The inductive McKay–Navarro condition for finite groups of Lie type

Birte Johansson

Vom Fachbereich Mathematik der Technischen
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1. Gutachter: Prof. Dr. G. Malle
2. Gutachterin: Prof. Dr. A. A. Schaeffer Fry

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Abstract

In the representation theory of finite groups, the so-called local-global conjectures assert a relation between the representation theory of a finite group and one of its local subgroups. The McKay–Navarro conjecture claims that the action of a set of Galois automorphisms on certain ordinary characters of the local and global group is equivariant. Navarro, Späth, and Vallejo reduced the conjecture to a problem about simple groups in 2019 and stated an inductive condition that has to be verified for all finite simple groups. In this work, we give an introduction to the character theory of finite groups and state the McKay–Navarro conjecture and its inductive condition. Furthermore, we recall the definition of finite groups of Lie type and present results regarding their structure and their representation theory. In the second part of this work, we verify the inductive McKay–Navarro condition for various families of finite groups of Lie type.

In defining characteristic, most groups have already been considered by Ruhstorfer. We show that the inductive condition also holds for the groups with exceptional graph automorphisms, the Suzuki and Ree groups, the groups $\mathsf{B}_n(2)$ for $n \geq 2$, as well as for the simple groups of Lie type with non-generic Schur multiplier in their defining characteristic. This completes the verification of the inductive McKay–Navarro condition in defining characteristic. We further consider the Suzuki and Ree groups and verify the inductive condition for all primes. On the way, we show that there exists a Galois-equivariant Jordan decomposition for their irreducible characters. Moreover, we consider some families of groups of Lie type that do not admit a generic choice of a local subgroup. We show that the inductive condition is satisfied for the prime $\ell = 3$ and the groups $\mathrm{PSL}_3(q)$ with $q \equiv 4, 7 \mod 9$, $\mathrm{PSU}_3(q)$ with $q \equiv 2, 5 \mod 9$, and $\mathsf{G}_2(q)$ with $q \equiv 2, 4, 5, 7 \mod 9$. Further, we verify the inductive condition for the prime $\ell = 2$ and $\mathsf{G}_2(3^f)$ for $f \geq 1$, ${}^3\mathsf{D}_4(q)$, and ${}^2\mathsf{E}_6(q)$ where q is an odd prime power.

Zusammenfassung

In der Darstellungstheorie endlicher Gruppen stellen sogenannte lokal-globale Vermutungen einen Zusammenhang zwischen den Darstellungen einer endlichen Gruppe und einer ihrer lokalen Untergruppen her. Die McKay–Navarro-Vermutung besagt, dass die Wirkung von bestimmten Galoisautomorphismen auf einigen gewöhnlichen irreduziblen Charakteren beider Gruppen equivariant ist. Navarro, Späth und Vallejo führten diese Vermutung 2019 auf ein Problem über endliche einfache Gruppen zurück und formulierten eine induktive Bedingung, die für alle endlichen einfachen Gruppen überprüft werden muss. In dieser Arbeit geben wir eine kurze Einführung in die Charaktertheorie endlicher Gruppen und formulieren die McKay–Navarro-Vermutung sowie ihre induktive Bedingung. Außerdem stellen wir endliche Gruppen vom Lie-Typ und wichtige Resultate über ihre Charaktertheorie vor. Im zweiten Teil wird die induktive McKay–Navarro-Bedingung für verschiedene Familien endlicher Gruppen vom Lie-Typ nachgewiesen.

In definierender Charakteristik wurde die Bedingung für die meisten Gruppen bereits von Ruhstorfer betrachtet. Wir zeigen, dass die Gruppen mit exzeptionellen Graphautomorphismen, die Suzuki- und Ree-Gruppen, $\mathsf{B}_n(2)$ für $n \geq 2$ und die einfachen Gruppen mit exzeptionellem Schur-Multiplikator die induktive Bedingung in ihrer definierenden Charakteristik erfüllen. Dies schließt den Fall der definierenden Charakteristik ab. Außerdem betrachten wir die Suzuki- und Ree-Gruppen und weisen die induktive Bedingung für alle Primzahlen nach. Dabei zeigen wir, dass es eine Galois-equivariante Jordan-Zerlegung für ihre irreduziblen Charaktere gibt. Zudem betrachten wir einige Familien von Gruppen, die keine generische Wahl einer lokalen Untergruppe zulassen. Wir zeigen, dass die Gruppen $\mathrm{PSL}_3(q)$ mit $q \equiv 4, 7 \mod 9$, $\mathrm{PSU}_3(q)$ mit $q \equiv 2, 5 \mod 9$ und $\mathsf{G}_2(q)$ mit $q \equiv 2, 4, 5, 7 \mod 9$ die induktive Bedingung für die Primzahl $\ell = 3$ erfüllen. Außerdem weisen wir die induktive Bedingung für $\ell = 2$ und die Gruppen $\mathsf{G}_2(3^f)$ mit $f \geq 1$, ${}^3\mathsf{D}_4(q)$ und ${}^2\mathsf{E}_6(q)$ für ungerade Primpotenzen q nach.

Contents

Introduction

The main object of this thesis is the study of the inductive McKay–Navarro condition for some finite groups of Lie type. With this work, we contribute to the verification of the McKay–Navarro conjecture for all finite groups.

Local-global conjectures. The McKay–Navarro conjecture is one of the so-called local-global conjectures in the representation theory of finite groups. Local-global conjectures assert a relation between the representation theory of a finite group and some of its subgroups. Thus, they describe a correspondence between a *global* setting that is the group itself and its *local* subgroups that are usually normalizers of subgroups of prime power order.

The most basic local-global conjecture is the McKay conjecture. It claims that there is a correspondence between certain irreducible characters of a group and of normalizers of Sylow subgroups.

Conjecture (McKay)**.** *Let G be a finite group, ℓ a prime, and R a Sylow ℓ-subgroup of G. There exists a bijection*

$$\{\chi \in \mathrm{Irr}(G) \mid \ell \nmid \chi(1)\} \to \{\psi \in \mathrm{Irr}(N_G(R)) \mid \ell \nmid \psi(1)\}.$$

This correspondence has originally been observed in [**McK72**] for the prime $\ell = 2$. In [**Isa73**], Isaacs proved the above claim for groups of odd order at any prime ℓ. The first formal statement of the McKay conjecture appeared in [**Alp76**] together with a refinement to the setting of blocks. In the following years, many generalizations, refinements, or related conjectures have been stated, e.g. the Isaacs–Navarro conjecture, Alperin's weight conjecture, or Dade's counting conjectures. For an overview of these related conjectures, see for example [**Nav18**].

In this thesis, we are interested in a refinement of the McKay conjecture that was proposed by Navarro in 2004, see [**Nav04**]. The values of characters of groups are sums of roots of unity in $\mathbb{C}$ and we can therefore consider the action of Galois automorphisms in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ on the character values. This induces an action on the irreducible characters of a group. Navarro observed that the actions of certain Galois automorphisms on the global and local characters in the McKay conjecture are permutation isomorphic. This yields the McKay–Navarro conjecture that is also known as the Galois–McKay conjecture.

Reduction to simple groups. The claim of the McKay conjecture is amazingly simple and can be understood right after introducing irreducible characters of groups as in Section 1.1. It has been investigated and verified for many families or classes of groups. However, so far no conceptual proof of it is known. In the last years, there has been a lot of progress on the proof of the McKay conjecture by using the Classification of Finite Simple Groups (CFSG).

As a first step, Isaacs, Malle, and Navarro proved the following reduction theorem for the McKay conjecture by introducing the so-called *inductive McKay condition.* Thereby, they reduced the McKay conjecture to a problem about finite simple groups.

Theorem. [**IMN07**] *Assume that the inductive McKay condition holds for all finite simple non-abelian groups and a fixed prime ℓ. Then, the McKay conjecture is true for ℓ.*

In a second and final step to prove the McKay conjecture, one now has to verify the inductive McKay condition for all non-abelian simple groups appearing in the CFSG. For the prime $\ell = 2$, this has already been completed in [**MS15**]. Therefore, the McKay conjecture holds for $\ell = 2$.

For the McKay–Navarro conjecture, Navarro, Späth, and Vallejo proved a similar reduction theorem in [**NSV20**]. Thus, they introduced the *inductive McKay–Navarro condition* in [**NSV20**, Definition 3.1] that now has to be verified for all finite simple groups.

We will state the McKay–Navarro conjecture and its inductive condition in the first chapter of this thesis after introducing ordinary representations and characters of finite groups and some preliminary character theoretic results and constructions.

Finite groups of Lie type. From the CFSG we know that, besides the alternating and sporadic groups, all non-abelian finite simple groups are finite groups of Lie type. Thus, if we want to verify inductive conditions for all finite simple groups, we naturally have to study finite groups of Lie type and their representation theory.

Starting with the work of Deligne and Lusztig, the representation theory of finite groups of Lie type has been investigated thoroughly in the last 50 years. Thus, there is an extensive and far-reaching theory available that allows us to study representations of a whole infinite family of groups of a certain type at the same time. We can use these tools to consider the inductive McKay–Navarro condition for finite groups of Lie type.

We present some basic results about linear algebraic groups and their structure in Chapter 2. Using this, we give the definition of finite groups of Lie type and mention some of their properties that we need in later chapters. In Chapter 3, we focus on their representation theory and introduce concepts such as Lusztig induction, Jordan decomposition of characters, Harish-Chandra induction, Gelfand–Graev characters, and a generalization of these ideas to finite disconnected reductive groups.

Main results. Starting from Chapter 4, we apply the theory that has been introduced in the previous chapters to the inductive McKay–Navarro condition. There, we first collect some results that will be useful in the proof of the following main results.

Finite groups of Lie type arise from connected reductive linear algebraic groups that are defined over an algebraically closed field of positive characteristic p. If we consider the inductive McKay–Navarro condition for a finite group of Lie type and a prime ℓ, then the situation depends on whether ℓ is the same as the characteristic p or not. If these primes are equal, then we say that we are in the case of defining characteristic.

In [**Ruh21**], Ruhstorfer verified the inductive McKay–Navarro condition for finite groups of Lie type in their defining characteristic with some exceptions. We considered these exceptions in [**Joh22**], yielding the following theorem.

Theorem A. *The inductive McKay–Navarro condition is satisfied in the defining characteristic for the groups* $\mathsf{B}_2(2^i)$, $\mathsf{G}_2(3^i)$, $\mathsf{F}_4(2^i)$, $\mathsf{B}_n(2)$ *for integers* $i \geq 1$, $n \geq 2$, *for the Suzuki and Ree groups, as well as for* $\mathsf{B}_2(2)'$, $\mathsf{G}_2(2)'$, ${}^2\mathsf{G}_2(3)'$, ${}^2\mathsf{F}_4(2)'$, *and the simple groups of Lie type with non-generic Schur multiplier.*

This completes the verification of the inductive McKay–Navarro condition in defining characteristic. We present these results in Chapter 5.

Inspired by the work on Suzuki and Ree groups in their defining characteristic, we also consider the inductive McKay–Navarro condition for the Suzuki and Ree groups and arbitrary primes. Since they are the only finite groups of Lie type that do not arise from Frobenius endomorphisms, it seems natural to study them separately. The following results can be found in [**Joh21**].

Theorem B. *The inductive McKay–Navarro condition is satisfied for the Suzuki and Ree groups for all primes ℓ.*

To prove this, we generalize the existence of a Galois-equivariant Jordan decomposition for connected reductive groups with connected center and Frobenius morphisms from [**SV20**] to the Suzuki and Ree groups. Note that the main work for the proof of the following theorem has already been done by Srinivasan and Vinroot and we only consider the case of Suzuki and Ree groups, see Proposition 6.6.

Theorem C. *Let $\mathbf{G}$ be a connected reductive group with connected center defined over $\overline{\mathbb{F}_q}$ for some prime power q, F a Steinberg endomorphism, and $(\mathbf{G}^*, F^*)$ in duality with $(\mathbf{G}, F)$. Let m be the exponent of $\mathbf{G}^F$ and ζ_m a primitive m-th root of unity. For a Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, let $b \in \mathbb{Z}$ with $\gcd(b, m) = 1$ such that $\zeta_m^\sigma = \zeta_m^b$. If $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ has Jordan decomposition (s, ν) for $s \in \mathbf{G}^{*F^*}$ semisimple and ν a unipotent character of $C_{\mathbf{G}^{*F^*}}(s)$, then χ^σ has Jordan decomposition (s^b, ν^σ).*

Theorem B and Theorem C are proven in Chapter 6 apart from some special cases that are considered separately in Chapter 7.

Similar to these special cases for the Suzuki and Ree groups, there are also other finite groups of Lie type and primes that cannot be considered together with the other groups of Lie type of the same type. In the inductive McKay–Navarro condition, we choose and investigate a normal subgroup that contains the normalizer of a Sylow subgroup. For most groups of Lie type, there is a canonical choice of this local subgroup that has been introduced in [**Mal07**], see Section 2.7. There are some exceptions to this canonical choice for the primes $\ell = 2$ and $\ell = 3$. For $\ell = 2$, these are the symplectic groups and it has already been shown that they satisfy the inductive McKay–Navarro condition in [**RSF22**]. In [**Joh21**], we considered the exceptions that occur for $\ell = 3$.

Theorem D. *The inductive McKay–Navarro condition holds for $\ell = 3$ and the groups*

(a) $\mathrm{PSL}_3(q)$ *with* $q \equiv 4, 7 \mod 9$*,*

(b) $\mathrm{PSU}_3(q)$ *with* $q \equiv 2, 5 \mod 9$*, and*

(c) $\mathsf{G}_2(q)$ *with* $q \equiv 2, 4, 5, 7 \mod 9$*.*

We present the proof of this theorem in Chapter 7.

Besides the work on the case of defining characteristic, most progress on the inductive McKay–Navarro condition has been made for the prime $\ell = 2$ in [**SF22**] and [**RSF22**]. We complete the verification of the inductive McKay–Navarro condition for $\ell = 2$ and some families of groups for which it has been already shown that parts of the inductive McKay–Navarro condition are true.

Theorem E. *The inductive McKay–Navarro condition holds for the prime $\ell = 2$ and the simple groups $\mathsf{G}_2(3^f)$ for $f \geq 1$, ${}^3\mathsf{D}_4(q)$, and ${}^2\mathsf{E}_6(q)$ for any odd prime power q.*

The proof of this theorem is contained in Chapter 8.

CHAPTER 1

Representations and the McKay–Navarro conjecture

In this chapter, we give a short introduction to the representation theory of finite groups and state the McKay–Navarro conjecture and its inductive condition. Throughout, G always denotes a finite group.

1.1. Representations, characters, and projective representations

We start by introducing some background theory from the representation theory of finite groups. The definitions and results from this section can be found in any introductory book on representation theory, see e.g. [**Isa76**].

In an attempt to be as self-contained as possible, we present the basic definitions and constructions that are most important in this work. However, we do not aim for completeness and assume that the reader is familiar with the basic notions.

Representation theory is a powerful tool to investigate the structure of groups. The idea is easy: We embed a given group G into a group of matrices in order to obtain a better understanding of the group G itself. As it turns out, considering different choices of such a group homomorphism for different matrix groups leads to a beautiful theory that is deeply connected to the structure of the group. In the following, we only study finite groups and their representation theory. Besides being an interesting concept itself, representation theory can also be used to prove results that are purely group-theoretic. Therefore, the representation theory of finite groups (and further also of all kinds of mathematical objects) has evolved into a huge research area.

Studying representations of finite groups highly depends on the considered matrix groups and the characteristics of the fields they are defined over. In this work, we are only concerned with ordinary representation theory, i.e. complex representations and characters. Therefore, we do not consider representations over other fields and all mentioned representations and characters are defined over the complex numbers.

Definition 1.1. Let V be any complex vector space of finite dimension n. A *complex representation* of a finite group G is a group homomorphism

$$\mathcal{R} : G \to \mathrm{GL}(V).$$

We say that $\mathcal{R}$ has *degree* n. Two representations $\mathcal{R}_1$ and $\mathcal{R}_2$ of G are called *similar* if there exists an invertible matrix M such that $\mathcal{R}_1(g) = M\mathcal{R}_2(g)M^{-1}$ for all $g \in G$. Then, we write $\mathcal{R}_1 \sim \mathcal{R}_2$.

Similarity defines an equivalence relation on the set of representations. We can also consider substructures of representations. In the following, we are often interested in representations that do not have any proper substructures.

Definition 1.2. Let $\{0\} \subsetneq W \subseteq V$ be a subspace such that $\mathcal{R}(g)(W) \subseteq W$ for all $g \in G$. We can define a representation by

$$\mathcal{R}' : G \to \mathrm{GL}(W), \quad g \mapsto \mathcal{R}(g)|_W.$$

This is called a *subrepresentation of* $\mathcal{R}$. A representation is called *irreducible* if it does not have a proper subrepresentation and has positive degree.

Every representation of a group corresponds to a module of the group algebra $\mathbb{C}G$. Many of the structures that are introduced in the following can also be considered in terms of modules. However, we only present them in the language of representations and characters here.

The following elementary result holds in much more generality and can also be stated for representations in any characteristic.

Proposition 1.3 (Schur's lemma)**.** *Let $\mathcal{R}$ be an irreducible representation of G of degree n. If $M \in M_n(\mathbb{C})$ satisfies*

$$M\mathcal{R}(g) = \mathcal{R}(g)M$$

for all $g \in G$, then $M = \zeta I_n$ for some $\zeta \in \mathbb{C}$.

Here and in the following, we write $M_n(\mathbb{C})$ for the set of all $n \times n$-matrices over $\mathbb{C}$ and I_n for the identity matrix of dimension n.

We now define characters of groups.

Definition 1.4. Let $\mathcal{R}$ be a complex representation of G. Then

$$\chi : G \to \mathbb{C}, \quad g \mapsto \mathrm{Tr}(\mathcal{R}(g))$$

is called the *character* of G afforded by $\mathcal{R}$. The *degree of* χ is the degree of $\mathcal{R}$. A character of degree 1 is called a *linear character*.

Characters of irreducible representations are called *irreducible characters*. We denote the set of irreducible characters of G by $\mathrm{Irr}(G)$. Let ℓ be a prime. We call a character of G an *ℓ'-character* if its degree is not divisible by ℓ and denote the subset of complex irreducible ℓ'-characters by $\mathrm{Irr}_{\ell'}(G)$.

It is easy to see that similar representations afford the same character. Conversely, it is more involved to show that two representations are only similar if they afford the same character. Thus, characters are a convenient way to study representations of a group.

Definition 1.5. A *class function* of a group G is a function from G to $\mathbb{C}$ that is constant on the conjugacy classes of G. We denote the vector space of class functions of G by $\mathrm{CF}(G)$.

The set of irreducible characters forms a $\mathbb{C}$-basis of the space of class functions of a group. This basis is even orthogonal with respect to the following inner product.

Definition 1.6. For $\chi, \psi \in \mathrm{CF}(G)$, we can define an inner product

$$\langle \chi, \psi \rangle_G := \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

We sometimes omit the group name in the index. Since the irreducible characters form a basis of the space of class functions, every class function can be written as a sum of irreducible characters. More precisely, the following holds.

Lemma 1.7. *For every $\psi \in \mathrm{CF}(G)$ we have*

$$\psi = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi, \psi \rangle \cdot \chi.$$

Definition 1.8. Let $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{CF}(G)$ such that we have $\langle \chi, \psi \rangle \neq 0$. Then, we say that χ is an *irreducible constituent* of ψ. The integer $\langle \chi, \psi \rangle$ is called the *multiplicity* of χ in ψ.

We have already mentioned in the introduction that we will be interested in the values of characters and the action of Galois automorphisms on them. Character values inherit some important properties from their affording representation.

Lemma 1.9. *Let χ be a character of G with degree n and $g \in G$.*

(a) $\chi(g) = \epsilon_1 + \cdots + \epsilon_n$ *where* $\epsilon_i \in \mathbb{C}$ *such that* $\epsilon_i^{\mathrm{ord}(g)} = 1$.

(b) $\chi(g) \in \mathbb{Q}(\zeta_m)$ *where m is the exponent of G and $\zeta_m \in \mathbb{C}$ is a primitive m-th root of unity.*

(c) $\chi(g^{-1}) = \overline{\chi(g)}$ *where $\overline{\chi(g)}$ denotes the complex conjugate of $\chi(g)$ in $\mathbb{C}$.*

These claims directly follow after diagonalizing the matrix image of the affording representation. In the next proposition, we collect some elementary results that are used in later chapters.

Proposition 1.10. (a) *The degree of a character divides the group order.*

(b) *The number of irreducible characters of G is the same as the number of conjugacy classes of G.*

(c) *(Degree formula) We have*

$$|G|^2 = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

In the following, we will need basic operations of characters like induction, restriction, inflation and deflation. We give the definitions here to introduce our notation.

Definition 1.11. Let $H \leq G$ and $\psi \in \mathrm{CF}(H)$, $\chi \in \mathrm{CF}(G)$.

(i) The *induction of ψ from H to G* is given by

$$\mathrm{Ind}_H^G : G \to \mathbb{C}, \quad g \mapsto \frac{1}{|H|} \sum_{h \in G} \psi^\circ(h^{-1}gh)$$

where for all $y \in G$

$$\psi^\circ(y) := \begin{cases} \psi(y) & \text{if } y \in H, \\ 0 & \text{else.} \end{cases}$$

(ii) The *restriction of χ from G to H* is given by

$$\mathrm{Res}_H^G(\chi) := \chi|_H.$$

Assume that N is a normal subgroup of G and let $\pi : G \to G/N$ be the quotient homomorphism.

(iii) For $\theta \in \mathrm{Irr}(G/N)$, the *inflation of θ from G/N to G* is

$$\mathrm{Inf}_{G/N}^G(\theta) := \theta \circ \pi \in \mathrm{Irr}(G).$$

(iv) For $\chi \in \mathrm{Irr}(G)$ with $N \subseteq \ker(\chi)$, the *deflation of χ from G to G/N* is given by

$$\mathrm{Def}_{G/N}^G(\chi)(gN) := \chi(g)$$

for all $gN \in G/N$. Here, the set $\ker(\chi) := \{g \in G \mid \chi(g) = \chi(1)\}$ is called the *kernel of χ*.

Induction and restriction behave very naturally with respect to the inner product of characters defined above.

Proposition 1.12 (Frobenius reciprocity)**.** *Let $H \leq G$, $\psi \in \mathrm{CF}(H)$, $\chi \in \mathrm{CF}(G)$. Then*

$$\langle \mathrm{Ind}_H^G(\psi), \chi \rangle_G = \langle \psi, \mathrm{Res}_H^G(\chi) \rangle_H.$$

Given two different subgroups of G, we can induce a class function of one of the subgroups to G and then restrict it to the other subgroup. This is described by Mackey's formula. Here, we only state a restricted version that can be generalized to arbitrary subgroups of G, see e.g. [**Isa76**, Problem 5.6].

Theorem 1.13 (Mackey formula)**.** *Let $H, K \leq G$ such that $G = HK$ and $\psi \in \mathrm{CF}(H)$. Then, we have*

$$\mathrm{Res}^G_K(\mathrm{Ind}^G_H(\psi)) = \mathrm{Ind}^K_{H\cap K}(\mathrm{Res}^H_{H\cap K}(\psi)).$$

Given a representation or character, we can also consider the determinant or its values on the center of G.

Definition 1.14. Let $\mathcal{R}$ be a representation of G affording χ.

(i) The *determinant of the character* χ *(or determinantal character)* is the linear character

$$\det(\chi) : G \to \mathbb{C}, \quad g \mapsto \det(\mathcal{R}(g)).$$

(ii) The *central character of* χ is the linear character $\theta : Z(G) \to \mathbb{C}$ given by

$$\mathcal{R}(z) = \theta(z) \cdot I_{\chi(1)}$$

for all $z \in Z(G)$. We can also extend this definition to the *center of* χ

$$Z(\chi) := \{g \in G \mid |\chi(g)| = \chi(1)\} = \{g \in G \mid \mathcal{R}(g) = \zeta I_{\chi(1)} \text{ for some } \zeta \in \mathbb{C}\}.$$

Note that the central character is well-defined by Schur's lemma.

We will also need the notion of projective representations that generalizes the concept of representations.

Definition 1.15. Let V be a finite-dimensional complex vector space. A *projective representation* is a map $\mathcal{P} : G \to \mathrm{GL}(V)$ such that for every $g, h \in G$ there exists some $\alpha(g, h) \in \mathbb{C}^\times$ satisfying

$$\mathcal{P}(g)\mathcal{P}(h) = \alpha(g, h)\mathcal{P}(gh).$$

The map $\alpha : G \times G \to \mathbb{C}^\times$ is called the *factor set* of $\mathcal{P}$.

1.2. Actions on characters

As already mentioned in the introduction, the McKay conjecture asserts that we can find bijections between subsets of irreducible characters of a group and one of its local subgroups. Navarro's refinement proclaims that this bijection is compatible with the action of certain Galois automorphisms on the character sets. In this section, we give a formal definition of this action and of the action of group automorphism on characters.

1.2.1. Action of group automorphisms. The automorphism group of G is denoted by $\mathrm{Aut}(G)$. For a subgroup $H \leq G$, we write $\mathrm{Aut}(G)_H$ for the setwise stabilizer of H in $\mathrm{Aut}(G)$, i.e.

$$\mathrm{Aut}(G)_H := \{\kappa \in \mathrm{Aut}(G) \mid \kappa(H) \subseteq H\}.$$

In particular, the inner automorphisms induced by elements of $N_G(H)$ are the inner automorphisms in $\mathrm{Aut}(G)_H$.

We often use the element $g \in G$ itself or c_g to denote the corresponding conjugator isomorphism and write

$$c_g(h) = {}^g h = h^{g^{-1}} = ghg^{-1}$$

for all $h \in H$.

Definition 1.16. Let N be a normal subgroup of G. We define an action of $\kappa \in \mathrm{Aut}(G)_N$ on the characters of N by setting

$$\chi^\kappa(x) := \chi(x^{\kappa^{-1}}) = \chi(\kappa(x))$$

for all $x \in N$ and all characters χ of N. In the same way, we obtain an action of $\mathrm{Aut}(G)_N$ on the (projective) representations of N.

In particular, this also gives us an action of the inner automorphisms of G on the characters of N and we write $\chi^{c_g}(x) = \chi^g(x) = \chi(x^{g^{-1}}) = \chi(gxg^{-1})$ for an element $g \in G$. We write $\mathrm{Inn}(G)$ for the set of inner automorphisms in $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G \mid H) \leq \mathrm{Inn}(G)$ for the inner automorphisms induced by the elements of a subgroup $H \leq G$.

The elements of $A \leq \mathrm{Aut}(G)$ fixing a character χ of N form the stabilizer A_χ. The stabilizer of χ under the action of G is also called the *inertia group* of χ in G and denoted by $I_G(\chi) = G_\chi$.

Since the action of $\mathrm{Aut}(G)_N$ preserves irreducibility and character degrees, it restricts to an action on $\mathrm{Irr}(N)$ and also on $\mathrm{Irr}_{\ell'}(N)$ for all primes ℓ.

1.2.2. Action of Galois automorphisms. Let $\zeta_{\exp(G)} \in \mathbb{C}$ be a fixed primitive $\exp(G)$-th root of unity. We know that all values of characters of G lie in $\mathbb{Q}(\zeta_{\exp(G)})$. Thus, $\mathcal{G} := \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ acts on the character values where $\mathbb{Q}^{\mathrm{ab}}$ is the subfield of $\mathbb{C}$ generated by $\mathbb{Q}$ and all roots of unity. This allows us to define an action of $\mathcal{G}$ on the characters of G.

