{\partial x^j} \longrightarrow {}^{\partial f_0}/_{\partial x^j}$ as $n \to \infty$ in $L^2(\mathbb{X}; \mathsf{x})$-sense for any chart $x = (x^j)_{j=1}^{d}$.

This convergence is maintained under passing to $\pi_0^* L^2(\mathbb{X}; \mathbf{x})$, i. e. weighting the manifold. Finally, we conclude that

$$L(f_n^{\mathsf{v}} \otimes g) \longrightarrow L(f_0^{\mathsf{v}} \otimes g) = L(f \otimes g) \qquad \text{in } H \text{ as } n \to \infty.$$

□

The final proof of this section virtually is the same as in [GS14, Theorem 4.7].

**Proposition 2.1.18** ((D2) for locally Lipschitzian potentials)**.** Let $\Psi$ be a loc-Lipschitzian potential bounded from below. Then, $(L, D)$ is essentially m-dissipative on $H$.

*Proof.* Without loss of generality, we assume that $\Psi \geq 0$. Let $\varepsilon \in (0,\infty)$ and fix some $g \in D \setminus \{0\}$. Choose $\varphi, \psi \in D$ such that

$$\varphi|_{\mathsf{supp}(g)} = 1, \quad \psi|_{\mathsf{supp}(\varphi)} = 1 \quad \text{and} \quad 0 \leq \varphi \leq \psi \leq 1.$$

Let $f \in D$ arbitrary. Throughout the proof, we add to the generators and invariant measures a subscript to indicate the corresponding potential, e. g. $\mu_0 = \lambda_{\mathsf{t}}$ in case of the zero potential. By construction and using dissipativity of $(L_{\psi\Psi}, D)$ on $L^2(\mathsf{T}\mathbb{X}; \mu_{\psi\Psi})$ we get that

$$\begin{aligned}
&\|(\mathsf{Id}_H - L_\Psi)(\varphi f) - g\|_{L^2(\mu_\Psi)} \\
&\leq \|\varphi\big((\mathsf{Id}_H - L_{\psi\Psi})f - g\big)\|_{L^2(\mu_{\psi\Psi})} + \|f\|_{L^2(\mu_{\psi\Psi})} \cdot \|\nabla_{\mathsf{x}}\varphi\|_{L^\infty(\lambda_{\mathsf{x}})} \\
&\leq \|(\mathsf{Id}_H - L_{\psi\Psi})f - g\|_{L^2(\mu_{\psi\Psi})} + \|(\mathsf{Id}_H - L_{\psi\Psi})f\|_{L^2(\mu_{\psi\Psi})} \cdot \|\nabla_{\mathsf{x}}\varphi\|_{L^\infty(\lambda_{\mathsf{x}})}.
\end{aligned}$$

Now, we tighten the requirements on $\varphi$ via additionally demanding that

$$\|\nabla_{\mathsf{x}}\varphi\|_{L^\infty(\mathbb{X}\to\mathsf{T}\mathbb{X};\mathsf{x})} = \|\nabla_{\mathsf{x}}\varphi\|_{L^\infty(\mathsf{x})} < \frac{\varepsilon}{4} \cdot \|g\|^{-1}_{L^2(\mu_0)} = \frac{\varepsilon}{4} \cdot \|g\|^{-1}_{L^2(\mathsf{t})}.$$

Due to Corollary 2.1.17 $(L_{\psi\Psi}, D)$ is essentially m-dissipative on $L^2(\mathsf{T}\mathbb{X}; \mu_{\psi\Psi})$, hence as a consequence of the Lumer-Philips Theorem, there is $f \in D$ such that simultaneously hold

$$\|(\mathsf{Id}_H - L_{\psi\Psi})f - g\|_{L^2(\mu_{\psi\Psi})} \leq \frac{\varepsilon}{2} \quad \text{and} \quad \|(\mathsf{Id}_H - L_{\psi\Psi})f\|_{L^2(\mu_{\psi\Psi})} \leq 2\|g\|_{L^2(\mu_{\psi\Psi})}.$$

For such an $f$ we end up with $\|(\mathsf{Id}_H - L_\Psi)(\varphi f) - g\|_{L^2(\mu_{\psi\Psi})} < \varepsilon$. In conclusion, we proved that $(\mathsf{Id}_H - L_\Psi)(D)$ is dense in $H = L^2(\mathsf{T}\mathbb{X}; \mu_\Psi)$. □

## Hypocoercivity conditions

In the course of verifying the set of conditions (H) the most demanding task is to derive an explicit expression for the operator $PA^2P$ on $D$. We need this to prove (H3) by means of [GS14, Corollary 2.13]. This requires a few auxiliary results that I did not find in the literature. Regarding (H4), one explicitly uses that $\mathscr{H}_{\mathsf{x}}$ is a full spray, not just a semispray.

**Lemma 2.1.19** (algebraic relation (H1)). Let condition (P1) hold. Then, we have $PAP|_D = 0$.

*Proof.* Recall Equation (2.8). Furthermore, we are going to apply the formula for Gaußian integrals from [GS14, Lemma 3.1]: Let $f \in D$ and consider polar coordinates in the fibre at $x \in \mathbb{X}$. Using [For12, Satz 14.8], which is an application of the transformation formula and Fubini, we get that

$$\begin{aligned}
&\int_{\mathsf{T}_x\mathbb{X}} APf \, \mathsf{d}\nu_x \\
&= \int_{(0,\infty)} \int_{\mathsf{U}_x\mathbb{X}} -\mathsf{x}_x\Big(v(r,u)\,,\, \nabla_{\mathsf{x}}\mathsf{E}_\nu[f](x)\Big) \cdot r^{d-1}\, \frac{d\nu(v(r,u))}{d\lambda \otimes \mathsf{S}} \quad \mathsf{S}_x(\mathsf{d}u)\, \lambda(\mathsf{d}r) \\
&= \int_{(0,\infty)} 0 \cdot r^{d-1}\, \frac{d\nu(v(r,u_0))}{d\lambda \otimes \mathsf{S}} \; \lambda(\mathsf{d}r) = 0,
\end{aligned}$$

where $u_0 \in \mathsf{U}_x\mathbb{X}$ is arbitrary, $\mathsf{S}$ denotes the surface measure of the sphere $\mathbb{S}^{d-1}$ and

$$v\colon\; (0,\infty) \times \mathbb{S}^{d-1} \to v((0,\infty) \times \mathbb{S}^{d-1}) \subseteq \mathsf{T}_x\mathbb{X},\; (r,u) \longmapsto v(r,u)$$

is the diffeomorphism corresponding to $(0,\infty) \times \mathbb{S}^{d-1} \to \mathbb{R}^d$, $(r,u) \mapsto ru$. We can freely choose $u_0$, since $\nu$ is invariant with respect to rotations. We have seen that $P_S AP$ is trivial an $D$, so we can use orthogonality of $P_S$ to obtain

$$0 = (P_S APf\,,1)_{L^2(\mathbb{X};\mathsf{x})} = (APf\,,1)_H \qquad \text{for all } f \in D.$$

Thus, $PAP|_D = 0$. □

**Lemma 2.1.20** (microscopic hypocoercivity (H2)). Let condition (P1) hold. Then, condition (H2) is fulfilled with $\Lambda_m = \alpha$.

*Proof.* Let $f \in D$. Using the Poincaré inequality for Gaußian measures, see [Bec89], we deduce that

$$\begin{aligned}(-Sf\,,f)_H &= \frac{\alpha}{\beta}\Big(\nabla_{\mathsf{v}} f\,,\nabla_{\mathsf{v}} f\Big)_{L^2(\mathrm{T}\mathbb{X}\to\mathrm{T}^2\mathbb{X};\mathbf{t})} = \frac{\alpha}{\beta}\,\|\nabla_{\mathsf{v}} f\|^2_{L^2(\mathrm{T}\mathbb{X}\to\mathrm{T}^2\mathbb{X};\mathbf{t})}\\ &\geq \alpha\left\| f - \left(v\mapsto \int_{\mathrm{T}_{\pi_0(v)}\mathbb{X}} f\;\mathrm{d}\nu\right)\right\|_H^2 = \alpha\|(\mathsf{Id}_H - P_S)f\|_H^2\end{aligned}$$

and the claim follows. □

Moving on, the strategy for proving condition (H3) relies on [GS14, Corollary 2.13]. Most importantly, one has to prove that $(\mathsf{Id}_H - PA^2P, D)$ is essentially m-dissipative. To do so, we characterise $\mathsf{Id}_H - PA^2P$ on $D$ starting from Equation (2.8) and show that the range $(\mathsf{Id}_H - PA^2P)(D)$ is dense in $H$. Before we calculate that in several smaller steps that $(\mathsf{Id}_H - PA^2P, D)$ actually is the weighted horizontal Laplace-Beltrami operator, we insert an auxiliary lemma.

**Lemma 2.1.21.** For all $\mathscr{X} \in \Gamma^\infty(\mathrm{T}\mathbb{X})$ holds that

$$\mathsf{h}\Big(\nabla^{\mathsf{t}}_{\mathscr{H}_{\mathsf{x}}}\,\mathsf{hl}(\mathscr{X})\,,\mathscr{H}_{\mathsf{x}}\Big) = \mathsf{x}_{\pi_0}\Big(\nabla^{\mathsf{x}}_{\mathsf{Id}_{\mathrm{T}\mathbb{X}}}(\mathscr{X}\circ\pi_0)\,,\mathsf{Id}_{\mathrm{T}\mathbb{X}}\Big).$$

*Proof.* Let $\mathscr{X} \in \Gamma^\infty(\mathrm{T}\mathbb{X})$. Then, the Koszul formula, see [Nic96, Equation 4.1.3], reads as

$$\begin{aligned}&2\,\mathsf{h}\Big(\nabla^{\mathsf{t}}_{\mathscr{H}_{\mathsf{x}}}\,\mathsf{hl}(\mathscr{X})\,,\mathscr{H}_{\mathsf{x}}\Big)\\ &= \mathscr{H}_{\mathsf{x}}\Big(\mathsf{h}\Big(\mathsf{hl}(\mathscr{X})\,,\mathscr{H}_{\mathsf{x}}\Big)\Big) + \mathsf{hl}(\mathscr{X})\Big(\mathsf{h}\Big(\mathscr{H}_{\mathsf{x}}\,,\mathscr{H}_{\mathsf{x}}\Big)\Big)\\ &\qquad - \mathscr{H}_{\mathsf{x}}\Big(\mathsf{h}\Big(\mathscr{H}_{\mathsf{x}}\,,\mathsf{hl}(\mathscr{X})\Big)\Big) - \mathsf{h}\Big(\mathsf{hl}(\mathscr{X})\,,\Big[\mathscr{H}_{\mathsf{x}}\,,\mathscr{H}_{\mathsf{x}}\Big]\Big) - \mathsf{h}\Big(\mathscr{H}_{\mathsf{x}}\,,\Big[\mathsf{hl}(\mathscr{X})\,,\mathscr{H}_{\mathsf{x}}\Big]\Big)\\ &\qquad + \mathsf{h}\Big(\mathscr{H}_{\mathsf{x}}\,,\Big[\mathscr{H}_{\mathsf{x}}\,,\mathsf{hl}(\mathscr{X})\Big]\Big)\\ &= \mathsf{hl}(\mathscr{X})\Big(|\mathsf{Id}_{\mathrm{T}\mathbb{X}}|^2_{\mathsf{x}}\Big) + 2\mathsf{h}\Big(\mathscr{H}_{\mathsf{x}}\,,\Big[\mathscr{H}_{\mathsf{x}}\,,\mathsf{hl}(\mathscr{X})\Big]\Big)\\ &= \mathsf{h}\Big(\mathsf{hl}(\mathscr{X})\,,\nabla_{\mathsf{h}}\Big(|\mathsf{Id}_{\mathrm{T}\mathbb{X}}|^2_{\mathsf{x}}\Big)\Big) + 2\mathsf{h}\Big(\mathscr{H}_{\mathsf{x}}\,,\Big[\mathscr{H}_{\mathsf{x}}\,,\mathsf{hl}(\mathscr{X})\Big]\Big).\end{aligned}$$

First, we note that

$$d\pi_0\Big[\mathscr{H}_{\mathsf{x}}\,,\mathsf{hl}(\mathscr{X})\Big] = \Big[d\pi_0\mathscr{H}_{\mathsf{x}}\,,d\pi_0\,\mathsf{hl}(\mathscr{X})\Big] = \Big[\mathsf{Id}_{\mathrm{T}\mathbb{X}}\,,\mathscr{X}\circ\pi_0\Big] = \nabla^{\mathsf{t}}_{\mathsf{Id}_{\mathrm{T}\mathbb{X}}}(\mathscr{X}\circ\pi_0).$$

Second, the value of $\Big|\mathsf{Id}_{\mathrm{T}\mathbb{X}}\Big|^2_{\mathsf{x}}$ does not specifically depend on the current position and therefore it could be approximated just by functions from $\kappa^* C^\infty_{\mathsf{c}}(\mathbb{X})$. For all intents and purposes, this function can be treated as a horizontal lift and the horizontal gradient of a horizontal lift equals 0 always. Hence, the claim is proven. □

Consider a curve $s\colon (-\delta,\delta)\to\mathrm{T}\mathbb{X}$ such that $s(0)=v$ and $s'(0)=\mathscr{H}_{\mathsf{x}}(v)$ for $v\in\mathrm{T}\mathbb{X}$ fixed and some small $\delta\in(0,\infty)$. Let $x := \pi_0(v)$. The following computation relies on $\pi_0\circ s$ being a geodesic of $\mathscr{H}_{\mathsf{x}}$ and the characterisation of the directional derivative in terms of parallel transport 'pt' along $s$ which is provided by the Levi-Civita connection:

$$\begin{aligned}-\mathscr{H}_{\mathsf{x}}APf(v) &\overset{(2.8)}{=} -\mathscr{H}_{\mathsf{x}}\Big(-\mathsf{x}_{\pi_0}\Big(\mathsf{Id}_{\mathrm{T}\mathbb{X}}\,,\nabla_{\mathsf{x}}\mathsf{E}_\nu[f]\circ\pi_0\Big)\Big)(v)\\ &= \mathsf{t}_v\Big(\lim_{t\to0}\frac{1}{t}\Big(\mathsf{pt}^{s(t)}_{s(t)}(\mathsf{Id}_{\mathrm{T}\mathbb{X}}\circ s(t)) - \mathsf{pt}^{s(t)}_{s(0)}(\mathsf{Id}_{\mathrm{T}\mathbb{X}}\circ s(0))\Big)\,,\nabla_{\mathsf{h}}P_Sf(v)\Big)\\ &\qquad\qquad + \mathsf{h}_v\Big(\mathscr{H}_{\mathsf{x}}(v)\,,\nabla^{\mathsf{t}}_{\mathscr{H}_{\mathsf{x}}}\Big(\nabla_{\mathsf{h}}P_Sf\Big)(v)\Big)\\ &= \mathsf{h}_v\Big(\mathscr{H}_{\mathsf{x}}(v)\,,\mathsf{E}_\nu\Big[\nabla_{\mathsf{h}}f\Big](v)\Big) + \mathsf{x}_x\Big(v\,,\nabla^{\mathsf{x}}_v\Big(\nabla_{\mathsf{x}}\mathsf{E}_\nu[f]\Big)(x)\Big)\\ &= \mathsf{x}_x\Big(v\,,\nabla_{\mathsf{x}}\mathsf{E}_\nu[f](x)\Big) + \mathsf{x}_x\Big(v\,,\nabla^{\mathsf{x}}_v\Big(\nabla_{\mathsf{x}}\mathsf{E}_\nu[f]\Big)(x)\Big).\end{aligned}\tag{2.10}$$

For the second equality we also used the metric compatibility of the Levi-Civita connection $\nabla^{\mathsf{t}}$. The last but one line is obtained applying Lemma 2.1.21 to the second summand. Now, we transform into polar coordinates in the fibre $\mathsf{T}_x\mathbb{X}$ similar as in the proof of Lemma 2.1.19. With this ansatz we calculate applying [GS14, Lemma 3.1] twice that

$$
\begin{aligned}
&\int_{\mathsf{T}_x\mathbb{X}} -\mathcal{H}_{\mathsf{x}}\Big(-\mathsf{x}_{\pi_0}\Big(\mathsf{Id}_{\mathsf{T}\mathbb{X}}\,,\nabla_{\mathsf{x}}\mathsf{E}_\nu[f]\circ\pi_0\Big)\Big)\;\mathsf{d}\nu_x\\
&=\int_{\mathsf{T}_x\mathbb{X}} \mathsf{x}_x\Big(v\,,\nabla_{\mathsf{x}}\mathsf{E}_\nu[f](x)\Big)\;\mathsf{d}\nu_x(v)+\int_{\mathsf{T}_x\mathbb{X}} \mathsf{x}_x\Big(v\,,\nabla^{\mathsf{x}}_v\Big(\nabla_{\mathsf{x}}\mathsf{E}_\nu[f]\Big)(x)\Big)\;\mathsf{d}\nu_x(v)\\
&=\int_{(0,\infty)}\int_{\mathsf{U}_x\mathbb{X}} \mathsf{x}_x\Big(v(r,u)\,,\nabla^{\mathsf{x}}_{v(r,u)}\Big(\nabla_{\mathsf{x}}\mathsf{E}_\nu[f]\Big)(x)\Big)\\
&\qquad\qquad\cdot r^{d-1}\,\frac{d\nu_x(v(r,u))}{d\lambda\otimes\mathsf{S}_x}\quad \mathsf{dS}_x(u)\,\lambda(\mathsf{d}r)\\
&=\frac{1}{d}\,\Delta_{\mathsf{x}}\mathsf{E}_\nu[f](x)\quad\cdot\int_{(0,\infty)} r^2\cdot\left(\frac{1}{\mathsf{S}(\mathbb{S}^{d-1})}\int_{\mathsf{U}_x\mathbb{X}}1\;\mathsf{dS}_x\right)\cdot r^{d-1}\,\frac{d\nu_x(v(r,u_0))}{d\lambda\otimes\mathsf{S}_x}\;\lambda(\mathsf{d}r)\\
&=\frac{1}{\beta}\,\Delta_{\mathsf{x}}\mathsf{E}_\nu[f](x)\quad\cdot 1,
\end{aligned}
\tag{2.11}
$$

where we have taken $u_0\in\mathsf{U}_x\mathbb{X}$ arbitrary, since $\nu_x$ is invariant with respect to rotations. In order to arrive at the last but one line, we consider some chart $\left(x^j\right)_{j=1}^d$ at $x\in\mathbb{X}$ providing normal coordinates; in such coordinates the Levi-Civita connection is understood in terms of directional derivatives as $\nabla^{\mathsf{x}}_{\mathcal{Y}}\mathcal{X}(x)=\sum_{i,j\in\{1,\ldots,d\}}\mathcal{Y}^i(x)\,\frac{\partial\mathcal{X}^j(x)}{\partial x^i}\,x^j$ for all $\mathcal{X},\mathcal{Y}\in\Gamma^\infty(\mathsf{T}\mathbb{X})$ with local coordinate expressions $\mathcal{X}=\sum_{j=1}^d\mathcal{X}^j\cdot\partial x^j$ and $\mathcal{Y}=\sum_{j=1}^d\mathcal{Y}^j\cdot\partial x^j$. Thus, we can understand the mapping $\mathsf{U}_x\mathbb{X}\to\mathsf{U}_x\mathbb{X}$, $u\longmapsto\left(\nabla^{\mathsf{x}}_u\nabla_{\mathsf{x}}\mathsf{E}[f]\right)(x)$ as the matrix in [GS14, Lemma 3.1]. The last step of Equation (2.11) is due to the fact that the mean of a chi-squared distribution equals the number of degrees of freedom, i. e. $d$ in the present case.

With the proof of part (ii) of Lemma 2.1.5 and with [GS14, Lemma 3.1] we similarly get that for every $v\in\mathsf{T}\mathbb{X}$ with $x:=\pi_0(v)$ holds

$$
\begin{aligned}
P_S\Big(\frac{1}{\beta}\,\nabla_{\mathsf{v}}\Phi^{\mathsf{h}}(APf)\Big)(v)&=\int_{\mathsf{T}_x\mathbb{X}}\mathcal{H}_{\mathsf{x}}\Phi^{\mathsf{v}}\cdot APf\;\mathsf{d}\nu_x\\
&\overset{(2.8)}{=}-\int_{\mathsf{T}_x\mathbb{X}}\mathsf{h}\Big(\mathcal{H}_{\mathsf{x}}\,,\nabla_{\mathsf{h}}\Phi^{\mathsf{v}}\Big)\cdot\mathsf{h}\Big(\mathcal{H}_{\mathsf{x}}\,,\nabla_{\mathsf{h}}(P_Sf)\Big)\;\mathsf{d}\nu_x\\
&=-\int_{\mathsf{T}_x\mathbb{X}}\mathsf{x}_x\Big(\mathsf{Id}_{\mathsf{T}\mathbb{X}}\,,\nabla_{\mathsf{x}}\Phi(x)\Big)\cdot\mathsf{x}_x\Big(\mathsf{Id}_{\mathsf{T}\mathbb{X}}\,,\nabla_{\mathsf{x}}\mathsf{E}_\nu[f](x)\Big)\;\mathsf{d}\nu_x\\
&=-\frac{1}{\beta}\,\mathsf{x}_x\Big(\nabla_{\mathsf{x}}\Phi(x)\,,\nabla_{\mathsf{x}}\mathsf{E}_\nu[f](x)\Big)=-\frac{1}{\beta}\,\partial_{\nabla_{\mathsf{x}}\Phi}(\mathsf{E}_\nu[f])(x).
\end{aligned}
\tag{2.12}
$$

Combining Equation (2.11) and Equation (2.12) we conclude that

$$
\begin{aligned}
PA^2Pf=P_SA^2Pf&=\frac{1}{\beta}\cdot\Big(\Delta_{\mathsf{x}}\mathsf{E}_\nu[f]\circ\pi_0-\partial_{\nabla_{\mathsf{x}}\Psi}(\mathsf{E}_\nu[f])\circ\pi_0\Big)\\
&=\frac{1}{\beta}\cdot\Delta_{\mathsf{h}}(\mathsf{E}_\nu[f]^{\mathsf{v}})=\frac{1}{\beta}\cdot\Delta_{\mathsf{h}}(P_Sf)
\end{aligned}
\tag{2.13}
$$

for all $f\in D$. Compare our result to [GS16, Equation (3.16)]. These preparations give shape to the following corollary, compare to [GS16, Proposition 3.9].

**Corollary 2.1.22** ($PA^2P$ is essentially m-dissipative). Let condition (P1) hold. Then, the range $(\mathsf{Id}_H-PA^2P)(D)$ is dense in $H$, thus $PA^2P$ is essentially m-dissipative on $D$.

*Proof.* Right away, we know that $(PA^2P,D)\overset{(2.13)}{=}\left({}^1\!/_\beta\,\Delta_{\mathsf{h}}\circ P_S,D\right)$ is essentially m-dissipative on $P_S(H)=\pi_0^*L^2(\mathbb{X};\mathbf{x})$, as $P_S(D)=\pi_0^*C_c^\infty(\mathbb{X})$ and $(\Delta_{\mathsf{h}},D)$ is essentially self-adjoint in $H$. The later is true, since $(\Delta_{\mathsf{h}},D_0)$ is essentially self-adjoint in $H$ as so is the Laplace-Beltrami on $(\mathsf{T}\mathbb{X},\mathbf{h})$, and due to the fact that $D_0$ is dense in $D$.

Let $g \in H$ such that

$$((\mathsf{Id}_H - PA^2P)f\,,g)_H = 0$$

for all $f \in D$ and we claim that $g = 0$. The assumption on $g$ immediately implies that

$$0 = ((\mathsf{Id}_H - PA^2P)f_0^{\mathsf{v}}\,, P_S g)_H = \left(f_0^{\mathsf{v}} - \frac{1}{\beta}\,\Delta_{\mathsf{h}} f_0^{\mathsf{v}}\,, P_S g\right)_{\pi_0^* L^2(\mathbb{X};\mathbf{x})}$$

for all $f_0 \in C_{\mathsf{c}}^\infty(\mathbb{X})$. Thus, $P_S g = 0$, since the range $\left(\mathsf{Id}_H - {}^1\!/\!_\beta\,\Delta_{\mathsf{h}}\right)(\pi_0^* C_{\mathsf{c}}^\infty(\mathbb{X}))$ is dense in $\pi_0^* L^2(\mathbb{X};\mathbf{x})$. Ultimately, this means that

$$(f\,,g)_H = (PA^2Pf\,,g)_H = \left({}^1\!/\!_\beta\,\Delta_{\mathsf{h}}((P_S f)\,, P_S g\right)_{\pi_0^* L^2(\mathbb{X};\mathbf{x})} = 0 \qquad \text{for all } f \in D,$$

which implies $(f\,,g)_H = 0$ for all $f \in D$, hence $g = 0$ as claimed. □

Finally, we are in the position to verify (H3).

**Proposition 2.1.23** (macroscopic hypocoercivity (H3)). Assume that the conditions (P1) and (P2) hold. Then, condition (H3) is fulfilled with $\Lambda_M = \frac{1}{\beta}\Lambda$.

*Proof.* Let $f \in D$. Since $(PA^2P, D) = \left({}^1\!/\!_\beta\,\Delta_{\mathsf{h}} P_S, D\right)$ pregenerates the weighted horizontal gradient form restricted to the range of $P_S$ in the sense that

$$(PA^2Pf\,,g)_H = -\frac{1}{\beta}\int_{\mathsf{T}\mathbb{X}} \mathsf{h}\left(\nabla_{\mathsf{h}} P_S f\,, \nabla_{\mathsf{h}} P_S g\right)\,\mathsf{d}\mu \qquad \text{for all } f, g \in D,$$

we easily compute that

$$\begin{aligned}\|APf\|_H^2 &= \frac{1}{\beta}\int_{\mathsf{T}\mathbb{X}} \left|\nabla_{\mathsf{h}} P_S f(v)\right|_{\mathsf{h}}^2 \quad \mu(\mathsf{d}v) = \frac{1}{\beta}\int_{\mathbb{X}} \left|\nabla_{\mathsf{x}} \mathsf{E}_\nu[f](x)\right|_{\mathsf{x}}^2 \quad \lambda_{\mathbf{x}}(\mathsf{d}x)\\ &= \frac{1}{\beta}\left\|\nabla_{\mathsf{x}}\mathsf{E}_\nu[f]\right\|_{L^2(\mathbb{X}\to\mathsf{T}\mathbb{X};\mathbf{x})}^2 \geq \frac{1}{\beta}\Lambda\left\|\mathsf{E}_\nu[f] - (\mathsf{E}_\nu[f]\,,1)_{L^2(\mathbb{X};\mathbf{x})}\right\|_{L^2(\mathbb{X};\mathbf{x})}^2\end{aligned}$$

by Poincaré inequality (2.4). Combining this estimates with the previous corollary, then [GS14, Corollary 2.13] finishes the proof. □

The remaining hypocoercivity condition (H4) is checked via a standard procedure relying on [GS14, Lemma 2.14] and [GS14, Proposition 2.15], compare to also [GS16, Proposition 3.11].

**Lemma 2.1.24** (boundedness of $(BS, D)$, first part of (H4)). Let condition (P1) hold. Then, with $c_1 := \frac{1}{2}\alpha$ it holds that

$$\|BSf\|_H \leq c_1\|(\mathsf{Id}_H - P_j)f\|_H \qquad \text{for all } f \in D$$

and $P_j \in \{P, P_S\}$, $j \in \{1, 2\}$.

*Proof.* First, we show that $SAP = \alpha AP$ on $D$. For the first time, it will become important here that $\mathcal{H}_{\mathsf{x}}$ is not just a semispray, but actually a spray, i. e. additionally we have that $\left[\mathcal{V}\,, \mathcal{H}_{\mathsf{x}}\right] = \mathcal{H}_{\mathsf{x}}$. Let $f \in D$ be fixed. Then, we immediately get that

$$SAPf = SAP_S f = -\alpha\mathcal{V}(-\mathcal{H}_{\mathbf{x}}(P_S f)) = \alpha\mathcal{V}\left(\mathcal{H}_{\mathsf{x}}(\mathsf{E}_\nu[f])^{\mathsf{v}}\right),$$

since we know from the proof of Lemma 2.1.5 that $(Ah\,,1)_H = 0$ for all $h \in D$. Using the Koszul formula, see [Nic96, Equation 4.1.3], we calculate that for all $\mathcal{X} \in \Gamma^\infty(\mathsf{T}\mathbb{X})$ holds

$$\mathcal{V}\mathsf{h}\left(\mathcal{H}_{\mathsf{x}}\,, \mathsf{hl}\,\mathcal{X}\right) = 0 - \mathsf{h}\left(\left[\mathcal{V}\,, \mathcal{H}_{\mathsf{x}}\right], \mathsf{hl}\,\mathcal{X}\right) + \mathsf{h}\left([\mathcal{V}\,, \mathsf{hl}\,\mathcal{X}]\,, \mathcal{H}_{\mathsf{x}}\right).$$

Similar to [GK02, Proposition 5.1] mentioned before in Lemma 2.1.11, one could use local coordinates for $\mathcal{V}$ in order to show that $[\mathcal{V}\,, \mathsf{hl}\,\mathcal{X}]$ is purely vertical. As $\mathcal{H}_{\mathsf{x}}$ even is a (full) spray, we gain that

$$\begin{aligned}SAPf &= \alpha\mathcal{V}\mathsf{h}\left(\mathcal{H}_{\mathsf{x}}\,, \nabla_{\mathsf{h}}(\mathsf{E}_\nu[f])^{\mathsf{v}}\right) = -\alpha\mathsf{h}\left(\mathcal{H}_{\mathsf{x}}\,, \nabla_{\mathsf{h}}(\mathsf{E}_\nu[f])^{\mathsf{v}}\right)\\ &= -\alpha\mathcal{H}_{\mathsf{x}}(\mathsf{E}_\nu[f])^{\mathsf{v}} = \alpha(-\mathcal{H}_{\mathbf{x}}(\mathsf{E}_\nu[f])^{\mathsf{v}}) = \alpha APf\end{aligned}$$

Setting $c_1 := \frac{\alpha}{2}$ the claim follows with [GS14, Lemma 2.14]. □

**Lemma 2.1.25** (boundedness of $(BA(\mathsf{Id}_H - P), D)$, second part of (H4))**.** Let the potential conditions (P) hold. Then, there exists a constant $c_2 \in (0, \infty)$ such that

$$\|BA(\mathsf{Id}_H - P)f\|_H \le c_2 \|(\mathsf{Id}_H - P_j)f\|_H \qquad \text{for all } f \in D$$

and $P_j \in \{P, P_S\}$, $j \in \{1, 2\}$.

*Proof.* Let $f \in D$ and $g := (\mathsf{Id}_H - PA^2P)f$. We know that $g \in D((BA)^*)$ with $(BA)^* g = -A^2Pf$, compare to [GS14, Proposition 2.15]. Using our knowledge from the proof of part (ii) of Lemma 2.1.5, furthermore Equation (2.8) and the CSBI we estimate that

$$\begin{aligned}
\|(BA)^* g\|_H &\le \Big\| v \longmapsto \mathcal{H}_{\mathsf{x}}{}^2(\mathsf{E}_\nu[f]^{\mathsf{v}})(v) \Big\|_H + \Big\| v \longmapsto \frac{1}{\beta}\, \nabla_{\mathsf{v}} \Phi^{\mathsf{h}}(APf)(v) \Big\|_H \\
&\le \Big\| \big\| \mathcal{H}_{\mathsf{x}}{}^2(\mathsf{E}[f]^{\mathsf{v}}) \big\| \Big\|_H + \Big\| \big|\mathcal{H}_{\mathsf{x}} \Phi^{\mathsf{v}}\big|_{\mathsf{h}} \cdot \big|\mathcal{H}_{\mathsf{x}}(\mathsf{E}_\nu[f]^{\mathsf{v}})\big|_{\mathsf{h}} \Big\|_H \\
&\le \Big\| v \mapsto |v|_{\mathsf{x}}^2 \Big\|_H \cdot \left( \Big\| \big\| \mathsf{Hess}_{\mathsf{t}}(\mathsf{E}_\nu[f]^{\mathsf{v}}) \big\| \Big\|_H + \Big\| \big|\nabla_{\mathsf{h}} \Phi^{\mathsf{v}}\big|_{\mathsf{h}} \cdot \big|\nabla_{\mathsf{h}} \mathsf{E}_\nu[f]^{\mathsf{v}}\big|_{\mathsf{h}} \Big\|_H \right) \\
&= \frac{d}{\beta} \left( \Big\| \big\| \mathsf{Hess}_{\mathsf{x}}(\mathsf{E}_\nu[f]) \big|^{\mathsf{v}} \Big\|_H + \Big\| \big|\nabla_{\mathsf{x}} \Phi\big|_{\mathsf{x}}^{\mathsf{v}} \cdot \big|\nabla_{\mathsf{x}} \mathsf{E}_\nu[f]\big|_{\mathsf{x}}^{\mathsf{v}} \Big\|_H \right).
\end{aligned}$$

Due to the form of $PA^2P$ we derived in Equation (2.13), we know that $u := Pf$ solves the elliptic equation

$$\begin{aligned}
&u - \frac{1}{\beta}\, \Delta_{\mathbf{h}} P_S u = g \\
\text{in} \qquad &\Big\{ u \in \pi_0^* L^2(\mathbb{X}; \mathbf{x}) \;\Big|\; \exists\, f_0 \in C_{\mathsf{c}}^\infty(\mathbb{X}) \colon\; u = f_0^{\mathsf{v}} - (f_0^{\mathsf{v}}, 1)_H \Big\}.
\end{aligned}$$

As we assumed the necessary potential conditions, the *a priori* estimates of Dolbeault, Mouhot and Schmeiser, compare to [GS14, Appendix], yield existence of a constant $c_2 \in (0, \infty)$ independent of $Pf$ and $g$ such that

$$\|(BA)^* g\|_H \le c_2 \cdot \|Pg\|_H \le c_2 \cdot \|g\|_H.$$

Now, [GS14, Propositions 2.15] does apply which finishes the proof. □

Collecting all the individual results so far we can infer Theorem 2.1.4 using the Hypocoercivity Theorem.

## 2.2 ▪ Geometric spherical velocity Langevin equation

As said in the introduction, we suppose that fibre filaments are both inextensible and extruded with constant speed, therefore we assume that velocities are normalised. See e. g. [KMW12a; GS13; Sti14] and references therein. Incorporating the 'polynomial-type' normalisation assumption on the velocities, one could think the fibre lay-down model as an SDAE on $\mathsf{T}\mathbb{X}$. However, we implement the algebraic side condition geometrically via passing from the tangent bundle to the unit tangent bundle $\pi_{0|\mathsf{U}} \colon \mathsf{U}\mathbb{X} \to \mathbb{X}$.

As the unit tangent bundle is a smooth subbundle and in particular a smooth submanifold, we have modify several objects used before in Section 2.1. By the choice of the Sasaki metric on $\mathsf{T}\mathbb{X}$, the normal bundle of $\mathsf{U}\mathbb{X}$, which is the quotient bundle $\mathsf{T}^2\mathbb{X}|_{\mathsf{U}\mathbb{X}}/\mathsf{TU}\mathbb{X}$ in the first place, can be realised as the orthogonal complement $\mathsf{NU}\mathbb{X} := \mathsf{TU}\mathbb{X}^\perp$ of $\mathsf{TU}\mathbb{X}$ with respect to Sasaki metric. Whilst the horizontal lift of unit tangent vector is again tangent to the unit tangent bundle, the same doesn't hold true for the vertical lift: Given $v \in \mathsf{U}\mathbb{X}$ and a vertical vector $a \in \mathsf{T}_v\mathsf{U}\mathbb{X}$, there is a $w \in \mathsf{T}_{\pi_0(v)}\mathbb{X}$ such that $a = \mathsf{vl}_v(w)$, but in general there is no vector field $\mathcal{X} \in \Gamma^\infty(\mathsf{T}\mathbb{X})$ – not even a local one – such that both $\mathsf{vl}_v(\mathcal{X}) = a$ and $\mathsf{vl}(\mathcal{X})\big|_{\mathsf{U}\mathbb{X}} \in \Gamma^\infty(\mathsf{TU}\mathbb{X})$ hold. Thus, one adapts the vertical lift slightly to guarantee that the lift of elements in $\mathsf{T}_{\pi_0(v)}\mathbb{X}$ are elements of $\mathsf{T}_v\mathsf{U}\mathbb{X}$. I first stumbled upon this phenomenon reading the diploma thesis [Fáb11, Abschnitt 2.1] which bases its discussion on the articles [BVA97; BV01]. The latter provide the following definition of so-called *tangential lifts*[2].

[2] This name seems highly unfortunate, it's established anyways.

2

**Definition 2.2.1** (tangential lift). Let $x \in \mathbb{X}$, $v \in \mathsf{T}_x\mathbb{X}$ and $u \in \mathsf{U}_x\mathbb{X}$. The *tangent lift of* $v$ is defined as

$$\mathsf{tl}_u(v) := \mathsf{vl}_u(v - \mathsf{x}(v\,,u)\cdot u) = \mathsf{vl}_u(v) - \mathsf{x}(v\,,u)\cdot \mathcal{N}_{\mathsf{t}}(u),$$

where the unit normal vector field $\mathcal{N}_{\mathsf{t}} \in \Gamma^\infty(\mathsf{NU}\mathbb{X})$ has the following properties:

$$\left\langle \mathcal{N}_{\mathsf{t}}\,, d\pi_0 \right\rangle = 0 \qquad \text{and} \qquad \left\langle \mathcal{N}_{\mathsf{t}}\,, d\kappa \right\rangle = \mathsf{Id}_{\mathsf{U}\mathbb{X}}.$$

We call $\mathcal{N}_{\mathsf{t}}$ the *Sasakian normal vector field*, since this normal vector field depends on our choice of the Sasaki metric on $\mathsf{T}\mathbb{X}$ as outlined above. ⌟

Thus, the restriction of the double tangent bundle to the unit tangent bundle splits into three parts, namely the vertical in terms of a tangential lift, the horizontal tangent bundle as well as the normal tangent bundle:

$$\mathsf{T}(\mathsf{T}\mathbb{X})|_{\mathsf{U}\mathbb{X}} = \underbrace{\mathsf{VU}\mathbb{X}}_{:=\mathsf{tl}(\mathsf{U}\mathbb{X})} \oplus \underbrace{\mathsf{HU}\mathbb{X}}_{=\mathsf{HT}\mathbb{X}|_{\mathsf{U}\mathbb{X}}} \oplus \underbrace{\mathsf{NU}\mathbb{X}}_{=\mathsf{TU}\mathbb{X}^\perp}. \tag{2.14}$$

Recall the unit Sasaki metric $\mathsf{u}$ introduced in Definition 1.2.29 on the configuration manifold $\mathsf{Q} = \mathsf{U}\mathbb{X}$ that renders $\mathsf{TQ} = \mathsf{HQ} \oplus \mathsf{VQ}$ into a fibrewise orthogonal sum.

Now we introduce some convenient shorthand notation.

*Notation* 2.2.2 (spherical notations). The vertical lift $f_0^{\mathsf{v}}$ of a function $f_0 \in C^\infty(\mathbb{X})$ should is to be read now as $f_0^{\mathsf{v}} = f_0 \circ \pi_{0|\mathsf{U}}$. Similarly, the symbol $\mathcal{H}_{\mathsf{x}}$ is to be interpreted as the restriction $\mathcal{H}_{\mathsf{x}}|_{\mathsf{Q}}$ of the Riemannian semispray to $\mathsf{Q}$. Note that it maps into $\mathsf{TQ}$ according to [Sak96, Lemma 4.2]. The horizontal lift of a function $f_0 \in C_{\mathsf{c}}^\infty(\mathbb{X})$ is up to constant summands characterised by $\langle a\,, df_0^{\mathsf{h}}\rangle = \langle a\,, df_0 \circ d\kappa\rangle$ for all $a \in \mathsf{TU}\mathbb{X}$. Compare to Remark 1.2.31. It is easily verified that $\nabla_{\mathsf{u}} f_0^{\mathsf{h}} = \mathsf{tl}(\nabla_{\mathsf{x}} f_0 \circ \pi_{0|\mathsf{U}})$ holds. ⌟

Another important side effect of the new standard fibre being a sphere relates to the equality

$$\Delta_{\mathbb{S}^{d-1}} \mathsf{Id}_{\mathbb{S}^{d-1}} = -(d-1)\,\mathsf{Id}_{\mathbb{S}^{d-1}}, \tag{2.15}$$

where the spherical Laplacian is taken componentwise in standard Euclidean coordinates. See [GS14, Lemma 3.2] or [GMS12, Appendix 7] and for a general proof on eigenvalues of the spherical Laplace-Beltrami we refer e. g. to [DX13, Theorem 1.4.5]. The following lemma states that the relation (2.15) carries over to the tensor Laplacian of the Riemannian semispray.

**Lemma 2.2.3.** It holds $\Delta_{\mathsf{u}} \mathcal{H}_{\mathsf{x}} = -(d-1)\mathcal{H}_{\mathsf{x}}$, where we denote by $\Delta_{\mathsf{u}}$ also the tensor Laplacian and think of the vector field $\mathcal{H}_{\mathsf{x}}$ as a $(1,0)$-tensor field.

*Proof.* For fixed $w \in \mathsf{U}\mathbb{X}$ with $x := \pi_{0|\mathsf{U}}(w)$ we get that

$$\begin{aligned} \mathsf{h}_w\left(\Delta_{\mathsf{u}} \mathcal{H}_{\mathsf{x}}\,|_{\mathsf{U}_x\mathbb{X}}\,, \mathsf{hl}(w)\right) &= \Delta_{\mathsf{u}} \mathsf{h}_w\left(\mathcal{H}_{\mathsf{x}}|_{\mathsf{U}\mathbb{X}x}\,, \mathsf{hl}(w)\right) \\ &= \Delta_{\mathsf{u}} \mathsf{x}_x\left(\mathsf{Id}_{\mathsf{U}_x\mathbb{X}}\,, w\right) = \mathsf{x}_x\left(\Delta_{\mathsf{u}} \mathsf{Id}_{\mathsf{U}_x\mathbb{X}}\,, w\right) = \mathsf{x}_x\left(\Delta_{\mathsf{v}|\mathsf{U}} \mathsf{Id}_{\mathsf{U}_x\mathbb{X}}\,, w\right). \end{aligned}$$

Use Equation (2.15) in the fibre $\mathsf{U}_x\mathbb{X} \simeq \mathbb{S}^{d-1}$. □

In its first form, the fibre lay-down model on $\mathbb{X}$ is formulated as the following Stratonovich SDE in $\mathsf{Q} = \mathsf{U}\mathbb{X}$:

$$\mathsf{d}\eta = \mathcal{H}_{\mathsf{x}}\;\mathsf{d}t + \mathsf{tl}_\eta(-\nabla_{\mathsf{x}}\Psi)\;\mathsf{d}t + \sigma \cdot \sum_{j=1}^{d} \mathsf{tl}_\eta\left(\frac{\partial}{\partial x_\eta^j}\right) \circ \mathsf{d}W_t^j, \tag{2.16}$$

where the chart $(x_\eta^1, x_\eta^2, \dots, x_\eta^d)$ at $\pi_{0|\mathsf{U}}(\eta)$ provides normal coordinates and $\sigma$ is a nonnegative diffusion parameter. Compared to Section 2.1, note that neither we rescale the potential nor we incorporate a friction term. Another name not tied to our industrial application would be *spherical velocity Langevin equation (on* $\mathbb{X}$*)* as used e. g. [BT18, Section 3]. Formally, the Kolmogorov backwards generator attains the form

$$L = \mathcal{H}_{\mathsf{x}} - \mathsf{tl}(\nabla_{\mathsf{x}}\Psi) + \frac{\sigma^2}{2}\,\Delta_{\mathsf{u}}\,. \tag{2.17}$$

We call it either *fibre lay-down generator* or *spherical velocity Langevin generator*. Per se, this operator is defined for all smooth functions on Q. However, one uses smooth functions with compact support as test functions since the model Hilbert space is of $L^2$-type. Some computations are done on the set

$$D_{0|\mathsf{U}} = \pi_{0|\mathsf{U}}^* C_{\mathsf{c}}^\infty(\mathbb{X}) \otimes \kappa^* C_{\mathsf{c}}^\infty(\mathbb{X}) := \mathsf{span}\{f_0^{\mathsf{v}} \otimes g_0^{\mathsf{h}} := f_0^{\mathsf{v}} \cdot g_0^{\mathsf{h}} \mid f_0, g_0 \in C_{\mathsf{c}}^\infty(\mathbb{X})\}$$

which is dense in $D := C_{\mathsf{c}}^\infty(\mathsf{Q})$ with respect to the local convex topology that implies uniform convergence of all derivatives on compact sets. Recall Lemma 1.2.34 and modify its proof.

Next, we formulate the main theorem regarding the fibre lay-down model and prove it in the same vein as in Section 2.1. We restrict ourselves to computations substantially different from the ones before, though. The potential conditions (P) are kept virtually the same with only the natural modifications. It turns out that we additionally need the Assumption (2.19) below. It has not been considered by others before, since it is always fulfilled in Euclidean space, compare to Example 2.2.5. One sees that Equation (2.16) transforms into

$$\mathrm{d}\eta = \mathcal{H}_{\mathsf{x}}\ \mathrm{d}t - \frac{1}{d-1}\nabla_{\mathsf{v}|\mathsf{U}}\big(\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\big)\ \mathrm{d}t + \sigma \cdot \sum_{j=1}^{d} \mathsf{tl}_\eta\left(\frac{\partial}{\partial x_\eta^j}\right) \circ \mathrm{d}W_t^j, \tag{2.18}$$

under Assumption (2.19). We discuss this assumption after the theorem is stated.

**Theorem 2.2.4** (Hypocoercivity of the geometric spherical velocity Langevin dynamic)**.** Consider the diffusion parameter $\sigma \in (0,\infty)$ and a Riemannian manifold $(\mathbb{X},\mathsf{x})$ satisfying (M). Assume that the potential $\Psi\colon\ \mathbb{X} \to \mathbb{R}$ fulfils the conditions (P), and additionally that the relation

$$\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}} = (d-1)\,\Psi^{\mathsf{h}} \qquad \text{on } \mathsf{U}\mathbb{X} \tag{2.19}$$

holds. Abbreviate by $\nu := \mathsf{S}_1^{d-1}$ the normalised surface measure on $(\mathbb{S}^{d-1}, \mathfrak{B}(\mathbb{S}^{d-1}))$ and define $\mu := \lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu$.

Then, the fibre lay-down generator

$$(L,D) = \left(\frac{\sigma^2}{2}\ \Delta_{\mathsf{v}|\mathsf{U}} + \mathcal{H}_{\mathsf{x}} - \frac{1}{d-1}\nabla_{\mathsf{v}|\mathsf{U}}\big(\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\big),\ C_{\mathsf{c}}^\infty(\mathsf{U}\mathbb{X})\right) \tag{2.20}$$

is closable in $H = L^2(\mathsf{U}\mathbb{X};\mu)$. Moreover, its closure $(L, D(L))$ generates a strongly continuous contraction semigroup $(T_t)_{t\in[0,\infty)}$. Finally, for all $\kappa_1 \in (1,\infty)$ there is a constant $\kappa_2 \in (0,\infty)$ such that for all $g \in H$ and times $t \in [0,\infty)$ holds

$$\| T_t g - (g\,,1)_H \|_H \le \kappa_1 \mathrm{e}^{-\kappa_2 t} \| g - (g\,,1)_H \|_H,$$

where $\kappa_2$ explicitly is given as

$$\kappa_2 = \frac{\kappa_1 - 1}{\kappa_1}\ \frac{\sigma^2}{n_1 + n_2\sigma^2 + n_3\sigma^4} \tag{2.21}$$

where $n_i \in (0,\infty)$ for $i \in \{1,2,3\}$ only depend on $\Psi$, compare to (B) in Section 1.1. ⌟

**Example 2.2.5** (Assumption (2.19) for Euclidean position space)**.** Let $\mathbb{X} = \mathbb{R}_x^d$ be endowed with standard Euclidean metric $\mathsf{x} = \big((\cdot\,,\cdot)_{\mathsf{euc}}\big)_{x\in\mathbb{R}^d}$. As discussed in the discussion subsequent to Definition 1.2.27, the Euclidean semispray $\mathcal{H}_{\mathsf{x}} = \mathcal{H}_{\mathsf{euc}}$ can be thought as the gradient of the Lagrangian $\mathsf{L} = \frac{1}{2}\,|\cdot|_{\mathsf{x}} = \frac{1}{2}\,|\cdot|_{\mathsf{euc}}$ Thus, it is basically the identity mapping: The mapping $w \mapsto \mathcal{H}_{\mathsf{euc}}(w)$ coincides with the gradient of $z \mapsto U_w(z) := (w\,,z)_{\mathsf{euc}}$. One comes to the same conclusion when starting with the canonical 2-form We investigate the restriction to the configuration manifold $\mathsf{Q} = \mathsf{U}\mathbb{X} = \mathsf{U}\mathbb{R}_x^d \simeq \mathbb{R}_x^d \times \mathbb{S}_v^{d-1}$ and find for every $w \in \mathbb{S}^{d-1}$ as well as $f \in C_{\mathsf{c}}^\infty(\mathsf{U}\mathbb{R}_x^d)$ that

$$\begin{aligned}
&\int_{\mathsf{U}_x\mathbb{R}^d} \mathcal{H}_{\mathsf{x}} f\ \mathrm{d}\nu_x \\
&= \int_{\mathsf{U}_x\mathbb{R}^d} \big(\nabla_{\mathsf{v}|\mathsf{U}} U_v(z)\,, \nabla_{\mathsf{v}|\mathsf{U}} f(v)\big)_{\mathsf{euc}}\ \nu_x(\mathrm{d}v) = \int_{\mathsf{U}_x\mathbb{R}^d} \big(\nabla_{\mathsf{s}} U_v(z)\,, \nabla_{\mathsf{s}} f(v)\big)_{\mathsf{euc}}\ \nu_x(\mathrm{d}v) \\
&= -\int_{\mathsf{U}_x\mathbb{R}^d} \Delta_{\mathsf{s}} U_v(z) \cdot f(v)\ \nu_x(\mathrm{d}v) \overset{(2.15)}{=} (d-1)\int_{\mathsf{U}_x\mathbb{R}^d} (v\,,z)_{\mathsf{euc}} \cdot f(v)\ \nu_x(\mathrm{d}v)
\end{aligned}$$

by integration by parts with some $z \in \mathbb{R}^d$, compare to [GS14, Lemma 3.3]. Here, 's' refers to the round metric on $\mathbb{S}^{d-1}$.

We derive an explicit expression for the horizontal lift of a given test function $f_0 \in C_c^\infty(\mathbb{R}^d_x)$. Any $a \in \mathsf{TQ}$ splits into vertical and horizontal part as $a = a_v \oplus a_x$, i.e. $\big\langle a\,, d\pi_{0|\mathsf{U}}\big\rangle = a_x$ and $\big\langle a\,, d\kappa|_{\mathsf{U}\mathbb{R}^d}\big\rangle = a_v = v := \pi_{0|\mathsf{U}}(a)$. Then, we find that

$$\langle a\,, df_0^{\mathsf{h}}\rangle = \Big\langle a_v \oplus a_x\,, df_0 \circ d\kappa|_{\mathsf{U}\mathbb{R}^d}\Big\rangle = \Big\langle v\,, df_0|_{\mathsf{U}\mathbb{R}^d}\Big\rangle = \left(v\,, \frac{\nabla_x f_0}{|\nabla_x f_0|} \circ \pi_{0|\mathsf{U}}(v)\right)_{\mathsf{euc}}.$$

Thus, $f_0^{\mathsf{h}}(v) = U_{\nabla_x f_0/|\nabla_x f_0|(x)}(v)$ for all $v \in \mathsf{U}_x\mathbb{R}^d$. Combining both intermediate results we arrive at

$$\begin{aligned}
\int_{\mathsf{U}_x\mathbb{R}^d} \mathcal{H}_{\mathsf{euc}}\Psi^{\mathsf{v}} \cdot g_0^{\mathsf{h}}\, \mathsf{d}\nu_x &= \int_{\mathsf{U}_x\mathbb{R}^d} \mathcal{H}_{\mathsf{euc}}(\Psi^{\mathsf{v}} \cdot g_0^{\mathsf{h}})\, \mathsf{d}\nu_x \\
&= (d-1)\int_{\mathsf{U}_x\mathbb{R}^d} \left(v\,, \frac{\nabla_{\mathsf{euc}}\Psi(x)}{\Psi(x)\,|\nabla_{\mathsf{euc}}\Psi(x)|}\right)_{\mathsf{euc}} \cdot \Psi^{\mathsf{v}}(v)\, g_0^{\mathsf{h}}(v)\, \mathsf{d}\nu_x(v) \\
&= (d-1)\int_{\mathsf{U}_x\mathbb{R}^d} \Psi^{\mathsf{h}}(v) \cdot g_0^{\mathsf{h}}(v)\, \mathsf{d}\nu_x(v)
\end{aligned}$$

holds for all $g_0 \in C_c^\infty(\mathbb{R}^d_x)$ with the particular choice of $z = \frac{\nabla_{\mathsf{euc}}\Psi(x)}{\Psi(x)\,|\nabla_{\mathsf{euc}}\Psi(x)|}$. Note that we assume $\Psi > 0$ with out loss of generality in view of (P1). Summarising, we have shown that Assumption (2.19) holds for $\mathbb{X} = \mathbb{R}^d_x$.
⌟

## Data conditions

We already impose Assumption (2.19) for the SAD-decomposition. This is a little odd, since the assertions of Lemma 2.2.6 below are true without this assumption when the antisymmetric operator is $\mathcal{H}_{\mathbf{x}} = \mathcal{H}_{\mathsf{x}} - \mathrm{tl}\big(\nabla_{\mathsf{x}}\Psi\big)$. This can be seen similarly as in Proposition 2.1.8: The configuration manifold Q is a smooth submanifold of $\mathsf{T}\mathbb{X}$. So, both the 2-form $\Omega$ and the almost complex structure $\mathcal{J}$ restrict to $\mathsf{T}^*\mathsf{Q}$ and $\mathsf{TQ}$ respectively. They are compatible with the unit Sasaki metric and generate basically the same Poisson bracket but on Q. In turn, the Poisson bracket defines Hamiltonian vector fields $\mathsf{H}_f \in \Gamma^\infty(\mathsf{TQ})$ for given $f \in C^\infty(\mathsf{Q})$. One finds that $\mathsf{H}_f(\rho) = -\rho\cdot\big(\mathcal{H}_{\mathsf{x}} f - \mathrm{tl}\big(\nabla_{\mathsf{x}}\Psi\big)(f)\big)$ for all $f \in D = C_c^\infty(\mathsf{Q})$, by investigating the action of the Hamiltonian vector field in local coordinates for Q that respect the Ehresmann connection and also provide a local trivialisation. Even though $(-\mathcal{H}_{\mathbf{x}}, D) = \big(-\mathcal{H}_{\mathsf{x}} + \mathrm{tl}\big(\nabla_{\mathsf{x}}\Psi\big), D\big)$ is the more natural formulation at first glance, we need the antisymmetric operator $A$ as in Lemma 2.2.6 for a suitable characterisation of $(PA^2P, D)$, specifically at the step (2.25).

**Lemma 2.2.6** (SAD-decomposition (D3), (D4), (D6)). Consider the loc-Lipschitzian potential $\Psi$ such that Assumption (2.19) is fulfilled. Consider the SAD-decomposition $L = S - A$ on $D$ with

$$S := \frac{\sigma^2}{2}\,\Delta_{\mathsf{v|U}} \quad \text{and} \quad A := -\mathcal{H}_{\mathbf{x}} = -\mathcal{H}_{\mathsf{x}} + \frac{1}{d-1}\,\nabla_{\mathsf{v|U}}\big(\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\big).$$

Then, the following assertions hold:

(i) $(S, D)$ is symmetric and nonpositive definite.

(ii) $(A, D)$ is antisymmetric.

(iii) For all $f \in D$ we have that $Lf \in L^1(\mathsf{U}\mathbb{X};\mu)$ and $\int_{\mathsf{U}\mathbb{X}} Lf\, \mathsf{d}\mu = 0$.

*Proof.*

(i) Using integration by parts we see that $(S, D)$ pregenerates the spherical gradient form on $\mathsf{U}\mathbb{X}$. Compare to the proof of part (i) of Lemma 2.1.5.

(ii) The adjoint of $\nabla_{\mathsf{v|U}}\Psi^{\mathsf{h}} \overset{(2.19)}{=} \frac{1}{d-1}\,\nabla_{\mathsf{v|U}}\big(\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\big)$ with respect to $L^2(\mathsf{U}\mathbb{X};\mathbf{u})$-scalar product is computed using Lemma 2.2.3 as

$$\left(\frac{1}{d-1}\,\nabla_{\mathsf{v|U}}\big(\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\big)\right)^* = -\frac{1}{d-1}\,\nabla_{\mathsf{v|U}}\big(\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\big) - \frac{1}{d-1}\,\Delta_{\mathsf{v|U}}\big(\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\big)$$

$$= -\frac{1}{d-1} \nabla_{\mathsf{v}|\mathsf{U}}\Big(\mathscr{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\Big) + \mathscr{H}_{\mathsf{x}} \Psi^{\mathsf{v}}.$$

The rest follows as in the proof of part (ii) of Lemma 2.1.5.

(iii) Follows with the parts (i) and (i). □

The fibrewise average is defined as in Section 2.1 just with '$\mathsf{U}_x\mathbb{X}$' instead of '$\mathsf{T}_x\mathbb{X}$'. Also, the form of the operator $(AP, D)$ given in Equation (2.8) just changes marginally:

$$APf = -\mathsf{x}\Big(\mathsf{Id}_{\mathsf{U}\mathbb{X}}\,,\nabla_{\mathsf{x}} \mathsf{E}_\nu[f] \circ \pi_{0|\mathsf{U}}\Big)\pi_{0|\mathsf{U}} \qquad \text{for all } f \in D. \tag{2.22}$$

Since the standard fibre $\mathbb{F} = \mathbb{S}^{d-1}$ is compact, one can simplify the proof Lemma 2.1.9 similar to [GS14, Lemma 3.8].

**Lemma 2.2.7.** Let condition (P1) hold. Then, we have $P(H) \subseteq D(S)$, $SP = 0$, $P(D) \subseteq D(A)$, and $AP(D) \subseteq D(A)$. Furthermore, $1 \in D(L)$ and $L1 = 0$.

*Proof.* The range $P(H)$ is identified with a subset of $L^2(\mathbb{X};\mathbf{x})$ via the vertical lift. For any $f_0 \in L^2(\mathbb{X};\mathbf{x})$ there is an $L^2$-approximating sequence $(f_n)_{n\in\mathbb{N}_+}$ in $C^\infty_{\mathsf{c}}(\mathbb{X})$. Then, it holds $f_n^{\mathsf{v}} \in D$ and $S(f_n^{\mathsf{v}}) = 0$ for all $n \in \mathbb{N}_+$. We conclude that $f_0^{\mathsf{v}} \in D(S)$ and $f_0^{\mathsf{v}} \in \mathsf{null}(S)$ as $f_n^{\mathsf{v}} \longrightarrow f_0^{\mathsf{v}}$ in $H$ as $n \to \infty$ and $(S, D(S))$ is closed.

We fix an $f \in D$. Choose $o \in \mathbb{X}$ and an open ball $\mathbb{U}(o,r)$ centred at $o$ with radius $r \in (0,\infty)$ with respect to the intrinsic metric on $(\mathbb{X},\mathsf{x})$ such that the support of $f$ is completely contained in $\pi_{0|\mathsf{U}}^{-1}(\mathbb{U}(o,r)) \subseteq \mathsf{U}\mathbb{X}$. Than, the support of $\mathsf{E}_\nu[f]$ is contained in $\mathbb{U}(o,r)$. Thus, $P_S f \in D$. Therefore, $P(D) \subseteq D \subseteq D(A)$.

Besides, we calculate via chain rule that

$$\begin{aligned} APf &= -\mathscr{H}_{\mathbf{x}}(\mathsf{E}_\nu[f]^{\mathsf{v}}) = -\Big\langle \mathscr{H}_{\mathsf{x}}\,, d\mathsf{E}_\nu[f] \circ d\pi_0 \Big\rangle \\ &= -\Big\langle \mathsf{Id}_{\mathsf{U}\mathbb{X}}\,, d\mathsf{E}_\nu[f] \Big\rangle = -d\mathsf{E}_\nu[f]. \end{aligned} \tag{2.23}$$

Consequently, $AP(D) \subseteq D \subseteq D(A)$, as the right-hand side of Equation (2.23) is smooth with compact support. Let $\varphi \in C^\infty_{\mathsf{c}}(\mathbb{X};[0,1])$ be a cut-off function such that $\varphi = 1$ on $\mathbb{U}(o,1)$ and $\varphi = 0$ outside of $\mathbb{U}(o,2)$. Define $\varphi_n := \varphi \circ \gamma_{\mathsf{Id}}(1/n)$ for all $n \in \mathbb{N}_+$, where $\gamma_x\colon [0,1] \to \mathbb{X}$ denotes the geodesic with $\gamma_x(0) = o$ and $\gamma_x(1) = x$. Note that $|\nabla_{\mathsf{x}}\varphi_n(x)|_{\mathsf{x}} \le \frac{1}{n}\|\nabla_{\mathsf{x}}\varphi(x)\|_{L^\infty(\mathsf{x})}$ for all $x \in \mathbb{X}$ and $n \in \mathbb{N}_+$. By construction, we have that

$$AP\varphi_n^{\mathsf{v}} = -\mathscr{H}_{\mathsf{x}}\varphi_n^{\mathsf{v}} = \Big\langle \mathscr{H}_{\mathsf{x}}\,, d\varphi_n \circ d\pi_0 \Big\rangle = -d\varphi_n \longrightarrow 0 \qquad \text{as } n \to \infty$$

pointwise and in $L^2$-sense. Since $(A, D(A))$ is closed, we have $1 \in D(A)$ and $A1 = 0$.

Since we know form part (i) of Lemma 2.2.6 that $S\varphi_n^{\mathsf{v}} = 0$ for all $n \in \mathbb{N}$, we have $L\varphi_n^{\mathsf{v}} = -A\varphi_n^{\mathsf{v}}$ for all $n \in \mathbb{N}$. The sequence $(L\varphi_n^{\mathsf{v}})_{n\in\mathbb{N}}$ converges in $H$ to 0 as $n \to \infty$. □

From here on, the rest of the data conditions is verified as in Section 2.1. Note regarding the Hörmander condition for $L$ that the Lie brackets of a semispray and a tangentially lifted local vector field evaluates to a local vector field with nontrivial horizontal part due to Lemma 2.1.12. Therefore, we get for the dimension of the spanned Lie algebra that

$$\dim\Big(\mathsf{Lie}_u\Big(\mathscr{H}_{\mathsf{x}}, \mathsf{tl}\Big(\nabla_{\mathsf{x}}\Psi\Big), \mathsf{tl}(\partial x^1), \ldots, \mathsf{tl}(\partial x^d)\Big)\Big) = \dim(\mathsf{U}\mathbb{X}) = 2d-1,$$

where $(x^j)_{j=1}^d$ are local coordinates at $\pi_{0|\mathsf{U}}(u)$ for given $u \in \mathsf{U}\mathbb{X}$. Now, the three step method for essential m-dissipativity of $(L, D)$ applies as outlined in Section 2.1.

## Hypocoercivity conditions

**Lemma 2.2.8** (algebraic relation (H1))**.** Let $\Psi$ be loc-Lipschitzian such that $\lambda_{\mathbf{x}} = \exp(-\Psi)\,\lambda_{\mathsf{x}}$ is a probability measure on $(\mathbb{X}, \mathfrak{B}(\mathbb{X}))$. Then, we have $PAP|_D = 0$. Compare to [GS14, Proposition 3.11].

*Proof.* Using only [GS14, Lemma 3.1] and Equation (2.22) we calculate that

$$\int_{\mathsf{U}_x\mathbb{X}} APf \,\mathrm{d}\nu_x \overset{(2.22)}{=} \int_{\mathsf{U}_x\mathbb{X}} -\mathsf{x}_x\left(\mathsf{Id}_{\mathsf{U}\mathbb{X}}\,,\nabla_{\mathsf{x}}\mathsf{E}_\nu[f](x)\right)\,\mathrm{d}\nu_x = 0$$

holds for all $f \in D$ and $x \in \mathbb{X}$. The rest of the proof works as in Lemma 2.2.8. □

**Lemma 2.2.9** (microscopic coercivity (H2))**.** Let $\Psi$ loc-Lipschitzian such that $\lambda_{\mathbf{x}} = \exp(-\Psi)\,\lambda_{\mathsf{x}}$ is a probability measure on $(\mathbb{X}, \mathfrak{B}(\mathbb{X}))$. Then, condition (H2) holds with $\Lambda_m = (d-1)\sigma^2/2$. Compare to [GS14, Proposition 3.12].

*Proof.* The proof works the same way as in Lemma 2.1.20 using the Poincaré inequality for the spherical measure, see [Bec89, Theorem 2]. □

Regarding condition (H3), we again characterise the operator $(PA^2P, D)$ as a weighted horizontal Laplace-Beltrami applied to a fibrewise average. Essential m-dissipativity of this operator is obtained as in Corollary 2.1.22. Mirroring the computations in Equation (2.11) we calculate that

$$\begin{aligned}
&\int_{\mathsf{U}_x\mathbb{X}} -\mathcal{H}_{\mathsf{x}}\left(-\mathsf{x}_{\pi_{0|\mathsf{U}}}\left(\mathsf{Id}_{\mathsf{U}\mathbb{X}}\,,\nabla_{\mathsf{x}}\mathsf{E}_\nu[f]\circ\pi_{0|\mathsf{U}}\right)\right)\,\mathrm{d}\nu_x \\
&\overset{(2.22)}{=} \int_{\mathsf{U}_x\mathbb{X}} \mathsf{x}_x\left(v\,,\nabla_{\mathsf{x}}\mathsf{E}_\nu[f](x)\right)\,\nu_x(\mathrm{d}v) + \int_{\mathsf{U}_x\mathbb{X}} \mathsf{x}_x\left(v\,,\nabla^{\mathsf{x}}_v\left(\nabla_{\mathsf{x}}\mathsf{E}_\nu[f]\right)(x)\right)\,\nu_x(\mathrm{d}v) \\
&= \frac{1}{d}\,\Delta_{\mathsf{x}}\mathsf{E}_\nu[f](x)
\end{aligned} \tag{2.24}$$

for all $v \in \mathsf{U}\mathbb{X}$ with $x := \pi_{0|\mathsf{U}}(v)$.

As for Equation (2.12), we use the proof of part (ii) of Lemma 2.2.6 to get that

$$\begin{aligned}
P_S\left(\frac{1}{d-1}\,\nabla_{\mathsf{v}|\mathsf{U}}\left(\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\right)(APf)\right)(v) &= \int_{\mathsf{U}_x\mathbb{X}} \mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\cdot APf\,\mathrm{d}\nu_x \\
= -\frac{1}{d}\,\mathsf{x}_x\left(\nabla_{\mathsf{x}}\Psi(x)\,,\nabla_{\mathsf{x}}\mathsf{E}_\nu[f](x)\right) &= -\frac{1}{d}\,\partial_{\nabla_{\mathsf{x}}\Psi}(\mathsf{E}_\nu[f])(x)
\end{aligned} \tag{2.25}$$

for all $v \in \mathsf{U}\mathbb{X}$ with $x := \pi_{0|\mathsf{U}}(v)$.

Combining now both Equation (2.24) and Equation (2.25) we infer the relation

$$\begin{aligned}
PA^2Pf = P_SA^2Pf &= \frac{1}{d}\cdot\left(\Delta_{\mathsf{x}}\mathsf{E}_\nu[f]\circ\pi_{0|\mathsf{U}} - \partial_{\nabla_{\mathsf{x}}\Psi}(\mathsf{E}_\nu[f])\circ\pi_{0|\mathsf{U}}\right) \\
&= \frac{1}{d}\cdot\Delta_{\mathbf{h}}(\mathsf{E}_\nu[f]^{\mathsf{v}}) = \frac{1}{d}\cdot\Delta_{\mathbf{h}}(P_Sf)
\end{aligned} \tag{2.26}$$

for all $f \in D$. Compare this to [GS14, Equation (3.27)]. Now, we are in the position to prove the next proposition.

**Proposition 2.2.10** (macroscopic coercivity (H3))**.** Let $\Psi$ loc-Lipschitzian such that $\lambda_{\mathbf{x}} = \exp(-\Psi)\,\lambda_{\mathsf{x}}$ is a probability measure on $(\mathbb{X}, \mathfrak{B}(\mathbb{X}))$ satisfying the Poincaré inequality (2.4). Then, condition (H3) is fulfilled with $\Lambda_M = {}^1\!/_d\,\Lambda$. Compare to [GS14, Proposition 3.14].

*Proof.* For all $f \in D$ we compute that

$$\begin{aligned}
\|APf\|_H^2 &= \int_{\mathbb{X}}\int_{\mathsf{U}_x\mathbb{X}} (APf)^2\big|_{\mathsf{U}_x\mathbb{X}}\,\mathrm{d}\nu_x\,\lambda_{\mathbf{x}}(\mathrm{d}x) \\
&\overset{(2.22)}{=} \int_{\mathbb{X}}\int_{\mathsf{U}_x\mathbb{X}} \mathsf{x}_x\left(v\,,\nabla_{\mathsf{x}}\mathsf{E}_\nu[f](x)\right)^2\,\nu_x(\mathrm{d}v)\,\lambda_{\mathbf{x}}(\mathrm{d}x) \\
&= \frac{1}{d}\int_{\mathbb{X}}\left|\nabla_{\mathsf{x}}\mathsf{E}_\nu[f](x)\right|_{\mathsf{x}}^2\,\lambda_{\mathbf{x}}(\mathrm{d}x) \\
&= \frac{1}{d}\left\|\nabla_{\mathsf{x}}\mathsf{E}_\nu[f]\right\|^2_{L^2(\mathbb{X}\to\mathsf{T}\mathbb{X};\mathbf{x})} \geq \frac{1}{d}\,\Lambda\left\|\mathsf{E}_\nu[f] - (\mathsf{E}_\nu[f]\,,1)_{L^2(\mathbb{X};\mathbf{x})}\right\|^2_{L^2(\mathbb{X};\mathbf{x})}
\end{aligned}$$

using the Poincaré inequality of the weighted base measure. The claim follows with [GS14, Corollary 2.13], since $(PA^2P, D)$ is essentially m-dissipative due to the modification of Corollary 2.1.22 to the case of $\mathsf{Q} = \mathsf{U}\mathbb{X}$. □



In contrast to checking the first part of condition (H4) in Section 2.1, we do not need $\mathscr{H}_{\mathsf{x}}$ to be a full spray. Compare the following Lemmas 2.2.11 and 2.2.12 as well as their proofs to [GS14, Proposition 3.15].

**Lemma 2.2.11** (boundedness of $(BS, D)$, first part of (H4)). Let $\Psi$ be loc-Lipschitzian such that $\lambda_{\mathbf{x}} = \exp(-\Phi)\,\lambda_{\mathsf{x}}$ is a probability measure on $(\mathbb{X}, \mathfrak{B}(\mathbb{X}))$. Then, with $c_1 := (\mathscr{d}-1)\,\sigma^2/4$ it holds that

$$\|BSf\|_H \leq c_1 \|(\mathrm{Id}_H - P_j)f\|_H \qquad \text{for all } f \in D$$

and $P_j \in \{P, P_S\}$, $j \in \{1, 2\}$.

*Proof.* Let $f \in D$ be fixed. Then, we observe that

$$\begin{aligned} SAPf = SAP_S f &= \frac{\sigma^2}{2}\, \Delta_{\mathsf{v}|\mathsf{U}}\big(-\mathscr{H}_{\mathbf{x}}(P_S f)\big) \\ &= \frac{\sigma^2}{2}\,(\mathscr{d}-1)\cdot \mathscr{H}_{\mathbf{x}}(P_S f) = -\frac{\sigma^2}{2}\,(\mathscr{d}-1)APf \end{aligned}$$

by Lemma 2.2.3, and since $(Ah\,, 1)_H = 0$ holds for all $h \in D$ by part (ii) of Lemma 2.2.6. □

**Lemma 2.2.12** (boundedness of $(BA(\mathrm{Id}_H - P), D)$, second part of (H4)). Let all the conditions of Theorem 2.2.4 on the potential hold. Then, there exists a constant $c_2 \in (0, \infty)$ such that

$$\|BA(\mathrm{Id}_H - P)f\|_H \leq c_2 \|(\mathrm{Id}_H - P_j)f\|_H \qquad \text{for all } f \in D$$

and $P_j \in \{P, P_S\}$, $j \in \{1, 2\}$.

*Proof.* Let $f \in D$ and $g := (\mathrm{Id} - PA^2P)f$ as in the proof of Lemma 2.1.25. Now, the relevant estimate reads as

$$\|(BA)^* g\|_H \leq \Big\| |\mathrm{Hess}_{\mathsf{x}}(\mathsf{E}_\nu[f])|^{\mathsf{v}} \Big\|_H + \frac{1}{\mathscr{d}} \Big\| |\nabla_{\mathsf{x}}\Phi|^{\mathsf{v}}_{\mathsf{x}} \cdot |\nabla_{\mathsf{x}}\mathsf{E}_\nu[f]|^{\mathsf{v}}_{\mathsf{x}} \Big\|_H.$$

In view of Equation (2.26) we have the solution $u := Pf$ of the elliptic equation

$$\begin{aligned} & u - \frac{1}{\mathscr{d}}\,\Delta_{\mathbf{h}} P_S u = g \\ \text{in} \qquad & \Big\{ u \in \pi^*_{0|\mathsf{U}} L^2(\mathbb{X}; \mathbf{x}) \;\Big|\; \exists\, f_0 \in C^\infty_{\mathsf{c}}(\mathbb{X}) \colon\; u = f_0^{\mathsf{v}} - (f_0^{\mathsf{v}}\,, 1)_H \Big\}. \end{aligned}$$

The proof is completed as in Lemma 2.1.25. □

## 2.3 ▪ Martingale solutions, $L^2$-exponential ergodicity, and limit of diffusion parameter approaching infinity

We start this section by showing existence of $L$-martingale solutions to the SDEs investigated in this chapter. The strong mixing of the corresponding semigroups with exponential rate of convergence then implies their $L^2$-exponential ergodicity, see Corollary 2.3.2 below. For basic notions used in the following theorem we refer to [Sta99], [Tru00], and [Tru03].

**Corollary 2.3.1** (existence of martingale solutions). Consider the configuration manifold $\mathsf{Q} \in \{\mathsf{T}\mathbb{X}, \mathsf{U}\mathbb{X}\}$ and let the assumptions of the respective main theorems hold, i. e. the assumptions of Theorem 2.1.4 for $\mathsf{Q} = \mathsf{T}\mathbb{X}$ and of Theorem 2.2.4 for $\mathsf{Q} = \mathsf{U}\mathbb{X}$.

Then, there is a Hunt process

$$\mathbf{HP} = \Big(\Omega, \mathfrak{A}, \mathfrak{F} = (\mathfrak{F}_t)_{t\in[0,\infty)}, \eta = (\eta_t)_{t\in[0,\infty)}, (\mathbb{P}_v)_{v\in\mathsf{Q}}\Big)$$

properly associated in the resolvent sense with $(L, D(L))$ having infinite life-time and continuous paths $\mathbb{P}_v$-almost surely for all $v \in \mathsf{Q}$. Note that 'properly associated in the resolvent sense' means the following:

If $(G_a)_{a\in(0,\infty)}$ denotes the resolvent corresponding to $(L, D(L))$, then the transition resolvent $(R_a)_{a\in(0,\infty)}$ yields a quasi-continuous version $R_a f$ of $G_a f$ for all $f \in L^2(\mathsf{Q};\mu)$ and $a \in (0,\infty)$. Recall that the transition resolvent is given as $R_a f(v) = \int_{(0,\infty)} \exp(-as)\mathsf{E}_v[f(\eta_s)]\,\lambda(\mathsf{d}s)$ with $\mathsf{E}_v[\cdot] = \mathsf{E}_{\mathbb{P}_v}[\cdot]$.

Moreover, for quasi every initial point $v \in \mathsf{Q}$ the probability measure $\mathbb{P}_v$ solves the martingale problem for $(L, C_c^2(\mathsf{Q}))$, i. e. $\mathbf{HP}$ is a martingale solution to either (2.2) if $\mathsf{Q} = \mathsf{T}\mathbb{X}$ or to (2.18) if $\mathsf{Q} = \mathsf{U}\mathbb{X}$ for quasi every initial point $v \in \mathsf{Q}$.

*Proof.* As said several times before, $D = C_c^\infty(\mathsf{Q})$ is a core of $(L, D(L))$. Observe that $D$ also is an algebra which separates the points of Q. Thus, $(L, D(L))$ defines a generalised Dirichlet form fulfilling the assumptions of [Sta99, Theorem IV.2.2]. This theorem provides a special standard process $\mathbf{HP}$ properly associated with $(L, D(L))$ in the resolvent sense. Now, infinite life-time follows from (D7), i. e. conservativity, together with [Sta99, Theorem IV.3.8 (ii)]. Moreover, continuous paths are obtained via [Tru03, Theorem 3.3]. Summarising, $\mathbf{HP}$ is a Hunt process indeed. For the statement concerning the martingale problem see [CG08, Corollary 1] and its proof. Note that there are finer statements on the martingale problem, see [Tru00]. ☐

Now, consider the probability measure $\mathbb{P}$ on $(\Omega, \mathfrak{A})$ defined as

$$\mathbb{P}(A) := \int_{\mathsf{Q}} \mathbb{P}_v(A)\ \mu(\mathsf{d}v) \qquad \text{for all } A \in \mathfrak{A}.$$

Furthermore, suppose some $g \in L^2(\mathsf{Q};\mu)$ with $\mathsf{E}_\mu[g] = 0$. Then, we estimate for all $t \in (0,\infty)$ that

$$\begin{aligned}
&\left\|\frac{1}{t}\int_{[0,t)} g(\eta_s)\ \lambda(\mathsf{d}s)\right\|^2_{L^2(\mathbb{P})} = \int_\Omega \frac{1}{t^2}\int_{[0,t)^2} g(\eta_s)g(\eta_u)\ \lambda(\mathsf{d}(s,u))\ \mathsf{d}\mathbb{P} \\
&= \frac{1}{t^2}\int_{[0,t)^2} \mathsf{E}_{\mathbb{P}}[g(\eta_s)g(\eta_u)]\ \lambda(\mathsf{d}(s,u)) = \frac{2}{t^2}\int_{[0,t)}\int_{[0,s)} \mathsf{E}_{\mathbb{P}}[g(\eta_s)g(\eta_u)]\ \lambda(\mathsf{d}u)\,\lambda(\mathsf{d}s) && (2.27)\\
&= \frac{2}{t^2}\int_{[0,t)}\int_{[0,s)} (T_{s-u}g\,,g)_{L^2(\mu)}\ \lambda(\mathsf{d}u)\,\lambda(\mathsf{d}s) && (2.28)\\
&= \frac{4}{t^2}\int_{[0,2t)}\int_{[0,t)} (T_v g\,,g)_{L^2(\mu)}\ \lambda(\mathsf{d}v)\,\lambda(\mathsf{d}w) = \frac{8}{t}\int_{[0,t)} (T_v g\,,g)_{L^2(\mu)}\ \lambda(\mathsf{d}v) && (2.29)\\
&\le \frac{8}{t}\,\|g\|_{L^2(\mu)}\int_{[0,t)} \|T_v g\|_{L^2(\mu)}\ \lambda(\mathsf{d}v).
\end{aligned}$$

At step (2.27) we use Fubini. Afterwards at step (2.28), we can ensure $u < s$ for symmetry reasons and transform expectation with respect to $\mathbb{P}$ using the (weak) Markov property. Then, at step (2.29) we apply the 2D-transformation formula with $v = s - u$ and $w = s + u$, and finish with the CSBI. By the previous estimate in combination with our main theorems we not only can infer convergence to 0 as $t \to \infty$, but also the rate of convergence is explicitly computable. Indeed, with $g$ as before we gain

$$\begin{aligned}
\left\|\frac{1}{t}\int_{[0,t)} g(\eta_s)\ \lambda(\mathsf{d}s)\right\|^2_{L^2(\mathbb{P})} &\le \frac{8}{t}\,\|g\|_{L^2(\mu)}\int_{[0,t)} \|T_v g\|_{L^2(\mu)}\ \lambda(\mathsf{d}v)\\
&\le \frac{8}{t}\,\|g\|^2_{L^2(\mu)}\int_{[0,t)} \kappa_1 \mathsf{e}^{-v\kappa_2}\ \lambda(\mathsf{d}v)\\
&= \frac{8}{t}\cdot\frac{\kappa_1}{\kappa_2}\,(1-\mathsf{e}^{-t\kappa_2})\cdot\|g\|^2_{L^2(\mu)}.
\end{aligned}$$

Thus after reducing everything to zero-mean functions with respect to $\mu$, we proved the following corollary.

**Corollary 2.3.2** ($L^2$-exponentially ergodicity with optimal rate and explicit constants)**.** Consider the configuration manifold $\mathsf{Q} \in \{\mathsf{T}\mathbb{X}, \mathsf{U}\mathbb{X}\}$ and let the assumptions of the respective main theorems hold. Moreover, let $\kappa_1$ and $\kappa_2$ be the constants form these respective theorems. Then, we have

$$\left\|\frac{1}{t}\int_{[0,t)} f(\eta_s)\,\lambda(\mathsf{d}s) - \mathsf{E}_\mu[f]\right\|_{L^2(\mathbb{P})} \le \frac{2}{\sqrt{t}}\cdot\sqrt{\frac{2\kappa_1}{\kappa_2}\,(1-\mathsf{e}^{-t\kappa_2})}\cdot\big\|f-\mathsf{E}_\mu[f]\big\|_{L^2(\mu)}$$

for all $t \in (0,\infty)$.

As in [CG10], we say that the martingale solutions are $L^2$-*exponential ergodic*, i. e. ergodic with a rate that corresponds to exponential convergence of the corresponding semigroups.

*Remark* 2.3.3. In the title of Corollary 2.3.2 we claim that the rate $t^{-1/2}$ is optimal. This is obvious in the case that the spectrum of the generator $(L, D(L))$ has, apart from the eigenvalue zero, the largest element $-\kappa < 0$ which is an eigenvalue of $(L, D(L))$. Evidently, all the inequalities in the estimates prior to Corollary 2.3.2 are equalities when choosing the function $g$ there as the eigenvector corresponding to $-\kappa$. Hence, the rate of convergence in Corollary 2.3.2 is sharp with $\kappa_1 = 1$ and $\kappa_2 = \kappa$.

In situations where $(L, D(L))$ can be controlled by a Lyapunov function, see e. g. [HM19] in case of purely Euclidean setting, one obtains also exponential rates of convergence for the corresponding semigroups; even in (weighted) total variation distance. This implies pointwise convergence of the semigroup applied to test functions at an exponential rate. But even this convergence with an exponential rate would not give a better rate as the one in Corollary 2.3.2. As we mentioned before in Section 1.1, M. R. Soloveitchik showed exponential convergence in total variation norm in [Sol95, Proposition 4.5], however the rate there is not very explicit. ⌟

### $\kappa_2 \to \Lambda$ as $\sigma \to \infty$?

We want to close the section with a few comments on the limiting behaviour for $\sigma \to \infty$. An anonymous reviewer formulated great interest in this topic pointing us to [BT18, Section 3.1]. There, the authors formulated the following conjecture regarding the spherical velocity Langevin dynamic: 'It is expected that the optimal rate converges, when the parameters go to infinity, to the spectral gap of the base manifold, but unfortunately the rate obtained converges to 0 and we do not reach the base manifold spectral gap.' The argument is based on the result [ABT15, Theorem 2.2] that the projection of the process in $\mathrm{U}\mathbb{X}$ to $\mathbb{X}$ converges in distribution to a Wiener process. In our two main theorems, we have given explicit formulae (2.5) and (2.21) for the constant $\kappa_2$ depending on the choice of $\kappa_1 \in (1, \infty)$ in terms of constants $n_i$, $i \in \{1, 2, 3\}$, that depend on the potential and maybe a rescaling of said potential. First and foremost, we observe that $\kappa_2 \to 0$ as $\sigma \to \infty$ for $\kappa_2$ as in Equation (2.21). Thus, we didn't prove the Baudoin-Tardif conjecture. In order to achieve at least some convergence, we could rescale time by the factor $\sigma^2$: If one replaces time $t$ by $\sigma^2 t$ and passes the scaling factor over to $\kappa_2$ in the hypocoercivity estimate, then we obtain that $\sigma^2\kappa_2 \to \frac{\kappa_1 - 1}{\kappa_1}\frac{1}{n_3}$ as $\sigma \to \infty$. The simple form of the spectral gap on the base manifold is not to be expected with our approach. For sake of clarity, we write out $n_3$ for $\mathsf{Q} = \mathrm{U}\mathbb{X}$ using the computation (B) in Section 1.1:

$$\begin{aligned} n_3 &= 4\,\frac{\Lambda + d}{\Lambda} \cdot a_3 \cdot \bar{\varepsilon}_{\Psi;\max} = 4\,\frac{\Lambda + d}{\Lambda}\,\frac{d-1}{2} \cdot \left(\frac{1}{4}\,(d-1)\right)^2 \frac{\Lambda + d}{2\Lambda} \cdot \bar{\varepsilon}_{\Psi;\max} \\ &= \left(\frac{1}{4}\,\frac{\Lambda + d}{\Lambda}\right)^2 (d-1)^3 \cdot \bar{\varepsilon}_{\Psi;\max}. \end{aligned}$$

## 2.4 ▪ Opportunities for further research

### Infinite dimensional cases

One might ask oneself about the case of infinite-dimensional position manifolds such as Fréchet, Banach, or Hilbert manifolds. A Fréchet/Banach/Hilbert manifold is a manifold *modelled on a Fréchet/Banach/Hilbert space* in the sense of [Lan95, § II.1]. Recently, B. Eisenhuth and M. Grothaus applied in [EG21] the AHHM to the Langevin equation on the Cartesian product of two real separable Hilbert spaces. As every Hilbert manifold is parallelisable due to Kuiper's Theorem, see [EE70], the previously mentioned study also covers (up to diffeomorphisms) Hilbert manifolds as 'position' manifolds. Even though D. W. Henderson achieved in [Hen70] a nice classification result for separable metric Fréchet manifolds and by extension Banach manifolds, this classification is given just in terms of homeomorphisms instead of diffeomorphisms. I don't know a stronger classification theorem, but once such a result is found, one can pull the constructions of B. Eisenhuth and M. Grothaus back along the classification diffeomorphism.

### Boundary value problems

In our industrial application the limits of the conveyor belt do not cause interesting boundary effects, hence we just looked at position manifolds without boundary. A possible direction for further research would be considering manifolds $\mathbb{X}$ with nontrivial boundary. But other boundaries might arise from the choice of the configuration manifold Q: Suppose the 'algebraic' requirement that the norms of tangent vectors are contained in $\mathsf{A} \subseteq [0,\infty)$ and define

$$\mathsf{A}_x\mathbb{X} := \left\{ v \in \mathsf{T}_x\mathbb{X} \,\middle|\, |v|_{\mathsf{x}} \in \mathsf{A} \right\} \quad \text{for all } x \in \mathbb{X} \qquad \text{as well as} \qquad \mathsf{A}\mathbb{X} := \bigsqcup_{x\in\mathbb{X}} \mathsf{A}_x\mathbb{X}.$$

Since the Levi-Civita connection is compatible with the Riemannian metric, the associated parallel transport leaves $\mathsf{A}\mathbb{X}$ invariant. Furthermore, each $\mathsf{A}_x\mathbb{X}$ is invariant with respect to rotations. Let us go through some of the possible constructions:

- Two cases we already covered: The choice $\mathsf{A} = [0,\infty)$ yields the tangent bundle again. For $\mathsf{A} = \{r\}$ with $r \in (0,\infty)$ the set $\mathsf{A}\mathbb{X}$ is the spherical tangent bundle of radius $r$, which we know as the unit tangent bundle for $r = 1$.
- For $\mathsf{A} = [0,r]$ with fixed $r \in (0,\infty)$ the space $\mathsf{A}\mathbb{X}$ can be viewed as 'globular' tangent bundle. It sometimes is also called disk tangent bundle, even though the standard fibre is a closed unit ball. This choice corresponds to the algebraic side condition of a sharp upper bound $r$ on the velocity magnitude.
- Consider $\mathsf{A} = [r_0, r_1]$ for $r_0, r_1 \in (0,\infty]$ with $r_0 < r_1$. Then, we think $\mathsf{A}\mathbb{X}$ as 'annular' tangent space as the standard fibre is an annulus if $d = 2$ or a spherical shell if $d > 2$. This choice corresponds to the algebraic side condition of sharp upper and lower bounds on the velocity magnitude.

Note that the choices $\mathsf{A} \in \{[0,r],[r_0,r_1]\}$ indeed yield a fibre bundle

$$\pi_{0|\mathsf{A}}\colon\ \mathsf{A}\mathbb{X} \to \mathbb{X},\ \mathsf{A}_x\mathbb{X} \ni v \longmapsto x$$

according to the Ehresmann Fibration Theorem, as the projection mapping $\pi_{0|\mathsf{A}}$ is proper. We can construct local product measures in the sense of A. Götz on $\mathsf{A}\mathbb{X}$ due to the fact that the standard fibre is invariant with respect to rotations. Thus, one can consider reasonable SDAEs by choosing $\mathsf{Q} = \mathsf{A}\mathbb{X}$ which has nontrivial boundary.

In the light of this observation, one might also want 'ellipsoidal' tangent spaces that also encode side conditions of 'preferred' directions. However, that is not immediately possible due to the general lack of parallelisability: If there was a smooth fibre bundle $\mathbb{E} \to \mathbb{X}$ with $\mathbb{E} \subseteq \mathsf{T}\mathbb{X}$ and the standard fibre being an ellipsoid, an ellipsoidal ball, or an ellipsoidal shell, then the principal axes of the fibres would form a global smooth frame. In Chapter 4 we present ideas that would pave the way for such side conditions: Suppose a locally Lipschitzian frame $E$ that shall yield the principal axes. Then, the set $S(E)$ on which $E$ fails to be smooth is of measure zero and its complement is dense, by Rademacher's Theorem, see [Fed96, Theorem 3.1.6]. Declare $S(E)$ to be the set of singularities and make $\mathbb{X}$ into a stratifold, compare to Section 4.1.2. Then, the top stratum $\mathbb{X} \setminus S(E)$ is a smooth parallelisable manifold with $E$ as global frame. Therefore, one can mostly work on that top stratum employing a global trivialisation.

## Foliations

Another interesting research topic is the more general framework of Riemannian foliations. As mentioned before in Section 1.1, F. Baudoin and C. Tardif established in [Bau16; BT18] the theoretical setup and already proved first hypocoercivity results.

## Control on the position space

Instead of a static position manifold $\mathbb{X}$ one could imagine a 'smoothly varying' family $(\mathbb{X}_c)_c$ of position manifolds indexed by a control parameter $c$. In this way, one could model e. g. stretching and letting loose the conveyor belt. For instance, the smoothly varying family can formally be realised as smooth fibre bundle over a manifold of control parameters. This might be interesting for further optimisation of nonwoven production processes. I want to thank Wolfgang Bock for proposing that idea to me.

## More general potentials

Recall that we rely on the *a priori* estimates originally derived in [DMS15], see [GS14, Section A.1], specifically in order to verify (H4). Hence, we either require the entirety (P) of potential conditions or replace (P3) by the assumptions (A3), (A4), and (A5) from [GS14, Assumption A.2], compare to [GS14, Remark 3.16]. Naturally, one is interested in less regular or even singular potentials like Lennard-Jones potentials used in

Statistical Mechanics. Unfortunately, it still is an object of ongoing research whether strong $L^2$-hypocoercivity can be established for singular potentials. Recall that one can alternatively employ the method of weak hypocoercivity, which we mentioned in Section 1.1.

Regarding $L^2$-ergodicity for singular potentials we refer to [Sti14, Chapter 3]. Therein, P. Stilgenbauer generalises ideas from [GK08; CG10] and develops an abstract ergodicity method. The ideas come from settings with even $L^2$-exponential ergodicity related to hypocoercivity, but the ergodicity method covers several new situations.

In the special infinite-dimensional situation of [EG21] one observes that the assumptions on the potential can be chosen a little different, but the differences are significant. Particularly see [EG21, Hypotheses 2, 3, 6, and 7]: The potential $\Phi$ there is convex, lower semicontinuous, bounded from below, an element of the Sobolev space $H^{1,2}(\mu_1)$ with respect to the Gaußian measure $\mu_1$ on the base Hilbert space, furthermore it has an essentially bounded weak gradient, the measure $1/\int \exp(-\Phi)\, \mathrm{d}\mu_1 \cdot \exp(-\Phi)\mu_1$ satisfies some kind of a Poincaré inequality, and finally one assumes a certain inequality used to prove the first part of (H4). The proofs of [EG21] might inspire future improvements of the results in finite dimension.

## Multiplicative noise

Very recently, A. Bertram and M. Grothaus published in [BG22] the first results regarding essential m-dissipativity and hypocoercivity for Langevin equations on position space $\mathbb{R}^d$ that are subject to multiplicative noise. Meaning that instead of the system (2.1) they study the system [BG22, Equation (1.1)] which we write here as

$$\begin{aligned} \mathrm{d}x_t &= v_t \,\mathrm{d}t \\ \mathrm{d}v_t &= b(v_t)\,\mathrm{d}t - \nabla\Psi(x_t)\,\mathrm{d}t + \sigma(v_t)\circ \mathrm{d}W_t. \end{aligned} \tag{2.30}$$

The mapping $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is assumed to have at least weakly differentiable coefficients. The mapping $b = (b_i)_{i=1}^d\colon \mathbb{R}^d \to \mathbb{R}^d$ is given by $b_i := \sum_{j=1}^d \partial_j a_{ij} - a_{ij}\,\mathrm{pr}_j$ with $(a_{ij})_{i,j\in\{1,\dots,d\}} := \sigma\sigma^\top$.

In order to formulate the problem (2.30) in the manifold setting, let us denote for a vector bundle $\mathbb{E}\to\mathbb{B}$ by $L(\mathbb{E})$ the space of endomorphisms of $\mathbb{E}$. By *endomorphism* we mean a map $F\colon \mathbb{E}\to\mathbb{E}$ that is fibrewise a vector space endomorphism[3] $F\colon \mathbb{E}_b \to \mathbb{E}_b$ for all $b\in\mathbb{X}$. If $\mathbb{X}$ as before is our position manifold, then we choose $\sigma$ as a section in the endomorphism bundle $L(\mathsf{VT}\mathbb{X})\to\mathsf{T}\mathbb{X}$. Of course, the section $\sigma$ needs to be at least weakly differentiable again. Furthermore, the term $b$ is defined as

$$b := \mathrm{div}_{\mathsf{v}}(\Sigma) - \Sigma\mathcal{V} \qquad \text{with } \Sigma := \sigma\sigma^*,$$

where the adjoint of $\sigma$ is taken with respect to vertical Sasaki metric. The vertical divergence of $\Sigma$ is computed row-wise when $\Sigma$ is represented in matrix form.

[3] Since the standard fibre is a finite dimensional vector space by definition of vector bundles, the endomorphism is continuous and has a matrix representation given a local frame with domain containing $b$.

# 3 Higher order fibre lay-down models

Simulated trajectories of particles described by the fibre lay-down model (2.18) appear coarser than the fibre filaments in real life. One of the ideas for dealing with this discrepancy is to let the stochastic perturbation act not on the velocity, but rather on the level of acceleration, compare to [Her+09]. Specifically, it was proposed in [KMW12b] to replace the Wiener process by an Ornstein-Uhlenbeck process. Meaning in the Euclidean case

$$\mathrm{d}x_t = v_t \,\mathrm{d}t \qquad \mathrm{d}v_t = -\nabla_{\mathsf{v}|\mathsf{U}}\Psi^{\mathsf{h}}(v_t)\,\mathrm{d}t + \tilde{\sigma} \circ \mathrm{d}\widetilde{W}_t, \tag{3.1}$$

with $\tilde{\sigma} \in (0,\infty)$, one replaces $\widetilde{W}$ by a solution of the system

$$\mathrm{d}v_t = a_t \,\mathrm{d}t \qquad \mathrm{d}a_t = -\alpha \cdot a_t \,\mathrm{d}t + \sigma \circ \mathrm{d}W_t \tag{3.2}$$

on $\mathsf{T}(\mathsf{U}\mathbb{R}^d_x) \simeq \mathsf{T}(\mathbb{R}^d_x \times \mathbb{S}^{d-1})$. This yields the new model

$$\mathrm{d}x_t = v_t \,\mathrm{d}t \tag{3.3a}$$

$$\mathrm{d}v_t = -\nabla_{\mathsf{v}|\mathsf{U}}\Psi^{\mathsf{h}}(v_t)\,\mathrm{d}t + a_t\,\mathrm{d}t \tag{3.3b}$$

$$\mathrm{d}a_t = -\alpha \cdot a_t \,\mathrm{d}t + \sigma \circ \mathrm{d}W_t. \tag{3.3c}$$

In fact, the new model given by Equation (3.3) causes smoother looking trajections in simulations. Therefore, we refer to model (3.3) as the 'smoothed' version of the fibre lay-down model (3.3). In [Her+09; KMW12b] the authors rather speak of 'smooth fibre lay-down models', which we avoid because the term 'smooth' is quite overloaded. For the sake of illustration, one finds in Figure 3.1 a few numerical experiments – random seed and window sizes are fixed of course. Also, compare to [KMW12b, Figure 2].

In this chapter, we have the very same agenda as in Chapter 2, but now for 'smoothed' modifications of Langevin-type models. As before, we formulate models over the position manifold $\mathbb{X}$ and attempt to prove hypocoercivity by means of the AHHM. It turns out though that we have to adapt the smoothed fibre lay-down equation by adding a bit of additional noise to the velocity component. Beforehand, we will discuss in Section 3.1 the technical setup for higher order equations. With these preparations, we are in the position to run the machinery used in Chapter 2 for the adapted smoothed fibre lay-down model in Section 3.2. We see there that the additional velocity noise is needed in order to get the microscopic coercivity estimate in (H2). The main result is Theorem 3.2.12. At the end of the section, we investigate how the hypocoercivity constants behave as the additional diffusion parameter approaches zero. Afterwards in Section 3.3, we briefly touch upon the smoothed Langevin equation focusing only on a few new aspects. We also compare it in Section 3.4 to the confusingly named generalised Langevin equation (GLE) as discussed for instance in [OP11; PSV21].

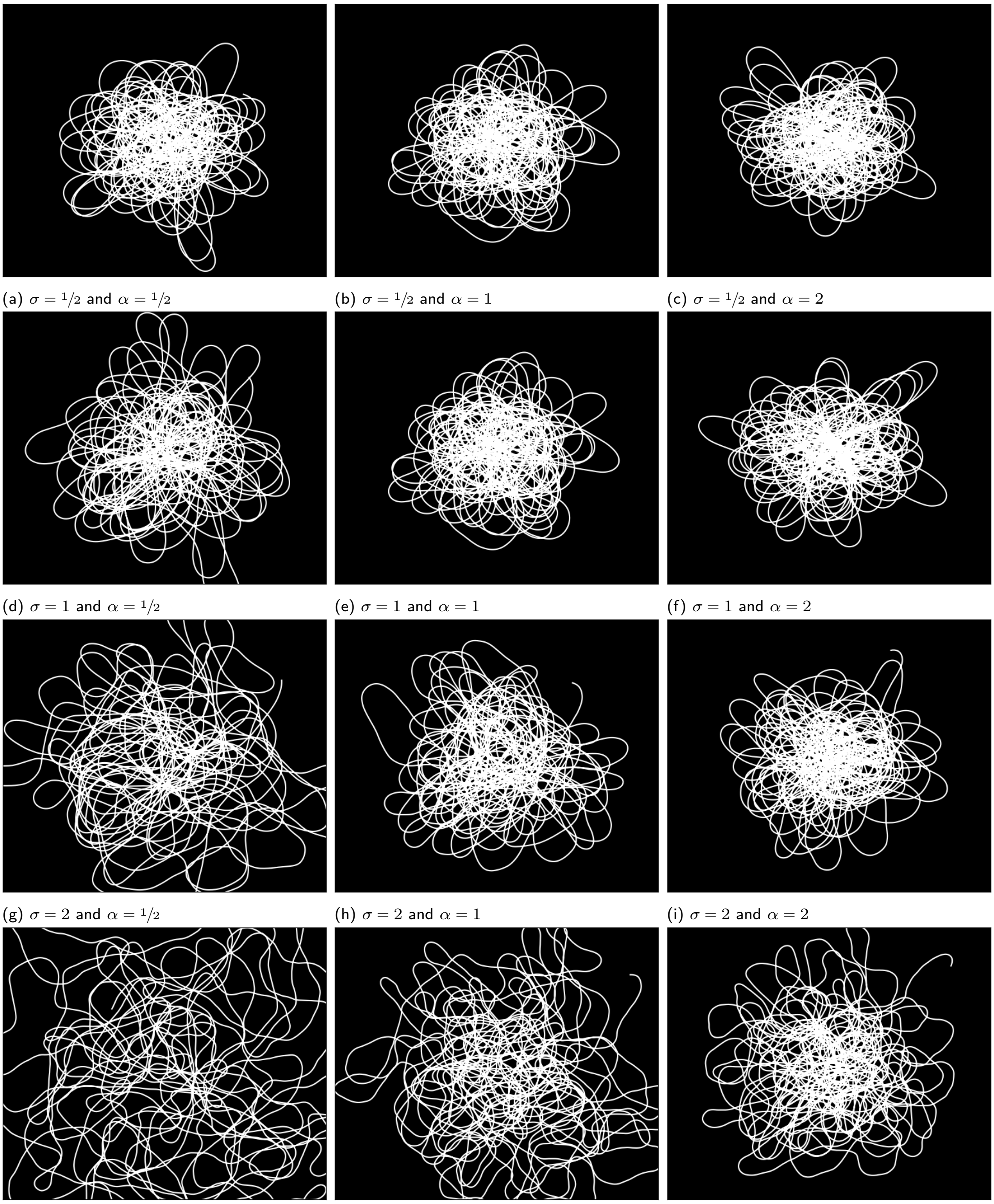
## 3.1 ▪ Higher order tangent spaces and related structures

In this section, we extend notions form Section 1.2 in regards to tangent bundles of higher order. We explore Ehresmannian Whitney sum decompositions, compatible Sasakian Riemannian metrics as well as associated differential operators. We encountered generalisations in this direction first in [BCD11, Section 3.1]. However, they are well-known in the course of higher order variational problems, see e. g. [Sau02]. Our aim here is to establish notions and notations that allow for easy computations and various degrees of freedom. In the end of Section 1.3, we mentioned other higher order objects that one could consider such as in the jet bundle formalism.

Let $\mathbb{B}$ be a $\mathscr{b}$-dimensional connected, smooth manifold. Recall Notation 1.2.8 on tangent bundles of order $k$. Since the case '$k = 1$' has been treated in Section 1.2 excessively, we assume that $k \in \mathbb{N} \setminus \{0,1\}$ for the rest of the section.

**Definition 3.1.1** (vertical distributions)**.** For a $k$th order tangent space there are $k$ many *vertical distributions* $\mathsf{V}^j\mathsf{T}^k\mathbb{B}$ for $j \in \{0,\dots,k-1\}$ characterised as the spaces tangent to the fibres of the bundle projection $\pi_{k,j}$. In formulae:

$$\mathsf{V}^j_u(\mathsf{T}^k\mathbb{B}) := \mathrm{null}(d_u\pi_{k,j}) \subseteq \mathsf{T}_u(\mathsf{T}^k\mathbb{B}) \qquad \text{for all } u \in \mathsf{T}^k\mathbb{B} \text{ and } j \in \{0,\dots,k-1\}.$$



(a) $\sigma = 1/2$ and $\alpha = 1/2$ (b) $\sigma = 1/2$ and $\alpha = 1$ (c) $\sigma = 1/2$ and $\alpha = 2$

(d) $\sigma = 1$ and $\alpha = 1/2$ (e) $\sigma = 1$ and $\alpha = 1$ (f) $\sigma = 1$ and $\alpha = 2$

(g) $\sigma = 2$ and $\alpha = 1/2$ (h) $\sigma = 2$ and $\alpha = 1$ (i) $\sigma = 2$ and $\alpha = 2$

(j) $\sigma = 5$ and $\alpha = 1/2$ (k) $\sigma = 5$ and $\alpha = 1$ (l) $\sigma = 5$ and $\alpha = 2$

Figure 3.1: Showcase of trajectories for the smoothed fibre lay-down model with $\sigma$ varying by row and $\alpha$ varying by column

For any $j \in \{0, \dots, k-1\}$ the dimension of the $j$th vertical distribution is computed as

$$\begin{aligned} \dim(\mathsf{V}^j\mathsf{T}^k\mathbb{B}) &= \dim(\mathsf{V}^{k-1}\mathsf{T}^k\mathbb{B}) + \sum_{\ell=j+1}^{k-1} \dim(\mathsf{T}^{k-\ell}\mathbb{B}) \\ &= \dim(\mathsf{V}^{k-1}\mathsf{T}^k\mathbb{B}) + \sum_{\ell=1}^{k-j-1} \dim(\mathsf{T}^{\ell}\mathbb{B}) = (2^k + 2^{k-j} - 2)\mathscr{b}. \end{aligned}$$

Furthermore, for $j \in \{0, \dots, k-1\}$ we define the $j$th vertical lift corresponding to the $j$th vertical distribution as follows: Let $u \in \mathsf{T}^k\mathbb{B}$ a base point and $w \in \mathsf{T}^j\mathbb{B}$ the vector to be lifted. The *$j$th vertical lift of $w$ at $u$ to $\mathsf{T}^{k+1}\mathbb{B}$* is the unique vector $\mathsf{v}_u\!\restriction_j^{k+1}(w) \in \mathsf{T}^{k+1}\mathbb{B}$ satisfying the relation

$$\left\langle \mathsf{v}_u\!\restriction_j^{k+1}(w)\,,\, d_u^{k+1-j}(d^j f)\right\rangle = \langle w\,,\, d^j f\rangle \qquad \text{for all } f \in C^\infty\mathbb{B}.$$

For the case of $k = 1$ one gets $\mathsf{v}\!\restriction_k^{k+1} = \mathsf{vl}$. Higher order vertical lifts can be constructed inductively:

$$\begin{aligned} \left\langle \mathsf{v}_u\!\restriction_j^{k+1}(w)\,,\, d_u^{k+1-j}(d^j f)\right\rangle &= \left\langle \mathsf{v}_u\!\restriction_j^{k+1}(w)\,,\, d_u\Big(d^{k-j}_{\pi_{k-1,k}(u)}(d^j f)\Big)\right\rangle = \langle w\,,\, d^j f\rangle \\ &\overset{\text{def}}{=} \left\langle \mathsf{vl}_u\Big(\mathsf{v}_{\pi_{k,k-1}(u)}\!\restriction_j^{k}(w)\Big)\,,\, d_u\Big(d^{k-j}_{\pi_{k-1,k}(u)}(d^j f)\Big)\right\rangle \\ &= \left\langle \mathsf{v}_{\pi_{k,k-1}(u)}\!\restriction_j^{k}(w)\,,\, d^{k-j}_{\pi_{k-1,k}(u)}(d^j f)\right\rangle, \end{aligned}$$

hence we have $\mathsf{v}_u\!\restriction_j^{k+1}(w) = \mathsf{vl}_u\Big(\mathsf{v}_{\pi_{k,k-1}(u)}\!\restriction_j^{k}(w)\Big) = \mathsf{v}_u\!\restriction_k^{k+1}\Big(\mathsf{v}_{\pi_{k,k-1}(u)}\!\restriction_j^{k}(w)\Big)$.

The assignment $\mathcal{V}^{(k,j)}(w) := \mathsf{v}_w\!\restriction_j^{k+1}(w)$ yields the analogue of the canonical vector field. One might call it *canonical $(k,j)$-vector field*. If $k = j$, then we just write $\mathcal{V}^{(k)} := \mathcal{V}^{(k,j)}$. $\lrcorner$

*Remark* 3.1.2 (motivation for Definition 3.1.3). The previously defined vertical distributions in $\mathsf{T}^{k+1}\mathbb{B}$ form a decreasing sequence:

$$\mathsf{V}^{k-1}\mathsf{T}^k\mathbb{B} \subseteq \mathsf{V}^{k-2}\mathsf{T}^k\mathbb{B} \subseteq \cdots \subseteq \mathsf{V}^1\mathsf{T}^k\mathbb{B} \subseteq \mathsf{V}^0\mathsf{T}^k\mathbb{B}.$$

In particular, $\mathsf{V}^{k-1}\mathsf{T}^k\mathbb{B} = \mathsf{V}\mathsf{T}^k\mathbb{B}$.

Consider an Ehresmann-type decomposition in terms of the Whitney sum $\mathsf{T}^{k+1}\mathbb{B} = \mathsf{V}^0\mathsf{T}^k\mathbb{B} \oplus \mathsf{H}^0\mathsf{T}^k\mathbb{B}$. Then, the vertical distributions $(\mathsf{V}^j\mathsf{T}^k\mathbb{B})_{j\in\{0,\dots,k-1\}}$ induce the following family of decompositions:

$$\begin{aligned} \mathsf{T}^{k+1}\mathbb{B} &= \mathsf{V}^1\mathsf{T}^k\mathbb{B} \oplus \mathsf{H}^1\mathsf{T}^k\mathbb{B} \oplus \mathsf{H}^0\mathsf{T}^k\mathbb{B}, \\ &\ \ \vdots \qquad\qquad \vdots \\ \mathsf{T}^{k+1}\mathbb{B} &= \mathsf{V}^{k-2}\mathsf{T}^k\mathbb{B} \oplus \bigoplus_{\ell\in\{0,\dots,k-2\}} \mathsf{H}^\ell\mathsf{T}^k\mathbb{B}, \\ \text{and}\qquad \mathsf{T}^{k+1}\mathbb{B} &= \mathsf{V}^{k-1}\mathsf{T}^k\mathbb{B} \oplus \bigoplus_{\ell\in\{0,\dots,k-1\}} \mathsf{H}^\ell\mathsf{T}^k\mathbb{B}. \end{aligned}$$

This observation gives rise to the following definition. $\lrcorner$

**Definition 3.1.3** (horizontal distribution). Consider an Ehresmann-type decomposition $\mathsf{T}^{k+1}\mathbb{B} = \mathsf{V}^0\mathsf{T}^k\mathbb{B} \oplus \mathsf{H}^0\mathsf{T}^k\mathbb{B}$. Then, we define recursively

$$\mathsf{H}^j\mathsf{T}^k\mathbb{B} := \mathsf{V}^0\mathsf{T}^k\mathbb{B} \setminus \left(\mathsf{V}^j\mathsf{T}^k\mathbb{B} \oplus \bigoplus_{\ell=1}^{j-1} \mathsf{H}^\ell\mathsf{T}^k\mathbb{B}\right) \cup \{0\} \quad \text{for all } j \in \{1, \dots, k-1\}. \tag{3.4}$$

The spaces $(\mathsf{H}^j\mathsf{T}^k\mathbb{B})_{j\in\{0,\dots,k-1\}}$ form a so-called *horizontal distribution*. $\lrcorner$

*Remark* 3.1.4. All the spaces $(\mathsf{H}^\ell\mathsf{T}^k\mathbb{B})_{\ell\in\{1,\dots,k-1\}}$ in Definition 3.1.3 originate from the space $\mathsf{H}^0\mathsf{T}^k\mathbb{B}$ by applying the tangent structure $J$ of $\mathsf{T}^k\mathbb{B}$ several times. The *tangent structure (or canonical endomorphism) of* $\mathsf{T}^k\mathbb{B}$ is the function

$$J\colon\ \mathsf{T}^{k+1}\mathbb{B} \to \mathsf{T}^{k+1}\mathbb{B},\ v \longmapsto \mathsf{vl}\big(\pi_{k+1,k}\big)_* v,$$

where we think of the vertical lift as vector bundle isomorphism $\mathsf{vl}\colon \left(\pi_{k+1,k}\right)_* \mathsf{T}^k\mathbb{B} \to \mathsf{VT}^k\mathbb{B}$ and therefore $\left(\pi_{k+1,k}\right)_* v$ denotes an element of the pullback bundle with respect to the projection $\pi_{k+1,k}$. The tangent structure is understood as a $(1,1)$-tensor field on $\mathsf{T}^k\mathbb{B}$ and has the properties that

$$\mathrm{null}(J^{k-j}) = \mathrm{ran}(J^{j+1}) \quad \text{and} \quad J^{k+1} = 0 \qquad \text{for } j \in \{0, \dots, k-1\}.$$

Eventually, it can be used to characterise both vertical and horizontal distribution as

$$\mathsf{V}^j\mathsf{T}^k\mathbb{B} = \mathrm{null}(J^{k-j}) = \mathrm{ran}(J^{j+1}) \quad \text{and} \quad \mathsf{H}^\ell\mathsf{T}^k\mathbb{B} = J^\ell(\mathsf{H}^0\mathsf{T}^k\mathbb{B})$$

for all $j \in \{0, \dots, k-1\}$ and $\ell \in \{0, \dots, k-1\}$. See [BCD11, Section 3.1]. ⌟

**Example 3.1.5.** For our purposes the most interesting example arises from $k$ 'stacked' Ehresmann connections

$$\mathsf{T}^{j+1}\mathbb{B} = \mathsf{V}^{j-1}\mathsf{T}^j\mathbb{B} \oplus \mathsf{HT}^j\mathbb{B} = \mathsf{VT}^j\mathbb{B} \oplus \mathsf{HT}^j\mathbb{B} \qquad \text{for all } j \in \{1, \dots, k\}.$$

The respective horizontal lifts $\mathsf{hl} = \mathsf{h}{\upharpoonright}_j^{j+1}\colon \ \mathsf{T}^j\mathbb{B} \to \mathsf{HT}^j\mathbb{B}$ for $j \in \{1, \dots, k\}$ can be concatenated yielding the lifts

$$\mathsf{h}{\upharpoonright}_\ell^{j+1}\colon \ \mathsf{T}^\ell\mathbb{B} \to \mathsf{HT}^j\mathbb{B}, \ w \longmapsto \mathsf{h}{\upharpoonright}_j^{j+1} \circ \dots \circ \mathsf{h}{\upharpoonright}_\ell^{\ell+1}(w) \qquad \text{for } \ell \in \{1, \dots, j\}.$$

Now, the horizontal distributions are given as

$$\begin{aligned} & \mathsf{H}^0\mathsf{T}^k\mathbb{B} = \mathsf{h}{\upharpoonright}_1^{k+1} \mathsf{T}\mathbb{B} \\ \text{and} \quad & \mathsf{H}^\ell\mathsf{T}^k\mathbb{B} = \mathsf{h}{\upharpoonright}_{k-(\ell-1)}^{k+1} \left(\mathsf{V}^{k-(\ell-1)-2}\mathsf{T}^{k-(\ell-1)-1}\mathbb{B}\right) = \mathsf{h}{\upharpoonright}_{k-\ell+1}^{k+1} \left(\mathsf{VT}^{k-\ell}\mathbb{B}\right) \end{aligned}$$

for all $\ell \in \{1, \dots, k-1\}$. ⌟

**Definition 3.1.6** (Ehresmann multiconnection)**.** Assume Ehresmann connections

$$\mathsf{T}^{j+1}\mathbb{B} = \mathsf{V}^{j-1}\mathsf{T}^j\mathbb{B} \oplus \mathsf{HT}^j\mathbb{B} = \mathsf{VT}^j\mathbb{B} \oplus \mathsf{HT}^j\mathbb{B} \qquad \text{for all } j \in \{1, \dots, k\}.$$

An *Ehresmann multiconnection* consists of the Whitney sum gradings

$$\mathsf{T}^{j+1}\mathbb{B} = \mathsf{V}^{j-1}\mathsf{T}^j\mathbb{B} \oplus \bigoplus_{\ell \in \{0, \dots, j-1\}} \mathsf{H}^\ell\mathsf{T}^j\mathbb{B} \qquad \text{for all } j \in \{1, \dots, k\}. \tag{3.5}$$

⌟

Horizontal distributions can be described in terms of semisprays and vice versa. The following definition of higher-order semisprays covers semisprays as in Definition 1.2.27 as 'semisprays of first order'.

**Definition 3.1.7** (higher-order semisprays)**.** A vector field $\mathcal{H}^{(k)} \in \Gamma^\infty(\mathsf{T}\mathbb{B}; \mathsf{T}^{k+1}\mathbb{B})$ is called *semispray of order* $k$ if it satisfies $\langle \mathcal{H}^{(k)}, d\pi_{k,0} \rangle = \mathrm{Id}_{\mathsf{T}\mathbb{B}}$. Equivalently, $\mathcal{H}^{(k)}$ is a semispray of order $k$ if any integral curve $s\colon I \to \mathsf{T}^k\mathbb{B}$ is of the form[1] $s = j^k(\pi_{k,0} \circ s)$, cf [BCD11, p. 11]. By analogy to the first order case, we call a curve $c\colon I \to \mathbb{B}$ *geodesic of* $\mathcal{H}^{(k)}$ if there is an integral curve $s\colon I \to \mathsf{T}^k\mathbb{B}$ of $\mathcal{H}^{(k)}$ such that $c = \pi_{k,0} \circ s$. ⌟

*Remark* 3.1.8 (local coefficients and generalised geodesic equations)**.** Higher-order semisprays, just as first-order ones, are determined by local coefficients. These coefficients appear in the $(k+1)$-order ordinary differential equation [BCD11, Equation (3.3)] that resembles a geodesic equation. See [BCD11, Section 3] for classification results on higher-order semisprays, reminiscent of the ones for the first-order semisprays, using these coefficients. ⌟

*Remark* 3.1.9 (horizontal distribution induced from higher order semispray)**.** It should be clear from Definition 3.1.7 that a $k$th order semispray $\mathcal{H}^{(k)}$ defines a space $\mathsf{H}^0\mathsf{T}^k\mathbb{B}$ as its range: $\mathsf{H}^0\mathsf{T}^k\mathbb{B} := \mathrm{ran}(\mathcal{H}^{(k)})$. From Remark 3.1.4 we know that a full horizontal distribution originates from this space applying the tangent structure repeatedly. ⌟

*Remark* 3.1.10. A $k$th order semispray can be built from $k$ semisprays of first order: Assume there are semisprays $\mathcal{H}^{(j+1,j)} \in \Gamma^\infty(\mathsf{T}^{j+1}\mathbb{B}; \mathsf{T}^{j+2}\mathbb{B})$ of first order for all $j \in \{0, \dots, k-1\}$. Define the vector field $\mathcal{H}^{(k)}$ as iterative composition of the given semispays: $\mathcal{H}^{(k)} := \mathcal{H}^{(k,k-1)} \circ \dots \circ \mathcal{H}^{(1,0)}$. Then, it's easy to check that

$$\left\langle \mathcal{H}^{(k)}, d\pi_{k,0} \right\rangle = \left\langle \mathcal{H}^{(k)}, d_u\left(\pi_{1,0} \circ \pi_{2,1} \circ \dots \circ \pi_{k-1,k-2} \circ \pi_{k,k-1}\right)\right\rangle$$

[1] Here '$j^k$' denotes the $k$th order jet, see [KMS93, Section IV.12].

$$
\begin{aligned}
&= \left\langle \mathcal{H}^{(k,k-1)} \circ \dots \circ \mathcal{H}^{(1,0)} \,,\, d_u\big(\pi_{1,0} \circ \pi_{2,1} \circ \dots \circ \pi_{k-1,k-2} \circ \pi_{k,k-1}\big)\right\rangle \\
&= \Big\langle \mathcal{H}^{(k,k-1)} \circ \dots \circ \mathcal{H}^{(1,0)} \,, \\
&\qquad d_{\pi_{k,k-1}(u)}\big(\pi_{1,0} \circ \pi_{2,1} \circ \dots \circ \pi_{k-1,k-2} \circ \pi_{k-1,k-2}\big) \circ d_u \pi_{k,k-1}\Big\rangle \\
&= \mathsf{Id}_{\mathsf{T}\mathbb{B}}.
\end{aligned}
$$

In case of an Ehresmann multiconnection the horizontal lifts yield higher-order semisprays via

$$
\mathcal{H}^{(k,\ell)}(w) := \mathsf{h}_w\!\upharpoonright_{\ell}^{k+1}(w) \qquad \text{for all } w \in \mathsf{T}^\ell\mathbb{B}.
$$

We might call these vector fields $(k,\ell)$*-semisprays on* $\mathbb{B}$. ⌟

**Example 3.1.11.** Consider the Riemannian semispray $\mathcal{H}_{\mathsf{x}}$ for $(\mathbb{X},\mathsf{x})$ and the tangent space $(\mathsf{T}\mathbb{X},\mathsf{t})$ endowed with Sasaki metric. Associated to the Sasaki metric there is the Riemannian semispray $\mathcal{H}_{\mathsf{t}}$, which might be called *Sasakian semispray* to avoid confusion. Then, the composition $\mathcal{H}_{\mathsf{x}}^{(2)} := \mathcal{H}_{\mathsf{t}} \circ \mathcal{H}_{\mathsf{x}}$ is a semispray of second order. As it depends just on the choice of the metric $\mathsf{x}$, it's reasonably called the *Riemannian semispray of second order*. ⌟

**Definition 3.1.12** (Sasakian multimetric).

(i) Assume an Ehresmann multiconnection as in Definition 3.1.6. Let a Riemannian metric $\mathsf{t}_{[k]}$ of the form

$$
\mathsf{t}_{[k]}\colon\ \mathsf{T}^{k+1}\mathbb{B} \times \mathsf{T}^{k+1}\mathbb{B} \to \mathbb{R},\ (a_1,a_2) \mapsto \sum_{\ell\in\{0,\dots,k-1\}} \mathsf{t}_{[k;\ell]}(a_1\,,a_2)
$$

build from a family $(\mathsf{t}_{[k;\ell]})_{\ell\in\{0,\dots,k-1\}}$ of Riemannian metrics

$$
\mathsf{t}_{[k;\ell]}\colon\ \mathsf{V}^{k-1}\mathsf{T}^k\mathbb{B} \oplus \mathsf{H}^\ell\mathsf{T}^k\mathbb{B} \times \mathsf{V}^{k-1}\mathsf{T}^k\mathbb{B} \oplus \mathsf{H}^\ell\mathsf{T}^k\mathbb{B} \to \mathbb{R}.
$$

Assume that for all $\ell \in \{0,\dots,k-1\}$ there are given Riemannian metrics $\mathsf{g}_\ell$ on $\mathsf{T}^\ell\mathbb{B}$. We call $\mathsf{t}_{[k]}$ *natural (with respect to* $(\mathsf{g}_\ell)_{\ell\in\{0,\dots,k-1\}}$ *)* if all elements of the family $(\mathsf{t}_{[k;\ell]})_{\ell\in\{0,\dots,k-1\}}$ are natural metrics in the sense that

$$
\begin{aligned}
\mathsf{t}_{[k;\ell]}(\mathcal{X}\,,\mathcal{Y}) = 0 &\qquad \text{and} \\
\mathsf{t}_{[k;0]}\big(\mathsf{h}\!\upharpoonright_1^{k+1}\mathcal{Z}_1\,,\mathsf{h}\!\upharpoonright_1^{k+1}\mathcal{Z}_2\big) = \mathsf{g}_0(\mathcal{Z}_1,\mathcal{Z}_2) &\quad \text{respectively} \\
\mathsf{t}_{[k;\ell]}\big(\mathsf{h}\!\upharpoonright_{k+1-\ell}^{k+1}\mathsf{v}\!\upharpoonright_{k-\ell}^{k+1-\ell}\mathcal{W}_1\,,\mathsf{h}\!\upharpoonright_{k+1-\ell}^{k+1}\mathsf{v}\!\upharpoonright_{k-\ell}^{k+1-\ell}\mathcal{W}_2\big) = \mathsf{g}_{k-\ell-1}(\mathcal{W}_1,\mathcal{W}_2) &\quad \text{if } \ell \neq 0
\end{aligned}
$$

for all $\mathcal{X} \in \Gamma^\infty(\mathsf{V}^{k-1}\mathsf{T}^k\mathbb{B})$, $\mathcal{Y} \in \Gamma^\infty(\mathsf{H}^\ell\mathsf{T}^k\mathbb{B})$, as well as $\mathcal{Z}_1,\mathcal{Z}_2 \in \Gamma^\infty(\mathsf{T}\mathbb{B})$, and $\mathcal{W}_1,\mathcal{W}_2 \in \Gamma^\infty(\mathsf{T}^{k-\ell}\mathbb{B})$. Compare to [GK02, Definition 6.1].

(ii) Assume a Riemannian metric $\mathsf{b}$ on $\mathbb{B}$ and that the Ehresmann multiconnection is induced by $\mathsf{b}$. Define $\mathsf{t}_{[0]} := \mathsf{b}$. Furthermore, let Riemannian metrics $(\mathsf{t}_{[j]})_{j\in\{0,\dots,k\}}$ which are of the form as in part (i) for $j \in \{1,\dots,k\}$.

The family $(\mathsf{t}_{[j]})_{j\in\{0,\dots,k\}}$ is called *Sasaki multimetric* if it respects the Ehresmann multiconnection by which we mean that

- it is *natural* in the recursive sense that for all $j \in \{1,\dots,k\}$ the metric $\mathsf{t}_{[j]}$ is natural with respect to $(\mathsf{t}_{[j']})_{j'\in\{0,\dots,j-1\}}$,
- and additionally holds

$$
\begin{aligned}
\mathsf{t}_{[k;0]}\big(\mathsf{v}\!\upharpoonright_1^{k+1}\mathcal{Z}_1\,,\mathsf{v}\!\upharpoonright_1^{k+1}\mathcal{Z}_2\big) &= \mathsf{t}_{[0]}(\mathcal{Z}_1\,,\mathcal{Z}_2) = \mathsf{b}(\mathcal{Z}_1\,,\mathcal{Z}_2) \\
\mathsf{t}_{[k;\ell]}\big(\mathsf{v}\!\upharpoonright_{k+1-\ell}^{k+1}\mathcal{W}_1\,,\mathsf{v}\!\upharpoonright_{k+1-\ell}^{k+1}\mathcal{W}_2\big) &= \mathsf{t}_{[k-\ell]}(\mathcal{W}_1\,,\mathcal{W}_2) \quad \text{if } \ell \neq 0
\end{aligned}
$$

  for all $\mathcal{Z}_1,\mathcal{Z}_2 \in \Gamma^\infty(\mathsf{T}\mathbb{B})$ and $\mathcal{W}_1,\mathcal{W}_2 \in \Gamma^\infty(\mathsf{T}^{k+1-\ell}\mathbb{B})$.

Consider the situation of part (ii). We see that $\mathsf{t}_{[1]}$ is the Sasaki metric with respect to $\mathsf{b}$ as in Definition 1.2.29, or in formulae $\mathsf{t}_{[1]} = \mathsf{t}_{[1;0]} = \mathsf{t} = \mathsf{v} + \mathsf{h}$. Just as for the standard Sasaki metric, a Sasakian multimetric naturally decomposes into a *vertical multimetric* $(\mathsf{v}_{[j]})_{j\in\{0,\dots,k\}}$ and a *horizontal multimetric*

$(\mathsf{h}_{[j]})_{j\in\{0,\dots,k\}}$: For all $j \in \{1,\dots,k\}$ and $\ell \in \{0,\dots,j-1\}$ the marginal metric $\mathsf{t}_{[j;\ell]}$ splits into a vertical and a horizontal part as $\mathsf{t}_{[j;\ell]} = \mathsf{v}_{[j]} + \mathsf{h}_{[j;\ell]}$; the marginal vertical metric and its horizontal counterpart are characterised by

$$
\begin{aligned}
\mathsf{h}_{[j;\ell]}(\mathcal{X}\,, \mathcal{Y}) &= 0 \\
\mathsf{h}_{[j;0]}(\mathsf{h}\!\upharpoonright_{1}^{j+1} \mathcal{Z}_1\,, \mathsf{h}\!\upharpoonright_{1}^{j+1} \mathcal{Z}_2) &= \mathsf{b}(\mathcal{Z}_1\,, \mathcal{Z}_2) \\
\mathsf{h}_{[j;\ell]}\big(\mathsf{h}\!\upharpoonright_{j+1-\ell}^{j+1} \mathsf{v}\!\upharpoonright_{j-\ell}^{j+1-\ell} \mathcal{W}_1\,, \mathsf{h}\!\upharpoonright_{j+1-\ell}^{j+1} \mathsf{v}\!\upharpoonright_{j-\ell}^{j+1-\ell} \mathcal{W}_2\big) &= \mathsf{t}_{[j-\ell]}(\mathcal{W}_1\,, \mathcal{W}_2) \quad \text{if } \ell \neq 0 \\
\mathsf{v}_{[j]}(\mathcal{X}\,, \mathcal{Y}) &= 0, \\
\text{and} \quad \mathsf{v}_{[j]}\Big(\mathsf{v}\!\upharpoonright_{j}^{j+1} \widetilde{\mathcal{W}}_1\,, \mathsf{v}\!\upharpoonright_{j}^{j+1} \widetilde{\mathcal{W}}_2\Big) &= \mathsf{t}_{[j-1]}\Big(\widetilde{\mathcal{W}}_1\,, \widetilde{\mathcal{W}}_2\Big)
\end{aligned}
\tag{3.6}
$$

for all $\mathcal{X} \in \Gamma^\infty(\mathsf{V}^{j-1}\mathbb{B})$, $\mathcal{Y} \in \Gamma^\infty(\mathsf{H}^{\ell}\mathsf{T}^{j}\mathbb{B})$, $\mathcal{W}_1, \mathcal{W}_2 \in \Gamma^\infty(\mathsf{T}^{j-\ell}\mathbb{B})$, $\widetilde{\mathcal{W}}_1, \widetilde{\mathcal{W}}_2 \in \Gamma^\infty(\mathsf{T}^{j}\mathbb{B})$, and $\mathcal{Z}_1, \mathcal{Z}_2 \in \Gamma^\infty(\mathsf{T}\mathbb{B})$. The horizontal part $\mathsf{h}_{[j]}$ reads as

$$
\mathsf{h}_{[j]} = \sum_{\ell\in\{0,\dots,j-1\}} \mathsf{h}_{[j;\ell]}.
$$

⌟

*Remark* 3.1.13. Assume a Riemannian metric $\mathsf{b}$ on $\mathbb{B}$, then a Sasakian multimetric is constructed iteratively: Then, $\mathsf{b}$ induces an Ehresmann connection $\mathsf{T}^2\mathbb{B} = \mathsf{VT}\mathbb{B} \oplus \mathsf{HT}\mathbb{B}$ and a corresponding Sasaki metric $\mathsf{t}$. We mentioned before in Example 3.1.11 that the Sasaki metric $\mathsf{t}_{[1]} = \mathsf{t}$ induces an associated semispray as a Riemannian metric on $\mathsf{T}\mathbb{B}$, which in turn induces an Ehresmann connection on $\mathsf{T}^3\mathbb{B}$ by means of Remark 3.1.4. The latter defines a Sasaki metric $\mathsf{t}_{[2]}$ on $\mathsf{T}^2\mathbb{B}$, and so on. In fact, there is no degree of freedom for another set of metrics that could meet the properties required by Definition 3.1.12 part (ii). ⌟

Henceforth, we consider a Riemannian manifold $(\mathbb{B}, \mathsf{b})$ with the induced Ehresmann multiconnection and Sasakian multimetric $(\mathsf{t}_{[j]})_{j\in\{1,\dots,k\}} = (\mathsf{v}_{[j]} + \mathsf{h}_{[j]})$ according to Remark 3.1.13. For the most part, we are interested in the case $k = 2$ for our third-order models and thus simplify the notation appropriately.

*Notation* 3.1.14 (Simplyfied notation for $k = 2$). Consider $k = 2$. Then, the Sasakian multimetric $(\mathsf{t}_{[j]})_{j\in\{0,1,2\}}$ on $\mathsf{T}^2\mathbb{B}$ consists of

$$
\mathsf{t}_{[0]} = \mathsf{b}, \quad \mathsf{t}_{[1]} = \mathsf{t} \quad \text{and} \quad \mathsf{t}_{[2]} = \mathsf{v}_{[2]} + \mathsf{h}_{[2;1]} + \mathsf{h}_{[2;0]}.
$$

The gradient with respect to $\mathsf{t}_{[2]}$ splits into three parts, two of which get a shorthand notation as well:

$$
\nabla_{\mathsf{h}1} := \nabla_{\mathsf{h}[2;1]} \quad \text{and} \quad \nabla_{\mathsf{h}0} := \nabla_{\mathsf{h}[2;0]}.
$$

Of course, we will proceed similarly for other objects induced by $\mathsf{t}_{[2]}$ such as the divergence and the Laplacian, e. g. $\Delta_{\mathsf{h}0} = \Delta_{\mathsf{h}[2;0]}$. ⌟

*Notation* 3.1.15 (vertical/horizontal lift of functions). We iterate on Remark 1.2.31 for higher order lifts of functions. Vertical lifts of $f_0$ are the pullbacks with respect to the projections in the tangent multibundle: The *$j$th vertical lift of $f_0$* with $j \in \{1,\dots,k\}$ is defined as $f_0^{\mathsf{v}[j]} := f_0 \circ \pi_{j,0}$. If we operate with particular numbers for $j$, then we omit square brackets for sake of readability, e. g. $f_0^{\mathsf{v}2} = f_0^{\mathsf{v}[2]}$. Similarly to Remark 1.2.31, the *$j$th horizontal lift $f_0^{\mathsf{h}[j]}$ of $f_0$* with $j \in \{1,\dots,k\}$ is defined by

$$
\left\langle a\,, df_0^{\mathsf{h}[j]} \right\rangle = \langle a\,, df \circ d\kappa_{[j,0]} \rangle \qquad \text{for all } a \in \mathsf{T}^{j+1}\mathbb{B}
$$

and might be abbreviated for explicit numbers e. g. as $f_0^{\mathsf{h}2} = f_0^{\mathsf{h}[2]}$. Here, the mapping $d\kappa_{[j,0]}$ is the concatenation of connector maps $d\kappa_{[j,j-1]}$, …, $d\kappa_{[2,1]}$, $d\kappa_{[1,0]}$, where $d\kappa_{[1,0]}$ is the connector map with respect to $\mathsf{b}$, $d\kappa_{[2,1]}$ is the connector map with respect to $\mathsf{t}$, $d\kappa_{[3,2]}$ is the connector map with respect to $\mathsf{t}_{[2]}$, and so on. ⌟

With the lifting procedure in place, we explore the action of differential operators and semisprays on suitable functions, similar to Lemma 1.2.35.

**Lemma 3.1.16** (Differential operators associated to a Sasaki multiconnection).

(i) Let $f \in C_{\mathsf{c}}^{\infty}(\mathbb{B})$ and denote by $f^{\mathsf{l}}\colon \mathsf{T}^{k-1}\mathbb{B} \to \mathbb{R}$ some lifting of $f$ involving vertical and horizontal liftings in a not specified order.

Then, for the vertical and horizontal gradients hold the following relations:

$$
\begin{aligned}
&\nabla_{\mathsf{v}[k]}\left(f^{\mathsf{l}}\right)^{\mathsf{h}} = \nabla_{\mathsf{t}[k-1]} f^{\mathsf{l}} \quad \text{and} \quad \nabla_{\mathsf{v}[k]}\left(f^{\mathsf{l}}\right)^{\mathsf{v}} = 0 \quad \text{as well as}\\
&\nabla_{\mathsf{h}[k]}\left(f^{\mathsf{l}}\right)^{\mathsf{v}} = \nabla_{\mathsf{t}[k-1]} f^{\mathsf{l}} \quad \text{and} \quad \nabla_{\mathsf{h}[k]}\left(f^{\mathsf{l}}\right)^{\mathsf{h}} = 0, \quad \text{in particular}\\
&\nabla_{\mathsf{v}[k]} f^{\mathsf{h}[k]} = \mathsf{v}{\restriction}_1^{k+1}\left(\nabla_{\mathsf{b}} f \circ \pi_{k,0}\right) \quad \text{and} \quad \nabla_{\mathsf{h}[k]} f^{\mathsf{v}[k]} = \mathsf{h}{\restriction}_1^{k+1}\left(\nabla_{\mathsf{b}} f \circ \pi_{k,0}\right).
\end{aligned}
$$

(ii) Let $\mathcal{X} \in \Gamma^{\infty}(\mathsf{T}\mathbb{B})$ and denote by $\mathcal{X}^{\mathsf{l}} \in \Gamma^{\infty}(\mathsf{T}^{k}\mathbb{B})$ some lifting of $\mathcal{X}$ involving vertical and horizontal liftings in a not specified order.

Then, for the vertical and horizontal divergences hold the following relations:

$$
\begin{aligned}
&\operatorname{div}_{\mathsf{v}[k]} \mathsf{h}{\restriction}_k^{k+1}\left(\mathcal{X}^{\mathsf{l}}\right) = \operatorname{div}_{\mathsf{t}[k-1]} \mathcal{X}^{\mathsf{l}} \quad \text{and} \quad \operatorname{div}_{\mathsf{v}[k]} \mathsf{v}{\restriction}_k^{k+1}\left(\mathcal{X}^{\mathsf{l}}\right) = 0 \quad \text{as well as}\\
&\operatorname{div}_{\mathsf{h}[k]} \mathsf{v}{\restriction}_k^{k+1}\left(\mathcal{X}^{\mathsf{l}}\right) = \operatorname{div}_{\mathsf{t}[k-1]} \mathcal{X}^{\mathsf{l}} \quad \text{and} \quad \operatorname{div}_{\mathsf{h}[k]} \mathsf{h}{\restriction}_k^{k+1}\left(\mathcal{X}^{\mathsf{l}}\right) = 0 \quad \text{in particular}\\
&\operatorname{div}_{\mathsf{v}[k]} \mathsf{h}{\restriction}_1^{k+1} \mathcal{X} = \operatorname{div}_{\mathsf{b}} \mathcal{X} \quad \text{and} \quad \operatorname{div}_{\mathsf{h}[k]} \mathsf{v}{\restriction}_1^{k+1} \mathcal{X} = \operatorname{div}_{\mathsf{b}} \mathcal{X}\,.
\end{aligned}
$$

(iii) Let $f \in C_{\mathsf{c}}^{\infty}(\mathbb{B})$ and denote by $f^{\mathsf{l}}\colon \mathsf{T}^{k-1}\mathbb{B} \to \mathbb{R}$ some lifting of $f$ involving vertical and horizontal liftings in a not specified order.