Definition 1.17. For $\sigma \in \mathcal{G}$ and a character χ of G, we set

$$\chi^\sigma(g) := (\chi(g))^\sigma$$

for all $g \in G$. This defines a character χ^σ. Analogously, σ acts on a (projective) representation $\mathcal{P}$ defined over $\mathbb{Q}^{\mathrm{ab}}$ by applying the automorphism to the matrix entries, i.e.

$$(\mathcal{P}^\sigma(g))_{ij} := \mathcal{P}(g)^\sigma_{ij}.$$

We always find an integer b that is coprime to $\exp(G)$ such that $\zeta^\sigma_{\exp(G)} = \zeta^b_{\exp(G)}$. This b fully determines the action of σ on the characters of G and we say that σ is *described by* b. Again, this restricts to an action on $\mathrm{Irr}(N)$ and $\mathrm{Irr}_{\ell'}(N)$ for all primes ℓ.

We denote the orbit of a character $\chi \in \mathrm{Irr}(G)$ under the action of $\mathcal{G}' \subseteq \mathcal{G}$ by $\chi^{\mathcal{G}'}$. Later, we will consider the setwise stabilizer of the orbit $\chi^{\mathcal{G}'}$ under the action of an automorphism group $A \leq \mathrm{Aut}(G)$. We denote it by $A_{\chi^{\mathcal{G}'}}$.

1.3. Clifford theory and character extensions

We have already seen above that we can use constructions like induction and restriction to relate the characters of a group to those of its subgroups. In the special situation of a normal subgroup, this relation is understood quite well and can be described by the so-called Clifford theory.

In this section, we state the main results from Clifford theory and introduce the concepts of character extensions and projective representations associated to some character of a normal subgroup.

1.3.1. Clifford theory. Clifford theory describes the relationship between the character theory of G and of a normal subgroup $N \leq G$. We state some basic results that can be found in many textbooks about character theory, see e.g. [**Isa76**, Chapter 6].

Proposition 1.18. *Let N be a normal subgroup of G and $\chi \in \mathrm{Irr}(G)$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of $\mathrm{Res}^G_N(\chi)$. Then, we have*

$$\mathrm{Res}^G_N(\chi) = m \cdot \sum_{i=1}^{t} \theta_i$$

where $\theta = \theta_1, \dots, \theta_t$ are the distinct G-conjugates of θ and $m = \langle \mathrm{Res}^G_N(\chi), \theta \rangle_N$.

In this case, we say that χ *lies above* θ and θ *is a character under* χ. It directly follows from Frobenius reciprocity that all characters over θ are a constituent of $\mathrm{Ind}^G_N(\theta)$. The set of all irreducible characters over a fixed $\theta \in \mathrm{Irr}(N)$ is denoted by $\mathrm{Irr}(G \mid \theta)$.

Proposition 1.19 (Clifford correspondence)**.** *Let N be a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$. Then, we have a bijection*

$$\mathrm{Irr}(I_G(\theta) \mid \theta) \to \mathrm{Irr}(G \mid \theta), \quad \psi \mapsto \mathrm{Ind}_{I_G(\theta)}^{G}(\psi).$$

The characters $\psi \in \mathrm{Irr}(I_G(\theta) \mid \theta)$ and $\mathrm{Ind}_{I_G(\theta)}^{G}(\psi) \in \mathrm{Irr}(G \mid \theta)$ are called *Clifford correspondents* in the situation of the theorem.

1.3.2. Character extensions. As we will see in the next section, extensions of characters and representations play an important role in the inductive McKay–Navarro condition.

Definition 1.20. Let H be a subgroup of G, $\psi \in \mathrm{Irr}(H)$ and χ a character of G. We say that χ *extends* ψ if we have $\chi|_H = \psi$. Similarly, a (projective) representation $\mathcal{P}$ of G *extends* a representation $\mathcal{R}$ of H if we have $\mathcal{P}|_H = \mathcal{R}$.

If $H \trianglelefteq G$, it is clear that ψ can only be extended to G if it is G-invariant. Even if this is the case, it does not ensure that an extension exists. The situation is different if G/H is cyclic.

Lemma 1.21. *Let $N \trianglelefteq G$ be a normal subgroup, $\psi \in \mathrm{Irr}(N)$ a G-invariant character and assume that G/N is cyclic. Then, ψ extends to G.*

This can be proven elementary, see e.g. [**Nav18**, Theorem 5.1] which even gives us a construction of the representation affording the extension.

The next proposition is due to Gallagher and tells us about the different extensions of an irreducible character.

Proposition 1.22. *Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$ be an extension of $\psi \in \mathrm{Irr}(N)$. For $\beta \in \mathrm{Irr}(G/N)$, the characters $\beta\chi$ are irreducible and distinct for different β. They form all of the irreducible constituents of $\mathrm{Ind}_N^G(\psi)$ and we have*

$$\mathrm{Ind}_N^G(\psi) = \sum_{\beta \in \mathrm{Irr}(G/N)} \beta(1) \cdot \beta\chi.$$

Proof. This is proven in [**Isa76**, Corollary 6.17] and the last claim follows from Frobenius reciprocity since we have $\langle \mathrm{Ind}_N^G(\psi), \beta\chi \rangle = \langle \psi, \beta(1)\psi \rangle = \beta(1)$. □

We need a possibility to generalize the notion of character extensions in the case of characters that cannot be extended. This is provided by projective representations.

Lemma 1.23. *Let $N \trianglelefteq G$ and ψ be an irreducible character of N. There exists a projective representation $\mathcal{P}$ of G_ψ such that its restriction to N affords ψ and*

$$\mathcal{P}(ng) = \mathcal{P}(n)\mathcal{P}(g), \quad \mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n)$$

for all $n \in N$ and $g \in G_\psi$. It can be chosen such that it is defined over $\mathbb{Q}^{\mathrm{ab}}$ and its factor set takes roots of unity values.

We say that such a projective representation $\mathcal{P}$ is *associated to* ψ. The first part of this lemma is well-known, see e.g. [**Isa76**, (11.2)]. The second part can be found in [**NSV20**, Corollary 1.2] and allows us to assume that all (projective) representations that occur in the following are defined over $\mathbb{Q}^{\mathrm{ab}}$.

To state the inductive McKay–Navarro condition, we have to consider the action of group and Galois automorphisms on projective representations associated to a character.

Lemma 1.24. [**NSV20**, Lemma 1.4] *In the setting of Lemma 1.23, let $\mathcal{P}$ be a projective representation associated to ψ and $(g,\sigma) \in G \times \mathcal{G}$ with $\psi^{g\sigma} := (\psi^g)^\sigma = \psi$. Then, there exists a unique function $\mu_{g\sigma} : G_\psi \to \mathbb{C}^\times$ that is constant on cosets of N with $\mu_{g\sigma}(1) = 1$ such that*

$$\mathcal{P}^{g\sigma} := (\mathcal{P}^g)^\sigma \sim \mu_{g\sigma}\mathcal{P}$$

where $\sim$ denotes similarity between projective representations.

In the following, we use $\mu_{g\sigma}$ to denote the transition function defined here. In the situation of the lemma, we can say more about the $\mu_{g\sigma}$ if we know that ψ extends to G and $\mathcal{P}$ is a representation of G. Then, $\mathcal{P}^{g\sigma}$ is again a representation affording an extension of ψ and it follows from Proposition 1.22 that $\mu_{g\sigma}$ is a linear character of G/N.

1.4. The McKay–Navarro conjecture and its inductive condition

In this section, we state the McKay conjecture and the McKay–Navarro conjecture. Further, we define universal covering groups and use this to present the inductive McKay and McKay–Navarro condition. Finally, we introduce the notion of character triples that can be used to rephrase the inductive conditions in a more condensed way.

1.4.1. The McKay–Navarro conjecture. We now present the McKay–Navarro conjecture that is the main object of this work. The McKay conjecture has already been stated in the introduction. We recall it here to emphasize its relation to the McKay–Navarro conjecture.

Conjecture (McKay)**.** *Let G be a finite group, ℓ a prime, and R a Sylow ℓ-subgroup of G. There exists a bijection*

$$\mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(N_G(R)).$$

To state Navarro's refinement of this conjecture, we have to specify a subset of $\mathcal{G}$ depending on a fixed prime ℓ.

Definition 1.25. Let $\mathcal{H}_\ell \subseteq \mathcal{G}$ be the subset of all Galois automorphisms that map all roots of unity $\zeta \in \mathbb{C}$ with $\ell \nmid \mathrm{ord}(\zeta)$ to some ℓ-th power ζ^{ℓ^k} for some integer k.

Definition 1.26. For any integer z and prime ℓ we define integers $z_\ell, z_{\ell'} \in \mathbb{Z}$ by $z = z_\ell z_{\ell'}$ where z_ℓ is an ℓ-power and $\ell \nmid z_{\ell'}$. We say that z_ℓ is the *ℓ-part of z* and $z_{\ell'}$ is the *ℓ'-part of z.*

We will write $\mathcal{H} := \mathcal{H}_\ell$ if the index is clear from the context. In particular, if G is given then $\sigma \in \mathcal{G}$ is contained in $\mathcal{H}_\ell$ if and only if σ is described by b and b is an ℓ-power modulo $\exp(G)_{\ell'}$.

In 2004, Navarro claimed that the bijection from the McKay conjecture can be chosen equivariant under the action of Galois automorphisms in $\mathcal{H}_\ell$ [**Nav04**].

Conjecture (McKay–Navarro)**.** *Let G be a finite group, ℓ a prime, and R a Sylow ℓ-subgroup of G. There exists an $\mathcal{H}_\ell$-equivariant bijection*

$$\mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(N_G(R)).$$

In the same work, Navarro also proposed a blockwise version of this conjecture. However, we are only concerned with this ordinary version of the McKay–Navarro conjecture in the following.

The conjecture itself has been verified for groups with cyclic Sylow ℓ-subgroups and for the sporadic groups by Navarro himself. Further, it has been checked for solvable groups by Dade and for symmetric groups by Fong, see [**Nav04**]. Nath and Brunat–Nath also showed that it holds for the alternating groups, see [**Nat09**] and [**BN21**].

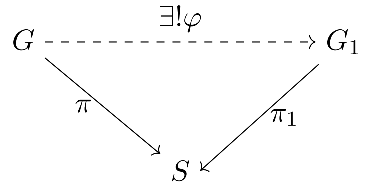
1.4.2. Universal covering groups. We later state the inductive McKay–Navarro condition in terms of finite simple groups. In order to do this, we first have to define universal covering groups. We follow [**Nav18**, Appendix B]. For more information, see also [**Asc00**, Section 33].

Definition 1.27. Let S be a finite group.

(i) A *central extension (or covering) of* S is a group G together with a surjective homomorphism $\pi : G \to S$ such that $\ker(\pi) \subseteq Z(G)$.

(ii) Let G_1, G_2 be central extensions of S with corresponding homomorphisms π_1, π_2. A *morphism of central extensions* is a group homomorphism $\varphi : G_1 \to G_2$ such that $\pi_1 = \pi_2 \circ \varphi$.

(iii) A central extension $\pi : G \to S$ is called *universal covering group of* S if for every central extension $\pi_1 : G_1 \to S$ there is a unique morphism of central extensions $\varphi : G \to G_1$.



We say that $\ker(\pi)$ is the *Schur multiplier* of S.

Lemma 1.28. [**Nav18**, Lemma B.3, Lemma B.4] *If a finite group S has a universal covering group and Schur multiplier, then they are unique up to isomorphism. If S is perfect, i.e. it equals its commutator subgroup $[S, S]$, then it possesses a universal covering group that is also perfect.*

In this work, we often have to consider universal coverings of finite simple groups. The following result allows us to identify automorphisms of a group and of its universal covering.

Lemma 1.29. [**Nav18**, Lemma B.8] *Let S be a perfect group and $\pi : G \to S$ a universal covering group. For every $\kappa \in \mathrm{Aut}(S)$, there exists a unique $\widehat{\kappa} \in \mathrm{Aut}(G)$ such that $\pi \circ \widehat{\kappa} = \kappa \circ \pi$. Further, $\widehat{\kappa}$ stabilizes* $\ker(\pi)$ *and every automorphism of G stabilizing* $\ker(\pi)$ *arises in this way from an automorphism of S.*

1.4.3. Inductive conditions. As mentioned in the introduction, one tries to prove local-global conjectures in two steps:

(1) Reduction of the conjecture to a problem about simple groups: If all simple groups involved in a finite group G satisfy a possibly stronger inductive condition, then the conjecture holds for G itself. A simple group S is involved in G if $S \cong K/N$ for some $N \lhd K \leq G$.

(2) Verification of the inductive condition for all finite simple groups (using the CFSG).

We now state the inductive condition for the McKay and McKay–Navarro conjecture.

For the McKay conjecture, the first step has already been completed in 2007 by Isaacs, Malle and Navarro [**IMN07**], yielding the inductive McKay condition.

Condition (Inductive McKay condition)**.** For a finite non-abelian simple group S and a prime ℓ dividing $|S|$, let G be a universal covering group of S. Let $R \in \mathrm{Syl}_\ell(G)$ and $\Gamma := \mathrm{Aut}(G)_R$. Then, S satisfies the *inductive McKay condition for* ℓ if the following holds:

(1) *(Equivariance condition)* There exists a Γ-stable subgroup $N_G(R) \subseteq N \subsetneq G$ and a Γ-equivariant bijection
$$\Omega : \mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(N).$$

(2) *(Extension condition)* For all $\chi \in \mathrm{Irr}_{\ell'}(G)$, there exist projective representations $\mathcal{P}$ of $G \rtimes \Gamma_\chi$ and $\mathcal{P}'$ of $N \rtimes \Gamma_{\Omega(\chi)}$ associated to χ respectively $\Omega(\chi)$ such that the respective factor sets α, α' coincide on $(N \rtimes \Gamma_\chi) \times (N \rtimes \Gamma_\chi)$ and the scalar matrices $\mathcal{P}(c)$, $\mathcal{P}'(c)$ correspond to the same scalar for all $c \in C_{G \rtimes \Gamma_\chi}(G)$.

Note that the matrix $\mathcal{P}(c)$ is indeed scalar for all $c \in C_{G \rtimes \Gamma_\chi}(G)$ by Schur's Lemma since it has to commute with all $\mathcal{P}(g)$ for $g \in G$.

This is not the original inductive McKay condition as stated in [**IMN07**] but an equivalent formulation due to Späth [**Spä12**]. The inductive McKay condition has been verified for many groups and group series by several authors. It was shown in [**Mal08b**] that it is true for the alternating and sporadic groups, leaving us with the simple groups of Lie type. In their defining characteristic (see Section 2.7), the inductive McKay condition holds by [**Spä12**] building on work in [**Bru09**], [**BH11**], and [**Mas10**]. For other primes, we know that the inductive condition holds for all series of groups of Lie type except possibly those of type D_n and ${}^2\mathsf{D}_n$: In [**Mal07**], [**Spä09**], and [**Spä10**], Malle and Späth constructed a suitable local subgroup and a bijection Ω. Some exceptions to these constructions have been settled in [**Mal08a**]. In an extensive series of papers, Cabanes and Späth showed that this bijection Ω satisfies the required equivariance and extension properties for type A, B, C, E, F_4, and G_2 in [**CS13**], [**CS17a**], [**CS17b**], and [**CS19**].

Using this work, Malle and Späth completed the proof of the McKay conjecture for $\ell = 2$ in [**MS15**]. This shows us that the strategy of reducing the conjecture and then verifying the inductive condition actually leads to results about the conjecture itself.

The McKay–Navarro conjecture has been reduced to a problem about simple groups in 2019 by Navarro, Späth, and Vallejo [**NSV20**]. The resulting inductive condition is the following [**NSV20**, Definition 3.1]:

Condition (Inductive McKay–Navarro condition)**.** For a finite non-abelian simple group S and a prime ℓ dividing $|S|$, let G be a universal covering group of S. Let $R \in \mathrm{Syl}_\ell(G)$ and $\Gamma := \mathrm{Aut}(G)_R$. Then, S satisfies the *inductive McKay–Navarro condition for* ℓ if the following holds:

(1) *(Equivariance condition)* There exists a Γ-stable subgroup $N_G(R) \subseteq N \subsetneq G$ and a $\Gamma \times \mathcal{H}_\ell$-equivariant bijection

$$\Omega : \mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(N).$$

(2A) *(Extension condition)* For all $\chi \in \mathrm{Irr}_{\ell'}(G)$, there exist projective representations $\mathcal{P}$ of $G \rtimes \Gamma_\chi$ and $\mathcal{P}'$ of $N \rtimes \Gamma_{\Omega(\chi)}$ with entries in $\mathbb{Q}^{\mathrm{ab}}$ associated to χ respectively $\Omega(\chi)$ such that the respective factor sets α, α' take roots of unity values, coincide on $(N \rtimes \Gamma_\chi) \times (N \rtimes \Gamma_\chi)$, and the scalar matrices $\mathcal{P}(c)$, $\mathcal{P}'(c)$ correspond to the same scalar for all $c \in C_{G \rtimes \Gamma_{\chi^{\mathcal{H}}}}(G)$.

(2B) Further, the transition functions μ_a and μ'_a arising from $\mathcal{P}$ and $\mathcal{P}'$ as in Lemma 1.24 agree on $N \rtimes \Gamma_\chi$ for all $a \in (\Gamma \times \mathcal{H}_\ell)_\chi$.

Theorem 1.30. [**NSV20**, Theorem A] *Let G be a finite group and ℓ a prime. If all simple groups involved in G satisfy the inductive McKay–Navarro condition for ℓ, then the McKay–Navarro conjecture holds for G and p.*

We can see that the inductive McKay–Navarro condition refines the inductive McKay condition: In (1), we require an additional $\mathcal{H}_\ell$-equivariance of the bijection; in (2A) we need projective representations over $\mathbb{Q}^{\mathrm{ab}}$; and (2B) is a new condition. Thus, we can often use the work that has already been done on the inductive McKay condition and extend it to verify the inductive McKay–Navarro condition.

1.4.4. Character triples and $\mathcal{H}$-triples. The inductive McKay–Navarro condition can also be stated in terms of $\mathcal{H}$-triples. We do not use this notion very often and write down the full corresponding conditions instead. Since the inductive McKay–Navarro condition has originally been stated in the language of $\mathcal{H}$-triples in [**NSV20**, Definition 3.1], we still give the basic definitions of character triples and their relations. This will also be useful to shorten the statement of some results in later chapters.

Definition 1.31. [**NSV20**, Section 1] Let H be a finite group and G a normal subgroup.

(i) If $\chi \in \mathrm{Irr}(G)$ is H-invariant, then we say that (H, G, χ) is a *character triple*.

(ii) If $\chi \in \mathrm{Irr}(G)$ satisfies
$$\{\chi^h \mid h \in H\} \subseteq \{\chi^\sigma \mid \sigma \in \mathcal{H}\},$$
then (H, G, χ) is called an *$\mathcal{H}$-triple* and denoted by $(H, G, \chi)_\mathcal{H}$.

Thus, in the statement of the inductive McKay–Navarro condition, $(G \rtimes \Gamma_{\chi^\mathcal{H}}, G, \chi)$ and $(N \rtimes \Gamma_{\psi^\mathcal{H}}, N, \psi)$ are $\mathcal{H}$-triples for all $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(N)$.

Character triples and $\mathcal{H}$-triples can be ordered. Since we only need an ordering of $\mathcal{H}$-triples in the following, we only give this definition.

Definition 1.32. [**NSV20**, Definition 1.5] Let $(H, G, \chi)_\mathcal{H}$ and $(M, N, \psi)_\mathcal{H}$ be $\mathcal{H}$-triples. Then, we write $(H, G, \chi)_\mathcal{H} \geq_c (M, N, \psi)_\mathcal{H}$ if

(i) $H = GM$, $G \cap M = N$, $C_H(G) \subseteq M$;

(ii) $(M \times \mathcal{H})_\chi = (M \times \mathcal{H})_\psi$;

(iii) there exist projective representations $\mathcal{P}$ of H_χ and $\mathcal{P}'$ of M_ψ over $\mathbb{Q}^{\mathrm{ab}}$ associated with χ and ψ such that the corresponding factor sets take roots of unity values, coincide on $M_\chi \times M_\chi$ and the scalar matrices $\mathcal{P}(c)$, $\mathcal{P}'(c)$ correspond to the same scalar for all $c \in C_H(G)$;

(iv) for every $a \in (M \times \mathcal{H})_\psi$ the functions μ_a and μ'_a agree on M_ψ.

Lemma 1.33. *In the setting of the inductive McKay–Navarro condition, the extension conditions* (2A) *and* (2B) *are equivalent to*
$$(G \rtimes \Gamma_{\chi^\mathcal{H}}, G, \chi)_\mathcal{H} \geq_c (N \rtimes \Gamma_{\chi^\mathcal{H}}, N, \Omega(\chi))_\mathcal{H}$$
for all $\chi \in \mathrm{Irr}_{\ell'}(G)$.

Proof. It is clear that this ordering of $\mathcal{H}$-triples implies (2A) and (2B). Conversely, we immediately know
$$G \rtimes \Gamma_{\chi^\mathcal{H}} = G(N \rtimes \Gamma_{\chi^\mathcal{H}}), \quad G \cap (N \rtimes \Gamma_{\chi^\mathcal{H}}) = N$$
since $N \leq G$. For $g \in G$, $\kappa \in \Gamma$ and with $\mathrm{id} \in \Gamma$ denoting the identity automorphism, we have
$$\begin{aligned}(g, \kappa) \in C_{G \rtimes \Gamma}(G) \quad &\Leftrightarrow \quad (g, \kappa)(h, \mathrm{id}) = (h, \mathrm{id})(g, \kappa) \quad \text{for all } h \in G\\ &\Leftrightarrow \quad (g\kappa(h), \kappa) = (hg, \kappa) \quad \text{for all } h \in G\\ &\Leftrightarrow \quad g\kappa(h) = hg \quad \text{for all } h \in G\\ &\Leftrightarrow \quad \kappa(h) = g^{-1}hg \quad \text{for all } h \in G.\end{aligned}$$
Since conjugation with elements in $N_G(R)$ forms the subgroup of inner automorphisms in Γ and acts trivially on the characters of G and N, we can conclude
$$C_{G \rtimes \Gamma_{\chi^\mathcal{H}}}(G) = \{(zg, c_g^{-1}) \mid g \in N_G(R), z \in Z(G)\} \subseteq N \rtimes \Gamma_{\chi^\mathcal{H}}.$$
For a $\Gamma \times \mathcal{H}$-equivariant bijection Ω,
$$((N \rtimes \Gamma_{\chi^\mathcal{H}}) \times \mathcal{H})_\chi = ((N \rtimes \Gamma_{\chi^\mathcal{H}}) \times \mathcal{H})_{\Omega(\chi)} = ((N \rtimes \Gamma) \times \mathcal{H})_\chi = ((N \rtimes \Gamma) \times \mathcal{H})_{\Omega(\chi)}$$
follows by equivariance and since N acts trivially on all characters of G and N. $\square$

CHAPTER 2

Finite groups of Lie type

In this chapter, we introduce finite groups of Lie type and some basic ideas about their structure. The Classification of Finite Simple Groups (CFSG) tells us that every finite simple group is a member of one of the following families:

- Cyclic groups of prime order.
- Alternating groups A_n of degree $n \geq 5$.
- The simple groups of Lie type.
- The 26 sporadic groups and the Tits group ${}^2\mathsf{F}_4(2)'$.

Cyclic groups and alternating groups each form one family of infinitely many finite simple groups. As we will see in the following chapter, finite groups of Lie type consist of many families of infinitely many finite simple groups. Therefore, the simple groups of Lie type form "most of" the finite simple groups. Since we want to verify the inductive conditions for all finite simple groups, we have to study finite groups of Lie type.

We start with the description of characteristic structures in algebraic groups and their associated Lie algebras. We continue with the definition of root data, the classification of connected reductive groups and the description of isogeny types. After this, we define parabolic and Levi subgroups and state the Chevalley relations for semisimple groups.

In the second part of this chapter, we define Steinberg endomorphisms, construct finite groups of Lie type, and give information about their orders and maximal tori. Next, we study simple groups of Lie type and state results about their universal coverings and automorphism groups. We continue with the concept of dual groups and construct a regular embedding of a connected reductive group into a group with connected center. Finally, we describe a canonical choice of the local group in the inductive McKay–Navarro condition for simple groups of Lie type.

2.1. Basics about linear algebraic groups

Linear algebraic groups are affine varieties that possess a group structure such that inversion and the group operation are morphisms of varieties. We assume that the reader is familiar with the basic concepts about linear algebraic groups as provided in [**Gec03**], [**MT11**], or [**GM20**, Chapter 1]. In this section, we recall the basic definitions and statements that we need in this thesis. We first consider the structure of linear algebraic groups themselves and then define the associated Lie algebra.

Let $\mathbf{G}$ be a linear algebraic group over an algebraically closed field $\mathbf{k}$. In the following, all algebraic groups that we are interested in are linear and we often do not explicitly mention this property. We follow [**MT11**].

2.1.1. Jordan decomposition, tori, and related structures. We first introduce possible properties of algebraic groups and their elements.

Due to the following theorem, we can understand algebraic groups as in the Zariski topology closed subgroups of general linear groups of finite dimensional vector spaces.

Theorem 2.1. [**Gec03**, Corollary 2.4.4] *Every linear algebraic group over* $\mathbf{k}$ *can be embedded as a closed subgroup of the general linear group*

$$\mathrm{GL}_n(\mathbf{k}) := \{A \in \mathbf{k}^{n\times n} \mid \det(A) \neq 0\} \quad \textit{for some } n \geq 1.$$

This provides an intuitive understanding of algebraic groups aside from their formal definition in terms of algebraic varieties. Naturally, $\mathrm{GL}_n(\mathbf{k})$ is an example of an algebraic group.

In the following, we often consider connected algebraic groups and are moreover interested in whether certain subgroups are also connected.

Definition 2.2. A linear algebraic group is called *connected* if it cannot be written as a disjoint union of two non-empty open subsets (with respect to the Zariski topology). Every linear algebraic group $\mathbf{G}$ can be written as the disjoint union of connected components. We denote the connected component containing the identity element by $\mathbf{G}^\circ$.

Note that $\mathbf{G}^\circ$ is a closed normal subgroup of finite index in $\mathbf{G}$.

In the following, we are only interested in algebraic groups defined over algebraically closed fields of positive characteristic. Since this simplifies some concepts, from now on we restrict our considerations to this case and assume $\mathrm{char}(\mathbf{k}) = p > 0$.

Definition 2.3. (i) An element $g \in \mathbf{G}$ is called *unipotent* if the order of g is a power of p. We write $\mathbf{G}_u$ for the set of unipotent elements in $\mathbf{G}$. We say that $\mathbf{G}$ *is unipotent* if $\mathbf{G}_u = \mathbf{G}$.

(ii) An element $g \in \mathbf{G}$ is called *semisimple* if its image under an embedding as in Theorem 2.1 is diagonalizable. We write $\mathbf{G}_s$ for the set of semisimple elements in $\mathbf{G}$.

Note that the definition of semisimple elements does not depend on the choice of the considered embedding. Every element of $\mathbf{G}$ can be written in terms of semisimple and unipotent elements.

Proposition 2.4 (Jordan decomposition of elements)**.** [**MT11**, Theorem 2.5] *Every element $g \in \mathbf{G}$ has a unique decomposition $g = us = su$ where $s \in \mathbf{G}$ is semisimple and $u \in \mathbf{G}$ is unipotent.*

We now define certain subgroups that occur in algebraic groups and tell us a lot about their structure.