Combining part (i) and (ii) we get that

$$
\begin{aligned}
&\Delta_{\mathsf{v}[k]}\left(f^{\mathsf{l}}\right)^{\mathsf{h}} = \Delta_{\mathsf{t}[k-1]} f^{\mathsf{l}} \quad \text{and} \quad \Delta_{\mathsf{v}[k]}\left(f^{\mathsf{l}}\right)^{\mathsf{v}} = 0 \quad \text{as well as}\\
&\Delta_{\mathsf{h}[k]}\left(f^{\mathsf{l}}\right)^{\mathsf{v}} = \Delta_{\mathsf{t}[k-1]} f^{\mathsf{l}} \quad \text{and} \quad \Delta_{\mathsf{h}[k]}\left(f^{\mathsf{l}}\right)^{\mathsf{h}} = 0, \quad \text{in particular}\\
&\Delta_{\mathsf{v}[k]} f^{\mathsf{h}[k]} = \Delta_{\mathsf{b}} f \circ \pi_{k,0} = \left(\Delta_{\mathsf{b}} f\right)^{\mathsf{v}[k]} \quad \text{and}\\
&\Delta_{\mathsf{h}[k]} f^{\mathsf{v}[k]} = \Delta_{\mathsf{b}} f \circ \pi_{k,0} = \left(\Delta_{\mathsf{b}} f\right)^{\mathsf{v}[k]}.
\end{aligned}
$$

**Corollary 3.1.17** ('Higher-order' Liouville Theorem)**.** For the Riemannian $k$th order semispray $\mathcal{H}_{\mathsf{t}[k-1]} = \mathcal{H}_{\mathsf{t}[k-2]} \circ \dots \circ \mathcal{H}_{\mathsf{b}}$ with respect to $(\mathbb{B}, \mathsf{b})$ holds $\operatorname{div}_{\mathsf{t}[k]} \mathcal{H}_{\mathsf{t}[k-1]} = \operatorname{div}_{\mathsf{t}} \mathcal{H}_{\mathsf{b}} = 0$. In other words, $\mathcal{H}_{\mathsf{t}[k-1]}$ is solenoidal.

**Example 3.1.18** (Dense set of lifted functions and Euclidean case)**.** The tensor product of the four spaces

$$
\begin{aligned}
\left(C_{\mathsf{c}}^{\infty}(\mathbb{B})\right)^{\mathsf{v2}} &:= \left\{ f^{\mathsf{v2}} \mid f \in C_{\mathsf{c}}^{\infty}(\mathbb{B}) \right\}, \quad \left(C_{\mathsf{c}}^{\infty}(\mathbb{B})\right)^{\mathsf{h1v1}} := \left\{ \left(f^{\mathsf{h1}}\right)^{\mathsf{v1}} \,\middle|\, f \in C_{\mathsf{c}}^{\infty}(\mathbb{B}) \right\},\\
\left(C_{\mathsf{c}}^{\infty}(\mathbb{B})\right)^{\mathsf{v1h1}} &:= \left\{ \left(f^{\mathsf{v1}}\right)^{\mathsf{h1}} \,\middle|\, f \in C_{\mathsf{c}}^{\infty}(\mathbb{B}) \right\}, \quad \left(C_{\mathsf{c}}^{\infty}(\mathbb{B})\right)^{\mathsf{h2}} := \left\{ f^{\mathsf{h2}} \mid f \in C_{\mathsf{c}}^{\infty}(\mathbb{B}) \right\}
\end{aligned}
$$

is a total subset of $C_{\mathsf{c}}^{\infty}(\mathsf{T}^2\mathbb{B})$ with respect to the usual local convex topology, where convergence also implies uniform convergence of all derivatives on compacts, compare to Lemma 1.2.34. This is significant as certain computations are done on this tensor product space

$$
D_0 := \left(C_{\mathsf{c}}^{\infty}(\mathbb{B})\right)^{\mathsf{v2}} \otimes \left(C_{\mathsf{c}}^{\infty}(\mathbb{B})\right)^{\mathsf{h1v1}} \otimes \left(C_{\mathsf{c}}^{\infty}(\mathbb{B})\right)^{\mathsf{v1h1}} \otimes \left(C_{\mathsf{c}}^{\infty}(\mathbb{B})\right)^{\mathsf{h2}}. \tag{3.7}
$$

Particularly, let $\mathbb{B} = \mathbb{R}^{\mathscr{d}}$ with standard Euclidean geometry, thus one has $\mathsf{T}^2\mathbb{B} \simeq \mathbb{R}^{\mathscr{d}}_x \oplus \mathbb{R}^{\mathscr{d}}_v \oplus \mathbb{R}^{2\mathscr{d}}_a$. Then, we write out what the possible lifts of $f \in C_{\mathsf{c}}^{\infty}(\mathbb{R}^{\mathscr{d}})$ to a function on $\mathsf{T}^2\mathbb{R}^{\mathscr{d}}$ could be:

- $f^{\mathsf{v2}}$ coincides with $\mathsf{T}^2\mathbb{R}^{\mathscr{d}} \to \mathbb{R},\ (x, v, a) \longmapsto f(x)$,
- $\left(f^{\mathsf{h1}}\right)^{\mathsf{v1}}$ coincides with $\mathsf{T}^2\mathbb{R}^{\mathscr{d}} \to \mathbb{R},\ (x, v, a) \longmapsto f(v)$ and,
- if we mirror the Ehresmann connection as $\mathbb{R}^{2\mathscr{d}}_a = \mathbb{R}^{\mathscr{d}}_{\mathsf{vpr}(a)} \oplus \mathbb{R}^{\mathscr{d}}_{\mathsf{hpr}(a)}$, then $f^{\mathsf{h2}}$ coincides with $\mathsf{T}^2\mathbb{R}^{\mathscr{d}} \to \mathbb{R},\ (x, v, a) \mapsto f(\mathsf{hpr}(a))$ as well as $\left(f^{\mathsf{v1}}\right)^{\mathsf{h1}}$ coincides with $\mathsf{T}^2\mathbb{R}^{\mathscr{d}} \to \mathbb{R},\ (x, v, a) \mapsto f(\mathsf{vpr}(a))$.

Computing the gradient with respect to $\mathsf{t}[2]$ shows that

- $\nabla_{\mathsf{t}[2]} f^{\mathsf{v}2} = \nabla_{\mathsf{h}0} f^{\mathsf{v}2} = \nabla_x((x,v,a) \mapsto f(x)) = \nabla_{\mathsf{euc}} f$,
- $\nabla_{\mathsf{t}[2]} (f^{\mathsf{h}1})^{\mathsf{v}1} = \nabla_{\mathsf{h}1} (f^{\mathsf{h}1})^{\mathsf{v}1} = \nabla_v((x,v,a) \mapsto f(v)) = \nabla_{\mathsf{euc}} f$,
- $\nabla_{\mathsf{t}[2]} (f^{\mathsf{v}1})^{\mathsf{h}1} = \nabla_{\mathsf{v}[2]} (f^{\mathsf{v}1})^{\mathsf{h}1} = \nabla_x(\mathsf{T}\mathbb{R}^{\mathscr{d}} \simeq \mathbb{R}^{2\mathscr{d}} \to \mathbb{R},\ (x,v) \mapsto f(x)) = \nabla_{\mathsf{euc}} f$,
- $\nabla_{\mathsf{t}[2]} f^{\mathsf{h}2} = \nabla_{\mathsf{v}[2]} f^{\mathsf{h}2} = \nabla_v(\mathsf{T}\mathbb{R}^{\mathscr{d}} \to \mathbb{R},\ (x,v) \mapsto f(v)) = \nabla_{\mathsf{euc}} f$. ⌟

3

## 3.2 ▪ Smoothed spherical velocity Langevin equation

Compared to Section 2.2, we consider the restriction of the triple tangent bundle to the tangent bundle of the unit tangent bundle over the position manifold. This restriction splits into four parts as follows:

$$\mathsf{T}(\mathsf{T}^2\mathbb{X})|_{\mathsf{TU}\mathbb{X}} = \mathsf{T}(\mathsf{TU}\mathbb{X}) \oplus \mathsf{T}(\mathsf{TU}\mathbb{X})^{\perp} = \mathsf{V}^1\mathsf{TU}\mathbb{X} \oplus \mathsf{H}^0\mathsf{TU}\mathbb{X} \oplus \mathsf{H}^1\mathsf{TU}\mathbb{X} \oplus \underbrace{\mathsf{NTU}\mathbb{X}}_{=\mathsf{TU}\mathbb{X}^{\perp}},$$

where one has that

$$\mathsf{V}^1\mathsf{TU}\mathbb{X} = \mathsf{v}{\restriction}^3_2\,(\mathsf{TU}\mathbb{X}), \quad \mathsf{H}^0\mathsf{TU}\mathbb{X} = \mathsf{h}{\restriction}^3_2\,(\mathsf{HU}\mathbb{X}), \quad \text{and} \quad \mathsf{H}^1\mathsf{TU}\mathbb{X} = \mathsf{h}{\restriction}^3_2\,(\mathsf{VU}\mathbb{X}).$$

For sake of readability, we might abbreviate the configuration manifold as $\mathsf{Q} := \mathsf{TU}\mathbb{X}$, so the previous decomposition reads as

$$\mathsf{T}(\mathsf{T}^2\mathbb{X})|_{\mathsf{Q}} := \mathsf{V}^1\mathsf{Q} \oplus \mathsf{H}^0\mathsf{Q} \oplus \mathsf{H}^1\mathsf{Q} \oplus \mathsf{NQ}. \tag{3.8}$$

As before in Section 2.2 several modifications subordinate to the choice of $\mathsf{Q}$ have to be made.

*Notation* 3.2.1 (more spherical notations). Similar to Notation 2.2.2, the restriction of the tangent bundle projection $\pi_{1,0}$ to $\mathsf{U}\mathbb{X}$ is denoted by $\pi_{1,0|\mathsf{U}}$. Compositions of tangent bundle projections with $\pi_{1,0|\mathsf{U}}$ get the adapted index notation as well, e. g. $\pi_{2,0|\mathsf{U}} = \pi_{2,1} \circ \pi_{1,0|\mathsf{U}}$. The restriction

$$\mathcal{V}_{\mathsf{U}} := \mathcal{V}^{(2)}|_{\mathsf{TU}\mathbb{X}}\colon\ \mathsf{TU}\mathbb{X} \to \mathsf{T}^2\mathsf{U}\mathbb{X},\, a \longmapsto \mathsf{v}_a{\restriction}^3_2\,(a) = \mathsf{vl}_a(a)$$

is the usual canonical vector field for $\mathsf{U}\mathbb{X}$ being the base manifold. Since the unit tangent bundle $\mathsf{U}\mathbb{X}$ is naturally endowed with unit Sasaki metric $\mathsf{u}$, we do not feel the need to change notation for the higher order semispray associated with that metric, so the symbol $\mathcal{H}^{(2)}_{\mathsf{x}}$ now should be read as $\mathcal{H}_{\mathsf{u}} \circ \left(\mathcal{H}_{\mathsf{x}}|_{\mathsf{U}\mathbb{X}}\right)$. Recall that the standard unit Sasaki metric decomposes as $\mathsf{u} = \mathsf{v}|\mathsf{U} + \mathsf{h}$ and we carry the restriction symbol '$|\mathsf{U}$' over to the higher-order objects, if it seems appropriate. Therefore, we write the metric on $\mathsf{TU}\mathbb{X}$ induced by $\mathsf{u}$ as $\mathsf{t}[2]|\mathsf{U} = \mathsf{v}|\mathsf{U}[2] + \mathsf{h}|\mathsf{U}[2,1] + \mathsf{h}[2,0]$. Also, the shorthands get the restriction symbol: For instance, for the gradient $\nabla_{\mathsf{h}|\mathsf{U}[2,1]}$ becomes $\nabla_{\mathsf{h}1|\mathsf{U}}$.

The vertical lift of functions $f_0$ on $\mathbb{X}$ is to be read as

$$f_0^{\mathsf{v}[j]} = f_0 \circ \pi_{j,0|\mathsf{U}} = f_0 \circ \pi_{j,1} \circ \pi_{1,0|\mathsf{U}}.$$

The horizontal lift of sufficiently regular functions $f_0$ on $\mathbb{X}$ is defined by the requirement

$$\left\langle a\,,df_0^{\mathsf{h}[j]}\right\rangle \;=\; \langle a\,,df \circ d\kappa[j,0]\rangle \qquad \text{for } \lambda_{\mathsf{t}[j]}\text{-almost all } a \in \mathsf{T}^j\mathsf{U}\mathbb{B}.$$

Compare to Notation 2.2.2. ⌟

Now, consider the following Stratonovich SDE on $\mathsf{Q}$:

$$\begin{aligned} \mathrm{d}\eta &= \mathcal{H}^{(2)}_{\mathsf{x}} \circ \pi_{2,1}\ \mathrm{d}t - \mathsf{h}{\restriction}^3_2\,\left(\mathsf{tl}_{\pi_{2,1}}\left(\nabla_{\mathsf{x}}\Psi\right)\right)\ \mathrm{d}t - \alpha \cdot \mathcal{V}_{\mathsf{U}}\ \mathrm{d}t + \sigma \sum_{i=1}^{2d-1} \mathsf{v}{\restriction}^3_2 \left(\frac{\partial}{\partial v^i_\eta}\right) \circ \mathrm{d}W^i_t \\ &= \mathcal{H}^{(2)}_{\mathsf{x}}\,\mathrm{d}t - \mathsf{h}{\restriction}^3_2\,\left(\mathsf{tl}\left(\nabla_{\mathsf{x}}\Psi\right)\right)\mathrm{d}t - \alpha \cdot \mathcal{V}_{\mathsf{U}}\,\mathrm{d}t + \sigma \sum_{i=1}^{2d-1} \mathsf{v}{\restriction}^3_2 \left(\frac{\partial}{\partial v^i_\eta}\right) \circ \mathrm{d}W^i_t, \end{aligned} \tag{3.9}$$

where $\left(v_\eta^1, v_\eta^2, \dots, v_\eta^{2d-1}\right)$ is a chart at $\pi_{2,1}(\eta)$ providing normal coordinates. For sake of readability, we dropped the bundle projection $\pi_{2,1}$ in the second line. We refer to Equation (3.9) as *smoothed fibre lay-down model on* $\mathbb{X}$. Let $\Psi^{\upharpoonright}$ be a shorthand for $\left(\Psi^{\mathsf{h}1}\right)^{\mathsf{v}1}\colon \mathsf{Q} \to \mathbb{R}$. The Kolmogorov generator $L$ corresponding to the smoothed fibre lay-down model reads as

$$L = \mathscr{H}_{\mathsf{x}}^{(2)} \circ \pi_{2,1} - \nabla_{\mathsf{h}1|\mathsf{U}} \Psi^{\upharpoonright} - \alpha \cdot \mathcal{V}_{\mathsf{U}} + \frac{\sigma^2}{2}\, \Delta_{\mathsf{v}|\mathsf{U}[2]} \qquad \text{on } C^\infty(\mathsf{Q}).$$



If one drops the projection $\pi_{2,1}$ in the first term, since it looks unpleasant, one should be careful to keep in mind how this operator acts on smooth functions: For $f \in C^\infty(\mathsf{Q})$ the action of $\mathscr{H}_{\mathsf{x}}^{(2)} \circ \pi_{2,1}$ on $f$ at $a \in \mathsf{Q}$ is prescribed as

$$\left(\mathscr{H}_{\mathsf{x}}^{(2)} \circ \pi_{2,1}\right)(f)\colon\ \mathsf{Q} \to \mathbb{R},\ a \longmapsto \mathsf{t}[2]_a\Big(\mathscr{H}_{\mathsf{u}}(\mathscr{H}_{\mathsf{x}}(\pi_{2,1}(a)))\,,\nabla_{\mathsf{t}[2]} f\Big).$$

In particular, one obtains $\left(\mathscr{H}_{\mathsf{x}}^{(2)} \circ \pi_{2,1}\right) f_0^{\mathsf{v}2} = \left(\mathscr{H}_{\mathsf{x}} f_0^{\mathsf{v}}\right)^{\mathsf{v}1}$. As we have seen in Section 2.2, the previous form of the generator $L$ wont carry us far when checking the hypocoercivity conditions (H), most importantly when determining $PA^2P$ on a suitable core domain $D$. However, under the additional Assumption (2.19) we get as before $\mathsf{tl}\,\nabla_{\mathsf{x}} \Psi = \frac{1}{d-1}\, \nabla_{\mathsf{u}}\left(\mathscr{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\right)$. Then, the generator attains the form

$$\begin{aligned} L &= \mathscr{H}_{\mathsf{x}}^{(2)} \circ \pi_{2,1} - \frac{1}{d-1}\, \mathsf{h}\!\upharpoonright_2^3 \left(\nabla_{\mathsf{v}|\mathsf{U}}\left(\mathscr{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\right)\right) - \alpha \cdot \mathcal{V}_{\mathsf{U}} + \frac{\sigma^2}{2}\, \Delta_{\mathsf{v}|\mathsf{U}[2]} \\ &= \mathscr{H}_{\mathsf{x}}^{(2)} \circ \pi_{2,1} - \frac{1}{d-1}\, \nabla_{\mathsf{h}1|\mathsf{U}}\left(\mathscr{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\right)^{\mathsf{v}1} - \alpha \cdot \mathcal{V}_{\mathsf{U}} + \frac{\sigma^2}{2}\, \Delta_{\mathsf{v}|\mathsf{U}[2]} \qquad \text{on } C^\infty(\mathsf{Q}). \end{aligned} \tag{3.10}$$

In fact, the local coordinate form of Equation (3.9) for the case Euclidean case, see Example A.2.3, coincides with the form derived by P. Stilgenbauer in his PhD thesis. But Equation (3.10) is to my best knowledge the first instance of the generator in invariant form.

As announced previously, it turns out that we want to add a bit Brownian noise to the action in $\mathsf{H}^1\mathsf{Q}$ in Equation (3.9), which could be thought as diffusion on the velocity level. Otherwise, we are not able to check the microcoercivity estimate from (H2) later on and hence the AHHM doesn't apply. All the other conditions could be proven comparatively easy. Introduce the parameter $\sigma_v \in (0,\infty)$ and add the term

$$\sqrt{2\alpha\, \frac{\sigma_v}{\sigma_v + 1}} \cdot \sum_{j=1}^{d} \mathsf{h}\!\upharpoonright_2^3 \mathsf{tl}\left(\frac{\partial}{\partial x_\eta^j}\right) \circ \mathsf{d}W_t^{(v;j)}$$

to Equation (3.9), where $\left(x_\eta^1, \dots, x_\eta^d\right)$ provides normal coordinates at $\pi_{2,0}(\eta)$ and $W^{(v)} = \left(W_t^{(v)}\right)_{t\in[0,\infty)}$ is a Wiener process independent of $W = (W_t)_{t\in[0,\infty)}$. Then, Equation (3.10) transforms into

$$\begin{aligned} L &= \mathscr{H}_{\mathsf{x}}^{(2)} \circ \pi_{2,1} - \frac{1}{d-1}\, \nabla_{\mathsf{h}1|\mathsf{U}}\left(\mathscr{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\right)^{\mathsf{v}1} - \alpha \cdot \mathcal{V}_{\mathsf{U}} + \frac{\sigma^2}{2}\, \Delta_{\mathsf{v}|\mathsf{U}[2]} + \alpha\, \frac{\sigma_v}{\sigma_v + 1}\, \Delta_{\mathsf{h}1|\mathsf{U}} \\ &= \mathscr{H}_{\mathsf{x}}^{(2)} \circ \pi_{2,1} - \frac{1}{d-1}\, \nabla_{\mathsf{h}1|\mathsf{U}}\left(\mathscr{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\right)^{\mathsf{v}1} + \frac{\alpha}{\beta}\, \Delta_{\mathsf{v}|\mathsf{U}[2]} + \alpha\, \frac{\sigma_v}{\sigma_v + 1}\, \Delta_{\mathsf{h}1|\mathsf{U}}, \end{aligned} \tag{3.11}$$

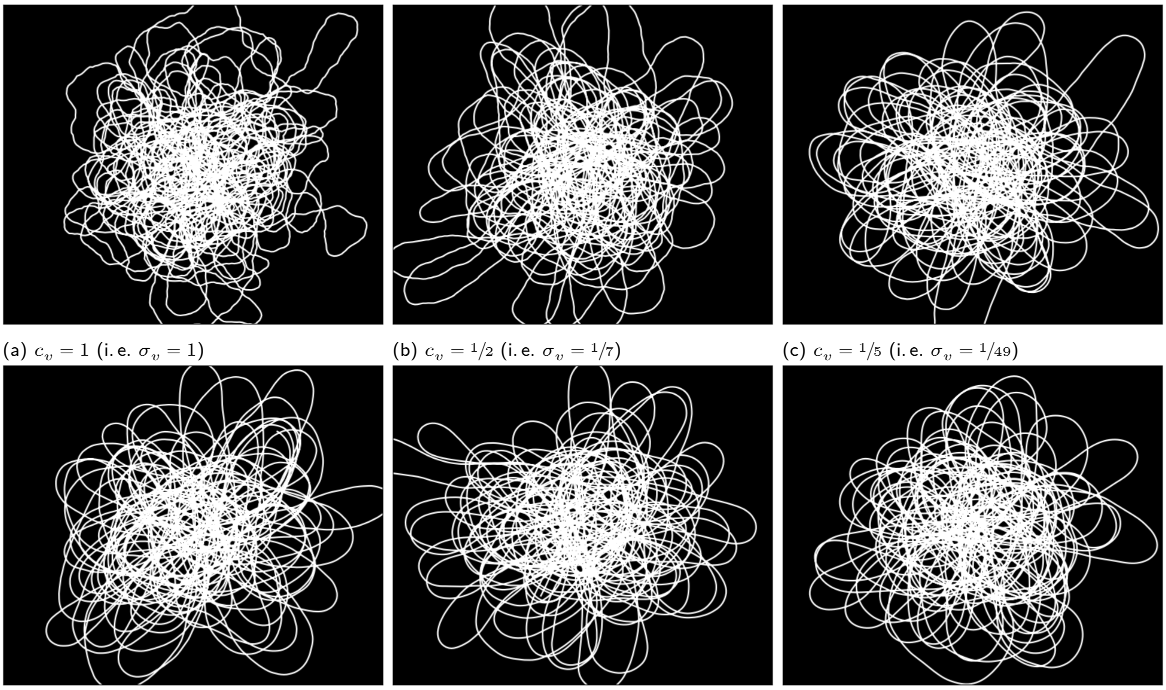
where $\beta \in (0,\infty)$ is an auxiliary variable, similar as in Section 2.1, such that $\frac{\sigma^2}{2} = \frac{\alpha}{\beta}$. The adaptation seems counterintuitive as the original idea of the smoothed model was to 'integrate' stochastic irregularity of the acceleration dynamic so that the velocity and subsequently the position trajectories look smoother. Fundamentally, the reason for introducing an additional diffusion part is that we have to choose the projection $P$ as basically an iterated fibrewise average vertically lifted twice – we do so in order to get a grasp of the $PA^2P$ on the core domain $D$ for verification of macroscopic hypocoercivity (H3). This projection maps into the nullspace of[2] $\Delta_{\mathsf{v}|\mathsf{U}[2]}$, the symmetric part of the generator of the smoothed fibre lay-down model, but *not onto* that nullspace. That goes against the grain of [DMS15, Section 1.3] where the choice of $P$ is motivated in the context of linear kinetic equations.[3] It seems that the smoothed fibre lay-down model

[2]To be precise, we are talking here about the closure of the densely defined, symmetric operator $\left(\Delta_{\mathsf{v}|\mathsf{U}[2]}, C_{\mathsf{c}}^\infty(\mathsf{Q})\right)$ in the Hilbert space $H$ defined in Definition 3.2.2 below.

[3]Actually, the motivation in both [DMS09] and [DMS15] is that this nullspace consists of local equilibria. We just think that [DMS15] pronounces it more clearly.

doesn't provide enough relaxation on either microscopic or macroscopic level for the AHHM to apply. Thus, we want to see what can be done, if we artificially switch to a setting for which the AHHM was designed for. Our hope is that the new parameter $\sigma_v$ can be chosen small without affecting the hypocoercivity constants too much. The final discussion is postponed to the end of this section.

First, we show some typical realisations for the adapted smoothed fibre lay-down model in the plane, see Figure 3.2. We fixed the diffusion parameter in order to investigate the effect of the additional velocity noise. One should keep in mind that the specific values of the parameter $\sigma_v$ are not too interesting there, since they are determined by our somewhat arbitrary choice of the intermediate function $\sigma_v \mapsto {}^{\sigma_v}/_{(\sigma_v+1)}$ and the prescribed values of the control $c_v := \sqrt{2\alpha\, {}^{\sigma_v}/_{(\sigma_v+1)}}$ on the velocity noise. In principle, the values of the control $c_v$ range from $\sqrt{2\alpha}$, which corresponds to similar amount of noise on velocity and acceleration level, to 0, which corresponds to the smoothed fibre lay-down model (3.9). In Figure 3.2, we observe that the trajectories already look fairly smooth for the 'large' value of $c_v = {}^1/_5$. To the naked eye, the trajectories for smaller values of $c_v$ appear as regular as the one for $c_v = 0$. Such observations indicate that the $\sigma_v$-adapted smoothed fibre lay-down model is a reasonable replacement of the smoothed fibre lay-down model, up to the time when a hypocoercivity result is found for the smoothed fibre lay-down model.



(a) $c_v = 1$ (i. e. $\sigma_v = 1$) (b) $c_v = {}^1/_2$ (i. e. $\sigma_v = {}^1/_7$) (c) $c_v = {}^1/_5$ (i. e. $\sigma_v = {}^1/_{49}$)

(d) $c_v = {}^1/_7$ (i. e. $\sigma_v = {}^1/_{97}$) (e) $c_v = {}^1/_{10}$ (i. e. $\sigma_v = {}^1/_{199}$) (f) $c_v = 0$ (i. e. $\sigma_v = 0$)

Figure 3.2: Realisations of the $\sigma_v$-adapted smoothed fibre lay-down model for fixed $\alpha = \sigma = 1$ and different values of $c_v := \sqrt{2\alpha\, {}^{\sigma_v}/_{(\sigma_v+1)}}$

## Data conditions

**Definition 3.2.2** (model Hilbert space (D1)). Consider the probability space $(\mathsf{Q}, \mathfrak{B}(\mathsf{Q}), \mu)$ with

$$\mu = \lambda_{\mathbf{t}[2]|\mathsf{U}} = \left(\lambda_{\mathbf{x}} \otimes_{\mathsf{loc}} \nu\right) \otimes_{\mathsf{loc}} \nu_\beta$$

with base weight $\rho_{\mathbb{X}} := \exp(-\Psi)$ such that $\lambda_{\mathbf{x}}$ is a probability measure, with $\nu$ being the normalised spherical surface measure and $\nu_\beta = \mathsf{N}(0; {}^1/_\beta \,\mathsf{Id}_{\mathbb{R}^{2d-1}})$ being the zero-mean normal distribution on $\mathbb{R}^{2d-1}$ with covariance matrix ${}^1/_\beta \,\mathsf{Id}_{\mathbb{R}^{2d-1}}$. The model Hilbert space is $H := L^2(\mathsf{Q}; \mu) = L^2(\mathsf{TU}\mathbb{X}; \mathbf{t}[2]|\mathsf{U})$. ⌟

**Lemma 3.2.3** (SAD-decomposition (D3), (D4), (D6)). Let the potential $\Psi$ be loc-Lipschitzian such that Assumption (2.19) holds on $\mathsf{U}\mathbb{X}$. Consider the decomposition $L = S - A$ on the core domain $D := C_{\mathsf{c}}^\infty(\mathsf{Q})$ with

$$Sf = \alpha\, \Delta_{\beta;\sigma_v} f := \frac{\alpha}{\beta}\, \Delta_{\mathsf{v}|\mathsf{U}[2]} f + \alpha\, \frac{\sigma_v}{\sigma_v + 1}\, \Delta_{\mathsf{h}1|\mathsf{U}} f$$

$$\text{and} \qquad Af = \mathcal{H}_{\mathsf{x}}^{(2)} f := \mathcal{H}_{\mathsf{x}}^{(2)} f - \frac{1}{d-1} \nabla_{\mathsf{h1|U}} \big(\mathcal{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\big)^{\mathsf{v1}} f$$

for all $f \in D$.

Then, the following assertions hold:

(i) $(S, D)$ is symmetric and nonpositive definite.

(ii) $(A, D)$ is antisymmetric.

(iii) For all $f \in D$ we have that $Lf \in L^1(\mathsf{Q}; \mu)$ and $\int_{\mathsf{Q}} Lf \, \mathrm{d}\mu = 0$.



*Proof.*

(i) We show that the two summands of $(S, D)$ are symmetric and nonpositive operators. As in the proof of part (i) of Lemma 2.1.5, we compute the vertical logarithmic derivative with the usual suggestive shorthands

$$\begin{aligned}\frac{1}{d\nu_\beta/d\lambda} \nabla_{\mathsf{v|U[2]}} \left(\frac{d\nu_\beta}{d\lambda}\right)(f_1^{\mathsf{h}}) &= \nabla_{\mathsf{euc}} \left(-\frac{\beta}{2} \left|\mathsf{Id}_{\mathbb{R}^{2d-1}}\right|^2_{\mathsf{euc}}\right) f_1 \\ &= -\beta \left\langle \mathsf{Id}_{\mathsf{U}\mathbb{X}} \, , df_1 \right\rangle = -\beta \cdot \mathcal{V}_{\mathsf{U}}(f_1^{\mathsf{h}})\end{aligned}$$

for all $f_1 \in C_{\mathsf{c}}^\infty(\mathsf{U}\mathbb{X})$. Regarding the last summand in $\Delta_{\beta;\sigma_v}$ note that it is up to the factor $\alpha \frac{\sigma_v}{\sigma_v+1}$ the Laplace-Beltrami operator with respect to the horizontal marginal metric $\mathsf{h}_{|\mathsf{U}[2;1]}$. As it doesn't interact with the base weight $\exp(-\Psi)$ nor the fibre weight $d\nu_\beta/d\lambda$, it is symmetric and nonpositive definite.

(ii) Use the computations from part (ii) of Lemma 2.1.5 as a basis: The adjoint of $\frac{1}{d-1} \nabla_{\mathsf{h1|U}} \big(\mathcal{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\big)^{\mathsf{v1}}$ with respect to $L^2(\mathsf{Q}; \mathbf{t}_{[2]|\mathsf{U}})$-scalar product is computed as

$$\begin{aligned}&\left(\frac{1}{d-1} \nabla_{\mathsf{h1|U}} \big(\mathcal{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\big)^{\mathsf{v1}}\right)^* \\ &= -\frac{1}{d-1} \nabla_{\mathsf{h1|U}} \big(\mathcal{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\big)^{\mathsf{v1}} - \frac{1}{d-1} \Delta_{\mathsf{h1|U}} \big(\mathcal{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\big)^{\mathsf{v1}} \\ &= -\frac{1}{d-1} \nabla_{\mathsf{u}} \big(\mathcal{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\big) \circ \pi_{2,1} - \frac{1}{d-1} \Delta_{\mathsf{u}} \big(\mathcal{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\big) \circ \pi_{2,1} \\ &= -\frac{1}{d-1} \Delta_{\mathsf{u}} \big(\mathcal{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\big) \circ \pi_{2,1} + \big(\mathcal{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\big)^{\mathsf{v1}}.\end{aligned}$$

(iii) Using the first two assertions of this lemma one has for all $f \in D$ that $(Lf\, , 1)_H = (Sf\, , 1)_H - (Af\, , 1)_H = 0$. □

*Notation* 3.2.4. From Lemma 3.2.3 it follows that $(S, D)$, $(A, D)$, and $(L, D)$ are closable. The closures of $(S, D)$, $(A, D)$ and $(L, D)$ are denoted by $(S, D(S))$, $(A, D(A))$ and $(L, D(L))$ respectively. ⌟

The projection $P_S$ again is defined by means of an average, specifically the doubled fibrewise average $\mathsf{E}_{\nu;\beta}[\cdot]$ given as

$$\mathsf{E}_{\nu;\beta}[f] \colon \ \mathbb{X} \to \mathbb{R}, \ x \longmapsto \int_{\mathsf{U}_x\mathbb{X}} \int_{\mathsf{T}_u\mathsf{U}\mathbb{X}} f(a) \quad \nu_\beta(\mathrm{d}a)\ \nu(\mathrm{d}u) \qquad \text{for all } f \in H.$$

Then, $P_S$ is defined as twofold vertical lifting as $P_S f := \big(\mathsf{E}_{\nu;\beta}[f]\big)^{\mathsf{v2}}$ for all $f \in H$ and furthermore define $P = P_S - (\cdot\, , 1)_H$.

**Lemma 3.2.5** (Properties of projection $P$ and semigroup conservativity (D5), (D7)). Let condition (P1) hold. Then, we have $P(H) \subseteq D(S)$, $SP = 0$, $P(D) \subseteq D(A)$ and $AP(D) \subseteq D(A)$. Moreover, $1 \in D(L)$ and $L1 = 0$.

*Proof.* Since the standard fibre $\mathbb{R}^{2d-1}$ of TQ is noncompact, we follow the strategy from the proof of Lemma 2.1.9, which is based on [GS16, Lemma 3.4 part (iv)] as the reader might recall.

Regarding the assertions involving just $S$ and $P$ note that there is nothing to prove for the last summand of $\Delta_{\beta;\sigma_v}$ as it reduces to the situation of the symmetric operator in Section 2.2 on the image of $P$. For the first summand we basically replace $(\mathbb{X}, \mathsf{x})$ by $(\mathsf{U}\mathbb{X}, \mathsf{u})$ in the first parts of the proof of Lemma 2.1.9: Let $u_o \in \mathsf{U}\mathbb{X}$ and $\chi \in C_\mathsf{c}^\infty(\mathsf{U}\mathbb{X}; [0,1])$ be a cut-off function such that $\chi = 1$ on $\mathbb{U}(u_o; r)$ for some $r \in (0, \pi/4)$ and $\chi = 0$ outside of $\mathbb{U}(u_o; 2r)$. Define $\chi_n := \chi(\gamma_{\mathsf{Id}}(1/n))$ for all $n \in \mathbb{N}_+$, where $\gamma_u$ refers to the geodesic connecting $u_0$ and $u$ with $\gamma_u(0) = u_0$ and $\gamma_u(1) = u$. There is some $c(\chi) \in (0, \infty)$ such that

$$|\nabla_\mathsf{u} \chi_n(x)|_\mathsf{u} \leq \frac{c(\chi)}{n} \quad \text{and} \quad |\mathsf{Hess}_\mathsf{u}(\chi_n)(x)|_\infty \leq \frac{c(\chi)}{n^2}$$

for all $x \in \mathbb{X}$ and $n \in \mathbb{N}_+$. Let $f_0 \in C_\mathsf{c}^\infty(\mathsf{U}\mathbb{X})$, then the sequence $(f_n)_{n\in\mathbb{N}_+} := (f_0^\mathsf{v} \otimes \chi_n^\mathsf{h})_n$ is in $D$ and converges to $f_0^\mathsf{v}$ in $H$. Due to dominated convergence we get that

$$\Delta_{\mathsf{v}|\mathsf{U}[2]} f_n = \frac{\sigma^2}{2}\Big(f_0^\mathsf{v} \cdot \Delta_{\mathsf{v}|\mathsf{U}[2]} \chi_n^\mathsf{h} - \beta f_0^\mathsf{v} \cdot \mathcal{V}_\mathsf{U} \chi_n^\mathsf{h}\Big) \longrightarrow 0 \qquad \text{in } H \text{ as } n \to \infty.$$

Combining this with our knowledge regarding $\Delta_{\mathsf{h}1|\mathsf{U}}$ and the fact that $(S, D(S))$ is closed, we infer that $f_0^\mathsf{v} \in D(S)$ and $S f_0^\mathsf{v} = 0$. Next on, since $\mathrm{ran}(P) \subseteq \pi_{2,1}^* L^2(\mathsf{U}\mathbb{X}; \lambda_\mathsf{x} \otimes_{\mathsf{loc}} \nu)$ we pick $g_0 \in L^2(\mathsf{U}\mathbb{X}; \lambda_\mathsf{x} \otimes_{\mathsf{loc}} \nu)$ and show that both $g_0^\mathsf{v} \in D(S)$ and $S g_0^\mathsf{v} = 0$. Since $C_\mathsf{c}^\infty(\mathsf{U}\mathbb{X})$ is dense in $L^2(\mathsf{U}\mathbb{X}; \lambda_\mathsf{x} \otimes_{\mathsf{loc}} \nu)$, we can choose a sequence $(g_n)_{n\in\mathbb{N}_+}$ in $C_\mathsf{c}^\infty(\mathsf{U}\mathbb{X})$ approximating $g_0$ in $L^2(\lambda_\mathsf{x} \otimes_{\mathsf{loc}} \nu)$. As seen before, $(g_n^\mathsf{v})_{n\in\mathbb{N}_+}$ is a sequence in $D(S)$ such that $S g_n^\mathsf{v} = 0$ for all $n \in \mathbb{N}_+$. Due to closedness of $(S, D(S))$, it follows $g_0^\mathsf{v} \in D(S)$ and $S g_0^\mathsf{v} = 0$.

We turn to the statements involving $A$. Let $o \in \mathbb{X}$ and $\varphi \in C_\mathsf{c}^\infty(\mathbb{X}; [0,1])$ be a cut-off function such that $\varphi = 1$ on $\mathbb{U}(o; 1)$ and $\varphi = 0$ outside of $\mathbb{U}(o; 2)$. Define $\varphi_n := \varphi(\gamma_{\mathsf{Id}}(1/n))$ for all $n \in \mathbb{N}_+$, where $\gamma_x$ refers to the geodesic connecting $o$ and $x$ with $\gamma_x(0) = o$ and $\gamma_x(1) = x$. There is some $c(\varphi) \in (0, \infty)$ such that

$$\Big|\nabla_\mathsf{x} \varphi_n(x)\Big|_\mathsf{x} \leq \frac{c(\varphi)}{n} \quad \text{and} \quad \Big|\mathsf{Hess}_\mathsf{x}(\varphi_n)(x)\Big|_\infty \leq \frac{c(\varphi)}{n^2}$$

for all $x \in \mathbb{X}$ and $n \in \mathbb{N}_+$. Let $f_0 \in C_\mathsf{c}^\infty(\mathbb{X})$ and define an approximating sequence for $f_0^{\mathsf{v}2}$ in $H$ via

$$f_n := f_0^{\mathsf{v}2} \otimes \big(\varphi_n^{\mathsf{h}1}\big)^{\mathsf{v}1} \otimes \big(\varphi_n^{\mathsf{v}1}\big)^{\mathsf{h}1} \otimes \varphi_n^{\mathsf{h}2} \qquad \text{for all } n \in \mathbb{N}_+.$$

Using dominated convergence we find that

$$\begin{aligned}
A f_n &= -\big(\varphi_n^{\mathsf{h}1}\big)^{\mathsf{v}1} \otimes \big(\varphi_n^{\mathsf{v}1}\big)^{\mathsf{h}1} \otimes \varphi_n^{\mathsf{h}2} \cdot \mathcal{H}_\mathsf{x}^{(2)} f_0^{\mathsf{v}2} \\
&\quad + \Bigg(\big(\varphi_n^{\mathsf{h}1}\big)^{\mathsf{v}1} \frac{1}{d-1} \nabla_\mathsf{u}\Big(\mathcal{H}_\mathsf{x}\Psi^\mathsf{v}\Big)\big(f_0^{\mathsf{v}1}\big) \\
&\qquad\qquad + f_0^{\mathsf{v}2} \frac{1}{d-1} \nabla_\mathsf{u}\Big(\mathcal{H}_\mathsf{x}\Psi^\mathsf{v}\Big)\big(\varphi_n^{\mathsf{h}1}\big)\Bigg) \big(\varphi_n^{\mathsf{v}1}\big)^{\mathsf{h}1} \otimes \varphi_n^{\mathsf{h}2} \\
&= -\big(\varphi_n^{\mathsf{h}1}\big)^{\mathsf{v}1} \otimes \big(\varphi_n^{\mathsf{v}1}\big)^{\mathsf{h}1} \otimes \varphi_n^{\mathsf{h}2} \cdot \mathcal{H}_\mathsf{x}^{(2)} f_0^{\mathsf{v}2} \\
&\quad + f_0^{\mathsf{v}2} \otimes \big(\varphi_n^{\mathsf{h}1}\big)^{\mathsf{v}1} \otimes \big(\varphi_n^{\mathsf{v}1}\big)^{\mathsf{h}1} \cdot \frac{1}{d-1} \nabla_\mathsf{u}\Big(\mathcal{H}_\mathsf{x}\Psi^\mathsf{v}\Big)\big(\varphi_n^{\mathsf{h}1}\big) \\
\longrightarrow &\quad -\mathcal{H}_\mathsf{x}^{(2)} f_0^{\mathsf{v}2} = -\Big(\mathcal{H}_\mathsf{x} f_0^\mathsf{v}\Big)^\mathsf{v} \qquad \text{in } H \text{ as } n \to \infty
\end{aligned}$$

We use this result in order to show $1 \in D(A)$ and $A1 = 0$; then, we can infer that $P(D) \subseteq D(A)$. From Lemma 2.1.9 we know that $\mathcal{H}_\mathsf{x}\varphi_n^\mathsf{v} \to 0$ in $H$ as $n \to \infty$ implying that $A\varphi_n^{\mathsf{v}2} = \Big(\mathcal{H}_\mathsf{x}\varphi_n^\mathsf{v}\Big)^\mathsf{v} \to 0$ in $H$ as $n \to \infty$, which is what we wanted.

Now, let us prove $AP(D) \subseteq D(A)$. Define the sequence $(h_n)_{n\in\mathbb{N}}$ in $D(A)$ via $h_n := \chi_n^\mathsf{h} \otimes A f_0^{\mathsf{v}2}$ for all $n \in \mathbb{N}_+$ and $f_0$ as well as the cut-off function $\chi_n$ as before. Recall that $\mathcal{H}_\mathsf{x}^{(2)} f_0^{\mathsf{v}2} \in H$. Then similar as in Lemma 2.1.9, using dominated convergence we get that

$$\begin{aligned}
A h_n &= \Big(\mathcal{H}_\mathsf{x} f_0^\mathsf{v}\Big)^\mathsf{v} \cdot \frac{1}{d-1} \nabla_{\mathsf{h}1|\mathsf{U}}\Big(\mathcal{H}_\mathsf{x}\Psi^\mathsf{v}\Big)^{\mathsf{v}1} \chi_n^\mathsf{h} - \chi_n^\mathsf{h} \cdot \mathcal{H}_\mathsf{x}^{(2)}\Big(\mathcal{H}_\mathsf{x} f_0^\mathsf{v}\Big)^\mathsf{v} \\
&= \Big(\mathcal{H}_\mathsf{x} f_0^\mathsf{v}\Big)^\mathsf{v} \cdot \frac{1}{d-1} \nabla_{\mathsf{v}|\mathsf{U}}\Big(\mathcal{H}_\mathsf{x}\Psi^\mathsf{v}\Big)\chi_n - \chi_n^\mathsf{h} \cdot \mathcal{H}_\mathsf{x}^{(2)}\Big(\mathcal{H}_\mathsf{x} f_0^\mathsf{v}\Big)^\mathsf{v}
\end{aligned}$$

$$= \big(\mathcal{H}_{\mathsf{x}} f_0^{\mathsf{v}}\big)^{\mathsf{v}} \cdot \frac{1}{d-1} \nabla_{\mathsf{v}|\mathsf{U}}\big(\mathcal{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\big)\chi_n - \chi_n^{\mathsf{h}} \cdot \big(\mathcal{H}_{\mathsf{x}}^2 f_0^{\mathsf{v}}\big)^{\mathsf{v}} \quad \longrightarrow -\big(\mathcal{H}_{\mathsf{x}}^2 f_0^{\mathsf{v}}\big)^{\mathsf{v}}$$

in $H$ as $n \to \infty$. The statements regarding $L$ follow as in previous works: The sequence $(\varphi_n^{\mathsf{v}2})_{n\in\mathbb{N}}$ approximates the constant function 1 in $H$ and $L\varphi_n^{\mathsf{v}2} = -A\varphi_n^{\mathsf{v}2} = \big(\mathcal{H}_{\mathsf{x}}\varphi_n^{\mathsf{v}}\big)^{\mathsf{v}}$ converges in $H$ to 0 as $n \to \infty$ which finishes the proof using closedness of $(L, D(L))$. □

*Remark* 3.2.6. From the proof of Lemma 3.2.5 we know for all $f \in D$ that $Pf \in D(A)$ and

$$\begin{aligned} APf(a) &= -\mathsf{t}[2]_a\Big(\mathcal{H}_{\mathsf{u}} \circ \mathcal{H}_{\mathsf{x}}(u)\, , \nabla_{\mathsf{t}[2]} P_S f(a)\Big) \\ &= -\mathsf{h}[2]_a\Big(\mathcal{H}_{\mathsf{u}} \circ \mathcal{H}_{\mathsf{x}}(u)\, , \mathsf{h}\!\restriction_2^3 \Big(\nabla_{\mathsf{t}[1]}\big(\mathsf{E}_{\nu;\beta}[f]\big)^{\mathsf{v}1}(u)\Big)\Big) \\ &= -\Big(\mathcal{H}_{\mathsf{x}}\big(\mathsf{E}_{\nu;\beta}[f]\big)^{\mathsf{v}}\Big)^{\mathsf{v}} = -\mathsf{x}_x\Big(u\, , \nabla_{\mathsf{x}}\big(\mathsf{E}_{\nu;\beta}[f]\big)(x)\Big) \end{aligned} \tag{3.12}$$

for all $a \in \mathsf{Q}$ with $u := \pi_{2,1}(a) \in \mathsf{U}\mathbb{X}$ and $x := \pi_{1,0|\mathsf{U}}(u)$. ⌟

What remains is the notorious question of essential m-dissipativity of the operator $(L, D)$ as in Equation (3.11). This can be verified with the three step strategy that we established in Section 2.1:

(i) For the case of a smooth potential $\Psi$, it's fairly easy to see that $L$ satisfies the Hörmander condition. One just has to iterate on Lemma 2.1.12 using the local coordinate forms of higher-order sprays. Then, essentially m-dissipativity is proven as in Proposition 2.1.13.

(ii) For globally Lipschitzian potentials we perturb the operator

$$(L_0, D_0) = \big(\alpha\, \Delta_{\beta;\sigma_v} + \mathcal{H}_{\mathsf{x}}^{(2)}, D_0\big)$$

corresponding to the case of $\Psi$ being the zero potential and apply the Kato Perturbation Theorem, Proposition 2.1.13. To this end, one first shows that $(L_0, D_0)$ is essentially m-dissipative on $L^2(\mathsf{Q}; (\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu) \otimes_{\mathsf{loc}} \nu_\beta)$ and then that

$$\Big(ULU^{-1}, \pi_{2,0}^* H^{1,\infty}(\mathbb{X}) \otimes (C_c^\infty(\mathbb{X}))^{\mathsf{h}1\mathsf{v}1} \otimes (C_c^\infty(\mathbb{X}))^{\mathsf{v}1\mathsf{h}1} \otimes (C_c^\infty(\mathbb{X}))^{\mathsf{h}2}\Big)$$

on $L^2(\mathsf{Q}; (\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu) \otimes_{\mathsf{loc}} \nu_\beta)$ is essentially m-dissipative, where $U$ is the unitary isomorphism

$$H \to L^2(\mathsf{Q}; (\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu) \otimes_{\mathsf{loc}} \nu_\beta), \;\; f \longmapsto (\exp(-\Psi/2))^{\mathsf{v}2} \cdot f.$$

See Lemma 2.1.14 and Lemma 2.1.16. Then, one proceeds as in Corollary 2.1.17.

(iii) We wrap thinks up as in Proposition 2.1.18 for the locally Lipschitz case.

## Hypocoercivity conditions

**Lemma 3.2.7** (algebraic relation (H1)). Assume that condition (P1) is satisfied. Then, it holds $PAP = 0$ on $D$.

*Proof.* Let $f \in D$ and $x \in \mathbb{X}$. Due to Equation (3.12) we know that

$$\begin{aligned} \int_{\mathsf{U}_x\mathbb{X}} \int_{\mathsf{T}_u\mathsf{U}\mathbb{X}} APf \quad \mathrm{d}\nu_\beta\; \nu(du) &= \int_{\mathsf{U}_x\mathbb{X}} \int_{\mathsf{T}_u\mathsf{U}\mathbb{X}} -\mathsf{x}_x\Big(u\, , \nabla_{\mathsf{x}}\mathsf{E}_{\nu;\beta}[f](x)\Big) \quad \mathrm{d}\nu_\beta\; \nu(du) \\ &= \int_{\mathsf{U}_x\mathbb{X}} -\mathsf{x}_x\Big(u\, , \nabla_{\mathsf{x}}\mathsf{E}_{\nu;\beta}[f](x)\Big)\; \nu(du) \end{aligned}$$

which is zero just as in the proof of Lemma 2.2.8. □

**Lemma 3.2.8** (microscopic coercivity (H2)). Let the potential $\Psi$ be loc-Lipschitz such that $\lambda_{\mathsf{x}} = \exp(-\Psi)\lambda_{\mathsf{x}}$ is a probability measure on $(\mathbb{X}, \mathfrak{B}(\mathbb{X}))$. Then, we have that

$$(-Sf\, , f)_H \geq \Lambda_m \|(\mathrm{Id}_H - P_S)f\|_H^2 := \alpha\, m_v \cdot \|(\mathrm{Id}_H - P_S)f\|_H^2 \qquad \text{for all } f \in D$$

with $m_v = m_v(\sigma_v) := \min\Big(1, (d-1)\, \frac{\sigma_v}{\sigma_v+1}\Big)$.

*Proof.* Instead of $D$ we consider $D_0$ as in Equation (3.7) which is dense in $D$ with respect to the usual local convex topology that implies uniform convergence of all derivatives on compact sets. Let $f \in D_0$ of the form $f := f_{00}^{\mathsf{v2}} \otimes \left(f_{01}^{\mathsf{h1}}\right)^{\mathsf{v1}} \otimes \left(f_{10}^{\mathsf{v1}}\right)^{\mathsf{h1}} \otimes f_{11}^{\mathsf{h2}}$ with $f_j \in C_{\mathsf{c}}^{\infty}(\mathbb{X})$ for $j \in \{00, 01, 10, 11\}$. Define

$$P_{\nu_\beta} f := f_{00}^{\mathsf{v2}} \otimes \left(f_{01}^{\mathsf{h1}}\right)^{\mathsf{v1}} \cdot \int_{\mathsf{T}_{\pi_{2,1}}\mathsf{U}\mathbb{X}} \left(f_{10}^{\mathsf{v1}}\right)^{\mathsf{h1}} \otimes f_{11}^{\mathsf{h2}} \quad \mathrm{d}\nu_\beta .$$

Note that $P_{\nu_\beta}$ on $D_0$ extends to a continuous linear operator on $D \subseteq H$, even more an orthogonal projection on $H$. For sake of readability, we write $u = \pi_{2,1}(a)$ given some $a \in \mathsf{TU}\mathbb{X}$ in the following calculation: We estimate that

$$\begin{aligned}
&\int_{\mathsf{TU}\mathbb{X}} (f - P_S f)^2 \, \mathrm{d}\mu \\
&\leq \int_{\mathsf{TU}\mathbb{X}} \left(f - P_{\nu_\beta} f\right)^2 \mathrm{d}\mu + \int_{\mathsf{TU}\mathbb{X}} \left(P_{\nu_\beta} f - P_S f\right)^2 \quad \mathrm{d}\mu \\
&\leq \frac{1}{\beta} \int_{\mathsf{TU}\mathbb{X}} \left|\nabla_{\mathsf{v}[2]} f\right|_{\mathsf{v}[2]}^2 \mathrm{d}\mu + \int_{\mathbb{X}} \int_{\mathsf{U}_x\mathbb{X}} \int_{\mathsf{T}_u\mathsf{U}\mathbb{X}} \left(P_{\nu_\beta} f - P_S f\right)^2 \quad \mathrm{d}\nu_\beta \, \mathrm{d}\nu \, \mathrm{d}\lambda_{\mathsf{x}} && (3.13) \\
&= \frac{1}{\beta} \int_{\mathsf{TU}\mathbb{X}} \left|\nabla_{\mathsf{v}[2]} f\right|_{\mathsf{v}[2]}^2 \mathrm{d}\mu \\
&\qquad + \int_{\mathbb{X}} f_{00}(x)^2 \cdot \int_{\mathsf{U}_x\mathbb{X}} \left(f_{01}^{\mathsf{h1}} - \int_{\mathsf{U}_x\mathbb{X}} f_{01}^{\mathsf{h1}} \, \mathrm{d}\nu\right)^2 (u) \\
&\qquad\qquad \cdot \left(\int_{\mathsf{T}_u\mathsf{U}\mathbb{X}} \left(f_{10}^{\mathsf{v1}}\right)^{\mathsf{h1}} \otimes f_{11}^{\mathsf{h2}} \, \mathrm{d}\nu_{\alpha;\sigma}\right)^2 \quad \nu(\mathrm{d}u) \ \lambda_{\mathsf{x}}(\mathrm{d}x) \\
&\leq \frac{1}{\beta} \int_{\mathsf{TU}\mathbb{X}} \left|\nabla_{\mathsf{v|U}[2]} f\right|_{\mathsf{v|U}[2]}^2 \mathrm{d}\mu \\
&\qquad + \int_{\mathbb{X}} f_{00}(x)^2 \\
&\qquad\qquad \cdot \frac{1}{d-1} \int_{\mathsf{U}_x\mathbb{X}} \left|\nabla_{\mathsf{v|U}[1]} \left(\left(\int_{\mathsf{T}_u\mathsf{U}\mathbb{X}} \left(f_{10}^{\mathsf{v1}}\right)^{\mathsf{h1}} \otimes f_{11}^{\mathsf{h2}} \, \mathrm{d}\nu_\beta\right) \otimes f_{01}^{\mathsf{h1}}\right)\right|_{\mathsf{v|U}[1]}^2 && (3.14) \\
&\qquad\qquad \nu(\mathrm{d}u) \ \lambda_{\mathsf{x}}(\mathrm{d}x) \\
&= \frac{1}{\beta} \int_{\mathsf{TU}\mathbb{X}} \left|\nabla_{\mathsf{v|U}[2]} f\right|_{\mathsf{v|U}[2]}^2 \mathrm{d}\mu + \frac{1}{d-1} \int_{\mathsf{TU}\mathbb{X}} \left|\nabla_{\mathsf{h1|U}} f\right|_{\mathsf{h|U}[1]}^2 \mathrm{d}\mu \\
&\leq \max\left(1, \frac{1}{d-1} \frac{\sigma_v + 1}{\sigma_v}\right) \int_{\mathsf{TU}\mathbb{X}} \left|\frac{1}{\beta} \nabla_{\mathsf{v|U}[2]}(f) + \frac{\sigma_v}{\sigma_v + 1} \nabla_{\mathsf{h1|U}} f\right|_{\mathsf{t}[2]|\mathsf{U}}^2 \mathrm{d}\mu \\
&= \max\left(1, \frac{1}{d-1} \frac{\sigma_v + 1}{\sigma_v}\right) \left(-\left(\frac{1}{\beta} \Delta_{\mathsf{v|U}[2]} f + \frac{\sigma_v}{\sigma_v + 1} \Delta_{\mathsf{h1|U}} f\right), f\right)_H . && (3.15)
\end{aligned}$$

We used the Poincaré inequality for the Gaußian measure at step (3.13), moreover the Poincaré inequality for the uniform spherical surface measure at step (3.14), and finally integration by parts to arrive at step (3.15). By bringing the constant to the left-hand side and then multiplying by $\alpha$, the previous estimates imply that

$$\begin{aligned}
(-Sf, f)_H &= \left(-\alpha\left(\frac{1}{\beta} \Delta_{\mathsf{v|U}[2]} f + \frac{\sigma_v}{\sigma_v + 1} \Delta_{\mathsf{h1|U}} f\right), f\right)_H \\
&\geq \alpha \min\left(1, (d-1) \frac{\sigma_v}{\sigma_v + 1}\right) \|(\mathrm{Id}_H - P_S) f\|_H^2 .
\end{aligned}$$

$\square$

Since we are again concerned with showing that the operator $(\mathsf{Id}_H - PA^2P, D)$ is essentially m-dissipative, we focus now on characterising $(PA^2P, D)$. Reminiscent of Equation (2.24) we calculate that

$$P_S\big(-\mathcal{H}_{\mathsf{x}}^{(2)} APf\big) = \frac{1}{d}\,\big(\Delta_{\mathsf{x}}(\mathsf{E}_{\nu;\beta}[f])\big)^{\mathsf{v}2},$$

and furthermore, we derive

$$\begin{aligned}
&P_S\Big(\frac{1}{d-1}\,\nabla_{\mathsf{h}1|\mathsf{U}}\big(\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\big)^{\mathsf{v}1}(APf)\Big)(a)\\
&= \int_{\mathsf{U}_x\mathbb{X}}\int_{\mathsf{T}_u\mathsf{U}\mathbb{X}} \big(\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\big)^{\mathsf{v}1}\cdot APf \quad \mathsf{d}\nu_\beta\ \nu(\mathsf{d}u) = -\int_{\mathsf{U}_x\mathbb{X}} \mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\cdot \mathcal{H}_{\mathsf{x}}\big(\mathsf{E}_{\nu;\beta}[f]\big)^{\mathsf{v}}\ \mathsf{d}\nu\\
&= -\frac{1}{d}\,\mathsf{x}_x\big(\nabla_{\mathsf{x}}\Psi(x)\,,\nabla_{\mathsf{x}}\big(\mathsf{E}_{\nu;\beta}[f]\big)(x)\big) = -\frac{1}{d}\,\nabla_{\nabla_{\mathsf{x}}\Psi}\big(\mathsf{E}_{\nu;\beta}[f]\big)(x)
\end{aligned}$$

for all $a \in \mathsf{Q}$ with $x := \pi_{2,0|\mathsf{U}}(a)$. Combining these results, we find for all $f \in D$ that

$$\begin{aligned}
PA^2Pf = P_SA^2Pf &= \frac{1}{d}\Big(\big(\Delta_{\mathsf{x}}\mathsf{E}_{\nu;\beta}[f]\big)^{\mathsf{v}2} - \nabla_{\nabla_{\mathsf{x}}\Psi}\big(\mathsf{E}_{\nu;\beta}[f]\big)\circ\pi_{2,0|\mathsf{U}}\Big)\\
&= \frac{1}{d}\,\Delta_{\mathsf{h}0|\mathsf{U}}\big(\mathsf{E}_{\nu;\beta}[f]\big)^{\mathsf{v}2} = \frac{1}{d}\,\Delta_{\mathsf{h}0|\mathsf{U}}(P_Sf)\,.
\end{aligned} \tag{3.16}$$

Now, it's clear that $(PA^2P, D) = \big({}^1\!/\!{}_d\cdot\Delta_{\mathsf{h}0|\mathsf{U}}, C_c^\infty(\mathsf{Q})\big)$ is essentially m-dissipative in $H$, as it is up to a factor the well-known Laplace-Beltrami on $\mathsf{Q}$ endowed with the horizontal metric induced by u with the usual predomain of smooth functions with compact support. This knowledge is quintessential for treating the remaining hypocoercivity conditions.

**Proposition 3.2.9** (macroscopic coercivity (H3)). Let $\Psi$ loc-Lipschitz such that $\lambda_{\mathbf{x}} = \exp(-\Psi)\lambda_{\mathsf{x}}$ is a probability measure $(\mathbb{X}, \mathfrak{B}(\mathbb{X}))$ satisfying the Poincaré inequality (2.4). Then, the macroscopic coercivity estimate is fulfilled with $\Lambda_M = \frac{1}{d}\,\Lambda$.

*Proof.* Let $f \in D$. Using Equation (3.12) and then applying the Poincaré inequality (2.4) we estimate that

$$\begin{aligned}
\|APf\|_H^2 &= \int_{\mathbb{X}}\int_{\mathsf{U}_x\mathbb{X}} \mathsf{x}_x\big(v\,,\nabla_{\mathsf{x}}\mathsf{E}_{\nu;\beta}[f]\big)^2 \quad \nu(\mathsf{d}v)\ \lambda_{\mathbf{x}}(\mathsf{d}x) = \frac{1}{d}\int_{\mathbb{X}} \big|\nabla_{\mathsf{x}}\mathsf{E}_{\nu;\beta}[f]\big|_{\mathsf{x}}^2\ \lambda_{\mathbf{x}}(\mathsf{d}x)\\
&= \frac{1}{d}\,\big\|\nabla_{\mathsf{x}}\mathsf{E}_{\nu;\beta}[f]\big\|^2_{L^2(\mathbb{X}\to\mathsf{T}\mathbb{X};\mathbf{x})} \geq \frac{1}{d}\,\Lambda\,\Big\|\mathsf{E}_{\nu;\beta}[f] - \big(\mathsf{E}_{\nu;\beta}[f]\,,1\big)_{L^2(\mathbb{X};\mathbf{x})}\Big\|^2_{L^2(\mathbb{X};\mathbf{x})}.
\end{aligned}$$

Since $(PA^2P, D)$ is essentially m-dissipative, the claim now follows by [GS14, Corollary 2.13]. ☐

Recall that the linear continuous operator $B$ on $H$ is the extension of

$$B := \big(\mathsf{Id}_H + (AP)^*AP\big)^{-1}(AP)^* \qquad \text{on } D((AP)^*).$$

**Lemma 3.2.10** (boundedness of $(BS, D)$, first part of (H4)). Let condition (P1) hold. Then, with $c_1 := \frac{\alpha}{2}(d-1)\,\frac{\sigma_v}{\sigma_v+1}$ it holds

$$\|BSf\|_H \leq c_1\|(\mathsf{Id}_H - P_j)f\|_H \qquad \text{for all } f \in D$$

and $P_j \in \{P, P_S\}$, $j \in \{1, 2\}$.

*Proof.* Let $f \in D$. Then, we find that $SAPf = -\alpha(d-1)\,\frac{\sigma_v}{\sigma_v+1}\,APf$. Indeed, using Equation (3.12) as well as $\Delta_{\mathsf{u}}\,\mathcal{H}_{\mathsf{x}} = -(d-1)\mathcal{H}_{\mathsf{x}}$ we calculate that

$$\begin{aligned}
SAPf &= \frac{\alpha}{\beta}\,\Delta_{\mathsf{v}|\mathsf{U}[2]}\,APf - \alpha\mathcal{V}_{\mathsf{U}}APf + \alpha\,\frac{\sigma_v}{\sigma_v+1}\,\Delta_{\mathsf{h}1|\mathsf{U}}\,APf\\
&= 0 - 0 + \alpha\,\frac{\sigma_v}{\sigma_v+1}\,\Big(\Delta_{\mathsf{v}|\mathsf{U}[1]}\big(-\mathcal{H}_{\mathsf{x}}\mathsf{E}_{\nu;\beta}[f]^{\mathsf{v}}\big)\Big)^{\mathsf{v}}\\
&= \alpha\,\frac{\sigma_v}{\sigma_v+1}\,(d-1)\big(\mathcal{H}_{\mathsf{x}}\mathsf{E}_{\nu;\beta}[f]^{\mathsf{v}}\big)^{\mathsf{v}} = -\alpha\,\frac{\sigma_v}{\sigma_v+1}\,(d-1)APf.
\end{aligned}$$

Thus, $PAS = \alpha\,\frac{\sigma_v}{\sigma_v+1}\,(d-1)\,PA$ on $D$ and the claim follows. Compare to the similar proof of [GS16, Proposition 3.11] for the Langevin equation on $\mathbb{R}^d$. ☐

3

**Lemma 3.2.11** (boundedness of $(BA(\mathrm{Id}_H - P), D)$, second part of (H4)). Let the potential conditions 2.1.3 hold. Then, there exists a constant $c_{\Phi;\beta} \in (0,\infty)$ such that

$$\|BA(\mathrm{Id}_H - P)f\|_H \le c_{\Phi;\beta}\|(\mathrm{Id}_H - P_j)f\|_H \qquad \text{for all } f \in D$$

and $P_j \in \{P, P_S\}$, $j \in \{1,2\}$.

*Proof.* In course of applying [GS14, Proposition 2.15], we want to estimate the quantity $\|(BA)^* g\|_H$ for $g := (\mathrm{Id}_H - PA^2P)f$ with $f \in D$. On the one hand, its well-known that $(BA)^* g = -A^2 P f$, see [GS14, Proposition 2.15]. On the other hand, using Equation (3.12) and the strategy as in Equation (2.10), we can see that

$$\begin{aligned}
-A^2Pf &= A\Big(\mathsf{x}_{\pi_{2,0|\mathsf{U}}}\big(\pi_{2,1|\mathsf{U}}\,,\nabla_{\mathsf{x}}\big(\mathsf{E}_{\nu;\beta}[f]\big)\circ\pi_{2,0|\mathsf{U}}\big)\Big)\\
&= \mathsf{x}_{\pi_{2,0|\mathsf{U}}}\big(\pi_{2,1|\mathsf{U}}\,,\nabla_{\mathsf{x}}\big(\mathsf{E}_{\nu;\beta}[f]\big)\circ\pi_{2,0|\mathsf{U}}\big)\\
&\qquad + \mathsf{x}_{\pi_{2,0|\mathsf{U}}}\Big(\pi_{2,1|\mathsf{U}}\,,\nabla^{\mathsf{x}}_{\pi_{2,1|\mathsf{U}}}\big(\nabla_{\mathsf{x}}\big(\mathsf{E}_{\nu;\beta}[f]\big)\circ\pi_{2,0|\mathsf{U}}\big)\Big).
\end{aligned}$$

Hence, we estimate that

$$\begin{aligned}
\big\|(BA)^* g\big\|_H &\le \Big\|a\mapsto \mathcal{H}^{(2)}_{\mathsf{x}}\big(\mathsf{E}_{\nu;\beta}[f]\big)^{\mathsf{v}2}(a)\Big\|_H + \frac{1}{d}\Big\||\nabla_{\mathsf{x}}\Psi|^{\mathsf{v}2}_{\mathsf{x}}\cdot|\nabla_{\mathsf{x}}\mathsf{E}_{\nu;\beta}[f]|^{\mathsf{v}2}_{\mathsf{x}}\Big\|_H\\
&\le \|v\mapsto |v|^2_{\mathsf{x}}\|_H\cdot\Big\|\mathrm{Hess}_{\mathsf{t}[2]|\mathsf{U}}\Big(\big(\mathsf{E}_{\nu;\beta}[f]\big)^{\mathsf{v}2}\Big)\Big\|_H\\
&\qquad + \frac{1}{d}\Big\||\nabla_{\mathsf{x}}\Psi|^{\mathsf{v}2}_{\mathsf{x}}\cdot|\nabla_{\mathsf{x}}\mathsf{E}_{\nu;\beta}[f]|^{\mathsf{v}2}_{\mathsf{x}}\Big\|_H\\
&= \frac{d}{\beta}\Big\|\mathrm{Hess}_{\mathsf{t}[2]|\mathsf{U}}\Big(\big(\mathsf{E}_{\nu;\beta}[f]\big)^{\mathsf{v}2}\Big)\Big\|_H + \frac{1}{d}\Big\||\nabla_{\mathsf{x}}\Psi|^{\mathsf{v}2}_{\mathsf{x}}\cdot|\nabla_{\mathsf{x}}\mathsf{E}_{\nu;\beta}[f]|^{\mathsf{v}2}_{\mathsf{x}}\Big\|_H.
\end{aligned}$$

Since we know that $PA^2P = \frac{1}{d}\,\Delta_{\mathbf{h}0|\mathsf{U}}(P_S)$ on $D$ by Equation (3.16), the function $u := Pf$ solves the elliptic problem

$$\begin{aligned}
&u - \frac{1}{d}\,\Delta_{\mathbf{h}0|\mathsf{U}}(P_S u) = g\\
\text{in}\qquad &\Big\{u \in {\pi_{2,0|\mathsf{U}}}^* L^2(\mathbb{X};\mathbf{x})\;\Big|\;\exists\, f_{00}\in C^\infty_{\mathsf{c}}(\mathbb{X})\colon\;\; u = f_{00}^{\mathsf{v}2} - (f_{00}^{\mathsf{v}2}\,,1)_H\Big\}.
\end{aligned}$$

In order to apply the *a priori* estimates of Dolbeault, Mouhot and Schmeiser, the potential conditions (P) are required, in particular $C^2$-regularity. With these estimates at hand, one can infer existence of a constant $c_{\Phi;\beta}\in(0,\infty)$ independent of $Pf$ and $g$ such that

$$\big\|(BA)^* g\big\|_H \le c_{\Phi;\beta}\|Pg\|_H \le c_{\Phi;\beta}\|g\|_H.$$

Finally, we use [GS14, Proposition 2.15] finishing the proof. □

Summarising plainly our result, the AHHM implies the following theorem.

**Theorem 3.2.12** (Hypocoercivity of the $\sigma_v$-perturbated smoothed fibre lay-down model). Consider a Riemannian manifold $(\mathbb{X},\mathsf{x})$ satisfying (M) and a potential $\Phi\colon\; \mathbb{X}\to\mathbb{R}$ satisfying (P) and $\mathcal{H}_{\mathsf{x}}\Phi^{\mathsf{v}} = (d-1)\Phi^{\mathsf{h}}$ on $\mathsf{U}\mathbb{X}$. Furthermore, assume the diffusion parameters $\sigma,\sigma_v\in(0,\infty)$, the 'friction' parameter $\alpha\in(0,\infty)$, and $\beta\in(0,\infty)$ such that $\sigma^2/2 = \alpha/\beta$. Let the measure $\mu$ and the Hilbert space $H = L^2(\mathsf{T}\mathsf{U}\mathbb{X};\mu)$ as in Definition 3.2.2. Then, the operator $(L, C^\infty_{\mathsf{c}}(\mathsf{T}\mathsf{U}\mathbb{X}))$ as in Equation (3.11) is closable in $H$. Its closure $(L, D(L))$ generates a SCCS $(T_t)_{t\in[0,\infty)}$. For all $\kappa_1\in(1,\infty)$ there is a constant $\kappa_2\in(0,\infty)$ such that for all $g\in H$ and times $t\in[0,\infty)$ holds

$$\|T_t g - (g\,,1)_H\|_H \le \kappa_1 \mathrm{e}^{-\kappa_2 t}\|g - (g\,,1)_H\|_H,$$

and $\kappa_2$ is given as

$$\kappa_2 = \frac{\kappa_1 - 1}{\kappa_1}\,\frac{\alpha}{n_1 + n_2\alpha + n_3\alpha^2}, \tag{3.17}$$

where $n_i\in(0,\infty)$ for $i\in\{1,2,3\}$ only depend on $\Phi$ and $\beta$. Compare to the calculations (A) and (B) in Section 1.1 with $\Lambda_M = \frac{\Lambda}{d}$, $\Lambda_m = \alpha\, m_v = \alpha\,\min\!\big(1,(d-1)\,\frac{\sigma_v}{\sigma_v+1}\big)$, $c_1 = \frac{\alpha}{2}\,\frac{\sigma_v}{\sigma_v+1}(d-1)$, and $c_2 = c_{\Phi;\beta}$ as in Lemma 3.2.11. ⌟

### Discussion of $\sigma_v \to 0$

Let us explicitly write down the constants $n_i$ in Equation (3.17) in order to explore what happens as $\sigma_v$ passes to 0. Computing the coefficients of the $\bar{\varepsilon}_{\Phi;\beta}(\alpha) = \alpha/(a_1 + a_2\alpha + a_3\alpha^2)$ corresponding to Theorem 3.2.12 yields

$$
\begin{aligned}
a_1 &= a_1(\sigma_v) = \frac{1}{m_v}\,(1 + c_{\Phi;\beta}) \cdot \left(1 + (1 + c_{\Phi;\beta})\frac{\Lambda + d}{2\Lambda}\right) + \frac{1}{2}\frac{\Lambda}{\Lambda + d},\\
a_2 &= a_2(\sigma_v) = \frac{1}{m_v}\,\frac{1}{2}\frac{\sigma_v}{\sigma_v + 1}(d-1)\left(1 + (1 + c_{\Phi;\beta})\frac{\Lambda + d}{2\Lambda}\right),\\
a_3 &= a_3(\sigma_v) = \frac{1}{m_v}\,\left(\frac{1}{2}\frac{\sigma_v}{\sigma_v + 1}(d-1)\right)^2\frac{\Lambda + \beta}{2\Lambda}.
\end{aligned}
$$

Recall the proofs at [GS16, p. 164-165] or [Sti14, pp. 109-110] respectively for the construction of $\bar{\varepsilon}_{\Phi;\beta}(\alpha)$. Again, the corresponding $n_i$ are defined by

$$
\begin{aligned}
n_i = n_i(\sigma_v) &:= 4\,\frac{\Lambda + d}{\Lambda}\,a_i \cdot \bar{\varepsilon}_{\Phi;\beta;\mathsf{max}} = 4\,\frac{\Lambda + d}{\Lambda}\,a_i(\sigma_v) \cdot \bar{\varepsilon}_{\Phi;\beta;\mathsf{max}}(\sigma_v) \qquad \text{with}\\
\bar{\varepsilon}_{\Phi;\beta;\mathsf{max}} = \bar{\varepsilon}_{\Phi;\beta;\mathsf{max}}(\sigma_v) &:= \mathsf{max}\left(1, \bar{\varepsilon}_{\Phi;\beta}\left(\sqrt{\frac{a_1}{a_3}}\right)\right) = \mathsf{max}\left(1, \bar{\varepsilon}_{\Phi;\beta}\left(\sqrt{\frac{a_1(\sigma_v)}{a_3(\sigma_v)}}\right)\right).
\end{aligned}
$$

Since $m_v$ converges to 1 as $\sigma_v$ tends to 0, $1/m_v = \mathsf{max}\left(1, \frac{1}{d-1}\frac{\sigma_v+1}{\sigma_v}\right)$, this factor converges to 1 as $\sigma_v$ tends to 0. Hence, the $a_i$ converge as well: We have $a_3 \longrightarrow 0$, $a_2 \longrightarrow 0$ and,

$$
a_1 \longrightarrow a_1(0) = (1 + c_{\Phi;\beta}) \cdot \left(1 + (1 + c_{\Phi;\beta})\frac{\Lambda + d}{2\Lambda}\right) + \frac{1}{2}\frac{\Lambda}{\Lambda + d}
$$

as $\sigma_v \to 0$. Therefore, for $\sigma_v$ large enough one has

$$
\bar{\varepsilon}_{\Phi;\beta;\mathsf{max}} = \bar{\varepsilon}_{\Phi;\beta;\mathsf{max}}(\sigma_v) = \bar{\varepsilon}_{\Phi;\beta}\left(\sqrt{\frac{a_1(\sigma_v)}{a_3(\sigma_v)}}\right) = \frac{1}{2\sqrt{a_1(\sigma_v) \cdot a_3(\sigma_v)} + a_2(\sigma_v)}
$$

and thus $\bar{\varepsilon}_{\Phi;\beta;\mathsf{max}} \longrightarrow \infty$ as $\sigma_v \to 0$. Immediately, we obtain $n_1 \to \infty$ as $\sigma_v \to 0$, which already is enough to conclude $\kappa_2 = \kappa_2(\sigma_v) \longrightarrow 0$ as $\sigma_v \to 0$ with $\kappa_2(\sigma_v)$ given by Equation (3.17). This is rather unfortunate, as we are not able to establish hypocoercivity for the smoothed fibre lay-down model asymptotically. Let us console ourselves with the fact that in applications one could choose some $\sigma_v$ small enough such that the simulated trajectories look smooth enough and stick with that choice. Nonetheless, further research is desirable extending the AHHM such that it also applies to the original smoothed fibre lay-down model and higher-order equations in general.

## 3.3 ▪ Smoothed Langevin equation

In the same vein as in the previous section, one could tackle the 'smoothed' version of the Langevin equation which for the position space $\mathbb{X} = \mathbb{R}^d_x$ reads as

$$
\begin{aligned}
\mathsf{d}x_t &= v_t\;\mathsf{d}t && (3.18a)\\
\mathsf{d}v_t &= -\nabla_{\mathsf{v}}\Psi(x_t)\;\mathsf{d}t + a_t\;\mathsf{d}t && (3.18b)\\
\mathsf{d}a_t &= -\alpha \cdot a_t\;\mathsf{d}t + \sigma \circ \mathsf{d}W_t. && (3.18c)
\end{aligned}
$$

Compare this to Equation (3.3). For a general position manifold $\mathbb{X}$ we would consider on $\mathsf{T}^2\mathbb{X}$ the Stratonovich SDE

$$
\begin{aligned}
\mathsf{d}\eta &= \mathcal{H}^{(2)}_{\mathsf{x}} \circ \pi_{2,1}\,\mathsf{d}t - \mathsf{h}\!\upharpoonright^3_2\left(\mathsf{v}_{\pi_{2,1}}\!\upharpoonright^2_1\left(\nabla_{\mathsf{x}}\Psi\right)\right)\mathsf{d}t - \alpha \cdot \mathcal{V}^{(2)}\,\mathsf{d}t + \sigma\sum_{i=1}^{2d}\mathsf{v}\!\upharpoonright^3_2\left(\frac{\partial}{\partial v^i_\eta}\right) \circ \mathsf{d}W^i_t\\
&= \mathcal{H}^{(2)}_{\mathsf{x}}\,\mathsf{d}t - \mathsf{h}\!\upharpoonright^3_2 \mathsf{v}\!\upharpoonright^2_1\left(\nabla_{\mathsf{x}}\Psi\right)\mathsf{d}t - \alpha \cdot \mathcal{V}^{(2)}\,\mathsf{d}t + \sigma\sum_{i=1}^{2d}\mathsf{v}\!\upharpoonright^3_2\left(\frac{\partial}{\partial v^i_\eta}\right) \circ \mathsf{d}W^i_t,
\end{aligned} \qquad (3.19)
$$

where $\left(v_\eta^1, v_\eta^2, \dots, v_\eta^{2d}\right)$ is a chart at $\pi_{2,1}(\eta)$ providing normal coordinates. Again, we dropped the bundle projection $\pi_{2,1}$ for sake of readability in the second line. Also, one replaces $\Psi$ by $\Phi = \beta\Psi$ with an auxiliary variable $\beta \in (0, \infty)$. For sake of readability, we define $\Psi^\upharpoonleft := \left(\Psi^{\mathsf{h}1}\right)^{\mathsf{v}1} \colon \mathsf{T}^2\mathbb{X} \to \mathbb{R}$ and subsequently $\Phi^\upharpoonleft$. Then, the corresponding generator reads as

$$L = \mathcal{H}_{\mathsf{x}}^{(2)} - \nabla_{\mathsf{h}1}\Psi^\upharpoonleft - \alpha \cdot \mathcal{V}^{(2)} + \frac{\sigma^2}{2}\,\Delta_{\mathsf{v}[2]} = \mathcal{H}_{\mathsf{x}}^{(2)} - \frac{1}{\beta}\,\nabla_{\mathsf{h}1}\Phi^\upharpoonleft - \alpha \cdot \mathcal{V}^{(2)} + \frac{\sigma^2}{2}\,\Delta_{\mathsf{v}[2]} \tag{3.20}$$

on $C_{\mathsf{c}}^\infty(\mathsf{T}^2\mathbb{X})$. We call Equation (3.19) as *smoothed Langevin equation on* $\mathbb{X}$ and consequently we refer to the operator $L$ by *smoothed Langevin generator*.

Before proceeding, we look a bit closer at the Euclidean case.

**Example 3.3.1** (Euclidean position space). Let $\mathbb{X} = \mathbb{R}_x^d$ be endowed with standard metric. We identify a double tangent vector with a triple

$$z = (x, v, a) \in \mathsf{T}^2\mathbb{X} \simeq \mathbb{R}_x^d \times \mathbb{R}_v^d \times \mathbb{R}_a^{2d}$$

as well as the two gradients

$$\left(\nabla_{\mathsf{t}[2]}, C_{\mathsf{c}}^\infty(\mathsf{T}^2\mathbb{X})\right) \quad \text{and} \quad ((\nabla_x, \nabla_v, \nabla_a)^\top, C_{\mathsf{c}}^\infty(\mathbb{R}_x^d \times \mathbb{R}_v^d \times \mathbb{R}_a^{2d})).$$

Then, the vector field $\nabla_{\mathsf{h}1}\Phi^\upharpoonleft$ acts nontrivially only on test functions in $D_0$ of the form $f = \left(f_0^{\mathsf{h}1}\right)^{\mathsf{v}1}$ with $f_0 \in C_{\mathsf{c}}^\infty(\mathbb{R}^d)$, i. e. that formally depend only on the velocity variable. We calculate that

$$\begin{aligned}\nabla_{\mathsf{h}1}\Phi^\upharpoonleft(f)(z) &= \mathsf{h}_{[2;1]}\left(\nabla_{\mathsf{h}1}\left(\Phi^{\mathsf{h}1}\right)^{\mathsf{v}1}(z)\,,\nabla_{\mathsf{h}1}\left(f_0^{\mathsf{h}1}\right)^{\mathsf{v}1}(z)\right)\\ &= \left(\nabla_v\Phi^{\mathsf{h}1}(v)\,,\nabla_v f_0^{\mathsf{h}1}(v)\right)_{\mathsf{euc}} = \left(\nabla_x\Phi(x)\,,\nabla_x f_0(x)\right)_{\mathsf{euc}}.\end{aligned}$$

For general functions $f \in C_{\mathsf{c}}^\infty(\mathsf{T}^2\mathbb{X})$, approximated via a sequence in $D_0$, the action reads as

$$\nabla_{\mathsf{h}1}\Phi^\upharpoonleft(f)(z) = \left(\nabla_v\Phi^{\mathsf{h}1}(v)\,,\nabla_v f(z)\right)_{\mathsf{euc}} = \left(\nabla_x\Phi(x)\,,\nabla_v f(z)\right)_{\mathsf{euc}}.$$

Hence, for all $f \in C_{\mathsf{c}}^\infty(\mathsf{T}^2\mathbb{X})$ we get that the action of $L$ on $f$ pointwisely reads as

$$Lf(z) = \left(v\,,\nabla_x f(z)\right)_{\mathsf{euc}} - \frac{1}{\beta}\left(\nabla_x\Phi(x)\,,\nabla_v f(z)\right)_{\mathsf{euc}} - \alpha\cdot\left(a\,,\nabla_a f(z)\right)_{\mathsf{euc}} + \frac{\sigma^2}{2}\,\Delta_a f(z).$$

⌟

We don't repeat the attempt on the AHHM procedure here. That again would require an additional $\sigma_v$-noise term as in Section 3.2, just for the sake of the microscopic coercivity estimate. Instead, we sketch in broad strokes some things worth mentioning when one considers just the smoothed Langevin equation. First, we come up with invariant measure and model Hilbert space. Consider the measure

$$\mu = \rho_\mu\lambda_{\mathsf{t}[2]} := \left(\mu_{\mathbb{X}} \otimes_{\mathsf{loc}} \nu_\beta\right) \otimes_{\mathsf{loc}} \nu_{\alpha;\sigma}$$

on $\mathsf{T}^2\mathbb{X}$, where $\mu_{\mathbb{X}} := \exp(-\Phi)\lambda_{\mathsf{x}}$, $\nu_\beta$ is the zero mean Gaußian measure with covariance matrix $\beta^{-1}\mathsf{Id}_{\mathbb{R}^d}$, and $\nu_{\alpha;\sigma}$ is the zero mean Gaußian measure with covariance matrix ${}^{2\alpha}/_{\sigma^2}\,\mathsf{Id}_{\mathbb{R}^{2d}}$. As usual, we assume that $\mu_{\mathbb{X}}$ is a probability measure, thus $\mu$ is a probability measure. Define $H := L^2(\mathsf{T}^2\mathbb{X};\mu)$. On $D := C_{\mathsf{c}}^\infty(\mathsf{T}^2\mathbb{X})$ the generator $L$ as in Equation (3.20) decomposes as

$$S := \frac{\alpha}{\beta}\,\Delta_{\mathbf{v}[2]} = \frac{\sigma^2}{2}\,\Delta_{\mathsf{v}[2]} - \alpha\cdot\mathcal{V}^{(2)} \qquad \text{and} \qquad A = \mathcal{H}_{\mathbf{x}}^{(2)} := \mathcal{H}_{\mathsf{x}}^{(2)} - \frac{1}{\beta}\,\nabla_{\mathsf{h}1}\Phi^\upharpoonleft,$$

where we assume that $\beta$ satisfies ${}^{\sigma^2}/_2 = {}^{\alpha}/_{\beta}$. Similar to Lemma 2.1.5, one can prove under the assumption of $\Phi$ being weakly harmonic that

- $(S, D)$ is symmetric in $H$ and nonpositive definite,
- $(A, D)$ is antisymmetric in $H$, and

- $Lf \in L^1(\mathsf{T}^2\mathbb{X};\mu)$ as well as $\int_{\mathsf{T}^2\mathbb{X}} Lf \,\mathsf{d}\mu = 0$ for all $f \in D$.

By employing a Poisson bracket construction we can get rid of the restrictive assumption on $\Phi$, compare to Proposition 2.1.8 and the discussion surrounding it. We want to give a few more details, since that Poisson bracket actually does not correspond to the almost complex structure of the first order tangent bundle $\mathsf{T}(\mathsf{T}\mathbb{X})$. For sake of illustration, we start investigating the example of Euclidean case.

**Example 3.3.2** (Almost complex structure for Euclidean position space). Let $\mathbb{X} = \mathbb{R}^d_x$ with standard Euclidean metric $\mathsf{x} = \left((\cdot\,,\cdot)_{\mathsf{euc}}\right)_{x\in\mathbb{R}^d}$ Define

$$\mathcal{J} := \begin{pmatrix} \begin{pmatrix} 0 & \mathsf{Id}_{\mathbb{R}^d} \\ -\mathsf{Id}_{\mathbb{R}^d} & 0 \end{pmatrix} & 0_{\mathbb{R}^{2d}} \\ 0_{\mathbb{R}^{2d}} & 0_{\mathbb{R}^{2d}} \end{pmatrix}.$$

Note that $(-\mathcal{J})\mathcal{J} = \mathsf{pr}_{12}$. This justifies the name *almost complex projection*, in contrast to the classical situation of an almost complex structure on a manifold $\mathbb{B}$ where an almost complex structure is a fibrewise linear mapping $J\colon \mathsf{T}\mathbb{B} \to \mathsf{T}\mathbb{B}$ with $J^2 = -\mathsf{Id}$. Corresponding to $\mathcal{J}$, we define a anticommutative bilinear form $\{\cdot\,,\cdot\}$ via

$$\{f\,,g\} = -(\nabla f\,,\mathcal{J}\nabla g)_{\mathsf{euc}} = (\nabla_x f\,,-\nabla_v g)_{\mathsf{euc}} + (\nabla_v f\,,\nabla_x g)_{\mathsf{euc}}$$

for $f,g \in C^\infty_{\mathsf{c}}(\mathsf{T}^2\mathbb{R}^d)$. Note that it fulfils the Leibniz rule right away and one can verify also the Jacobi identity. Hence, $\{\cdot\,,\cdot\}$ is a Poisson bracket on $\mathsf{T}^2\mathbb{R}^d$. The corresponding Hamiltonian vector fields are denoted by the letter 'H' for instance $\mathsf{H}_f(g) = \{f\,,g\}$. We define the bilinear form $\mathcal{A}$ by integrating minus the Poisson bracket with respect to $\mu$: $\mathcal{A}(f,g) := \int_{\mathsf{T}^2\mathbb{R}^d} -\{f\,,g\}\,\mathsf{d}\mu$ for $f,g \in C^\infty_{\mathsf{c}}(\mathsf{T}^2\mathbb{R}^d)$. Using integration by parts we find that

$$\mathcal{A}(f,g) = \int_{\mathsf{T}^2\mathbb{R}^d} \mathsf{H}_f(\rho_\mu)\cdot g\,\mathsf{d}\lambda = (\beta A f\,,g)_H.$$

Therefore, $(A,D)$ is antisymmetric without the assumption of a weakly harmonic potential. ⌟

**Definition 3.3.3** (almost complex projection). Let $J\colon \mathsf{T}^k\mathbb{B} \to \mathsf{T}^k\mathbb{B}$ be a fibrewise linear mapping as well as a fibrewise subspace $V \subseteq \mathsf{T}^k\mathbb{B}$, i.e. $V \cap \mathsf{T}^k_a\mathbb{B} \subseteq \mathsf{T}^k_a\mathbb{B}$ is a linear subspace for all $a \in \mathsf{T}^{k-1}\mathbb{B}$. We call $J$ an *almost complex projection* if $J^2 = -\mathsf{pr}_V$ holds, where the projection $\mathsf{pr}_V$ to $V$ is defined fibrewise. ⌟

*Remark* 3.3.4. The usual almost complex structure $J$ on a tangent bundle $\pi_0\colon \mathsf{T}\mathbb{B} \to \mathbb{B}$ is characterised by the equations

$$d\pi_0 \circ J = -d\kappa \qquad \text{and} \qquad d\kappa \circ J = d\pi_0,$$

see [Dom62, Equation (12)]. In our current situation of $\mathbb{B} = \mathsf{T}\mathbb{X}$ this characterisation reads as

$$d\pi_{2,1} \circ J = -d\kappa_{[2,1]} \qquad \text{and} \qquad d\kappa_{[2,1]} \circ J = d\pi_{2,1}. \tag{3.21}$$

Imposing the two following relations

$$d\pi_{2,0} \circ \mathcal{J} = -d\kappa_{[1,0]} \circ d\pi_{2,1} \qquad \text{and} \qquad d\kappa_{[1,0]} \circ d\pi_{2,1} \circ \mathcal{J} = d\pi_{2,0} \tag{3.22}$$

we define another linear mapping $\mathcal{J}$ on $\mathsf{T}^3\mathbb{X}$. We observe how $\mathcal{J}$ acts on lifted vector fields: Let $\mathcal{X} \in \Gamma^\infty(\mathsf{T}\mathbb{X})$, then Equation (3.22) implies that

$$\mathcal{J}(\mathsf{h}\!\restriction^3_1 \mathcal{X}) = \mathsf{h}\!\restriction^3_2 (\mathsf{v}\!\restriction^2_1 \mathcal{X}), \quad \mathcal{J}(\mathsf{v}\!\restriction^3_1 \mathcal{X}) = \mathcal{J}(\mathsf{v}\!\restriction^3_2 \mathsf{h}\!\restriction^2_1 \mathcal{X}) = 0,$$
$$\text{and} \quad \mathcal{J}(\mathsf{h}\!\restriction^3_2 \mathsf{v}\!\restriction^2_1 \mathcal{X}) = -\mathsf{h}\!\restriction^3_1 \mathcal{X}.$$

Thus, $\mathcal{J}^2 = -\mathsf{pr}_{\mathsf{HT}^2\mathbb{X}}$. The almost complex projection $\mathcal{J}$ can be understood as the 'horizontal' lifting of the standard almost complex structure on $\mathsf{T}\mathbb{X}$.

With this almost complex projection we define a Poisson bracket $\{\cdot\,,\cdot\}$ via

$$\{f\,,g\} := \mathsf{t}_{[2]}\left(\nabla_{\mathsf{t}[2]} f\,,-\mathcal{J}\nabla_{\mathsf{t}[2]} g\right) \qquad \text{for all } f,g \in C^\infty_{\mathsf{c}}(\mathsf{T}^2\mathbb{X}).$$

There are corresponding Hamiltonian vector fields denoted by the letter 'H' such that $\mathsf{H}_f(g) = \{f\,,g\}$ for all $f,g \in C^\infty_{\mathsf{c}}(\mathsf{T}^2\mathbb{X})$. We derive the local coordinate expression as well as solenoidality of these Hamiltonians from the standard almost complex structure on $\mathrm{T}\mathbb{X}$: Denote Hamiltonians with respect to the aforementioned almost complex structure by '$\mathsf{H}^{(1)}$' and let $(v^j)_{j=1}^{2d}$ be chart of $\mathrm{T}\mathbb{X}$ such that $(\partial v^i)_{i=1}^{d}$ provides locally a basis of $\mathsf{V}\mathrm{T}\mathbb{X}$ and $(\partial v^{d+k})_{k=1}^{d}$ locally a basis of $\mathsf{H}\mathrm{T}\mathbb{X}$. These coordinates lifted to $\mathsf{T}^3\mathbb{X}$ yield coordinates respecting the decomposition $\mathsf{T}^3\mathbb{X} = \mathsf{V}^1\mathsf{T}^2\mathbb{X} \oplus \mathsf{H}^1\mathsf{T}^2\mathbb{X} \oplus \mathsf{H}^0\mathsf{T}^2\mathbb{X}$, and the Hamiltonians with respect to $\mathcal{J}$ take the form

$$\mathsf{H}_f = -\sum_{k=1}^{d} \partial_{\mathsf{hl}\,\partial v^k} f \cdot \partial_{\mathsf{hl}\,\partial v^{d+k}} + \sum_{i=1}^{d} \partial_{\mathsf{hl}\,\partial v^{d+i}} f \cdot \partial_{\mathsf{hl}\,\partial v^i} \qquad \text{for all } f \in C_{\mathsf{c}}^{\infty}(\mathsf{T}^2\mathbb{X}).$$

Furthermore, the nontrivial divergences of $\mathcal{J}$-Hamiltonians are computed to be the divergence of corresponding Hamiltonians $\mathsf{H}^{(1)}$:

$$\mathrm{div}_{\mathsf{t}[2]}\mathsf{H}_{f^{\mathsf{v}}} = \mathrm{div}_{\mathsf{h}[2]}\mathsf{H}_{f^{\mathsf{v}}} = \mathrm{div}_{\mathsf{t}}\mathsf{H}_f^{(1)} = 0 \qquad \text{and} \qquad \mathrm{div}_{\mathsf{t}[2]}\mathsf{H}_{f^{\mathsf{h}}} = 0$$

for all $f \in C_{\mathsf{c}}^{\infty}(\mathrm{T}\mathbb{X})$.

By the Divergence Theorem, the antisymmetric bilinear form $\mathcal{A}$ defined by integrating minus the Poisson bracket as $\mathcal{A}(f,g) := -\int_{\mathsf{T}^2\mathbb{X}} \{f\,,g\}\,\mathrm{d}\mu$ can be alternatively written as

$$\mathcal{A}(f,g) = -\int_{\mathsf{T}^2\mathbb{X}} \mathsf{H}_f(g)\,\mathrm{d}\mu = \int_{\mathsf{T}^2\mathbb{X}} \mathsf{H}_f(\rho_\mu)\cdot g\,\mathrm{d}\lambda_{\mathsf{t}[2]} \qquad \text{for all } f,g \in C_{\mathsf{c}}^{\infty}(\mathsf{T}^2\mathbb{X}).$$

With the local coordinate form for $\mathsf{H}_f$ one calculates for all $f \in D_0$ that $\mathsf{H}_f(\rho_\mu) = -\rho_\mu\beta\cdot\left(\mathcal{H}_{\mathsf{x}}^{(2)} - \nabla_{\mathsf{h}1}\Psi^{\upharpoonright}\right)f$ and that assertion also holds for all $f \in D$, as $D_0$ is dense in $D$. Thus, $\mathcal{A} = (\beta A\,\cdot\,,\cdot)_{L^2(\mu)}$ holds on $D \times D$. Hence, $(A, D)$ is antisymmetric without the assumption of weakly harmonic potential. ⌟

## 3.4 ▪ Generalised Langevin equation

Finally, we touch upon the so-called generalised Langevin equation (GLE). G. Stoltz brought the hypocoercivity discussion of this model in [PSV21] to our attention. For a derivation of the generic form [PSV21, (1.1c)] of GLE we refer to [Zwa61; Zwa73]. For sake of completeness, we briefly write it down here in the usual form, but don't go into any details:

$$\ddot{q}(t) = -\nabla V(q(t)) - \int_0^t \gamma(t-s)\cdot\dot{q}(s)\,\mathrm{d}s + F(t)$$

with position variable $q$, smooth potential $V$, $(F(t))_{t\in[0,\infty)}$ a zero-mean, stationary Gaußian process such that $\mathsf{E}[F(s)\cdot F(t)] = \mathrm{const}\cdot\gamma(|t-s|)$, where the unnamed constant would be $1/\beta$ for $\beta$ as in Section 2.1. In the introduction of [OP11] one finds a short discussion on GLE-subclasses that are equivalent to finite-dimensional Markovian systems – it depends on the point of view, but general GLE are notoriously either finite-dimensional non-Markovian or infinite-dimensional Hamiltonian systems. G. A. Pavliotis, G. Stoltz, and U. Vaes consider the stochastic Hamiltonian system [PSV21, (2.2)] obtained via a quasi-Markovian approximation using [Pav14, Proposition 8.1]. As far as we know, GLE on general manifolds have not been considered yet, so we want to propose generalisations of the nice finite-dimensional systems in the spirit of this thesis.

For sake of illustration, we start of with basically the system [OP11, (7)]. Consider the position space $\mathbb{X} \in \{\mathbb{R}^d, \mathbb{T}^d\}$ with standard notation of this thesis ($x$ position variable, $v$ velocity variable, $\Psi$ potential on $\mathbb{X}$ etc.) and $\mathrm{T}\mathbb{X} \simeq \mathbb{X}\times\mathbb{R}_v^d$ due to parallelisability. Suppose that $\gamma$ can be approximated by the finite sum of exponentials $t \longmapsto \sum_{n=1}^{N} \lambda_n^2\cdot\mathrm{e}^{-\alpha_n\cdot|t|}$ for real numbers $\lambda_1,\dots,\lambda_N \in \mathbb{R}$ as well as $\alpha_1,\dots,\alpha_N \in (0,\infty)$, where $N \in \mathbb{N}_+$, compare to [Kup04, Section 5]. Then, the approximating system of SDEs [OP11, (7)] reads as

$$\begin{aligned} \mathrm{d}x_t &= v_t\,\mathrm{d}t \\ \mathrm{d}v_t &= -\nabla_x\Psi(x_t)\,\mathrm{d}t + \sum_{n=1}^{N}\lambda_n\cdot z_t^n\,\mathrm{d}t \\ \mathrm{d}z_t^n &= -\lambda_n v_t\,\mathrm{d}t - \alpha_n z_t^n\,\mathrm{d}t + \sqrt{\frac{2\alpha_n}{\beta}}\circ\mathrm{d}W_t^n \qquad \text{for all } n \in \{1,\dots,N\}, \end{aligned} \tag{3.23}$$

where $\beta \in (0,\infty)$ and $z_t^1, \dots, z_t^N \in \mathbb{R}^d$. As in Section 2.1, we define $\Phi := \beta\Psi$ and henceforth use $\Phi$ instead of $\Psi$. We point the reader to [OP11, Assumption 2.1] on the potential $\Psi$; in particular, $\Psi$ is smooth and [OP11, Assumption 2.1 (ii)] implies a Poincaré inequality for the measure $\exp(-\Phi)\lambda$. The Markovian approximation (3.23) has the configuration manifold $\mathsf{Q}^N := \mathsf{T}\mathbb{X} \times (\mathbb{R}^d)^N \simeq \mathbb{X} \times \mathbb{R}_v^d \times (\mathbb{R}^d)^N$ and the generator $L^N$ reading as

$$\begin{aligned} & L^N f(x,v,z) \\ &= (v\,, \nabla_x f(x,v,z))_{\mathsf{euc}} - \frac{1}{\beta}\,(\nabla_x \Phi(x)\,, \nabla_v f(x,v,z))_{\mathsf{euc}} + \sum_{n=1}^N \lambda_n (z^n\,, \nabla_v f(x,v,z))_{\mathsf{euc}} \\ &\quad + \sum_{n=1}^N -\lambda_n (v\,, \nabla_{z^n} f(x,v,z))_{\mathsf{euc}} - \alpha_n (z^n\,, \nabla_{z^n} f(x,v,z))_{\mathsf{euc}} + \frac{\alpha_n}{\beta}\,\Delta_{z^n} f(x,v,z) \end{aligned} \tag{3.24}$$

for all smooth functions $f \in C^\infty(\mathsf{Q}^N)$ and $(x,v,z) = \left(x, v, (z^1, \dots, z^N)^\top\right) \in \mathsf{Q}^N$. Compare to [OP11, Equation (8)]. We also know the invariant measure $\mu^N$ on $\mathsf{Q}^N$, see [OP11, Equation (10)] or [PSV21, (2.3)]:

$$\begin{aligned} \mu^N &= \left(\exp(-\Phi)\lambda \otimes_{\mathsf{loc}} \mathsf{N}(0;\beta^{-1}\mathsf{Id}_d)\right) \;\otimes\; \bigotimes_{n=1}^N \mathsf{N}(0;\beta^{-1}\mathsf{Id}_d) \\ &= \exp(-\Phi)\lambda \;\otimes\; \bigotimes_{n=1}^{N+1} \mathsf{N}(0;\beta^{-1}\mathsf{Id}_d). \end{aligned}$$

In this situation M. Ottobre and G. A. Pavliotis proved under [OP11, Assumption 2.1] ergodicity and exponentially fast convergence to equilibrium in relative entropy norm, see [OP11, Theorems 2.1, 2.2, 2.3]. The latter is done in the hypocoercivity framework of C. Villani. Such results but for more general forms of GLE are summarised in [PSV21, Section 2.2]. We would like to contribute our modelling approach to allow for more general position manifolds $\mathbb{X}$. Afterwards, we would like to discuss what assumptions of the AHHM hold right away and which ones cause trouble. Inspired by the inspection of GLE, we propose another smoothed fibre lay-down model.

Consider a Riemannian manifold $(\mathbb{X}, \mathsf{x})$ satisfying (M) and $\Psi\colon\ \mathbb{X} \to \mathbb{R}$ satisfying (P). As configuration manifold we pick $\mathsf{Q} := \mathsf{T}^2\mathbb{X}$ endowed with the Sasaki multimetric $\mathsf{t}[2] = \mathsf{v}[2] + \mathsf{h}[2;0] + \mathsf{h}[2;1]$. Then, we have the Ehresmannian decomposition

$$\mathsf{TQ} = \mathsf{V}^1\mathsf{Q} \oplus \mathsf{H}^0\mathsf{Q} \oplus \mathsf{H}^1\mathsf{Q}.$$

We interpret the Markovian GLE-approximation described by Equation (3.23) as geodesic motion of a particle in presence of $\Psi$ that additionally is subjected to a velocity motion that consists of $N$ separate Langevin dynamics. The geodesic velocity motion is influenced by a Gaußian potential and scaled by the factor $\lambda_n$, whilst $\alpha_n$ is the friction parameter and $\sqrt{2\alpha_n/\beta}$ is the diffusion parameter, $n \in \{1, \dots, N\}$, respectively. Define $\mathcal{H}_{\mathsf{hx}} := \mathcal{H}_{\mathsf{h}} \circ \mathcal{H}_{\mathsf{x}}$. Recall from Definition 3.1.1 the higher order canonical vector fields

$$\begin{aligned} \mathcal{V}^{(2)} = \mathcal{V}^{(2,2)}\colon\ & \mathsf{T}^2\mathbb{X} \to \mathsf{T}^3\mathbb{X}, w \longmapsto \mathsf{v}_w\!\upharpoonright_2^3(w) \\ \text{and} \qquad \mathcal{V}^{(2,1)}\colon\ & \mathsf{T}^1\mathbb{X} \to \mathsf{T}^3\mathbb{X}, w \longmapsto \mathsf{v}_w\!\upharpoonright_1^3(w). \end{aligned}$$

Now, we write the system from Equation (3.23) as the system of Stratonovich SDEs

$$\mathsf{d}\eta = \eta_0 \;\mathsf{d}t + \sum_{n=1}^N \eta_n \;\mathsf{d}t \tag{3.25a}$$

$$\mathsf{d}\eta_0 = \mathcal{H}_{\mathsf{hx}} \;\mathsf{d}t - \frac{1}{\beta}\,\mathsf{h}\!\upharpoonright_2^3\left(\mathsf{v}\!\upharpoonright_1^2\left(\nabla_{\mathsf{x}}\Phi\right)\right)\;\mathsf{d}t \tag{3.25b}$$

$$\mathsf{d}\eta_n = \lambda_n \cdot \left(\mathcal{H}_{\mathsf{v}} - \mathcal{V}^{(2,1)}\right) - \alpha_n \mathcal{V}^{(2)} \;\mathsf{d}t + \sqrt{\frac{2\alpha_n}{\beta}} \cdot \sum_{i=1}^{2d} \mathsf{v}\!\upharpoonright_2^3\left(\frac{\partial}{\partial v^i_{\eta_n}}\right) \circ \mathsf{d}W_t^i \tag{3.25c}$$

for all $n \in \{1, \dots, N\}$.

Here we dropped again eventual tangent bundle projections and time indices for sake of readability. Furthermore, $\left(v^i_{\eta_n}\right)_{i=1}^{2d}$ refers to normal coordinates at $\eta_n$. The aforementioned geodesic motion in presence of $\Psi$ is described by Equation (3.25b). In contrast, each one of the dynamics described by Equation (3.25c) can be viewed as Langevin equation (2.2) on the Riemannian manifold $(\mathsf{T}\mathbb{X}, \mathsf{v})$ with Gaußian potential $\frac{1}{2}\left|\mathsf{Id}_{\mathsf{T}^2\mathbb{X}}\right|^2_{\mathsf{v}}$ and the additional scaling factor $\lambda_n$. The first line, Equation (3.25a), yields a curve $\eta\colon I \to \mathsf{T}\mathbb{X}$ and the trajectory of the position dynamic is $t \mapsto x_t := \pi_0 \circ \eta_t$. The generator corresponding to the system (3.25) is

$$L = \mathcal{H}_{\mathsf{hx}} - \frac{1}{\beta}\mathsf{h}\!\restriction^3_2\left(\mathsf{v}\!\restriction^2_1\left(\nabla_{\mathsf{x}}\Phi\right)\right) + \sum_{n=1}^{N}\lambda_n\cdot\left(\mathcal{H}_{\mathsf{v}} - \mathcal{V}^{(2,1)}\right) - \alpha_n\mathcal{V}^{(2)}\ \mathsf{d}t + \frac{\alpha_n}{\beta}\ \Delta_{\mathsf{v}[2]} \tag{3.26}$$

which is defined for all smooth functions on $\mathsf{Q}$.

**Definition 3.4.1** (model Hilbert space (D1)). Consider the probability space $(\mathsf{Q}, \mathfrak{B}(\mathsf{Q}), \mu)$ with

$$\mu = \lambda_{\mathbf{t}[2]} = \left(\lambda_{\mathbf{x}} \otimes_{\mathsf{loc}} \nu_1\right) \otimes_{\mathsf{loc}} \nu_\beta$$

with base weight $\rho_{\mathbb{X}} := \exp(-\Phi)$ such that $\lambda_{\mathbf{x}} = \rho_{\mathbb{X}}\lambda_{\mathsf{x}}$ is a probability measure, and Gaußian measures $\nu_1 := \mathsf{N}(0; \mathsf{Id}_d)$ as well as $\nu_\beta := \mathsf{N}(0; {}^1\!/\!\beta\, \mathsf{Id}_{2d})$. The model Hilbert space is $H := L^2(\mathsf{Q}; \mu) = L^2(\mathsf{T}^2\mathbb{X}; \mathbf{t}[2])$. ⌟

**Example 3.4.2.** For $\mathbb{X} \in \{\mathbb{R}^d, \mathbb{T}^d\}$ the system (3.25) and the generator $L$ as in Equation (3.26) reduce to the system (3.23) and the operator $L^N$ as in Equation (3.24). Indeed: Due to parallelisability, we can write

$$\begin{aligned} &\mathsf{H}^0\mathsf{Q} \simeq \mathsf{HT}\mathbb{X} \simeq \mathbb{X}\times\mathbb{R}^d_v, \quad \mathsf{H}^1\mathsf{Q} \simeq \mathsf{VT}\mathbb{X} \simeq (\mathbb{R}^d_v \times \mathbb{R}^d_a) \\ \text{and}\quad &\mathsf{V}^1\mathsf{Q} \simeq \mathsf{T}(\mathbb{R}^d_v\times\mathbb{R}^d_a) \simeq (\mathbb{R}^d_v\times\mathbb{R}^d_a)\times(\mathbb{R}^d_a\times\mathbb{R}^d_\jmath), \end{aligned}$$

where the subscripts '$v$', '$a$', and '$\jmath$' signify $v$elocity, $a$cceleration, and $\jmath$erk respectively. Hence, the $\mathsf{H}^0\mathsf{Q}$-valued vector field $\mathcal{H}_{\mathsf{hx}}$ is understood just as $\mathcal{H}_{\mathsf{x}}$. In contrast, the $\mathsf{H}^1\mathsf{Q}$-valued vector field $\mathcal{H}_{\mathsf{v}}$ acts as

$$\begin{aligned} \mathcal{H}_{\mathsf{v}} f_0^{\mathsf{v}}(x,v,a) &= \mathsf{h}_{[2;1]}\left(\mathcal{H}_{\mathsf{v}}\ ,\ \nabla_{\mathsf{h}1} f_0^{\mathsf{v}}\right)(x,v,a) \\ &= \mathsf{v}_{(x,v)}\left(\mathsf{Id}_{\mathsf{T}^2\mathbb{X}}(a)\ ,\ \nabla_{\mathsf{v}} f_0(x,v)\right) = \left(a\ ,\ \nabla_v f(x,v)\right)_{\mathsf{euc}} \end{aligned}$$

for functions $f_0 \in C^\infty(\mathsf{T}\mathbb{X}) \simeq C^\infty(\mathbb{X}\times\mathbb{R}^d_v)$. The two canonical vector fields $\mathcal{V}^{(2,1)}$ and $\mathcal{V}^{(2)}$ act as

$$\begin{aligned} &\mathcal{V}^{(2,1)} f(x,v,a) = \mathsf{v}_{[2]}\left(\mathcal{V}^{(2,1)}\ ,\ \nabla_{\mathsf{v}[2]} f\right)(x,v,a) = \left(v\ ,\ \nabla_a f(x,v,a)\right)_{\mathsf{euc}}, \\ \text{and}\qquad &\mathcal{V}^{(2)} f(x,v,a) = \mathsf{v}_{[2]}\left(\mathcal{V}^{(2)}\ ,\ \nabla_{\mathsf{v}[2]} f\right)(x,v,a) = \left(a\ ,\ \nabla_a f(x,v,a)\right)_{\mathsf{euc}} \end{aligned}$$

on functions $f \in C^\infty(\mathsf{Q}) \simeq C^\infty(\mathbb{X}\times\mathbb{R}^d_v\times\mathbb{R}^d_a)$. Clearly, the Laplace-Beltrami $\Delta_{\mathsf{v}[2]}$ is read as the Laplacian $\Delta_a$. Instead of the dynamics $z^1 = (z^1_t)_t, \dots, z^N = (z^N_t)_t$ on copies of $\mathbb{R}^d_a$ we go with independent dynamics $\eta_1 = (\eta_{1;t})_t, \dots, \eta_N = (\eta_{N;t})_t$ living in $\mathsf{T}^2\mathbb{X}$. ⌟

*Remark* 3.4.3 (SAD-decomposition for $L$). The generator $L$ as in Equation (3.26) decomposes on $D := C^\infty_{\mathsf{c}}(\mathsf{Q})$ into

$$L = -A_0 + L_{1:N} := -A_0 + S_{1:N} - A_{1:N}$$

with the operators

$$\begin{aligned} S_{1:N} &= \frac{\alpha_{1:N}}{\beta}\ \Delta_{\mathbf{v}[2]} = \sum_{n=1}^{N}\frac{\alpha_n}{\beta}\ \Delta_{\mathbf{v}[2]} \\ &:= \frac{\alpha_{1:N}}{\beta}\ \Delta_{\mathsf{v}[2]} - \alpha_{1:N}\mathcal{V}^{(2)} = \sum_{n=1}^{N}\frac{\alpha_n}{\beta}\ \Delta_{\mathsf{v}[2]} - \alpha_n\mathcal{V}^{(2)} && \text{with } \alpha_{1:N} := \sum_{n=1}^{N}\alpha_n, \\ A_0 &:= -\mathcal{H}_{\mathsf{hx}} + \frac{1}{\beta}\mathsf{h}\!\restriction^3_2\left(\mathsf{v}\!\restriction^2_1\left(\nabla_{\mathsf{x}}\Phi\right)\right), && \text{and} \\ A_{1:N} &:= -\lambda_{1:N}\left(\mathcal{H}_{\mathsf{v}} - \mathcal{V}^{(2,1)}\right) = -\sum_{n=1}^{N}\lambda_n\cdot\left(\mathcal{H}_{\mathsf{v}} - \mathcal{V}^{(2,1)}\right) && \text{with } \lambda_{1:N} := \sum_{n=1}^{N}\lambda_n. \end{aligned}$$

Clearly, the operator $(S_{1:N}, D)$ on $H$ is symmetric and nonpositive definite; also, the operators $(A_0, D)$ and $(A_{1:N}, D)$ are antisymmetric. Compare to Lemma 2.1.5. ⌟

*Remark* 3.4.4 (essential m-dissipativity of $(L, D)$). In [OP11, Proposition 3.1] it is proven for $\mathbb{X} \in \{\mathbb{R}^d, \mathbb{T}^d\}$ and smooth potential $\Phi$ that $(L, D)$ is essentially m-dissipative. The strategy of the proof is taken from [HN05, Proposition 5.5], therefore it shares the same spirit with Proposition 2.1.13. The key to the manifold setting is to obtain the Hörmander condition for $L$ using Lemma 2.1.12 with $\mathcal{H} = \mathcal{H}_\mathsf{t} = \mathcal{H}_\mathsf{v} + \mathcal{H}_\mathsf{h}$. Afterwards, one considers the situation without base weight on $\mathbb{X}$, i. e. on the space $L^2(\mathsf{Q}; \big(\lambda_\mathsf{x} \otimes_{\mathsf{loc}} \nu_1\big) \otimes_{\mathsf{loc}} \nu_\beta)$ the operator $L_0 = -\mathcal{H}_{\mathsf{hx}} + L_{1:N}$ with predomain $D$ is found to be essentially m-dissipative as in Lemma 2.1.14. Then, introduce the unitary transformation

$$U\colon\ H \to L^2(\mathsf{Q}; \big(\lambda_\mathsf{x} \otimes_{\mathsf{loc}} \nu_1\big) \otimes_{\mathsf{loc}} \nu_\beta),\ f \longmapsto \Big(\exp\Big(-\frac{1}{2}\Phi\Big)\Big)^{\mathsf{v}2} \cdot f = \exp\Big(-\frac{1}{2}\Phi^{\mathsf{v}2}\Big) \cdot f$$

and look at

$$\widetilde{L} := ULU^{-1} = -UA_0U^{-1} + UL_{1:N}U^{-1} =: -\widetilde{A}_0 + \widetilde{L}_{1:N}$$

on a suitable domain. Since $\widetilde{A}_0$ acts on twice vertically lifted functions basically the same way as $\widetilde{A}$ from Section 2.1 acts on vertically lifted functions, one can deal with a globally Lipschitzian potential via the almost identical perturbation argument as in the proof of Lemma 2.1.16 and then arguments as in Corollary 2.1.17. The case of a locally Lipschitzian potential is covered as in Proposition 2.1.18. ⌟

Unfortunately, the question of $L^2$-hypocoercivity of the (approximate) GLE remains open. From point of view of the AHHM, the crux is the lack of a symmetric operator $S_0$ complementing $A_0$ in terms of a SAD-decomposition. In other words, the geodesic motion corresponding to the base Riemannian metric x is not perturbed by a diffusion that provides the necessary amount of microscopic coercivity. As we did in Section 3.2, one can add a 'small noise on the velocity level' by introducing $S_0$ as weighted version of the Laplace-Beltrami $\Delta_{\mathsf{h}1}$ times friction parameter $\alpha_{1:N}$ as well as some factor involving a scaling parameter $\sigma_v$, e. g. ${}^{\sigma_v}/_{\sigma_v+1}$. The weighted Laplacian is required, rather than the unweighted one, due to the fact that the covariance matrix of the Gaußian measure $\nu_1$ is to be rescaled by the reciprocal of the $\sigma_v$-diffusion factor, in order for $\mu$ to still be the invariant measure. Not to forget that $\mathcal{V}^{(2,1)}$-term also gets the $\sigma_v$-diffusion factor as consequence of the change of measure. Still, the hypocoercivity constant $\kappa_2 = \kappa_2(\sigma_v)$ converges to 0 when taking the limit $\sigma_v \to 0$. Compare to the discussion at the end of Section 3.2.

To end the chapter, we propose another fibre lay-down model that yields smoother looking paths and might benefit from future research achievements in the GLE-field. Instead of the smoothed fibre lay-down model (3.9) we suggest basically the GLE but with the algebraic side condition of normalised velocities. The side condition is incorporated via the choice of $\mathsf{Q} := \mathsf{TU}\mathbb{X}$ and accordingly of the Sasaki multimetric on $\mathsf{Q}$ induced by the unit Sasaki metric $\mathsf{u} = \mathsf{v}|\mathsf{U} + \mathsf{h}$.

$$\mathrm{d}\eta = \eta_0\ \mathrm{d}t + \eta_1\ \mathrm{d}t \tag{3.27a}$$

$$\mathrm{d}\eta_0 = \mathcal{H}_{\mathsf{hx}}\ \mathrm{d}t - \mathsf{h}\!\upharpoonright^3_2 \Big(\mathsf{tl}\big(\nabla_\mathsf{x}\Psi\big)\Big)\ \mathrm{d}t \tag{3.27b}$$

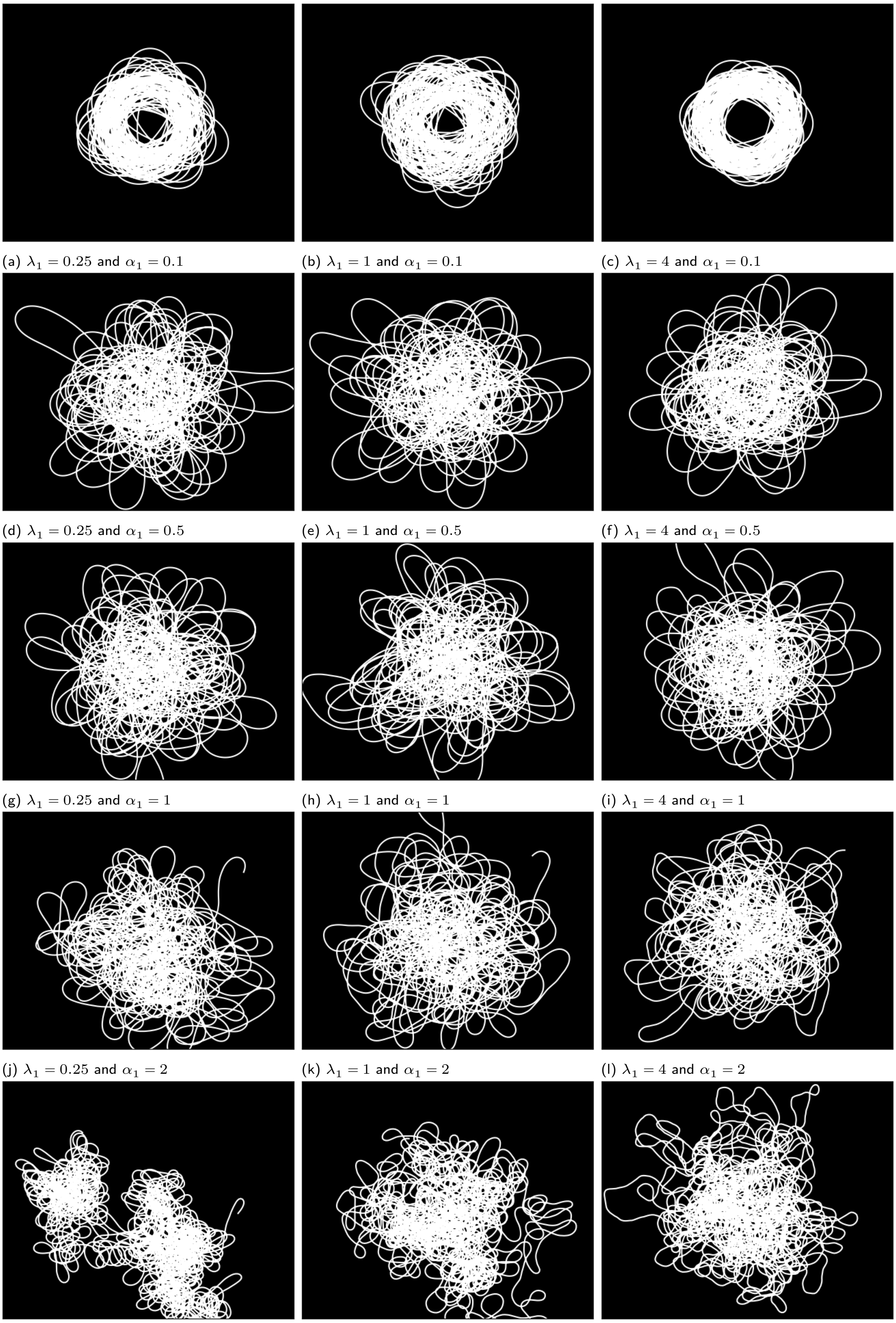
$$\mathrm{d}\eta_1 = \lambda_1 \cdot \big(\mathcal{H}_{\mathsf{v}|\mathsf{U}} - \mathcal{V}^{(2,1)}|_{\mathsf{U}\mathbb{X}}\big) - \alpha_1 \mathcal{V}^{(2)}|_{\mathsf{TU}\mathbb{X}}\ \mathrm{d}t + \sqrt{\frac{2\alpha_1}{\beta}} \cdot \sum_{i=1}^{2d-1} \mathsf{v}\!\upharpoonright^3_2 \left(\frac{\partial}{\partial u^i_{\eta_1}}\right) \circ \mathrm{d}W^i_t. \tag{3.27c}$$

A certainly fitting name for Equation (3.27), not related to fibre lay-down dynamics, is *spherical velocity GLE*. Again as in Section 3.2, we replace the expression $\mathsf{h}\!\upharpoonright^3_2 \Big(\mathsf{tl}\big(\nabla_\mathsf{x}\Psi\big)\Big)$ by the vector field $\frac{1}{d-1}\nabla_{\mathsf{h}1|\mathsf{U}}\big(\mathcal{H}_\mathsf{x}\Psi^\mathsf{v}\big)^{\mathsf{v}1}$. Furthermore, we simplify notation for the restricted canonical vector fields as $\mathcal{V}^{(2,1)}_\mathsf{U} := \mathcal{V}^{(2,1)}|_{\mathsf{U}\mathbb{X}}$ and $\mathcal{V}^{(2)}_\mathsf{U} := \mathcal{V}^{(2)}|_{\mathsf{TU}\mathbb{X}}$. Then, the so-called *spherical velocity GLE-generator* reads as

$$L = \mathcal{H}_{\mathsf{hx}} - \frac{1}{d-1}\nabla_{\mathsf{h}1|\mathsf{U}}\big(\mathcal{H}_\mathsf{x}\Psi^\mathsf{v}\big)^{\mathsf{v}1} + \lambda_1 \cdot \big(\mathcal{H}_\mathsf{v} - \mathcal{V}^{(2,1)}_\mathsf{U}\big) - \alpha_1 \mathcal{V}^{(2)}_\mathsf{U}\ \mathrm{d}t + \frac{\alpha_1}{\beta}\ \Delta_{\mathsf{v}|\mathsf{U}[2]} \tag{3.28}$$

and decomposes on $D := C^\infty_\mathsf{c}(\mathsf{Q})$ in virtually the same way as in Remark 3.4.3 with $N = 1$. The corresponding invariant measure on $\mathsf{Q}$ is $\mu := \big(\lambda_\mathsf{x} \otimes_{\mathsf{loc}} \nu\big) \otimes_{\mathsf{loc}} \nu_\beta$ with $\nu := \mathsf{S}^{d-1}_1$ being the normalised spherical surface measure. The operator $(L, D)$ from Equation (3.28) is essentially m-dissipative on $H := L^2(\mathsf{Q}; \mu)$, as the arguments from Remark 3.4.4 translate just with slight modifications. But the question whether the generated semigroup is $H$-hypocoercive is a topic for further research.

We close the section by presenting a few numerical experiments: In Figure 3.3 one finds trajectories for Equation (3.27) in the plane. We fixed $\beta = 1$ in order to investigate the influence of the parameters $\alpha_1$ and $\lambda_1$.



(a) $\lambda_1 = 0.25$ and $\alpha_1 = 0.1$
(b) $\lambda_1 = 1$ and $\alpha_1 = 0.1$
(c) $\lambda_1 = 4$ and $\alpha_1 = 0.1$
(d) $\lambda_1 = 0.25$ and $\alpha_1 = 0.5$
(e) $\lambda_1 = 1$ and $\alpha_1 = 0.5$
(f) $\lambda_1 = 4$ and $\alpha_1 = 0.5$
(g) $\lambda_1 = 0.25$ and $\alpha_1 = 1$
(h) $\lambda_1 = 1$ and $\alpha_1 = 1$
(i) $\lambda_1 = 4$ and $\alpha_1 = 1$
(j) $\lambda_1 = 0.25$ and $\alpha_1 = 2$
(k) $\lambda_1 = 1$ and $\alpha_1 = 2$
(l) $\lambda_1 = 4$ and $\alpha_1 = 2$
(m) $\lambda_1 = 0.25$ and $\alpha_1 = 4$
(n) $\lambda_1 = 1$ and $\alpha_1 = 4$
(o) $\lambda_1 = 4$ and $\alpha_1 = 4$

Figure 3.3: Spherical velocity GLE with $\lambda_1$ varying by row and $\alpha_1$ varying by column

# 4 Dimension reduction by means of scaling limits

Until now we assumed a general finite dimension $d$ of the Riemannian manifold $(\mathbb{X}, \mathrm{x})$, which in most applications would not be the most desirable degree of freedom. In fibre lay-down applications we surely consider $d \in \{2, 3\}$, as on the one hand the conveyor belt is two-dimensional by nature whereas on the other hand the nonwoven fleece has microscopically a three-dimensional structure. Clearly, the three-dimensional fibre lay-down models discussed so far don't fit the reality of industrial production well, since the velocities are uniformly distributed on the sphere $\mathbb{S}^{d-1} = \mathbb{S}^2$ and thus defy gravity as well as other possible influences that might favour a particular set of directions. Thus, in contrast to the former *isotropic* models we look for *anisotropic* variations.

Proposals to achieve this goal have been made in [KMW12a; KMW12b]. The idea of A. Klar et al. is to introduce scaling parameters in special local coordinate equations to weight certain velocity directions differently. The pool of scaling limit techniques is immensely broad and many of them by design answer specific questions in specific situations. We mention some techniques from this multifarious zoo and results in Section B.1, but that exhibition is by no means exhaustive.

As we formalise our scaling procedure as geometric as possible, we recall in Section 4.1.1 common notions of convergence for spaces that carry measurable, metric, metric-measure, or so-called spectral structures. We don't attempt to cover everything in this thesis, but a few more details are given in Section B.2. Though, we interject an introduction to stratifolds in Section 4.1.2. This concept was introduced by Matthias Kreck and I think it provides the perfect framework compared to other generalisations of smooth manifolds: The power of stratifolds comes from generalising properties of manifolds within the context of so-called differential spaces, rather than relying on group actions, quotient space constructions etc. Furthermore, we can allow for singularities by proximity to stratified spaces in topology. In fact, several quite intuitive operations naturally result in stratifolds like taking the cone over or the suspension of a manifold. Afterwards in Section 4.2, we deliver on the geometric convergence results. First in Section 4.2.1, we look at the sphere as isolated standard fibre of the unit tangent bundle and make it collapsing to a sphere of lower dimension. See Corollary 4.2.6, Corollary 4.2.9, and Corollary 4.2.10. Then in Section 4.2.2, we adapt the configuration manifold of isotropic fibre lay-down models on $\mathbb{X}$ to the configuration stratifold[1] for anisotropic fibre lay-down models, and construct the limit space when the standard fibre collapses.

Eventually, in Section 4.3 we formulate our anisotropic fibre-lay down models, once the configuration stratifold bundle is well-established. Then, we investigate hypocoercivity of the anisotropic as well as the collapsed dynamics. See Theorem 4.3.1, Corollary 4.3.3, and Theorem 4.3.9. Having generalised Dirichlet forms at the back of one's mind, one naturally asks whether or not the anisotropic fibre lay-down models converge in Mosco sense. In case the reader is not familiar with Mosco convergence, we provide a short discussion in the appendix, see Section B.3. Mosco convergence is rather sensible in regards to dimension reduction. Therefore, the final result Theorem 4.3.10 establishing Mosco convergence of the anisotropic fibre lay-down models seems quite remarkable.

Before we get into the details, we sketch our take on the scaling problem. Let us suppose that the unit tangent bundle is trivial: $\mathrm{U}\mathbb{X} \simeq \mathbb{X} \times \mathbb{S}^{d-1}$. Furthermore, we think $\mathbb{S}^{d-1}$ as isometrically embedded submanifold of $\mathbb{R}^d$ of the form

$$\mathbb{S}^{d-1} = \left\{ u = \sum_{j=1}^{d} u_j e_j \in \mathbb{R}^d \;\middle|\; \sum_{j=1}^{d} u_j^2 = 1 \right\},$$

where $(e_j)_{j=1}^d$ is the standard basis of $\mathbb{R}^d$. Without loss of generality, we choose $e_d$ as the direction that we want to get rid off. The sphere $\mathbb{S}^{d-1}$ carries the normalised surface measure $\mathsf{S}_1^{d-1}$ as canonical probability measure. For every scaling parameter $\varepsilon \in (0,1)$ we consider a probability density $\rho_{\mathbb{S}}^{\varepsilon}$ such that on the one hand $\rho_{\mathbb{S}}^{\varepsilon}$ is invariant with respect to rotations around the $e_d$-axis and on the other hand the probability

[1] To be more precise, it is the total space of a stratifold bundle, as introduced in Definition 4.2.14 later.

measures $\rho^{\varepsilon}_{\mathbb{S}}\mathsf{S}^{d-1}_1$ weakly converge to $\mathsf{S}^{d-2}_1 \otimes \delta_{\pi/2}$ as $\varepsilon \downarrow 0$. The Dirac measure $\delta_{\pi/2}$ detects the 'equator' in standard polar coordinates: One measures the 'latitudinal angle' for the highest order 'meridian' starting with 0 at the 'north pole' $e_d$, thus the angle $\pi/2$ corresponds to points in the 'equator'

$$\mathbb{S}^{d-2} = \left\{ u = \sum_{j=1}^{d} u_j e_j \in \mathbb{R}^d \,\middle|\, \sum_{j=1}^{d-1} u_j^2 = 1 \wedge u_d = 0 \right\}.$$

We construct an explicit example for such densities $(\rho^{\varepsilon}_{\mathbb{S}})_{\varepsilon}$ in Section 4.2.1.

The poles are to be treated somewhat separately. E. g. in the case of $d = 3$ the density $\rho^{\varepsilon}_{\mathbb{S}}$, $\varepsilon \in (0, 1)$, can not be globally smooth and have globally nonvanishing derivative at the same time. Otherwise due to its rotational invariance, it would determine a smooth section $\mathbb{S}^2 \to \mathsf{T}\mathbb{S}^2$, $u \mapsto E_2(u)$ that always points into direction of $e_3$. But such a section is not even continuous by the Hairy Ball Theorem. Be that as it may, the density can easily be smooth with nonvanishing derivative 'far away' from the poles $\pm e_3$. Back in the general case, we desire some kind of stratification separating poles from the rest of the sphere. Although so-called Alexandrov spaces come with such a stratification, we choose to rely on the fairly new concept of stratifolds. On the one hand it gives us a bit more structure to work with and on the other hand that structure is easier accessible in terms of a certain subalgebra of continuous functions.

Beyond that, we also use stratifolds to get into a situation of 'quasi-parallelisability'. Meaning, we transform the position space $\mathbb{X}$ into a stratifold such that the top stratum is a parallelisable smooth manifold and the remaining strata can basically be neglected. E. g. if one makes the poles $\pm e_3$ of $\mathbb{X} = \mathbb{S}^2$ into one stratum and the complement into the second stratum $\mathbb{X}_{\oslash}$, then $\mathbb{X}_{\oslash}$ is diffeomorphic to an open cylinder over $\mathbb{S}^1$ (with finite height) and therefore is parallelisable.

## 4.1 ▪ Preface

We suppose that the reader is familiar with neither varifolds, nor Gromov-Hausdorff-type convergences, nor convergence of spectral structures. Therefore, we give basic definitions, examples, references etc. in Section 4.1.1. The convergence of varifolds is the weakest geometric notion of convergence in this thesis and one does not strictly need it for our purposes. Nevertheless, we recommend to consider it anyways: As varifolds are a measure theoretic generalisation of manifolds, this versatile notion allows for all kinds of manipulations. We use varifolds for a first construction of the limit spaces under our scaling limits. Though, we see in the process that varifold convergence is not appropriate to cover reduction of dimension correctly. In lieu, the central notion of geometric convergence is Gromov-Hausdorff convergence of (pointed) metric measure spaces.

Afterwards, we introduce the concept of stratifolds in Section 4.1.2. This might surprise some, since in the context of Gromov-Hausdorff convergence the category of so-called Alexandrov spaces is the natrual one. Indeed, as an implication of the Gromov Compactness Theorem for metric spaces, see [Gro07, Proposition 5.2], the set of pointed finite dimensional Riemannian manifolds with Ricci curvature bounded from below is precompact with respect to Gromov-Hausdorff distance, see [Gro07, Theorem 5.3]. The limit spaces of such Riemannian manifolds are Alexandrov spaces. However, I prefer stratifolds for all intents and purposes of this thesis, as stratifolds by their very design yield a nicer stratification compared to Alexandrov spaces. In the end of Section 4.1.2, we briefly discuss well-known results concerning Alexandrov spaces contrasting them to stratifolds.

### 4.1.1 ▪ Overview of several notions of geometric convergence

#### Weak convergence of varifolds

Anticipating definitions below, varifolds are certain measures, thus one would naturally say that a sequence $(\mathbb{V}_n)_{n\in\mathbb{N}}$ of varifolds in a manifold $\mathbb{Y}$ converges to a varifold $\mathbb{V}$ in $\mathbb{Y}$ if $\mathbb{V}_n \longrightarrow \mathbb{V}$ weakly as $n \to \infty$. The term 'varifold' was coined by F. J. Almgren Jr. for a substitute of manifolds in geometric measure theory, see [Alm66, Section 3]. In its first form the concept appeared before in the works of Laurence Chisholm Young as 'generalised surface'. The slightly different varifold formalism has proven itself by now for solving variational problems arising in real life applications. See e. g. [Tay76; Tay78] for applications to soap-film-like respectively soap-bubble-like surfaces or crystalline structures. We present a more 'geometrical' formulation from [All72, Section 3] which describes varifolds as Radon measures on Graßmannians. As a side note, the interested reader is also referred to [Sim83, Chapter 4 and 8] which approaches the well-behaved so-called integral varifolds from a different direction that might be easier to grasp. Rectifiability theorems like [Sim83, Theorem 38.3] and [All72, Theorems 3.5.1 and 3.5.2] relate integral varifolds in the sense of W. Allard and L. Simon to each other.

**Definition 4.1.1** (varifolds). Consider a smooth manifold $\mathbb{Y}$ as submanifold of $\mathbb{R}^{n}$. Denote by $\mathbb{G}(k, \mathbb{Y})$ the (unoriented) Graßmannian of $\mathbb{Y}$: If $\mathbb{G}(k, n)$ is the space of $k$-dimensional (unoriented) subspaces of $\mathbb{R}^{n}$ with its natural topology[2], then one defines

$$\mathbb{G}(k, \mathbb{Y}) := (\mathbb{Y} \times \mathbb{G}(k, n)) \cap \big\{(y, S) \mid y \in \mathbb{Y} \wedge S \subseteq \mathsf{T}_y \mathbb{Y}\big\}.$$

A *$k$-dimensional varifold $\mathbb{V}$ (in the ambient space $\mathbb{Y}$)* is a Radon measure on $\mathbb{G}(k, \mathbb{Y})$. Furthermore, we define via $\|\mathbb{V}\|(A) := \mathbb{V}(\{(y, S) \in \mathbb{G}(k, \mathbb{Y}) \mid y \in A\})$ for $A \in \mathfrak{B}(\mathbb{Y})$ a Radon measure on $\mathbb{Y}$, which we call *weight measure*. If $\|\mathbb{V}\|(\mathbb{Y})$ is finite, this number is called *weight of* $\mathbb{V}$.

**Example 4.1.2.** Consider the situation of Definition 4.1.1. If $\mathbb{Y}$ is compact and $B \in \mathfrak{B}(\mathbb{Y})$, then the following assignment defines a varifold $\mathbb{V}_B$ due to the Riesz Representation Theorem:

$$\int_{\mathbb{G}(k,\mathbb{Y})} f \, \mathrm{d}\mathbb{V}_B := \int_B f(y, \mathsf{T}_y \mathbb{Y}) \, \mathscr{H}_k(\mathrm{d}y)$$

for all continuous functions $f\colon \mathbb{G}(k, \mathbb{Y}) \to \mathbb{R}$ with compact support. Here, '$\mathscr{H}_k$' denotes the $k$-dimensional Hausdorff measure. The weight measure $\|\mathbb{V}_B\|$ equals the restriction $\mathscr{H}_k|_B$ to $B$.