Definition 2.5. (i) An algebraic group is called a *torus* if it is isomorphic to a direct product of a finite number of copies of $\mathbf{k}^\times$.

(ii) A *Borel subgroup of* $\mathbf{G}$ is a maximal closed, connected, solvable subgroup of $\mathbf{G}$.

Tori are therefore connected, abelian, and isomorphic to a group of diagonal matrices. Further, they consist of semisimple elements. Since tori are closed, connected and solvable, every torus of $\mathbf{G}$ is contained in a Borel subgroup of $\mathbf{G}$. We are often interested in the *maximal tori* of a group that are maximal with respect to inclusion.

The following results show that maximal tori and Borel subgroups are characteristic for the algebraic group they are contained in.

Proposition 2.6. (a) [**MT11**, Theorem 6.4] *All Borel subgroups of* $\mathbf{G}$ *are conjugate.*

(b) [**MT11**, Corollary 6.5] *All maximal tori of* $\mathbf{G}$ *are conjugate.*

(c) [**MT11**, Corollary 6.11(a)] *If* $\mathbf{G}$ *is connected, every semisimple element of* $\mathbf{G}$ *lies in a maximal torus of* $\mathbf{G}$.

Example 2.7. We consider the general linear group $\mathrm{GL}_n := \mathrm{GL}_n(\mathbf{k})$. Let $T_n := T_n(\mathbf{k})$ be the subgroup of diagonal matrices of GL_n and $B_n := B_n(\mathbf{k})$ be the subgroup of upper triangular matrices of GL_n. It is obvious that T_n is a maximal torus of GL_n and one can easily show that B_n is a Borel subgroup of GL_n, see [**MT11**, Example 6.7].

We now define the character group of an algebraic group. Note that this notion differs from the definitions in Chapter 1. However, $\mathbf{G}$ is in general not a finite group and the different usage of the same expression should thereby not lead to confusion.

Definition 2.8. Let $\mathbf{G}$, $\mathbf{H}$ be algebraic groups.

(i) A map $\varphi : \mathbf{G} \to \mathbf{H}$ is called a morphism of algebraic groups if it is a morphism of varieties, i.e. it can be defined by polynomial functions in the coordinates of the varieties, and a group homomorphism.

(ii) A morphism of algebraic groups $\mathbf{G} \to \mathbf{k}^\times$ is called a *character of* $\mathbf{G}$. The set of characters of $\mathbf{G}$ is called the *character group* of $\mathbf{G}$ and denoted by $X(\mathbf{G})$.

(iii) A morphism of algebraic groups $\mathbf{k}^\times \to \mathbf{G}$ is called a *cocharacter of* $\mathbf{G}$. The set of cocharacters of $\mathbf{G}$ is denoted by $Y(\mathbf{G})$.

We can define a so-called *perfect pairing* between the character and cocharacter group of a torus.

Lemma 2.9. [**MT11**, Proposition 3.6] *Let $\mathbf{T}$ be a torus with character group $X := X(\mathbf{T})$ and set of cocharacters $Y := Y(\mathbf{T})$. Then, the map*

$$\langle\ ,\ \rangle : X \times Y \to \mathbb{Z}, \quad \chi(\varphi(t)) = t^{\langle \chi, \varphi \rangle}$$

defines a perfect pairing between X and Y, i.e. any homomorphism $X \to \mathbb{Z}$ is of the form $\chi \mapsto \langle \chi, \varphi \rangle$ for some $\varphi \in Y$ and any homomorphism $Y \to \mathbb{Z}$ is of the form $\varphi \mapsto \langle \chi, \varphi \rangle$ for some $\chi \in X$.

We now define the radical of a group and the related notions of being semisimple and reductive. Note that this is not analogous to the definition of unipotent groups, i.e. in general semisimple groups do not consist of semisimple elements.

Definition 2.10. (i) The maximal closed connected solvable normal subgroup of $\mathbf{G}$ is called the *radical* $R(\mathbf{G})$ *of* $\mathbf{G}$.

(ii) The maximal closed connected normal unipotent subgroup of $\mathbf{G}$ is called the *unipotent radical* $R_u(\mathbf{G})$ *of* $\mathbf{G}$.

(iii) The group $\mathbf{G}$ is called *semisimple* if it is connected and we have $R(\mathbf{G}) = \{1\}$.

(iv) The group $\mathbf{G}$ is called *reductive* if we have $R_u(\mathbf{G}) = \{1\}$.

2.1.2. Lie algebra and root spaces. To every algebraic group, we can associate a Lie algebra. We now define the Lie algebra of $\mathbf{G}$, its weight spaces and roots. This also allows us to define subgroups in the algebraic group itself, the so-called root subgroups.

Definition 2.11. Let $\mathbf{k}[\mathbf{G}]$ be the coordinate ring of $\mathbf{G}$.

(i) A *(point) derivation* of $\mathbf{k}[\mathbf{G}]$ at $x \in \mathbf{G}$ is a $\mathbf{k}$-linear map $D : \mathbf{k}[\mathbf{G}] \to \mathbf{k}$ such that

$$D(fg) = f(x)D(g) + g(x)D(f)$$

for all $f, g \in \mathbf{k}[\mathbf{G}]$.

(ii) All point derivations at x form the *tangent space* $T_x(\mathbf{G})$ *of* $\mathbf{G}$ *at* x.

(iii) The *Lie algebra* $\mathrm{Lie}(\mathbf{G}) := T_1(\mathbf{G})$ of $\mathbf{G}$ is the tangent space of $\mathbf{G}$ at the identity.

(iv) Let $\mathbf{H}$ be an algebraic group over $\mathbf{k}$ and $\varphi : \mathbf{G} \to \mathbf{H}$ a morphism of algebraic groups. For any $x \in \mathbf{G}$, there is a natural linear map

$$d_x\varphi : T_x(\mathbf{G}) \to T_{\varphi(x)}(\mathbf{H})$$

that is called the *differential of* φ *at* x.

(v) For $g \in \mathbf{G}$, let c_g be the inner automorphism of $\mathbf{G}$ defined by $c_g(x) = gxg^{-1}$ for all $x \in \mathbf{G}$. We define the *adjoint representation of* $\mathbf{G}$ by

$$\mathrm{Ad} : \mathbf{G} \to \mathrm{GL}(\mathrm{Lie}(\mathbf{G})), \quad g \mapsto (\mathrm{Ad}(g) := d_1 c_g : \mathrm{Lie}(\mathbf{G}) \to \mathrm{Lie}(\mathbf{G})).$$

The Lie algebra of $\mathrm{GL}_n(\mathbf{k})$ is naturally isomorphic to the vector space of all $n \times n$-matrices $M_n(\mathbf{k})$. For an algebraic group $\mathbf{G}$ we can therefore use the embedding from Theorem 2.1 to obtain an embedding $\mathrm{Lie}(\mathbf{G}) \subseteq M_n(\mathbf{k})$. We can define a *Lie product* on $\mathrm{Lie}(\mathbf{G})$ via $[A, B] = AB - BA$ for all $A, B \in \mathrm{Lie}(\mathbf{G})$.

Definition 2.12. Let $\mathbf{T} \subseteq \mathbf{G}$ be a maximal torus and $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})$.

(i) The *weight space of* $\chi \in X(\mathbf{T})$ *under the action of* $\mathbf{G}$ *on* $\mathfrak{g}$ is the set

$$\mathfrak{g}_\chi := \{x \in \mathfrak{g} \mid \mathrm{Ad}(t)(x) = \chi(t)x \text{ for all } t \in \mathbf{T}\}.$$

(ii) The *roots of* $\mathbf{G}$ *with respect to* $\mathbf{T}$ are the non-zero characters that have non-zero weight spaces. Thus, the set of roots of $\mathbf{G}$ with respect to $\mathbf{T}$ is given by

$$\Phi(\mathbf{G}) := \{\chi \in X(\mathbf{T}) \mid \chi \neq 0,\ \mathfrak{g}_\chi \neq 0\}.$$

(iii) The *Weyl group of* $\mathbf{G}$ *with respect to* $\mathbf{T}$ is $\mathbf{W}_\mathbf{G}(\mathbf{T}) := N_\mathbf{G}(\mathbf{T})/C_\mathbf{G}(\mathbf{T})$. If $\mathbf{G}$ is connected reductive, we have $C_\mathbf{G}(\mathbf{T}) = \mathbf{T}$ and thereby $\mathbf{W}_\mathbf{G}(\mathbf{T}) = N_\mathbf{G}(\mathbf{T})/\mathbf{T}$.

We fix a maximal torus $\mathbf{T} \subseteq \mathbf{G}$ and write $\Phi := \Phi(\mathbf{G})$ for the set of roots with respect to $\mathbf{T}$. We can now state a structure theorem for connected reductive groups.

Theorem 2.13. [**MT11**, Theorem 8.17] *Let* $\mathbf{G}$ *be connected reductive and* $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})$.

(a) *With* $\mathfrak{g}_0 := \mathrm{Lie}(\mathbf{T})$*, we have the root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

and $\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Phi$.

(b) *For each* $\alpha \in \Phi$*, there exists a morphism of algebraic groups*

$$u_\alpha : \mathbf{k}^+ \to \mathbf{G} \quad \textit{with} \quad t u_\alpha(c) t^{-1} = u_\alpha(\alpha(t)c)$$

for all $t \in \mathbf{T}$, $c \in \mathbf{k}$ *such that the restriction onto its image is an isomorphism. It is unique up to multiplication with some constant* $c' \in \mathbf{k}^\times$.

(c) *We have* $\mathbf{G} = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Phi \rangle$ *for* $\mathbf{U}_\alpha := u_\alpha(\mathbf{k}^+)$.

Definition 2.14. For $\alpha \in \Phi$, the groups $\mathbf{U}_\alpha$ from Theorem 2.13 are called the *root subgroups of* $\mathbf{G}$ with respect to $\mathbf{T}$.

We can construct a set of generators of the Weyl group that we will refer to in the following.

Proposition 2.15. [**MT11**, Proposition 8.20] *For* $\alpha \in \Phi$*, we set* $\mathbf{T}_\alpha := (\ker \alpha)^\circ \subseteq \mathbf{T}$ *and* $\mathbf{C}_\alpha := C_\mathbf{G}(\mathbf{T}_\alpha)$*. Choose an element* $n_\alpha \in N_{\mathbf{C}_\alpha}(\mathbf{T}) \setminus C_{\mathbf{C}_\alpha}(\mathbf{T})$ *and let* s_α *be the image of* n_α *in* $N_{\mathbf{C}_\alpha}(\mathbf{T})/C_{\mathbf{C}_\alpha}(\mathbf{T}) \subseteq \mathbf{W}_\mathbf{G}(\mathbf{T})$*. Then, the Weyl group is generated by the reflections* s_α.

Note that the reflections s_α act naturally on the character group $X(\mathbf{T})$.

Example 2.16. We illustrate the definitions from this section with an example that can be found in more detail and with the omitted computations in [**MT11**, Example 3.5, 7.13, 8.2] and [**GM20**, Example 1.3.7].

Let $\mathrm{GL}_n = \mathrm{GL}_n(\mathbf{k})$ be the general linear group and $T_n \subseteq \mathrm{GL}_n$ the subgroup of diagonal matrices of GL_n. We already know that T_n is a maximal torus of GL_n. We write $\mathrm{diag}(t_1, \ldots, t_n)$ for any diagonal $n \times n$-matrix with entries $t_1, \ldots, t_n \in \mathbf{k}$ on its diagonal. We now determine the **characters** of T_n. For all $1 \leq i \leq n$, we have a character given by

$$\chi_i : T_n \to \mathbf{k}^\times, \quad \mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i.$$

In fact, every character of T_n can be written as a sum of these characters χ_i and we have

$$X(T_n) := \{a_1\chi_1 + \cdots + a_n\chi_n \mid a_i \in \mathbb{Z}\} \cong \mathbb{Z}^n.$$

Similarly, the **cocharacters** are given by

$$\varphi_{(b_1,\ldots,b_n)} : \mathbf{k}^\times \to T_n, \quad c \mapsto \mathrm{diag}(c^{b_1}, \ldots, c^{b_n})$$

for tuples $(b_1, \ldots, b_n) \in \mathbb{Z}^n$. Thus, we have $Y(T_n) \cong \mathbb{Z}^n \cong X(T_n)$. The perfect pairing between $X(T_n)$ and $Y(T_n)$ is given by

$$\langle a_1\chi_1 + \cdots + a_n\chi_n, \varphi_{(b_1,\ldots,b_n)}\rangle = \sum_{i=1}^{n} a_i b_i.$$

The **Lie algebra** of GL_n is the $\mathbf{k}$-vector space of all $n \times n$-matrices $\mathfrak{gl}_n = M_n(\mathbf{k})$. The **adjoint representation** is given by matrix conjugation, i.e.

$$\mathrm{Ad} : \mathrm{GL}_n \to \mathrm{GL}(\mathfrak{gl}_n), \quad X \mapsto \left(\mathrm{Ad}(X) = c_X : \mathfrak{gl}_n \to \mathfrak{gl}_n, \quad M \mapsto XMX^{-1}\right).$$

We now determine the weight spaces of characters. For $1 \leq i, j \leq n$, let E_{ij} be the matrix that has a 1 at the position (i, j) and zeros otherwise. Then, we have

$$\mathrm{Ad}(\mathrm{diag}(t_1, \ldots, t_n))(E_{ij}) = t_i t_j^{-1} E_{ij}.$$

Since conjugation is linear and every element in $\mathfrak{gl}_n$ can be written as a linear combination of the E_{ij}, the only non-zero weight spaces occur for the characters

$$\chi_{ij} := \chi_i - \chi_j : T_n \to \mathbf{k}^\times, \quad \mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i t_j^{-1}.$$

If we choose $i = j$, then the character χ_{ij} is trivial. Thus, the **roots** of GL_n with respect to T_n with corresponding **weight spaces** are

$$\Phi = \{\chi_{ij} \mid 1 \leq i, j \leq n,\ i \neq j\}, \quad (\mathfrak{gl}_n)_{\chi_{ij}} = \langle E_{ij}\rangle_{\mathbf{k}}.$$

This yields the root space decomposition

$$\mathfrak{gl}_n = \mathrm{Lie}(T_n) \oplus \bigoplus_{i \neq j} \langle E_{ij}\rangle_{\mathbf{k}}$$

where $\mathrm{Lie}(T_n)$ is the Lie algebra of T_n. It consists of all diagonal matrices in $M_n(\mathbf{k})$.

To determine the **root subgroups**, let

$$u_{\chi_{ij}} : \mathbf{k}^+ \to \mathrm{GL}_n, \quad c \mapsto I_n + cE_{ij}.$$

This map satisfies

$$t u_{\chi_{ij}}(c) t^{-1} = t(I_n + cE_{ij})t^{-1} = I_n + c t_i t_j^{-1} E_{ij} = u_{\chi_{ij}}(t_i t_j^{-1} c) = u_{\chi_{ij}}(\chi_{ij}(t)c)$$

for all $t = \mathrm{diag}(t_1, \ldots, t_n) \in T_n$, $c \in \mathbf{k}$. Thus, the root subgroups are given by

$$\mathbf{U}_{\chi_{ij}} = \{I_n + cE_{ij} \mid c \in \mathbf{k}\}.$$

We see that we have

$$\mathrm{GL}_n = \langle T_n, (I_n + cE_{ij}) \mid c \in \mathbf{k},\ 1 \leq i, j \leq n,\ i \neq j\rangle$$

as claimed.

The **Weyl group** of GL_n with respect to T_n is $\mathbf{W}_{\mathrm{GL}_n}(T_n) = N_{\mathrm{GL}_n}(T_n)/T_n$ since GL_n is connected reductive. The normalizer of the group of diagonal matrices is given by all monomial matrices that permute and scale the entries on the diagonal. Taking the quotient by T_n, the representatives of $\mathbf{W}_{\mathrm{GL}_n}(T_n)$ in $N_{\mathrm{GL}_n}(T_n)$ are given by the permutation matrices in GL_n. Thus, $\mathbf{W}_{\mathrm{GL}_n}(T_n)$ is isomorphic to the symmetric group on n letters $\mathrm{Sym}(n)$.

2.2. Root data and connected reductive groups

As we have seen in the previous section, every algebraic group is associated to a set of roots with respect to a maximal torus. The roots of a group contain a lot of information about the structure of a group and can be used to classify connected reductive groups.

In this section, we define abstract root systems and root data and describe their connection to connected reductive groups. We further use these tools to state Chevalley's classification of connected reductive groups.

2.2.1. Root systems. We now define the abstract notion of root systems that is independent of the previous notion of roots of algebraic groups. Note that Φ does not denote the set of roots of an algebraic group anymore unless it is explicitly defined like this.

Definition 2.17. A subset Φ of a finite-dimensional real vector space E is called an *abstract root system* in E if

(R1) Φ is finite, $0 \notin \Phi$, Φ generates E;

(R2) if $c \in \mathbb{R}$ such that $\alpha, c\alpha \in \Phi$, then $c = \pm 1$;

(R3) for each $\alpha \in \Phi$ there exists a reflection $s_\alpha \in \mathrm{GL}(E)$ along α that stabilizes Φ;

(R4) $s_\alpha.\beta - \beta \in \mathbb{Z}\alpha$ for all $\alpha, \beta \in \Phi$.

The group $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$ is called the *Weyl group* of Φ and the dimension of E is called the rank of Φ.

(R4) is also called the crystallographic condition and is not always required. The elements of Φ are also called *roots*.

The Weyl group W of an abstract root system is always finite. Thus, we can choose a W-invariant positive definite bilinear form on E that is unique up to non-zero scalars on each irreducible W-submodule of E. We fix such a form and can therefore consider lengths and angles of elements of E.

We define the notion of a base and thereby divide the roots into negative and positive roots.

Definition 2.18. A subset $\Delta \subseteq \Phi$ is called a *base of a root system* Φ if it is a vector space basis of E and any $\alpha \in \Phi$ can be written as $\alpha = \sum_{\beta \in \Delta} a_\beta \beta$ with integers a_β that are either all non-negative or non-positive.

If Δ is a base of Φ, we say that a root α is *positive* if all a_α are non-negative. The set of positive roots of Φ with respect to Δ is denoted by Φ^+.

A root system with base Δ is called *indecomposable* if there is no partition of Δ into non-empty orthogonal subsets. We can describe and visualize indecomposable root systems with the help of Dynkin diagrams.

Definition 2.19. Let Φ be a root system with base Δ. The *Dynkin diagram associated to* Φ has $|\Delta|$ nodes that are indexed by the elements of Δ. Two different nodes $\alpha, \beta \in \Delta$ are connected by $\mathrm{ord}(s_\alpha s_\beta) - 2$ edges if $\mathrm{ord}(s_\alpha s_\beta) \in \{2, 3, 4\}$ and by 3 edges if $\mathrm{ord}(s_\alpha s_\beta) = 6$. If α and β have different lengths, then these edges are directed with an arrow pointing to the shorter root.

If we do not put the arrows between roots of different lengths, we obtain the *Coxeter diagram* of Φ that does not uniquely determine Φ. It will be important in later sections.

We can classify the indecomposable root systems in real vector spaces.

Theorem 2.20. [**MT11**, Theorem 9.6] *The indecomposable root systems in real vector spaces have the following types:*

$$\mathsf{A}_n (n \geq 1), \quad \mathsf{B}_n (n \geq 2), \quad \mathsf{C}_n (n \geq 3), \quad \mathsf{D}_n (n \geq 4), \quad \mathsf{E}_6, \quad \mathsf{E}_7, \quad \mathsf{E}_8, \quad \mathsf{F}_4, \quad \mathsf{G}_2.$$

The associated Dynkin diagrams can be found in Figure 2.1.

The root systems on the left side of Figure 2.1 belong to infinite families and are of so-called *classical type*. The root systems on the right side are of *exceptional type*. In all those root systems, the roots have at most two different lengths. Thus, we can talk about short and long roots.

We can now see how the set of roots and the Weyl group attached to a reductive group are related to the analogous notions for abstract root systems.

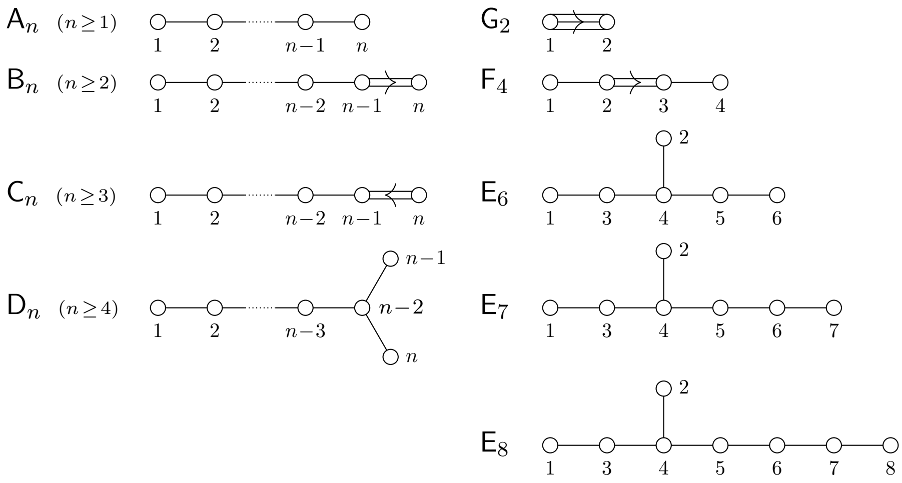


FIGURE 2.1. Dynkin diagrams of the indecomposable root systems.

Proposition 2.21. [**MT11**, Proposition 9.2] *Let* $\mathbf{G}$ *be a reductive group and* $\Phi := \Phi(\mathbf{G})$ *the set of roots with respect to a maximal torus* $\mathbf{T}$*. We consider* Φ *as a subsystem of* $E := X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$*. Then* Φ *is an abstract root system in* $\langle \Phi \rangle_{\mathbb{R}}$ *with reflections* s_α *for* $\alpha \in \Phi$ *and Weyl group* $\mathbf{W} = \mathbf{W}_{\mathbf{G}}(\mathbf{T})$*. If* $\mathbf{G}$ *is semisimple, we have* $E = \langle \Phi \rangle_{\mathbb{R}}$.

One can further see that the root system of a reductive group is indecomposable if and only if the group is simple. Thus, Theorem 2.20 describes all possible root systems of simple algebraic groups.

2.2.2. Root data and isogeny types. We now extend the notion of root systems to abstract root data. This will be useful to distinguish connected reductive groups with the same root system.

Definition 2.22. A tuple $(X, \Phi, Y, \Phi^\vee)$ is called a *root datum* if

(RD1) $X \cong \mathbb{Z}^n \cong Y$ and there exists a perfect pairing $\langle\ ,\ \rangle : X \times Y \to \mathbb{Z}$;

(RD2) $\Phi \subseteq X$ and $\Phi^\vee \subseteq Y$ are abstract root systems in $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{Z}\Phi^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, respectively;

(RD3) there exists a bijection $\Phi \to \Phi^\vee, \alpha \mapsto \alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Phi$;

(RD4) the reflections s_α of the root system Φ and $s_{\alpha^\vee}$ of the root system $\Phi^\vee$ satisfy

$$s_\alpha . \chi = \chi - \langle \chi, \alpha^\vee \rangle \alpha, \quad s_{\alpha^\vee} . \gamma = \gamma - \langle \alpha, \gamma \rangle \alpha^\vee,$$

for all $\chi \in X, \gamma \in Y$.

For a given root datum, we can define the so-called Cartan matrix that encodes information about the root datum.

Definition 2.23. Let $(X, \Phi, Y, \Phi^\vee)$ be a root datum and Δ a base of Φ. For $1 \leq i, j \leq |\Delta|$, the integers $C_{ij} := \langle \alpha_j, \alpha_i^\vee \rangle$ form the *Cartan matrix* $(C_{ij})_{1 \leq i,j \leq |\Delta|}$ of Φ.

The condition (RD3) ensures that the diagonal entries of a Cartan matrix are 2. The Cartan matrix encodes a lot of important information about the associated root datum.

Again, the structure of root data naturally occurs for algebraic groups.

Lemma 2.24. [**MT11**, Lemma 8.19] *Let* Φ *be the root system of* $\mathbf{G}$ *with respect to a maximal torus* $\mathbf{T}$*. There exists a unique map* $\Phi \to Y(\mathbf{T})$, $\alpha \mapsto \alpha^\vee$ *such that (RD3) and (RD4) are satisfied for* $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\} \subseteq Y(\mathbf{T})$.

The elements of $\Phi^\vee$ are also called the *coroots* of $\mathbf{G}$.

Proposition 2.25. [**MT11**, Proposition 9.11] *Let Φ be the root system of a connected reductive group $\mathbf{G}$ with respect to a maximal torus $\mathbf{T}$ with Weyl group $\mathbf{W}$ and $\Phi^\vee$ as in Lemma 2.24. Then, $(X(\mathbf{T}), \Phi, Y(\mathbf{T}), \Phi^\vee)$ is a root datum.*

This now can be used to classify connected reductive groups. For semisimple groups, the following theorem is due to Chevalley and therefore also known as Chevalley's classification theorem.

Theorem 2.26. [**GM20**, Corollary 1.3.13, Theorem 1.3.14] *For a given root datum, there exist a connected reductive algebraic group over $\mathbf{k}$ and a maximal torus that realize the root datum. Conversely, if two connected reductive groups have the same root data, then they are isomorphic.*

Every simple connected reductive group is associated to an indecomposable root system of one of the types listed in Theorem 2.20. If we do not only look at the root system but also at the root datum, then this correspondence is unique. We now study the possibilities for connected reductive groups that have the same root system but are not isomorphic. Thus, we have to investigate different root data that correspond to the same root system.

For an abstract root datum $(X, \Phi, Y, \Phi^\vee)$, there is a natural injective homomorphism

$$X \cong \operatorname{Hom}(Y, \mathbb{Z}) \to \Omega := \operatorname{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$$

given by restriction. Thus, we can identify X with its image in Ω and obtain the inclusion $\mathbb{Z}\Phi \subseteq X \subseteq \Omega$. The root data with fixed root system Φ are classified by the subgroups X of Ω containing $\mathbb{Z}\Phi$. Taking the quotient, this can be described in a more condensed way.

Definition 2.27. The *fundamental group of the root system* Φ is given by $\Lambda(\Phi) := \Omega/\mathbb{Z}\Phi$.

Therefore, the root data with fixed root system Φ correspond to subgroups $X/\mathbb{Z}\Phi$ of the fundamental group $\Lambda(\Phi)$. The fundamental groups of the indecomposable root systems are displayed in Table 2.1.

Type of Φ	A_n	B_n	C_n	D_{2n+1}	D_{2n}	E_6	E_7	E_8	F_4	G_2
$\Lambda(\Phi)$	C_{n+1}	C_2	C_2	C_4	$C_2 \times C_2$	C_3	C_2	1	1	1

Table 2.1. Fundamental groups of the indecomposable root systems [**MT11**, Table 9.2].

We now define a similar notion for a semisimple group that, together with its root system, sufficiently describes the corresponding root datum.

Definition 2.28. Let $\mathbf{G}$ be a semisimple algebraic group with maximal torus $\mathbf{T}$, root system $\Phi := \Phi(\mathbf{G})$, and Ω as defined above.

(i) The *fundamental group of* $\mathbf{G}$ is given by $\Lambda(\mathbf{G}) := \Omega/X(\mathbf{T})$.
(ii) The group $\mathbf{G}$ is called *simply connected* if we have $X(\mathbf{T}) = \Omega$.
(iii) The group $\mathbf{G}$ is *of adjoint type* if we have $X(\mathbf{T}) = \mathbb{Z}\Phi$.