**Example 4.1.3** (Riemannian manifolds as varifolds). Consider a $\ell$-dimensional Riemannian manifold $(\mathbb{B}, \mathsf{b})$ as a submanifold of the ambient space $\mathbb{R}^{n}$. For the moment, assume that $\mathbb{B}$ is compact; hence the Riemannian volume measure $\lambda_{\mathsf{b}}$ could be viewed as probability measure up to a normalisation constant that we are going to suppress henceforth. Define

$$k_{\mathbb{B}}\colon\ \mathbb{B} \times \mathfrak{B}(\mathbb{G}(\ell, n)) \to \mathbb{R},\ \ (b, \bar{A}) \longmapsto \int_{\mathbb{G}(\ell,n)} \mathbb{1}_{\bar{A}}(S)\ \delta_{\mathsf{T}_b\mathbb{B}}(\mathrm{d}S).$$

Then, $k_{\mathbb{B}}(b, \cdot)$ is a probability measure for all $b \in \mathbb{B}$ and $k_{\mathbb{B}}(\cdot, \bar{A})$ is measurable for all $\bar{A} \in \mathfrak{B}(\mathbb{G}(\ell, n))$, thus $k_{\mathbb{B}}$ is a Markov kernel. Hence, there exists a unique probability measure $\mathbb{V}_{\mathsf{b}}$ satisfying

$$\begin{aligned} \mathbb{V}_{\mathsf{b}}(A) &= \int_{\mathbb{B}} k_{\mathbb{B}}(b, \mathrm{pr}_2(A)) \cdot \mathbb{1}_{\mathrm{pr}_1(A)}(b)\ \lambda_{\mathsf{b}}(\mathrm{d}b) = \int_{\mathrm{pr}_1(A)} k_{\mathbb{B}}(b, \mathrm{pr}_2(A))\ \lambda_{\mathsf{b}}(\mathrm{d}b) \\ &= \int_{\mathrm{pr}_1(A)} \int_{\mathbb{G}(\ell,n)} \mathbb{1}_{\mathrm{pr}_2(A)}(S) \quad \delta_{\mathsf{T}_b\mathbb{B}}(\mathrm{d}S)\ \lambda_{\mathsf{b}}(\mathrm{d}b) \end{aligned} \tag{4.1}$$

for all $A \in \mathfrak{B}(\mathbb{G}(\ell, \mathbb{R}^{n})) = \mathfrak{B}(\mathbb{R}^{n} \times \mathbb{G}(\ell, n))$. Similarly, if the not necessarily compact $\mathbb{B}$ is endowed with a weight $\rho_{\mathbb{B}}$ such that $\lambda_{\mathbf{b}} = \rho_{\mathbb{B}} \lambda_{\mathsf{b}}$ is a probability measure, then there is a probability measure $\mathbb{V}_{\mathbf{b}}$ that satisfies Equation (4.1) with $\lambda_{\mathsf{b}}$ replaced by $\lambda_{\mathbf{b}}$.

## Gromov-Hausdorff convergence

One fairly basic notion of geometric convergence is provided via the Gromov-Hausdorff (GH) distance between metric spaces. Originally, it was defined in [Gro81, Section 7]; see [Gro07, Section 3.A] for concise definitions. Here, we restrict ourselves to a few examples to just give an impression. What we really need are the extensions of Gromov-Hausdorff convergence that include for instance measure convergence.

**Example 4.1.4** (collapsing of manifolds).

(i) Let $(e_1, e_2)$ an orthonormal base of $\mathbb{R}^2$. For $c \in (0, 1]$ we consider the torus $\mathbb{T}_c$ corresponding to the quotient space $\mathbb{R}^2/\mathbb{L}_c$ where $\mathbb{L}_c$ denotes the lattice generated by $(e_1, ce_2)$. Then, the Ricci curvature of every $\mathbb{T}_c$ is zero and $(\mathbb{T}_c)_{c\in(0,1]}$ GH-converges to the sphere $\mathbb{S}^1$ as $c \downarrow 0$ corresponding to $\mathbb{R}e_1/\mathbb{Z}e_1$. See [Gro07, Counterexamples 5.5].

[2] It comes from the identification of $\mathbb{G}(k, n)$ with the coset $\mathrm{O}(n)/\mathrm{O}(k) \times \mathrm{O}(n - k)$.

(ii) Consider for parameters $c \in (0,1]$ the ellipsoids $\mathbb{Y}_c$ defined as

$$\mathbb{Y}_c := \left\{ x \in \mathbb{R}^3 \,\middle|\, x_1^2 + x_2^2 + \left(\frac{x_3}{c}\right)^2 = 1 \right\}.$$

Then, the sequence $(\mathbb{Y}_c)_{c\in(0,1]}$ convergences as $c \downarrow 0$ to the space $\mathbb{DD}$ constructed as follows: Take different closed disks $\mathbb{D}_-$ and $\mathbb{D}_+$ of radius 1 and glue them together along their respective boundary which forms the 'sutura' $S$ of points that lay on both glued disks; the quotient space centred at 0 is now denoted by $\mathbb{DD}$ and endowed with the metric

$$\begin{aligned} &\mathbb{DD} \times \mathbb{DD} \to [0,\infty): \\ &(x,y) \longmapsto \begin{cases} |x-y|_{\text{euc}} & x,y \in \mathbb{D}_\pm \\ \inf_{z\in S}(|x-z|_{\text{euc}} + |z-y|_{\text{euc}}) & x \in \mathbb{D}_\pm, y \in \mathbb{D}_\mp \end{cases}. \end{aligned}$$

That metric ensures that points on the same disk have Euclidean distance to each other, whilst for points on different disks one measures the shortest path travelling from the point $x \in \mathbb{D}_+$ to a point $z$ on the sutura and from $z$ to $y \in \mathbb{D}_-$. Compare to [Sor07, Example 2.3] and Figure 4.1.

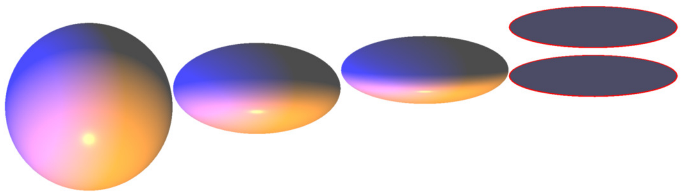


Figure 4.1: Ellipsoids collapsing from left to right, where the red circles are glued together

**Example 4.1.5** (appearing singularities). Again, this example is taken from [Gro07, Counterexamples 5.5]. Consider the hemisphere $\mathbb{Y}_1$ and deform it towards a cone as in Figure 4.2. The cone is not a Riemannian

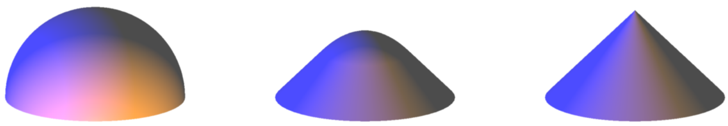


Figure 4.2: Deformation of a hemisphere to a cone

manifold. However, it can be seen as a stratifold, see Example 4.1.22.

**Example 4.1.6** (vanishing holes). Consider the sequence $(\mathbb{Y}_c)_{c\in(0,1]}$ constructed as follows: For $\mathbb{Y}_1$ glue a pair of handles to the sphere $\mathbb{S}^2$. As the parameter $c$ decreases to zero the handles become smaller and smaller such that the Gromov-Hausdorff limit is exactly the sphere $\mathbb{S}^2$. See [Sor07, Example 2.2]

These examples show that dimension, curvature, and several more geometric data are not preserved by Gromov-Hausdorff convergence. As mentioned and seen before, the limit of a GH-convergent sequence of (Riemannian) manifolds is not necessarily a (Riemannian) manifold – even under nice curvature bounds. However, there are several closedness results available many of which can be found in the expository article [Fuk90].

### (Pointed) Measured Gromov(-Hausdorff) convergence

We adopt notions from [GMS15] which are contrasted in [GMS15, Section 1.1.5] specifically for compact metric measure spaces: *Pointed measured Gromov (pmG) convergence* just asks for weak convergence of measures, whilst *pointed measured Gromov-Hausdorff convergence (pmGH)* requires both weak convergence of measures and Hausdorff convergence of the metric spaces. There are several equivalent definitions of pmG-convergence, see [GMS15, Section 3.2] and Definition B.2.6 in the appendix. We want to spotlight the *extrinsic* definition given below in Definition 4.1.7. In [GMS15, Section 3.4] it's explained very neatly that the extra information of an *effective realisation* in the extrinsic approach enables us to make a precise definition, how a sequence of points might converge when each point is in a (different) space of a converging sequence of spaces, see [GMS15, Definition 3.23].

**Definition 4.1.7** (Extrinsic pmG-convergence). Let $J$ be a directed set and a net of pointed metric measure spaces $\big(\mathbb{Y}_j, \mathsf{dist}_j, \mu_j, \bar{y}_j\big)_{j\in J}$. We say that this net *converges extrinsically to* $(\mathbb{Y}_\infty, \mathsf{dist}_\infty, \mu_\infty, \bar{y}_\infty)$, if there exists a so-called *effective realisation* $\big(\mathbb{Y}, \mathsf{dist}_{\mathbb{Y}}, \iota_j\big)_{j\in J\cup\{\infty\}}$, consisting of a complete, separable metric space $(\mathbb{Y}, \mathsf{dist}_{\mathbb{Y}})$ and isometric embeddings $\iota_j\colon\ \mathbb{Y}_j \to \mathbb{Y}$ for all $j \in J$ as well as $\iota_\infty\colon\ \mathbb{Y}_\infty \to \mathbb{Y}$, such that the net of image measures weakly converges, i. e. $\big(\iota_j\big)_*\mu_j \rightharpoonup (\iota_\infty)_*\mu_\infty$, and the net of designated points also converges, i. e. $\iota_j(\bar{y}_j) \to \iota_\infty(\bar{y}_\infty) \in \mathsf{supp}((\iota_\infty)_*\mu_\infty)$. This is [GMS15, Definition 3.9]. ⌟

For the precise definition of *pmGH-convergence* we refer to [GMS15, Definition 3.24 and Proposition 3.28]; also see the discussion in [GMS15, Remarks 3.25-27].[3] In general, pmG-convergence doesn't imply pmGH-convergence. Wherefore, one desires powerful criteria for this implication to hold. We briefly present here [GMS15, Theorem 3.33] which crucially requires a common doubling constant for the converging spaces.

**Theorem 4.1.8** (from pmG- to pmGH-convergence). Let $J$ be a directed net. Assume pointed metric measure spaces $\big(\mathbb{Y}_j, \mathsf{dist}_j, \mu_j, \bar{y}_j\big)$, $j \in J \cup \{\infty\}$, such that $\big(\mathbb{Y}_j, \mathsf{dist}_j\big)$ is complete as well as separable, $\mu_j$ is a nonzero Borel measure that is finite on every bounded subset and, $\bar{y}_j \in \mathsf{supp}\big(\mu_j\big)$ for all $j \in J \cup \{\infty\}$. Furthermore, assume that the sequence $\big(\mathbb{Y}_j, \mathsf{dist}_j, \mu_j, \bar{y}_j\big)_{j\in J}$ is uniformly $c$-doubling for some constant $c \in (0,\infty)$ and $\mathsf{supp}(\mu_\infty) = \mathbb{Y}_\infty$. If $\big(\mathbb{Y}_j, \mathsf{dist}_j, \mu_j, \bar{y}_j\big)_{j\in J}$ converges in pmG-sense to $(\mathbb{Y}_\infty, \mathsf{dist}_\infty, \mu_\infty, \bar{y}_\infty)$, then $\big(\mathbb{Y}_j, \mathsf{dist}_j, \mu_j, \bar{y}_j\big)_{j\in J}$ converges to $(\mathbb{Y}_\infty, \mathsf{dist}_\infty, \mu_\infty, \bar{y}_\infty)$ in pmGH-sense. ⌟

One of the prominent features of pmGH-convergence is that the set $\mathsf{CD}(K,N)$ of metric measure spaces with lower Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded by $N \in [1,\infty]$ from above is stable under this type of convergence. Here the notion of lower Ricci curvatures for metric measure spaces is due to K.-T. Sturm as well as J. Lott and C. Villani, see [Stu06a; Stu06b] and [LV09]. In a nutshell, the construction of $\mathsf{CD}(K,N)$-spaces is the geometric side of a coin with the Bakry-Émery (curvature-dimension) condition on the analytic side. Recall from the famous paper [BÉ85] that the Bakry-Émery condition is rooted in $\Gamma$-calculus in contrast to the ideas from optimal (mass) transport employed by Sturm and Lott-Villani. We highly recommend the papers [AGS14; AGS15] that are devoted to establish equivalence between these curvature-dimension conditions in a certain sense. Specifically, in [AGS14, Section 5] the subclass $\mathsf{RCD}(K,\infty)$ of $\mathsf{CD}(K,\infty)$-spaces that behave somewhat 'Riemannian' was introduced and the authors showed that these spaces satisfy the Bakry-Émerey condition with lower bound $K$. Afterwards, [AGS14] investigated when the converse implication holds. Furthermore, the $\mathsf{RCD}(K,\infty)$-property is stable under pmG-convergence according to [GMS15, Theorem 7.2].

Let us close this subsection with a few additional references. In the series of papers [CC97; CC00a; CC00b] J. Cheeger and T. H. Colding provide a classical extensive study of sequences of Riemannian manifolds converging. Moreover, the variation of eigenvalues of the Laplacians under the measured Gromov-Hausdorff convergence is the topic of [Fuk87]. In particular, K. Fukaya provides several explicit examples and counter examples for his results which is much appreciated.

## Convergence of spectral structures

We briefly present a central result of the famous paper [KS03] by K. Kuwae and T. Shioya. It states that if one has a sequence of in some sense converging Hilbert spaces and self-adjoint nonnegative definite linear operators on these Hilbert spaces, then several different convergences of objects associated to the self-adjoint operators are equivalent. Thus, each one of the equivalent convergence statements can be used to define the so-called *convergence of spectral structures*.

**Definition 4.1.9** (spectral structures). Consider a Hilbert space $H$. A quintuple $\Sigma = (\Lambda, \mathcal{E}, E, T, R)$ is called a *spectral structure on* $H$ if $(\Lambda, D(\Lambda))$ is a self-adjoint nonnegative definite linear operator on $H$ that has spectral measure $E$ and generates the densely defined closed bilinear form $(\mathcal{E}, D(\mathcal{E}))$ as well as the SCCS $T = (T_t)_{t\in[0,\infty)}$ and the SCCR $R = (R_a)_{a\in\rho(\Lambda)}$. ⌟

[3] The difference are the asymptotic approximation mappings denoted by $f_j^{R,\varepsilon}$ in [GMS15].

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From now on we assume a net $(H^j)_{j\in J}$ of Hilbert spaces convergent to $H$ in the sense of [KS03, Section 2.2] as well as special structures $\Sigma^j = (\Lambda^j, \mathcal{E}^j, E^j, T^j, R^j)$ on $H^j$ for any $j \in J$ and $\Sigma = (\Lambda, \mathcal{E}, E, T, R)$ on $H$. Alternatively, the convergent net of Hilbert spaces can be understood in terms of asymptotic relations of Hilbert spaces, see the appendix Section B.2.

**Definition 4.1.10** (strong graph limit). The *strong graph limit* $\Gamma[\Lambda^i]$ *of* $(\Lambda^j)_{j\in J}$ is defined as the set of of pairs $(u, v) \in H \times H$ such that there is a net $(u^j)_{j\in J}$ with both $u^j \to u$ and $\Lambda^j u^j \to v^j$ strongly. This is [KS03, Definition 2.7]. ⌟

**Theorem 4.1.11** (various equivalent characterisations of spectral convergence). The following statements are all equivalent and therefore all of them can equally adduced in order to define the convergence $\Sigma^j \to \Sigma$:

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(i) $R^j_a \to R_a$ strongly for some $a \in \rho(\Lambda)$,

(ii) $T^j_t \to T_t$ strongly for some $t \in (0, \infty)$,

(iii) $E^j((x, y]) \to E((x, y])$ strongly for any $x, y \in \mathbb{R}$ with $x < y$,

(iv) $(E^j u^j\,, v^j)_{H^j} \to (Eu\,, v)_H$ vaguely for any nets $(u^j)_{j\in J}$, $(v^j)_{j\in J}$, and any $u, v \in H$ such that $H^j \ni u^j \to u \in H$ strongly and $H^j \ni u^j \to u \in H$ weakly

(v) $\mathcal{E}^j \to \mathcal{E}$ in the sense of Mosco convergence, see Definition B.3.2 in the appendix. ⌟

In case of convergence of spectral structures, one also has the following asymptotical statement for the spectra, which is [KS03, Proposition 2.5].

**Theorem 4.1.12** (asymptotical behaviour of spectra under spectral convergence). If one has that $\Sigma^j \to \Sigma$, then it holds $\sigma(\Lambda) \subseteq \lim_{j\in J} \sigma(\Lambda^j)$ in the sense that for all $s \in \sigma(\Lambda)$ there are $s^j \in \sigma(\Lambda^j)$ such that $s = \lim_{j\in J} s^j$. ⌟

### 4.1.2 ▪ Introduction to Stratifolds

As said in the introduction to this chapter, the concept of stratifolds is due to M. Kreck. His book [Kre10] is the holy scripture of stratifold theory and our main reference on the topic. However, the primal ideas of local detectability and subsequently differential spaces originally are formulated in [Sik71].

**Definition 4.1.13** (local detectability). Let $E$ be a topological space and $\mathbf{C}$ a set of functions with domain in $E$. A real-valued function $f$ with $\mathrm{dom}(f) \subseteq E$ is said to be *locally* $\mathbf{C}$*-detectable* if for every $x \in \mathrm{dom}(f)$ there exists a open neighbourhood $U$ of $x$ in $\mathrm{dom}(f)$ and a $g \in \mathbf{C}$ such that $f\mathbb{1}_U = g\mathbb{1}_U$. For given $V \subseteq E$ we denote by $\mathbf{C}(V)$ the set of all locally $\mathbf{C}$-detectable functions with domain $V$. We call $\mathbf{C}$ *locally detectable* if $\mathbf{C} = \mathbf{C}(E)$. ⌟

**Definition 4.1.14** (differential space). Consider a pair $(E, \mathbf{C})$ consisting of a topological space $E$ and a locally detectable subalgebra $\mathbf{C} \subseteq C^0(E; \mathbb{R})$ that is closed with respect to 'composition with smooth functions' by which we mean that for all $k \in \mathbb{N}_+$, $f_1, \dots, f_k \in \mathbf{C}$ and $g \in C^\infty(\mathbb{R}^k; \mathbb{R})$ the composition $E \to \mathbb{R}$, $x \mapsto g(f_1(x), \dots, f_k(x))$ belongs to $\mathbf{C}$ as well. Then, one calls $\mathbf{C}$ the *differential structure of* $E$ and the pair $(E, \mathbf{C})$ is called a *differential space.*

For two differential spaces $(E_1, \mathbf{C}(E_1))$ and $(E_2, \mathbf{C}(E_2))$ a continuous mapping $\varphi$ from $E_1$ to $E_2$ is a *morphism of differential spaces* if $f \circ \varphi \in \mathbf{C}(E_1)$ for all $f \in \mathbf{C}(E_2)$. Consequently, a homeomorphism $i$ from $E_1$ to $E_2$ is called an *isomorphism (of differential spaces)* if for all $f_j \in \mathbf{C}(E_j)$, $j \in \{1, 2\}$, both $f_2 \circ i \in \mathbf{C}(E_1)$ and $f_1 \circ i^{-1} \in \mathbf{C}(E_2)$ hold. ⌟

Compare the definition of local detectability and differential spaces to [Sik71, Section 1]. We feel that a few examples are necessary. In particular, Example 4.1.16 is of major significance for our considerations, since it explains how to make a point into a 'singularity'.

**Example 4.1.15** (compatibility with manifold formalism). As explained in [Kre10, Section 1.3], we can understand a $\mathscr{y}$-dimensional smooth manifold $\mathbb{Y}$ equivalently as a differential space $(\mathbb{Y}, \mathbf{C})$ that is locally isomorphic to $\mathbb{R}^{\mathscr{y}}$ in the sense that for each $y \in \mathbb{Y}$ there is an open neighbourhood $U$ of $y$, an open set $V \subseteq \mathbb{R}^{\mathscr{y}}$, and an isomorphism $(V, C^\infty(V)) \to (U, \mathbf{C}(U))$ between differential spaces. Naturally, for every open subset $Y \subseteq \mathbb{Y}$ the pair $(Y, \mathbf{C}(Y))$ is a smooth manifold in this sense. ⌟

**Example 4.1.16** (smooth functions locally constant at north pole). Consider the standard round sphere $\mathbb{S}^{d-1}$ supplemented with the set $\mathbf{C} \subseteq C^\infty(\mathbb{S}^{d-1})$ of functions such that any $f \in \mathbf{C}$ is locally constant near the 'north pole'[4] $N = (0, \dots, 0, 1)^\top$ meaning that there is a neighbourhood $U(f)$ of $N$ such that $f$ is constant on $U(f)$. Then, $(\mathbb{S}^{d-1}, \mathbf{C})$ is a differential space, but not a smooth manifold. Indeed, suppose an open neighbourhood $U$ of $N$, a homeomorphism $i\colon V \to U$ for some $V \subseteq \mathbb{R}^{d-1}$ open, and $f \in C^\infty(V)$ not locally constant at $i^{-1}(N)$, then $f \circ i^{-1} \notin \mathbf{C}(U)$, thus $i$ is not an isomorphism between differential spaces. See [Kre10, Exercise 1.4.10]. ⌟

*Remark* 4.1.17 (restriction of differential spaces). For a differential space $(E, \mathbf{C})$ and a subset $V \subseteq E$ the set $\mathbf{C}(V)$ in general does not coincide with the set of restrictions of $\mathbf{C}$-elements to $V$, compare [Kre10, Exercise 1.4.2]. However, if $V$ is closed and $E$ is a stratifold, see Definition 4.1.20 below, then $\mathbf{C}(V)$ coincides with the set of restrictions of $\mathbf{C}$-elements to $V$, see [Kre10, Proposition 2.4.5] and [Kre10, Exercise 1.4.9] for the instance of a closed submanifold $V$ of a manifold $E$. ⌟

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Recall the the algebraic construction of tangent spaces of manifolds from part (i) of Example 1.2.7. As the construction can be employed for basically any set of continuous functions, we can use it for differential spaces. Similarly, the logic for defining the differential of a smooth function carries over to the world of differential spaces, so their name really is justified.

**Definition 4.1.18** (tangent spaces of differential spaces). Consider a differential space $(\mathbb{Y}, \mathbf{C})$, functions $f, g \in \mathbf{C}$, and a point $y \in \mathbb{Y}$. We view $f$ and $g$ as equivalent if there is an open neighbourhood of $y$ on which both functions coincide; the equivalence class of $f$ is called *germ of $f$ at $y$* and is denoted by $[f]_y$. The set of all germs at $y$ is denoted by $\mathbf{C}_y$. We define the tangent space $\mathsf{T}_y\mathbb{Y}$ as the vector space of derivations at $y$ that are linear functions $v\colon \mathbf{C}_y \to \mathbb{R}$ obeying the Leibniz rule $v\big([f]_y \cdot [g]_y\big) = v\big([f]_y\big) \cdot g(y) + f(y) \cdot v\big([g]_y\big)$ for all $f, g \in \mathbf{C}$ and $y \in \mathbb{Y}$. ⌟

**Definition 4.1.19** (differentials of morphism between differential spaces). Consider two differential spaces $(\mathbb{Y}_1, \mathbf{C}(\mathbb{Y}_1))$ and $(\mathbb{Y}_2, \mathbf{C}(\mathbb{Y}_2))$. A function $f$ from $\mathbb{Y}_1$ to $\mathbb{Y}_2$ is a *morphism* if $g \circ f \in \mathbf{C}(\mathbb{Y}_1)$ holds for all $g \in \mathbf{C}(\mathbb{Y}_2)$. For any point $y \in \mathbb{Y}_1$ the *differential $d_x f$ of a morphism $f$ at $y$* is defined as the mapping

$$\begin{aligned} d_x f\colon\ & \mathsf{T}_x\mathbb{Y}_1 \to \mathsf{T}_{f(x)}\mathbb{Y}_2, \\ & v \longmapsto \Big( [\mathbf{C}(\mathbb{Y}_2)]_{f(x)} \to \mathbb{R},\ [g]_{f(x)} \mapsto v([g \circ f]_x) \Big). \end{aligned}$$

⌟

If the differential spaces $\mathbb{Y}_1$ and $\mathbb{Y}_2$ in Definition 4.1.19 are smooth manifolds, then one obtains the same notion of differentials as in Definition 1.2.10.

**Definition 4.1.20** (stratifolds). Consider a differential space $(\mathbb{Y}, \mathbf{C})$ decomposed into the disjoint union $\mathbb{Y} = \bigcup_{k \in \mathbb{N}} \mathsf{strat}_k(\mathbb{Y})$ where $\mathsf{strat}_k(\mathbb{Y})$ is the so-called *$k$th stratum* defined as

$$\mathsf{strat}_k(\mathbb{Y}) := \big\{ y \in \mathbb{Y} \;\big|\; \dim\big(\mathsf{T}_y\mathbb{Y}\big) = k \big\} \qquad \text{for all } k \in \mathbb{N}.$$

In particular, this implies that the tangent space at any point $y \in \mathbb{Y}$ has finite dimension. We say that $(\mathbb{Y}, \mathbf{C})$ is a *$\mathscr{y}$-dimensional stratifold* if it has the following properties:

(Strat0) The space $\mathbb{Y}$ is a locally compact, second countable Hausdorff space and the so-called *$k$-skeleta* $\bigcup_{\tilde{k} \leq k} \mathsf{strat}_{\tilde{k}}(\mathbb{Y})$ are closed (topological) subspaces.

(Strat1) The number $\mathscr{y} \in \mathbb{N}$ is minimal such that all tangent spaces have dimension at most $\mathscr{y}$, i.e. $\dim(\mathsf{T}_y\mathbb{Y}) \leq \mathscr{y}$ for all points $y \in \mathbb{Y}$.

(Strat2) Any restriction of $\mathbf{C}$ to a stratum $\mathsf{strat}_k(\mathbb{Y})$ yields a smooth manifold

$$(\mathsf{strat}_k(\mathbb{Y}), C^\infty(\mathsf{strat}_k(\mathbb{Y}))) := (\mathsf{strat}_k(\mathbb{Y}), \mathbf{C}(\mathsf{strat}_k(\mathbb{Y}))).$$

(Strat3) For all $y \in \mathsf{strat}_k(\mathbb{Y})$ the mapping

$$\mathbf{C}_y \to [C^\infty(\mathsf{strat}_k(\mathbb{Y}))]_y,\ [f]_y \longmapsto \big[f|_{\mathsf{strat}_k(\mathbb{Y})}\big]_y$$

is a vector space isomorphism.

[4]Here the sphere is viewed as a subset of $\mathbb{R}^d$ just for sake of illustration, as the specific choice of $N$ doesn't matter.

(Strat4) For every point $y \in \mathbb{Y}$ and open neighbourhood $U \subseteq \mathbb{Y}$ of $y$ there is a so-called *bump function* $\eta \in \mathbf{C}$ such that $\eta(y) \neq 0$ and $\operatorname{supp}(\eta) \subseteq U$.

$\lrcorner$

Before giving examples, we want to discuss some of the defining properties as well as immediate properties. First and foremost, stratifolds in this thesis are finite dimensional, even though the definition could be extended to allow infinite dimensions as well. Furthermore, by property (Strat1) all strata $\operatorname{strat}_k(\mathbb{Y})$ are empty for $k \in \{\ell \in \mathbb{N} \mid \ell > \mathscr{y}\}$. This doesn't conflict with (Strat2) as the definition of manifolds doesn't exclude the empty space.[5] The stratum $\operatorname{strat}_{\mathscr{y}}(\mathbb{Y})$ is referred to as the *top stratum*. It coincides with the complement of the $(\mathscr{y}-1)$-skeleton in $\mathbb{Y}$, thus the skeleton might be thought as the *singular set of* $\mathbb{Y}$ in our applications. Moreover, the mapping in property (Strat3) is always surjective, so injectivity is the new piece of information formalising the idea that one could locally extend a smooth function on a stratum to an element of $\mathbf{C}$ somewhat uniquely. However, this is to be taken with a grain of salt as there might be smooth functions on strata that do not extend to the whole stratifold. Furthermore, the property implies that the $k$th stratum is in fact a $k$-dimensional manifold. The property (Strat4) is the essential ingredient for constructing a partition of unity of smooth functions subordinate to a given open cover, see [Kre10, Proposition 2.2.3]. Prior and form now on in the context of stratifolds the term 'smooth function' refers to a morphism in the sense of Definition 4.1.19 from a stratifold to a smooth manifold.



**Example 4.1.21.** Every $\mathscr{d}$-dimensional manifold is a stratifold that coincides with its top stratum, or in other words its $(\mathscr{y}-1)$-skeleton is empty. A characterisation of algebraic varieties and (topological) stratified spaces in terms of certain stratifolds[6] is possible, however not obvious. Therefore we refer to the thesis [Gri03] of A. Grinberg, as so does M. Kreck. $\lrcorner$

**Example 4.1.22** (open cone over a manifold)**.** Quite accessible nonmanifold examples are cones over manifolds: Consider some $\mathscr{b}$-dimensional compact[7] manifold $\mathbb{B}$ as the 'base'. Then, the open cone over (the topological space) $\mathbb{B}$ is defined as the quotient

$$^{\circ}\mathrm{C}\mathbb{B} := (\mathbb{B} \times (0,1]) \,/\, (\mathbb{B} \times \{1\}).$$

See Figure 4.3. This cone is endowed with the subalgebra $\mathbf{C} \subseteq C^0({}^{\circ}\mathrm{C}\mathbb{B})$ of continuous functions that are

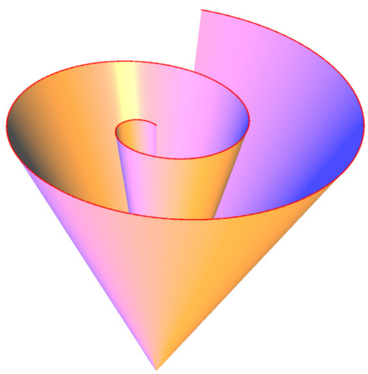

Figure 4.3: Open cone 'over' the red spiral

both constant on an open neighbourhood of the vertex point $(\mathbb{B} \times \{1\})/(\mathbb{B} \times \{1\})$ and whose restriction to the smooth manifold $\mathbb{B} \times (0,1)/(\mathbb{B} \times \{1\})$ is smooth. The pair $({}^{\circ}\mathrm{C}\mathbb{B}, \mathbf{C})$ is a differential space indeed. The topological conditions (Strat0) are clear and there are exactly two nonempty strata, namely the singleton consisting of the vertex and the lateral area $\mathbb{B} \times (0,1)/(\mathbb{B} \times \{1\})$. See [Kre10, Example 2.3.1] for more details, but we also apply the same techniques to suspensions of manifolds in Example 4.1.23 below.

In a similar way, one can choose an algebra of functions that renders one-point compactifications of a noncompact smooth manifold into a stratifold: It consists of continuous functions on the one-point compactification that are both constant on some open neighbourhood of the point at infinity and smooth when restricted to the underlying manifold. $\lrcorner$

[5] But the dimension of the empty manifold is not well-defined, rather it has arbitrary dimension.
[6] Cornered parametrised stratifolds to be precise.
[7] I. e. compact as a topological space.

**Example 4.1.23** (suspension of manifolds). Intuitively, a (topological) suspension can be thought as two (closed) (topological) cones with different vertices over the same base that are glued together at their shared base. More formally, consider some $\mathscr{b}$-dimensional compact manifold $\mathbb{B}$, then the suspension of $\mathbb{B}$ is defined as the quotient space

$$\begin{aligned} \mathsf{Sus}\mathbb{B} &= (\mathbb{B} \times [-1,1]) \,/\, \sim \\ &:= (\mathbb{B} \times [-1,1]) \,/\, \{((b_1,t_1),(b_2,t_2)) \in (\mathbb{B} \times [-1,1])^2 \mid t_1 = t_2 \in \{\pm 1\}\} \end{aligned}$$

with vertices denoted as $p_- := \mathbb{B} \times \{-1\}/\sim$ and $p_+ := \mathbb{B} \times \{1\}/\sim$. See Figure 4.4. It is endowed with

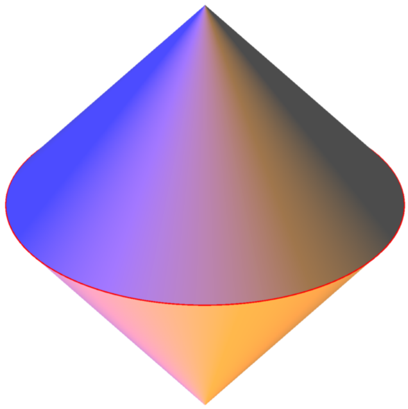

Figure 4.4: Suspension of the red circle

the subalgebra $\mathbf{C} \subseteq C^0(\mathsf{Sus}\mathbb{B})$ of continuous functions $f$ that are both constant on some open neighbourhoods $U_-$ and $U_+$ of the vertex points $p_-$ and $p_+$ respectively and the restriction of $f$ to either of the smooth manifolds $S_1 := \mathbb{B} \times \{0\}/\sim$ or $S_2 := \mathbb{B} \times ((-1,0) \cup (0,1))/\sim$ yields a smooth mapping. The space $\mathsf{Sus}\mathbb{B}$ is a second countable Hausdorff space and locally compact as $\mathbb{B}$ is compact. The stratification formally is determined as follows: For $y \in \mathsf{Sus}\mathbb{B} \setminus \{p_\pm\}$ there are two possibilities, either $y$ lies in the 'sutura' $S_1$ or in the one of the two parts of the 'cap' $S_2$. As both sets are smooth manifolds, the set $\mathbf{C}_y$ of $\mathbf{C}$-germs at $y$ coincides with the set of germs of smooth functions on $S_j$ if $y \in S_j$, $j \in \{1,2\}$. Hence, we have that

$$\mathsf{T}_y(S_j) = \begin{cases} \mathsf{T}_y(\mathbb{B} \times \{0\}) & \text{if } j = 1 \\ \mathsf{T}_y(\mathbb{B} \times ((-1,0) \cup (0,1))) & \text{if } j = 2 \end{cases}$$

$$\text{and} \qquad \dim \mathsf{T}_y(S_j) = \begin{cases} \mathscr{b} & \text{if } j = 1 \\ \mathscr{b} + 1 & \text{if } j = 2 \end{cases}.$$

The tangent space at $y$ for $y \in \{p_\pm\}$ contains only the zero derivation and thus is 0-dimensional. In conclusion, the nonempty strata are exactly

$$\begin{aligned} &\mathsf{Sus}_{\mathscr{b}}\mathbb{B} := \mathsf{strat}_{\mathscr{b}}(\mathsf{Sus}\mathbb{B}) = S_1, \quad \mathsf{Sus}_{\mathscr{b}+1}\mathbb{B} := \mathsf{strat}_{\mathscr{b}+1}(\mathsf{Sus}\mathbb{B}) = S_2, \\ \text{and} \quad &\mathsf{Sus}_0\mathbb{B} := \mathsf{strat}_0(\mathsf{Sus}\mathbb{B}) = \{p_\pm\}. \end{aligned}$$

Except for (Strat4) the remaining conditions are clearly satisfied. For all $y \in \mathsf{Sus}_{\mathscr{b}+1}\mathbb{B}$ there is a bump function at $y$ since there is a corresponding bump function in $\mathbb{B} \times ((-1,0) \cup (0,1))$. Let $y \in \mathsf{Sus}_0\mathbb{B}$ be a vertex point, without lost of generality $y = p_+$. An open neighbourhood $U$ of $y$ necessarily contains an open neighbourhood of $y$ of the form $(\mathbb{B} \times (\delta, 1])/\sim$ for some $\delta \in (0,1)$. Let $g\colon (0,1] \to [0,\infty)$ be a smooth function such that $g = 1$ in a neighbourhood of 1 and $g = 0$ on $(0,\delta]$. Then, the assignment $(b,t)/\sim \longmapsto g(t)$ is a bump function for $y$. Similarly for the sutura: For given $y \in \mathsf{Sus}_{\mathscr{b}}\mathbb{B}$ any open neighbourhood of $y$ contains an open neighbourhood of $y$ of the form $(U \times (-\delta,\delta))/\sim$ for some $U \subseteq \mathbb{B}$ open and $\delta \in (0,1)$. Take a bump function $g$ for the point corresponding to $y$ in $\mathbb{B}$ with domain in $U$ and a bump function $h$ for 0 in $(-\delta,\delta)$. Then, the assignment $(b,t)/\sim \longmapsto g(b)h(t)$ yields a bump function for $y$. Summarising, the suspension $\mathsf{Sus}\mathbb{B}$ is a $(\mathscr{b}+1)$-dimensional stratifold. ⌟

Apart from [Kre10], A. Grinberg provides even more useful theory on stratifolds in her PhD thesis. E. g. she defines stratifold bundles where the standard fibre is allowed to be a stratifold, see [Gri03, Section 1.2] and Definition 4.2.14 later on, and she discusses resolutions of singularities of stratifolds in [Gri03, Chapter 2]. from the introduction to this section

As we now know what structure a stratifold has to offer, we came back to the more general concept of Alexandrov spaces for comparison. Recall that we mentioned Alexandrov spaces in the introduction to this section, as they appear as Gromov-Hausdorff limits of nice Riemannian manifolds. An *Alexandrov space* is a complete length space with curvature bounded from below and finite Hausdorff dimension. A length space is a metric space where the distance function is given by the infimum of the length of continuous curves connecting two given points. Curvature bounds in length spaces are defined in terms of comparison triangles. One the one hand, an Alexandrov space possesses a topological stratification into topological manifolds, see [BGP92, Remark 6.9]. One the other hand, there are some useful results on the respective sets of regular and singular points. The set of singular points of a $y$-dimensional Alexandrov space $\mathbb{Y}$ might be dense, however it has at most dimension $(y-1)$, compare to [OS94, Theorem A, Section 2]. The set of regular points is locally path-connected, has a differentiable structure, and carries a continuous Riemannian metric that coincides with the metric on $\mathbb{Y}$ measuring the angle between shortest geodesics, see [OS94, Sections 5 and 6]. We encountered similar but stronger properties in the framework of stratifolds: By design, the stratification consists even of smooth manifolds and in particular the $(y-1)$-skeleton subsumes all singular points. Aside from the references on Alexandrov spaces that we already mentioned, the interested reader is pointed to the famous works [Ale51; Ale57; Per93] by A. D. Alexandrov and G. Y. Perelman.

*Remark* 4.1.24 (strativarifolds). We think that a synthesis of the concepts of stratifolds and varifolds promises to be useful for solving variational problems with singularities. We recommend more research in this direction and propose the name *strativarifolds*. However, that is by no means the point of this thesis, thus we defer our ideas to the appendix, see Appendix C. Though, the configuration stratifolds $(\mathsf{Q}^\varepsilon\mathbb{X})_{\varepsilon\in[0,1]}$ constructed in Section 4.2.2 later on can be viewed as examples. ⌟

## 4.2 ▪ Geometric convergence of configuration stratifold

After the preparations of the previous sections, we have the vocabulary to formulate our new technique. One of the central ideas is that we declare certain 'directions' in the unit tangent bundle to be singular. Therefore, we leave the category of smooth manifolds in favour of stratifolds. This is a critical step, that happens before we attempt any scaling at all.

We not only manipulate the differential structure of the standard fibre, i. e. the sphere. But also, we declare some singularities in the position manifold. We do so mindful of the fact that the position manifold is in general not parallelisable: The choice of 'singular directions' is formalised via the choice of a measurable section $\oslash\colon \mathbb{X} \to \mathsf{U}\mathbb{X}$. We call $\oslash$ *compass (on* $\mathbb{X}$*)*, as $\oslash(x)$ shall tell us where 'north' is in the fibre $\mathsf{U}_x\mathbb{X}$ for $x \in \mathbb{X}$. The set of 'singular directions' in the fibre $\mathsf{U}_x\mathbb{X}$ is given by the set $\{\pm\oslash(x)\}$ of 'north' and 'south pole'. If $\mathbb{X}$ is not assumed to be parallelisable, the compass can not be chosen as a smooth section. In fact, we can't even ensure continuity. The best we could do is choosing the compass as a locally Lipschitzian section. Then, it fails to be differentiable on a set of measure zero and the complement $\mathsf{dom}(\partial\oslash)$, i. e. the set on which the compass is differentiable, is dense. Recall Rademacher's Theorem, see [Fed96, Theorem 3.1.6] and [AFL05, Theorem 5.7]. Additionally, we require that the set $\mathbb{X} \setminus \mathsf{dom}(\partial\oslash)$ of singular points for $\oslash$ consists of finitely many isolated points. More could be done and we recommend more research in that regard, but we decide to keep things simple here. In lieu of the entire position manifold $\mathbb{X}$, we render $\mathbb{X}$ into a stratifold with exactly two nonempty strata $\mathsf{strat}_0(\mathbb{X}) = \mathsf{dom}(\partial\oslash)^\mathsf{c}$ and $\mathsf{strat}_d(\mathbb{X}) = \mathsf{dom}(\partial\oslash)$. For the most part, the top stratum $\mathsf{strat}_d(\mathbb{X})$ serves as position manifold rather than the whole space $\mathbb{X}$. The unit tangent bundle $\mathsf{U}\mathbb{X}$ as configuration manifold is replaced by a stratifold $\mathsf{Q}^1\mathbb{X}$ that is the total space of a fibre bundle-like construction over some base stratifold and with a stratifold as standard fibre. As a consequence, we are almost in the situation of a parallelisable position manifold: For given $x \in \mathbb{X}$ we can add unit vectors $e_1(x), \dots, e_{d-1}(x)$ to the northwards pointing unit vector $e_d(x) := \oslash(x)$ such that $(e_j(x))_{j=1}^d$ is an orthonormal basis of $\mathsf{T}_x\mathbb{X}$ with respect to the underlying Riemannian metric on $\mathbb{X}$. This yields a measurable orthonormal frame $(e_j)_{j=1}^d$ which we call *compass frame*. The compass frame can be chosen in such a way that it restricts to a smooth frame on $\mathsf{strat}_d(\mathbb{X})$ just by keeping track of the indices between fibres.

After all this is done, we actually start scaling. We create a sequence of metric measure spaces $(\mathsf{Q}^\varepsilon\mathbb{X})_\varepsilon$ by endowing $\mathsf{Q}^1\mathbb{X}$ with suitable weight functions $(\rho^\varepsilon)_{\varepsilon\in(0,1]}$ and adapting the distance metric accordingly.

Assume that the weight functions are chosen in such a way that $(\mathsf{Q}^\varepsilon\mathbb{X})_{\varepsilon\in(0,1]}$ converges to a stratifold in pmGH-sense as $\varepsilon \downarrow 0$. Intuitively, the collapsed space $\mathsf{Q}^0\mathbb{X}$ restricts to a smooth fibre bundle over $\mathsf{strat}_d(\mathbb{X})$ with fibres

$$\mathsf{Q}^0_x\mathbb{X} = \left\{ u = \sum_{j=1}^{d} u_j e_j(x) \in \mathsf{U}_x\mathbb{X} \;\middle|\; \sum_{k=1}^{d-1} u_k^2 = 1 \wedge u_d = 0 \right\} \cup \{\pm e_d\}.$$

Now, the spherical velocity Langevin equation can be reformulated to an equation on $\mathsf{Q}^1\mathbb{X}$ instead of $\mathsf{U}\mathbb{X}$. Then, by introducing the weight function $\rho^\varepsilon$ we get a weighted spherical velocity Langevin equation.

### 4.2.1 ▪ Collaps of spheres in different senses

#### Gromov-Hausdorff-type convergence



Consider the sphere $\mathbb{S}^{d-1}$, $d \in \mathbb{N} \setminus \{0,1,2\}$, in polar coordinates, see Notation A.2.1. We construct an explicit family of probability densities $(\bar{\rho}^\varepsilon_{\mathbb{S}})_{\varepsilon\in(0,1)}$ on $\mathbb{S}^{d-1}$ such that the sphere experiences convergence to $\mathbb{S}^{d-2}$ as $\varepsilon \downarrow 0$ in a suitable sense. To this end, we look at the sequence $(\beta_n)_{n\in\mathbb{N}}$ given by

$$\beta_n\colon\ (0,\pi) \to \mathbb{R},\ \theta \longmapsto \frac{1}{c_\beta(n)} \sin(\theta)^n := \left( \int_{(0,\pi)} \sin^n \,\mathrm{d}\lambda \right)^{-1} \cdot \sin(\theta)^n.$$

By means of the following two lemmas we show that the sequence of measures $(\beta_n\lambda)_{n\in\mathbb{N}}$ on $((0,\pi),\mathfrak{B}((0,\pi)))$ converges weakly to the Dirac measure $\delta_{\pi/2}$. I don't know a reference for the next lemma, thus a proof is given. Whereas, the lemma after next is more number theoretic in nature and proven by straight forward calculation.

**Lemma 4.2.1** (Dirac sequences on finite intervals). Assume that a sequence $(\kappa_n)_{n\in\mathbb{N}}$ in $L^1((-1,1);\lambda)$ is given such that $\kappa_n \geq 0$ $\lambda$-almost everywhere, $\int_{(-1,1)} \kappa_n \,\mathrm{d}\lambda = 1$ and we have the following $\lambda$-almost everywhere uniform convergence: For all $\delta \in (0,1)$ the '$(1-\delta)$-outskirts' sequence $\left(\kappa_n \cdot \mathbb{1}_{(-1,-\delta)\cup(\delta,1)}\right)_{n\in\mathbb{N}}$ uniformly converges to 0 as $n \to \infty$ $\lambda$-almost everywhere, i. e. for all $\bar{\varepsilon} \in (0,\infty)$ there is a $n_{\bar{\varepsilon}} \in \mathbb{N}$ such that

$$\left\| \kappa_n \cdot \mathbb{1}_{(-1,-\delta)\cup(\delta,1)} \right\|_{L^\infty} < \bar{\varepsilon} \qquad \text{for all } n \geq n_{\bar{\varepsilon}}. \tag{4.2}$$

Then, for all $f \in C^0(\mathbb{R})$ and compact intervals $[a,b] \subseteq \mathbb{R}$ we have

$$\int_{[-1,1]} f(\mathrm{Id} - y)\kappa_n(y)\ \lambda(\mathrm{d}y) \longrightarrow f \qquad \text{uniformly on } [a,b] \text{ as } n \to \infty.$$

*Proof.* Let $f \in C^0(\mathbb{R})$ and $[a,b] \subseteq \mathbb{R}$ as well as $\bar{\varepsilon} \in (0,\infty)$. Without loss of generality, assume that $f|_{[a,b]}$ does not equal the zero function. Note that for all $x \in [a,b]$ holds $f(x) = \int_{[-1,1]} f(x)\kappa_n(y)\,\lambda(\mathrm{d}y)$ for all $n \in \mathbb{N}$ by the assumptions on $(\kappa_n)_n$. Thus, we get the estimate

$$\begin{aligned}
&\left| \int_{[-1,1]} f(x-y)\kappa_n(y)\ \lambda(\mathrm{d}y) - f(x) \right| \leq \int_{[-1,1]} |f(x-y) - f(x)|\kappa_n(y)\ \lambda(\mathrm{d}y) \\
&= \int_{[-1,-\delta)\cup(\delta,1]} |f(x-y) - f(x)|\kappa_n(y)\,\lambda(\mathrm{d}y) + \int_{[-\delta,\delta]} |f(x-y) - f(x)|\kappa_n(y)\,\lambda(\mathrm{d}y) \\
&=: I_1 + I_2.
\end{aligned}$$

Define the nonnegative constant $M := \|f \cdot \mathbb{1}_{[a-1,b+1]}\|_\infty$. Since $f\colon [a-1,b+1] \to \mathbb{R}$ is uniformly continuous, we can choose $\delta \in (0,1)$ such that for all $z, z' \in [a-1,b+1]$ with $|z-z'| < \delta$ holds $|f(z) - f(z')| < \bar{\varepsilon}/2$. By assumption, there is $N_{\bar{\varepsilon}} := n_{\bar{\varepsilon}/8M} \in \mathbb{N}$ such that the estimate (4.2) attains the form of $\|\kappa_n \cdot \mathbb{1}_{(-1,-\delta)\cup(\delta,1)}\|_{L^\infty} < \bar{\varepsilon}/8M$ for all $n \geq N_{\bar{\varepsilon}}$. With these preparations we estimate that

$$I_2 \leq \frac{\bar{\varepsilon}}{2} \int_{[-\delta,\delta]} \kappa_n \,\mathrm{d}\lambda \leq \frac{\bar{\varepsilon}}{2} \int_{[-1,1]} \kappa_n \,\mathrm{d}\lambda = \frac{\bar{\varepsilon}}{2}$$

and for all $n \geq N_{\bar{\varepsilon}}$ that

$$I_1 \leq 2M \cdot \frac{\bar{\varepsilon}}{8M} \int_{[-1,-\delta)\cup(\delta,1]} 1 \,\mathrm{d}\lambda \leq \frac{\bar{\varepsilon}}{4} \int_{[-1,1]} 1 \,\mathrm{d}\lambda = 2 \cdot \frac{\bar{\varepsilon}}{4} = \frac{\bar{\varepsilon}}{2}.$$

All together, we get

$$\left| \int_{[-1,1]} f(x-y)\kappa_n(y) \;\lambda(\mathrm{d}y) - f(x) \right| \leq I_1 + I_2 \leq 2 \cdot \frac{\bar{\varepsilon}}{2} = \bar{\varepsilon},$$

and thus the statement is proven. □

4

**Lemma 4.2.2** (Explicit form of the reciprocal normalisation constansts). Consider a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_0, a_1 \in \mathbb{R} \setminus \{0\}$ and such that $a_n = \frac{n-1}{n}\, a_{n-2}$ holds for all $n \in \mathbb{N} \setminus \{0,1\}$. Then, for all $k \in \mathbb{N}$ we have the explicit expression

$$a_{2k} = a_0 \prod_{j=1}^{k} \Big(1 - \frac{1}{2j}\Big) = a_0\, \pi^{-\frac{1}{2}} \cdot \frac{\Gamma\big(k+\frac{1}{2}\big)}{\Gamma(k+1)},$$

$$a_{2k+1} = a_1 \prod_{j=1}^{k} \Big(1 - \frac{1}{2j+1}\Big) = a_1\, \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(k+1)}{\big(k+\frac{1}{2}\big)\cdot\Gamma\big(k+\frac{1}{2}\big)} = a_1\, \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(k+1)}{\Gamma\big(k+\frac{3}{2}\big)}.$$

In particular, $c_\beta(n) = \frac{n-1}{n}\, c_\beta(n-2)$ holds for all $n \in \mathbb{N} \setminus \{0,1\}$ and since $c_\beta(0) = \pi$ as well as $c_\beta(1) = 2$, we have

$$c_\beta(2k) = \sqrt{\pi}\, \frac{\Gamma\big(k+\frac{1}{2}\big)}{\Gamma(k+1)} \quad \text{and} \quad c_\beta(2k+1) = \sqrt{\pi}\, \frac{\Gamma(k+1)}{\Gamma\big(k+\frac{3}{2}\big)} \qquad \text{for all } k \in \mathbb{N}.$$

*Proof.* Based on the well-known identity $\Gamma\big(k+\frac{1}{2}\big) = \pi^{\frac{1}{2}} 4^{-k} \frac{(2k)!}{k!}$ we easily calculate that

$$\begin{aligned}\frac{a_{2k}}{a_0\, \pi^{-\frac{1}{2}} \cdot \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}} &= \pi^{\frac{1}{2}} k! \cdot \pi^{-\frac{1}{2}} k! \cdot ((2k)!)^{-1} 4^k \prod_{j=1}^{k} \Big( \frac{(2j-1)}{2j} \cdot \frac{2j}{2j} \Big) \\ &= 2^{2k} (k!)^2 \left( \prod_{j=1}^{k} \frac{1}{2j} \right)^2 = 1\end{aligned}$$

and similarly

$$\begin{aligned}\frac{a_{2k+1}}{a_1\, \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(k+1)}{(k+\frac{1}{2})\cdot\Gamma(k+\frac{1}{2})}} &= \pi^{-\frac{1}{2}}\, \frac{2k+1}{k!} \cdot \Gamma\Big(k+\frac{1}{2}\Big) \cdot \prod_{j=1}^{k} \Big( \frac{2j}{2j+1} \cdot \frac{2j}{2j} \Big) \\ &= \frac{1}{2^{2k}(k!)^2} \cdot \frac{(2k+1)!}{(2k+1)!} \cdot \left( \prod_{j=1}^{k} 2j \right)^2 = 1.\end{aligned}$$

It remains to show the recursion formula for $(c_\beta(n))_{n\in\mathbb{N}}$. The recursion formula was found by P. Stilgenbauer, but not published; we present his argument here: For all $n \in \mathbb{N} \setminus \{0,1\}$ we calculate using integration by parts and Pythagoras' Theorem that

$$\begin{aligned}c_\beta(n) &= -\int_{(0,\pi)} \sin^{n-1} \cdot \cos' \;\mathrm{d}\lambda = (n-1) \int_{(0,\pi)} \sin^{n-2} \cdot \cos^2 \;\mathrm{d}\lambda \\ &= (n-1) \int_{(0,\pi)} \sin^{n-2} \;\mathrm{d}\lambda - (n-1) \int_{(0,\pi)} \sin^{n} \;\mathrm{d}\lambda.\end{aligned}$$

This implies $c_\beta(n) = \frac{n-1}{n}\, c_\beta(n-2)$ finishing the proof. □

**Lemma 4.2.3** (Convergence of spherical surface measure along highest order meridian). For all $\delta \in (0,1)$ the sequence

$$\left(\beta_n \cdot \mathbb{1}_{(0,\frac{\pi}{2}(1-\delta))\cup(\frac{\pi}{2}\cdot(1+\delta),\pi)}\right)_{n\in\mathbb{N}}$$

uniformly converges as $n \to \infty$ $\lambda$-almost everywhere. Thus, combining Lemma 4.2.1 and the Portemanteau Theorem we get that

$$\beta_n\lambda \longrightarrow \delta_{\frac{\pi}{2}} \quad \text{weakly in } ((0,\pi), \mathfrak{B}((0,\pi))) \text{ as } n \to \infty.$$

*Proof.* Let $\delta \in (0,1)$. We are going to distinguish the cases of even and odd indices. For symmetry reasons we restrict ourselves to the interval $(0, {}^{\pi}\!/\!_2(1-\delta))$. Furthermore, for sake of brev'ty we define $\nu(\delta) := \sin({}^{\pi}\!/\!_2(1-\delta)) \in (0,1)$. By monotonicity, we have $\left\|\beta_n \mathbb{1}_{(0,\pi/2(1-\delta))}\right\|_\infty = \beta_n({}^{\pi}\!/\!_2(1-\delta)) = c_\beta(n)^{-1}\nu(\delta)^n$ for all $n \in \mathbb{N}_+$. Consider the case of even indices, then one gets

$$\frac{\nu(\delta)^{2k}}{c_\beta(2k)} = \pi^{-\frac{1}{2}} \frac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2})} \cdot \nu(\delta)^{2k} = \frac{1}{\pi} \cdot \frac{(k!)^2}{(2k)!} \cdot \left(\frac{\nu(\delta)}{2}\right)^{2k}$$

for $k \in \mathbb{N}$ by Lemma 4.2.2. Thus, $\left\|\beta_{2k}\mathbb{1}_{(0,\frac{\pi}{2}(1-\delta))}\right\|_\infty \longrightarrow 0$ as $k \to \infty$. Similarly for the case of odd indices, we first note by Lemma 4.2.2 that

$$\begin{aligned}\frac{\nu(\delta)^{2k+1}}{c_\beta(2k+1)} &= \pi^{-\frac{1}{2}} \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+1)} \cdot \nu(\delta)^{2k+1} = \frac{(2k)!}{(k!)^2 4^k} \cdot \left(k+\frac{1}{2}\right) \cdot \nu(\delta)^{2k+1} \\ &= \frac{(2k+1)!}{(k!)^2} \cdot \left(\frac{\nu(\delta)}{2}\right)^{2k+1}\end{aligned}$$

for $k \in \mathbb{N}$. Using $\prod_{j=k+1}^{2k+1} j \in \mathcal{O}(k^k)$ as $k \to \infty$, we get that

$$\lim_{k\to\infty} c_\beta(2k+1)^{-1}\nu(\delta)^{2k+1} = \frac{1}{2} \lim_{k\to\infty} \frac{k^k}{k! \cdot 2^{2k}} \nu(\delta)^{2k+1}.$$

The latter limit equals 0, since by the quotient criterion the series $\sum_{k\in\mathbb{N}} \frac{k^k}{k! \cdot 2^{2k}}$ converges:

$$\frac{(k+1)^{k+1} \cdot k! \cdot 2^{2k}}{(k+1)! \cdot 2^{2(k+1)} \cdot k^k} = \frac{1}{4} \frac{(k+1)^k}{k^k} = \frac{1}{4}\left(1+\frac{1}{k}\right)^k \longrightarrow \frac{\mathrm{e}}{4} \approx 0.67957 < 1 \qquad \text{as } k \to \infty.$$

□

*Remark* 4.2.4 (Byproducts of Lemma 4.2.3). We already mentioned that P. Stilgenbauer also considered the scaling by means of $\beta_n$ – the idea naturally comes to mind looking at the spherical surface measure in polar coordinates. He pointed out that the weak convergence in Lemma 4.2.3 is equivalent to

$$\lim_{n\to\infty} \int_{(0,\pi/2)} \mathsf{Id}\, \mathrm{d}\beta_n\lambda = \lim_{n\to\infty} \int_{(0,\pi/2)} \theta \cdot \beta_n(\theta)\ \lambda(\mathrm{d}\theta) = \frac{\pi}{4}. \tag{4.3}$$

Indeed: By the Portemanteau Theorem weak convergence $\beta_n\lambda \to \delta_{\pi/2}$ is equivalent to $\lim_{n\to\infty}\int_{(0,\pi)} f\, \mathrm{d}\beta_n\lambda = \int_{(0,\pi)} f\, \mathrm{d}\delta_{\pi/2} = f({}^{\pi}\!/\!_2)$ for all Lipschitz continuous functions $f$ on $(0,\pi)$. With such a function $f$ with Lipschitz constant $\mathsf{Lip}(f)$ we estimate that

$$\begin{aligned}\left|\int_{(0,\pi)} f - f({}^{\pi}\!/\!_2)\, \mathrm{d}\beta_n\lambda\right| &\le \int_{(0,\pi)} |f - f({}^{\pi}\!/\!_2)|\, \mathrm{d}\beta_n\lambda \le \mathsf{Lip}(f) \int_{(0,\pi)} \left|\mathsf{Id} - \frac{\pi}{2}\right| \mathrm{d}\beta_n\lambda \\ &= 2 \cdot \mathsf{Lip}(f) \int_{(0,\pi/2)} \mathsf{Id} - \frac{\pi}{2}\, \mathrm{d}\beta_n\lambda.\end{aligned}$$

Since $\int_{(0,\pi/2)} \beta_n\, \mathrm{d}\lambda = {}^1\!/\!_2$, one can infer convergence of the right-hand side if (4.3) holds. Vice versa, the convergence in (4.3) even is necessary as one could choose $f = |\mathsf{Id} - {}^{\pi}\!/\!_2|$. By investigating recursive as well as explicit formulae for the sequence $\left(\int_{(0,\pi/2)} \mathsf{Id}\, \mathrm{d}\beta_n\lambda\right)_{n\in\mathbb{N}}$, one gets two assertions both equivalent to (4.3):

$$\sum_{k\in\mathbb{N}\ \{0\}} \frac{1}{(2k)^2} \cdot \frac{1}{c_\beta(2k)} = \frac{\pi}{8} \quad \text{or} \quad \sum_{k\in\mathbb{N}} \frac{1}{(2k+1)^2} \cdot \frac{1}{c_\beta(2k+1)} = \frac{\pi}{4}.$$

Thus, we proved these formulae for $\pi$ that P. Stilgenbauer conjectured. I couldn't find these series expansions in the literature. ⌟

Now, define the sequence of probability measures $(\nu_n)_{n\in\mathbb{N}}$ on $\mathbb{S}^{d-1}$ via

$$\nu_n(A) := \int_{\mathbb{S}^{d-2}} \int_{(0,\pi)} \mathbb{1}_A(u, \theta_{d-2})\ (\beta_{d-2+n}\lambda)(\mathrm{d}\theta_{d-2})\ \mathsf{S}_1^{d-2}(\mathrm{d}u)$$

for all $A \in \mathfrak{B}(\mathbb{S}^{d-1})$ and $n \in \mathbb{N}$. From Lemma 4.2.3 we can infer that

$$\nu_n \longrightarrow \nu_\infty \quad \text{weakly on } \big(\mathbb{S}^{d-1}, \mathfrak{B}(\mathbb{S}^{d-1})\big) \text{ as } n \to \infty$$

where the limit $\nu_\infty := (\iota_{\mathbb{S}^{d-1}})_* \mathsf{S}_1^{d-2}$ is the pushforward of the normalised surface measure on $\mathbb{S}^{d-2}$ with respect to the standard set-level embedding $\iota_{\mathbb{S}^{d-1}}\colon\ \mathbb{S}^{d-2} \hookrightarrow \mathbb{S}^{d-1}$. The measures $(\nu_n)_{n\in\mathbb{N}}$ all are absolutely continuous with respect to $\nu_0 = \mathsf{S}_1^{d-1}$. For all $n \in \mathbb{N}_+$ we take the Radon-Nikodým derivative $\bar{\rho}_{\mathbb{S}}^{1/n} := {}^{\nu_n}/_{\nu_0}$ as a weight function for the round metric $\mathsf{s}$ on $\mathbb{S}^{d-1}$. To be more precise, we choose $\bar{\rho}_{\mathbb{S}}^{1/n}$ as the representative of ${}^{\nu_n}/_{\nu_0}$ that is both invariant with respect to rotations around $e_d$-axis and 0 at the poles. This function is smooth everywhere and positive except at the poles.

Recall Example 4.1.3 for the varifold representation of manifolds. Denote by $\mathbb{V}_{\mathbb{S};1/n}$ the varifold corresponding to the measure space[8] $\big(\mathbb{S}^{d-1}, \mathfrak{B}(\mathbb{S}^{d-1}), \nu_n\big)$ for all $n \in \mathbb{N}_+$. Already, we have shown that $\mathbb{V}_{\mathbb{S};1/n}$ converges as $n \to \infty$ to a varifold $\mathbb{V}_{\mathbb{S};0}$ that corresponds to the measure space $\big(\mathbb{S}^{d-2}, \mathfrak{B}(\mathbb{S}^{d-2}), \nu_\infty\big)$. But this limit varifold is still $(d-1)$-dimensional: Due to the weak convergence $\nu_n \to \nu_\infty$ we get that for all $f \in C_c^0(\mathbb{G}(d-1, \mathbb{R}^d))$ holds

$$\begin{aligned}\big\langle f\,,\,\mathbb{V}_{\mathbb{S};1/n}\big\rangle &= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{G}(d-1,d)} f(y,S)\quad \delta_{\mathsf{T}_u\mathbb{S}^{d-1}}(\mathrm{d}S)\ \nu_n(\mathrm{d}u)\\ &\longrightarrow \int_{\mathbb{S}^{d-2}} \int_{\mathbb{G}(d-1,d)} f(y,S)\quad \delta_{\mathsf{T}_u\mathbb{S}^{d-1}}(\mathrm{d}S)\ \nu_\infty(\mathrm{d}u) = \big\langle f\,,\,\mathbb{V}_{\mathbb{S};0}\big\rangle \qquad \text{as } n\to\infty.\end{aligned}$$

Hence, the varifold $\mathbb{V}_{\mathbb{S};0}$ is $(d-1)$-dimensional in terms of Definition 4.1.1. This effect is a consequence of the varifold approach itself: Backed into the classical definition of varifolds is the fixed dimension of the 'tangent space information', hence the concept is not destined to capture dimension reduction. Even the generalisation of varifolds proposed in Remark 4.1.24 wouldn't fix that. Thus, we turn to matters of pmGH-convergence.

The oncoming corollary shows that the metric measure spaces naturally associated to the weighted manifolds $\big(\mathbb{S}^{d-1}, \bar{\rho}_{\mathbb{S}}^{1/n}\mathsf{s}\big)$ converge in pmGH-sense. To this end, we should determine how the induced distance metric looks like. In a nutshell, keep the set of geodesics on $\mathbb{S}^{d-1}$ as it is and only add a 'penalty' to the length measure. In the same vein as in Definition 1.2.23, for a continuously differentiable path $\gamma\colon\ I \to \mathbb{S}^{d-1}$ its velocity at time $t \in I$ in the weighted manifold $\big(\mathbb{S}^{d-1}, \bar{\rho}_{\mathbb{S}}^{1/n}\mathsf{s}\big)$ is defined as

$$\begin{aligned}\dot{\gamma}_{1/n}(t)\colon\ C^\infty(\mathbb{S}^{d-1}) \to \mathbb{R}\cup\{\pm\infty\},\ f \longmapsto \frac{1}{\bar{\rho}_{\mathbb{S}}^{1/n}(\gamma(t))} \cdot \frac{d}{ds}\left(\bar{\rho}_{\mathbb{S}}^{1/n} f\right)(\gamma(s))\Big|_{s=t}\\ = \langle f\,,\,\dot{\gamma}(t)\rangle + f(\gamma(t)) \cdot \frac{\big\langle \bar{\rho}_{\mathbb{S}}^{1/n}\,,\,\dot{\gamma}(t)\big\rangle}{\bar{\rho}_{\mathbb{S}}^{1/n}(\gamma(t))},\end{aligned}$$

where we read in the logarithmic derivative term the quotient as infinity times sign of the numerator whenever $\gamma(t)$ is one of the poles. If points $u_1, u_2 \in \mathbb{S}^{d-1}$ are given, then the new distance metric is

$$\begin{aligned}\mathrm{dist}_{\mathsf{s};1/n}(u_1,u_2) &= \inf_{\substack{\gamma \text{ geodesic in } (\mathbb{S}^{d-1},\mathsf{s}):\\ \gamma(0)=u_1,\ \gamma(1)=u_2}} \int_0^1 \sqrt{|\dot{\gamma}(t)|_{\mathsf{s}}^2 + \left|\frac{\big\langle \bar{\rho}_{\mathbb{S}}^{1/n}\,,\,\dot{\gamma}(t)\big\rangle}{\bar{\rho}_{\mathbb{S}}^{1/n}(\gamma(t))}\right|^2}\ \mathrm{d}t\\ &= \inf_{\gamma\ \ldots} \int_0^1 \sqrt{|\dot{\gamma}(t)|_{\mathsf{s}}^2 + \left({}^1/_{\bar{\rho}_{\mathbb{S}}^{1/n}(\gamma(t))}\right)^2 \left|\partial_{\dot{\gamma}(t)}\bar{\rho}_{\mathbb{S}}^{1/n}(\gamma(t))\right|^2}\ \mathrm{d}t.\end{aligned}$$

[8] That additionally carries 'tangential' information when we talk about varifolds.

This construction for sure belongs to mathematical folklore, I am just not aware of a specific reference at the time of writing. Intuitively, connecting geodesics are brutally penalised if they pass through the poles. But really they are penalised for moving any closer to the closest pole in case there is one. If the minimising geodesic $\gamma$ maps into the domain of $\theta^{d-2}$ and $\theta^{d-2}(\gamma)$ is constant, then we can think $\mathrm{dist}_{\mathsf{s};1/n}(u_1,u_2) = \int_0^1 \sqrt{|\dot\gamma(t)|_{\mathsf{s}}^2 + (h(\theta^{d-2}(\gamma)))^2}\,\mathrm{d}t$ for some smooth function $h\colon(0,\pi)\to[0,\infty)$ with $h(\pi/2)=0$ and $\lim_{\theta\to 0} h(\theta) = \lim_{\theta\to\pi} h(\theta) = \infty$. Note that the restriction of $\mathrm{dist}_{\mathsf{s};1/n}$ to $\mathbb{S}^{d-2}$ coincides with $\mathrm{dist}_{\mathbb{S}^{d-1}}$ restricted to $\mathbb{S}^{d-2}$. As we guessed and have seen by now, the poles $\pm e_d$ are to be interpreted as points at infinity in the limit $\varepsilon\downarrow 0$ from the point of view of the distance metrics. Therefore, we extend $\mathrm{dist}_{\mathbb{S}^{d-2}}$ to $\cdot\mathbb{S}\cdot := \mathbb{S}^{d-2}\cup\{\pm e_d\}$ by

$$\mathrm{dist}_{\cdot\mathbb{S}\cdot}(u_1,u_2) := \begin{cases} \mathrm{dist}_{\mathbb{S}^{d-2}}(u_1,u_2) & \text{if } u_1,u_2\in\mathbb{S}^{d-2} \\ 0 & \text{if } u_1=u_2\in\{\pm e_d\} \\ \infty & \text{if } u_1\neq u_2 \text{ and } u_j\in\{\pm e_d\} \text{ for some } j\in\{1,2\} \end{cases}.$$

Naturally, one extends the normalised surface measure $\mathsf{S}_1^{d-2}$ to the Borel-$\sigma$-algebra of $(\cdot\mathbb{S}\cdot,\mathrm{dist}_{\cdot\mathbb{S}\cdot})$ by charging the singletons containing the poles with measure 0 and denotes the extension by $\mathsf{S}_1^{d-2}$ again.

*Remark* 4.2.5 (uniform doubling of weighted spheres). Assume $d=3$. Then, one calculates that the weighted spheres are $(d-1)$-dimensional Riemannian manifolds with nonnegative Ricci curvature. As a consequence, they are $2^{d-1}$-doubling by the Bishop-Gromov Comparison Theorem[9], see [Cha93, Theorem 3.10]. The computation is postponed to Remark A.2.4.

Now, assume $d>3$. Unfortunately, computing the weighted Ricci curvatures either in local coordinates or in terms of [Bes87, Theorem 1.159 part d)] does not yield lower bounds independent of $n$. It is easy to show the curvature-dimension conditions $\left(\mathbb{S}^{d-1},\mathrm{dist}_{\mathsf{s};1/n},\nu_n\right)\in\mathrm{CD}(1,N)$ for all $n\in\mathbb{N}_+$ and $N\in\mathbb{N}\cup\{\infty\}$ with $N\geq(d-1)(n+1)$. See [BGL14, Section C.6]. Using [Stu06b, Corollary 2.4] on doubling of $\mathrm{CD}(K,N)$-spaces, we get doubling, but a strictly divergent sequence $\left(2^{(d-1)(n+1)}\right)_{n\in\mathbb{N}_+}$ of upper bounds on the respective doubling constants. So we are left to just conjecture uniform doubling in the next corollary. A possibility for still proving uniform doubling might be showing a so-called '$(\alpha,\beta)$-polynomial growth condition in $\delta$-asymptotical $\{\pm e_d\}$-direction' as introduced in [TD15] and used later on in [TSD15]. ⌟

**Corollary 4.2.6** (pmGH-convergence of spheres along the highest order meridian). Pick a point $\bar u\in\mathbb{S}^{d-2}$. If $d>3$, then asssume that $(\nu_n)_{n\in\mathbb{N}}$ are doubling with a shared doubling constant independent of $n$. Then, the sequence $\left(\mathbb{S}^{d-1},\mathrm{dist}_{\mathsf{s};1/n},\nu_n,\bar u\right)_{n\in\mathbb{N}_+}$ of pointed $(d-1)$-dimensional spheres converges to the lower dimensional space $(\cdot\mathbb{S}\cdot,\mathrm{dist}_{\cdot\mathbb{S}\cdot},\nu_\infty,\bar u)$ in pmGH-sense.

*Proof.* We show extrinsic pmG-convergence in the sense of Definition 4.1.7: Consider

$$\mathbb{Y} = \bigsqcup_{k\in\mathbb{N}_+}\mathbb{Y}_k \sqcup \mathbb{Y}_\infty := \bigsqcup_{k\in\mathbb{N}_+}(\mathbb{S}^{d-1}\cup\{0\}) \sqcup (\cdot\mathbb{S}\cdot\cup\{0\}).$$

Here, we treat the point $0\in\mathbb{Y}_{\bar k}$ as artificial point at infinity for $\bar k\in\mathbb{N}_+\cup\{\infty\}$: For $k\in\mathbb{N}_+$ and $y_{1,k},y_{2,k}\in\mathbb{Y}_k\setminus\{0\}$ we define $\mathrm{dist}_{\mathbb{Y}_k}(y_{1,k},y_{2,k}) := \mathrm{dist}_{\mathsf{s};1/k}(y_{1,k},y_{2,k})$ as well as $\mathrm{dist}_{\mathbb{Y}_k}(y_{1,k},0):=\infty$ and $\mathrm{dist}_{\mathbb{Y}_k}(0,0):=0$. Similarly, for all $y_{1,\infty},y_{2,\infty}\in\mathbb{Y}_\infty\setminus\{0\}$ we define $\mathrm{dist}_{\mathbb{Y}_\infty}(y_{1,\infty},y_{2,\infty}) := \mathrm{dist}_{\cdot\mathbb{S}\cdot}(y_{1,\infty},y_{2,\infty})$ as well as $\mathrm{dist}_{\mathbb{Y}_\infty}(y_{1,\infty},0):=\infty$ and $\mathrm{dist}_{\mathbb{Y}_\infty}(0,0):=0$. Then, we endow $\mathbb{Y}$ with the distance metric

$$\begin{aligned}\mathrm{dist}_{\mathbb{Y}}\colon\ \mathbb{Y}\to[0,\infty],\ (y_1,y_2) = \left((\bar k,y_{1,\bar k})_{\bar k\in\mathbb{N}_+\cup\{\infty\}},(\bar k,y_{2,\bar k})_{\bar k\in\mathbb{N}_+\cup\{\infty\}}\right)\\ \longmapsto \sum_{\bar k\in\mathbb{N}_+\cup\{\infty\}}\mathrm{dist}_{\mathbb{Y}_{\bar k}}(y_{1,\bar k},y_{2,\bar k}).\end{aligned}$$

Moreover, we consider the embeddings

$$\iota_n\colon\ \left(\mathbb{S}^{d-1},\mathrm{dist}_{\mathsf{s};1/n},\nu_n\right)\to\mathbb{Y},\ u\longmapsto(\bar k,\delta_{n,\bar k}u)_{\bar k\in\mathbb{N}_+\cup\{\infty\}}$$

[9] The comparison theorem as proven by R. Bishop can be found in [BC64, Corollary 4], whereas result and idea of the proof have been outlined in the short note [Bis63] before.

$$\text{and} \qquad \iota_\infty\colon\ (\cdot\mathbb{S}\cdot, \mathsf{dist}_{\cdot\mathbb{S}\cdot}, \nu_\infty) \to \mathbb{Y},\ u \longmapsto \big(\bar{k}, \delta_{\infty,\bar{k}} u\big)_{\bar{k}\in\mathbb{N}_+\cup\{\infty\}},$$

where '$\delta$' refers to a Kronecker delta. Then, $(\mathbb{Y}, \mathsf{dist}_\mathbb{Y})$ and $(\iota_{\bar{n}})_{\bar{n}\in\mathbb{N}_+\cup\{\infty\}}$ form an effective realisation. Alternatively, one can show pmG-convergence by finding a measure approximation in the sense of Definition B.2.6: Consider the projection $\mathsf{pr}_{\mathbb{S}^{d-2}}$ from $\mathbb{S}^{d-1}\setminus\{\pm e_d\}$ to $\mathbb{S}^{d-2}$. Then, the family $(\mathsf{pr}_{\mathbb{S}^{d-2}})_{n\in\mathbb{N}_+}$ is a measure approximation.

Once the uniform doubling condition is satisfied, we can infer pmGH-convergence from pmG-convergence, which finishes the proof. Recall Theorem 4.1.8 and Remark 4.2.5. □

By setting $\bar{\rho}^\varepsilon_\mathbb{S} := \bar{\rho}_\mathbb{S}^{1/n[\varepsilon]}$ with $n[\varepsilon] := \lceil 1/\varepsilon\rceil + 1$ for $\varepsilon\in(0,1)$, we find the desired sequence of explicit weight functions and hence achieve our goal for this subsection. However, we are not content yet and therefore turn to convergence of spectral structures.

4

## Spectral convergence

Define the Hilbert spaces

$$H_{\mathbb{S};1/n} := L^2(\mathbb{S}^{d-1};\nu_n) \quad\text{and}\quad H_{\mathbb{S};0} := L^2(\cdot\mathbb{S}\cdot;\mathsf{s}_1^{d-2}) = L^2(\mathbb{S}^{d-2};\mathsf{s}_1^{d-2}).$$

Now, on to a very important step with serious implications: In the spirit of Section 4.1.2 and specifically Example 4.1.16, we declare the poles $\pm e_d$ to be singularities and therefore adapt the standard differential structure on $\mathbb{S}^{d-1}$ to the differential structure $\mathbf{C}$ of smooth functions that are constant in an open environment of a singularity. Then, the two nonempty strata are

$$\mathbb{S}^{d-1}_{\mathsf{top}} := \mathsf{strat}_{d-1}(\mathbb{S}^{d-1}) \quad\text{and}\quad \mathsf{strat}_0(\mathbb{S}^{d-1}) = \{\pm e_d\}.$$

For sake of brevity, we call the stratifold $(\mathbb{S}^{d-1},\mathbf{C})$ the *stratified sphere*. We already built some intuition that actually $\mathbb{S}^{d-1}_{\mathsf{top}}$ is the object whose dimension is reduced. In case we can show both pmGH-convergence and spectral convergence, the limit object consists on set level of the sphere $\mathbb{S}^{d-2}$, the reduction of $\mathbb{S}^{d-1}_{\mathsf{top}}$, and the remaining nonempty stratum. I. e. the limit set is $\cdot\mathbb{S}\cdot$ fitting our expectations from the previous part on pmGH-convergence. The set $\cdot\mathbb{S}\cdot$ is rendered naturally into a differential space by extending the standard differential structure on $\mathbb{S}^{d-2}$: The differential structure $\mathbf{C}(\cdot\mathbb{S}\cdot)$ on $\cdot\mathbb{S}\cdot$ is defined by the assertion that $f\in\mathbf{C}(\cdot\mathbb{S}\cdot)$ if and only if $f\in C^0(\cdot\mathbb{S}\cdot)$ and $f|_{\mathbb{S}^{d-2}}\in C^\infty(\mathbb{S}^{d-2})$. This yields $\mathsf{strat}_{d-2}(\cdot\mathbb{S}\cdot) = \mathbb{S}^{d-2}$ and $\mathsf{strat}_0(\cdot\mathbb{S}\cdot) = \{\pm e_d\}$.

Next, we make the stratified sphere into a 'Riemannian stratifold' in a natural way starting from the classical round metric $\mathsf{s}$ on $\mathbb{S}^{d-1}$. Consider a chart $h$ of the smooth manifold $\big(\mathbb{S}^{d-1}_{\mathsf{top}}, \mathbf{C}|_{\mathbb{S}^{d-1}_{\mathsf{top}}}\big) = \big(\mathbb{S}^{d-1}_{\mathsf{top}}, C^\infty(\mathbb{S}^{d-1}_{\mathsf{top}})\big)$ that provides local coordinates $(u^j)_{j=1}^{d-1}$ in 'close proximity' to one of the poles $e\in\{\pm e_d\}$. By the latter we mean that the coordinate functions locally extend to $e$ as an element of $\mathbf{C}$; this extension is unique due to the defining property (Strat3) of stratifolds. Thus, we must have that all the germs at $e$ are zero, in formulae: $[u^j]_e = 0$ for all $j\in\{1,\dots,d-1\}$. This justifies the definition

$$\mathsf{s}^1_u(\partial u^i(u)\,,\partial u^j(u)) := \begin{cases}\mathsf{s}_u(\partial u^i(u)\,,\partial u^j(u)) & ,\ \text{if } u\neq e\\ 0 & ,\ \text{if } u = e\end{cases}$$

for all $u\in\mathsf{dom}(h)\cup\{e\}$ and $i,j\in\{1,\dots,d-1\}$. As the pole $e$ and the chart $h$ can be chosen arbitrarily, this procedure defines a Riemmanian metric $\mathsf{s}^1$ on $\big(\mathbb{S}^{d-1}_{\mathsf{top}}, C^\infty(\mathbb{S}^{d-1}_{\mathsf{top}})\big)$ that smoothly[10] extends to the poles by zero and coincides with the standard round metric on $\mathbb{S}^{d-1}$ far away from the poles. The following example shows that all local coordinate expressions for objects on the standard sphere are still true on $\mathbb{S}^{d-1}_{\mathsf{top}}$ but in an approximate sense.

**Example 4.2.7** (approximate charts)**.** Consider any chart $h$ of the manifold $\mathbb{S}^{d-1}$ providing local coordinates $(u^j)_{j=1}^{d-1}$. Assume that the set $S_h := \{\pm e_d\}\cap\mathsf{dom}(h)$ of poles in the chart domain is nonempty. Then, the restriction of $h$ to $\mathbb{S}^{d-1}_{\mathsf{top}}$ can't be a chart for the manifold $\big(\mathbb{S}^{d-1}_{\mathsf{top}}, C^\infty(\mathbb{S}^{d-1}_{\mathsf{top}})\big)$. Supposing otherwise, the restrictions of the coordinate functions would extend to $S_h$ as elements of $\mathbf{C}$; meaning that there are constant in some open environments of the poles in $S_h$ and in particular they are not injective, contradiction.

[10]Recall that the term 'smoothness' really has a strict formal meaning for functions that map from stratifolds to manifolds.

For every $e \in S_h$ and some large number $k \in \mathbb{N}_+$ we consider a cut-off function $\varphi_{e;k}$ that constantly is 1 on $\mathbb{U}(e, 1/k)$ as well as 0 outside of $\mathbb{U}(e, 2/k)$, compare to Remark 1.2.33. Then, we define approximate coordinate functions by

$$u_k^j := u^j \cdot \prod_{e \in S_h} (1 - \varphi_{e;k}) \qquad \text{on } \mathsf{dom}(h) \text{ for all } j \in \{1, \dots, d-1\}.$$

These approximate coordinate functions are elements of $\mathbf{C}$ and provide on $\mathsf{dom}(h) \setminus \mathbb{U}(e, 2/k)$ the original coordinates. Hence, the family $\left(u_k^j\right)_{j=1}^{d-1}$ gives rise to a chart $h_k$ for $\left(\mathbb{S}^{d-1}_{\mathsf{top}}, C^\infty(\mathbb{S}^{d-1}_{\mathsf{top}})\right)$ with $\mathsf{dom}(h) \setminus \mathbb{U}(e, 2/k) \subseteq \mathsf{dom}(h_k)$.

The same approximation scheme is performed for the chart $h$ of standard spherical coordinates $(\theta^j)_{j=0}^{d-2}$, see Notation A.2.1, even though the poles are not contained in the chart domain. We observe that the restriction of the polar coordinate chart can not be a chart of $\left(\mathbb{S}^{d-1}_{\mathsf{top}}, C^\infty(\mathbb{S}^{d-1}_{\mathsf{top}})\right)$ for the same reason as before. Fortunately, the assignment

$$\theta_k^j := \theta^j \cdot (1 - \varphi_{+e_d;k})(1 - \varphi_{-e_d;k}) \qquad \text{on } \mathsf{dom}(h) \text{ for all } j \in \{0, \dots, d-2\}$$

defines a chart of approximate spherical coordinates on $\left(\mathbb{S}^{d-1}_{\mathsf{top}}, C^\infty(\mathbb{S}^{d-1}_{\mathsf{top}})\right)$. ⌟

As a consequence, objects associated to the new Riemannian metric $\mathsf{s}^1$ coincide with the objects associated to $\mathsf{s}$ in the a 'large' subdomain of approximate charts derived from the charts of $\mathbb{S}^{d-1}$. For instance, this applies to the Christoffel symbols, hence subsequently to the Levi-Civita connection, geodesic equation, and Euler-Lagrange equations.

*Remark* 4.2.8 (Curvature at the poles). Every smooth vector field $\mathcal{X} \in \Gamma^\infty(\mathsf{T}\mathbb{S}^{d-1}_{\mathsf{top}})$ smoothly extends to the stratified sphere by zero, since if it is represented as $\mathcal{X} = \sum_j \mathcal{X}^j \cdot \partial/\partial u^j$ in local coordinates $(u^j)_j$ that locally extend to elements of $\mathbf{C}$, then every coefficient function $\mathcal{X}^j$ smoothly extends to an element of $\mathbf{C}$ and every $\partial/\partial u^j$ extends by zero. Even more, we can infer that for every pole $e$ there is an open neighbourhood $U_e$ of $e$ such that $\mathcal{X}$ is constant on $U_e \cap \mathbb{S}^{d-1}_{\mathsf{top}}$. That means that Lie and covariant derivatives extend to poles by zero. Therefore, the Riemannian curvature tensor on $\left(\mathbb{S}^{d-1}_{\mathsf{top}}, \mathsf{s}^1\right)$ extends to the poles by zero and so does the Ricci curvature tensor. ⌟

Furthermore, we observe that the induced distance metric $\mathsf{dist}_{\mathsf{s}^1}$ on the stratifold as a length space coincides with $\mathsf{dist}_{\mathsf{s}}$. But, as geodesics in $\mathbb{S}^{d-1}_{\mathsf{top}}$ can't pass through the poles, two points can not necessarily be connected by a length minimising geodesic. Once we weight $\mathsf{s}^1$ by $\bar{\rho}_{\mathbb{S}}^{-1/n}$ the corresponding distance metric also coincides with $\mathsf{dist}_{\mathsf{s};1/n}$, since in order to define the latter distance metric we penalised connecting curves for passing through poles. Moreover, we get for the Riemannian volume measure that $\lambda_{\mathsf{s}^1} = \mathsf{S}_1^{d-1}\big(= \lambda_{\mathsf{s}}\big)$ in all approximate charts, which implies that this equality holds even globally on $(\mathbb{S}^{d-1}, \mathbf{C})$.

We also render $(\cdot\mathbb{S}\cdot, \mathbf{C}(\cdot\mathbb{S}\cdot))$ into a 'Riemannian' stratifold. To this end, denote by 'g' the round metric on $\mathbb{S}^{d-2}$ and naturally extend it to $\cdot\mathbb{S}\cdot$ by setting

$$\mathsf{g}(u_1\,, u_2) := \begin{cases} \mathsf{g}(u_1\,, u_2) & u_1, u_2 \in \mathbb{S}^{d-2} \\ 0 & u_1, u_2 \in \{\pm e_d\} \\ 0 & u_{1/2} \in \mathbb{S}^{d-2},\ u_{2/1} \in \{\pm e_d\} \end{cases} \qquad \text{for all } u_1, u_2 \in \cdot\mathbb{S}\cdot.$$

Clearly, both distance metric and volume measure on the top stratum $\mathbb{S}^{d-2}$ remain unchanged. The previously used extensions $\mathsf{dist}_{\cdot\mathbb{S}\cdot}$ and $\mathsf{S}_1^{d-2}\big(= \lambda_{\mathsf{g}}\big)$ on $\cdot\mathbb{S}\cdot$ are compatible with g in the expected way. Consequently, our convergence result Corollary 4.2.6 for weighted spheres as metric measure spaces translates to the stratified spheres.

**Corollary 4.2.9.** The sequence $\left(\left(\mathbb{S}^{d-1}, \mathbf{C}, \bar{\rho}_{\mathbb{S}}^{-1/n}\mathsf{s}^1\right)\right)_{n \in \mathbb{N}_+}$ of weighted stratified spheres with associated distance metric and volume measure pmGH-converges to $(\cdot\mathbb{S}\cdot, \mathbf{C}(\cdot\mathbb{S}\cdot), \mathsf{g})$ with associated distance metric and volume measure.

The Laplace-Beltrami on the Riemannian manifold $\left(\mathbb{S}^{d-1}_{\mathsf{top}}, \mathsf{s}^1\right)$ can be viewed as a self-adjoint operator $\left(\Delta_{\mathsf{s}^1}, D(\Delta_{\mathsf{s}^1})\right)$ on $L^2(\mathbb{S}^{d-1}, \mathsf{S}_1^{d-1})$, since $\mathsf{strat}_0(\mathbb{S}^{d-1}) = \{\pm e_d\}$ has measure zero. The operator

$\left(\Delta_{s^1}, D(\Delta_{s^1})\right)$ defines a spectral structure on the unweighted stratifold $(\mathbb{S}^{d-1}, \mathbf{C})$. For given $n \in \mathbb{N}_+$ the spectral structure on the weighted sphere is given by the weighted Laplace-Beltrami operator of the form

$$\Delta_{s;1/n} f := \Delta_{s^1} f + \frac{1}{\bar{\rho}_{\mathbb{S}}^{1/n}} \nabla_{s^1} \bar{\rho}_{\mathbb{S}}^{1/n}(f) = \Delta_{s^1} f + n \cdot \cot \theta^{d-2} \partial_{\theta^{d-2}} f$$

$$\text{for} \qquad f \in \left\{ g \in \mathbf{C} \,\middle|\, \cot \theta^{d-2} \partial_{\theta^{d-2}} g \in L^2(\mathbb{S}^{d-1}; \nu_n) = H_{\mathbb{S};1/n} \right\} \subseteq D\left(\Delta_{s;1/n}\right) \subseteq H_{\mathbb{S};1/n}.$$

For sake of illustration, we opted for writing the logarithmic derivative term in polar coordinates, see Notation A.2.1. More formally, the weighted Laplace-Beltrami is defined first on the top stratum and then considered as an operator on $H_{\mathbb{S};1/n}$. Furthermore, the spectral structure on $\cdot\mathbb{S}\cdot$ is given by the Laplace-Beltrami operator $(\Delta_{\mathbb{S}^{d-2}}, D(\Delta_{\mathbb{S}^{d-2}}))$ on $H_{\mathbb{S};0}$.

Note that $\mathbf{C}$ is dense both in $L^2(\mathbb{S}^{d-1}; \mathsf{S}_1^{d-1})$ and in $H_{\mathbb{S};1/n}$. Indeed: Let $f \in C^\infty(\mathbb{S}^{d-1})$ and assume without lost of generality that the support is contained in the upper hemisphere, i. e.

$$\operatorname{supp}(f) \subseteq \left\{ \sum_{j=1}^{d} u_j e_j \in \mathbb{S}^{d-1} \,\middle|\, u_d > 0 \right\}.$$

Consider for $k \in \mathbb{N}_+$ large enough a smooth cut-off function $\varphi_k$ with $\varphi_k = 1$ on $\mathbb{U}(e_d, 1/k)$ and $\varphi_k = 0$ outside of $\mathbb{U}(e_d, 2/k)$. Define the approximation function

$$f_k := f \cdot (1 - \varphi_k) + \left(f \,,\, \mathbb{1}_{\mathbb{U}(e_d, 1/k)}\right)_{H_{\mathbb{S};1/n}} \cdot \varphi_k. \tag{4.4}$$

Then, it holds that $f_k \in \mathbf{C}$ and we get that $f_k \to f$ both in $L^2(\mathbb{S}^{d-1}; \mathsf{S}_1^{d-1})$ and in $H_{\mathbb{S};1/n}$ as $k \to \infty$. As $C^\infty(\mathbb{S}^{d-1})$ is dense both in $L^2(\mathbb{S}^{d-1}; \mathsf{S}_1^{d-1})$ and in $H_{\mathbb{S};1/n}$, the claim is proven.

Moving on, we observe that $H_{\mathbb{S};0}$ is continuously embedded into $H_{\mathbb{S};1/n}$ as a closed subspace for all $n \in \mathbb{N}$ by the canonical embedding

$$\iota_{\mathbb{S};1/n} \colon \; H_{\mathbb{S};0} \to H_{\mathbb{S};1/n}, \; f \mapsto f \circ \mathrm{pr}_{\mathbb{S}^{d-2}}.$$

Abusing notation, one might view the operator $(\Delta_{\mathbb{S}^{d-2}}, D(\Delta_{\mathbb{S}^{d-2}}))$ really as operator in $H_{\mathbb{S};1/n}$ given by

$$\Delta_{\mathbb{S}^{d-2}} f := \iota_{\mathbb{S};1/n}(\Delta_{\mathbb{S}^{d-2}} f_0) \qquad \text{for all } f = \iota_{\mathbb{S};1/n} f_0 \in \iota_{\mathbb{S};1/n}(D(\Delta_{\mathbb{S}^{d-2}})).$$

Consider the left inverse $\Phi_{\mathbb{S};1/n} \colon \; H_{\mathbb{S};1/n} \to H_{\mathbb{S};0}$ of $\iota_{\mathbb{S};1/n}$, which can be thought as continuous projection in $H_{\mathbb{S};1/n}$ to the range of $\iota_{\mathbb{S};1/n}$. Then, we arrive at the situation of a convergent net of Hilbert spaces as in [KS03, Section 2.2]. Recall that alternatively one could think in terms of asymptotic relations of Hilbert spaces, see specifically Definition B.2.7 in the appendix. Now, the following corollary arises from the observation that $\Delta_{\mathbb{S}^{d-2}} = \Phi_{\mathbb{S};1/n} \circ \Delta_{s;1/n} \circ \iota_{\mathbb{S};1/n}$ first on $C^\infty(\mathbb{S}^{d-2})$ and hence also on $D(\Delta_{\mathbb{S}^{d-2}})$.

**Corollary 4.2.10.** The graph of $(\Delta_{\mathbb{S}^{d-2}}, D(\Delta_{\mathbb{S}^{d-2}}))$ coincides with the strong graph limit $\Gamma\left[\Delta_{s;1/n}\right]$, compare to Definition 4.1.10. Hence, the spectral structures on the weighted spheres converge to the spectral structure on $\mathbb{S}^{d-2}$ (or $\cdot\mathbb{S}\cdot$ respectively) according to Theorem 4.1.11.

*Proof.* Consider a pair $(u_0, v_0)$ in the graph of the operator $(\Delta_{\mathbb{S}^{d-2}}, D(\Delta_{\mathbb{S}^{d-2}}))$ on $H_{\mathbb{S};0}$. Then, for the net $\left(\iota_{\mathbb{S};1/n} u_0\right)_{n \in \mathbb{N}_+}$ we observe that

$$\iota_{\mathbb{S};1/n} u_0 \longrightarrow u_0 \circ \mathrm{pr}_{\mathbb{S}^{d-2}} = u_0$$

$$\text{and} \qquad \Delta_{s;1/n}\left(\iota_{\mathbb{S};1/n} u_0\right) = \Delta_{\mathbb{S}^{d-2}}\left(\iota_{\mathbb{S};1/n} u_0\right) = \iota_{\mathbb{S};1/n} v_0 \longrightarrow v_0 \circ \mathrm{pr}_{\mathbb{S}^{d-2}} = v_0$$

strongly as $n \to \infty$. Thus, $(u_0, v_0) \in \Gamma\left[\Delta_{s;1/n}\right]$.

Vice versa, consider now some given pair $(u_0, v_0) \in \Gamma\left[\Delta_{s;1/n}\right]$. By definition of the strong graph limit, there is a net $(\tilde{u}_n)_{n \in \mathbb{N}_+}$ with $\tilde{u}_n \in H_{\mathbb{S};1/n}$ for all $n \in \mathbb{N}_+$ and

$$\tilde{u}_n \longrightarrow u_0 \quad \text{as well as} \quad \Delta_{s;1/n} \tilde{u}_n \longrightarrow v_0 \qquad \text{strongly as } n \to \infty.$$

Passing to the image sequence $(u_n)_{n \in \mathbb{N}_+} := \left(\Phi_{\mathbb{S};1/n} \tilde{u}_n\right)_n$, this implies the convergences in $H_{\mathbb{S};0}$

$$u_n \longrightarrow u_0$$

$$\text{and} \qquad \Delta_{\mathbb{S}^{d-2}} u_n = \Phi_{\mathbb{S};1/n}\Big(\Delta_{\mathsf{s};1/n}\big(\iota_{\mathbb{S};1/n} u_n\big)\Big) = \Phi_{\mathbb{S};1/n}\big(\Delta_{\mathsf{s};1/n}\tilde{u}_n\big) \longrightarrow v_0 \qquad \text{as } n \to \infty.$$

Since $(\Delta_{\mathbb{S}^{d-2}}, D(\Delta_{\mathbb{S}^{d-2}}))$ is closed in $H_{\mathbb{S};0}$, it follows that $v_0 = \Delta_{\mathbb{S}^{d-2}} u_0$. Therefore, $(u_0, v_0)$ is in the graph of $(\Delta_{\mathbb{S}^{d-2}}, D(\Delta_{\mathbb{S}^{d-2}}))$. □

We have to spare a minute for considering the spectra as this turns out to be central for the argument of Mosco convergence. From Theorem 4.1.12, we know that $\sigma(\Delta_{\mathbb{S}^{d-2}}) \subseteq \lim_{n\in\mathbb{N}_+} \sigma\big(\Delta_{\mathsf{s};1/n}\big)$. Recalling [DX13, Theorem 1.4.5], we also know that $-(d-2)$ is the eigenvalue of $\Delta_{\mathbb{S}^{d-2}}$ on $\mathbb{S}^{d-2}$ corresponding to the 1-homogeneous spherical harmonic $\chi_0$. So, there must be a sequence $(s_n)_{n\in\mathbb{N}_+}$ such that $s_n \in \sigma\big(\Delta_{\mathsf{s};1/n}\big)$ and $s_n \to -(d-2)$ as $n \to \infty$. In fact, note that $\chi := \iota_{\mathbb{S};1/n}(\chi_0) \in D\big(\Delta_{\mathsf{s};1/n}\big)$ for fixed $n \in \mathbb{N}_+$. Indeed, approximate $\chi$ by a sequence $(\chi_k)_k$ in $\mathbf{C}$ with functions $\chi_k$ as in Equation (4.4) and indices $k \in \mathbb{N}_+$ large enough. In that construction, we used the functions $(\varphi_k)_k$, for which there is a constant $c \in (0,\infty)$

$$|\nabla_{\mathsf{s}^1}\varphi_k(u)|_{\mathsf{s}^1} \leq \frac{c}{k} \quad \text{and} \quad |\mathsf{Hess}_{\mathsf{s}^1}(\varphi_k)(u)|_\infty \leq \frac{c}{k^2}$$

for all $k$ and $u \in \mathbb{S}^{d-1}$. Then, we get that the images $\big(\Delta_{\mathsf{s};1/n}\chi_k\big)_k$ converge and moreover that $\Delta_{\mathsf{s};1/n}\chi_k \longrightarrow \Delta_{\mathsf{s};1/n}\chi$ as $k \to \infty$, since $\big(\Delta_{\mathsf{s};1/n}, D(\Delta_{\mathsf{s};1/n})\big)$ is closed. We notice that

$$\Delta_{\mathsf{s};1/n}\chi = \Delta_{\mathsf{s};1/n}\big(\iota_{\mathbb{S};1/n}\chi_0\big) = \Delta_{\mathbb{S}^{d-2}}\big(\iota_{\mathbb{S};1/n}\chi_0\big) = -(d-2)\cdot\iota_{\mathbb{S};1/n}\chi_0 = -(d-2)\,\chi.$$

Beyond that, we also get along the same lines that $\chi \in D\big(\Delta_{\mathsf{s}^1}\big)$ with $\Delta_{\mathsf{s}^1}\chi = -(d-2)\chi$. To this end, one considers the continuous embedding

$$\iota_{\mathbf{C}}\colon\ H_{\mathbb{S};0} \to L^2(\mathbb{S}^{d-1}; \mathsf{S}_1^{d-1}),\ f \longmapsto f \circ \mathsf{pr}_{\mathbb{S}^{d-2}},$$

its left inverse $\Phi_{\mathbf{C}}$, and the relation $\Delta_{\mathsf{s}^1} \circ \iota_{\mathbf{C}} = \Delta_{\mathbb{S}^{d-2}} \circ \iota_{\mathbf{C}}$. We summarise these findings in a lemma.

**Lemma 4.2.11.** For all $n \in \mathbb{N}_+$ it holds $-(d-2) \in \sigma\big(\Delta_{\mathsf{s};1/n}\big)$. More precisely, $-(d-2)$ is an eigenvalue and the lifting $\iota_{\mathbb{S};1/n}(\chi_0)$ of the 1-homogeneous spherical harmonic $\chi_0$ on $\mathbb{S}^{d-2}$ is the corresponding eigenvector. Note that this result does not depend on the specific choice of $\rho_{\mathbb{S}}^{1/n}$. Furthermore, $-(d-2) \in \sigma\big(\Delta_{\mathsf{s}^1}\big)$ is an eigenvalue of $\Delta_{\mathsf{s}^1}$ for $\chi$.

*Remark* 4.2.12 (spherical harmonics). The deeper theoretical reason for the previously observed effect is that the set of spherical harmonics differs for the two Laplacians $\Delta_{\mathsf{s}}$ and $\Delta_{\mathsf{s}^1}$. Let us recall the definitions, compare to [DX13, Section 1.1]: A real homogeneous polynomial $p$ of degree $k$ on $\mathbb{R}^d$ is a polynomial of the form

$$p(x) = \sum_{\substack{\ell\in\mathbb{N}^d\\|\ell|=k}} c_\ell \cdot x^\ell = \sum_{\substack{\ell_1,\dots,\ell_d\in\mathbb{N}\\ \ell_1+\dots+\ell_d=k}} c_{(\ell_1,\dots,\ell_d)} \cdot x_1^{\ell_1}\cdots x_d^{\ell_d} \qquad \text{with } c_\ell = c_{(\ell_1,\dots,\ell_d)} \in \mathbb{R}$$

using multiindex notation. Such a homogeneous polynomial $p$ is harmonic if additionally $\Delta_{\mathbb{R}^d}p = 0$ holds for the standard Laplacian $\Delta_{\mathbb{R}^d}$ on $\mathbb{R}^d$. The restriction of such a harmonic polynomial to $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ is a spherical harmonic. But since one has $\mathsf{T}_{\pm e_d}(\mathbb{S}^{d-1}) = \{0\}$ for the stratified sphere $(\mathbb{S}^{d-1}, \mathbf{C})$, the reasonable requirement for a spherical harmonic $p$ to be a *spherical harmonic with respect to* $\Delta_{\mathsf{s}^1}$ is to additionally demand that the germs of $p|_{\mathbb{S}^{d-1}_{\mathrm{top}}}$ 'continuously' extend to the poles by zero in the following sense: For all $e \in \{\pm e_d\}$ there is a neighbourhood $U_e$ of $e$ such that $[p]_y = 0$ holds for all $y \in U_e \cap \mathbb{S}^{d-1}_{\mathrm{top}}$.[11] Of course, this is the case if and only if $c_{(\ell_1,\dots,\ell_d)} = 0$ holds for all the coefficients with $\ell_d \in \mathbb{N}_+$. Equivalently, we can say that $p$ is a spherical harmonic with respect to $\Delta_{\mathsf{s}^1}$ if and only if there is a spherical harmonic $p_0$ on $\mathbb{S}^{d-2} \subseteq \mathbb{R}^{d-1}$ such that $p = \iota_{\mathbf{C}}(p_0)$. That means that one has to be careful regarding statements made for $\Delta_{\mathsf{s}}$ relying on spherical harmonics to the operator $\Delta_{\mathsf{s}^1}$. ⌟

In contrast, W. Beckners proof of the standard Poincaré inequality for the spherical surface measure, see [Bec89, Theorem 2], heavily relies on the usual spherical harmonics, but we do not need to adapt

[11] Note that this does not necessarily mean that $p$ is smooth in the stratifold sense that $p \in \mathbf{C}$. However, the requirement is akin to continuous differentiability at the poles, although $p$ as a function on the stratified sphere turns out to be not even continuous.

statement or proof for the stratified sphere: Let $f \in \mathbf{C}$, then the gradient of $f$ equals zero in open neighbourhoods $U(\pm e_d)$ of the poles and outside of these neighbourhoods the standard sphere and stratified sphere are indistinguishable. Thus, the inequality

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} |\nabla_{\mathsf{s}^1} f|^2_{\mathsf{s}^1}\, \mathrm{d}\lambda_{\mathsf{s}^1} = \int_{\mathbb{S}^{d-1}\setminus(U(+e_d)\cup U(-e_d))} |\nabla_{\mathsf{s}^1} f|^2_{\mathsf{s}^1}\, \mathrm{d}\lambda_{\mathsf{s}^1} \\
&= \int_{\mathbb{S}^{d-1}\setminus(U(+e_d)\cup U(-e_d))} |\nabla_{\mathsf{s}} f|^2_{\mathsf{s}}\, \mathrm{d}\lambda_{\mathsf{s}} = \int_{\mathbb{S}^{d-1}} |\nabla_{\mathsf{s}} f|^2_{\mathsf{s}}\, \mathrm{d}\lambda_{\mathsf{s}} \\
&\geq (d-1) \int_{\mathbb{S}^{d-1}} \left(f - (f\,,1)_{L^2(\mathbb{S}^{d-1};\mathsf{s})}\right)^2 \mathrm{d}\lambda_{\mathsf{s}} \\
&= (d-1) \int_{\mathbb{S}^{d-1}\setminus(U(+e_d)\cup U(-e_d))} \left(f - (f\,,1)_{L^2(\mathbb{S}^{d-1};\mathsf{s})}\right)^2 \mathrm{d}\lambda_{\mathsf{s}} + 0 \\
&= (d-1) \int_{\mathbb{S}^{d-1}\setminus(U(+e_d)\cup U(-e_d))} \left(f - (f\,,1)_{L^2(\mathbb{S}^{d-1};\mathsf{s}^1)}\right)^2 \mathrm{d}\lambda_{\mathsf{s}^1} \\
&= (d-1) \int_{\mathbb{S}^{d-1}} \left(f - (f\,,1)_{L^2(\mathbb{S}^{d-1};\mathsf{s}^1)}\right)^2 \mathrm{d}\lambda_{\mathsf{s}^1}
\end{aligned}$$

holds by [Bec89, Theorem 2].

### 4.2.2 ▪ Collaps of unit tangent bundles based on convergence of the standard fibre

Consider the disjoint union of the spaces

$$\mathsf{Q}^0_x\mathbb{X} := \left\{ u = \sum_{j=1}^{d} u_j e_j(x) \in \mathsf{U}_x\mathbb{X} \,\middle|\, u_d = 0 \right\} \qquad \text{for } x \in \mathbb{X},$$

defined in terms of the compass frame $(e_j)_{j=1}^d$, and the projection

$$\pi_{\mathsf{Q}^0}\colon\ \mathsf{Q}^0\mathbb{X} := \bigsqcup_{x\in\mathbb{X}} \mathsf{Q}^0_x\mathbb{X} \to \mathbb{X},\ \mathsf{Q}^0_x\mathbb{X} \ni u \longmapsto x.$$

As alluded to before, we don't get a smooth or even continuous fibre bundle, due to lack of parallelisability of the base manifold. Be that as it may, we refer to $\pi_{\mathsf{Q}^0}\colon \mathsf{Q}^0\mathbb{X} \to \mathbb{X}$ as *equatorial tangent bundle* event though we still have to make sense of the term 'bundle' here. In the following construction, we opt for describing $\mathsf{Q}^0\mathbb{X}$ as a limit varifold first. Overall, we assume a sequence of probability measures $(\nu^\varepsilon)_{\varepsilon\in(0,1]}$ on $\mathbb{S}^{d-1}$ such that

- $\nu^1 = \mathsf{S}^{d-1}_1$,
- every $\nu^\varepsilon$ is invariant with respect to rotations around the axis passing through the poles $\pm e_d$,
- there is the Radon-Nikodým derivative $\rho^\varepsilon_{\mathbb{S}} := {}^{d\nu^\varepsilon}/_{d\nu^1}$ for all $\varepsilon \in (0,1]$,
- the sequence $(\mathbb{S}^{d-1}, \nu^\varepsilon, \bar{u})_{\varepsilon\in(0,1]}$ for given $\bar{u} \in \mathbb{S}^{d-2}$ converges in pmG-sense to $(\cdot\mathbb{S}\cdot, \nu^0 := \mathsf{S}^{d-2}_1, \bar{u})$ (and thus also in pmGH-sense),
- and the spectral structures of the weighted spheres $(\mathbb{S}^{d-1}, \mathbf{s}^\varepsilon := \rho^\varepsilon_{\mathbb{S}}\mathsf{s})_{\varepsilon\in(0,1]}$ converge to the spectral structure corresponding to the Laplace-Beltrami operator $(\Delta_{\mathbb{S}^{d-2}}, D(\Delta_{\mathbb{S}^{d-2}}))$ on $L^2(\cdot\mathbb{S}\cdot;\nu^0) = L^2(\mathbb{S}^{d-2};\nu^0)$.

We have seen previously that such a sequence $(\nu^\varepsilon)_{\varepsilon\in(0,1]}$ exists. Suppose the embedding $\mathsf{U}\mathbb{X} \hookrightarrow \mathbb{R}^n$ to view the unit tangent bundle of $\mathbb{X}$ as submanifold of the $n$-dimensional Euclidean space. We construct a weighted version of unit tangent bundle as varifold $\mathbb{V}_\varepsilon$: Introduce the transition kernel

$$k_\varepsilon\colon\ \mathbb{X}\times\mathfrak{B}(\mathbb{R}^n),\ (x,A) \longmapsto \int_{\mathsf{U}_x\mathbb{X}} \mathbb{1}_{A\cap\mathsf{U}_x\mathbb{X}}(u) \quad \nu^\varepsilon_x(\mathrm{d}u),$$

where $\nu_x^\varepsilon$ is a copy of $\nu^\varepsilon$ on $\mathsf{U}_x\mathbb{X}$ that is compass aligned by which we mean that ${}^{d\nu_x^\varepsilon}/_{d\nu_x^1}(\pm\text{⦻}(x)) = 0$. This kernel induces a measure $\mu[k_\varepsilon]$ on $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$ by the assignment $\mu[k_\varepsilon](A) := \int_{\mathbb{X}} k_\varepsilon(x, A)\ \lambda_{\mathsf{x}}(\mathrm{d}x)$ for $A \in \mathfrak{B}(\mathbb{R}^n)$. Note that $\mu[k_\varepsilon]$ is supported on $\mathsf{U}\mathbb{X} \subseteq \mathbb{R}^n$, but it depends on the chosen embedding and can generally not be realised as a bundle measure in the sense of Definition 1.2.19. Now, the varifold $\mathbb{V}_\varepsilon$ is defined via

$$
\begin{aligned}
\langle f\,,\mathbb{V}_\varepsilon\rangle &= \int_{\mathbb{R}^n\times\mathbb{G}(2d-1,n)} f\ \mathrm{d}\mathbb{V}_\varepsilon \\
&:= \int_{\mathbb{R}^n}\int_{\mathbb{G}(2d-1,n)} f(u,S) \qquad \delta_{\mathsf{T}_u\mathsf{U}\mathbb{X}}(\mathrm{d}S)\quad \mathbb{1}_{\mathsf{U}\mathbb{X}}(u)\ \mu[k_\varepsilon](\mathrm{d}u) \\
&= \int_{\mathbb{X}}\int_{\mathsf{U}_x\mathbb{X}}\int_{\mathbb{G}(2d-1,n)} f(u,S) \qquad \delta_{\mathsf{T}_u\mathsf{U}\mathbb{X}}(\mathrm{d}S)\ \nu_x^\varepsilon(\mathrm{d}u)\ \lambda_{\mathsf{x}}(\mathrm{d}x)
\end{aligned}
$$

for all $f \in C_{\mathsf{c}}^\infty(\mathbb{R}^n \times \mathbb{G}(2d-1, n))$. In view of Example 4.1.3, one might be even more inclined to think that $\mu[k_\varepsilon] = \lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu^\varepsilon$ would hold. We want to see that the sequence $(\mathbb{V}_\varepsilon)_{\varepsilon\in(0,1]}$ of varifolds converges to a varifold $\mathbb{V}_0$ as $\varepsilon \downarrow 0$. By assumption on $(\nu^\varepsilon)_{\varepsilon\in(0,1]}$, we can define the transition kernel

$$
k_0\colon\ \mathbb{X}\times\mathfrak{B}(\mathbb{R}^n),\ (x,A)\longmapsto \lim_{\varepsilon\downarrow 0}\int_{\mathsf{U}_x\mathbb{X}} \mathbb{1}_{A\cap\mathsf{U}_x\mathbb{X}}(u)\quad \nu_x^\varepsilon(\mathrm{d}u) = \int_{\mathsf{U}_x\mathbb{X}} \mathbb{1}_{A\cap\mathsf{Q}_x^0\mathbb{X}}(u)\quad \nu_x^0(\mathrm{d}u),
$$

where $\nu_x^0$ is a copy of $\nu^0$ on $\mathsf{Q}_x^0\mathbb{X}$ that also is compass aligned but now in the sense that $\nu_x^0$ is a measure on $\mathsf{U}_x\mathbb{X}$ supported on $\mathsf{Q}_x^0\mathbb{X}$. Consequently, we get a measure $\mu[k_0]$ given by $\mu[k_0](A) := \int_{\mathbb{X}} k_0(x, A)\ \lambda_{\mathsf{x}}(\mathrm{d}x)$ for all $A \in \mathfrak{B}(\mathbb{R}^n)$. By construction, we arrive at

$$
\begin{aligned}
\langle f\,,\mathbb{V}_\varepsilon\rangle &= \int_{\mathbb{X}}\int_{\mathsf{U}_x\mathbb{X}}\int_{\mathbb{G}(2d-1,n)} f(u,S)\quad \delta_{\mathsf{T}_u\mathsf{U}\mathbb{X}}(\mathrm{d}S)\ \nu_x^\varepsilon(\mathrm{d}u)\ \lambda_{\mathsf{x}}(\mathrm{d}x) \\
&\longrightarrow \int_{\mathbb{X}}\int_{\mathsf{U}_x\mathbb{X}}\int_{\mathbb{G}(2d-1,n)} f(u,S)\quad \delta_{\mathsf{T}_u\mathsf{U}\mathbb{X}}(\mathrm{d}S)\ \nu_x^0(\mathrm{d}u)\ \lambda_{\mathsf{x}}(\mathrm{d}x) \\
&= \int_{\mathbb{X}}\int_{\mathsf{Q}_x^0\mathbb{X}}\int_{\mathbb{G}(2d-1,n)} f(u,S)\quad \delta_{\mathsf{T}_u\mathsf{U}\mathbb{X}}(\mathrm{d}S)\ \nu_x^0(\mathrm{d}u)\ \lambda_{\mathsf{x}}(\mathrm{d}x) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{G}(2d-1,n)} f(u,S) \qquad \delta_{\mathsf{T}_u\mathsf{U}\mathbb{X}}(\mathrm{d}S)\quad \mathbb{1}_{\mathsf{Q}^0\mathbb{X}}(u)\ \mu[k_0](\mathrm{d}u) \qquad \text{as } \varepsilon\downarrow 0
\end{aligned}
$$

for all $f \in C_{\mathsf{c}}^0(\mathbb{R}^n \times \mathbb{G}(2d-1, n))$. Define the varifold $\mathbb{V}_0$ by

$$
\langle f\,,\mathbb{V}_0\rangle := \int_{\mathbb{R}^n}\int_{\mathbb{G}(2d-1,n)} f(u,S)\quad \delta_{\mathsf{T}_u\mathsf{U}\mathbb{X}}(\mathrm{d}S)\quad \mathbb{1}_{\mathsf{Q}^0\mathbb{X}}(u)\ \mu[k_0](\mathrm{d}u)
$$

for all $f \in C_{\mathsf{c}}^0(\mathbb{R}^n \times \mathbb{G}(2d-1, n))$. At the cost of relying on a specific embedding $\mathsf{U}\mathbb{X} \hookrightarrow \mathbb{R}^n$, we can now view the mere set $\mathsf{Q}^0\mathbb{X}$ as the varifold $\mathbb{V}_0$. Again, one should note that $\mathbb{V}_0$ is by construction a $(2d-1)$-dimensional varifold as every $\mathbb{V}_\varepsilon$ is.

Certainly, our next step has to be establishing $\mathsf{Q}^0\mathbb{X}$ as an object independent of an ambient space. We recollect the assumptions on the compass mentioned in the introduction to this chapter. Once more, we stress that these assumptions and subsequent stratifold constructions are not needed if $\mathbb{X}$ is parallelisable e. g. in the Euclidean case $\mathbb{X} = \mathbb{R}^d$.

**Condition 4.2.13** (Compass conditions (⦻)).

(⦻ 1) Assume that the compass ⦻: $\mathbb{X} \to \mathsf{U}\mathbb{X}$ is a locally Lipschitzian vector field. According to Rademacher's Theorem, the set $\mathsf{dom}(\partial⦻) \subseteq \mathbb{X}$ on which ⦻ is differentiable has full measure and is dense.

(⦻ 2) Assume that $\mathbb{X} \setminus \mathsf{dom}(\partial⦻)$ consists of finitely many points which we declare as singular points. Consistently, we say that the set $\mathbb{X}_{⦻} := \mathsf{dom}(\partial⦻)$ consists of regular points. ⌟

Now, we replace the differentiable structure on $\mathbb{X}$ by the differential structure $\mathbf{C}(⦻) \subseteq C^\infty(\mathbb{X})$ such that $f \in \mathbf{C}(⦻)$ holds if and only if for every $x \in \mathbb{X}_{⦻}^{\mathsf{c}}$ there is an open neighbourhood $U_x \subseteq \mathbb{X}$ of $x$ such that $f$ is constant on $U_x$. We note that the top stratum of $(\mathbb{X}, \mathbf{C}(\oslash))$ is $\mathsf{strat}_d(\mathbb{X}) = \mathbb{X}_\oslash$ and the only other nonempty stratum is the complement $\mathsf{strat}_0(\mathbb{X}) = \mathbb{X}_\oslash^\mathsf{c}$. Furthermore, $\mathbb{X}_\oslash$ is a parallelisable manifold. Next, we replace the differentiable structure on $\mathsf{U}\mathbb{X}$ by the differential structure $\mathbf{C}(\mathsf{Q}^1\mathbb{X}) \subseteq C^\infty(\mathsf{U}\mathbb{X})$ such that $f \in \mathbf{C}(\mathsf{Q}^1\mathbb{X})$ if and only if

- the fibrewise average $\mathsf{E}_{\nu^\varepsilon}[f] = \left(x \mapsto \int_{\mathsf{U}_x\mathbb{X}} f \,\mathrm{d}\nu_x^\varepsilon\right)$ is in $\mathbf{C}(\oslash)$,
- and for all $x \in \mathbb{X}_\oslash$ there are two open neighbourhoods $U_\pm$ of $\pm\oslash(x)$ respectively such that $f|_{\mathsf{U}_x\mathbb{X}}$ is constant on $U_\pm \cap \mathsf{U}_x\mathbb{X}$.

Note that vertical lifts of functions form $C^\infty(\mathbb{X}_\oslash)$ are contained in $\mathbf{C}(\mathsf{Q}^1\mathbb{X})$ automatically, since they are fibrewise constant. The set $\mathsf{U}\mathbb{X}$ endowed with the differential structure $\mathbf{C}(\mathsf{Q}^1\mathbb{X})$ is denoted by $\mathsf{Q}^1\mathbb{X}$. We want to see that $\mathsf{Q}^1\mathbb{X}$ still is a bundle in a certain sense. Technically, the following definition of so-called stratifold bundles from [Gri03, Section 1.2] does not suffice, but is our starting point nonetheless.

**Definition 4.2.14** (stratifold bundles)**.** Consider a stratifold $\mathbb{E}$ and a smooth manifold $\mathbb{B}$. A morphism $\pi\colon \mathbb{E} \to \mathbb{B}$ is a *stratifold bundle* if for each $b \in \mathbb{B}$ there is an open neighbourhood $U_b$ of $b$, a stratifold $\mathbb{F}_b$, and an isomorphism $\varphi\colon \pi^{-1}(U_b) \to U_b \times \mathbb{F}_b$ that renders the following diagram commutative:

$$\begin{tikzcd} \pi^{-1}(U_b) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U_b \times \mathbb{F}_b \arrow[dl, "\mathsf{pr}_1"] \\ U_b & \end{tikzcd}$$

Besides, one uses terminology as for fibre bundles: We call $\mathbb{B}$ the base space, $\mathbb{E}$ the total space, and the mapping $\pi$, as part of the stratifold, the bundle projection. ⌟

**Lemma 4.2.15.** The mapping $\mathsf{Q}^1\mathbb{X}_\oslash = \mathsf{Q}^1\mathbb{X}|_{\mathsf{U}\mathbb{X}_\oslash} \to \mathbb{X}_\oslash$, $\mathsf{Q}^1_x\mathbb{X} \ni u \longmapsto x$ is a stratifold bundle with the fibre $\mathbb{F}_x$ for $x \in \mathbb{X}_\oslash$ being the stratifold $\mathbb{F}_x = (\mathbb{S}^{d-1}, \mathbf{C})$ with singularities at $\{\pm e_d\}$.

*Proof.* The short version of the proof is that the unit tangent bundle $\mathsf{U}\mathbb{X}_\oslash$ over $\mathbb{X}_\oslash$ is a trivial bundle. Now, we can manipulate the standard fibre in $\mathsf{U}\mathbb{X}_\oslash \simeq \mathbb{X}_\oslash \times \mathbb{S}^{d-1}$ as we like and find that $\mathbb{X}_\oslash \times (\mathbb{S}^{d-1}, \mathbf{C}) \simeq \mathsf{Q}^1\mathbb{X}_\oslash$. So the statement of the lemma can be strengthened to the effect that we even have a *trivial* stratifold bundle.

Alternatively, a version grounded in the definition of stratifold bundles goes like this: Consider $x \in \mathbb{X}_\oslash$ and a local trivialisation at $x$ of the smooth fibre bundle $\pi_{0|\mathsf{U}}\colon \mathsf{U}\mathbb{X} \to \mathbb{X}$ consisting of the trivialisation chart $\chi$ and domain of trivialisation $U$ such that $U \cap \mathbb{X}_\oslash^\mathsf{c} = \emptyset$. Explicitly, we can choose $U$ as domain of a chart $h$ from the differentiable structure of $\mathbb{X}$. Then, the mapping

$$\tilde{\chi}\colon\ \pi_{0|\mathsf{U}}^{-1}(U) \to h(U) \times \mathbb{S}^{d-1},\ u \longmapsto (h(\pi_{0|\mathsf{U}}(u)), u_h)$$

is a diffeomorphism. Here, $u_h$ is the representation of $u$ in the chart $h$, compare to Example 1.2.7 part (iii). Then, one directly obtains $\chi$ from $\tilde{\chi}$. Now, we replace the standard sphere in the target set of $\chi$ by the stratifold $(\mathbb{S}^{d-1}, \mathbf{C})$. We show that the mapping $\chi\colon \pi_{0|\mathsf{U}}^{-1}(U) \to U \times (\mathbb{S}^{d-1}, \mathbf{C})$ is an isomorphism of stratifolds: Let $f_1$ be a function from the restriction $\mathbf{C}_1$ of $\mathbf{C}(\mathsf{Q}^1\mathbb{X})$ to $\pi_{0|\mathsf{U}}^{-1}(U)$ and $f_2$ be a function from the canonical differential structure $\mathbf{C}_2$ of the product stratifold composed of $U$ and $(\mathbb{S}^{d-1}, \mathbf{C})$. Then, we have both that $f_1 \circ \chi^{-1} \in \mathbf{C}_2$ and that $f_2 \circ \chi \in \mathbf{C}_1$. So, the choice $\varphi = \chi$ yields the local trivialisation of a stratifold bundle. □

Due to the fact that $(\mathbb{X}, \mathbf{C}(\oslash))$ has just two rather simple nonempty strata, we don't have to think to hard on how to extend Definition 4.2.14 to the situation of a stratifold as base space instead of a smooth manifold. A more general extension of Definition 4.2.14 would be a topic for further research. We finally call the mapping

$$\pi_{\mathsf{Q}^1}\colon\ \mathsf{Q}^1\mathbb{X} \to \mathbb{X},\ \mathsf{Q}^1_x\mathbb{X} \ni u \longmapsto x$$

a (stratifold) bundle in the sense that the restrictions of $\pi_{\mathsf{Q}^1}$ as

$$\mathsf{Q}^1\mathbb{X}_\oslash = \mathsf{Q}^1\mathbb{X}|_{\mathsf{U}\mathbb{X}_\oslash} \to \mathbb{X}_\oslash \qquad \text{and} \qquad \mathsf{Q}^1\mathbb{X}_\oslash^\mathsf{c} = \mathsf{Q}^1\mathbb{X}|_{\mathsf{U}\mathbb{X}_\oslash^\mathsf{c}} \to \mathbb{X}_\oslash^\mathsf{c} \tag{4.5}$$

to the nonempty strata $\mathbb{X}_\oslash$ and $\mathbb{X}_\oslash^\mathsf{c}$ are stratifold bundles. The former part of the assertion was addressed in the previous lemma, but the latter part follows directly from ($\oslash$ 2). Indeed, as $\mathbb{X}_\oslash^\mathsf{c}$ consists of finitely many isolated points the tangent bundle over the manifold $\left(\mathbb{X}_{\oslash}^{\mathsf{c}}, C^\infty(\mathbb{X}_{\oslash}^{\mathsf{c}}) = \mathbf{C}(\oslash)|_{\mathbb{X}_{\oslash}^{\mathsf{c}}}\right)$ is just $\mathsf{T}\mathbb{X}_{\oslash}^{\mathsf{c}} = \bigsqcup_{x\in\mathbb{X}_{\oslash}^{\mathsf{c}}}\{0\}$. Hence, $\mathsf{U}\mathbb{X}_{\oslash}^{\mathsf{c}} = \emptyset$, meaning that the second mapping in Equation (4.5) has empty domain: $\mathsf{Q}^1\mathbb{X}_{\oslash}^{\mathsf{c}} = \emptyset$. The first stratifold bundle in Equation (4.5) is trivial in the sense that there is an isomorphism of stratifolds

$$\Upsilon_{\oslash} \colon\ \mathsf{Q}^1\mathbb{X}_{\oslash} \to \mathbb{X}_{\oslash} \times (\mathbb{S}^{d-1}, \mathbf{C})$$

that rotates the fibres $\mathsf{Q}^1_x\mathbb{X}_{\oslash}$ with singularities $\pm\oslash(x)$ to align with the strata of $(\mathbb{S}^{d-1}, \mathbf{C})$. In particular, we now know the dimensions of the possibly interesting strata of $\mathsf{Q}^1\mathbb{X}$: The nonempty strata are

$$\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X}) \simeq_{\Upsilon_{\oslash}} \mathbb{X}_{\oslash} \times \mathbb{S}^{d-1}_{\mathsf{top}} \quad \text{and} \quad \mathsf{strat}_{d}(\mathsf{Q}^1\mathbb{X}) \simeq_{\Upsilon_{\oslash}} \mathbb{X}_{\oslash} \times \{\pm e_d\}.$$

Similarly, $\pi_{\mathsf{Q}^0} \colon\ \mathsf{Q}^0\mathbb{X} \to \mathbb{X}$ is understood as a stratifold bundle in the sense that the restrictions to $\mathsf{U}\mathbb{X}_{\oslash}$ and $\mathsf{U}\mathbb{X}_{\oslash}^{\mathsf{c}} = \emptyset$ respectively yield stratifold bundles analogously to Equation (4.5). Then, we have $\mathsf{Q}^0\mathbb{X}_{\oslash}^{\mathsf{c}} = \emptyset$ and the nonempty strata are

$$\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X}) \simeq \mathbb{X}_{\oslash} \times \mathbb{S}^{d-2} \quad \text{and} \quad \mathsf{strat}_{d}(\mathsf{Q}^0\mathbb{X}) \simeq \mathbb{X}_{\oslash} \times \{\pm e_d\}.$$

In the stratifold sense, the Riemannian metric $\mathsf{x}$ on the manifold $\left(\mathbb{X}_{\oslash}, \mathbf{C}(\oslash)|_{\mathbb{X}_{\oslash}}\right) = (\mathbb{X}_{\oslash}, C^\infty(\mathbb{X}_{\oslash}))$ extends smoothly to the stratifold $(\mathbb{X}, \mathbf{C}(\oslash))$ by zero. As we did for the round metric on the stratified sphere, we can justify this in terms of approximate charts: Consider local coordinates $(x^j)_{j=1}^d$ for the manifold $\mathbb{X}$ such that the chart domain contains at least one singularity. For every singular point $o \in \mathbb{X}_{\oslash}^{\mathsf{c}}$ we choose cut-off functions $(\varphi_{o;k})_k$ that are constantly 1 on $\mathbb{U}(o, 1/k)$ and 0 outside of $\mathbb{U}(o, 2/k)$, where the indices $k$ are some large natural numbers. Define for each $k$ approximate local coordinates $\left(x_k^j\right)_{j=1}^d$ by

$$x_k^j := x^j \cdot \prod_{o\in\mathbb{X}_{\oslash}^{\mathsf{c}}} (1 - \varphi_{o;k}) \in \mathbf{C}(\oslash) \qquad \text{for all } j \in \{1, \dots, d\}.$$

Then, we get that $\mathsf{x}_o\left(\partial x_k^i, \partial x_k^j\right) = 0 \longrightarrow 0$ as $k \to \infty$ for all $i, j \in \{1, \dots, d\}$. We denote the smooth extension of $\mathsf{x}$ on $(\mathbb{X}_{\oslash}, C^\infty(\mathbb{X}_{\oslash}))$ to $(\mathbb{X}, \mathbf{C}(\oslash))$ by $\mathsf{x}^1$.

Again, local coordinate expressions in terms of $\mathsf{x}$ are still valid ‘far away’ from singularities, meaning in a large subdomains of approximate charts. One of the consequences is that the stratifold $\mathbb{X}$ is still complete as a length space, i. e. with respect to the distance metric induced by measuring the length of connecting paths in terms of $\mathsf{x}^1$, but for two given points there might not be a minimising geodesic connecting them, since geodesics with respect to $\mathsf{x}^1$ do not pass through singularities. Furthermore, we have $\lambda_{\mathsf{x}^1} = \lambda_{\mathsf{x}}$, since $\mathbb{X}_{\oslash}^{\mathsf{c}}$ is a $\lambda_{\mathsf{x}}$-null set. We also use the notation $\mathsf{x}^1$ for the restriction of $\mathsf{x}$ on $(\mathbb{X}_{\oslash}, C^\infty(\mathbb{X}_{\oslash}))$ in order to emphasise the different differential structure, even in places where it would not be strictly necessary.

With all this work done, we can enjoy how the several pieces finally come together to form the configuration stratifolds. First for the top stratum as the most important part: On $\mathsf{Q}^1\mathbb{X}_{\oslash}$ the measure $\lambda_{\mathsf{x}^1} \otimes_{\mathsf{loc}} \nu^1 = \left(\exp(-\Psi)\,\lambda_{\mathsf{x}^1}\right) \otimes_{\mathsf{loc}} \mathsf{S}_1^{d-1}$ can be defined as the pushforward measure with respect to $\Upsilon_{\oslash}^{-1}$ of the actual product measure $\lambda_{\mathsf{x}^1} \otimes \mathsf{S}_1^{d-1}$, in formulae:

$$\mu^1 = \lambda_{\mathsf{x}^1} \otimes_{\mathsf{loc}} \nu^1 := \left(\Upsilon_{\oslash}^{-1}\right)_* \left(\lambda_{\mathsf{x}^1} \otimes \mathsf{S}_1^{d-1}\right).$$

We keep the notation ‘$\otimes_{\mathsf{loc}}$’ albeit the construction is substantially different to Definition 1.2.19. Similarly, we can define the fibre weighted measures

$$\begin{aligned}\mu^\varepsilon &= \lambda_{\mathsf{x}^1} \otimes_{\mathsf{loc}} \nu^\varepsilon \\ &:= \left(\Upsilon_{\oslash}^{-1}\right)_* \left(\lambda_{\mathsf{x}^1} \otimes \nu^\varepsilon\right) = \left(\Upsilon_{\oslash}^{-1}\right)_* \left(\lambda_{\mathsf{x}^1} \otimes \left(\rho_{\mathbb{S}}^\varepsilon\, \mathsf{S}_1^{d-1}\right)\right) \quad \text{for all } \varepsilon \in (0,1) \text{ on } \mathsf{Q}^1\mathbb{X}_{\oslash}\end{aligned}$$

which are absolutely continuous with respect to $\lambda_{\mathsf{x}^1} \otimes_{\mathsf{loc}} \nu^1$ with Radon-Nikodým-derivative $\rho^\varepsilon := {d\mu^\varepsilon}/{d\nu^1}$. The pushforward with respect to $\Upsilon_{\oslash}$ of the unit Sasaki metric restricted to $\mathsf{Q}^1\mathbb{X}_{\oslash}$ trivialises as

$$\left(\Upsilon_{\oslash}\right)_* \mathsf{u}^1 = \left(\Upsilon_{\oslash}\right)_* (\mathsf{v}^1|_{\mathsf{U}} + \mathsf{h}^1) = \mathsf{s}^1 + \mathsf{x}^1.$$

As for $\mathsf{x}$ and $\mathsf{x}^1$, the upper index 1 just signifies that we are working with the differential structures differing from the standard differentiable structures of the involved manifolds. One defines $\mathbf{u}^1 = \mathsf{v}^1|_{\mathsf{U}} + \mathbf{h}^1 :=$

$(\Upsilon_{\oslash}^{-1})_*(\mathsf{s}^1 + \mathbf{x}^1)$ on $\mathsf{Q}^1\mathbb{X}_{\oslash}$. Note that $\mu^1$ on $\mathsf{Q}^1\mathbb{X}_{\oslash}$ coincides with the volume measure $\lambda_{\mathbf{u}^1}$. Unsurprisingly, weighted versions of yonder restricted unit Sasaki metric $\mathbf{u}^1$ are straightforwardly defined as the pushforward of the weighted product metric with respect to $\Upsilon_{\oslash}^{-1}$:

$$\mathbf{u}^\varepsilon = \mathbf{v}^\varepsilon|_{\mathsf{U}} + \mathbf{h}^1 := (\Upsilon_{\oslash}^{-1})_*(\mathbf{s}^\varepsilon + \mathbf{x}^1) = (\Upsilon_{\oslash}^{-1})_*(\rho_{\mathbb{S}}^\varepsilon\, \mathsf{s}^1 + \mathbf{x}^1) \qquad \text{for all } \varepsilon \in (0,1).$$

The associated distance metric $\mathsf{dist}_{\mathbf{u}^\varepsilon}$ coincides with $(\Upsilon_{\oslash}^{-1})_*\big(\mathsf{dist}_{\mathbf{s};\varepsilon} + \mathsf{dist}_{\mathbf{x}^1}\big)$, where $\mathsf{dist}_{\mathbf{s};\varepsilon}$ is either constructed as $\mathsf{dist}_{\mathbf{s};1/n}$ in Section 4.2.1 directly or the distance metric induced by $\mathbf{s}^\varepsilon$. Observe that for $\varepsilon \in (0,1]$ the tangent spaces of the strata split in Ehresmannian fashion as

$$\begin{aligned} \mathsf{T}(\mathsf{strat}_{2d-1}(\mathsf{Q}^\varepsilon\mathbb{X})) &= \mathsf{V}(\mathsf{strat}_{2d-1}(\mathsf{Q}^\varepsilon\mathbb{X})) \oplus \mathsf{H}(\mathsf{strat}_{2d-1}(\mathsf{Q}^\varepsilon\mathbb{X})) \\ &\simeq_{\Upsilon_{\oslash}} \mathsf{T}\mathbb{S}^{d-1}_{\mathsf{top}} \oplus \mathsf{T}\mathbb{X}_{\oslash} \\ \text{and} \quad \mathsf{T}(\mathsf{strat}_{d}(\mathsf{Q}^\varepsilon\mathbb{X})) &= \mathsf{V}(\mathsf{strat}_{d}(\mathsf{Q}^\varepsilon\mathbb{X})) \oplus \mathsf{H}(\mathsf{strat}_{d}(\mathsf{Q}^\varepsilon\mathbb{X})) \\ &\simeq_{\Upsilon_{\oslash}} \mathsf{T}\{\pm e_d\} \oplus \mathsf{T}\mathbb{X}_{\oslash}. \end{aligned}$$

In the same vein as before, we construct a measure $\mu^0$ and a Riemannian metric $\mathbf{u}^0$ on the preimage $\Upsilon_{\oslash}^{-1}(\mathbb{X}_{\oslash} \times \cdot\mathbb{S}\cdot) \subseteq \mathsf{Q}^1\mathbb{X}_{\oslash}$. The resulting space is then denoted by $\mathsf{Q}^0\mathbb{X}_{\oslash}$. Clearly, we choose $\mu^0 = \lambda_{\mathbf{x}^1} \otimes_{\mathsf{loc}} \nu^0 := (\Upsilon_{\oslash}^{-1})_*\big(\lambda_{\mathbf{x}^1} \otimes \nu^0\big)$ and denote by $\nu_x^0$ the fibre measure $\big(\Upsilon_{\oslash}^{-1}(x,\cdot)\big)_*\nu^0$ on $\Upsilon_{\oslash}^{-1}(\{x\} \times \cdot\mathbb{S}\cdot)$. Endow the set $\mathsf{Q}^0\mathbb{X}_{\oslash}$ with the topology induced from the pushforward distance metric $(\Upsilon_{\oslash}^{-1})_*\big(\mathsf{dist}_{\mathbf{x}^1} + \mathsf{dist}_{\cdot\mathbb{S}\cdot}\big)$. This metric measure space is supplemented with the differential structure $\mathbf{C}(\mathsf{Q}^0\mathbb{X}) \subseteq C^0(\mathsf{Q}^0\mathbb{X}_{\oslash})$ defined by demanding $f \in \mathbf{C}(\mathsf{Q}^0\mathbb{X})$ if and only if there is a function $g \in C^\infty(\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X}))$ such that $f = g$ on the set $\Upsilon_{\oslash}^{-1}(\mathbb{X}_{\oslash} \times \mathbb{S}^{d-2})$. Then, $(\mathsf{Q}^0\mathbb{X}_{\oslash}, \mathbf{C}(\mathsf{Q}^0\mathbb{X}))$ is a stratifold with nonempty strata

$$\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X}_{\oslash}) \simeq_{\Upsilon_{\oslash}} \mathbb{X}_{\oslash} \times \mathbb{S}^{d-2} \qquad \text{and} \qquad \mathsf{strat}_{d}(\mathsf{Q}^0\mathbb{X}_{\oslash}) \simeq_{\Upsilon_{\oslash}} \mathbb{X}_{\oslash} \times \{\pm e_d\}.$$

Now, one endows $\mathsf{Q}^0\mathbb{X}_{\oslash}$ with the mapping $\mathbf{u}^0 = \mathsf{v}^0|_{\mathsf{U}} + \mathbf{h}^1 := (\Upsilon_{\oslash}^{-1})_*(\mathsf{g} + \mathbf{x}^1)$ that is in fact a smooth Riemannian metric on the top stratum. Note that

- the induced distance metric $\mathsf{dist}_{\mathbf{u}^0}$ coincides with $(\Upsilon_{\oslash}^{-1})_*\big(\mathsf{dist}_{\mathbf{x}^1} + \mathsf{dist}_{\cdot\mathbb{S}\cdot}\big)$, and
- the associated Laplace-Beltrami operator $\Delta_{\mathbf{u}^0}$ coincides with $(\Upsilon_{\oslash}^{-1})_*\big(\Delta_{\mathsf{g}} + \Delta_{\mathbf{x}^1}\big)$ on $\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X}_{\oslash})$.