The fundamental group of a semisimple algebraic group $\mathbf{G}$ is always finite. The group

$$\Lambda(\mathbf{G}) \cong \Lambda(\Phi(\mathbf{G}))/(X(\mathbf{T})/\mathbb{Z}\Phi)$$

corresponds to the choice of the subgroup $X(\mathbf{T})/\mathbb{Z}\Phi \subseteq \Lambda(\Phi(\mathbf{G}))$ described above. Therefore, a semisimple group $\mathbf{G}$ not of type D_{2n} is, up to isomorphism, uniquely determined by its root system $\Phi := \Phi(\mathbf{G})$ and its fundamental group. We say that the different semisimple algebraic groups corresponding to the same root system Φ are the *isogeny types* corresponding to Φ. An *isogeny* is a surjective homomorphism π of algebraic groups such that its kernel $\ker(\pi)$ is finite. This name is due to the following result.

Proposition 2.29. [**MT11**, Proposition 9.15] *Let* $\mathbf{G}$ *be a semisimple group with root system* Φ *and* $\mathbf{G}_{sc}$, $\mathbf{G}_{ad}$ *the simply connected and the adjoint group associated to the same root system* Φ*. Then, there exist isogenies*

$$\mathbf{G}_{sc} \xrightarrow{\pi_{sc}} \mathbf{G} \xrightarrow{\pi_{ad}} \mathbf{G}_{ad}$$

that satisfy $\ker(\pi_{sc}) \cong \Lambda(\mathbf{G}_{sc})_{p'}$ *and* $\ker(\pi_{ad}) \cong (\Lambda(\mathbf{G}_{ad})/\Lambda(\mathbf{G}))_{p'}$*. In particular, we have* $Z(\mathbf{G}_{sc}) \cong \Lambda(\Phi)_{p'}$ *and* $Z(\mathbf{G}_{ad}) = 1$.

For a semisimple group $\mathbf{G}$, we can determine all possibilities for $\Lambda(\mathbf{G})$ using Table 2.1. For all root systems except those of type A_n and D_n, the fundamental group $\Lambda(\Phi)$ is simple and $\mathbf{G}$ is either simply connected or of adjoint type. This gives us a list of all simple connected reductive groups with their corresponding root systems, see e.g. [**MT11**, Table 9.2].

Again, we give an example to illustrate the introduced concepts. Since many of the above results only hold for semisimple groups, we do not consider GL_n but two related algebraic groups.

Example 2.30. The following example can be found in more detail in [**GM20**, Example 1.3.8 and 1.3.9].

(a) We consider the special linear group

$$\mathbf{G} := \mathrm{SL}_n(\mathbf{k}) := \{A \in \mathrm{GL}_n(\mathbf{k}) \mid \det A = 1\}.$$

We continue to use the notation from Example 2.16 and identify some occurring maps with their restrictions. Let $\mathbf{T} := T_n \cap \mathbf{G}$ be the maximal torus consisting of the diagonal matrices in $\mathbf{G}$. The characters and roots of $\mathbf{T}$ are given by the restricted characters of T_n, i.e.

$$X(\mathbf{T}) = \langle \chi_1, \dots, \chi_n \rangle_{\mathbb{Z}}, \quad \Phi = \{\chi_{ij} \mid 1 \leq i, j \leq n, i \neq j\}.$$

Again, we have a base of Φ of size $n-1$. Further, one can see that Φ is a root system of type A_{n-1}. The cocharacters and coroots are given by

$$Y(\mathbf{T}) = \left\{\varphi_{(b_1,\dots,b_n)} \mid b_i \in \mathbb{Z},\ \textstyle\sum_{i=1}^n b_i = 0\right\}, \quad \Phi^\vee = \{\chi_{ij}^\vee \mid 1 \leq i, j \leq n, i \neq j\}$$

with $\chi_{ij}^\vee := \varphi_{(b_1,\dots,b_n)}$ where $b_i = 1, b_j = -1$ and all other b_k are 0. The additional restrictions compared to the cocharacters of GL_n occur since we only allow maps that have an image with determinant 1.

We can directly see that we have $Y(\mathbf{T}) = \mathbb{Z}\Phi^\vee$. This implies $X(\mathbf{T}) = \Omega$ and therefore $\mathbf{G}$ is simply connected and we have $\Lambda(\mathbf{G}) = 1$.

(b) We now consider the projective general linear group

$$\widetilde{\mathbf{G}} := \mathrm{PGL}_n(\mathbf{k}) = \mathrm{GL}_n / Z(\mathrm{GL}_n).$$

Then, $\widetilde{\mathbf{T}} := T_n/Z(\mathrm{GL}_n)$ is a maximal torus of $\widetilde{\mathbf{G}}$. Similarly, we can use the quotient map π and its universal property to translate the characters of GL_n with $Z(\mathrm{GL}_n)$ in their kernel to $\widetilde{\mathbf{G}}$. To simplify notation, we denote these lifted characters by the same symbols as for GL_n. Then, we have

$$X(\widetilde{\mathbf{T}}) = \left\{ \textstyle\sum_{i=1}^n a_i\chi_i \mid a_i \in \mathbb{Z},\ \sum_{i=1}^n a_i = 0\right\}, \quad \Phi = \{\chi_{ij} \mid 1 \leq i, j \leq n, i \neq j\},$$

$$Y(\widetilde{\mathbf{T}}) = \{\pi \circ \varphi_{(b_1,\dots,b_n)} \mid b_i \in \mathbb{Z}\}, \quad \Phi^\vee = \{\pi \circ \chi_{ij}^\vee \mid 1 \leq i, j \leq n, i \neq j\}.$$

The root system Φ is again of type A_{n-1}. We see that we have $X(\widetilde{\mathbf{T}}) = \mathbb{Z}\Phi$ and therefore $\Lambda(\widetilde{\mathbf{G}}) = \Lambda(\Phi)$. Thus, $\widetilde{\mathbf{G}}$ is of adjoint type.

Example 2.31. We consider the symplectic groups of rank 4

$$\mathbf{G} := \mathrm{Sp}_4(\mathbf{k}) := \left\{A \in \mathrm{GL}_4(\mathbf{k}) \mid A^T J A = J\right\} \quad \text{for } J := \begin{pmatrix} 0 & J_2 \\ -J_2 & 0 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Again, a maximal torus $\mathbf{T}$ of $\mathbf{G}$ is given by the diagonal matrices in $\mathbf{G}$. They are of the form $\operatorname{diag}(x, y, y^{-1}, x^{-1})$ for some $x, y \in \mathbf{k}^\times$. The normalizer of $\mathbf{T}$ in $\mathbf{G}$ is generated by the permutation matrices in $\mathbf{G}$ that cyclically permute the diagonal entries and the one interchanging x with its inverse. Thus, the Weyl group is isomorphic to D_8, the dihedral group of order 8.
The characters of $\mathbf{T}$ are integer linear combinations of $\chi_1, \chi_2 \in X(\mathbf{T})$ defined by

$$\chi_1(\operatorname{diag}(x, y, y^{-1}, x^{-1})) = x, \quad \chi_2(\operatorname{diag}(x, y, y^{-1}, x^{-1})) = y.$$

By computing the adjoint map, we can determine the roots. With a corresponding base $\Delta := \{\alpha, \beta\}$ where $\alpha := 2\chi_2$, $\beta := \chi_1 - \chi_2$, we have

$$\Phi(\mathbf{G}) = \Phi^+ \cup -\Phi^+, \quad \Phi^+ := \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta\}.$$

One can see that this is a root system of type B_2 where α is a long root and β is a short root. For more details, see e.g. [**MT11**, Example 9.5].

2.3. Levi decomposition and structure of semisimple groups

In this section, we introduce the Levi decomposition of parabolic subgroups and the Chevalley relations for semisimple algebraic groups.

Proposition 2.32. [**MT11**, Theorem 11.1] *Let $\mathbf{G}$ be connected reductive and $\mathbf{B} \supseteq \mathbf{T}$ a Borel subgroup containing a maximal torus of $\mathbf{G}$. Then, there exists a base Δ of Φ such that*

$$\mathbf{B} = \mathbf{T} \cdot \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$$

for any fixed order of the $\alpha \in \Phi^+$. The expression is unique with respect to this fixed order. Moreover, we have

$$\mathbf{W}_{\mathbf{G}}(\mathbf{T}) = \langle s_\alpha \mid \alpha \in \Delta \rangle, \quad \mathbf{G} = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \pm\Delta \rangle.$$

In the situation of the proposition, the base Δ is called the set of simple roots with respect to $\mathbf{T} \subseteq \mathbf{B}$. In the remaining section, let $\mathbf{G}$ be a connected reductive algebraic group, $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$ a maximal torus of $\mathbf{G}$ contained in a Borel subgroup of $\mathbf{G}$. Let $\Phi \supseteq \Delta$ be the corresponding root system and set of simple roots and $\mathbf{W}$ the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}$ with simple reflections $S := \{s_\alpha \mid \alpha \in \Delta\}$. We now define families of subgroups of $\mathbf{W}$ and $\mathbf{G}$ that are indexed by subsets of S or simple roots.

Definition 2.33. Let $I \subseteq S$ and $\Delta_I := \{\alpha \in \Delta \mid s_\alpha \in I\}$ be the corresponding subset of simple roots. For $w \in \mathbf{W}$, we denote a preimage of w in $N_{\mathbf{G}}(\mathbf{T})$ by n_w.

(i) The subgroup $\mathbf{W}_I := \langle s \in I \rangle$ is called a *standard parabolic subgroup of* $\mathbf{W}$. Any conjugate of a standard parabolic subgroup is called a *parabolic subgroup of* $\mathbf{W}$.

(ii) The set $\Phi_I := \Phi \cap \sum_{\alpha \in \Delta_I} \mathbb{Z}\alpha$ is the corresponding *parabolic subsystem of roots.*

(iii) The group

$$\mathbf{P}_I := \bigsqcup_{w \in \mathbf{W}_I} \mathbf{B} n_w \mathbf{B}$$

is called a *standard parabolic subgroup of* $\mathbf{G}$. The conjugates of standard parabolic subgroups of $\mathbf{G}$ are called *parabolic subgroups of* $\mathbf{G}$.

Note that, although it is not clear from the definition above, the P_I are in fact closed connected subgroups of $\mathbf{G}$ for all $I \subseteq S$, see [**MT11**, Proposition 12.2]. The following proposition already gives us an idea why it is interesting to study parabolic subgroups.

Proposition 2.34. [**MT11**, Proposition 12.2] *Let $I, J \subseteq S$.*

(a) *For $I \neq J$, the standard parabolic subgroups $\mathbf{P}_I$ and $\mathbf{P}_J$ are non-conjugate.*

(b) *Any subgroup of $\mathbf{G}$ containing a Borel subgroup is a parabolic subgroup.*

(c) *We have* $\mathbf{P}_I = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Phi^+ \cup \Phi_I \rangle$.

Parabolic subgroups play an important role in the study of connected reductive groups. Later, we will need them to investigate the representation theory of finite groups arising from them. For this, we also need the Levi decomposition of parabolic subgroups.

Definition 2.35. Let $I \subseteq S$. We set

$$\mathbf{U}_I := \langle \mathbf{U}_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_I \rangle, \quad \mathbf{L}_I := \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Phi_I \rangle.$$

Then, we have $\mathbf{P}_I = \mathbf{U}_I \rtimes \mathbf{L}_I$ [**MT11**, Proposition 12.6]. This is called the *Levi decomposition of* $\mathbf{P}_I$ and the group $\mathbf{L}_I$ is called the *standard Levi complement of* $\mathbf{P}_I$. Conjugates of standard Levi complements are called *Levi subgroups of* $\mathbf{G}$.

For semisimple algebraic groups, we have more information about their structure. We can define a distinguished set of generators that are subject to the so-called Steinberg or Chevalley relations.

Theorem 2.36. [**GLS98**, Theorem 1.12.1] *Let* $\mathbf{G}$ *be semisimple. For* $\alpha \in \Phi$, *we set*

$$n_\alpha(t) := u_\alpha(t) u_{-\alpha}(-t^{-1}) u_\alpha(t), \quad h_\alpha(t) := n_\alpha(t) n_\alpha(1)^{-1}$$

for all $t \in \mathbf{k}^\times$. *Then the following properties hold:*

(a) $u_\alpha(t) u_\alpha(s) = u_\alpha(t+s)$ *for all* $\alpha \in \Phi$ *and* $t, s \in \mathbf{k}$;

(b) *for linearly independent* $\alpha, \beta \in \Phi$ *we have*

$$[u_\alpha(t), u_\beta(s)] = \prod_{i,j>0,\ i\alpha+j\beta \in \Phi} u_{i\alpha+j\beta}(c_{ij\alpha\beta} t^i s^j)$$

for all $t, s \in \mathbf{k}^\times$ *with scalars* $c_{ij\alpha\beta} \in \{\pm 1, \pm 2, \pm 3\}$ *independent of* t, s;

(c) $h_\alpha(t) h_\alpha(s) = h_\alpha(ts)$ *and* $[h_\alpha(t), h_\beta(s)] = 1$ *for all* $\alpha \in \Phi$ *and* $t, s \in \mathbf{k}^\times$;

(d) $\mathbf{T} = \langle h_\alpha(t) \mid \alpha \in \Phi, t \in \mathbf{k}^\times \rangle$.

The relations (a)-(c) form the Steinberg or Chevalley relations and (b) is called the commutator formula. The constants $c_{ij\alpha\beta}$ are well-known and can be found for example in [**GLS98**, Theorem 1.12.1]. Note that the h_α are cocharacters.

These relations can also be used to construct a semisimple algebraic group of a certain type. If Φ is a root system and we have abstract symbols $u_\alpha(t)$ for all $\alpha \in \Phi$, $t \in \mathbf{k}$ satisfying the Steinberg relations, these symbols generate a semisimple algebraic group with root system Φ. This is called the Steinberg presentation [**Ste68**, Chapter 6]. We can require additional relations to determine the isogeny type of the group, see e.g. [**GLS98**, Theorem 1.12.4].

2.4. Steinberg endomorphisms and finite groups of Lie type

As mentioned earlier, we are interested in finite groups arising from algebraic groups. In this section, we define Steinberg endomorphisms and use them to construct finite groups of Lie type from algebraic groups. We follow [**MT11**].

Let $\mathbf{G}$ be a linear algebraic group over an algebraically closed field $\mathbf{k} = \overline{\mathbb{F}_p}$ of characteristic p.

Definition 2.37. Let $F : \mathbf{G} \to \mathbf{G}$ be an endomorphism of $\mathbf{G}$ and $q = p^f$ for some $f \geq 1$.

(i) The *standard Frobenius map on* $\mathrm{GL}_n(\mathbf{k})$ is given by mapping the matrix entries to their q-th power $F_q : (a_{i,j}) \mapsto (a_{i,j}^q)$.

(ii) The map F is called a *Frobenius map* with respect to an $\mathbb{F}_q$-structure if there exists an embedding $\iota : \mathbf{G} \to \mathrm{GL}_n(\mathbf{k})$ such that $F_q \circ \iota = \iota \circ F$.

(iii) The map F is called a *Steinberg map* if some power of F is a Frobenius map.

Steinberg maps are sometimes also called Frobenius roots. We can use them to construct finite groups from algebraic groups.

Definition 2.38. Let $\mathbf{G}$ be a connected reductive group and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism. The set of fixed points of $\mathbf{G}$ under F is denoted by

$$\mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}$$

and called a *finite group of Lie type*.

Note that $\mathbf{G}^F$ is in fact a finite group. Indeed, every endomorphism F of a simple group $\mathbf{G}$ that gives rise to a finite group $\mathbf{G}^F$ is a Steinberg endomorphism [**MT11**, Theorem 21.5].

Example 2.39. Let $\mathbf{G} := \mathrm{SL}_n(\mathbf{k})$ be the special linear group over $\mathbf{k}$ and $F := F_q$ the standard Frobenius map restricted to $\mathbf{G}$. Then, $\mathbf{G}^F = \mathrm{SL}_n(\mathbb{F}_q) = \mathrm{SL}_n(q)$ consists of all $n \times n$-matrices with entries in $\mathbb{F}_q$ and determinant 1.

Now, let

$$\gamma : \mathrm{SL}_n(\mathbf{k}) \to \mathrm{SL}_n(\mathbf{k}), \quad A \mapsto A^{-T}$$

be the transpose inverse homomorphism and set $F' := F_q \circ \gamma$. It is not obvious that F' is a Frobenius map as in Definition 2.37, but we can directly see that $F'^2 = F_{q^2}$. The fixed point group $\mathbf{G}^{F'} =: \mathrm{SU}_n(q)$ is called the special unitary group. It is a subgroup of $\mathrm{SL}_n(q^2)$.

We now state the Lang–Steinberg theorem. It is very important as it allows us to transfer many results about algebraic groups to the corresponding finite groups.

Theorem 2.40. [**MT11**, Theorem 21.7] *Let $\mathbf{G}$ be connected and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism. Then, the Lang map*

$$\mathscr{L} : \mathbf{G} \to \mathbf{G}, \quad g \mapsto F(g)g^{-1}$$

is a surjective morphism.

From now, let $\mathbf{G}$ be connected reductive and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism. We are often interested in F-stable subgroups of $\mathbf{G}$, i.e. subgroups $\mathbf{H} \leq \mathbf{G}$ such that $F(\mathbf{H}) \subseteq \mathbf{H}$. Then, we can also consider the finite group $\mathbf{H}^F$.

Proposition 2.41. *There exists an F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$ and an F-stable Borel subgroup $\mathbf{B} \subseteq \mathbf{G}$ containing $\mathbf{T}$. All such pairs are $\mathbf{G}^F$-conjugate.*

Definition 2.42. If $\mathbf{T}_0 \subseteq \mathbf{G}$ is an F-stable maximal torus such that it is contained in an F-stable Borel subgroup $\mathbf{B}_0$, then $\mathbf{T}_0$ is called a *maximally split torus*.

Let $\mathbf{W} = \mathbf{W}_{\mathbf{G}}(\mathbf{T}_0)$ be the Weyl group with respect to a maximally split torus $\mathbf{T}_0 \subseteq \mathbf{B}_0$ contained in an F-stable Borel subgroup. Let Φ be the corresponding root system and Δ the set of simple roots. Then, F induces an automorphism of $\mathbf{W}$

$$\sigma : \mathbf{W} \to \mathbf{W}, \quad n\mathbf{T}_0 \mapsto F(n)\mathbf{T}_0.$$

In the following, we denote this automorphism by σ and write $\mathbf{W}^\sigma = \{w \in \mathbf{W} \mid \sigma(w) = w\}$.

One can show that σ restricts to a graph automorphism of the corresponding Coxeter diagram and use this to classify all possible Steinberg endomorphisms:

Proposition 2.43. [**MT11**, Proposition 22.2, Theorem 22.5]

(a) *There exists a permutation ρ of Φ^+ such that for every $\alpha \in \Phi^+$ there are $a_\alpha \in \mathbf{k}^\times$ and $k_\alpha \in \mathbb{N}$ with*

$$F(u_\alpha(c)) = u_{\rho(\alpha)}\left(a_\alpha c^{p^{k_\alpha}}\right)$$

for all $c \in \mathbf{k}$. Moreover, ρ stabilizes the set of simple roots Δ and induces a graph automorphism of the corresponding Coxeter diagram.

(b) *If* $\mathbf{G}$ *is simple and not of type* $\mathsf{B}_2, \mathsf{G}_2,$ *or* F_4 *where* $p = 2, p = 3$ *or* $p = 2$*, respectively, then* ρ *preserves root lengths and all* k_α *coincide, i.e.*

$$F(u_\alpha(c)) = u_{\rho(\alpha)}(a_\alpha c^q)$$

for some p*-power* q *and all* $c \in \mathbf{k}$.

(c) *If* $\mathbf{G}$ *is of type* $\mathsf{B}_2, \mathsf{G}_2,$ *or* F_4 *with* $p = 2, p = 3,$ *or* $p = 2$*, respectively, then there exists an exceptional Coxeter diagram automorphism* ρ *that interchanges root lengths. It is induced by an exceptional endomorphism*

$$\gamma : \mathbf{G} \to \mathbf{G}, \quad \gamma(u_\alpha(c)) = \begin{cases} u_{\rho(\alpha)}(c) & \text{if } \alpha \text{ is long,} \\ u_{\rho(\alpha)}(c^p) & \text{if } \alpha \text{ is short.} \end{cases}$$

Conversely, for every prime power q *of* p *and graph automorphism of the Coxeter diagram* ρ *as above, there exists a Steinberg endomorphism that corresponds to these parameters.*

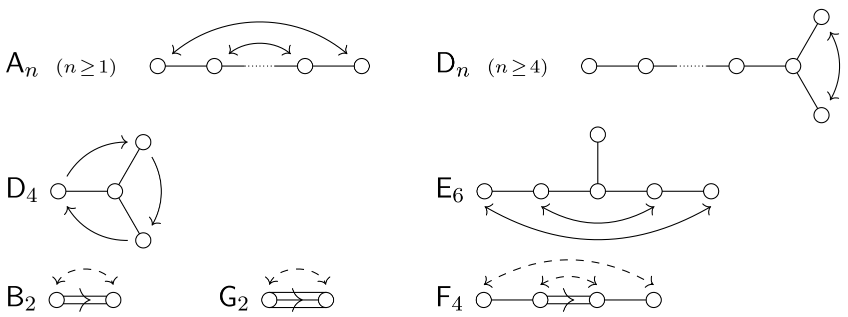


FIGURE 2.2. Non-trivial graph automorphisms of the Coxeter diagrams of indecomposable root systems. The dashed arrows only give rise to a Steinberg endomorphism of the corresponding algebraic group in characteristic 2 or 3, respectively.

Thus, we can use the Coxeter diagrams to determine the possible choices of ρ and thereby the possible Steinberg endomorphisms of a group $\mathbf{G}$. All non-trivial graph automorphisms of connected Coxeter diagrams can be found in Figure 2.2. The Frobenius endomorphism corresponding to $\rho = \mathrm{id}$ and $q = p$ is also called *standard Frobenius endomorphism* and denoted by F_p.

We now introduce a more convenient way to denote a finite group of Lie type.

Definition 2.44. (i) The pair $(\mathbf{G}, F)$ or the group $\mathbf{G}^F$ is called *untwisted* if F induces the trivial graph automorphism on the Coxeter diagram of Φ. Otherwise, it is called *twisted.*

(ii) Let $f \geq 1$ be the smallest integer such that F^f is a Frobenius map with respect to an $\mathbb{F}_{q_0}$-structure. Then, we say that $q \in \mathbb{R}_{>0}$ with $q^f = q_0$ is *attached to* F.

Let $\mathbf{G}$ be connected reductive of type L_n and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism that is attached to the positive real number q.

- If $\mathbf{G}^F$ is untwisted, then we write $\mathbf{G}^F = \mathsf{L}_n(q)$.
- If $\mathbf{G}^F$ is twisted, then we write $\mathbf{G}^F = {}^d\mathsf{L}_n(q_0)$ where d is the order of the automorphism that F induces on the Coxeter diagram.

We sometimes add a subscript *ad* or *sc* to the type of $\mathbf{G}$ if it is of adjoint or simply connected type. A list of all finite groups of Lie type can be found in [**MT11**, Table 22.1].

Example 2.45. The twisted groups ${}^2\mathsf{B}_2(q^2)$, ${}^2\mathsf{G}_2(q^2)$, and ${}^2\mathsf{F}_4(q^2)$ only occur if $\mathbf{k}$ has characteristic $p = 2$, $p = 3$, and $p = 2$, respectively, and we have $q^2 = p^{2f+1}$ for some $f \in \mathbb{N}_0$. Thus, q is not an integer which means that the groups are not defined over $\mathbb{F}_q$.

They come from the exceptional Coxeter diagram automorphism described in Proposition 2.43(c). Further, they are the only simple finite groups of Lie type that arise from a simple group $\mathbf{G}$ and a Steinberg map F that is not a Frobenius map. The groups ${}^2\mathsf{B}_2(q^2)$ are also called Suzuki groups and the groups ${}^2\mathsf{G}_2(q^2)$ and ${}^2\mathsf{F}_4(q^2)$ are called the small and large Ree groups.

2.5. Properties of finite groups of Lie type

In this section, we study properties of finite groups of Lie type that can be characterized through the underlying root system and Steinberg endomorphism. We consider the orders of finite groups of Lie type, F-stable maximal tori, simple groups of Lie type, their Schur covers and automorphism groups, and end the section with the definition of duality for groups of Lie type.

Throughout the section, let $\mathbf{G}$ be connected reductive and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism.

2.5.1. Orders of finite groups of Lie type. There is a general formula for the order of finite groups of Lie type that only depends on a positive real number q and the root system of the corresponding simple algebraic group. In order to state the formula, we need to define a length function on the elements of the Weyl group.

Definition 2.46. Let Φ be a root system with base Δ and Weyl group W. The *length* $l(w)$ *of an element* $w \in W$ is the minimal integer such that w is the product of $l(w)$ elements in $\{s_\alpha \mid \alpha \in \Delta\}$.

Proposition 2.47. [**MT11**, Proposition 24.3] *Let* $\mathbf{G}$ *be simple,* F *a Steinberg endomorphism of* $\mathbf{G}$ *and* q *attached to* F*. Let* $\mathbf{T} \subseteq \mathbf{B}$ *be an* F*-stable maximal torus contained in an* F*-stable Borel subgroup and* σ *the automorphism of* $\mathbf{W}$ *induced by* F*. Then, we have*

$$|\mathbf{G}^F| = |\mathbf{B}^F| \sum_{w \in \mathbf{W}^\sigma} q^{l(w)}.$$

This can be written as a product of a power of q and cyclotomic polynomials in q. A formula given in this way that holds for all connected reductive groups $\mathbf{G}$ can be found in [**MT11**, Corollary 24.6].

In the following, we continue to write q for the positive real number attached to F. The polynomial in q giving the order of $\mathbf{G}^F$ is also called the *order polynomial* of $\mathbf{G}$ with respect to F. A full list of order polynomials can be found for example in [**MT11**, Table 24.1].

Example 2.48. The Suzuki and Ree groups have the order polynomials given in Table 2.2 where we denote the d-th cyclotomic polynomial by Φ_d. Note that for the Suzuki and Ree groups q is not an integer. Therefore, some of the cyclotomic polynomials evaluated at q are not rational and some of the polynomials can be factorized over $\mathbb{Q}(q)$ into polynomials with non-rational coefficients but rational values at q. With this in mind, we write

- $(\Phi_1\Phi_2)(q) := \Phi_1(q)\Phi_2(q)$
- $\Phi_8^{\pm}(q) := q^2 \pm \sqrt{2}q + 1$, i.e. $\Phi_8^{+}(q)\Phi_8^{-}(q) = \Phi_8(q)$;
- $\Phi_{12}^{\pm}(q) := q^2 \pm \sqrt{3}q + 1$, i.e. $\Phi_{12}^{+}(q)\Phi_{12}^{-}(q) = \Phi_{12}(q)$;
- $\Phi_{24}^{\pm}(q) := q^4 \pm \sqrt{2}q^3 + q^2 \pm \sqrt{2}q + 1$, i.e. $\Phi_{24}^{+}(q)\Phi_{24}^{-}(q) = \Phi_{24}(q)$.

2.5.2. F-stable maximal tori. In the following, we often consider F-stable maximal tori of $\mathbf{G}$. Let $\mathbf{T}_0$ be a maximally split torus of $\mathbf{G}$ and $\mathbf{W}$ the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$. We already know that all F-stable maximal tori of $\mathbf{G}$ are $\mathbf{G}$-conjugate and want to obtain a more detailed description. We follow [**GM20**, Section 1.6.4].