Furthermore, we get Ehresmannian decompositions with respect to $\mathbf{u}^0$ for the tangent bundles over the strata:

$$\begin{aligned} \mathsf{T}(\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X}_{\oslash})) &= \mathsf{V}(\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X}_{\oslash})) \oplus \mathsf{H}(\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X}_{\oslash})) \\ &\simeq_{\Upsilon_{\oslash}} \mathsf{T}\mathbb{S}^{d-2} \oplus \mathsf{T}\mathbb{X}_{\oslash} \\ \text{and} \qquad \mathsf{T}(\mathsf{strat}_{d}(\mathsf{Q}^0\mathbb{X})) &= \mathsf{V}(\mathsf{strat}_{d}(\mathsf{Q}^0\mathbb{X})) \oplus \mathsf{H}(\mathsf{strat}_{d}(\mathsf{Q}^0\mathbb{X})) \\ &\simeq_{\Upsilon_{\oslash}} \mathsf{T}\{\pm e_d\} \oplus \mathsf{T}\mathbb{X}_{\oslash}. \end{aligned}$$

Turning to matters of convergence, observe that the statement of Corollary 4.2.6 also holds if we replace the sequence $(\nu_n)_{n\in\mathbb{N}}$ therein by $(\nu^\varepsilon)_{\varepsilon\in(0,1]}$, since the only fact that matters there is the assumption of pmG-convergence and that the distance metrics can be constructed from the spherical weight functions. Thence, for all $\bar{u} \in \mathsf{Q}^1\mathbb{X}_{\oslash}$ with $\mathsf{pr}_2 \circ \Upsilon_{\oslash}(\bar{u}) \in \mathbb{S}^{d-2}$ the sequence of pointed metric measure spaces

$$(\mathsf{Q}^\varepsilon\mathbb{X}_{\oslash})_{\varepsilon\in(0,1]} := \big(\mathsf{Q}^1\mathbb{X}_{\oslash}, \mathsf{dist}_{\mathbf{u}^\varepsilon}, \mu^\varepsilon, \bar{u}\big)_{\varepsilon\in(0,1]}$$

converges in pmGH-sense to $\big(\mathsf{Q}^0\mathbb{X}_{\oslash}, \mathsf{dist}_{\mathbf{u}^0}, \mu^0, \bar{u}\big)$ as $\varepsilon \downarrow 0$. In turn, from Corollary 4.2.10 we infer convergence of the spectral structures on $\mathsf{Q}^\varepsilon\mathbb{X}_{\oslash}$ given by the Laplacians $\big(\Delta_{\mathbf{u}^\varepsilon}, D(\Delta_{\mathbf{u}^\varepsilon})\big)$ on $L^2(\mathsf{Q}^\varepsilon\mathbb{X}_{\oslash}; \mu^\varepsilon)$ to the spectral structure on $\mathsf{Q}^0\mathbb{X}_{\oslash}$ given by $\big(\Delta_{\mathbf{u}^0}, D(\Delta_{\mathbf{u}^0})\big)$ on $L^2(\mathsf{Q}^0\mathbb{X}_{\oslash}; \mu^0)$.

We don't have to spent any time dealing with $\mathsf{Q}^\varepsilon\mathbb{X}_{\oslash}^{\mathsf{c}}$ for $\varepsilon \in [0,1]$, which always is empty as so is $\mathsf{U}\mathbb{X}_{\oslash}^{\mathsf{c}}$. Therefore, we write $\mathsf{Q}^\varepsilon\mathbb{X}$ instead of $\mathsf{Q}^\varepsilon\mathbb{X}_{\oslash}$ oftentimes. Of course, if we don't make an assumption like ($\oslash$ 2) that makes $\mathsf{U}\mathbb{X}_{\oslash}^{\mathsf{c}}$ empty, then we have do be more careful. Sometimes we choose to draw attention to the hypothetical case, where $\mathsf{U}\mathbb{X}_{\oslash}^{\mathsf{c}}$ is nonempty, by ostentatiously writing $\mathsf{Q}^\varepsilon\mathbb{X}_{\oslash}$. This should help, if future researchers attempt to extend our results.

In conclusion, we have completely characterised the spaces $\mathsf{Q}^\varepsilon\mathbb{X}$ for $\varepsilon \in (0,1]$ as well as $\mathsf{Q}^0\mathbb{X}$ and the convergence $\mathsf{Q}^\varepsilon\mathbb{X} \longrightarrow \mathsf{Q}^0\mathbb{X}$ as $\varepsilon \downarrow 0$ on geometric level.

## 4.3 ▪ Hypocoercivity and Mosco convergence

In this section, we show hypocoercivity of the anisotropic fibre lay-down models that we derive from fibrewise weighting in the configuration stratifold. The same is done for the corresponding limit model. Furthermore, we will see that the anisotropic models do converge in Mosco sense to the limit model.

Let us very briefly summarise the situation established in Section 4.2.2: We suppose that the manifold $\mathbb{X}$, with $d = \dim(\mathbb{X}) \in \mathbb{N} \setminus \{0,1,2\}$, is not parallelisable and therefore we assume that the compass conditions ($\oslash$) hold. By declaring both sets $\mathbb{X}^{\mathsf{c}}_{\oslash}$ and $\{\pm\oslash(x) \mid x \in \mathbb{X}\}$ to be singular sets, one constructs the stratifold bundle $\pi_{\mathsf{Q}^1}\colon \mathsf{Q}^1\mathbb{X} \to \mathbb{X}$ from the unit tangent bundle $\mathsf{U}\mathbb{X}$. The total space is endowed with a measure $\mu^1$ and a weighted Sasaki-type metric $\mathbf{u}^1$ such that $\mu^1 = \lambda_{\mathbf{u}^1}$. Both objects are constructed primarily on $\mathsf{U}\mathbb{X}_{\oslash}$, since $\mathbb{X}_{\oslash}$ is parallelisable and $\mathsf{U}\mathbb{X}^{\mathsf{c}}_{\oslash}$ is empty. Introducing the fibre weights $\rho^{\varepsilon}_{\mathbb{S}}$ for $\varepsilon \in (0,1]$ results in the spaces $\mathsf{Q}^{\varepsilon}\,\mathbb{X}$ carrying the measure $\mu^{\varepsilon}$ and the metric $\mathbf{u}^{\varepsilon}$ respectively with $\mu^{\varepsilon} = \lambda_{\mathbf{u}^{\varepsilon}}$. The spaces $\mathsf{Q}^{\varepsilon}\mathbb{X}$, $\varepsilon \in (0,1]$, are endowed with the distance metric corresponding to $\mathbf{u}^{\varepsilon}$ as well as the spectral structure given by $\big(\Delta_{\mathbf{u}^{\varepsilon}}, D(\Delta_{\mathbf{u}^{\varepsilon}})\big)$ on $H^{\varepsilon} := L^2(\mathsf{Q}^{\varepsilon}\mathbb{X};\mu^{\varepsilon})$. Moreover, the equatorial tangent bundle $\pi_{\mathsf{Q}^0}\colon \mathsf{Q}^0\mathbb{X} \to \mathbb{X}$ is constructed as stratifold bundle with measure $\mu^0$, metric $\mathbf{u}^0$, distance metric corresponding to $\mathbf{u}^0$, and spectral structure given by $\big(\Delta_{\mathbf{u}^0}, D(\Delta_{\mathbf{u}^0})\big)$ on $H^0 := L^2(\mathsf{Q}^0\mathbb{X};\mu^0)$. We have seen pmGH-convergence and spectral convergence as $\varepsilon \downarrow 0$. Henceforth, we assume that the weighted volume measure $\lambda_{\mathbf{x}^1} = \exp(-\Psi)\lambda_{\mathsf{x}^1}$ on $\mathbb{X}$ is a probability measure, thus so are $\mu^{\varepsilon}$ and $\mu^0$.

### 4.3.1 ▪ Hypocoercivity under fibrewise scaling

Let us turn to the anisotropic adaptations of the spherical velocity Langevin equation. By passing over to a configuration stratifold bundle, we even get for $\varepsilon = 1$ an anisotropic model, which we treat first. For $\mathsf{Q}^1\mathbb{X}_{\oslash}$ we consider *almost* the situation of Section 2.2 and the proofs of data conditions (D) as well as hypocoercivity conditions (H) translate with a few notable changes, that we are discuss now. The role of smooth functions is played now by $\mathbf{C}(\mathsf{Q}^1\mathbb{X}_{\oslash}) = \mathbf{C}(\mathsf{Q}^1\mathbb{X})|_{\mathsf{Q}^1\mathbb{X}_{\oslash}}$ replacing the standard differential structure of $\mathsf{U}\mathbb{X}_{\oslash}$. Thus, one should be aware that the natural core domain $D^1_{\oslash}$ of smooth functions with compact support reads as

$$D^1_{\oslash} := \mathbf{C}_{\mathsf{c}}(\mathsf{Q}^1\mathbb{X}_{\oslash}) = \Big\{ f \in \mathbf{C}(\mathsf{Q}^1\mathbb{X})|_{\mathsf{Q}^1\mathbb{X}_{\oslash}} \;\Big|\; (\mathrm{supp}(f) \text{ is compact.}) \Big\}$$

In fact, $\mathbf{C}_{\mathsf{c}}(\mathsf{Q}^1\mathbb{X}_{\oslash})$ is dense in $L^2(\mathsf{Q}^1\mathbb{X}_{\oslash};\mu^1) = L^2(\mathsf{U}\mathbb{X}_{\oslash};\lambda_{\mathsf{x}} \otimes_{\mathsf{loc}} \nu^1)$. One infers this from the two facts that $C^{\infty}_{\mathsf{c}}(\mathrm{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X}))$ is dense in $L^2(\mathrm{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X});\mu^1)$ and $\mathrm{strat}_{d}(\mathsf{Q}^1\mathbb{X})$ is a $\mu^1$-null set. Due to the presence of a global trivialisation, we obtain a dense subset $D^1_{\oslash;0} \subseteq D^1_{\oslash}$ as the span of the pushforward of $C^{\infty}_{\mathsf{c}}(\mathbb{X}_{\oslash}) \otimes \mathbf{C}$ with respect to the mapping $\Upsilon^{-1}_{\oslash}$. In fact, working on $D^1_{\oslash;0} := \mathrm{span}(\Upsilon^{-1}_{\oslash})_*(C^{\infty}_{\mathsf{c}}(\mathbb{X}_{\oslash}) \otimes \mathbf{C})$ is more comfortable than on the span of tensor products of vertical and horizontal lifts. E. g. suppose $f = f_0 \otimes f_1$ with $f_0 \in C^{\infty}_{\mathsf{c}}(\mathbb{X}_{\oslash})$ and $f_1 \in \mathbf{C}$, then

$$\begin{aligned}\nabla_{\mathsf{u}^1}\big((\Upsilon^{-1}_{\oslash})_* f\big) &= \nabla_{\mathsf{v}^1|_{\mathsf{U}}}\big((\Upsilon^{-1}_{\oslash})_* f\big) + \nabla_{\mathsf{h}^1}\big((\Upsilon^{-1}_{\oslash})_* f\big)\\ &= \nabla_{\mathsf{v}^1|_{\mathsf{U}}}(f_1 \circ \mathrm{pr}_2) + \nabla_{\mathsf{h}^1}(f_0 \circ \mathrm{pr}_1) = (\Upsilon^{-1}_{\oslash})_*\big(\nabla_{\mathsf{s}^1} f_1 + \nabla_{\mathsf{x}^1} f_0\big).\end{aligned}$$

Now, we study the following Stratonovich SDE on the top stratum $\mathrm{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X}_{\oslash})$, which we afterwards extend to an equation on $\mathsf{Q}^1\mathbb{X}_{\oslash}$:

$$\mathrm{d}\eta = \mathcal{H}_{\mathsf{x}^1}\,\mathrm{d}t - \frac{1}{d-2}\,\nabla_{\mathsf{v}^1|_{\mathsf{U}}}\big(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\big)\,\mathrm{d}t + \sigma \cdot \sum_{j=1}^{d} \mathrm{tl}_{\eta}\big(v^j_{\eta}\big) \circ \mathrm{d}W^j_t, \tag{4.6}$$

where $\beta \in (0,\infty)$ is such that $\alpha/\beta = \sigma^2/2$. Here, we consider normal coordinates $\big(x^j_{\eta}\big)_{j\in\{1,\dots,d\}}$ at $\pi_{\mathsf{Q}^1}(\eta_t)$ and time $t$ for the smooth manifold $(\mathbb{X}_{\oslash}, C^{\infty}(\mathbb{X}_{\oslash}))$. Since the tangential lifts of $(\partial x^j)_{j\in\{1,\dots,d\}}$ might not lie in the tangent space of $\mathrm{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X})$ at $\eta_t$, we construct $\big(v^j_{\eta}\big)_{j\in\{1,\dots,d\}}$ as approximate chart derived from $(\partial x^j)_{j\in\{1,\dots,d\}}$, compare to Example 4.2.7, and restrict its domain such that the coordinate functions are smooth on $\mathrm{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X})$. The sharp sighted reader already noticed the factor $1/(d-2)$, in contrast to $1/(d-1)$ as in Section 2.2. That is a consequence of Lemma 4.2.11, which we discuss in a moment.

The Kolmogorov backwards generator associated to Equation (4.6) reads as

$$L^1 = \frac{\alpha}{\beta}\,\Delta_{\mathsf{v}^1|_{\mathsf{U}}} + \mathcal{H}_{\mathsf{x}^1} := \frac{\sigma^2}{2}\,\Delta_{\mathsf{v}^1|_{\mathsf{U}}} + \mathcal{H}_{\mathsf{x}^1} - \frac{1}{d-2}\,\nabla_{\mathsf{v}^1|_{\mathsf{U}}}\big(\mathcal{H}_{\mathsf{x}}\Psi^{\mathsf{v}}\big). \tag{4.7}$$

We realise the reason for the factor $1/(d-2)$ in the context of the SAD-decomposition. First, the symmetric part $\left(\frac{\alpha}{\beta}\,\Delta_{\mathsf{v}^1|\mathsf{U}}, C_{\mathsf{c}}^\infty(\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X}_{\circledcirc}))\right)$ is essentially self-adjoint and nonpositive definite on the Hilbert space $L^2(\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X}_{\circledcirc});\mu^1)$, as it is the Laplace-Beltrami operator on the Riemannian manifold $(\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X}_{\circledcirc}), \mathsf{v}^1|\mathsf{U})$. Second, we want to show that $\left(-\mathcal{H}_{\mathsf{x}^1}, C_{\mathsf{c}}^\infty(\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X}_{\circledcirc}))\right)$ is an antisymmetric operator on that same Hilbert space $L^2(\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X}_{\circledcirc});\mu^1)$. Indeed, one observes that

$$\begin{aligned}\left(\nabla_{\mathsf{v}^1|\mathsf{U}}\left(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\right)\right)^* &= -\nabla_{\mathsf{v}^1|\mathsf{U}}\left(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\right) - \Delta_{\mathsf{v}^1|\mathsf{U}}\left(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\right)\\ &= -\nabla_{\mathsf{v}^1|\mathsf{U}}\left(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\right) + (d-2)\cdot\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}.\end{aligned} \tag{4.8}$$

We arrive at step (4.8) by adapting Lemma 2.2.3 using the eigenvalue information from Lemma 4.2.11: If we apply the Laplacian $\Delta_{\mathsf{s}^1}$ componentwise to the identity $\mathsf{Id}_{\mathbb{S}^{d-1}_{\mathrm{top}}}$, we end up with

$$\Delta_{\mathsf{s}^1}\mathsf{Id}_{\mathbb{S}^{d-1}_{\mathrm{top}}} = -(d-2)\,\mathsf{Id}_{\mathbb{S}^{d-1}_{\mathrm{top}}} \qquad \text{by Lemma 4.2.11},$$

instead of Equation (2.15). Then, this implies that $\Delta_{\mathsf{s}^1}\mathcal{H}_{\mathsf{x}^1} = -(d-2)\,\mathcal{H}_{\mathsf{x}^1}$ justifying step (4.8). Hence, we conclude antisymmetry of $\left(-\mathcal{H}_{\mathsf{x}^1}, C_{\mathsf{c}}^\infty(\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X}_{\circledcirc}))\right)$ as claimed. Note that this whole argument is linked just to the geometry of the fibres and not to the regularity of the compass.

Even though the remaining nonempty stratum $\mathsf{strat}_d(\mathsf{Q}^1\mathbb{X})$ is a $\mu^1$-null set and therefore doesn't matter for the AHHM, we think about how to extend the involved vector fields and differential operators to $\mathsf{strat}_d(\mathsf{Q}^1\mathbb{X})$. In that geometric sense, both SDE (4.6) and associated generator extend to the entire configuration stratifold. Obviously, vertical vector fields on $\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X})$ extend to $\mathsf{Q}^1\mathbb{X}_{\circledcirc}$ by zero, as the vertical tangent space of $\mathsf{strat}_d(\mathsf{Q}^1\mathbb{X})$ contains only the zero vector. Per se, the semispray $\mathcal{H}_{\mathsf{x}^1}$ is defined on the entire tangent space of the Riemannian manifold $(\mathbb{X}_{\circledcirc}, \mathsf{x}^1)$ and is purely horizontal. All our fibrewise manipulations affect just the vertical tangent spaces, thus we can just take $\mathcal{H}_{\mathsf{x}^1}$ as already defined on $\mathsf{strat}_d(\mathsf{Q}^1\mathbb{X})$. Thus, we developed geometric meaning to viewing both the SDE (4.6) and the generator of the form (4.7) as given on $\mathsf{Q}^1\mathbb{X}_{\circledcirc}$. We call them *$\circledcirc$-anisotropical fibre lay-down model* and *$\circledcirc$-anisotropical fibre lay-down generator* respectively.[12] As announced before, we write $\mathsf{Q}^1\mathbb{X}$ instead of $\mathsf{Q}^1\mathbb{X}_{\circledcirc}$ from now on, since $\mathsf{Q}^1\mathbb{X}^{\mathsf{c}}_{\circledcirc}$ is empty and a formal distinction between $\mathsf{Q}^1\mathbb{X}$ and $\mathsf{Q}^1\mathbb{X}_{\circledcirc}$ serves no purpose for what follows.

All things considered, the AHHM applies to the $\circledcirc$-anisotropical fibre lay-down equation on $\mathsf{Q}^1\mathbb{X}$ similar to Section 2.2, but with the discussed changes. In particular, essential m-dissipativity of the generator is achieved via the by now well-known three step procedure. We touch upon the Hörmander condition to make the previous statement lucid: Consider $u \in \mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X})$ and local coordinates $(x^j)_{j=1}^{d}$ at $\pi_{\mathsf{Q}^1}(u)$. Construct the local coordinates $(v^j)_{j=1}^{d}$ at $u$ from $(\partial x^j)_{j=1}^{d}$ in terms of approximate charts such that every coordinate function $v^j$ is in $C^\infty(\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X})) = \mathbf{C}(\mathsf{Q}^1\mathbb{X})|_{\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X})}$. Then, one gets

$$\dim\left(\mathsf{Lie}_u\left(\mathcal{H}_{\mathsf{x}^1}, \mathsf{tl}\left(\nabla_{\mathsf{x}^1}\Psi\right), \mathsf{tl}\,v^1, \dots, \mathsf{tl}\,v^d\right)\right) = \dim(\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X})) = 2d-1.$$

Thus, the Hörmander condition is satisfied on $\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X})$. For $\mathsf{strat}_d(\mathsf{Q}^1\mathbb{X})$ we just have the values of $\mathcal{H}_{\mathsf{x}^1}$ at hand, but that does not matter too much, as $\mathsf{strat}_d(\mathsf{Q}^1\mathbb{X})$ is a $\mu^1$-null set. The rest of the essential m-dissipativity argument runs through now. Thence, we obtain a hypocoercive semigroup $T^1 = (T^1_t)_{t\in[0,\infty)}$ on $H^1 = L^2(\mathsf{Q}^1\mathbb{X};\mu^1)$ as summarised in the following theorem.

**Theorem 4.3.1** (hypocoercivity of $\circledcirc$-anisotropic fibre lay-down dynamic)**.** Consider the compass $\circledcirc$ fulfilling the conditions ($\circledcirc$) on the Riemannian manifold $(\mathbb{X}, \mathsf{x})$ satisfying (M) with $d = \dim(\mathbb{X}) \in \mathbb{N}\setminus\{0,1,2\}$. Assume that the conditions (P) on the potential $\Psi$ hold. Construct the Riemannian stratifold $(\mathbb{X}, \mathbf{C}(\circledcirc), \mathsf{x}^1)$ and the configuration stratifold bundle $\mathsf{Q}^1\mathbb{X}$ via replacing the standard fibre of the unit tangent bundle by the stratifold $(\mathbb{S}^{d-1}, \mathbf{C})$. Then, the data conditions (D) and the hypocoercivity conditions (H) hold for the $\circledcirc$-anisotropic fibre lay-down generator $L^1$ on the model Hilbert space $H^1$ with constants $\Lambda^1_m = (d-1)\,\sigma^2/2$, $\Lambda^1_M = 1/d\,\Lambda$, $c^1_1 = (d-2)\,\sigma^2/2$, and $c^1_2 = c_\Psi$ depending only on $\Psi$. By the Hypocoercivity Theorem, the semigroup $T^1$ associated to $L^1$ is hypocoercive: For all $g \in H^1$ and times $t \in [0,\infty)$ holds

$$\left\|T^1_t g - (g\,,1)_{H^1}\right\|_{H^1} \le \kappa^1_1 \mathrm{e}^{-\kappa^1_2 t}\left\|g - (g\,,1)_{H^1}\right\|_{H^1},$$

[12] An even longer name for the equation would be *$\circledcirc$-anisotropical spherical velocity Langevin equation*, that on the upside is not related to a specific application.

where the choice of $\kappa_1^1 \in (1,\infty)$ determines $\kappa_2^1$ as

$$\kappa_2^1 = \frac{\kappa_1^1 - 1}{\kappa_1^1} \frac{\sigma^2}{n_1^1 + n_2^1\sigma^2 + n_3^1\sigma^4}$$

where $n_i^1 \in (0,\infty)$ for $i \in \{1,2,3\}$ are computable in terms of $\Lambda_m^1$, $\Lambda_M^1$, $c_1^1$, and $c_2^1$. ⌟

Next, we obtain more anisotropic equations via weighting $\mathsf{Q}^1\mathbb{X}$ by $\rho^\varepsilon = 1 \otimes_{\mathsf{loc}} \rho_{\mathbb{S}}^\varepsilon := {}^{d\mu^\varepsilon}/_{d\mu^1}$ for $\varepsilon \in (0,1)$: We make the following additional assumption on $\rho_{\mathbb{S}}^\varepsilon$ which ensures that the weighted vertical Laplace-Beltrami $\Delta_{\mathbf{v}^1|_{\mathsf{U}}}$ attains a nice form.

**Assumption 4.3.2** (logarithmic derivative). For all $\varepsilon \in (0,1)$ there is a smooth function $\psi_{\mathbb{S}}^\varepsilon$ on $\mathbb{S}_{\mathsf{top}}^{d-1}$ and some $\beta \in (0,\infty)$ such that ${}^1/_{\rho_{\mathbb{S}}^\varepsilon} \nabla_{\mathsf{s}} \rho_{\mathbb{S}}^\varepsilon = -{}^1/_\beta \nabla_{\mathsf{s}} \psi_{\mathbb{S}}^\varepsilon$ holds on $\mathbb{S}_{\mathsf{top}}^{d-1}$. ⌟

Note that Assumption 4.3.2 is satisfied for the explicit densities from Section 4.2.1. Under Assumption 4.3.2 we get the form

$$\Delta_{\mathbf{v}^\varepsilon|_{\mathsf{U}}} = \frac{1}{\rho^\varepsilon} \operatorname{div}_{\mathsf{v}^1|_{\mathsf{U}}}\left(\rho^\varepsilon \nabla_{\mathsf{v}^1|_{\mathsf{U}}}\right) = \Delta_{\mathsf{v}^1|_{\mathsf{U}}} - \frac{1}{\beta} \cdot \nabla_{\mathsf{v}^1|_{\mathsf{U}}} \psi_{\mathbb{S}}^\varepsilon$$

on the restriction of $\mathbf{C}(\mathsf{Q}^1\mathbb{X})$ to the top stratum, where $\nabla_{\mathsf{v}^1|_{\mathsf{U}}} \psi_{\mathbb{S}}^\varepsilon$ is a short hand for the pushforward $(\Upsilon_{⦸}^{-1})_*\left(\nabla_{\mathsf{s}^1} \psi_{\mathbb{S}}^\varepsilon\right)$, i.e. it holds $\nabla_{\mathsf{v}^1|_{\mathsf{U}}} \psi_{\mathbb{S}}^\varepsilon(f) = \nabla_{\mathsf{s}^1} \psi_{\mathbb{S}}^\varepsilon(f \circ \Upsilon_{⦸}^{-1})$ for all $f \in C^\infty(\mathsf{strat}_{2d-1}(\mathsf{Q}^1\mathbb{X}_{⦸}))$. Then, the ⦸-anisotropic fibre lay-down model in Equation (4.6) on $\mathsf{Q}^1\mathbb{X}$ is transformed into the following SDE on $\mathsf{Q}^\varepsilon\mathbb{X}$:

$$\mathrm{d}\eta = \mathcal{H}_{\mathsf{x}^1}\, \mathrm{d}t - \frac{1}{d-2} \nabla_{\mathsf{v}^1|_{\mathsf{U}}}\left(\mathcal{H}_{\mathsf{x}^1} \Psi^{\mathsf{v}}\right) \mathrm{d}t - \frac{\alpha}{\beta} \cdot \nabla_{\mathsf{v}^1|_{\mathsf{U}}} \psi_{\mathbb{S}}^\varepsilon\, \mathrm{d}t + \sigma \cdot \sum_{j=1}^{d} \mathsf{tl}_\eta\left(v_\eta^j\right) \circ \mathrm{d}W_t^j. \tag{4.9}$$

We refer to Equation (4.9) as *$\rho^\varepsilon$-anisotropic fibre lay-down equation (on $\mathsf{Q}^\varepsilon\mathbb{X}$)* , since $\rho^\varepsilon$ encapsulates the dependency on the compass ⦸ and the weighting density $\rho_{\mathbb{S}}^\varepsilon$. The corresponding Kolmogorov backward generator reads as

$$L^\varepsilon = \frac{\alpha}{\beta} \Delta_{\mathbf{v}^\varepsilon|_{\mathsf{U}}} + \mathcal{H}_{\mathbf{x}^1} = \frac{\sigma^2}{2} \Delta_{\mathsf{v}^1|_{\mathsf{U}}} - \frac{\alpha}{\beta} \cdot \nabla_{\mathsf{v}^1|_{\mathsf{U}}} \psi_{\mathbb{S}}^\varepsilon + \mathcal{H}_{\mathsf{x}^1} - \frac{1}{d-2} \nabla_{\mathsf{v}^1|_{\mathsf{U}}}\left(\mathcal{H}_{\mathsf{x}} \Psi^{\mathsf{v}}\right).$$

Note that $L^\varepsilon f$ is defined for all $f \in \mathbf{C}(\mathsf{Q}^1\mathbb{X})$, but its natural predomain as an operator on $H^\varepsilon = L^2(\mathsf{Q}^\varepsilon\mathbb{X}; \mu^\varepsilon)$ is the space $\mathbf{C}_{\mathsf{c}}(\mathsf{Q}^1\mathbb{X})$ of smooth functions $f$ with compact support. By construction the SAD-decomposition holds with

$$S^\varepsilon := \frac{\alpha}{\beta} \Delta_{\mathbf{v}^\varepsilon|_{\mathsf{U}}}, \quad A^\varepsilon := -\mathcal{H}_{\mathbf{x}^1}, \quad \text{and} \quad D^\varepsilon := \mathbf{C}_{\mathsf{c}}(\mathsf{Q}^1\mathbb{X}).$$

One could translate the proofs for employing the AHHM from Chapter 2. E. g. one constructs from fibrewise averages with respect to $\nu^\varepsilon$ the projections $P^\varepsilon$ as well as $P_S^\varepsilon$ and computes that $P^\varepsilon(A^\varepsilon)^2 P^\varepsilon = {}^1/_d \cdot \Delta_{\mathbf{h}^1} P_S^\varepsilon$ on $D^\varepsilon$. To shake things up a little and for sake of brevity, we pursue an alternative course that only requires existence of semigroups $T^\varepsilon$ generated by $L^\varepsilon$ for all $\varepsilon \in (0,1]$. Again, we employ the three step procedure for essential m-dissipativity used before. In particular, the generator $L^\varepsilon$ satisfies the Hörmander condition, since there is just the additional vertical vector field $\nabla_{\mathsf{v}^1|_{\mathsf{U}}} \psi_{\mathbb{S}}^\varepsilon$ involved, compared to the case of $\varepsilon = 1$.

In order to conclude hypocoercivity for all the semigroups $T^\varepsilon$, we employ a result from [PV20]. In this recently published paper, P. Patie and A. Vaidyanathan looked at so-called *intertwining* semigroups. Two SCCSs $\tau = (\tau_t)_{t\in[0,\infty)}$ and $\tilde{\tau} = (\tilde{\tau}_t)_{t\in[0,\infty)}$ on Hilbert spaces $X$ and $\widetilde{X}$ are said to intertwine if there is a linear bounded so-called *intertwining operator* $V\colon \widetilde{X} \to X$ such that $\tau_t V = V \tilde{\tau}_t$ holds for all $t \in [0,\infty)$. If intertwining operators are additionally required to be bijective, then intertwining forms an equivalence relation for SCCSs on Hilbert spaces. The corresponding equivalence classes are called *similarity orbits*. Now, [PV20, Proposition 2.1] asserts that if one representative of a similarity orbit experiences hypocoercivity, so do all representatives. Beyond just that, the respective hypocoercivity constants can be determined in terms of the respective intertwining operators via the condition number $\kappa_V := \|V\| \cdot \|V^{-1}\|$. Employing this machinery, we obtain the following corollary.

**Corollary 4.3.3** (hypocoercivity of anisotropic fibre lay-down dynamics). and also the configuration stratifolds $(\mathsf{Q}^\varepsilon\mathbb{X})_{\varepsilon\in(0,1]}$ and weighting it by the densities $(\rho^\varepsilon_{\mathbb{S}})_{\varepsilon\in(0,1]}$.

Consider the transformation $U^\varepsilon\colon\ H^\varepsilon \to H^1,\ f \longmapsto \sqrt{\rho^\varepsilon}\cdot f$ for fixed $\varepsilon \in (0,1)$. Note that $U^\varepsilon$ is a unitary operator, thus it has the condition number $\kappa_{U^\varepsilon} = 1$. Furthermore, $U^\varepsilon$ satisfies $T^1_t U^\varepsilon = U^\varepsilon T^\varepsilon_t$. Hence, $T^1$ and $T^\varepsilon$ belong to the same similarity orbit and $T^\varepsilon$ is hypocoercive with constants $\kappa_{U^\varepsilon}\kappa_1 = \kappa_1$ and $\kappa_2$, since $T^1$ is hypocoercive with constants $\kappa_1$ and $\kappa_2$. Compare to [PV20, Proposition 2.1].

In applications one would fix some positive $\varepsilon$ small enough for realistic anisotropic behaviour. The previous corollary guarantees hypocoercivity without any change in the hypocoercivity constants compared to the $\oslash$-anisotropic fibre lay-down model. Note that Corollary 4.3.3 does not yield optimal hypocoercivity constants for the weighted dynamics though. E. g. if one checks for microscopic coercivity (H2) of the $\rho^\varepsilon$-anisotropic fibre lay-down model, then one would fibrewise employ a Poincaré inequality for $\lambda_{\mathsf{s}^\varepsilon}$, wherefore the derived constant $\Lambda^\varepsilon_m$ would differ from $\Lambda^1_m$.

## 4.3.2 ▪ Hypocoercivity of collapsed fibre lay-down models and Mosco convergence

We start this subsection by discussing the collapsed fibre lay-down dynamic on the equatorial tangent bundle $\mathsf{Q}^0\mathbb{X}$. In particular, we still obtain hypocoercivity. Again, the way to go is translating results from [GS14] or respectively Section 2.2. Giving all the details would be repetitive and redundant, but we take the time where it seems necessary. Once more, an important piece of information is the global trivialisation $\Upsilon_\oslash$ of $\mathsf{Q}^1\mathbb{X}$ which restricts to $\Upsilon_\oslash\colon\ \mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X}) \to \mathbb{X}\times\mathbb{S}^{d-2}$. Recall that $\mathbf{u}^0 = \mathsf{v}^0|\mathsf{U} + \mathbf{h}^1 = (\Upsilon_\oslash^{-1})_*(\mathsf{g}+\mathbf{x}^1)$ is a smooth Riemannian metric on the manifold $\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})$ and extends in the stratifold sense smoothly to $\mathsf{Q}^0\mathbb{X}$. As the fibres collapsed as $\varepsilon\downarrow 0$, we need to understand the various vertical objects, whilst the horizontal ones stay unchanged.

Vertical vector fields on $\mathsf{Q}^0\mathbb{X}$, the vertical gradient $\nabla_{\mathsf{v}^0|\mathsf{U}}$, and the vertical Laplacian $\Delta_{\mathsf{v}^0|\mathsf{U}}$ characteristically act on functions of the form

$$f_1 = (\Upsilon_\oslash)_*\big(1\otimes\tilde f_1\big) = \big(1\otimes\tilde f_1\big)\circ\Upsilon_\oslash \qquad \text{with } \tilde f_1\in C^\infty(\mathbb{S}^{d-2})$$

as follows: Suppose a vertical vector field $\mathcal{X}\in\Gamma^\infty(\mathsf{V}(\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})))$, then one computes

$$\mathcal{X}f_1 = \Big\langle \mathcal{X}\,,d\Big(\big(1\otimes\tilde f_1\big)\circ\Upsilon_\oslash\Big)\Big\rangle = \Big\langle d\Upsilon_\oslash(\mathcal{X})\,,d\tilde f_1\Big\rangle.$$

Therefore, the vertical gradient is found to be the pushforward of $\nabla_{\mathsf{g}}\tilde f_1$ by $\Upsilon_\oslash^{-1}$:

$$\mathcal{X}f_1 = \mathsf{v}^0|\mathsf{U}\Big(\mathcal{X}\,,\nabla_{\mathsf{v}^0|\mathsf{U}}f_1\Big) = \mathsf{g}\big(d\Upsilon_\oslash(\mathcal{X})\,,\nabla_{\mathsf{g}}\tilde f_1\big) = \mathsf{v}^0|\mathsf{U}\Big(\mathcal{X}\,,d\Upsilon_\oslash^{-1}\big(\nabla_{\mathsf{g}}\tilde f_1\big)\Big),$$

where $\mathcal{X}$ is a vertical vector field as before. For the Laplace-Beltrami $\Delta_{\mathsf{v}^0|\mathsf{U}}$ we get

$$\Delta_{\mathsf{v}^0|\mathsf{U}}f_1 = \mathrm{div}_{\mathsf{v}^0|\mathsf{U}}\Big(\nabla_{\mathsf{v}^0|\mathsf{U}}f_1\Big) = \mathrm{div}_{\mathsf{v}^0|\mathsf{U}}\Big(d\Upsilon_\oslash^{-1}\big(\nabla_{\mathsf{g}}\tilde f_1\big)\Big) = \Delta_{\mathsf{g}}\tilde f_1\circ\Upsilon_\oslash,$$

since for every vertical vector field $\mathcal{X}$ as before the divergence reads as $\mathrm{div}_{\mathsf{v}^0|\mathsf{U}}\mathcal{X} = \mathrm{div}_{\mathsf{g}}(d\Upsilon_\oslash(\mathcal{X}))$. Note that the set

$$\begin{aligned}&(\Upsilon_\oslash)_*(C^\infty(\mathbb{X}_\oslash)\otimes C^\infty(\mathbb{S}^{d-2}))\\ &= \mathrm{span}\{(f_0\circ\Upsilon_\oslash)\otimes(f_1\circ\Upsilon_\oslash)\ \big|\ f_0\in C^\infty(\mathbb{X}_\oslash), f_1\in C^\infty(\mathbb{S}^{d-2})\}\end{aligned}$$

is dense in $C^\infty(\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X}))$, recall [Hor66, Example 2.4.10] and [Gro+01, Section 3.1.3]. Vertical vector fields, vertical gradient, and vertical Laplacian-Beltrami are extended to $\mathbf{C}(\mathsf{Q}^0\mathbb{X})$ by zero: For all $f_1\in C^\infty(\mathsf{strat}_d(\mathsf{Q}^0\mathbb{X}))$ one naturally defines

$$\mathcal{X}f_1 := 0, \qquad \nabla_{\mathsf{v}^0|\mathsf{U}}f_1 := 0, \qquad \text{and} \qquad \Delta_{\mathsf{v}^0|\mathsf{U}}f_1 = 0,$$

where $\mathcal{X}\in\Gamma^\infty(\mathsf{V}(\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})))$. Due to the fact that

$$\Big(\Delta_{\mathsf{v}^0|\mathsf{U}}, C^\infty(\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X}))\Big) \qquad \text{on } L^2(\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X});\mu^0)$$

is essentially self-adjoint by Theorem 1.2.22, so is the operator $\left(\Delta_{\mathsf{v}^0|_{\mathsf{U}}}, \mathbf{C}(\mathrm{Q}^0\mathbb{X})\right)$ on $H^0$, as $\mathsf{strat}_{d}(\mathrm{Q}^0\mathbb{X})$ is a $\mu^0$-null set.

Now that we have studied vertical gradient and Laplace-Beltrami on $\mathrm{Q}^0\mathbb{X}$, we show that

$$\left(\mathcal{H}_{\mathsf{x}^1} - \frac{1}{d-2}\nabla_{\mathsf{v}^0|_{\mathsf{U}}}\left(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\right),\ C^\infty(\mathsf{strat}_{2d-2}(\mathrm{Q}^0\mathbb{X}))\right) \qquad \text{on } L^2(\mathsf{strat}_{2d-2}(\mathrm{Q}^0\mathbb{X});\mu^0)$$

is antisymmetric. Indeed, the adjoint of $\nabla_{\mathsf{v}^0|_{\mathsf{U}}}\left(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\right)$ with respect to $L^2(\mu^0)$-scalar product amounts to

$$\left(\nabla_{\mathsf{v}^0|_{\mathsf{U}}}\left(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\right)\right)^* = -\nabla_{\mathsf{v}^0|_{\mathsf{U}}}\left(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\right) - \Delta_{\mathsf{v}^0|_{\mathsf{U}}}\left(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\right)$$

and we verify next that $\Delta_{\mathsf{v}^0|_{\mathsf{U}}}\left(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\right) = -(d-2)\,\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}$ holds. That follows from Equation (2.15), since we get similar to Lemma 2.2.3 that

$$\begin{aligned}\mathsf{h}^1_w\left(\Delta_{\mathsf{v}^0|_{\mathsf{U}}}\mathcal{H}_{\mathsf{x}^1}, \mathsf{hl}(w)\right) &= \Delta_{\mathsf{v}^0|_{\mathsf{U}}}\mathsf{h}^1_w\left(\mathcal{H}_{\mathsf{x}^1}, \mathsf{hl}(w)\right) = \Delta_{\mathsf{v}^0|_{\mathsf{U}}}\mathsf{x}^1_x\left(\mathsf{Id}_{\mathrm{Q}^0_x\mathbb{X}\cap\mathsf{strat}_{2d-2}(\mathrm{Q}^0\mathbb{X})}, w\right)\\ &= \mathsf{x}^1_x\left(\Delta_{\mathsf{v}^0|_{\mathsf{U}}}\mathsf{Id}_{\mathrm{Q}^0_x\mathbb{X}\cap\mathsf{strat}_{2d-2}(\mathrm{Q}^0\mathbb{X})}, w\right) = -(d-2)\,\mathsf{h}^1_w\left(\mathcal{H}_{\mathsf{x}^1}, \mathsf{hl}(w)\right)\end{aligned}$$

for fixed $w \in \mathsf{strat}_{2d-2}(\mathrm{Q}^0\mathbb{X})$ with $x := \pi_{\mathrm{Q}^0}(w)$. Hence, we already have the SAD-decomposition for the operator $L^0$ that we now introduce alongside with the collapsed fibre lay-down equation.

We study the Stratonovich SDE on the top stratum $\mathsf{strat}_{2d-2}(\mathrm{Q}^0\mathbb{X})$ of the equatorial tangent bundle:

$$\mathsf{d}\eta = \mathcal{H}_{\mathsf{x}^1}\,\mathsf{d}t - \frac{1}{d-2}\nabla_{\mathsf{v}^0|_{\mathsf{U}}}\left(\mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\right)\mathsf{d}t + \sigma\cdot\sum_{j=1}^{d-1}\mathsf{tl}_\eta\left(\frac{\partial}{\partial x^j_\eta}\right)\circ\mathsf{d}W^j_t, \tag{4.10}$$

where $\left(x^j_\eta\right)_{j=1}^{d}$ are normal coordinates at $\pi_{\mathrm{Q}^0}(\eta)$ such that $\partial/\partial x^d_\eta(\eta) = \oslash(\pi_{\mathrm{Q}^0}(\eta))$. We call Equation (4.10) the *collapsed fibre lay-down equation*. The corresponding Kolmogorov backwards generator is

$$L^0 = \frac{\sigma^2}{2}\Delta_{\mathsf{v}^0|_{\mathsf{U}}} + \mathcal{H}_{\mathsf{x}^1} = \frac{\sigma^2}{2}\Delta_{\mathsf{v}^0|_{\mathsf{U}}} + \mathcal{H}_{\mathsf{x}^1} - \frac{1}{d-2}\nabla_{\mathsf{v}^0|_{\mathsf{U}}}\left(\mathcal{H}^0_{\mathsf{x}}\Psi^{\mathsf{v}}\right), \tag{4.11}$$

which is defined on $\mathbf{C}(\mathrm{Q}^0\mathbb{X})|_{\mathsf{strat}_{2d-2}(\mathrm{Q}^0\mathbb{X})} = C^\infty(\mathsf{strat}_{2d-2}(\mathrm{Q}^0\mathbb{X}))$. As we know how the vertical vector fields and the vertical Laplacian in the equations (4.10) and (4.11) extend to $\mathrm{Q}^0\mathbb{X}$, both the collapsed fibre lay-down equation and its generator also extend to $\mathrm{Q}^0\mathbb{X}$.

In order to get a hypocoercivity result, we quickly run down the essential arguments for the data and hypocoercivity conditions to hold. The model Hilbert space $H^0 = L^2(\mathrm{Q}^0\mathbb{X};\mu^0)$ is already established and we also have proven the SAD-decomposition.

**Corollary 4.3.4** (SAD-decomposition for $L^0$). Consider

$$L^0 = S^0 - A^0 \quad \text{with} \quad S^0 := \frac{\sigma^2}{2}\Delta_{\mathsf{v}^0|_{\mathsf{U}}} \quad \text{and} \quad A^0 := \mathcal{H}_{\mathsf{x}^1} \qquad \text{on} \quad D^0 := \mathbf{C}_{\mathsf{c}}(\mathrm{Q}^0\mathbb{X}).$$

The following assertions hold:

(i) $(S^0, D^0)$ is symmetric and nonpositive definite on $H^0$.

(ii) $(A^0, D^0)$ is antisymmetric on $H^0$.

(iii) For all $f \in D^0$ we have that $L^0 f \in L^1(\mathrm{Q}^0\mathbb{X};\mu^0)$ and $\int_{\mathrm{Q}^0\mathbb{X}} L^0 f\,\mathsf{d}\mu^0 = 0$.

Again, the projections $P^0 := P^0_S - (\cdot\,,1)_{L^2(\mu^0)}$ and $P^0_S$ are derived from the fibrewise average with respect to $\mu^0$ and subsequently vertical lift:

$$\mathsf{E}_{\mu^0}[f]\colon\ \mathbb{X}\to\mathbb{R},\ x\longmapsto\int_{\mathrm{Q}^0_x\mathbb{X}} f\,\mathsf{d}\nu^0_x = \int_{\mathrm{Q}^0_x\mathbb{X}\cap\mathsf{strat}_{2d-2}(\mathrm{Q}^0\mathbb{X})} f\quad\mathsf{d}\nu^0_x$$

$$= \begin{cases} \int_{\mathbb{S}^\cdot} f \circ \Upsilon_{\oslash}^{-1}(x,\cdot) \quad \mathrm{d}\nu^0 = \int_{\mathbb{S}^{d-2}} f \circ \Upsilon_{\oslash}^{-1}(x,\cdot) \quad \mathrm{d}\nu^0 & \text{if } x \in \mathbb{X}_{\oslash} \\ 0 & \text{if } x \in \mathbb{X}_{\oslash}^{\mathsf{c}} \end{cases}$$

and $P_S^0 f := \big(\mathsf{E}_{\mu^0}[f]\big)^{\mathsf{v}} = \mathsf{E}_{\mu^0}[f] \circ \pi_{\mathsf{Q}^0}$ for all $f \in L^1(\mathsf{Q}^0\mathbb{X};\mu^0)$. Then, one computes that

$$A^0 P^0 f(u) = -\mathsf{x}_x^1\Big(\mathsf{Id}_{\mathsf{Q}_x^0\mathbb{X}\cap\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})}\,,\,\nabla_{\mathsf{x}}\big(\mathsf{E}_{\mu^0}[f]\big)(x)\Big) \tag{4.12}$$

for all $f \in D^0$, $u \in \mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})$, and $x := \pi_{\mathsf{Q}^0}(u) \in \mathbb{X}_{\oslash}$. The remaining data conditions can be proven as for the spherical velocity Langevin equation with minor modifications. In particular, $L^0$ really satisfies the Hörmander condition. The argument is almost exactly as in Section 2.2 just with one tangentially lifted vector field less to pair with the semispray. We proceed with the hypocoercivity conditions.

**Lemma 4.3.5** (algebraic relation (H1))**.** It holds $P^0 A^0 P^0 = 0$ on $D^0$.

4

*Proof.* For all $f \in D^0$ and $x \in \mathbb{X}_{\oslash}$ one finds that

$$\begin{aligned}\int_{\mathsf{Q}_x^0\mathbb{X}\cap\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})} A^0P^0 f \quad \mathrm{d}\nu_x^0 &= \int_{\mathbb{S}^\cdot} A^0P^0 f \circ \Upsilon_{\oslash}^{-1}(x,\cdot) \quad \mathrm{d}\nu^0 \\ = \int_{\mathbb{S}^{d-2}} A^0P^0 f\circ\Upsilon_{\oslash}^{-1}(x,\cdot) \quad \mathrm{d}\nu^0 &\overset{(4.12)}{=} \int_{\mathbb{S}^{d-2}} -\mathsf{x}_x\Big(u\,,\,\nabla_{\mathsf{x}}\big(\mathsf{E}_{\mu^0}[f]\big)(x)\Big) \quad \nu^0(\mathrm{d}u) = 0.\end{aligned}$$

□

**Lemma 4.3.6** (microscopic coercivity (H2))**.** Then, condition (H2) holds with constant $\Lambda_m^0 = (d-2)\frac{\sigma^2}{2}$. Compare to Lemma 2.2.9.

*Proof.* Let $f \in D^0$. Then, one estimates

$$\begin{aligned}\big(-S^0 f\,,\,f\big)_{H^0} &= \frac{\sigma^2}{2}\Big\|\nabla_{\mathsf{v}^0|\mathsf{U}} f\Big\|_{H^0}^2 = \frac{\sigma^2}{2}\big\|\nabla_{\mathsf{g}}(f\circ\Upsilon_{\oslash}^{-1})\big\|^2_{L^2(\mathbb{X}_{\oslash}\times\mathbb{S}^{d-2}\to\mathsf{T}\mathbb{X}_{\oslash}\times\mathsf{T}\mathbb{S}^{d-2};\lambda_{\mathsf{x}}\otimes\nu^0)} \\ &\geq \frac{\sigma^2}{2}(d-2)\big\|(\mathsf{Id}_{H^0}-P_S^0)f\big\|_{H^0}^2\end{aligned}$$

using integration by parts and afterwards the Poincaré inequality for the spherical measure, see [Bec89, Theorem 2]. □

We characterise $P^0(A^0)^2P^0$ on $D^0$ now. Let $f \in D^0$ and $x \in \mathbb{X}_{\oslash}$. Then, we calculate as in Equation (2.24) that

$$\begin{aligned}&\int_{\mathsf{Q}_x^0\mathbb{X}\cap\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})} -\mathcal{H}_{\mathsf{x}^1}\big(A^0P^0 f\big) \quad \mathrm{d}\nu_x^0 \\ &= \int_{\mathsf{Q}_x^0\mathbb{X}\cap\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})} \mathsf{x}_x^1\Big(u\,,\,\nabla_{\mathsf{x}^1}\big(\mathsf{E}_{\mu^0}[f]\big)(x)\Big) \quad \nu_x^0(\mathrm{d}u) \\ &\qquad + \int_{\mathsf{Q}_x^0\mathbb{X}\cap\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})} \mathsf{x}_x^1\Big(u\,,\,\nabla_u^{\mathsf{x}^1}\Big(\nabla_{\mathsf{x}^1}\big(\mathsf{E}_{\mu^0}[f]\big)\Big)(x)\Big) \quad \nu_x^0(\mathrm{d}u) \\ &= \int_{\mathbb{S}^{d-2}} \mathsf{x}_x^1\Big(\Upsilon_{\oslash}^{-1}(x,u)\,,\,\nabla_{\mathsf{x}^1}\big(\mathsf{E}_{\mu^0}[f]\big)(x)\Big) \quad \nu^0(\mathrm{d}u) \\ &\qquad + \int_{\mathbb{S}^{d-2}} \mathsf{x}_x^1\Big(\Upsilon_{\oslash}^{-1}(x,u)\,,\,\nabla^{\mathsf{x}^1}_{\Upsilon_{\oslash}^{-1}(x,u)}\Big(\nabla_{\mathsf{x}^1}\big(\mathsf{E}_{\mu^0}[f]\big)\Big)(x)\Big) \quad \nu^0(\mathrm{d}u) \\ &= \frac{1}{d-1}\,\Delta_{\mathsf{x}^1}\big(\mathsf{E}_{\mu^0}[f]\big)(x) = \frac{1}{d-1}\,\Delta_{\mathsf{h}^1}(P_S^0 f)(x).\end{aligned} \tag{4.13}$$

Similar to Equation (2.12) and Equation (2.25), we derive for $v \in \mathsf{Q}_x^0\mathbb{X}\cap\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})$ that

$$\begin{aligned}P_S^0\Big(\frac{1}{d-2}\,\nabla_{\mathsf{v}^0|\mathsf{U}}\big(A^0P^0 f\big)\Big)(v) &= \int_{\mathsf{Q}_x^0\mathbb{X}\cap\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})} \mathcal{H}_{\mathsf{x}^1}\Psi^{\mathsf{v}}\cdot A^0P^0 f \quad \nu_x^0 \\ = -\int_{\mathbb{S}^{d-2}} \mathsf{x}_x^1\Big(\Upsilon_{\oslash}^{-1}(x,u)\,,\,\nabla_{\mathsf{x}^1}\Psi\Big)&\cdot\mathsf{x}_x^1\Big(\Upsilon_{\oslash}^{-1}(x,u)\,,\,\nabla_{\mathsf{x}^1}\mathsf{E}_{\mu^0}[f](x)\Big) \quad \nu^0(\mathrm{d}u) \\ = -\frac{1}{d-1}\,\mathsf{x}_x^1\Big(\nabla_{\mathsf{x}^1}\Psi\,,\,\nabla_{\mathsf{x}^1}\big(\mathsf{E}_{\mu^0}[f]\big)(x)\Big) &= -\frac{1}{d-1}\,\partial_{\nabla_{\mathsf{x}^1}\Psi}\big(\mathsf{E}_{\mu^0}[f]\big)(x).\end{aligned} \tag{4.14}$$

Putting the equations (4.13) and (4.14) together, we conclude that

$$P^0(A^0)^2P^0 f = P^0_S(A^0)^2P^0 f = \frac{1}{d-1}\left(\Delta_{\mathsf{h}^1}(P^0_S f) - \partial_{\nabla_{\mathsf{x}^1}\Psi}(\mathsf{E}_{\mu^0}[f])\circ\pi_{\mathsf{Q}^0}\right) \\ = \frac{1}{d-1}\,\Delta_{\mathsf{h}^1}(P^0_S f)\,. \tag{4.15}$$

Again, we have seen that $\left(P^0(A^0)^2P^0, D^0\right)$ is the weighted horizontal Laplace-Beltrami operator just with the factor $1/(d-1)$ instead of $1/d$ as Section 2.2. In particular, it is an essentially m-dissipative operator on $H^0$ by adapting Corollary 2.1.22.

**Lemma 4.3.7** (macroscopic coercivity (H3)). Assume that $\lambda_{\mathsf{x}^1} = \exp(-\Psi)\lambda_{\mathsf{x}^1}$ satisfies the Poincaré inequality (2.4). Then, condition (H3) is fulfilled with $\Lambda^0_M = 1/(d-1)\cdot\Lambda$.

*Proof.* One has that

$$\|A^0P^0 f\|^2_{H^0} \overset{(4.15)}{=} \frac{1}{d-1}\int_{\mathbb{X}} \left|\nabla_{\mathsf{x}^1}(\mathsf{E}_{\mu^0}[f])(x)\right|^2_{\mathsf{x}^1}\quad\lambda_{\mathsf{x}^1}(\mathrm{d}x) \\ \geq \frac{1}{d-1}\,\Lambda\left\|\mathsf{E}_{\mu^0}[f] - \left(\mathsf{E}_{\mu^0}[f]\,,1\right)_{L^2(\mathbb{X};\mathsf{x}^1)}\right\|^2_{L^2(\mathbb{X};\mathsf{x}^1)} \qquad \text{for all } f\in D^0.$$

This estimate with the fact that $(P^0(A^0)^2P^0, D^0)$ is essentially m-dissipative imply the claim by [GS14, Corollary 2.13]. ☐

Denote by $B^0$ the operator '$B$' from (H4) corresponding to $A^0$ and $P^0$.

**Lemma 4.3.8** (boundedness of $(B^0S^0, D^0)$, first part of (H4)). It holds that

$$\|B^0S^0 f\|_{H^0} \leq c^0_1\|(\mathrm{Id}_{H^0} - P_j)f\|_{H^0} \qquad \text{for all } f\in D$$

and $P_j\in\{P^0, P^0_S\}$, $j\in\{1,2\}$ with $c^0_1 := (d-2)\,\sigma^2/4$.

*Proof.* For $f\in D^0$ one computes that

$$S^0A^0P^0 f = S^0A^0P^0_S f = \frac{\sigma^2}{2}\,\Delta_{\mathsf{v}^0|\mathsf{U}}\left(-\mathcal{H}_{\mathsf{x}^1}P^0_S f\right) \\ = \frac{\sigma^2}{2}\,(d-2)\cdot\mathcal{H}_{\mathsf{x}^1}P^0_S f = -\frac{\sigma^2}{2}\,(d-2)A^0P^0 f$$

and the claim follows with [GS14, Lemma 2.14]. Compare to Lemma 2.2.11. ☐

The second boundedness assertion of (H4) is proven as in Lemma 2.2.12. All in all, the operator $(L^0, D(L^0))$ generates a hypocoercive semigroup $T^0$ by the Hypocoercivity Theorem, which we summarise in the following theorem.

**Theorem 4.3.9** (hypocoercivity of collapsed fibre lay-down dynamic). Consider the Riemannian manifold $(\mathbb{X},\mathsf{x})$, the potential $\Psi$, the compass ⊘ as in Theorem 4.3.1. Construct the Riemannian stratifold $(\mathbb{X},\mathbf{C}(⊘),\mathsf{x}^1)$ and the equatorial tangent bundle $\mathsf{Q}^0\mathbb{X}$ as configuration stratifold bundle. Then, the collapsed fibre lay-down generator $(L^0, D^0) = (L^0,\mathbf{C}(\mathsf{Q}^0\mathbb{X}))$ as in Equation (4.11) is closable in $H^0 = L^2(\mathsf{Q}^0\mathbb{X};\mathbf{u}^0)$. Its closure generates a SCCS $T^0 = (T^0_t)_{t\in[0,\infty)}$ that is hypocoercive: For all $\kappa^0_1\in(1,\infty)$ there is a constant $\kappa^0_2\in(0,\infty)$ such that such that for all $g\in H^0$ and times $t\in[0,\infty)$ holds

$$\left\|T^0_t g - (g\,,1)_{H^0}\right\|_{H^0} \leq \kappa^0_1\mathrm{e}^{-\kappa^0_2 t}\|g - (g\,,1)_{H^0}\|_{H^0},$$

where $\kappa^0_2$ explicitly is given as

$$\kappa^0_2 = \frac{\kappa^0_1 - 1}{\kappa^0_1}\,\frac{\sigma^2}{n^0_1 + n^0_2\,\sigma^2 + n^0_3\,\sigma^4}$$

where $n^0_i\in(0,\infty)$ for $i\in\{1,2,3\}$ only depend on $\Psi$, compare to (B) as in Section 1.1. ⌟

Let us turn to matters of Mosco convergence, see in the appendix Section B.3 for a short outline and notation. Recall from [CG08, Theorem 3] or Corollary 2.3.1 respectively that we consider the generalised Dirichlet forms $\mathcal{GE}^\varepsilon$, $\varepsilon \in [0,1]$, built from the constant zero Dirichlet form $(\mathcal{E}^\varepsilon, D(\mathcal{E}^\varepsilon)) = (0, H^\varepsilon)$ and the Dirichlet operator $(\mathcal{A}^\varepsilon, \mathcal{F}^\varepsilon) = (L^\varepsilon, D(L^\varepsilon))$ in order to obtain Hunt processes properly associated in the resolvent sense to $(L^\varepsilon, D(L^\varepsilon))$ that also solve the respective martingale problems. In formulae:

$$\mathcal{GE}^\varepsilon\colon\ D(L^\varepsilon) \times H^\varepsilon \cup H^\varepsilon \times D\big((L^\varepsilon)^*\big) \to \mathbb{R},$$
$$(f,g) \longmapsto \begin{cases} (-L^\varepsilon f\,,g)_{H^\varepsilon} & \text{if } f \in D(L^\varepsilon), g \in H^\varepsilon \\ \big(-(L^\varepsilon)^* g\,, f\big)_{H^\varepsilon} & \text{if } g \in D\big((L^\varepsilon)^*\big), f \in H^\varepsilon \end{cases},$$

compare to [Sta99, Example I.4.9 part (ii)]. The forms are extended to $H^\varepsilon$ by $+\infty$. Due to Theorem B.3.4, we know that one has to check either condition (F2), (F2'b), or (F2'); alternatively, we can verify the equivalent condition from Lemma B.3.5. In fact, the linear space $W := D^0 \subseteq H^0$ is dense in $(\mathcal{F}^0, \|\cdot\|_{\mathcal{F}^0}) = \big(D(L^0), \|\cdot\|_{D(L^0)}\big)$. We fix $w^0 \in D^0$ and $u^0 \in H^0$ as well as a net $(u^\varepsilon)_{\varepsilon\in(0,1]}$ with $u^\varepsilon \in H^\varepsilon$ weakly convergent to $u^0$. We construct a sequence $(w^\varepsilon)_{\varepsilon\in(0,1]}$ with $w^\varepsilon \in H^\varepsilon$ and strongly convergent to $w^0$ as $\varepsilon \downarrow 0$ such that $\lim_{\varepsilon\downarrow 0} \mathcal{GE}^\varepsilon(w^\varepsilon, u^\varepsilon) = \mathcal{GE}^0(w^0, u^0)$ holds.[13]

For all $\varepsilon \in (0,1]$ and $x \in \mathbb{X}_{\oslash}$ consider the fibrewise embeddings for the top strata

$$\iota_{\mathsf{Q}^\varepsilon_x\mathbb{X}}\colon\ L^2(\mathsf{Q}^0_x\mathbb{X} \cap \mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X}); \nu^0_x) \to L^2(\mathsf{Q}^\varepsilon_x\mathbb{X} \cap \mathsf{strat}_{2d-1}(\mathsf{Q}^\varepsilon\mathbb{X}); \nu^\varepsilon_x),$$
$$f_x \longmapsto f_x \circ \mathsf{pr}_{\mathsf{Q}^0_x\mathbb{X}\cap\mathsf{strat}_{2d-2}(\mathsf{Q}^0\mathbb{X})}.$$

Of course, the remaining parts of the fibres at $x$, i. e. the sets $\mathsf{Q}^0_x\mathbb{X} \cap \mathsf{strat}_d(\mathsf{Q}^0_x\mathbb{X})$ and $\mathsf{Q}^\varepsilon_x\mathbb{X} \cap \mathsf{strat}_d(\mathsf{Q}^\varepsilon_x\mathbb{X})$, are null sets with respect to $\nu^0_x$ and $\nu^\varepsilon_x$ respectively, so we could have written the $L^2$-spaces just as well as $L^2(\mathsf{Q}^0_x\mathbb{X}; \nu^0_x)$ and $L^2(\mathsf{Q}^\varepsilon_x\mathbb{X}; \nu^\varepsilon_x)$ instead. But the projection used in the definition of $\iota_{\mathsf{Q}^\varepsilon_x\mathbb{X}}$ a priori is well-defined just for the fibres of the top strata. Note that all the embeddings $\iota_{\mathsf{Q}^\varepsilon_x\mathbb{X}}$ are continuous. For every $f \in \mathbf{C}(\mathsf{Q}^0\mathbb{X})$ we define its embedding $\iota_{\mathsf{Q}^\varepsilon\mathbb{X}}(f)$ into $H^\varepsilon$ fibrewise: Define by $f_x := f|_{\mathsf{Q}^0_x\mathbb{X}}$ the fibre restriction of $f$ at $x \in \mathbb{X}_{\oslash}$, which can be identified with a function $g \in \mathbf{C}(\cdot\mathbb{S}\cdot)$ in terms of $\Upsilon^{-1}_{\oslash}(x,\cdot)$ as $g = f_x \circ \Upsilon^{-1}_{\oslash}(x,\cdot)$. Furthermore, define the function $\tilde{f}$ as

$$\tilde{f}\colon\ \mathsf{strat}_{2d-1}(\mathsf{Q}^\varepsilon\mathbb{X}) \to \mathbb{R},\ u \longmapsto \tilde{f}(u) := \Big(\iota_{\mathsf{Q}^\varepsilon_{(\pi_{\mathsf{Q}^1}(u))}\mathbb{X}}\Big(f_{\pi_{\mathsf{Q}}(u)}\Big)\Big)(u).$$

Then, note that for every $x \in \mathbb{X}_{\oslash}$ we can identify the function $\tilde{f}_x := \tilde{f}|_{\mathsf{Q}^\varepsilon_x\mathbb{X}}$ with $g \circ \mathsf{pr}_{\mathbb{S}^{d-2}}$ for some $g \in C^\infty(\mathbb{S}^{d-2})$ in terms of $\Upsilon^{-1}_{\oslash}(x,\cdot)$ as $g \circ \mathsf{pr}_{\mathbb{S}^{d-2}} = \tilde{f}_x \circ \Upsilon^{-1}_{\oslash}(x,\cdot)$ with equality in $L^2(\mathbb{S}^{d-1}; \nu^\varepsilon)$. Thus, $\tilde{f} \in L^2(\mathsf{strat}_{2d-1}(\mathsf{Q}^\varepsilon\mathbb{X}); \mu^\varepsilon)$ and even $\tilde{f} \in H^\varepsilon = L^2(\mathsf{Q}^\varepsilon\mathbb{X}; \mu^\varepsilon)$, since the remaining stratum $\mathsf{strat}_d(\mathsf{Q}^\varepsilon\mathbb{X})$ is a $\mu^\varepsilon$-null set. Hence, we eventually define $\iota_{\mathsf{Q}^\varepsilon\mathbb{X}}(f) := \tilde{f} \in H^\varepsilon$ which basically is an elongation of $f$ on $\mathsf{Q}^\varepsilon\mathbb{X}$. Note that $\iota_{\mathsf{Q}^\varepsilon\mathbb{X}}$ can be defined similarly for any function in $L^2(\mathsf{Q}^0\mathbb{X}; \mu^0)$ using representatives – we just loose the nice intuition in terms of the global trivialisation $\Upsilon_{\oslash}$. Then, the mapping $\iota_{\mathsf{Q}^\varepsilon\mathbb{X}}\colon H^0 \to H^\varepsilon$ is a continuous embedding. By construction, the sequence $(\iota_{\mathsf{Q}^\varepsilon\mathbb{X}}(f))_{\varepsilon\in(0,1]}$ strongly converges to $f$ as $\varepsilon \downarrow 0$. Thence, we choose $(w^\varepsilon)_{\varepsilon\in(0,1]} = \big(\iota_{\mathsf{Q}^\varepsilon\mathbb{X}}(w^0)\big)_{\varepsilon\in(0,1]}$.

Since every $w^\varepsilon$ can be approximated by a function $w^\varepsilon_k \in \mathbf{C}(\mathsf{Q}^\varepsilon\mathbb{X})$ that is constructed fibrewise similar to Equation (4.4) and also $(L^\varepsilon, D(L^\varepsilon))$ is closed, we can infer that $w^\varepsilon \in D(L^\varepsilon)$. Furthermore, since the SAD-decomposition for $L^\varepsilon$ even holds on $\mathbf{C}(\mathsf{Q}^\varepsilon\mathbb{X})$ and both $(S^\varepsilon, D(S^\varepsilon))$ and $(A^\varepsilon, D(A^\varepsilon))$ are closed, we get that $L^\varepsilon w^\varepsilon = S^\varepsilon w^\varepsilon - A^\varepsilon w^\varepsilon$. Then, we observe that

$$S^\varepsilon w^\varepsilon = \Delta_{\mathsf{v}^\varepsilon|_{\mathsf{U}}}\big(\iota_{\mathsf{Q}^\varepsilon\mathbb{X}}(w^0)\big) = \iota_{\mathsf{Q}^\varepsilon\mathbb{X}}\Big(\Delta_{\mathsf{v}^0|_{\mathsf{U}}}\, w^0\Big) = \iota_{\mathsf{Q}^\varepsilon\mathbb{X}}(S^0 w^0)$$
$$\text{and} \qquad A^\varepsilon w^\varepsilon = \iota_{\mathsf{Q}^\varepsilon\mathbb{X}}(A^0 w^0).$$

Recalling our assumptions, we conclude that

$$\mathcal{GE}^\varepsilon(w^\varepsilon, u^\varepsilon) = (-L^\varepsilon w^\varepsilon\,, u^\varepsilon)_{H^\varepsilon} = \Big(-\iota_{\mathsf{Q}^\varepsilon\mathbb{X}}(L^0 w^0)\,, u^\varepsilon\Big)_{H^\varepsilon}$$

[13] The basic idea for the construction of $(w^\varepsilon)_{\varepsilon\in(0,1]}$ is to use fibrewise the measure approximation, see Definition B.2.6, that we already used in Section 4.2.1.

$$\longrightarrow \left(-L^0 w^0\,, u^0\right)_{H^0} = \mathcal{G}\mathcal{E}^0(w^0, u^0)$$

as $\varepsilon \downarrow 0$. By Lemma B.3.5, this result implies the following theorem.

**Theorem 4.3.10** (Mosco convergence of $\rho^\varepsilon$-anisotropic fibre lay-down dynamics)**.** The $\rho^\varepsilon$-anisotropic fibre lay-down models from Equation (4.9) converge to the collapsed one from Equation (4.10) in the sense of Mosco as $\varepsilon \downarrow 0$.