We first introduce a different notion of conjugacy for elements of $\mathbf{W}$.

$\mathbf{G}^F$	$\lvert\mathbf{G}^F\rvert$
${}^2\mathsf{B}_2(q^2)$ $(q^2 = 2^{2f+1})$	$q^4(q^2-1)(q^4+1) = q^4(\Phi_1\Phi_2)(q)\Phi_8^+(q)\Phi_8^-(q)$
${}^2\mathsf{G}_2(q^2)$ $(q^2 = 3^{2f+1})$	$q^6(q^2-1)(q^2+1)(q^4-q^2+1) = q^6(\Phi_1\Phi_2)(q)\Phi_4(q)\Phi_{12}^+(q)\Phi_{12}^-(q)$
${}^2\mathsf{F}_4(q^2)$ $(q^2 = 2^{2f+1})$	$q^{24}(q^2-1)^2(q^2+1)^2(q^4+1)^2(q^4-q^2+1)(q^8-q^4+1)$ $= q^{24}(\Phi_1\Phi_2)(q)^2\Phi_4(q)^2\Phi_8^+(q)^2\Phi_8^-(q)^2\Phi_{12}(q)\Phi_{24}^+(q)\Phi_{24}^-(q)$

Table 2.2. Order polynomials of the Suzuki and Ree groups.

Definition 2.49. Let σ be the automorphism of $\mathbf{W}$ induced by F. Then, $w_1, w_2 \in \mathbf{W}$ are called *σ-conjugate* if we have $w_1 = \sigma(w)w_2w^{-1}$ for some $w \in \mathbf{W}$.

Let $\mathbf{T} = g\mathbf{T}_0g^{-1}$ be an F-stable maximal torus with $g \in \mathbf{G}$. Since $\mathbf{T}$ and $\mathbf{T}_0$ are F-stable, we have $F(g)\mathbf{T}_0F(g)^{-1} = g\mathbf{T}_0g^{-1}$ and can conclude $g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T}_0)$. Let $w \in \mathbf{W}$ the image of $g^{-1}F(g)$ in $\mathbf{W}$. We say that $\mathbf{T}$ is a *torus of type w*.

Proposition 2.50. *There is a bijection between the $\mathbf{G}^F$-conjugacy classes of F-stable maximal tori of $\mathbf{G}$ and the σ-conjugacy classes of $\mathbf{W}$ given by*

$$g\mathbf{T}_0g^{-1} \mapsto g^{-1}F(g)\mathbf{T}_0 \in \mathbf{W} = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0.$$

Further, if $\mathbf{T} := g\mathbf{T}_0g^{-1}$ is of type $w \in \mathbf{W}$, we have

$$\mathbf{T}_0[w] := \{t \in \mathbf{T}_0 \mid F(t) = n_w^{-1}tn_w\} \cong \mathbf{T}^F$$

where $n_w \in N_{\mathbf{G}}(\mathbf{T}_0)$ is any representative of w.

This allows us to parametrize the F-stable maximal tori in terms of $\mathbf{W}$ and to describe their structure in a generic way.

Example 2.51. [**Gec03**, Section 4.6] We determine the F-stable maximal tori of the Suzuki groups. Let $\mathbf{G}$ be of type B_2 defined over an algebraically closed field $\mathbf{k}$ of characteristic 2 and $F = F_p^f \circ \gamma$ where F_p is the standard Frobenius map and γ the exceptional graph endomorphism. Let $\mathbf{T}_0 := \langle h_1(t), h_2(s) \mid s, t \in \mathbf{k}^\times\rangle$ where h_1, h_2 are as given in the Steinberg presentation.

The Weyl group $\mathbf{W} = \langle s_1, s_2\rangle$ is the dihedral group of order 8. Since the automorphism σ of $\mathbf{W}$ induced by F interchanges s_1 and s_2, the σ-conjugacy classes of $\mathbf{W}$ are given by

$$\{1, s_1s_2, s_2s_1, s_1s_2s_1s_2\}, \quad \{s_1, s_2\}, \quad \{s_1s_2s_1, s_2s_1s_2\}.$$

Thus, the F-stable maximal tori of $\mathbf{G}$ are of type 1, s_1 and $s_1s_2s_1$.

Since $h_1(t)$ and $h_2(s)$ commute and we have

$$F(h_1(t)) = h_2(t^{2^{f+1}}), \quad F(h_2(t)) = h_1(t^{2^f}),$$

we can easily determine the F-stable points of $\mathbf{T}_0$ and obtain

$$\mathbf{T}_0[1] = \mathbf{T}_0^F = \{h_1(t)h_2(t^{2^{f+1}}) \mid t \in \mathbf{k}^\times, t^{2^{2f+1}} = t\} \cong C_{q^2-1}$$

where $q^2 := 2^{2f+1}$. We know from Theorem 2.36 that we have

$$n_{s_1}h_1(t)n_{s_1} = h_1(t^{-1}), \quad n_{s_1}h_2(t)n_{s_1} = h_1(t)h_2(t),$$
$$n_{s_2}h_1(t)n_{s_2} = h_1(t)h_2(t^2), \quad n_{s_2}h_2(t)n_{s_2} = h_2(t^{-1}).$$

From this, we can deduce

$$\mathbf{T}_0[s_1] = \{h_1(t^{2^f})h_2(t) \mid t \in \mathbf{k}^\times, t^{2^{2f+1}-2^{f+1}+1} = 1\} \cong C_{q^2-\sqrt{2}q+1},$$
$$\mathbf{T}_0[s_1s_2s_1] = \{h_1(t^{-2^f})h_2(t) \mid t \in \mathbf{k}^\times, t^{2^{2f+1}+2^{f+1}+1} = 1\} \cong C_{q^2+\sqrt{2}q+1}.$$

This is a complete set of representatives of $\mathbf{G}^F$-conjugacy classes of the F-stable maximal tori of $\mathbf{G}$.

We complete the investigation of F-stable maximal tori with the following useful observation.

Lemma 2.52. [**DM20**, Proposition 4.2.23] *Every semisimple $s \in \mathbf{G}^F$ lies in some F-stable maximal torus of $\mathbf{G}$.*

2.5.3. Simplicity and universal coverings. In order to prove one of the local-global conjectures that have been reduced to a problem about simple groups, we have to verify the corresponding inductive condition for all finite simple groups. If we want to verify it for a certain family of finite groups of Lie type, we first have to know which groups are actually simple.

Proposition 2.53. [**MT11**, Theorem 24.17, Remark 24.18, Proposition 24.21] *Let $\mathbf{G}_{sc}$ be a simple simply connected algebraic group and $\mathbf{G}_{ad}$ simple of adjoint type with the same underlying root system. Then, $\mathbf{G}_{sc}^F$ is perfect and $\mathbf{G}_{sc}^F/Z(\mathbf{G}_{sc}^F) \cong [\mathbf{G}_{ad}^F, \mathbf{G}_{ad}^F]$ is simple except for the following groups:*

- $\mathrm{SL}_2(2)$, $\mathrm{SL}_2(3)$, $\mathrm{SU}_3(2)$, *and* ${}^2\mathsf{B}_2(2)$ *are solvable;*
- $\mathsf{B}_2(2)$, $\mathsf{G}_2(2)$, *and* ${}^2\mathsf{G}_2(3)$ *are almost simple;*
- ${}^2\mathsf{F}_4(2)$ *contains the simple Tits group* ${}^2\mathsf{F}_4(2)'$ *as a normal subgroup of index* 2.

Since $\mathsf{B}_2(2)' \cong \mathrm{PSL}_2(9)$, $\mathsf{G}_2(2)' \cong \mathrm{PSU}_3(3)$, and ${}^2\mathsf{G}_2(3)' \cong \mathrm{PSL}_2(8)$, we have to consider the inductive McKay–Navarro condition for ${}^2\mathsf{F}_4(2)'$ and $\mathbf{G}^F/Z(\mathbf{G}^F)$ for every simple simply connected group $\mathbf{G}$ and Steinberg endomorphism F. We have to consider a universal covering group of these simple groups.

Proposition 2.54. [**GLS98**, Theorem 6.1.4] *Let $\mathbf{G}$ be simple of simply connected type and assume that $\mathbf{G}^F$ is perfect. Then, $\mathbf{G}^F$ is a universal covering group of $S = \mathbf{G}^F/Z(\mathbf{G}^F)$ except for the groups occurring in Table 2.3. In particular, the Schur multiplier of S is $Z(\mathbf{G}^F)$. The groups in Table 2.3 have Schur multiplier $Z(\mathbf{G}^F) \times M_e(\mathbf{G}^F)$.*

$\mathbf{G}^F$	$M_e(\mathbf{G}^F)$	$\mathbf{G}^F$	$M_e(\mathbf{G}^F)$	$\mathbf{G}^F$	$M_e(\mathbf{G}^F)$
$\mathrm{PSL}_2(4)$	C_2	$\mathsf{B}_3(2)$	C_2	${}^2\mathsf{E}_6(2)$	$C_2 \times C_2$
$\mathrm{PSL}_3(2)$	C_2	${}^2\mathsf{B}_2(8)$	$C_2 \times C_2$	$\mathrm{PSL}_2(9)$	C_3
$\mathrm{PSL}_3(4)$	$C_4 \times C_4$	$\mathsf{D}_4(2)$	$C_2 \times C_2$	$\mathrm{PSU}_4(3)$	$C_3 \times C_3$
$\mathrm{PSL}_4(2)$	C_2	$\mathsf{G}_2(4)$	C_2	$\mathsf{B}_3(3)$	C_3
$\mathrm{PSU}_4(2)$	C_2	$\mathsf{F}_4(2)$	C_2	$\mathsf{G}_2(3)$	C_3
$\mathrm{PSU}_6(2)$	$C_2 \times C_2$				

Table 2.3. Exceptional parts of the Schur multipliers of simple finite groups of Lie type [**MT11**, Table 24.3].

The groups $Z(\mathbf{G}^F)$ and $M_e(\mathbf{G}^F)$ are also called the *canonical* and *exceptional part* of the Schur multiplier of S. We say that a simple group has a *generic Schur multiplier* if the exceptional Schur multiplier is trivial.

Since we have $Z(\mathbf{G}^F) = Z(\mathbf{G})^F$ for simple algebraic groups by [**MT11**, Corollary 24.2], we can easily deduce the centers of $\mathbf{G}^F$ from $Z(\mathbf{G})$. In particular, the center of a simple group $\mathbf{G}$ of simply connected type is the p'-part of the fundamental group of the underlying root system and we can easily compute $Z(\mathbf{G}^F)$ as given in [**MT11**, Table 24.2].

2.5.4. Group automorphisms. In this section, we assume that $\mathbf{G}_{sc}$ is simple and $\mathbf{G}_{sc}^F$ is perfect. We consider the automorphisms of a finite simple group

$$S = \mathbf{G}_{sc}^F/Z(\mathbf{G}_{sc}^F) \cong [\mathbf{G}_{ad}^F, \mathbf{G}_{ad}^F].$$

Since we have to consider the action of group automorphisms on irreducible characters in the inductive McKay–Navarro condition, we need a good description of these automorphisms. We first describe certain classes of group automorphisms.

Definition 2.55. An automorphism of S is called

(i) a *diagonal automorphism* if it is induced by conjugation with an element of $\mathbf{G}_{ad}^F$ and not an inner automorphism of S;

(ii) a *field automorphism* if it arises from an automorphism f of $\mathbf{G}_{ad}$ such that for all positive roots α of $\mathbf{G}$ and we have $f(u_\alpha(c)) = u_\alpha(c^{p^k})$ for all $c \in \mathbf{k}$ with some fixed integer $k \geq 1$;

(iii) a *graph automorphism* if $\mathbf{G}_{sc}^F$ is untwisted and it arises from an automorphism f of $\mathbf{G}_{ad}$ with $f(u_\alpha(t)) = u_{\rho(\alpha)}(t)$ for all $c \in \mathbf{k}$ and all positive roots α where ρ is a non-trivial Dynkin diagram automorphism;

(iv) an *exceptional graph automorphism* if it arises from an exceptional endomorphism γ of $\mathbf{G}_{ad}$ as described in Proposition 2.43(c).

Note that twisted groups have no graph automorphisms since every graph endomorphism of $\mathbf{G}_{sc}$ acts on $\mathbf{G}_{sc}^F$ in the same way as some field automorphism. Together with the inner automorphisms, the described automorphisms already generate the whole automorphism group of S.

Proposition 2.56. [**GLS98**, Theorem 2.5.12] *The automorphism group of S is a split extension of the automorphisms induced by conjugation with elements of $\mathbf{G}_{ad}^F$ and the group of field and (possibly exceptional) graph automorphisms.*

More information on the structure of the automorphism group of S depending on its type can be looked up for example in [**GLS98**, Theorem 2.5.12].

Example 2.57. Let $\mathbf{G}$ be of type $\mathsf{B}_2, \mathsf{G}_2$, or F_4 with $p = 2$, $p = 3$, or $p = 2$ and γ the exceptional graph automorphism from Proposition 2.43(c). Then, we can easily see $\gamma^2 = F_p$. Since the p'-part of the fundamental group of $\mathbf{G}$ is trivial, there are no diagonal automorphisms of $\mathbf{G}^F$.

(a) If $F = F_p^f$ is a power of the standard Frobenius endomorphism, then $\mathrm{Out}(\mathbf{G}^F) \cong \langle\gamma\rangle$ is cyclic of order $2f$.

(b) If $F = F_p^f \circ \gamma$ is not a Frobenius map, then $\mathrm{Out}(\mathbf{G}^F) \cong \langle\gamma\rangle$ is cyclic of order $2f + 1$.

We know from Lemma 1.29 that we can identify the automorphisms of the universal covering $\mathbf{G}_{sc}^F$ of S with the automorphisms of S. Thus, we can use the results from this section in the inductive McKay–Navarro condition.

2.5.5. Dual groups. For finite groups of Lie type, there is a notion of duality that we introduce in this section. We will see later that this so-called dual group of a finite group of Lie type plays a crucial role in the description of its irreducible characters.

Definition 2.58. Let $\mathbf{T}_0$ be a maximally split torus of $\mathbf{G}$. Let $(X(\mathbf{T}_0), \Phi, Y(\mathbf{T}_0), \Phi^\vee)$ be the root datum of $\mathbf{G}$ with respect to $\mathbf{T}_0$. A pair $(\mathbf{G}^*, F^*)$ is called *dual to* $(\mathbf{G}, F)$ if

(i) there exists a maximally split torus $\mathbf{T}_0^* \subseteq \mathbf{G}^*$ such that the root datum of $\mathbf{G}^*$ with respect to $\mathbf{T}_0^*$ is isomorphic to $(Y(\mathbf{T}_0), \Phi^\vee, X(\mathbf{T}_0), \Phi)$ and

(ii) this isomorphism is compatible with the actions of F and F^*.

In this situation, we also say that $\mathbf{G}^F$ is dual to $\mathbf{G}^{*F^*}$. For a more detailed definition of isomorphisms of root data and the compatibility in (ii), see e.g. [**GM20**, Definition 1.5.17].

Example 2.59. Some examples of finite groups of Lie type and their dual groups are displayed in Table 2.4. In particular, the groups $\mathsf{G}_2(q)$, $\mathsf{F}_4(q)$, ${}^2\mathsf{B}_2(q^2)$, ${}^2\mathsf{G}_2(q^2)$, and ${}^2\mathsf{F}_4(q^2)$ are all dual to themselves [**Car85**, p.120].

The description of the root data of $\mathrm{SL}_n(\mathbf{k})$ and $\mathrm{PGL}_n(\mathbf{k})$ in Example 2.30 gives us a good impression how the roots and coroots of groups in duality correspond to each other.

$\mathbf{G}^F$	$\mathbf{G}^{*F^*}$
$(\mathsf{A}_n)_{sc}(q) = \mathrm{SL}_{n+1}(q)$	$(\mathsf{A}_n)_{ad}(q) = \mathrm{PGL}_{n+1}(q)$
$({}^2\mathsf{A}_n)_{sc}(q) = \mathrm{SU}_{n+1}(q)$	$({}^2\mathsf{A}_n)_{ad}(q) = \mathrm{PGU}_{n+1}(q)$
$(\mathsf{B}_n)_{sc}(q) = \mathrm{Spin}_{2n+1}(q)$	$(\mathsf{C}_n)_{ad}(q) = \mathrm{PCSp}_{2n}(q)$
$(\mathsf{C}_n)_{sc}(q) = \mathrm{Sp}_{2n}(q)$	$(\mathsf{B}_n)_{ad}(q) = \mathrm{SO}_{2n+1}(q)$

Table 2.4. Some finite groups of Lie type and their dual groups [**Car85**, p.120].

We sometimes need a correspondence between automorphisms of $\mathbf{G}^F$ and automorphisms of its dual. The following is described in more detail in [**Ruh22**, Remark 2.12] and [**NTT08**, Section 2].

Remark 2.60. Let $(\mathbf{G}^*, F^*)$ be dual to $(\mathbf{G}, F)$ and $\pi : \mathbf{G} \to \mathbf{G}$ a bijective morphism commuting with F. Then, the isogeny theorem (see [**GM20**, Lemma 1.4.24]) implies that π induces a bijective morphism

$$\pi^* : \mathbf{G}^* \to \mathbf{G}^* \quad \text{with} \quad F^* \circ \pi^* = \pi^* \circ F^*.$$

It is unique up to conjugation with elements in $\mathscr{L}_*^{-1}(Z(\mathbf{G}^*))$ where $\mathscr{L}_*$ denotes the Lang map of $\mathbf{G}^*$ with respect to F^*.

Both maps π and π^* restrict to automorphisms of $\mathbf{G}^F$ and $\mathbf{G}^{*F^*}$, respectively. If $\mathbf{G}$ is a simple algebraic group of simply connected type, then all elements in $\mathrm{Aut}(\mathbf{G}^F)$ arise in this way. Thus, we can define a dual automorphism $\kappa^* \in \mathrm{Aut}(\mathbf{G}^{*F^*})$ for every $\kappa \in \mathrm{Aut}(\mathbf{G}^F)$. Since $\mathbf{G}^*$ is of adjoint type, it has trivial center and κ^* is thereby even unique up to conjugation in $\mathscr{L}_*^{-1}(1) = \mathbf{G}^{*F^*}$.

2.6. A regular embedding and dual fundamental weights

We will see later that it is much easier to construct ordinary characters of a finite group of Lie type if the center of the corresponding algebraic group is connected. In this section, we introduce a construction that allows us to embed an algebraic group $\mathbf{G}$ of simply connected type into a connected reductive group with connected center. We follow [**GM20**, Section 1.7] and modify the constructions in [**Mas10**, Section 2].

First, we choose a suitable torus $\mathbf{S}$ containing $Z(\mathbf{G})$ that has a small rank as follows. Note that the group $Z(\mathbf{G})$ depends on the characteristic of $\mathbf{k}$. Let d be the minimal integer such that $Z(\mathbf{G})$ is generated by d elements. We set $\mathbf{S} := (\mathbf{k}^\times)^d$. Then, we have an embedding $\iota : Z(\mathbf{G}) \hookrightarrow \mathbf{S}$ and can consider $Z(\mathbf{G})$ as a subgroup of $\mathbf{S}$. We define the group $\widetilde{\mathbf{G}}$ by

$$\widetilde{\mathbf{G}} := (\mathbf{G} \times \mathbf{S})/Z(\mathbf{G}) = (\mathbf{G} \times \mathbf{S})/\{(z, z^{-1}) \mid z \in Z(\mathbf{G})\}$$

where $Z(\mathbf{G})$ acts by multiplication with (z, z^{-1}) on the direct product. Naturally, we have embeddings

$$\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}, \quad \mathbf{S} \hookrightarrow \widetilde{\mathbf{G}},$$

and we can identify $\mathbf{G}$ and $\mathbf{S}$ with their images in $\widetilde{\mathbf{G}}$. Let $\mathbf{T} \subseteq \mathbf{B}$ be a maximally split torus contained in an F-stable Borel subgroup with unipotent radical $\mathbf{U}$. Then, the group $\widetilde{\mathbf{T}} := \mathbf{TS}$ is a maximal torus and $\widetilde{\mathbf{B}} := \widetilde{\mathbf{T}}\mathbf{U}$ a Borel subgroup of $\widetilde{\mathbf{G}}$ with $R_u(\widetilde{\mathbf{B}}) = \mathbf{U}$.

Note that the group $\widetilde{\mathbf{G}}$ depends on the choice of the embedding ι. We can extend the field and graph automorphisms of $\mathbf{G}$ to $\widetilde{\mathbf{G}}$. A possible choice of these extensions can be found in [**Mas10**, Section 3 and 4]. We use the name of the automorphism of $\mathbf{G}$ itself to denote these extensions to $\widetilde{\mathbf{G}}$.

2.6.1. Dual fundamental weights. Let $\mathbf{G}$ be a simply connected group with maximally split torus $\mathbf{T}$ and Φ its root system with base $\Delta = \{\alpha_1, \dots, \alpha_m\}$ with respect to $\mathbf{T}$.

Definition 2.61. The *fundamental weights* of $\mathbf{T}$ with respect to Δ are a basis $\lambda_1, \dots, \lambda_m$ of $X(\mathbf{T})$ such that we have $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ for all $1 \leq i, j \leq m$.

Thus, the fundamental weights are a basis of $X(\mathbf{T})$ that is dual to the $\alpha_i^\vee$ with respect to the perfect pairing from Lemma 2.9. We want to have a similar basis $\omega_1^\vee, \dots, \omega_m^\vee$ of $Y(\mathbf{T})$ such that we have $\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}$ for all $1 \leq i, j \leq m$. However, such a basis does not always exist and we have to consider the universal group. We follow [**Mas10**, Section 6] and adjust the constructions to our setting.

Definition 2.62. For every $i \in \{1, \dots, m\}$ we define

$$\beta_i : \mathbf{k}^\times \to \mathbf{T}, \quad \beta_i(c) := \prod_{j=1}^{m} h_{\alpha_j}(c^{lD_{ji}})$$

where $(D_{ji})_{1 \leq j,i \leq m}$ is the inverse of the Cartan matrix of $\mathbf{G}$ and l is the exponent of $Z(\mathbf{G})$.

Note that due to the adjustment of the definition in [**Mas10**, Section 6], lD_{ji} does not have to be an integer. However, $lD_{ji} \cdot p^k$ is an integer for some $k \in \mathbb{N}$. Since the choice of a p-th root in a field of characteristic p is unique, the β_i are well-defined.

Definition 2.63. The *dual fundamental weights* are given by

$$\omega_i^\vee : \mathbf{k}^\times \to \widetilde{\mathbf{T}}, \quad \omega_i^\vee(c) := \beta_i(c_l) \prod_{j \in \mathfrak{I}_d} c_l^{\langle \omega_j, \beta_i \rangle}$$

for every $i \in \{1, \dots, m\}$. Here, d is the number of generators of $Z(\mathbf{G})$ as above, c_l is an l-th root of c, and $\mathfrak{I}_d \subseteq \mathfrak{I}$ consists of d elements of the index set $\mathfrak{I}$ given in Table 2.5.

Type	A_n	B_n	C_n	D_{2n+1}	D_{2n}	E_6	E_7	$\mathsf{G}_2, \mathsf{F}_4, \mathsf{E}_8$
$\exp(\Lambda(\Phi))$	$n+1$	2	2	4	2	3	2	1
$\mathfrak{I}$	$\{n\}$	$\{n\}$	$\{1\}$	$\{2n+1\}$	$\{2n, 2n-1\}$	$\{1\}$	$\{2\}$	$\emptyset$

Table 2.5. Exponents of the fundamental group of the root system $\Lambda(\Phi)$ and possible choices of $\mathfrak{I}$ from [**Mas10**, Table 6.9]. The exponent l of $Z(\mathbf{G})$ is the p'-part of $\exp(\Lambda(\Phi))$, see Proposition 2.29.

The dual fundamental weights are well-defined, see [**Mas10**, p. 18]. With the help of the index set $\mathfrak{I}$ we can also explicitly define an embedding $\iota : Z(\mathbf{G}) \hookrightarrow \mathbf{S}$. We assume that this embedding has been chosen as in [**Mas10**, Section 6].

Proposition 2.64. [**Mas10**, Proposition 6.11] *For all $\alpha_i \in \Delta$, let $\widetilde{\alpha}_i \in X(\widetilde{\mathbf{T}})$ be the unique extension of α_i that is trivial on $\mathbf{S}$. Then, we have*

$$\langle \widetilde{\alpha}_i, \omega_j^\vee \rangle = \delta_{ij}$$

for all $1 \leq i, j \leq m$. Further, the set $\{\omega_i^\vee \mid 1 \leq i \leq m\}$ forms a basis of $Y(\mathbf{T})$.

We now explicitly compute the dual fundamental weights for type G_2 and B_2.

Example 2.65. (a) Let $\mathbf{G}$ be of type G_2. Then, the inverse of the Cartan matrix is

$$D = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix},$$

see e.g. [**Mas10**, Appendix], and we have $l = 1$. Thus, the dual fundamental weights are given by

$$\omega_1^\vee(c) = \beta_1(c) = h_{\alpha_1}(c^2)h_{\alpha_2}(c^3), \quad \omega_2^\vee(c) = \beta_2(c) = h_{\alpha_1}(c)h_{\alpha_2}(c^2)$$

for all $c \in \mathbf{k}^\times$.

(b) Let $\mathbf{G}$ be of type B_2. Then the inverse of the Cartan matrix is

$$D = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{pmatrix},$$

see e.g. [**Mas10**, Appendix], and we have $l = 2$ if p is odd and $l = 1$ if p is even. First, assume that p is odd. From the definition of the β_j we know $\omega_i(\beta_j(c)) = c^{lD_{ij}}$ for all $c \in \mathbf{k}^\times$. Thus, the dual fundamental weights are given by

$$\omega_1^\vee(c) = \beta_1(\sqrt{c})\sqrt{c}^{\langle\omega_2,\beta_1\rangle} = h_{\alpha_1}(\sqrt{c}^2)h_{\alpha_2}(\sqrt{c}^1)\sqrt{c}^1 = h_{\alpha_1}(c)h_{\alpha_2}(\sqrt{c})\sqrt{c},$$
$$\omega_2^\vee(c) = \beta_2(\sqrt{c})\sqrt{c}^{\langle\omega_2,\beta_2\rangle} = h_{\alpha_1}(\sqrt{c}^2)h_{\alpha_2}(\sqrt{c}^2)\sqrt{c}^2 = h_{\alpha_1}(c)h_{\alpha_2}(c)c.$$

where $\sqrt{c}$ denotes a fixed square root of c in $\mathbf{k}^\times$.

If $p = 2$, we similarly obtain

$$\omega_1^\vee(c) = \beta_1(c) = h_{\alpha_1}(c)h_{\alpha_2}(\sqrt{c}), \quad \omega_2^\vee(c) = \beta_2(c) = h_{\alpha_1}(c)h_{\alpha_2}(c).$$

Remark 2.66. As already mentioned above, we slightly changed the constructions presented in this section from the setup in [**Mas10**, Section 6]. There, Maslowski did not work with the group $\widetilde{\mathbf{G}}$ as defined above but constructed a group that can be used for all groups of a certain type independently of their characteristic. This can be obtained by choosing d to be the maximal number of generators of $Z(\mathbf{G})$ for any characteristic of $\mathbf{k}$. Similarly, the construction of the dual fundamental weights always needs the whole set $\mathfrak{I}$ and l does not depend on p anymore.

This approach has the advantage that we can consider all simply connected groups of a certain type at the same time. However, in order to work with these universal groups, we have to extend the Steinberg endomorphisms of $\mathbf{G}$ to them. As mentioned above, Maslowski did this for the graph and field endomorphisms. Naturally, this is not possible for exceptional graph endomorphisms since they depend on the characteristic of $\mathbf{k}$. For example, in type B_2 over a field of characteristic 2 we do not see any natural way of extending the exceptional graph automorphism to $\mathbf{S}$. Since we later only need the dual fundamental weights for finite groups of Lie type in a fixed characteristic, we decided to modify the constructions as presented above.

2.7. Choosing the local subgroup

In the inductive McKay–Navarro condition, we choose a proper subgroup N of a finite group G containing the normalizer of a Sylow ℓ-subgroup of G and study its irreducible ℓ'-characters for a given prime ℓ. In order to treat infinite series of finite groups of Lie type, we need a generic way of choosing N. In this section, we present such choices. Let $\mathbf{G}$ be a connected reductive group defined over $\mathbf{k}$ and F a Steinberg endomorphism of $\mathbf{G}$.

If the prime ℓ is equal to $\mathrm{char}(\mathbf{k}) = p$, then the situation is quite easy.

Proposition 2.67. [**MT11**, Corollary 24.11] *Let $\mathbf{T}$ be a maximally split torus contained in an F-stable Borel subgroup $\mathbf{B}$ and $\mathbf{U} := R_u(\mathbf{B})$. Then $\mathbf{U}^F$ is a Sylow p-subgroup of $\mathbf{G}^F$ with normalizer $N_{\mathbf{G}^F}(\mathbf{U}^F) = \mathbf{B}^F$.*

Thus, with the notation of the previous proposition, we can choose $\mathbf{B}^F$ as the local subgroup N in the inductive McKay–Navarro condition.

If the characteristic of $\mathbf{k}$ is different from ℓ, we cannot give a generic description of the normalizer of a Sylow ℓ-subgroup. Nevertheless, we can construct a generic group containing the normalizer of a Sylow ℓ-subgroup for all finite groups of Lie type with some known exceptions for $\ell = 2$ and $\ell = 3$.

Definition 2.68. Let $\mathbf{S}$ be an F-stable torus of $\mathbf{G}$.

(i) Assume that F is a Frobenius map. We say that $\mathbf{S}$ is a *d-torus* if its order polynomial with respect to F is a power of the d-th cyclotomic polynomial Φ_d.

(ii) The centralizers of d-tori in $\mathbf{G}$ are called *d-split Levi subgroups* of $\mathbf{G}$.

(iii) A d-torus is called a *Sylow d-torus* of $\mathbf{G}$ if its order polynomial with respect to F is the full Φ_d-part of the order polynomial of $\mathbf{G}$ with respect to F.

(iv) Assume that $\mathbf{G}^F$ is a Suzuki or Ree group over a field of characteristic p. Let

$$\Psi := \{\Phi_1\Phi_2\} \cup \{\text{irreducible factors of } \Phi_d \text{ over } \mathbb{Z}[\sqrt{p}] \text{ for } d \geq 3\}.$$

For $\phi \in \Psi$ we say that $\mathbf{S}$ is a *ϕ-torus* if its order polynomial with respect to F is a power of ϕ.

We define *ϕ-split Levi subgroups* and *Sylow ϕ-tori* analogously to (ii) and (iii).

Note that the polynomials occurring in Table 2.2 as factors of the orders of the Suzuki and Ree groups are contained in Ψ.

Sylow tori have properties that are similar to those of Sylow subgroups. We need the following results from the Sylow theorem for Sylow tori.

Theorem 2.69. [**BM92**, Theorem 3.4] *Let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism and $d \geq 1$.*

(a) *There exist Sylow d-tori in $\mathbf{G}$.*

(b) *Any d-torus of $\mathbf{G}$ is contained in a Sylow d-torus of $\mathbf{G}$.*

(c) *All Sylow d-tori of $\mathbf{G}$ are $\mathbf{G}^F$-conjugate.*

An analogous result for Sylow ϕ-tori of Suzuki and Ree groups holds by [**BM92**, Section 3F].

Example 2.70. We have already seen in Example 2.48 that the Suzuki groups ${}^2\mathsf{B}_2(q^2)$ have the order polynomial

$$q^4(\Phi_1\Phi_2)(q)\Phi_8^+(q)\Phi_8^-(q).$$

From Example 2.51 we know that the F-stable maximal tori of ${}^2\mathsf{B}_2(q^2)$ have the order polynomials

$$(\Phi_1\Phi_2)(q), \quad \Phi_8^+(q), \quad \Phi_8^-(q)$$

which are polynomials contained in Ψ. Thus, all three F-stable maximal tori of ${}^2\mathsf{B}_2(q^2)$ are Sylow tori.

The Ree groups ${}^2\mathsf{F}_4(q^2)$ have the order polynomial

$$q^{24}(\Phi_1\Phi_2)(q)^2\Phi_4(q)^2\Phi_8^+(q)^2\Phi_8^-(q)^2\Phi_{12}(q)\Phi_{24}^+(q)\Phi_{24}^-(q).$$

From [**SI75**], we know that their F-stable maximal tori $T(1), \ldots, T(11)$ have the respective order polynomials

$$(\Phi_1\Phi_2)(q)^2, \quad (\Phi_1\Phi_2)(q)\Phi_4(q), \quad (\Phi_1\Phi_2)(q)\Phi_8^+(q), \quad (\Phi_1\Phi_2)(q)\Phi_8^-(q), \quad \Phi_8^+(q)\Phi_8^-(q),$$
$$\Phi_8^+(q)^2, \quad \Phi_8^-(q)^2, \quad \Phi_4(q)^2, \quad \Phi_{12}(q), \quad \Phi_{24}^+(q), \quad \Phi_{24}^-(q).$$

Thus, the Sylow tori of ${}^2\mathsf{F}_4(q^2)$ are $T(1)$, $T(6)$, $T(7)$, $T(8)$, $T(9)$, $T(10)$, and $T(11)$.

We see for both ${}^2\mathsf{B}_2(q^2)$ and ${}^2\mathsf{F}_4(q^2)$ that, for every $\phi \in \Psi$ dividing the order polynomial of the group, there is a Sylow ϕ-torus that is unique up to $\mathbf{G}^F$-conjugacy.

As already mentioned before, we can use Sylow tori to choose the local subgroup N in a generic way. This is due to the following theorem.

Theorem 2.71. [**Mal07**, Theorem 5.14, 5.19, and 8.4] *Let* $\mathbf{G}$ *be simple and defined over* $\mathbb{F}_q$ *where* q *is a power of* p. *Let* ℓ *be a prime dividing* $|\mathbf{G}^F|$ *different from* p *and* R *a Sylow* ℓ*-subgroup of* $\mathbf{G}^F$.

(1) *If* F *is a Frobenius map, we write* d *for the multiplicative order of* q *modulo* ℓ *if* $\ell \neq 2$, *and modulo* 4 *if* $\ell = 2$. *Then, there exists a Sylow* d*-torus* $\mathbf{S}$ *of* $\mathbf{G}^F$ *such that* $N_{\mathbf{G}^F}(R) \leq N_{\mathbf{G}^F}(\mathbf{S})$ *unless we are in one of the following cases:*
   (a) $\ell = 2$ *and* $\mathbf{G}^F = \mathrm{Sp}_{2n}(q)$ *for* $n \geq 1$ *with* $q \equiv 3, 5 \mod 8$*;*
   (b) $\ell = 3$ *and* $\mathbf{G}^F = \mathrm{SL}_3(q)$ *with* $q \equiv 4, 7 \mod 9$*;*
   (c) $\ell = 3$ *and* $\mathbf{G}^F = \mathrm{SU}_3(q)$ *with* $q \equiv 2, 5 \mod 9$*;*
   (d) $\ell = 3$ *and* $\mathbf{G}^F = \mathsf{G}_2(q)$ *with* $q \equiv 2, 4, 5, 7 \mod 9$.

(2) *If* $\mathbf{G}^F$ *is a Suzuki or Ree group, we write* $\phi^{(\ell)}$ *for the cyclotomic polynomial over* $\mathbb{Z}[\sqrt{p}]$ *dividing the generic order of* $\mathbf{G}$ *such that* ℓ *divides* $\phi^{(\ell)}(q)$. *This is uniquely determined except for* $\mathbf{G}^F = {}^2\mathsf{G}_2(q^2)$ *and* $\ell = 2$ *or* $\mathbf{G}^F = {}^2\mathsf{F}_4(q^2)$ *and* $\ell = 3$. *In both cases, we set* $\phi^{(\ell)} := q^2 + 1$. *Then, there exists a Sylow* $\phi^{(\ell)}$*-torus* $\mathbf{S}$ *of* $\mathbf{G}^F$ *such that* $N_{\mathbf{G}^F}(R) \leq N_{\mathbf{G}^F}(\mathbf{S})$ *unless we are in one of the following cases:*
   (a) $\ell = 2$ *and* $\mathbf{G}^F = {}^2\mathsf{G}_2(q^2)$*;*
   (b) $\ell = 3$ *and* $\mathbf{G}^F = {}^2\mathsf{F}_4(q^2)$ *with* $q \equiv 2, 5 \mod 9$.

If we are not in one of the mentioned cases, we can choose N as the normalizer of a Sylow torus in $\mathbf{G}^F$. The inductive McKay–Navarro condition also requires N to be stable under all automorphisms that stabilize the corresponding Sylow subgroup. This follows as in [**CS13**, Proposition 2.5] from the fact that all Sylow tori of the same order are conjugate under $\mathbf{G}^F$.

Proposition 2.72. *In the setting of Theorem 2.71, assume that* $\mathbf{G}$ *is simply connected and we are not in one of the cases listed in* (1) *or* (2). *Then, the Sylow torus* $\mathbf{S}$ *is stable under* $\mathrm{Aut}(\mathbf{G}^F)_R$.

CHAPTER 3

Representation theory of finite groups of Lie type

In this chapter, we introduce some basic concepts about the ordinary character theory of finite groups of Lie type. As we have seen in the previous chapter, finite groups of Lie type are constructed from a connected reductive group and a Steinberg endomorphism that corresponds to some prime power q and a (possibly trivial) graph automorphism of the corresponding Coxeter diagram. Therefore, each family of finite groups of Lie type consists of infinitely many finite groups. We are interested in a way to describe the irreducible characters of all finite groups of Lie type of a certain type at the same time. Since the fundamental work of Deligne and Lusztig in the 1970s, an extensive theory has been developed that provides us with a lot of information about the character theory of a family of groups of Lie type.

We give an overview of the parts of this theory that we need in this work. First, we define Lusztig induction as well as Deligne–Lusztig characters and state some of their properties that allow us to present the very important Jordan decomposition of characters. In the next section, we introduce Harish-Chandra induction and use this to define a duality operation on characters of finite groups of Lie type. We further give a definition of Gelfand–Graev characters and finally generalize some concepts from Deligne–Lusztig theory to the setting of disconnected groups. More details about the contents of this chapter can be found in [**GM20**], [**DM20**], or [**Car85**].

Let $\mathbf{G}$ be a connected reductive group over $\mathbf{k} := \overline{\mathbb{F}_p}$ and F a Steinberg endomorphism of $\mathbf{G}$.

3.1. Lusztig induction and Jordan decomposition

In this section, we present a parametrization of the irreducible characters of the finite group of Lie type $\mathbf{G}^F$. We first define Lusztig induction and Deligne–Lusztig characters.

Before we can do this, we have to introduce some related concepts. We follow [**GM20**, Section 2.2.2]. Let $\mathbf{X}$ be an algebraic variety defined over $\mathbf{k}$ and $F' : \mathbf{X} \to \mathbf{X}$ a Frobenius map. Let S be a finite group acting on $\mathbf{X}$ as algebraic automorphisms. For $g \in S$ commuting with F', the map $F'^n \circ g$ has only finitely many fixed points on $\mathbf{X}$ for all positive $n \in \mathbb{Z}$. We define the formal power series

$$R(t,g) := -\sum_{n=1}^{\infty} |\mathbf{X}^{F'^n \circ g}| t^n \ \in \mathbb{Z}[[t]].$$

One can show that $R(t,g)$ is independent of F'. It is a rational function in t, has only simple poles, and no pole at ∞.

Definition 3.1. In the above setting, the *Lefschetz number* of g on $\mathbf{X}$ is

$$\mathfrak{L}(g,\mathbf{X}) := \lim_{t\to\infty} R(t,g).$$

Lefschetz numbers are usually defined in more generality by considering the action of S on the so-called ℓ-adic cohomology groups with compact support $\mathrm{H}^i_c(\mathbf{X}, \overline{\mathbb{Q}_\ell})$ for $i \in \mathbb{Z}$ and a prime $\ell \neq p$, see e.g. [**Car85**, Appendix]. In this setting, we have

$$\mathfrak{L}(g,\mathbf{X}) = \sum_{i\geq 0} (-1)^i \operatorname{Tr}(g, H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)).$$

Since we do not have to work directly with these objects, it suffices to consider the definition given above.

We now specify the choice of $\mathbf{X}$ and S that we need for the definition of Lusztig induction. For $\mathbf{G}$ and F as defined above, let $\mathbf{Y} \subseteq \mathbf{G}$ be a closed subset and $\mathscr{L}$ the Lang map as in Theorem 2.40. We consider the algebraic variety $\mathbf{X} := \mathscr{L}^{-1}(\mathbf{Y})$.

Let H be a finite subgroup of $\mathbf{G}$ such that $h^{-1}\mathbf{Y}F(h) \subseteq \mathbf{Y}$ for all $h \in H$. Then, $S := \mathbf{G}^F \times H$ acts on $\mathscr{L}^{-1}(\mathbf{Y})$ via $(g,h).x = gxh^{-1}$ for all $(g,h) \in S$ and $x \in \mathscr{L}^{-1}(\mathbf{Y})$.

Definition 3.2. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ contained in a parabolic subgroup $\mathbf{P}$ with Levi decomposition $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$. We consider the action of $S := \mathbf{G}^F \times \mathbf{L}^F$ on $\mathbf{X} := \mathscr{L}^{-1}(\mathbf{U})$ as described above. Then, we define *Lusztig induction* by

$$R^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}} \colon \mathbb{Z}\operatorname{Irr}(\mathbf{L}^F) \to \mathbb{Z}\operatorname{Irr}(\mathbf{G}^F),\ \theta \mapsto \left(R^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}}(\theta) : g \mapsto \frac{1}{|\mathbf{L}^F|}\sum_{l\in\mathbf{L}^F} \mathfrak{L}((g,l),\mathscr{L}^{-1}(\mathbf{U}))\theta(l)\right),$$

and *Lusztig restriction* by

$${}^*R^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}} \colon \mathbb{Z}\operatorname{Irr}(\mathbf{G}^F) \to \mathbb{Z}\operatorname{Irr}(\mathbf{L}^F),\ \chi \mapsto \left({}^*R^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}}(\chi) : l \mapsto \frac{1}{|\mathbf{G}^F|}\sum_{g\in\mathbf{G}^F} \mathfrak{L}((g,l),\mathscr{L}^{-1}(\mathbf{U}))\chi(g)\right).$$

Lusztig induction and restriction are adjoint to each other with respect to the usual inner product of class functions [**GM20**, Definition 3.3.2]. In the remaining part of this section, we are interested in a special case of Lusztig induction that occurs if the F-stable Levi subgroup $\mathbf{L}$ is a torus.

Definition 3.3. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, $\theta \in \operatorname{Irr}(\mathbf{T}^F)$, and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. Then, we define the *Deligne–Lusztig character*

$$R^{\mathbf{G}}_{\mathbf{T}}(\theta) := R^{\mathbf{G}}_{\mathbf{T}\leq\mathbf{B}}(\theta) \in \mathbb{Z}\operatorname{Irr}(\mathbf{G}^F).$$

Although it is not clear from the definition, one can show that the Deligne–Lusztig characters do not depend on the choice of the Borel subgroup, see [**GM20**, Corollary 2.2.9]. Therefore, we usually denote them by $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ and do not specify a group $\mathbf{B}$.

Despite their name, Deligne–Lusztig characters are not characters but integer linear combinations of characters.

Definition 3.4. A *virtual character* of a group is a $\mathbb{Z}$-linear combination of characters of the group.

Thus, Deligne–Lusztig characters are virtual characters. In the following, $\mathbf{T}$ always denotes an F-stable maximal torus and θ an irreducible character of $\mathbf{T}^F$.

Lusztig induction is sometimes also called twisted induction since it generalizes the usual induction of characters:

Proposition 3.5. [**GM20**, Proposition 2.2.7] *Assume that $\mathbf{T}_0 \subseteq \mathbf{G}$ is a maximally split torus contained in an F-stable Borel subgroup $\mathbf{B}_0$ and $\theta \in \operatorname{Irr}(\mathbf{T}_0^F)$. We have*

$$R^{\mathbf{G}}_{\mathbf{T}_0}(\theta) = \operatorname{Ind}^{\mathbf{G}^F}_{\mathbf{B}_0^F} \circ \operatorname{Inf}^{\mathbf{B}_0^F}_{\mathbf{T}_0^F}(\theta).$$

In this special case, the Deligne–Lusztig character is clearly a character. For maximal tori that are not maximally split, we do not have such a construction and need the definition of Lusztig induction given above.

Deligne–Lusztig characters satisfy many convenient properties. For example, it is very useful that different Deligne–Lusztig characters are orthogonal to each other with respect to the usual scalar product of class functions.

Proposition 3.6. [**GM20**, Proposition 2.2.8] *Let* $\mathbf{T}, \mathbf{T}' \subseteq \mathbf{G}$ *be* F*-stable maximal tori,* $\theta \in \mathrm{Irr}(\mathbf{T}^F)$*, and* $\theta' \in \mathrm{Irr}(\mathbf{T}'^F)$*. Then*

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle = \frac{1}{|\mathbf{T}^F|} \cdot |\{g \in \mathbf{G}^F \mid g\mathbf{T}g^{-1} = \mathbf{T}' \text{ and } \theta^g = \theta'\}|.$$

In particular, $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ *if* $(\mathbf{T}, \theta)$ *and* $(\mathbf{T}', \theta')$ *are conjugate in* $\mathbf{T}^F$ *and they are orthogonal otherwise.*

Therefore, it suffices to consider Deligne–Lusztig characters for $\mathbf{G}^F$-conjugacy classes of pairs $(\mathbf{T}, \theta)$.

In the following, we often state formulas that hold up to a sign that is determined by the relative F-rank of $\mathbf{G}$ or one of its tori.

Definition 3.7. [**GM20**, Definition 2.2.11, Proposition 2.5.6]

(i) The *relative* F*-rank of* $\mathbf{G}$ is the maximal $i \geq 0$ such that $(q-1)^i$ divides the order polynomial of $\mathbf{G}$ with respect to F. If r is the relative F-rank of $\mathbf{G}$, we write $\varepsilon_{\mathbf{G}} := (-1)^r$.

(ii) The *semisimple* F*-rank of* $\mathbf{G}$ is the relative F-rank of $\mathbf{G}/R(\mathbf{G})$. We denote it by $r_{ss}(\mathbf{G})$.

We can now describe the degree of Deligne–Lusztig characters.

Proposition 3.8. [**GM20**, Theorem 2.2.12] *Let* $\mathbf{T}$ *be an* F*-stable maximal torus of* $\mathbf{G}$ *and* $\theta \in \mathrm{Irr}(\mathbf{T}^F)$*. Then, we have*

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(1) = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}|\mathbf{G}^F : \mathbf{T}^F|_{p'}.$$

In the following, we are also interested in other values of Deligne–Lusztig characters. They can be described with the help of the Green function.

Definition 3.9. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$. We define the *Green function* of $\mathbf{G}$ with respect to $\mathbf{T}$ by

$$Q_{\mathbf{T}}^{\mathbf{G}} : \mathbf{G}_u^F \to \mathbb{C}, \quad u \mapsto R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})(u)$$

where $\mathbf{G}_u$ consists of the unipotent elements in $\mathbf{G}$ as in Definition 2.3.

Proposition 3.10. [**GM20**, Proposition 2.2.5] *The Green function has integer values.*

We now express the values of Deligne–Lusztig characters in terms of the corresponding irreducible character of $\mathbf{T}^F$ and Green functions. However, we do not only need the Green function of the group itself but of $C_{\mathbf{G}}^{\circ}(s) := C_{\mathbf{G}}(s)^{\circ}$ where $s \in \mathbf{G}^F$ is semisimple. We first convince ourselves that we can still define Deligne–Lusztig characters and Green functions for $C_{\mathbf{G}}^{\circ}(s)$.

Lemma 3.11. [**DM20**, Propositions 3.5.1, 3.5.3] *Let* $s \in \mathbf{G}^F$ *be semisimple.*

(a) *The identity component* $C_{\mathbf{G}}^{\circ}(s)$ *is connected reductive.*

(b) *If* $g \in \mathbf{G}$ *has Jordan decomposition* $g = us$ *with* u *unipotent, then we have* $u \in C_{\mathbf{G}}^{\circ}(s)$*.*

Note that $C_{\mathbf{G}}^{\circ}(s)$ and the Jordan decomposition of an element in $\mathbf{G}^F$ are F-stable. Now we can state the so-called character formula:

Proposition 3.12. [**GM20**, Proposition 2.2.16] *Let* $g \in \mathbf{G}^F$ *and* $g = su = us$ *where* u *is unipotent and* s *is semisimple. Then, we have*

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = \frac{1}{|C_{\mathbf{G}}^{\circ}(s)^F|} \sum_{x \in \mathbf{G}^F,\, x^{-1}sx \in \mathbf{T}} Q_{x\mathbf{T}x^{-1}}^{C_{\mathbf{G}}^{\circ}(s)}(u)\theta(x^{-1}sx).$$

We can already assume from the definition of the Green functions that it is useful to study the Deligne–Lusztig characters arising from the trivial characters of maximal tori. As it turns out, their irreducible constituents are of importance.

Definition 3.13. An irreducible character χ of $\mathbf{G}$ is called *unipotent* if there exists an F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$ such that χ is a constituent of $R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})$.

We eventually want to determine the irreducible characters of $\mathbf{G}^F$. The next proposition tells us why we study Deligne–Lusztig characters in order to obtain those.

Proposition 3.14. [**GM20**, Corollary 2.2.19] *For every $\chi \in \operatorname{Irr}(\mathbf{G}^F)$ there exists an F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$ and a character $\theta \in \operatorname{Irr}(\mathbf{T}^F)$ such that χ is a constituent of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$.*

This already tells us that the union of all irreducible constituents of Deligne–Lusztig characters is $\operatorname{Irr}(\mathbf{G}^F)$. As we have already seen in Section 2.5.2, there is a convenient way to describe the $\mathbf{G}^F$-conjugacy classes of F-stable maximal tori of $\mathbf{G}$. However, we have to change our point of view on Deligne–Lusztig characters in order to get a partition of the irreducible characters of $\mathbf{G}^F$. Let $(\mathbf{G}^*, F^*)$ be dual to $(\mathbf{G}, F)$.

Proposition 3.15. [**GM20**, Corollary 2.5.14] *The $\mathbf{G}^F$-conjugacy classes of the set*

$$\{(\mathbf{T}, \theta) \mid \mathbf{T} \textit{ an } F\textit{-stable maximal torus of } \mathbf{G},\ \theta \in \operatorname{Irr}(\mathbf{T}^F)\}$$

are in bijection with the $\mathbf{G}^{*F^*}$-conjugacy classes of

$$\{(\mathbf{T}^*, s) \mid s \in \mathbf{G}^{*F^*} \textit{ semisimple, } \mathbf{T}^* \textit{ an } F^*\textit{-stable maximal torus of } \mathbf{G}^* \textit{ containing } s\}.$$

Remark 3.16. Note that this bijection is compatible with taking powers of θ and s, respectively. Thus, if the $\mathbf{G}^F$-conjugacy class of $(\mathbf{T}, \theta)$ corresponds to the $\mathbf{G}^{*F^*}$-conjugacy class of $(\mathbf{T}^*, s)$, then the $\mathbf{G}^F$-conjugacy class of $(\mathbf{T}, \theta^r)$ corresponds to the $\mathbf{G}^{*F^*}$-conjugacy class of $(\mathbf{T}^*, s^r)$ for any $r \in \mathbb{Z}$ [**GM20**, Remark 2.5.15].

To prove the existence of the bijection from Proposition 3.15, one has to introduce geometric conjugacy classes and other related concepts, see e.g. [**GM20**, Section 2.5]. Since we only need the result itself in the following, we do not say more about this. We can now use this bijection to introduce a different point of view on Deligne–Lusztig characters and change the associated parameters.

Definition 3.17. If $(\mathbf{T}^*, s)$ corresponds to $(\mathbf{T}, \theta)$ as in Proposition 3.15, we write

$$R_{\mathbf{T}^*}^{\mathbf{G}}(s) := R_{\mathbf{T}}^{\mathbf{G}}(\theta).$$

For every semisimple $s \in \mathbf{G}^{*F^*}$ we define the *Lusztig series (or rational series)*

$$\mathcal{E}(\mathbf{G}^F, s) := \{\chi \in \operatorname{Irr}(\mathbf{G}^F) \mid \langle \chi, R_{\mathbf{T}^*}^{\mathbf{G}}(s) \rangle \neq 0 \text{ for an } F\text{-stable max. torus with } s \in \mathbf{T}^{*F^*}\}.$$

For example, the unipotent characters of $\mathbf{G}^F$ form the set $\mathcal{E}(\mathbf{G}^F, 1)$. As it turns out, this parametrization of Deligne–Lusztig characters in terms of semisimple elements of the dual group has better properties than the original definition. It requires some work to obtain the following crucial result due to Lusztig.

Proposition 3.18. [**GM20**, Theorem 2.5.24] *Let $s, s' \in \mathbf{G}^{*F^*}$ be semisimple. Then, we have*

$$\mathcal{E}(\mathbf{G}^F, s) = \mathcal{E}(\mathbf{G}^F, s')$$

if s and s' are conjugate in $\mathbf{G}^{*F^*}$. Otherwise, we have

$$\mathcal{E}(\mathbf{G}^F, s) \cap \mathcal{E}(\mathbf{G}^F, s') = \emptyset.$$

Together with Proposition 3.14 we can conclude that we can choose a set of semisimple elements in $\mathbf{G}^{*F^*}$ such that the corresponding Lusztig series form a partition of the irreducible characters of $\mathbf{G}^F$.

Theorem 3.19. [**GM20**, Theorem 2.6.2] *We have*

$$\operatorname{Irr}(\mathbf{G}^F) = \bigsqcup_{s \in \mathbf{G}^{*F^*} \text{ semisimple}/\sim_{\mathbf{G}^{*F^*}}} \mathcal{E}(\mathbf{G}^F, s).$$

In order to understand $\operatorname{Irr}(\mathbf{G}^F)$, it remains to find a better description of $\mathcal{E}(\mathbf{G}^F, s)$ for semisimple $s \in \mathbf{G}^{*F^*}$. This is much easier if the center of $\mathbf{G}$ is connected.

Proposition 3.20. [**GM20**, Theorem 2.6.4] *Assume that $Z(\mathbf{G})$ is connected and let $s \in \mathbf{G}^{*F^*}$ be semisimple. Then, $C_{\mathbf{G}^*}(s)$ is connected and we have a bijection*

$$\mathcal{E}(\mathbf{G}^F, s) \to \mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, 1), \quad \chi \mapsto \chi_u$$

such that

$$\langle R^{\mathbf{G}}_{\mathbf{T}^*}(s), \chi \rangle = \pm \langle R^{C_{\mathbf{G}^*}(s)}_{\mathbf{T}^*}(1), \chi_u \rangle$$

for all F^*-stable maximal tori $\mathbf{T}^*$ of $\mathbf{G}^*$ containing s.