# A Local coordinate forms

## A.1 ▪ Abstract local coordinates

A chart $x = (x^j)_{j=1}^{\mathscr{b}}$ on a $\mathscr{b}$-dimensional manifold $\mathbb{B}$ with domain $U \subseteq \mathbb{B}$ induces local coordinates in a natural way local coordinates $(\bar{x}^j, v^j)_{j=1}^{\mathscr{b}}$ for the preimage $V := \pi_0^{-1}(U)$:

$$\bar{x}^j := x^j \circ \pi_0 \quad \text{and} \quad v^j := \frac{\partial}{\partial x^j} = \partial_{x^j} \qquad \text{for } j \in \{1, \dots, \mathscr{b}\}.$$

For short we might write $(\bar{x}, v) = (\bar{x}, \partial x) := (\bar{x}^j, v^j)_{j=1}^{\mathscr{b}}$ if confusion is not to be expected. Iterating this procedure, we get in the same vein an induced chart on $\pi_{2,0}^{-1}(U)$ consisting of

$$\bar{\bar{x}}^j := x^j \circ \pi_{2,0}, \quad \bar{v}^j := v^j \circ \pi_{2,1}, \quad \frac{\partial}{\partial \bar{x}^j} = \partial_{\bar{x}^j}, \quad \text{and} \quad a^j := \frac{\partial}{\partial v^j} = \partial_{v^j}$$

for $j \in \{1, \dots, \mathscr{b}\}$. As before, we might shorthand $(\bar{\bar{x}}, \bar{v}, \partial\bar{x}, a) = (\bar{\bar{x}}, \bar{v}, \partial\bar{x}, \partial v) := (\bar{\bar{x}}^j, \bar{v}^j, \partial_{\bar{x}^j}, a^j)_{j=1}^{\mathscr{b}}$. Similar charts could be used in higher order tangent bundles. It's clearly reasonable to overload the symbols that already got a bar, like $\bar{x}^j$ here.

We proceed to write in these abstract local coordinates important objects like vector fields, Riemannian metrics, and Ehresmann connections. A vector field $\mathscr{X} \in \Gamma^\infty(\mathrm{T}\mathbb{B})$ it can be locally represented by a collection of smooth functions $(\mathscr{X}_j)_{j=1}^{\mathscr{b}}$ via

$$\mathscr{X} = \sum_{j=1}^{\mathscr{b}} \mathscr{X}_j v^j = \sum_{j=1}^{\mathscr{b}} \mathscr{X}_j \frac{\partial}{\partial x^j}.$$

Given a Riemannian metric $\mathsf{b}$ on $\mathbb{B}$ it is locally represented by the matrix $(\mathsf{b}_{ij})_{ij}$ in the sense of the tensor

$$\mathsf{b} = \sum_{i,j \in \{1,\dots,\mathscr{b}\}} \mathsf{b}_{ij}\, dx^i \otimes dx^j.$$

The inverse of the matrix $(\mathsf{b}_{ij})_{ij}$ is signified by upper indices as $(\mathsf{b}^{ij})_{ij}$. By $\sqrt{\det \mathsf{b}}$ we denote the determinant of any local coordinate representation $(\mathsf{b}_{ij})_{ij}$ of $\mathsf{b}$. The Riemannian volume form $\mathrm{d}\lambda_{\mathsf{b}}$ of an oriented Riemannian manifold $(\mathbb{B}, \mathsf{b})$ locally reads as

$$\lambda_{\mathsf{b}} = \sqrt{\det \mathsf{b}} \bigwedge_{j=1}^{\mathscr{b}} dx^j = \sqrt{\det \mathsf{b}} \cdot dx^1 \wedge \dots \wedge dx^{\mathscr{b}}.$$

See the proof of [Nic96, Proposition 3.4.3] for the local construction of the Riemannian density field $|\mathrm{d}\lambda_{\mathsf{b}}|$.

Associated to the metric $\mathsf{b}$ – or to be more precise to the Levi-Civita connection – are the so-called *Christoffel symbols* that are defined as

$$\Gamma_{ij}^k := \frac{1}{2} \sum_{\ell=1}^{\mathscr{b}} \mathsf{b}^{k\ell} \left( \frac{\partial \mathsf{b}_{\ell i}}{\partial x^j} + \frac{\partial \mathsf{b}_{\ell j}}{\partial x^i} - \frac{\partial \mathsf{b}_{ij}}{\partial x^\ell} \right) \qquad \text{for all } i, j, k \in \{1, \dots, \mathscr{b}\}.$$

See [Nic96, § 3.3.6, § 4.1.2]. The local coordinate forms of gradient, divergence, and Laplacian are well-known, compare e. g. to [Lee13, Problem 14.11]:

$$\nabla_{\mathsf{b}} f = \sum_{i=1}^{\mathscr{b}} \left( \sum_{j=1}^{\mathscr{b}} \mathsf{b}^{ij} \frac{\partial f}{\partial x^j} \right) v^i = \sum_{i,j \in \{1,\dots,\mathscr{b}\}} \mathsf{b}^{ij} \frac{\partial f}{\partial x^j} v^i = \sum_{i,j \in \{1,\dots,\mathscr{b}\}} \mathsf{b}^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i},$$

A

$$\begin{aligned}
\operatorname{div}_{\mathsf{b}} \mathcal{X} = \operatorname{div}_{\mathsf{b}} \left( \sum_{j=1}^{\mathscr{b}} \mathcal{X}_j v^j \right) &= \frac{1}{\sqrt{\det \mathsf{b}}} \sum_{j=1}^{\mathscr{b}} \frac{\partial}{\partial x^j} \left( \sqrt{\det \mathsf{b}} \cdot \mathcal{X}_j \right) \\
&= \sum_{j=1}^{\mathscr{b}} \frac{\partial \mathcal{X}_j}{\partial x^j} + \mathcal{X}_j \frac{\partial}{\partial x^j} \left( \ln \sqrt{\det \mathsf{b}} \right),
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{\mathsf{b}} f &= \frac{1}{\sqrt{\det \mathsf{b}}} \sum_{i,j \in \{1,\dots,\mathscr{b}\}} \frac{\partial}{\partial x^i} \left( \mathsf{b}^{ij} \sqrt{\det \mathsf{b}} \frac{\partial f}{\partial x^j} \right) \\
&= \sum_{i,j \in \{1,\dots,\mathscr{b}\}} \mathsf{b}^{ij} \left( \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{k=1}^{\mathscr{b}} \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right)
\end{aligned}$$

for $f \in C^\infty(\mathbb{B})$ and $\mathcal{X} \in \Gamma^\infty(\mathrm{T}\mathbb{B})$. Sometimes, the local equation is used as definition of the divergence. If $(x^j)_{j=1}^{\mathscr{b}}$ is a chart of normal coordinates[1], then the Laplace-Beltrami operator simplifies to $\Delta_{\mathsf{b}} = \sum_j \partial^2/\partial x^j \partial x^j$ reminiscent of the Euclidean case. If the metric is weighted by a function $\rho_{\mathbb{B}}$ as in Definition 1.2.23, then the weighted gradient, divergence and Laplace-Beltrami read as

$$\begin{aligned}
\nabla_{\mathbf{b}} f &= \frac{1}{\rho_{\mathbb{B}}} \sum_{i,j \in \{1,\dots,\mathscr{b}\}} \mathsf{b}^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i}, \\
\operatorname{div}_{\mathbf{b}} \mathcal{X} &= \frac{1}{\rho_{\mathbb{B}} \sqrt{\det \mathsf{b}}} \sum_{j=1}^{\mathscr{b}} \frac{\partial}{\partial x^j} \left( \rho_{\mathbb{B}}^2 \mathcal{X}_j \sqrt{\det \mathsf{b}} \right),
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{\mathbf{b}} f &= \frac{1}{\rho_{\mathbb{B}} \sqrt{\det \mathsf{b}}} \sum_{i,j \in \{1,\dots,\mathscr{b}\}} \frac{\partial}{\partial x^i} \left( \rho_{\mathbb{B}} \mathsf{b}^{ij} \sqrt{\det \mathsf{b}} \frac{\partial f}{\partial x^j} \right) \\
&= \sum_{i,j \in \{1,\dots,\mathscr{b}\}} \mathsf{b}^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \left( \frac{1}{\rho_{\mathbb{B}}} \frac{\partial \rho_{\mathbb{B}}}{\partial x^i} \mathsf{b}^{ij} + \frac{\partial \mathsf{b}^{ij}}{\partial x^i} \right) \frac{\partial f}{\partial x^j}
\end{aligned}$$

for $f \in C^\infty(\mathbb{B})$ and $\mathcal{X} \in \Gamma^\infty(\mathrm{T}\mathbb{B})$. Again reminiscent of the Euclidean case, if $(x^j)_{j=1}^{\mathscr{b}}$ is a chart of normal coordinates, then the weighted Laplace-Beltrami operator attains the simple form

$$\Delta_{\mathbf{b}} f = \sum_j \frac{\partial^2 f}{\partial x^j \partial x^j} + \frac{1}{\rho_{\mathbb{B}}} \frac{\partial \rho_{\mathbb{B}}}{\partial x^j} \frac{\partial f}{\partial x^j} = \Delta_{\mathsf{b}} f + \frac{1}{\rho_{\mathbb{B}}} \left( \nabla_{\mathsf{b}} f \right)(\rho_{\mathbb{B}}) \quad \text{for } f \in C^\infty(\mathbb{B}).$$

We turn to the matters of the standard Riemannian Ehresmann connection associated to the Levi-Civita connection. From [Dom62, Equations (24) and (25)] or [GK02, Lemma 4.1 and Corollary 4.2] respectively we know some particular vertical and horizontal lifts:

$$\operatorname{vl}_{\mathcal{X}} v^i = \operatorname{vl}_{\mathcal{X}} \left( \frac{\partial}{\partial x^i} \right) = \mathcal{X}_i \frac{\partial}{\partial v^i} = \mathcal{X}_i a^i$$

and

$$\begin{aligned}
\operatorname{hl}_{\mathcal{X}} v^i = \operatorname{hl}_{\mathcal{X}} \left( \frac{\partial}{\partial x^i} \right) &= \mathcal{X}_i \frac{\partial}{\partial \bar{x}^i} - \sum_{j,k \in \{1,\dots,\mathscr{b}\}} \left( \Gamma^i_{jk} \circ \pi_0 \right) \bar{v}^j \mathcal{X}_k \cdot \frac{\partial}{\partial v^i} \\
&= \mathcal{X}_i \frac{\partial}{\partial \bar{x}^i} - \sum_{j,k \in \{1,\dots,\mathscr{b}\}} \left( \Gamma^i_{jk} \circ \pi_0 \right) \bar{v}^j \mathcal{X}_k \cdot a^i
\end{aligned}$$

for all $i \in \{1,\dots,\mathscr{b}\}$ and $\mathcal{X} = \sum_j \mathcal{X}_j v^j \in \Gamma^\infty(\mathrm{T}\mathbb{B})$. Compare the following local coordinate expressions to [Sak96, Section II.4.1]: Note that the differential $d\pi_{2,1}$ locally reads as

$$d\pi_{2,1}(\bar{x}, \bar{v}, \partial \bar{x}, a = \partial v) = (\bar{x}, \partial x) = (\bar{x}, v)$$

thinking $(\bar{x}, \bar{v}, \partial \bar{x}, a)$ as point in the double tangent bundle rather than a collection of mappings and similarly for $(\bar{x}, v)$ in the tangent bundle. The vertical tangent bundle at $(\bar{x}, v)$ is understood as $\mathrm{V}_{(\bar{x},v)}\mathrm{T}\mathbb{B} =$

[1] Recall [Nic96, § 4.1.3] and in particular [Nic96, Proposition 4.1.18].

$\{(\bar{x}, \bar{v}, 0, a) \mid a \in \mathbb{R}^{\ell}\}$ and identified with $\mathsf{T}_x\mathbb{B}$ via the mapping $(\bar{x}, \bar{v}, 0, a) \mapsto (\bar{x}, a)$. In contrast, the horizontal tangent bundle at $(\bar{x}, v)$ is

$$\mathsf{H}_{(\bar{x},v)}\mathsf{T}\mathbb{B} = \left\{ \left( \bar{x}^i, \bar{v}^i, \partial\bar{x}^i, -\sum_{j,k} \Gamma^i_{jk}\, \bar{v}^j\, \partial\bar{x}^k \right)_{i=1}^{\ell} \,\middle|\, \partial\bar{x}^1, \dots, \partial\bar{x}^{\ell} \in \mathbb{R} \right\}.$$

Therefore, the vertical and horizontal projections of $(\bar{x}, \bar{v}, \partial\bar{x}, a)$ read as

$$\mathsf{vpr}(\bar{x}, \bar{v}, \partial\bar{x}, a) = \left( \bar{x}^i, \bar{v}^i, 0, a^i + \sum_{j,k} \Gamma^i_{jk}\, \bar{v}^j\, \partial\bar{x}^k \right)_{i=1}^{\ell}$$

$$\text{and} \qquad \mathsf{hpr}(\bar{x}, \bar{v}, \partial\bar{x}, a) = \left( \bar{x}^i, \bar{v}^i, \partial\bar{x}^i, -\sum_{j,k} \Gamma^i_{jk}\, \bar{v}^j\, \partial\bar{x}^k \right)_{i=1}^{\ell}.$$

From the proof of [Sak96, Proposition II.4.1] that the connection map $d\kappa$ locally reads as

$$d\kappa(\bar{x}, \bar{v}, \partial\bar{x}, a) = \left( \bar{x}^i, a^i + \sum_{j,k} \Gamma^i_{jk}\, \bar{v}^j\, \partial\bar{x}^k \right)_{i=1}^{\ell}.$$

*Remark* A.1.1 (canonical flip). The mapping 'flip' given by $(\bar{x}, \bar{v}, \partial\bar{x}, a) \longmapsto (\bar{x}, \partial\bar{x}, \bar{v}, a)$ is independent of the chart. It is referred to as the *canonical flip* and sometimes used to define a semispray as vector field $\mathcal{H}$ that is a fix point for the canonical flip: $\mathsf{flip}(\mathcal{H}) = \mathcal{H}$.
⌟

*Remark* A.1.2 (general coordinate form of semisprays). The Riemannian/geodesic semispray locally reads as

$$\mathcal{H}_{\mathsf{b}}(\bar{x}, v) = \left( \bar{x}^i, \bar{v}^i, \bar{v}^i, -\sum_{j,k \in \{1,\dots,\ell\}} \Gamma^i_{jk}\, \bar{v}^j\, \bar{v}^k \right)_{i=1}^{\ell} \qquad \text{for } (\bar{x}, v) \in \mathsf{T}_x\mathbb{B}.$$

Integral curves $t \mapsto \big(\xi(t), \dot{\xi}(t)\big)$ of $\mathcal{H}_{\mathsf{b}}$ satisfy the *geodesic equation*

$$\ddot{\xi}^i + \sum_{j,k} \Gamma^i_{jk}\, \dot{\xi}^j\, \dot{\xi}^k = 0 \qquad \text{for all } i \in \{1, \dots, \ell\},$$

compare to [Sak96, Section 4.2 item (II)] and [MR99, Section 7.5]. In general, a semispray $\mathcal{H}$ is described by its so-called local coefficients $(G^i)_{i=1}^{\ell}$ via

$$\mathcal{H}(\bar{x}, v) = \left(\bar{x}^i, \bar{v}^i, \bar{v}^i, -2G^i(\bar{x}, v)\right)_{i=1}^{\ell} \qquad \text{for } (\bar{x}, v) \in \mathsf{T}_x\mathbb{B}$$

and hence, an integral curve $t \mapsto \big(\xi(t), \dot{\xi}(t)\big)$ of $\mathcal{H}$ satisfies

$$\ddot{\xi}^i + 2G^i\big(\xi(t), \dot{\xi}(t)\big) = 0 \qquad \text{for all } i \in \{1, \dots, \ell\}. \tag{A.1}$$

To the semispray $\mathcal{H}$ corresponds an Ehresmann connection such that the horizontal lift has the form

$$\mathsf{hl}_{\mathcal{X}}\, v^i = \mathcal{X}_i \frac{\partial}{\partial\bar{x}^i} - \sum_{j=1}^{\ell} N^i_j(\mathcal{X}) \mathcal{X}_j \cdot a^i \qquad \text{with } \mathcal{X} \in \Gamma^{\infty}(\mathsf{T}\mathbb{X})$$

and the local coefficients $\big(N^i_j\big)_{i,j \in \{1,\dots,\ell\}}$ satisfy $N^i_j = {}^{\partial G^i}\!/_{\partial v^j}$. If we consider a Lagrange space, i. e. a regular Lagrangian $\mathsf{L}\colon \mathsf{T}\mathbb{B} \to \mathbb{R}$, the *Euler-Lagrange equations*

$$\frac{\partial \mathsf{L}}{\partial\bar{x}^i} - \frac{d}{dt}\left(\frac{\partial \mathsf{L}}{\partial v^i}\right) = 0$$

are equivalent to Equation (A.1) and the local coefficients $(G^i)_{i=1}^{\ell}$ are given as

$$G^i = \frac{1}{4} \sum_{j,k \in \{1,\dots,\ell\}} g^{ij} \left( \frac{\partial^2 \mathsf{L}}{\partial v^j \partial\bar{x}^k}\, v^k - \frac{\partial \mathsf{L}}{\partial\bar{x}^j} \right),$$

where $(g_{ij})_{i,j} = \left({}^1\!/_2 \cdot {}^{\partial^2 \mathsf{L}}\!/_{\partial v^i \partial v^j}\right)_{i,j}$ are the component matrix of the Lagrangian metric and $\left(g^{ij}\right)_{i,j}$ is the inverse matrix. See [Buc06] and references therein for details.
⌟

A

## A.2 ▪ Particular coordinate forms

We fix the standard notation used for polar coordinates, see e. g. [DX13, Section 1.5].

*Notation* A.2.1 (polar coordinates). Consider polar coordinates recursively defined via

$$\begin{aligned} &\mathrm{pol}_d\colon\ (0,\infty)\times(0,2\pi)\times(0,\pi)^{d-2}\to \mathrm{ran}(\mathrm{pol}_d),\\ &(r,\theta_0,\theta_1,\dots,\theta_{d-2})\mapsto r\cdot\begin{pmatrix}\mathrm{pol}_{d-1}(r,\theta_0,\theta_1,\dots,\theta_{d-3})\cdot\sin\theta_{d-2}\\ \cos\theta_{d-2}\end{pmatrix}\quad\text{for } d>2\\ \text{and}\quad &\mathrm{pol}_2\colon\ (0,\infty)\times(0,2\pi)\to\mathbb{R}^2,\ (r,\theta_0)\mapsto r\cdot(\cos\theta_0,\sin\theta_0)^\top. \end{aligned}$$

The inverse $\mathrm{pol}_d^{-1}$ yields the chart of polar coordinates on $\mathbb{R}^d$; in particular, the restriction $(\mathrm{pol}_d(1,\cdot))^{-1}$ gives the chart $(\theta^j)_{j=0}^{d-2}$ on the sphere $\mathbb{S}^{d-1}$.

In polar coordinates, the matrix form $(\mathrm{s}_{ij})_{i,j\in\{0,1\}}$ of the round metric $\mathrm{s}$ on $\mathbb{S}^{d-1}$ has the only nonzero components $\mathrm{s}_{00}=(\sin\theta^1)^2$ and $\mathrm{s}_{11}=1$. Clearly, the polar coordinates are not normal coordinates. The only nonzero Christoffel symbols for $(\mathrm{s}_{ij})_{i,j}$ are $\Gamma^0_{01}=\Gamma^0_{10}=\cot\theta^1$ and $\Gamma^1_{00}=-\cos\theta^1\sin\theta^1$. Oftentimes, we use the normalisation of the tangent vectors $\partial_{\theta^i}$ given as $n^i:={}^{\partial_{\theta^i}}/_{|\partial_{\theta^i}|}=\sqrt{\mathrm{s}^{ii}}\partial_{\theta^i}$ for $i\in\{0,1\}$. We calculate the semispray $\mathcal{H}_\mathrm{s}$ at $v=\sum_{k=0}^1 w_k\partial_{\theta^k}=\sum_{k=0}^1\nu_k n^k$ as

$$\begin{aligned}\mathcal{H}_\mathrm{s}(v)&=w_0\ \partial_{\bar\theta^0}+w_1\ \partial_{\bar\theta^1}-2\cot\theta^1 w_0w_1\ \partial_{\partial_{\theta^0}}+\cos\theta^1\sin\theta^1 w_0^2\ \partial_{\partial_{\theta^1}}\\ &=\nu_0\ \bar n^0+\nu_1\ \bar n^1-2\cot\theta^1\ \nu_0\,\nu_1\ \partial_{n^0}+\cot\theta^1\nu_0^2\ \partial_{n^1},\end{aligned}\tag{A.2}$$

where $\bar n^i:=n^i\circ d\pi_{2,1}$. The Riemannian volume measures on $\mathbb{R}^d$ and $\mathbb{S}^{d-1}$ read in polar coordinates as product measures:

- on $\mathbb{R}^d$: $\lambda_{\mathrm{euc}}=r^{d-1}\mathrm{d}r\otimes\bigotimes_{j=0}^{d-2}(\sin\theta^j)^j\mathrm{d}\theta^j$,
- on $\mathbb{S}^{d-1}$: $\mathrm{S}^{d-1}=\lambda_\mathrm{s}=\bigotimes_{j=0}^{d-2}(\sin\theta^j)^j\mathrm{d}\theta^j$ where $\mathrm{S}^{d-1}$ denotes the spherical surface measure on $\mathbb{S}^{d-1}$.

⌟

### Fibre lay-down model on the sphere

We write the fibre lay-down model (2.16) on $\mathbb{X}=\mathbb{S}^2$ in polar coordinates.

**Example A.2.2** (fibre lay-down model on sphere). The chart $(\theta^0,\theta^1)$ on $\mathbb{S}^2$ induces the chart

$$\left(\bar\theta^0=\theta^0\circ\pi_{2,1},\ \bar\theta^1=\theta^1\circ\pi_{2,1},\ \partial_{\theta^0}=\frac{\partial}{\partial\theta^0},\ \partial_{\theta^1}=\frac{\partial}{\partial\theta^1}\right)$$

on $T\mathbb{S}^2$. Furthermore, $\Psi^\mathrm{h}=1\cdot\Psi(\bar\theta^0,\bar\theta^1)$. Then, Equation (2.16) becomes

$$\begin{aligned}\mathrm{d}\bar\theta^i_t&=w_{i;t}\ \mathrm{d}t\\ \mathrm{d}w_{0;t}&=-2\cot\bar\theta^1_t\cdot w_{0;t}w_{1;t}\ \mathrm{d}t-\frac{1}{(\sin\bar\theta^1_t)^2}\frac{\partial\Psi}{\partial\bar\theta^0}(\bar\theta^0_t,\bar\theta^1_t)\ \mathrm{d}t-\alpha w_{0;t}\ \mathrm{d}t+\sigma\frac{1}{\sin\bar\theta^1_t}\circ\mathrm{d}W^0_t\\ \mathrm{d}w_{1;t}&=+\cos\bar\theta^1_t\sin\bar\theta^1_t\cdot w_{0;t}^2\ \mathrm{d}t-\frac{\partial\Psi}{\partial\bar\theta^1}(\bar\theta^0_t,\bar\theta^1_t)\ \mathrm{d}t-\alpha w_{1;t}\ \mathrm{d}t+\sigma\circ\mathrm{d}W^1_t,\end{aligned}\tag{A.3}$$

which coincides with [Sti14, Equation (1.48)] indeed. Switching over to normalised tangent vectors, Equation (A.3) transforms into

$$\begin{aligned}\mathrm{d}\bar\theta^0_t&=\frac{\nu_{0;t}}{\sin\bar\theta^1_t}\ \mathrm{d}t\\ \mathrm{d}\bar\theta^1_t&=\nu_{1;t}\ \mathrm{d}t\\ \mathrm{d}\nu_{0;t}&=-2\cos\bar\theta^1_t\cdot\nu_{0;t}\nu_{1;t}\ \mathrm{d}t-\frac{1}{\sin\bar\theta^1_t}\frac{\partial\Psi}{\partial\bar\theta^0}(\bar\theta^0_t,\bar\theta^1_t)\ \mathrm{d}t-\alpha\nu_{0;t}\ \mathrm{d}t+\sigma\circ\mathrm{d}W^0_t\\ \mathrm{d}\nu_{1;t}&=\cot\bar\theta^1_t\cdot\nu_{0;t}^2\ \mathrm{d}t-\frac{\partial\Psi}{\partial\bar\theta^1}(\bar\theta^0_t,\bar\theta^1_t)\ \mathrm{d}t-\alpha\nu_{1;t}\ \mathrm{d}t+\sigma\circ\mathrm{d}W^1_t.\end{aligned}\tag{A.4}$$

⌟

A

## Higher order fibre lay-down models

We turn to the smoothed models of Chapter 3. P. Stilgenbauer was the first to determine necessary compensation terms that ensure that the process doesn't leave the configuration manifold – previous attempts did not contain these compensation terms. However, beyond that these compensation terms had neither physical nor geometric interpretation due to the overall extrinsic approach. In our intrinsic formulation these terms just appear naturally.

**Example A.2.3** (smoothed fibre lay-down model on Euclidean position space). Let $\mathbb{X} = \mathbb{R}^3_x$, then $\mathsf{Q} \simeq \mathbb{R}^3 \times \mathsf{T}\mathbb{S}^2_v$. The horizontal part of the unit Sasaki metric $\mathsf{u}$ is just the Euclidean metric, whilst the the vertical part is the round metric $\mathsf{s}$ on the sphere. We calculate for $b = \sum_{k=0}^{1} \beta_k \partial_{\theta^k} = \sum_{k=0}^{1} \nu_{\beta;k} n^k$ using [GK02, Lemma 4.1] that

$$\begin{aligned}
\mathsf{h}_b\restriction^3_2 (\alpha_i \partial_{\theta^i}) = \mathsf{h}_b\restriction^3_2 (\nu_{\alpha;i} n^i) &= \alpha_i \partial_{\bar\theta^i} - \sum_{j,\ell\in\{0,1\}} \big(\Gamma^i_{j\ell} \circ \pi_{2,1|\mathsf{U}}\big)\alpha_j \beta_\ell \,\frac{\partial}{\partial\partial_{\theta^i}} \\
&= \begin{cases} \alpha_0\; \partial_{\bar\theta^0} - \cot\theta^1 (\alpha_0\beta_1 + \alpha_1\beta_0)\; \partial_{\partial_{\theta^0}} & i=0 \\ \alpha_1\; \partial_{\bar\theta^1} + \cos\theta^1 \sin\theta^1 \alpha_0\beta_0\; \partial_{\partial_{\theta^1}} & i=1 \end{cases} \qquad \text{(A.5)}\\
&= \begin{cases} \nu_{\alpha;0}\; n^0 \circ d\pi_{2,1|\mathsf{U}} - \cot\theta^1 \left(\nu_{\alpha;0}\, \nu_{\beta;1} + \nu_{\alpha;1}\, \nu_{\beta;0}\right) \partial_{n^0} & i=0 \\ \nu_{\alpha;1}\; n^1 \circ d\pi_{2,1|\mathsf{U}} + \cot\theta^1 \nu_{\alpha;0}\nu_{\beta;0}\; \partial_{n^1} & i=1 \end{cases}.
\end{aligned}$$

The semispray $\mathcal{H}_{\mathsf{euc}}$ with respect to Euclidean metric can roughly be thought as the identity mapping. In this sense, the second order semispray $\mathcal{H}^{(2)}_{\mathsf{euc}}$ coincides with $\mathcal{H}_{\mathsf{s}}$. In the Euclidean case we know that Assumption (2.19) holds, see Example 2.2.5, and therefore one has $\mathsf{tl}\,\nabla_{\mathsf{euc}}\Psi = {}^1\!/_2\,\nabla_{\mathsf{s}}\Psi$. Now, we have gathered all the ingredients and write the smoothed fibre lay-down model (3.9) as the system

$$\begin{aligned}
\mathsf{d}x_t &= u_t\;\mathsf{d}t = \mathsf{pol}(\theta^0_t, \theta^1_t)\;\mathsf{d}t \\
\mathsf{d}\bar\theta^i_t &= -\frac{1}{2}\left(n^i\,,\nabla_{\mathsf{euc}}\Psi\right)_{\mathsf{euc}}\;\mathsf{d}t + \nu_{i;t}\;\mathsf{d}t \\
\mathsf{d}\nu_{0;t} &= -2\cot\theta^1 \nu_{0;t}\nu_{1;t} + \frac{1}{2}\cot\theta^1\Big(\left(n^0\,,\nabla_{\mathsf{euc}}\Psi\right)_{\mathsf{euc}}\nu_{1;t} + \left(n^1\,,\nabla_{\mathsf{euc}}\Psi\right)_{\mathsf{euc}}\nu_{0;t}\Big) \qquad \text{(A.6)}\\
&\qquad - \alpha\nu_{0;t}\;\mathsf{d}t + \sigma\;\mathsf{d}W^0_t \\
\mathsf{d}\nu_{1;t} &= \cot\theta^1 \nu^2_{0;t} - \frac{1}{2}\cot\theta^1 \left(n^0\,,\nabla_{\mathsf{euc}}\Psi\right)_{\mathsf{euc}}\nu_{0;t} - \alpha\nu_{1;t}\;\mathsf{d}t + \sigma\;\mathsf{d}W^1_t.
\end{aligned}$$

In [Sti14, Proposition 1.7] P. Stilgenbauer basically gives the same local coordinate form [Sti14, Equation (1.56)] for the smoothed fibre lay-down model on $\mathbb{R}^d$ for arbitrary dimensions $d$. Of course, we have chosen here $d = 3$ just for sake of simplicity and the case of higher dimensions is derived along the very same lines. So, the only difference is that [Sti14, Equation (1.56)] doesn't contain the factor ${}^1\!/_{d-1}$ in front of the potential terms. Our derivation seems more transparent and insightful as every term traces back to one of the four intrinsic terms from Equation (3.9). ⌟

## Curvature of weighted spheres $\mathbb{S}^2$

We discuss Ricci curvature of the weighted spheres $\left(\mathbb{S}^{d-1}, \mathsf{dist}_{\mathsf{s};1/n}, \nu_n, \bar u\right)_{n\in\mathbb{N}_+}$ as considered in Section 4.2 for the special case of $d = 3$. The implication of uniform doubling plays an important role in Corollary 4.2.6 and therefore we already devoted Remark 4.2.5 to the matter.

*Remark* A.2.4. Assume $d = 3$. We compute in polar coordinates the Christoffel symbols $\Gamma^k_{ij} = \Gamma^k_{ij}(n)$ with respect to $\mathsf{s}^{1/n} = \bar\rho_{\mathbb{S}}^{-1/n}\mathsf{s}$ and afterwards the components $R_{ij} = R_{ij}(n)$ of the Ricci tensor. The nonzero Christoffel symbols are

$$\Gamma^0_{01} = \Gamma^0_{10} = (n+1)\cot\theta^1, \quad \Gamma^1_{00} = -(n+1)\cos\theta^1\sin\theta^1, \quad \text{and} \quad \Gamma^1_{11} = n\cot\theta^1.$$

Using the expression

$$R_{ij} = \sum_{a=0}^{d-2} \frac{\partial\Gamma^a_{ij}}{\partial\theta^a} - \frac{\partial\Gamma^a_{ai}}{\partial\theta^j} + \sum_{b=0}^{d-2} \Gamma^a_{ab}\Gamma^b_{ij} - \Gamma^a_{ib}\Gamma^b_{aj} \qquad \text{with } i,j\in\{0,\dots,d-2\}$$

we compute that

$$R_{01} = R_{10} = 0, \quad R_{00} = (n+1)\big(\sin\theta^1\big)^2, \quad \text{and} \quad R_{11} = (n+1).$$

Thus, we have found that the weighted Ricci curvatures are nonnegative for all $n \in \mathbb{N}$. ⌟

# B | Terminology on scaling limits and geometric convergence

The objective of this appendix is twofold. First in Section B.1, we complement the picture painted in Chapter 4 by commenting on two particular scaling techniques used in literature. Afterwards, we provide vocabulary from the theory of converging Banach spaces. General definitions are given in Section B.2 for context. Then, we specifically recall notions regarding Mosco convergence in Section B.3.

## B.1 ▪ Overview of different methods

### White noise limits

The authors of [Her+09; KMW12a; KMW12b] investigate different limits for three-dimensional fibre lay-down models such as for large diffusions, large coiling forces, vanishing or prevalent anisotropicity effects, and balanced parameter convergence yielding a model reduction. While results regarding extreme cases of anisotropicity are obtained by considering specific values for the corresponding parameter, the model reduction from the smoothed fibre lay-down model to the original one is achieved by a so-called white noise limit. This argument can be found in [Her+09, Proposition 1.1], [KMW12b, Section 5.2], and [Mar13, Proposition 4.6]: Assume that for the smoothed model (3.3) the parameters $\alpha$ and $\sigma$ approach zero such that the quotient $\bar{\sigma} := \sigma/\alpha$ is fixed. Then, the smoothed model converges to the original fibre lay-down model with diffusion parameter $\bar{\sigma}$. Such limiting arguments are widespread in applied mathematics and physics. For instance, M. Ottobre and G. A. Pavliotis showed that the generalised Langevin equation converges in white noise limit sense to the Langevin equation. See [OP11, Theorem 2.6 and Section 6]. Though, it seems by no means trivial to rigorously justify white noise limit statements in more singular situations.

### Effective limits in homogeneous spaces

The frighteningly dense paper [Li15] was my starting point to the topic of scaling limits. X. M. Li discusses a homogenisation procedure on so-called homogeneous spaces. Among other influences, this study was inspired by collapsing of Riemannian manifolds, see [CG86; CG90]. A famous example of such collapsings are the Berger's spheres that converge to the 2-dimensional sphere $\frac{1}{2}\mathbb{S}^2$ of radius $\frac{1}{2}$ while the sectional curvatures are bounded, see the first paragraph of the introduction of [Li15]. A manifold $\mathbb{Y}$ is homogeneous if there is a transitive action by a Lie group $G$. In this case, denote by $H$ the isotropy group at a point $y$ and represent the manifold as coset space $G/H$. To this end, $G/H$ gets the unique smooth structure such that all mappings $G/H \to \mathbb{Y}$, $gH \mapsto \ell_g y$ are diffeomorphisms, where $\ell_g$ denotes the left action of $g$. X. M. Li's technique makes use of a so-called reductive structure that determines an Ehresmann connection in the principal bundle $\pi\colon G \to G/H$. Then, the slow motion $(x_t)_t$ is described as horizontal lift $(u_t)_t$ of curves in $G/H$ to $G$ in the sense that $\pi(u_t) = x_t$, compare to [Li15, Lemma 4.1], while the fast motions are considered as curves in $H$. A solution $(g_t^\varepsilon)_t$ of the SDE [Li15, Equation (1.1)] is viewed as perturbation of a conservation law where the projection $\pi\colon G \to G/H$ represents this conservation law. The equations for the slow motions for $(g_t^\varepsilon)_t$, i. e. the horizontal lift of $(\pi(g_t^\varepsilon))_t$ with respect to the Ehresmann connection prescribed by the reductive structure, are derived as well as for the fast motions in $H$. Under certain conditions convergence of the slow motions in weak topology on the path space and in Wasserstein metric is proven and in some instances the limiting process can be characterised as a scaled Brownian motion. The machinery is algebraic in nature, so [Li15, § 10] illuminates differential geometric context and [Li15, § 11] contains a few examples. One of these examples, [Li15, Example 11.4], considers the sphere $\mathbb{S}^{d-1}$ for $d \geq 3$ as homogeneous space $\mathsf{SO}(d)/\mathsf{SO}(d-1)$ and showcases explicitly the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$: Identify $\mathsf{SO}(d-1)$ with

$$\left\{ \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, R \in \mathsf{SO}(d-1) \right\},$$

then the choice

$$\mathfrak{m} := \left\{ \begin{pmatrix} 0 & c \\ -c^\top & 0 \end{pmatrix} \,\middle|\, c \in \mathbb{R}^{d-1} \right\}$$

yields indeed a reductive decomposition. There are a many different ways to think about a sphere as a homogeneous space, another one would e. g. use $G = \mathsf{SU}(2)$ and $H = \mathsf{U}(1)$, but the one given earlier is the right one for matters of Section 4.2.

In the end, we departed from the technique of X. M. Li, because we felt that its algebraic nature is somewhat restrictive and might obfuscate the intuition of applicationists.

## B.2 ▪ Terminology from geometric convergences

For this and the next section we primarily refer to the PhD thesis [Töl10] of J. M. Tölle. He consolidates many convergence notions and results in great generality, specifically for varying Banach spaces. Therefore, we can keep the discussion concise here.

**Definition B.2.1** (subnet). Let two directed sets $(I, \lhd_I)$ and $(J, \lhd_J)$, a nonempty set $E$, as well as a net $(x_i)_{i\in I}$ in $E$. If there is a cofinal mapping $h\colon J \to I$, i. e. for all $\tilde{i} \in I$ there is some $\tilde{j} \in J$ such that for all $j \in J$ with $j \rhd_J \tilde{j}$ holds $h(j) \rhd_I \tilde{i}$, we call $(x_{h(j)})_{j\in J}$ a *subnet of* $(x_i)_{i\in I}$. ⌟

For the following definition see [KS08, Definition 3.1] or [Töl10, Definition 5.8] respectively. See also [Töl10, Section 5.2.1] for a comparison to the 'Variable Banach spaces' of V. V. Zhikov and S. E. Pastukhova, which turns out to be the same concept.

**Definition B.2.2** ((Strong) asymptotic relation). Let $J$ be a directed set as well as metric spaces $(\mathbb{Y}_\infty, \mathsf{dist}_\infty)$ and $\big(\mathbb{Y}_j, \mathsf{dist}_j\big)$ for all $j \in J$. Define the disjoint union $\mathbb{Y} := \bigsqcup_{j\in J} \mathbb{Y}_j \sqcup \mathbb{Y}_\infty$. A topology $\mathfrak{T}(\mathbb{Y})$ on $\mathbb{Y}$ is called *asymptotic relation (between* $(\mathbb{Y}_j)_{j\in J}$ *and* $\mathbb{Y}_\infty$*)* if it's satisfying the following properties:

(A1) The restriction of $\mathfrak{T}(\mathbb{Y})$ to each $\mathbb{Y}_j$ or to $\mathbb{Y}_\infty$ coincides with its respective original topology; moreover, all $\mathbb{Y}_j$ and $\mathbb{Y}_\infty$ are closed in $\mathbb{Y}$.

(A2) For any $y \in \mathbb{Y}_\infty$ there is a net $(y_j)_{j\in J}$ with $y_j \in \mathbb{Y}_j$ converging to $y$ in $\mathbb{Y}$.

(A3) For any subnet $(j)_{j\in\tilde{J}}$ of $(j)_{j\in J}$ and all nets $(x_j)_{j\in\tilde{J}}$, $(y_j)_{j\in\tilde{J}}$ with $x_j, y_j \in \mathbb{Y}_j$ and limits $x = \lim_{j\in\tilde{J}} x_j, y = \lim_{j\in\tilde{J}} y_j \in \mathbb{Y}_\infty$ in $\mathbb{Y}$ one has that $\lim_{j\in\tilde{J}} \mathsf{dist}_j(x_j, y_j) = \mathsf{dist}_\infty(x, y)$.

(A4) For any subnet $(j)_{j\in\tilde{J}}$ of $(j)_{j\in J}$ and all nets $(x_j)_{j\in\tilde{J}}$, $(y_j)_{j\in\tilde{J}}$ with $x_j, y_j \in \mathbb{Y}_j$, limit $x = \lim_{j\in\tilde{J}} x_j \in \mathbb{Y}_\infty$ and $\lim_{j\in\tilde{J}} \mathsf{dist}_j(x_j, y_j) = 0$ it holds $\lim_{j\in\tilde{J}} y_j = x$.

In the instance that $(\mathbb{Y}_j)_{j\in J}$ and $\mathbb{Y}_\infty$ are topological linear spaces over a common coefficient field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ such that the topologies are compatible with the metric structures, an asymptotic relation is called *linear* if $\mathbb{Y}$-convergence is stable with respect to linear combination. I. e. for all nets $(x_j)_{j\in J}$, $(y_j)_{j\in J}$ with $x_j, y_j \in \mathbb{Y}_j$ and $\mathbb{Y}$-limits $x = \lim_{j\in J} x_j, y = \lim_{j\in J} y_j \in \mathbb{Y}_\infty$, and scalars $c_x, c_y \in \mathbb{K}$ the net $(c_x x_j + c_y y_j)_{j\in J}$ converge in $\mathbb{Y}$ to $c_x x + c_y y \in \mathbb{Y}_\infty$. Compare to [KS08, Definition 3.1]. If the topological linear spaces are normed spaces, then one requires that additionally the norms $\big(\|y_j\|_{\mathbb{Y}_j}\big)_{j\in\tilde{J}}$ for any convergent net $(y_j)_{j\in\tilde{J}}$ converge to $\|\lim_{j\in\tilde{J}} y_j\|_{\mathbb{Y}_\infty}$, where $(j)_{j\in\tilde{J}}$ of $(j)_{j\in J}$ is a subnet. Compare to [Töl10, Definition 5.6]. Similarly for the case that the topological linear spaces carry an inner product. ⌟

*Remark* B.2.3 (Weak convergence). The general notion of weak (and weak$*$) convergence is defined as in [Töl10, Definition 5.33]. If we consider a linear asymptotic relation between Hilbert spaces $(H_j)_{j\in J}$ and $H_0$, then that notion of weak convergence basically reduces to [KS03, Definition 2.5]: Let $(j)_{j\in\tilde{J}}$ be a subnet of $(j)_{j\in J}$; then, a net $(u_j)_{j\in\tilde{J}}$ with $u_j \in H_j$ *weakly converges to* $u_0 \in H_0$ if for any net $(w_j)_{j\in\tilde{J}}$ with $w_j \in H_j$ converging to $w_0 \in H_0$ it holds $\lim_{j\in\tilde{J}} \big(u_j\,,w_j\big)_{H_j} = (u_0\,,w_0)_{H_0}$. ⌟

**Definition B.2.4** (Convergence of bounded operators along linear asymptotic relations). Assume linear asymptotic relations between $(\mathbb{B}_j)_{j\in J}$ and $\mathbb{B}_\infty$ as well as between $(\mathbb{E}_j)_{j\in J}$ and $\mathbb{E}_\infty$ such that all involved spaces are Banach spaces. Furthermore, assume a subnet $(j)_{j\in\tilde{J}}$ of $(j)_{j\in J}$ and a net $(T_j)_{j\in\tilde{J}}$ of linear bounded operators with $T_j \in L(\mathbb{B}_j; \mathbb{E}_j)$, $j \in \tilde{J}$. We say that $(T_j)_{j\in\tilde{J}}$ *(strongly) converges to* $T_\infty \in L(\mathbb{B}_\infty; \mathbb{E}_\infty)$ if for any net $(b_j)_{j\in\tilde{J}}$ with $b_j \in \mathbb{B}_j$ and $b_\infty := \lim_{j\in\tilde{J}} b_j \in \mathbb{B}_\infty$ the net $(T_j b_j)_{j\in\tilde{J}}$ converges to $T_\infty b_\infty \in \mathbb{E}_\infty$. ⌟

**Definition B.2.5** (Metric approximation). Assume a asymptotic relation between $(\mathbb{Y}_j)_{j\in J}$ and $\mathbb{Y}_\infty$. Moreover, let $\big(m_j\colon \mathsf{dom}(m_j) \to \mathbb{Y}_j\big)_{j\in J}$ be a net of maps with domains $\mathsf{dom}(m_j) \subseteq \mathbb{Y}_\infty$ for all $j \in J$. It is called *metric approximation (for* $(\mathbb{Y}_j)_{j\in J}$ *and* $\mathbb{Y}_\infty$*)* if it satisfies the following properties:

(Met1) The net of domains $\big(\mathsf{dom}(m_j)\big)_{j\in J}$ is monotone nondecreasing and their union is dense in $\mathbb{Y}_\infty$.

(Met2) For any two points $x, y \in \bigcup_{j\in J} \mathsf{dom}(m_j)$ the distances of images under $m_j$ converge as

$$\lim_{j\in J} \mathsf{dist}_j(m_j(x), m_j(y)) = \mathsf{dist}_\infty(x, y).$$

If the (strong) asymptotic relation is linear, the domains are linear subspaces and the metric approximation consists of linear mappings, then the metric approximation is called *linear*. A metric approximation $\big(m_j\colon \mathsf{dom}(m_j) \to \mathbb{Y}_j\big)_{j\in J}$ is said to be *compatible with a given asymptotic relation on* $\mathbb{Y}$ if for all $y \in \bigcup_{j\in J} \mathsf{dom}(m_j)$ the net $\big(m_j(y)\big)_{j\in J}$ converges to $y$ in $\mathbb{Y}$. Compare to [KS08, Definitions 3.3 and 3.4] as well as [Töl10, Definition 5.24 and 5.26]. ⌟

Due to [KS08, Lemma 3.5] we know that (strong) asymptotic relations and metric approximations walk hand in hand: If either of them exists, there exists an instance of the alternative concept such that they are compatible. This also holds true if you add the word 'linear' everywhere in the previous definition.

Until the end of this subsection, we consider a net of pointed metric measure spaces $\big(\mathbb{Y}_j, \mathsf{dist}_j, \mu_j, \bar{x}_j\big)_{j\in J}$, and a pointed metric measure space $(\mathbb{Y}_\infty, \mathsf{dist}_\infty, \mu_\infty, \bar{x}_\infty)$. Assume that the measure spaces are locally compact Polish spaces with full supported Radon measures as in [KS08, Section 2].

**Definition B.2.6** (pmG-convergence). A net $\big(\varphi_j\colon \mathsf{dom}(\varphi_j) \to \mathbb{Y}_\infty\big)_{j\in J}$ of maps with $\mathsf{dom}(\varphi_j) \subseteq \mathbb{Y}_j$ is called *measure approximation* if it satisfies the properties

(Meas1) For all $j \in J$ the domain of $\varphi_j$ is a Borel set, i. e. $\mathsf{dom}(\varphi_j) \in \mathfrak{B}(\mathbb{Y}_j)$, and $\varphi_j$ is measurable.

(Meas2) The pushforward measures converge weakly, i. e.

$$\lim_{j\in J} \int_{\mathsf{dom}(\varphi_j)} f \circ \varphi_j \, \mathsf{d}\mu_j = \int_{\mathbb{Y}_\infty} f \, \mathsf{d}\mu_\infty \qquad \text{for all } f \in C^0_{\mathsf{c}}(\mathbb{Y}_\infty).$$

Assume the spaces $\mathbb{Y}_j$, $j \in J$, and $\mathbb{Y}_\infty$ are proper in the sense that bounded subsets are relatively compact. We say that they converge in *pointed measured Gromov (pmG) sense* if there is a measure approximation. In this case, the measure approximation is referred to as *pointed measured Gromov approximation*. See [KS08, Definitions 2.4 and 2.5]. Contrast that to [Fuk87, Definition 0.3] where also metric convergence is required. ⌟

**Definition B.2.7** ($L^p$-topology). Let $p \in [1, \infty)$ and $\big(\varphi_j\colon \mathsf{dom}(\varphi_j) \to \mathbb{Y}_\infty\big)_{j\in J}$ be a measure approximation between $(\mathbb{Y}_j)_{j\in J}$ and $\mathbb{Y}_\infty$. For all $j \in J$, $u \in C^0_{\mathsf{c}}(\mathbb{Y}_\infty; \mathbb{R})$ define

$$\Phi_j u := \begin{cases} u \circ \varphi_j & \text{if } u \in \mathsf{dom}(\varphi_j) \\ 0 = 0_{\mathbb{Y}_j} & \text{otherwise} \end{cases}$$
$$\text{and} \quad D(\Phi_j) := \Big\{ u \in C^0_{\mathsf{c}}(\mathbb{Y}_\infty; \mathbb{R}) \;\Big|\; \|\Phi_j u\|_{L^p(\mathbb{Y}_\infty)} < \infty \Big\}.$$

According to [KS08, Proposition 3.13], the net $\big(\Phi_j\colon \mathsf{dom}(\Phi_j) \to L^p(\mathbb{Y}_j)\big)_{j\in J}$ is a linear metric approximation. Hence, it induces on $\bigsqcup_{j\in J} L^p(\mathbb{Y}_j) \;\sqcup\; L^p(\mathbb{Y}_\infty)$ a unique linear asymptotic relation which one calls the *$L^p$-topology*. See [KS08, Definition 3.14]. ⌟

## B.3 ▪ Mosco convergence

As we use generalised Dirichlet form techniques to construct a nice stochastic process associated the Langevin-type generators in this thesis, we present the framework of generalised forms as introduced by W. Stannat, compare to [Sta99, Section I.2] and the summary in [Töl10, Section 2.2.3].

B

*Notation* B.3.1 (generalised forms). Consider a coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ on a Hilbert space $H$ with sector constant $K \in [1, \infty)$. Moreover, we consider a linear operator[1] $(\mathcal{A}, D(\mathcal{A}, H))$ that generates a SCCS on $H$ which restricts to a SCCS on $D(\mathcal{E})$. Denote the generator of the restricted semigroup by $(\mathcal{A}, D(\mathcal{A}, D(\mathcal{E})))$. The adjoint operator $(\mathcal{A}^*, D(\mathcal{A}^*, D(\mathcal{E})'))$ of $(\mathcal{A}, D(\mathcal{A}, D(\mathcal{E})))$ satisfies the same assumptions as $(\mathcal{A}, D(\mathcal{A}, D(\mathcal{E})))$ and furthermore, $D(\mathcal{A}, H) \cap D(\mathcal{E})$ is dense in $D(\mathcal{E})$. Therefore according to [Sta99, Lemma I.2.3], the operator $\mathcal{A}\colon D(\mathcal{A}, H) \cap D(\mathcal{E}) \to D(\mathcal{E})'$ is closable. Denote its closure by $(\mathcal{A}, \mathcal{F})$. The spaces $\mathcal{F}$ and $\mathcal{F}^* := D(\mathcal{A}^*, D(\mathcal{E})') \cap D(\mathcal{E})$ are endowed with graph inner product with respect to $\mathcal{A}$ or $\mathcal{A}^*$ respectively:

$$\begin{aligned} & (u_1\,, u_2)_{\mathcal{F}} := (u_1\,, u_2)_{D(\mathcal{E})} + (\mathcal{A}u_1\,, \mathcal{A}u_2)_{D(\mathcal{E})'} && \text{for all } u_1, u_2 \in \mathcal{F} \\ \text{and}\quad & (w_1\,, w_2)_{\mathcal{F}^*} := (w_1\,, w_2)_{D(\mathcal{E})} + (\mathcal{A}^*w_1\,, \mathcal{A}^*w_2)_{D(\mathcal{E})'} && \text{for all } w_1, w_2 \in \mathcal{F}^*. \end{aligned}$$

Then, the following assertions hold: $\mathcal{F}$ is a Hilbert space, $D(\mathcal{A}, D(\mathcal{E}))$ is dense in $\mathcal{F}$, both $\mathcal{F}$ and $\mathcal{F}^*$ are dense in $D(\mathcal{E})$, and one has the two estimates of dual pairings

$$_{D(\mathcal{E})}\langle u\,, \mathcal{A}u\rangle_{D(\mathcal{E})'} \le 0 \quad \text{and} \quad _{D(\mathcal{E})}\langle w\,, \mathcal{A}^*w\rangle_{D(\mathcal{E})'} \le 0 \qquad \text{for all } u \in \mathcal{F}, w \in \mathcal{F}^*.$$

Compare to [Sta99, Lemmas I.2.5 and I.2.6].

The *generalised form* $\mathcal{GE}$ *(associated to* $(\mathcal{E}, D(\mathcal{E}))$ *and* $(\mathcal{A}, \mathcal{F})$*)* is defined by

$$\mathcal{GE}(u, w) := \begin{cases} \mathcal{E}(u, w) - {}_{D(\mathcal{E})}\langle w\,, \mathcal{A}u\rangle_{D(\mathcal{E})'} & \text{if } u \in \mathcal{F}, w \in D(\mathcal{E}) \\ \mathcal{E}(w, u) - {}_{D(\mathcal{E})}\langle u\,, \mathcal{A}^*w\rangle_{D(\mathcal{E})'} & \text{if } u \in D(\mathcal{E}), w \in \mathcal{F}^* \end{cases}. \tag{B.1}$$

⌟

Consider a net $(H^j)_{j\in J}$ of Hilbert spaces and a Hilbert space $H^0$. Assume that an asymptotic relation on $\mathcal{H} := \bigsqcup_{j\in J} H^j \sqcup H^0$. besides the strong convergence Furthermore, let $(\mathcal{GE}^j)_{j\in J}$ be a net of generalised forms on $(H^j)_{j\in J}$ meaning that $\mathcal{GE}^j$ is a form on $H^j$ respectively for $j \in J$. Moreover, let $\mathcal{GE}^0$ be a generalised form on $H^0$. For every generalised form $\mathcal{GE}^j$, $j \in J \cup \{0\}$, denote by $(\mathcal{E}^j, D(\mathcal{E}^j))$ the associated coercive closed form and by $(\mathcal{A}^j, \mathcal{F}^j)$ the corresponding linear operator. Henceforth, every generalised Dirichlet form is extend to the entire Hilbert space by the artificial value $+\infty$.

The following definition of Mosco convergence of closed quadratic forms is [Töl10, Definition 7.11], compare to [KS03, Definition 2.11]. Afterwards, we show how that notion is translated to the net $(\mathcal{GE}^j)_{j\in J}$ of generalised forms.

**Definition B.3.2** (Mosco convergence). Consider nonnegative, closed quadratic forms $\mathcal{B}^j\colon H^j \to [0, \infty]$ for $j \in J \cup \{0\}$. One says that $(\mathcal{B}^j)_{j\in J}$ *Mosco converges to* $\mathcal{B}^0$ if the following two assertions hold:

(Mosco 1) For every net $(u^j)_{j\in J}$ with $u^j \in H^j$ for $j \in J$ weakly $\mathcal{H}$-convergent to $u^0 \in H^0$ it holds $\mathcal{B}^0(u^0, u^0) \le \liminf_{j\in J} \mathcal{B}^j(u^j, u^j)$.

(Mosco 2) For every $u^0 \in H^0$ there is a net $(u^j)_{j\in J}$ with $u^j \in H^j$ for $j \in J$ strongly $\mathcal{H}$-convergent to $u^0$ such that $\mathcal{B}^0(u^0, u^0) = \lim_{j\in J} \mathcal{B}^j(u^j, u^j)$.

⌟

Recall Theorem 4.1.11 linking Mosco convergence and convergence of spectral structures. In a certain sense this link shall also hold in the nonsymmetric case, i. e. we say that $(\mathcal{GE}^j)_{j\in J}$ Mosco converges to $\mathcal{GE}^0$ if the associated resolvents converge, see (R) below. Therefore, one introduces the following conditions and identifies mutual implication relations.

**Condition B.3.3** (Standard conditions for Mosco convergence). Define the 'asymmetry rate norms' as

$$\Theta^j(u^j) := \left\| \mathcal{GE}^j(\cdot, u^j) + (\cdot\,, u^j)_{H^j} \right\|_{D(\mathcal{E}^j)'} = \sup_{\substack{u \in D(\mathcal{E}^j) \\ \|u\|_{D(\mathcal{E}^j)} = 1}} \left| \mathcal{GE}^j(u, u^j) + (u\,, u^j)_{H^j} \right|$$

for all $u^j \in (\mathcal{F}^j)^*$ and $j \in J \cup \{0\}$. Extend these functionals $\Theta^j$ to $H^j$ artificially by $+\infty$.

[1] If $\mathcal{A} = 0$, then we end up with the classical Dirichlet form theory, see [Sta99, Example I.4.9 part (i)].

(F1) For every net $(u^j)_{j\in J}$ with $u^j \in H^j$ weakly $\mathscr{H}$-convergent to $u^0 \in H^0$ such that $\liminf_{j\in J} \Theta^j(u^j) < \infty$ holds that $u^0 \in D(\mathcal{E}^0)$.

(F1b) (F1), but with $\Theta^j$ replaced by $\|\cdot\|_{H^j}$.

(F2) For any $w^0 \in \mathscr{F}^0$ and every net $(u^j)_{j\in J}$ with $u^j \in D(\mathcal{E}^j)$ weakly $\mathscr{H}$-convergent to $u^0 \in D(\mathcal{E}^0)$ there is a net $(w^j)_{j\in J}$ with $w^0 \in H^0$ strongly $\mathscr{H}$-convergent to $w$ such that $\lim_{j\in J} \mathcal{GE}^j(w^j, u^j) = \mathcal{GE}^0(w^0, u^0)$.

(F2') There is a subspace $W \subseteq H^0$ dense in $(\mathscr{F}^0, \|\cdot\|_{\mathscr{F}^0})$ such that the following assertion holds: For all $w^0 \in W$, $u^0 \in D(\mathcal{E}^0)$, and any net $(u^j)_{j\in\widetilde{J}}$, where $\widetilde{J} \subseteq J$ is a directed subset, with $u^j \in H^j$ for all $j \in \widetilde{J}$ which is weakly $\mathscr{H}$-convergent to $u^0$ and satisfies $\sup_{j\in\widetilde{J}} \Theta^j(u^j) < \infty$ there is a net $(w^j)_{j\in\widetilde{J}}$ with $w^j \in H^j$ strongly $\mathscr{H}$-convergent to $w^0$ with $\liminf_{j\in\widetilde{J}} \mathcal{GE}^j(w^j, u^j) \leq \mathcal{GE}^0(w^0, u^0)$.

(F2'b) (F2'), but with $\Theta^j$ replaced by $\|\cdot\|_{H^j}$.

(R) The net of resolvent operators $\left(G_a^j\right)_{j\in J}$ corresponding to $(\mathcal{GE}^j, D(\mathcal{GE}^j))_{j\in J}$ strongly converges to the resolvent operator $G_a^0$ corresponding to $(\mathcal{GE}^0, D(\mathcal{GE}^0))$. ⌟

Since $\|\cdot\|_{H^j} \leq \Theta^j$ by [Hin98, Lemma 2.2 (ii)], one has both that (F1b) implies (F1) and that (F2'b) implies (F2'). The following theorem was proven originally by M. Hino for $H^{j_1} = H^{j_2}$, $j_1, j_2 \in J \cup \{0\}$, see [Hin98, Theorem 3.1, Corollary 3.3 (i)], and then extended by J. M. Tölle to the varying Hilbert space situation, see [Töl10, Theorem 7.15].

**Theorem B.3.4.**

(i) (F2) implies (F2').

(ii) The following equivalence holds: (F1) $\wedge$ (F2') $\iff$ (F1) $\wedge$ (F2) $\iff$ (R). If $(\mathcal{E}^j, D(\mathcal{E}^j)) = (0, H^j)$ for all $j \in J \cup \{0\}$, then (F1) can be omitted, see [Töl10, Remark 7.19].

(iii) (F1b) and (F2'b) imply (R). ⌟

The next lemma is [Töl10, Proposition 7.18], compare to J. M. Tölle's diploma thesis [Töl06, Proposition 2.45] and [Mat99, Proposition 2.5] for a constant asymptotic relation of Hilbert spaces.

**Lemma B.3.5.** Assume a dense subspace $W$ of $(\mathscr{F}^0, \|\cdot\|_{\mathscr{F}^0})$. Then, the following seemingly weaker assertion is actually equivalent to (F2): For all $w^0 \in W$ and nets $(u^j)_{j\in J}$ with $u^j \in H^j$, $j \in J$, weakly $\mathscr{H}$-convergent to $u^0 \in H^0$ there is a net $(w^j)_{j\in J}$ with $w^j \in H^j$, $j \in J$, strongly $\mathscr{H}$-convergent to $w^0 \in H^0$ such that

$$\lim_{j\in J} \mathcal{GE}^j(w^j, u^j) = \mathcal{GE}^0(w^0, u^0).$$

Coincidentally, we came across [Av10, Theorem 3.5], where S. Andres and M.-K. von Renesse prove Mosco convergence of Dirichlet forms assuming a similar replacement '(Mosco II')' of (F2) that works with a core of the target Dirichlet form. Apparently, the authors derived their result independently of Lemma B.3.5.

# C Miscellaneous

In Remark 4.1.24, we proposed a combination of stratifolds and varifolds. Assume a $\mathscr{b}$-dimensional stratifold $\mathbb{B}$ that is embedded into some $\mathbb{R}^{\mathscr{n}}$ as ambient space and carries a probability measure $\mu_{\mathbb{B}}$. We want to represent $\mathbb{B}$ as measure on a suitable Graßmannian, in the same way as a Riemannian manifold can be represented as varifold, see Example 4.1.3. Define the Graßmannians $\mathbb{G}(0{:}\mathscr{b}, \mathscr{n}) := \bigcup_{j=0}^{\mathscr{b}} \mathbb{G}(j, \mathscr{n})$ and $\mathbb{G}(0{:}\mathscr{b}, \mathbb{R}^{\mathscr{n}}) := \mathbb{R}^{\mathscr{n}} \times \mathbb{G}(0{:}\mathscr{b}, \mathscr{n})$. Furthermore, we define the selector function $\diamond\colon \mathbb{B} \to \mathbb{N}$ that assigns to a point $b$ the index of the stratum containing $b$, i. e. $b \in \mathsf{strat}_{\diamond(b)}(\mathbb{B})$. Moreover, let $\mathfrak{A} := \sigma\Big(\bigcup_{j=0}^{\mathscr{b}} \mathfrak{B}(\mathbb{G}(j, \mathscr{n}))\Big)$ be the $\sigma$-algebra on $\mathbb{G}(0{:}\mathscr{b}, \mathscr{n})$ generated by the union of Borel-$\sigma$-algebras of the $j$-dimensional Graßmannians. Now, consider the transition kernel

$$k_{\mathbb{B}}\colon\ \mathbb{B} \times \mathfrak{A} \to \mathbb{R},\ \left(b, \bar{A}\right) \longmapsto \int_{\mathbb{G}(0:\mathscr{b},\mathscr{n})} \mathbb{1}_{\bar{A}}(S) \quad \delta_{\mathsf{T}_b\left(\mathsf{strat}_{\diamond(b)}(\mathbb{B})\right)}(\mathsf{d}S).$$

Since $k_{\mathbb{B}}(b, \cdot)$ is a probability measure for all $b \in \mathbb{B}$ and $k_{\mathbb{B}}\left(\cdot, \bar{A}\right)$ is measurable for all $\bar{A} \in \mathfrak{A}$, the mapping $k_{\mathbb{B}}$ is a Markov kernel. Then, the assignment

$$\begin{aligned}
\mathbb{V}_{\mathbb{B}}(A) &:= \int_{\mathbb{B}} k_{\mathbb{B}}(b, \mathsf{pr}_2(A)) \cdot \mathbb{1}_{\mathsf{pr}_1(A)}(b)\ \mu_{\mathbb{B}}(\mathsf{d}b) = \int_{\mathsf{pr}_1(A)} k_{\mathbb{B}}(b, \mathsf{pr}_2(A))\ \mu_{\mathbb{B}}(\mathsf{d}b) \\
&= \int_{\mathsf{pr}_1(A)} \int_{\mathbb{G}(0:\mathscr{b},\mathscr{n})} \mathbb{1}_{\mathsf{pr}_2(A)}(S) \quad \delta_{\mathsf{T}_b\left(\mathsf{strat}_{\diamond(b)}(\mathbb{B})\right)}(\mathsf{d}S)\ \mu_{\mathbb{B}}(\mathsf{d}b)
\end{aligned}$$

for all $A \in \mathfrak{B}(\mathbb{G}(0{:}\mathscr{b}, \mathbb{R}^{\mathscr{n}}))$ defines a probability measure on $\mathbb{G}(0{:}\mathscr{b}, \mathbb{R}^{\mathscr{n}})$. The measure $\mathbb{V}_{\mathbb{B}}$ is the *stratlvarifold representing* $\mathbb{B}$ *(in* $\mathbb{R}^{\mathscr{n}}$*)*.

Now, it should be clear that the ambient space $\mathbb{R}^{\mathscr{n}}$ can be replaced by a smooth manifold $\mathbb{Y}$ as in Definition 4.1.1. Also in the spirit of Definition 4.1.1, one defines general $\mathscr{k}$-dimensional stratlvarifolds as Radon measures on $\mathbb{G}(0{:}\mathscr{k}, \mathbb{Y})$.

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C

# Acronyms

**AHHM** Abstract Hilbert space Hypocoercivity Method. 1–3, 23, 45, 49, 57, 58, 64–66, 69, 71, 98, 99

**CSBI** Cauchy-Bunyakovsky-Schwarz inequality. 28, 32, 37, 44

**GH** Gromov-Hausdorff convergence. 75, 76

**GLE** generalised Langevin equation. 49, 68, 69, 71, 72, 113, 137

**pmG** pointed measured Gromov convergence. 77, 87, 88, 92, 96, 115

**pmGH** pointed measured Gromov-Hausdorff convergence. 76, 77, 83, 86–89, 92, 96, 97

**SAD** decomposition $L = S - A$ on the core $D$. 25, 40, 58, 70, 71, 98, 99, 101, 104

**SCCR** strongly continuous contraction resolvent. 77

**SCCS** strongly continuous contraction semigroup. 25, 29, 30, 64, 77, 99, 103, 116

**SDAE** stochastic differential-algebraic equation. 37, 46

**SDE** stochastic differential equation. 19, 20, 23, 24, 38, 43, 56, 65, 68, 69, 97–99, 101, 113

# Index

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# Symbols

C

Q configuration manifold. 23, 38, 56, 69

$\mathsf{Q}^{\varepsilon}\mathbb{X}$ configuration stratifold bundle. 83, 97

$\mathsf{Ric}_{\mathsf{b}}$ Ricci curvature tensor of the Riemannian manifold $(\mathbb{B}, \mathsf{b})$. 19

$\cdot\mathbb{S}\cdot$ sphere $\mathbb{S}^{d-2}$ with two additional points $\pm e_d$. 87

$\mathsf{strat}_k(\mathbb{Y})$ $k$th stratum of the stratifold $\mathbb{Y}$. 79

$\mathsf{T}\mathbb{B}$ tangent bundle over the manifold $\mathbb{B}$. 8

$\mathsf{T}^k\mathbb{B}$ $k$th order tangent bundle over the manifold $\mathbb{B}$. 9

t Sasaki metric associated to an underlying Riemannian manifold. 16

tl tangential lift of a vector or vector field. 38

$\mathbb{U}$ an open ball in some metric space, $\mathbb{U}(o, r)$ when centered at $o$ with radius $r$. 1

u unit Sasaki metric associated to an underlying Riemannian manifold. 16

$\mathsf{U}\mathbb{B}$ unit tangent bundle over the manifold $\mathbb{B}$. 11

$\mathcal{V}$ canonical vector field. 14

$\mathsf{V}\mathbb{E}$ vertical bundle in the fibre bundle $\mathbb{E} \to \mathbb{B}$. 14

v vertical Sasaki metric associated to an underlying Riemannian manifold. 16

vl vertical lift of a vector or vector field. 14

v|u vertical unit Sasaki metric associated to an underlying Riemannian manifold. 16

$\mathbb{X}_{\oslash}$ top stratum of the stratifold $(\mathbb{X}, \mathbf{C}(\oslash))$. 93

# List of Figures



Copyright for all the pictures in the figures 1 and 2 belongs to Fraunhofer ITWM. Of course, the figures 1.1 and 1.3 are done via Ti*k*Z. The figures 1.2, 4.1, 4.2, 4.3, and 4.4. are rendered via `Asymptote`. The trajectories in figures 3.1, 3.2, and 3.3. are generated via a semi-implicit Euler-Maruyama scheme implemented in `Python`. They all share the same fixed random seed. The Figure 1.4 is generated employing the `Julia`-package `Manifolds.jl`, see [Axe+21].

# Scientific career

**Maximilian Constantin Mertin**

## Publications

- Grothaus M., Mertin M.; Hypocoercivity of Langevin-type dynamics on abstract smooth manifolds. *Stochastic Processes and their Applications*. 2021; https://doi.org/10.1016/j.spa.2021.12.007
- Grothaus M., Mertin M., Stilgenbauer P.; Hypocoercivity for geometric Langevin equations motivated by fibre lay-down models arising in industrial application. *GAMM - Mitteilungen*. 2018; https://doi.org/10.1002/gamm.201800011

## Academic background

| | | |
|---|---|---|
| since April 2018 | PhD studies<br>Technische Universität Kaiserslautern | |
| December 2017 | Master thesis<br>Technische Universität Kaiserslautern | |
| | *Title:* | »On a manifold-valued spherical Langevin equation« |
| | *Supervisor:* | Prof. Dr. M. Grothaus |
| 2015–2017 | Master studies in mathematics with minor in computer science<br>Technische Universität Kaiserslautern | |
| September 2015 | Bachelor thesis | |
| | *Title:* | »Diffusions on two-component configuration spaces« |
| | *Supervisor:* | Prof. Dr. M. Grothaus and Dr. T. Fattler |
| 2012–2015 | Bachelor studies in mathematics with minor in computer science<br>Technische Universität Kaiserslautern | |
| summer term 2012 | Bachelor studies in computer science with minor in mathematics<br>Universität Koblenz-Landau | |
| 2012 | Abitur at Gymnasium auf dem Asterstein, Koblenz | |