Together with the previous theorem, this immediately implies that we have the following Jordan decomposition of characters.

Theorem 3.21. *Let $Z(\mathbf{G})$ be connected. Then* $\operatorname{Irr}(\mathbf{G}^F)$ *can be parametrized by*

$$\{(s, \nu) \mid s \in \mathbf{G}^{*F^*} \text{semisimple}/\sim_{\mathbf{G}^{*F^*}},\ \nu \in \mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, 1)\}.$$

Note that the bijection in Proposition 3.20 and thereby also the Jordan decomposition itself is not unique.

We already know from Section 2.5.2 how we can describe $\mathbf{G}^{*F^*}$-conjugacy classes of F-stable maximal tori of $\mathbf{G}^*$. Since every semisimple element of $\mathbf{G}^{*F^*}$ lies in such a torus, we can also determine the $\mathbf{G}^{*F^*}$-conjugacy classes of semisimple elements.

The Jordan decomposition also gives a relation between the degrees of the corresponding characters.

Corollary 3.22. [**GM20**, Corollary 2.6.6] *In the setting of Proposition 3.20, we have*

$$\chi(1) = |\mathbf{G}^{*F^*} : C_{\mathbf{G}^*}(s)^{F^*}|_{p'} \cdot \chi_u(1).$$

If the center of $\mathbf{G}$ is not connected, the theory gets more complicated. However, one can use the regular embedding from Section 2.6 and investigate the relation between the irreducible characters of the group $\widetilde{\mathbf{G}}^F$ with connected center and of the group $\mathbf{G}^F$ itself. This yields a generalized version of the Jordan decomposition, see [**GM20**, Theorem 2.6.22].

Example 3.23. We use Deligne–Lusztig characters to determine the irreducible characters of the Suzuki groups. Note that they have already been determined with purely character-theoretical methods by Suzuki in [**Suz62**]. Let $\mathbf{G}$ be of type B_2 defined over an algebraically closed field $\mathbf{k}$ of characteristic 2 and $F = F_2^f \circ \gamma$ where F_2 is the standard Frobenius map and γ the exceptional graph endomorphism. We have

$$|\mathbf{G}^F| = q^4(q^2-1)(q^2-\sqrt{2}q+1)(q^2+\sqrt{2}q+1)$$

where $q^2 = 2^{2f+1}$. Let $\mathbf{T}_0$ be a maximally split torus of $\mathbf{G}$ and $\mathbf{T}_1$, $\mathbf{T}_2$ be representatives of the $\mathbf{G}^F$-conjugacy classes of F-stable maximal tori of $\mathbf{G}$ as determined in Example 2.51.

Since $Z(\mathbf{G})$ is connected, we can use the Jordan decomposition of characters to determine the irreducible characters of $\mathbf{G}^F$. We know that $(\mathbf{G}, F)$ is self-dual and we can thereby consider maximal tori and semisimple elements of the group itself to determine the Lusztig series of $\mathbf{G}^F$. We know that we have

$$\mathbf{T}_0^F \cong C_{q^2-1}, \quad \mathbf{T}_1^F \cong C_{q^2-\sqrt{2}q+1}, \quad \mathbf{T}_2^F \cong C_{q^2+\sqrt{2}q+1}.$$

Let $s \in \mathbf{G}^F$ be a non-trivial semisimple element and $\mathbf{T}$ an F-stable maximal torus of $\mathbf{G}$ such that $s \in \mathbf{T}^F$. In [**HB82**, Theorem XI.3.10] it has been shown that we have

$$N_{\mathbf{G}^F}(\mathbf{T}_0^F) = \mathbf{T}_0^F \rtimes C_2, \quad N_{\mathbf{G}^F}(\mathbf{T}_1^F) = \mathbf{T}_1^F \rtimes C_4, \quad N_{\mathbf{G}^F}(\mathbf{T}_2^F) = \mathbf{T}_2^F \rtimes C_4$$

and that $C_{\mathbf{G}}(s)^F = \mathbf{T}^F$. Thus, $C_{\mathbf{G}}(s)^F$ is cyclic and its unique unipotent character is the trivial character. By Proposition 3.20 and Corollary 3.22, the Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$ consists of only one character of degree $|\mathbf{G}^F : \mathbf{T}^F|_{2'}$. Since the element s is $\mathbf{G}^F$-conjugate to $|N_{\mathbf{G}^F}(\mathbf{T}) : \mathbf{T}^F|$ other elements in $\mathbf{T}^F$, this determines all non-unipotent irreducible characters of $\mathbf{G}^F$ as listed in Table 3.1.

Now we consider the unipotent characters of $\mathbf{G}^F$. The Lusztig series partition $\mathrm{Irr}(\mathbf{G}^F)$ and we can determine the number of irreducible characters of $\mathbf{G}^F$ by considering the conjugacy classes of $\mathbf{G}^F$. Since we already know the number of non-unipotent characters, we can easily conclude that $\mathbf{G}^F$ has four unipotent characters.

By Proposition 3.6, we have

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F}), R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})\rangle = |N_{\mathbf{G}^F}(\mathbf{T}) : \mathbf{T}^F| \in \{2, 4\}$$

for any F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$. Further, we know from Proposition 3.8 that we have

$$R_{\mathbf{T}_0}^{\mathbf{G}}(1_{\mathbf{T}_0^F})(1) = (-1) \cdot (-1) \cdot (q^4 + 1),$$

$$R_{\mathbf{T}_1}^{\mathbf{G}}(1_{\mathbf{T}_1^F})(1) = (-1) \cdot (q^2 - 1)(q^2 + \sqrt{2}q + 1),$$

$$R_{\mathbf{T}_2}^{\mathbf{G}}(1_{\mathbf{T}_2^F})(1) = (-1) \cdot (q^2 - 1)(q^2 - \sqrt{2}q + 1).$$

By Proposition 3.5 and Frobenius reciprocity, the trivial character is a constituent of $R_{\mathbf{T}_0}^{\mathbf{G}}(1_{\mathbf{T}_0^F})$. Therefore, we have

$$R_{\mathbf{T}_0}^{\mathbf{G}}(1_{\mathbf{T}_0^F}) = 1_{\mathbf{G}^F} + \chi_{\mathrm{St}}$$

for some $\chi_{\mathrm{St}} \in \mathrm{Irr}(\mathbf{G}^F)$ of degree q^4. The two remaining Deligne–Lusztig characters have odd degree and therefore consist of four irreducible constituents of multiplicity ± 1. Since the inner product with $R_{\mathbf{T}_0}^{\mathbf{G}}(1_{\mathbf{T}_0^F})$ vanishes, we can conclude

$$R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F}) = \pm(1_{\mathbf{G}^F} - \chi_{\mathrm{St}}) \pm \chi_1 \pm \chi_2$$

for $\mathbf{T} \in \{\mathbf{T}_1, \mathbf{T}_2\}$ with possibly different signs. Here, χ_1 and χ_2 are the two remaining unipotent characters. Using the degrees of the Deligne–Lusztig characters that we determined above and the degree formula from Proposition 1.10, we can conclude that both χ_1 and χ_2 have degree $(q^2 - 1)q/\sqrt{2}$ and that we have

$$R_{\mathbf{T}_1}^{\mathbf{G}}(1_{\mathbf{T}_1^F}) = 1_{\mathbf{G}^F} - \chi_{\mathrm{St}} - \chi_1 - \chi_2,$$

$$R_{\mathbf{T}_2}^{\mathbf{G}}(1_{\mathbf{T}_2^F}) = 1_{\mathbf{G}^F} - \chi_{\mathrm{St}} + \chi_1 + \chi_2.$$

We can use further knowledge about the conjugacy classes of $\mathbf{G}^F$ to show that χ_1 is the complex conjugate of χ_2. With the Green functions, we can also determine the values of the characters, see [**Gec03**, Section 4.6].

The description of the irreducible characters of a finite group of Lie type ${}^d\mathsf{L}_n(q)$ and their values in terms of the parameter q can be displayed in a character table where the conjugacy classes and irreducible characters are grouped in families. This is called the *generic character table* of ${}^d\mathsf{L}_n(q)$. Generic character tables are explicitly known for some families of groups of Lie type with small rank. Many of them are available in CHEVIE [**GHL**$^+$**96**].

s	Number of $\mathbf{G}^F$-conjugacy classes of this type	$\lvert\mathcal{E}(\mathbf{G}^F, s)\rvert$	Degrees of $\chi \in \mathcal{E}(\mathbf{G}^F, s)$
$1 \neq s \in \mathbf{T}_0^F$	$(q^2-2)/2$	1	(q^4+1)
$1 \neq s \in \mathbf{T}_1^F$	$(q^2-\sqrt{2}q)/4$	1	$(q^2-1)(q^2+\sqrt{2}q+1)$
$1 \neq s \in \mathbf{T}_2^F$	$(q^2+\sqrt{2}q)/4$	1	$(q^2-1)(q^2-\sqrt{2}q+1)$
$s=1$	1	4	1, q^4, $(q^2-1)q/\sqrt{2}$, $(q^2-1)q/\sqrt{2}$

TABLE 3.1. Lusztig series and character degrees for the Suzuki groups $\mathbf{G}^F$.

3.2. Harish-Chandra theory

In this section, we introduce the basic notions of Harish-Chandra theory. It provides us with a different partition of the irreducible characters of $\mathbf{G}^F$ and new tools to investigate these.

We first present a new construction for induction of characters that is called Harish-Chandra induction. Originally, it was defined for all finite groups that have a so-called BN-pair. Finite groups of Lie type always have a BN-pair consisting of the normalizer of a maximally split torus in $\mathbf{G}^F$ and the F-fixed points of the corresponding F-stable Borel group. Since we only consider finite groups of Lie type in the following, we restrict ourselves to this special case.

Definition 3.24. Let $\mathbf{P}$ be an F-stable parabolic subgroup of $\mathbf{G}$ and $\mathbf{L}$ an F-stable Levi subgroup of $\mathbf{P}$ such that we have the Levi decomposition $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$.

(i) The map

$$R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{N}_0 \operatorname{Irr}(\mathbf{L}^F) \to \mathbb{N}_0 \operatorname{Irr}(\mathbf{G}^F), \quad \theta \mapsto R_{\mathbf{L}}^{\mathbf{G}}(\theta) := \left(\operatorname{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F} \circ \operatorname{Inf}_{\mathbf{L}^F}^{\mathbf{P}^F}\right)(\theta)$$

is called *Harish-Chandra induction.*

(ii) Let ψ be a character of $\mathbf{P}^F$ afforded by a representation $\mathcal{R}$. Let $\mathcal{R}'$ be a subrepresentation of $\mathcal{R}$ of maximal degree such that $\mathbf{U}^F$ is in its kernel. Then, we define the map $\operatorname{Fix}_{\mathbf{U}^F} : \mathbb{N}_0 \operatorname{Irr}(\mathbf{P}^F) \to \mathbb{N}_0 \operatorname{Irr}(\mathbf{L}^F)$ by letting $\operatorname{Fix}_{\mathbf{U}^F}(\psi)$ be the character afforded by $\mathcal{R}'$. The map

$${}^*R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{N}_0 \operatorname{Irr}(\mathbf{G}^F) \to \mathbb{N}_0 \operatorname{Irr}(\mathbf{L}^F), \quad \chi \mapsto {}^*R_{\mathbf{L}}^{\mathbf{G}}(\chi) := \left(\operatorname{Fix}_{\mathbf{U}^F} \circ \operatorname{Res}_{\mathbf{P}^F}^{\mathbf{G}^F}\right)(\chi)$$

is called *Harish-Chandra restriction.*

Although the definition of Harish-Chandra induction and restriction depends on $\mathbf{P}$, the resulting functors are independent of its choice [**DM20**, Theorem 5.3.1]. As already implied by the notation, the maps are adjoint to each other with respect to the standard scalar product of characters.

Note that we already know from Proposition 3.5 that Harish-Chandra induction for a maximally split torus $\mathbf{T}_0$ coincides with Lusztig induction. This indeed holds in general.

Proposition 3.25. [**GM20**, Proposition 3.3.3] *Let $\mathbf{L} \subseteq \mathbf{P}$ be an F-stable Levi subgroup contained in an F-stable parabolic subgroup of $\mathbf{G}$. Then, Harish-Chandra induction is the same as Lusztig induction and Harish-Chandra restriction is the same as Lusztig restriction.*

This justifies the use of the same symbol for Lusztig and Harish-Chandra induction.

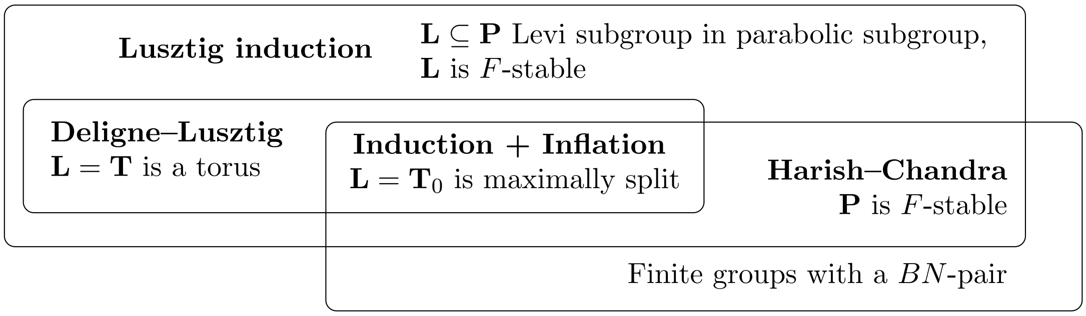


FIGURE 3.1. Overview of the settings of Lusztig induction, Deligne–Lusztig induction, and Harish-Chandra induction.

In the remaining section, we present some important properties of Harish-Chandra-induced characters and their irreducible constituents. We start with the definition of cuspidality.

Definition 3.26. Let $\mathbf{H} \subseteq \mathbf{G}$ be an F-stable connected reductive subgroup. The character $\chi \in \mathrm{Irr}(\mathbf{H}^F)$ is called a *cuspidal character* if we have ${}^*R_{\mathbf{L}}^{\mathbf{H}}(\chi) = 0$ for any proper F-stable Levi subgroup $\mathbf{L}$ contained in an F-stable parabolic subgroup of $\mathbf{H}$. The pair $(\mathbf{H}, \chi)$ is called a *cuspidal pair*.

If $\mathbf{L}$ is an F-stable Levi subgroup contained in an F-stable parabolic subgroup of $\mathbf{G}$ and $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ is cuspidal, then we also say that $(\mathbf{L}, \lambda)$ is a *cuspidal pair of* $\mathbf{G}^F$.

Cuspidal pairs of $\mathbf{G}^F$ are an important tool in the study of irreducible characters of $\mathbf{G}^F$. Similar to the Lusztig series, we consider sets of irreducible characters arising from Harish–Chandra induction from a cuspidal pair of $\mathbf{G}^F$.

Definition 3.27. For a cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}^F$, we define the *Harish-Chandra series*

$$\mathcal{E}(\mathbf{G}^F, \mathbf{L}^F, \lambda) := \{\chi \in \mathrm{Irr}(\mathbf{G}^F) \mid \chi \text{ is a constituent of } R_{\mathbf{L}}^{\mathbf{G}}(\lambda)\}.$$

The union of Harish-Chandra series $\mathcal{E}(\mathbf{G}^F, \mathbf{T}_0^F, \lambda)$ where $\mathbf{T}_0$ is a maximally split torus of $\mathbf{G}$ and $\lambda \in \mathrm{Irr}(\mathbf{T}_0^F)$ is called the *principal series* of $\mathbf{G}^F$.

We obtain a partition of the irreducible characters of $\mathbf{G}^F$ in terms of Harish-Chandra series.

Theorem 3.28. [**GM20**, Corollary 3.1.17] *We have*

$$\mathrm{Irr}(\mathbf{G}^F) = \bigsqcup_{(\mathbf{L},\lambda)} \mathcal{E}(\mathbf{G}^F, \mathbf{L}^F, \lambda)$$

where the sum runs over all cuspidal pairs of $\mathbf{G}^F$ *up to* $\mathbf{G}^F$*-conjugacy.*

We can describe Harish–Chandra series explicitly by considering the new concept of a Weyl group relative to the cuspidal pair.

Definition 3.29. Let $(\mathbf{L}, \lambda)$ be a cuspidal pair of $\mathbf{G}^F$. The *relative Weyl group* of $\mathbf{G}^F$ with respect to $(\mathbf{L}, \lambda)$ is the group

$$\mathbf{W}_{\mathbf{G}^F}(\mathbf{L}, \lambda) := N_{\mathbf{G}^F}(\mathbf{L}, \lambda)/\mathbf{L}^F.$$

We also write $\mathbf{W}_{\mathbf{G}^F}(\mathbf{L}) := \mathbf{W}_{\mathbf{G}^F}(\mathbf{L}, 1_{\mathbf{L}^F})$.

Theorem 3.30. [**GM20**, Theorem 3.2.5] *For a cuspidal pair* $(\mathbf{L}, \lambda)$ *of* $\mathbf{G}^F$*, there exists a bijection*

$$\mathcal{I} : \mathrm{Irr}(\mathbf{W}_{\mathbf{G}^F}(\mathbf{L}, \lambda)) \to \mathcal{E}(\mathbf{G}^F, \mathbf{L}^F, \lambda).$$

For $\eta \in \mathrm{Irr}(\mathbf{W}_{\mathbf{G}^F}(\mathbf{L},\lambda))$ we also write $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)_\eta := \mathcal{I}(\eta)$. Note that the relative Weyl group is a Coxeter group if $\mathbf{G}$ has connected center or λ is a unipotent character of $\mathbf{L}$. We can use this knowledge to obtain more information about its irreducible characters and thereby also about the irreducible constituents of the characters obtained by Harish-Chandra induction from cuspidal pairs.

Example 3.31. Let $\mathbf{G}$, F be such that $\mathbf{G}^F$ is the Suzuki group as in Example 3.23 and $\mathbf{T}_0$ a maximally split torus of $\mathbf{G}$. For every non-trivial $\lambda \in \mathrm{Irr}(\mathbf{T}_0^F)$, we already observed that its inertia group in $\mathbf{G}^F$ is just $\mathbf{T}_0^F$ itself. We thereby have a bijection

$$W_{\mathbf{G}^F}(\mathbf{T}_0,\lambda) = \{\mathrm{id}\} \to \mathcal{E}(\mathbf{G}^F,\mathbf{T}_0^F,\lambda) = \{\pm R_{\mathbf{T}_0}^{\mathbf{G}}(\lambda)\} \cap \mathrm{Irr}(\mathbf{G}^F).$$

For the trivial character, we have also seen that there is a bijection

$$W_{\mathbf{G}^F}(\mathbf{T}_0,1_{\mathbf{T}_0^F}) \cong C_2 \to \mathcal{E}(\mathbf{G}^F,\mathbf{T}_0^F,1_{\mathbf{T}_0^F}) = \{1_{\mathbf{G}^F},\chi_{\mathrm{St}}\}.$$

We conclude this chapter with a generalization of the concepts of Harish-Chandra theory to the so-called d-Harish-Chandra theory. Instead of F-stable Levi subgroups contained in F-stable parabolic subgroups of $\mathbf{G}$, we now consider d-split Levi subgroups as in Definition 2.68. Although we only give them in terms of d-split Levi subgroups, the following definitions can also be translated to the Suzuki and Ree groups and ϕ-split Levi subgroups. One can show that the 1-split Levi subgroups are always contained in an F-stable parabolic subgroup of $\mathbf{G}$.

Although we are not in the setting of Harish-Chandra induction anymore, we can still apply Lusztig induction to a d-split Levi subgroup $\mathbf{L}$ and a character $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$. We can now generalize some concepts from Harish-Chandra theory to this setting.

Definition 3.32. Let $d \geq 1$.

(i) Let $\mathbf{H} \subseteq \mathbf{G}$ be an F-stable connected reductive subgroup and $\chi \in \mathrm{Irr}(\mathbf{H}^F)$. The character χ is called a *d-cuspidal character* if we have ${}^*R_{\mathbf{L}}^{\mathbf{H}}(\chi) = 0$ for any proper d-split Levi subgroup $\mathbf{L}$ of $\mathbf{H}$. If $\mathbf{H}$ is additionally a d-split Levi subgroup, then the pair $(\mathbf{H},\chi)$ is called a *d-cuspidal pair of* $\mathbf{G}^F$.

(ii) The *d-Harish-Chandra series* $\mathcal{E}(\mathbf{G}^F,\mathbf{L}^F,\lambda)$ above a d-cuspidal pair $(\mathbf{L},\lambda)$ of $\mathbf{G}^F$ consists of all $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ that are irreducible constituents of $R_{\mathbf{L}\leq\mathbf{P}}^{\mathbf{G}}(\lambda)$ for some parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with Levi complement $\mathbf{L}$.

(iii) The *relative Weyl group* of a d-cuspidal pair $(\mathbf{L},\lambda)$ of $\mathbf{G}^F$ is given by the group $\mathbf{W}_{\mathbf{G}^F}(\mathbf{L}) := N_{\mathbf{G}^F}(\mathbf{L},\lambda)/\mathbf{L}^F$.

The statement from Theorem 3.30 can be generalized to d-cuspidal unipotent pairs.

Theorem 3.33. [**BMM93**, Theorem 3.2] *For a d-cuspidal unipotent pair $(\mathbf{L},\lambda)$ of $\mathbf{G}^F$, there exists a bijection*

$$\mathcal{I} : \mathrm{Irr}(\mathbf{W}_{\mathbf{G}^F}(\mathbf{L},\lambda)) \to \mathcal{E}(\mathbf{G}^F,\mathbf{L}^F,\lambda).$$

As we will see in Section 6.1, d-Harish-Chandra series can be used to describe the unipotent ℓ'-characters of $\mathbf{G}^F$.

3.3. Gelfand–Graev characters

In this section, we construct an important family of characters of $\mathbf{G}^F$, the Gelfand–Graev characters. In order to do this, we first have to consider subgroups of the unipotent radical of a Borel subgroup of $\mathbf{G}^F$. We follow [**DM20**, Section 12.3] and [**Ruh21**, Section 3.2].

Let $\mathbf{B}$ be an F-stable Borel subgroup containing an F-stable maximal torus $\mathbf{T}$ and $\mathbf{U} := R_u(\mathbf{B})$. As before, let Δ be a base of the root system of $\mathbf{G}$ with respect to $\mathbf{T} \subseteq \mathbf{B}$ and ρ the permutation of Δ induced by F. Then, ρ yields a partition of Δ into its r orbits $\delta_1,\dots\delta_r$. For $1 \leq i \leq r$, we write $\mathbf{U}_{\delta_i}$ for the image of $\prod_{\alpha\in\delta_i}\mathbf{U}_\alpha$ in $\mathbf{U}/[\mathbf{U},\mathbf{U}]$.

Every root subgroup is isomorphic to the additive group of $\mathbf{k}$ and we have an isomorphism $\mathbf{U}_{\delta_i} \cong (\mathbf{k}^+)^{|\delta_i|}$. We have $F(\mathbf{U}_\alpha) = \mathbf{U}_{F(\alpha)}$. Therefore, every $\mathbf{U}_\alpha$ with $\alpha \in \delta_i$ is stabilized by $F^{|\delta_i|}$. This gives us an isomorphism

$$\mathbf{U}_{\delta_i}^F \cong \mathbf{U}_\alpha^{F^{|\delta_i|}}.$$

We can choose the root isomorphisms $u_\alpha : \mathbf{k}^+ \to \mathbf{U}_\alpha$ such that $u_\alpha(1) \in \mathbf{U}_\alpha^{F^{|\delta_i|}}$. This induces an isomorphism

$$\mathbf{U}_{\delta_i}^F \cong \mathbb{F}_{q^{|\delta_i|}}^+$$

and it follows

$$\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F \cong \prod_{i=1}^r \mathbf{U}_{\delta_i}^F \cong \prod_{i=1}^r \mathbb{F}_{q^{|\delta_i|}}^+.$$

Let $\phi_0 \in \operatorname{Irr}(\mathbb{F}_p^+)$ be a fixed non-trivial character. By composing it with the trace, we get a character $\phi_0^{(N)}$ of $\mathbb{F}_{q^N}^+$ for every $N \in \mathbb{N}$.

Lemma 3.34. *The set* $\operatorname{Irr}(\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F)$ *can be parametrized by* $\prod_{i=1}^r \mathbb{F}_{q^{|\delta_i|}}$.

Proof. Every irreducible character ϕ_i of $\mathbf{U}_{\delta_i}^F \cong \mathbb{F}_{q^{|\delta_i|}}^+$ is given by $\phi_i(x) = \phi_0^{(|\delta_i|)}(a_i x)$ for some $a_i \in \mathbb{F}_{q^{|\delta_i|}}$. Thus, it is uniquely described by an element of $\mathbb{F}_{q^{|\delta_i|}}$.

On the other hand, we already know from the isomorphisms above that every character $\phi \in \operatorname{Irr}(\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F)$ is of the form $\phi = \prod_{i=1}^r \phi_i$ with characters $\phi_i \in \operatorname{Irr}(\mathbf{U}_{\delta_i}^F)$. This shows the claim. □

From now, we denote the linear character corresponding to the tuple $(1,\dots,1)$ by $\xi \in \operatorname{Irr}(\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F)$. By construction, $\xi|_{\delta_i}$ is not trivial for any $1 \leq i \leq r$ and it is invariant under the action of field and graph automorphisms of $\mathbf{G}^F$.

Definition 3.35. We define the *Gelfand–Graev character* $\Gamma_1 = \operatorname{Ind}_{\mathbf{U}^F}^{\mathbf{G}^F} \circ \operatorname{Inf}_{\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F}^{\mathbf{U}^F}(\xi)$.

If the center of $\mathbf{G}$ is connected, this is the only Gelfand–Graev character. Otherwise, we can define more Gelfand–Graev characters by inducing other characters of $\mathbf{U}^F$ that are trivial on $[\mathbf{U},\mathbf{U}]^F$ and non-trivial on all $\mathbf{U}_{\delta_i}^F$.

Proposition 3.36. [**DM20**, Theorem 12.3.4] *The Gelfand–Graev characters are multiplicity free, i.e. all of their irreducible constituents have multiplicity* 1.

We now illustrate these constructions by computing the character ξ for groups of type B_2 and a Steinberg endomorphism F.

Example 3.37. Let $\mathbf{G}$ be a connected reductive group of type B_2 and $\Delta = \{\alpha, \beta\}$ a base of its root system as in Example 2.31. Let $f \geq 1$.

(a) Let $F := F_p^f$ be a Frobenius endomorphism of $\mathbf{G}$. Then, the orbits of the action induced by F on Δ are given by $\{\alpha\}$ and $\{\beta\}$. With the notation from above, we have

$$(\mathbf{U}_\alpha/[\mathbf{U},\mathbf{U}])^F \cong \mathbb{F}_q^+ \cong (\mathbf{U}_\beta/[\mathbf{U},\mathbf{U}])^F$$

and

$$\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F \cong (\mathbf{U}_\alpha/[\mathbf{U},\mathbf{U}])^F(\mathbf{U}_\beta/[\mathbf{U},\mathbf{U}])^F.$$

We fix a non-trivial $\phi_0 \in \operatorname{Irr}(\mathbb{F}_p^+)$. Using the above isomorphisms, every character $\phi \in \operatorname{Irr}(\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F)$ is given by

$$\phi(x_\alpha x_\beta) = \phi_0(a_\alpha x_\alpha)\phi_0(a_\beta x_\beta)$$

for some $(a_\alpha, a_\beta) \in (\mathbb{F}_q)^2$ where $x_\alpha \in (\mathbf{U}_\alpha/[\mathbf{U},\mathbf{U}])^F$, $x_\beta \in (\mathbf{U}_\beta/[\mathbf{U},\mathbf{U}])^F$. In particular, we have $\xi(x_\alpha x_\beta) = \phi_0(x_\alpha)\phi_0(x_\beta)$.

(b) Assume that $\mathbf{k}$ has characteristic 2 and let $F := F_2^f \circ \gamma$ where γ is the exceptional graph endomorphism as defined in Proposition 2.43(c) and $q^2 := 2^{2f+1}$. Then, Δ is one orbit under the action of F and we have

$$(\mathbf{U}/[\mathbf{U},\mathbf{U}])^F \cong \mathbb{F}_{q^2}^+ \cong (\mathbb{F}_2^+)^{2f+1}.$$

The non-trivial character of $\mathbb{F}_2^+$ is given by $\phi_0(1) = -1$ and we can write

$$\phi_0^{(2)}(a_1,\dots,a_{2f+1}) = \phi_0(a_1+\dots+a_{2f+1}), \quad (a_1,\dots,a_{2f+1}) \in (\mathbb{F}_2^+)^{2f+1}.$$

Now, the irreducible characters of $\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F$ are already given by the choice of some $a \in \mathbb{F}_{q^2}$ and we have $\xi = \phi_0^{(2)}$.

Harish-Chandra induction and restriction allow us to define a duality map for characters of finite groups of Lie type. We will later apply this duality map to Gelfand–Graev characters and their constituents.

Definition 3.38. Let $\mathbf{B}$ be an F-stable Borel subgroup of $\mathbf{G}$. We define the *(Alvis–Curtis) duality map* by

$$D_{\mathbf{G}} = \sum_{\mathbf{P} \supset \mathbf{B}} (-1)^{r_{ss}(\mathbf{P})} R_{\mathbf{L}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}}^{\mathbf{G}} : \mathrm{CF}\left(\mathbf{G}^F\right) \to \mathrm{CF}\left(\mathbf{G}^F\right)$$

where the sum runs over the F-stable parabolic subgroups of $\mathbf{G}$ containing $\mathbf{B}$ and $\mathbf{L}$ denotes an arbitrarily chosen F-stable Levi subgroup of $\mathbf{P}$.

We can now define the notion of semisimple and regular characters.

Definition 3.39. (i) An irreducible character of $\mathbf{G}^F$ is called *regular* if it is a constituent of some Gelfand–Graev character.

(ii) An irreducible character of $\mathbf{G}^F$ is called *semisimple* if its dual is up to sign a constituent of some Gelfand–Graev character.

If the center of $\mathbf{G}$ is connected, there is only one Gelfand–Graev character and we also know that it is multiplicity free. This gives us the following statement.

Proposition 3.40. [**DM20**, Corollary 12.4.10] *Assume that $Z(\mathbf{G})$ is connected. The unique Gelfand–Graev character of $\mathbf{G}^F$ is the sum of all regular characters of $\mathbf{G}^F$. The dual of the Gelfand–Graev character is the sum of all the semisimple characters of $\mathbf{G}^F$ up to signs.*

3.4. Representation theory of disconnected groups

In the previous sections, we have only considered finite groups of Lie type that arise from a connected reductive group. In order to verify the inductive McKay–Navarro condition, we have to consider irreducible characters of subgroups of $\mathbf{G}^F \rtimes \mathrm{Aut}(\mathbf{G}^F)$. This finite group arises from a disconnected algebraic group. In [**DM94**], Digne and Michel extended parts of the theory that has already been established for connected reductive groups to certain disconnected groups. Although the theory is not as well-developed as in the connected case and crucial results like the Jordan decomposition of characters are not yet available, it allows us to consider Lusztig induction for disconnected groups and the relation to Lusztig induction for its connected component. With some additional constructions, we will be able to use these methods in Section 4.3.

In the following, let $\mathbf{G}$ be a reductive group. We follow [**GM20**, Section 4.8] and [**DM94**]. First, we have to extend the definitions of the various subgroups to the disconnected case.

Lemma 3.41. [**GM20**, Section 4.8.1] *Let $\mathbf{P} \leq \mathbf{G}$ be a closed subgroup containing a Borel subgroup of $\mathbf{G}^\circ$. Then, $\mathbf{P}^\circ$ is a parabolic subgroup of $\mathbf{G}^\circ$. If $\mathbf{L}^\circ$ is a Levi complement of $\mathbf{P}^\circ$, we have $\mathbf{P} = R_u(\mathbf{P}) \rtimes \mathbf{L}$ for $\mathbf{L} = N_{\mathbf{G}}(\mathbf{L}^\circ)$.*

Definition 3.42. The groups $\mathbf{P}$ and $\mathbf{L}$ as in Lemma 3.41 are called *quasi-parabolic* and *quasi-Levi subgroups of* $\mathbf{G}$.
A group $\mathbf{B} = N_{\mathbf{G}}(\mathbf{B}^\circ)$ where $\mathbf{B}^\circ$ is a Borel subgroup of $\mathbf{G}^\circ$ is called a *quasi-Borel subgroup of* $\mathbf{G}$. Their quasi-Levi subgroups are called *maximal quasi-tori.*

As already mentioned above, most of the theory is only developed for disconnected reductive groups that possess a special structure. To describe this structure, we first define a certain property of group automorphisms.

Definition 3.43. An automorphism $\tau \in \operatorname{Aut}(\mathbf{G}^\circ)$ is called *quasi-semisimple* if it stabilizes a pair $\mathbf{T}^\circ \subseteq \mathbf{B}^\circ$ of a maximal torus of $\mathbf{G}^\circ$ contained in a Borel subgroup of $\mathbf{G}^\circ$. It is called *quasi-central* if, additionally,

$$\dim C_{\mathbf{G}^\circ}(\tau) = \max\{\dim C_{\mathbf{G}^\circ}(\tau') \mid \tau' \in \operatorname{Inn}(\mathbf{G}^\circ)\tau\}.$$

Note that, in this situation, τ acts on $\mathbf{W}^\circ := N_{\mathbf{G}^\circ}(\mathbf{T}^\circ)/\mathbf{T}^\circ$. From now, we consider the reductive group $\mathbf{G} := \mathbf{G}^\circ \rtimes \langle\tau\rangle$ where τ is a quasi-central automorphism of $\mathbf{G}^\circ$. Let $F : \mathbf{G} \to \mathbf{G}$ be a Steinberg map commuting with τ. Note that [**DM94**] only considers Frobenius maps F. We checked that the results that we present here are still correct for Steinberg maps.

We first convince ourselves that we can choose quasi-tori and quasi-Levi subgroups such that they are stable under the actions of τ and F.

Proposition 3.44. [**DM94**, Proposition 1.36 and 1.38]

(a) *There is a pair* $\mathbf{T} \subseteq \mathbf{B}$ *of a quasi-torus of* $\mathbf{G}$ *contained in a quasi-Borel subgroup of* $\mathbf{G}$ *that is* F*-stable and* τ*-stable.*

(b) *Let* $\mathbf{L} \subseteq \mathbf{P}$ *be an* F*-stable quasi-Levi subgroup and a quasi-parabolic subgroup of* $\mathbf{G}$*. Then, there is a* $\mathbf{G}^F$*-conjugate of the pair* $\mathbf{L} \subseteq \mathbf{P}$ *that is* τ*-stable.*

We can now define a generalization of Lusztig induction to the disconnected group $\mathbf{G}$.

Definition 3.45. [**DM94**, Definition 2.2] Let $\mathbf{P}$ be a quasi-parabolic subgroup of $\mathbf{G}$ with $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$ where $\mathbf{L}$ is an F-stable quasi-Levi subgroup. We define a *generalized Lusztig induction and restriction*

$$R^{\mathbf{G}}_{\mathbf{L}\le\mathbf{P}} : \mathbb{Z}\operatorname{Irr}(\mathbf{L}^F) \to \mathbb{Z}\operatorname{Irr}(\mathbf{G}^F),\ \theta \mapsto \left(R^{\mathbf{G}}_{\mathbf{L}\le\mathbf{P}}(\theta) : g \mapsto \frac{1}{|\mathbf{L}^F|}\sum_{l\in\mathbf{L}^F} \mathfrak{L}((g,l),\mathscr{L}^{-1}(\mathbf{U}))\theta(l)\right),$$

$$^*R^{\mathbf{G}}_{\mathbf{L}\le\mathbf{P}} : \mathbb{Z}\operatorname{Irr}(\mathbf{G}^F) \to \mathbb{Z}\operatorname{Irr}(\mathbf{L}^F),\ \chi \mapsto \left({}^*R^{\mathbf{G}}_{\mathbf{L}\le\mathbf{P}}(\chi) : l \mapsto \frac{1}{|\mathbf{G}^F|}\sum_{g\in\mathbf{G}^F} \mathfrak{L}((g,l),\mathscr{L}^{-1}(\mathbf{U}))\chi(g)\right).$$

We see that the definition actually generalizes ordinary Lusztig induction and restriction for connected groups. Therefore, it also generalizes Harish-Chandra induction and, in this case, we see that it coincides with the natural construction of Harish-Chandra induction.

Lemma 3.46. [**DM94**, Corollaire 2.4] *Assume that* $\mathbf{L}$ *is an* F*-stable quasi-Levi subgroup containing* τ *that lies in a quasi-parabolic subgroup* $\mathbf{P}$*. Then the following are true:*

(a) $\operatorname{Res}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F}} \circ R^{\mathbf{G}}_{\mathbf{L}\le\mathbf{P}} = R^{\mathbf{G}^\circ}_{\mathbf{L}^\circ\le\mathbf{P}^\circ} \circ \operatorname{Res}^{\mathbf{L}^F}_{\mathbf{L}^{\circ F}}$.

(b) $\operatorname{Res}^{\mathbf{L}^F}_{\mathbf{L}^{\circ F}} \circ {}^*R^{\mathbf{G}}_{\mathbf{L}\le\mathbf{P}} = {}^*R^{\mathbf{G}^\circ}_{\mathbf{L}^\circ\le\mathbf{P}^\circ} \circ \operatorname{Res}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F}}$.

(c) *If* $\mathbf{P}$ *is* F*-stable, then we have* $R^{\mathbf{G}}_{\mathbf{L}\le\mathbf{P}} = \operatorname{Ind}^{\mathbf{G}^F}_{\mathbf{P}^F} \circ \operatorname{Inf}^{\mathbf{P}^F}_{\mathbf{L}^F}$.

As in the ordinary case, for certain choices of the quasi-Levi subgroup Lusztig induction does not depend on the quasi-parabolic subgroup.

Lemma 3.47. [**GM20**, Theorem 4.8.13] *Lusztig induction is independent of the choice of the quasi-parabolic subgroup if*

(a) *the quasi-Levi subgroup is a quasi-torus or*

(b) *the quasi-Levi subgroup is contained in an F-stable quasi-parabolic subgroup of* $\mathbf{G}$.

Then, we also write $R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$. We now define a duality map for characters in the disconnected setting.

Definition 3.48. [**DM94**, Definition 3.10] We define the *duality operator* on $\mathbf{G}^F$ by

$$D_{\mathbf{G}} := \sum_{\mathbf{P} \supseteq \mathbf{B}} (-1)^{r_{ss}(\mathbf{P}^\circ \rtimes \langle \tau \rangle)} R_{\mathbf{L}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}}^{\mathbf{G}}$$

where $\mathbf{B}$ is a fixed F-stable quasi-Borel subgroup containing τ. The sum runs over all F-stable quasi-parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{B}$ and $\mathbf{L}$ is any F-stable quasi-Levi subgroup of $\mathbf{P}$ containing τ.

We have some information about the scalar product of Deligne–Lusztig characters of disconnected groups. Since in later chapters we only need to know the number of irreducible constituents of a Deligne–Lusztig character, we give the result only for this special case.

Proposition 3.49. [**DM94**, Proposition 4.8] *Let $\mathbf{T}$ be an F-stable quasi-torus of $\mathbf{G}$ containing τ and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$. Then, we have*

$$\left\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta) \right\rangle_{\mathbf{G}^F} = \left\langle R_{(\mathbf{T}^\tau)^\circ}^{(\mathbf{G}^\tau)^\circ}(\theta|_{(\mathbf{T}^\tau)^\circ}), R_{(\mathbf{T}^\tau)^\circ}^{(\mathbf{G}^\tau)^\circ}(\theta|_{(\mathbf{T}^\tau)^\circ}) \right\rangle_{((\mathbf{G}^\tau)^\circ)^F}.$$

CHAPTER 4

Towards the verification of the inductive McKay–Navarro condition

In this chapter, we consider the inductive McKay–Navarro condition and state some general results that will help us to verify the condition later. We first consider the action of group and Galois automorphisms on Lusztig series. Further, we study the extension of characters and introduce a construction that allows us to use Deligne–Lusztig theory for certain semidirect products that occur in the inductive condition. Moreover, we show that we do not have to consider the full universal covering group in the inductive condition. We use the notation of the inductive McKay–Navarro condition from Section 1.4.3.

Parts of this chapter already appeared in [**Joh22**] and [**Joh21**].

4.1. Actions on Lusztig series

We have seen in the previous chapter that Lusztig series are a powerful tool to study the irreducible characters of finite groups of Lie type. Therefore, it will be useful to consider the actions of group and Galois automorphisms on Lusztig series. Let $\mathbf{G}$ be a connected reductive group and F a Steinberg endomorphism of $\mathbf{G}$. Let $(\mathbf{G}^*, F^*)$ be in duality with $(\mathbf{G}, F)$.

We first consider the action of group automorphisms on the Lusztig series of a group.

Lemma 4.1. [**Bru09**, Proposition 1] *Let $\mathbf{G}$ be simply connected, $s \in \mathbf{G}^{*F^*}$ semisimple, and κ an automorphism of $\mathbf{G}^F$. Then, κ maps the Lusztig series corresponding to s to*

$$\mathcal{E}(\mathbf{G}^F, s)^\kappa = \mathcal{E}(\mathbf{G}^F, \kappa^*(s))$$

where κ^* denotes the automorphism of $\mathbf{G}^{*F^*}$ corresponding to κ as in Remark 2.60.

The action of Galois automorphisms on the Lusztig series of a group can be described in a similar way.

Lemma 4.2. [**GM20**, Proposition 3.3.15] *Let $s \in \mathbf{G}^{*F^*}$ be semisimple and $\sigma \in \mathcal{G}$ a Galois automorphism described by $b \in \mathbb{Z}$. Then, σ maps the Lusztig series corresponding to s to*

$$\mathcal{E}(\mathbf{G}^F, s)^\sigma = \mathcal{E}(\mathbf{G}^F, s^b).$$

If $\mathbf{G}$ has connected center, we can strengthen this statement by considering the action on the Jordan decomposition of characters.

Theorem 4.3. [**SV20**, Theorem 5.1] *Assume that $Z(\mathbf{G})$ is connected and F is a Frobenius endomorphism. Let $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ with Jordan decomposition (s, ν) for $s \in \mathbf{G}^{*F^*}$ semisimple and ν a unipotent character of $C_{\mathbf{G}^{*F^*}}(s)$. Let $\sigma \in \mathcal{G}$ be described by $b \in \mathbb{Z}$. Then, χ^σ has Jordan decomposition (s^b, ν^σ).*

In Proposition 6.6, we will generalize this statement to Steinberg endomorphisms.

4.2. Character extensions

In the setting of the inductive McKay–Navarro condition, we have to find projective representations of $G \rtimes \Gamma_\chi$ associated to irreducible ℓ'-characters χ of G (respectively $N \rtimes \Gamma_\chi$ and N). In this section, we present some methods to construct character extensions or projective representations such that we can control the actions of Galois and group automorphisms on them. We will see that we are often interested in the construction of character extensions that are invariant under these actions. Further, we state some results and constructions that allow us to consider extensions to groups that we can control better than the groups $G \rtimes \Gamma_\chi$ and $N \rtimes \Gamma_\chi$.

If not specified otherwise, we continue to use the notation of the inductive McKay–Navarro condition.

As already mentioned in Section 1.3.2, we only know that a character of a normal subgroup can be extended to its inertia group if the corresponding factor group is cyclic. In our setting, this is the case for a character χ if

$$(G \rtimes \Gamma_\chi)/G \cong (N \rtimes \Gamma_\chi)/N \cong \Gamma_\chi$$

is cyclic. Luckily, there is a canonical way to extend representations to inner automorphisms.

Lemma 4.4. *Let $\mathcal{R}$ be a representation of G or N affording χ. There exists a canonical $(\Gamma \times \mathcal{G})_\chi$-invariant extension of $\mathcal{R}$ to the inner automorphisms in Γ_χ.*

Proof. If $c_h \in \Gamma_\chi$ is the inner automorphism associated to $h \in N_G(R)$, we can set

$$\widetilde{\mathcal{R}}(g, c_h) = \mathcal{R}(g)\mathcal{R}(h)$$

for all $g \in G$ (or N) and all $h \in N_G(R)$. Then, $\widetilde{\mathcal{R}}$ is $(\Gamma \times \mathcal{G})_\chi$-invariant since

$$\widetilde{\mathcal{R}}(g, c_h)^{\kappa\sigma} = \widetilde{\mathcal{R}}(\kappa(g), c_{\kappa(h)})^\sigma = \mathcal{R}(\kappa(g))^\sigma \mathcal{R}(\kappa(h))^\sigma \sim \mathcal{R}(g)\mathcal{R}(h) = \widetilde{\mathcal{R}}(g, c_h)$$

for all $(\kappa, \sigma) \in (\Gamma \times \mathcal{G})_\chi$. □

Remark 4.5. Note that this canonical extension yields trivial scalars associated to $\widetilde{\mathcal{R}}(c)$ for the elements

$$c \in C_{G \times \Gamma_\chi \mathcal{H}}(G) = \{(n, c_{n^{-1}}) \mid n \in N_G(R)\}.$$

As soon as we consider other extensions of characters, we also have to keep track of the character values on this centralizer.

In contrast to the extension condition in the inductive McKay condition, this cannot be dismissed directly: Assume that $\kappa \in \mathrm{Aut}(G)$ stabilizes N and there is a non-trivial inner automorphism in $\langle \kappa \rangle$. Let $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(N)$ be κ-invariant and extended by $\widehat{\chi} \in \mathrm{Irr}(G \rtimes \langle \kappa \rangle)$ and $\widehat{\psi} \in \mathrm{Irr}(N \rtimes \langle \kappa \rangle)$, respectively. Let $C := C_{G \rtimes \langle \kappa \rangle}(G)$ and

$$\varepsilon_\chi : C \to \mathbb{Q}^{\mathrm{ab}}, \quad \varepsilon_\psi : C \to \mathbb{Q}^{\mathrm{ab}}$$

such that

$$\widehat{\chi}(c) = \varepsilon_\chi(c)\widehat{\chi}(1), \quad \widehat{\psi}(c) = \varepsilon_\psi(c)\widehat{\psi}(1)$$

for all $c \in C$. We can easily see that $\varepsilon_\chi, \varepsilon_\psi$ are linear characters of C. Then, there is a linear character $\beta \in \mathrm{Irr}(C)$ such that $\beta\varepsilon_\chi = \varepsilon_\psi$. Now, C can be identified with a subgroup of $Z(G) \times \langle \kappa \rangle$ and we know that there exists $\widehat{\beta} \in \mathrm{Irr}(Z(G) \times \langle \kappa \rangle)$ extending β that is trivial on $Z(G)$. By Proposition 1.22, $\widehat{\beta}\widehat{\chi} \in \mathrm{Irr}(G \rtimes \langle \kappa \rangle)$ is an extension of χ and it is clear that the scalars corresponding to $\widehat{\beta}\widehat{\chi}$ and $\widehat{\psi}$ coincide for all $c \in C$. This suffices for the inductive McKay condition.

However, the construction does not always work for the inductive McKay–Navarro condition. Assume that $\widehat{\chi}$ and $\widehat{\psi}$ satisfy the second part of the extension condition (2B), i.e.

$$\widehat{\chi}^a = \mu_a \widehat{\chi}, \quad \widehat{\psi}^a = \mu_a \widehat{\psi}$$

for all $a \in (\Gamma \times \mathcal{H})_\chi$. Then we have

$$(\widehat{\beta}\widehat{\chi})^a = \widetilde{\mu}_a \mu_a \widehat{\beta}\widehat{\chi}$$

where $\widehat{\beta}^a = \widetilde{\mu}_a\widehat{\beta}$. One can easily think of examples for ℓ, $|C|$ and $\mathrm{ord}(\kappa)$ where $\widetilde{\mu}_a$ is not trivial.

The situation is more convenient if we consider extensions of characters χ that are invariant under the action of $(\Gamma \times \mathcal{H})_\chi$. The following lemma will be very useful if we have a cyclic group of outer automorphisms and consider their representatives in Γ.

Lemma 4.6. *Let $X \in \{G, N\}$ and $\chi \in \mathrm{Irr}(X)$ such that there exists a $(\Gamma \times \mathcal{H})_\chi$-invariant extension $\widehat{\chi} \in \mathrm{Irr}(X \rtimes \langle\kappa\rangle)$ of χ where $\kappa \in \Gamma_\chi$. Then, we find a $(\Gamma \times \mathcal{H})_\chi$-invariant extension of χ to $X \rtimes \langle\kappa\rangle$ such that the associated scalars on $C_{G\rtimes\langle\kappa\rangle}(G)$ are given by the trivial extension of the central character of χ on $Z(G)$.*

Proof. We can identify $C := C_{G\rtimes\langle\kappa\rangle}(G)$ with the direct product of $Z(G)$ and the subgroup $A \subseteq \langle\kappa\rangle$ of all inner automorphisms in $\langle\kappa\rangle$. Let $\varepsilon \in \mathrm{Irr}(C)$ such that we have $\widehat{\chi}(c) = \varepsilon(c)\widehat{\chi}(1)$ for all $c \in C$. Then, we have

$$\varepsilon(c)\widehat{\chi}(1) = \widehat{\chi}(c) = \widehat{\chi}(c)^a = \varepsilon(c)^a\widehat{\chi}(1)^a = \varepsilon(c)^a\widehat{\chi}(1)$$

for every $a \in (\Gamma \times \mathcal{H})_\chi$. Thus, ε is invariant under the action of $(\Gamma \times \mathcal{H})_\chi$. As a character of a direct product of groups, ε is given by $\varepsilon_1 \in \mathrm{Irr}(Z(G))$ and $\varepsilon_2 \in \mathrm{Irr}(A)$ where ε_1 is the central character of χ on $Z(G)$.

Let $\widehat{\beta} \in \mathrm{Irr}(\langle\kappa\rangle)$ be the inflation of $\varepsilon_2^{-1} \in \mathrm{Irr}(A)$. Then, $\widehat{\beta}$ is also $(\Gamma \times \mathcal{H})_\chi$-invariant and $\widehat{\beta}\widehat{\chi} \in \mathrm{Irr}(X \rtimes \langle\kappa\rangle)$ extends χ by Proposition 1.22. We further have

$$(\widehat{\beta}\widehat{\chi})^a = \widehat{\beta}^a\widehat{\chi}^a = \widehat{\beta}\widehat{\chi}$$

for all $a \in (\Gamma \times \mathcal{H})_\chi$ and

$$(\widehat{\beta}\widehat{\chi})(c) = \widehat{\beta}(c)\widehat{\chi}(c) = \varepsilon_2^{-1}(c)\varepsilon(c)\widehat{\chi}(1) = \varepsilon_1(c)\widehat{\chi}(1)$$

for every $c \in C$. This proves the claim. □

Next, we state some situations where we can easily find $(\Gamma \times \mathcal{G})_\chi$-invariant character extensions of a character χ.

Lemma 4.7. [**Joh21**, Remark 4.1] *Let M be a finite group, $\chi \in \mathrm{Irr}(M)$ and $A \leq \mathrm{Aut}(M)$.*

(a) *If χ is linear, it can be trivially extended to an $(A\times\mathcal{G})_\chi$-invariant character of $M \rtimes A_\chi$.*

(b) *Assume that χ can be extended to $\widehat{\chi} \in \mathrm{Irr}(M \rtimes A_\chi)$ and $\langle \chi, \mathrm{Ind}_X^M(\tau)\rangle = 1$ for some character $\tau \in \mathrm{Irr}(X)$ and a subgroup $X \leq M$. Let $\widehat{\tau} \in \mathrm{Irr}(X \rtimes A_\chi)$ be an $(A \times \mathcal{G})_\chi$-invariant extension of τ. Then, there is a unique extension $\widetilde{\chi} \in \mathrm{Irr}(M \rtimes A_\chi)$ of χ such that*

$$\left\langle \widetilde{\chi}, \mathrm{Ind}_{X\rtimes A_\chi}^{M\rtimes A_\chi}(\widehat{\tau})\right\rangle \neq 0$$

that is $(A \times \mathcal{G})_\chi$-invariant.

Proof. (a) Since χ is a group homomorphism, we can define a character of X by setting $\chi(g,a) = \chi(g)$ for all $(g,a) \in M \rtimes A_\chi$. This is obviously $(A \times \mathcal{G})_\chi$-invariant.

(b) By Proposition 1.22, we have

$$\mathrm{Ind}_X^{M\rtimes A_\chi}(\tau) = \mathrm{Ind}_{X\rtimes A_\chi}^{M\rtimes A_\chi}\left(\sum_{\beta\in\mathrm{Irr}(A_\chi)} \beta(1)\cdot\beta\,\widehat{\tau}\right) = \sum_{\beta\in\mathrm{Irr}(A_\chi)} \beta(1)\cdot\beta\,\mathrm{Ind}_{X\rtimes A_\chi}^{M\rtimes A_\chi}(\widehat{\tau}).$$

Now, $\beta'\widehat{\chi} \in \mathrm{Irr}(X \rtimes A_\chi)$ is an extension of χ for every linear $\beta' \in \mathrm{Irr}(A_\chi)$. By Frobenius reciprocity we have

$$\left\langle \beta'\widehat{\chi}, \mathrm{Ind}_X^{M\rtimes A_\chi}(\tau)\right\rangle = \left\langle \chi, \mathrm{Ind}_X^M(\tau)\right\rangle = 1.$$

Comparing this with the constituents of $\mathrm{Ind}_X^{M\rtimes A_\chi}(\tau)$ gives us a unique linear $\beta' \in \mathrm{Irr}(A_\chi)$ such that

$$\left\langle \beta'\widehat{\chi}, \mathrm{Ind}_{X\rtimes A_\chi}^{M\rtimes A_\chi}(\widehat{\tau})\right\rangle = 1.$$

Since $\widehat{\tau}$ is $(A \times \mathcal{H})_\chi$-invariant, it is clear from the induction formula that $\mathrm{Ind}_{X \rtimes A_\chi}^{M \rtimes A_\chi}(\widehat{\tau})$ is also $(A \times \mathcal{H})_\chi$-invariant. The uniqueness of $\beta' \widehat{\chi}$ yields the claim. □

For real characters, we have some easy general criteria that ensure the existence of a suitable extension.

Proposition 4.8. *Let $X \trianglelefteq M$ be normal and let $\chi \in \mathrm{Irr}(X)$ be a real character. Assume that either*

(a) *X has odd index in M or*

(b) *χ has odd degree and $\det(\chi)$ is the trivial character.*

Then, there exists an $(M \times \mathcal{G})_\chi$-invariant real extension of χ to M_χ.

Proof. This can be proven as in [**NT08**, Corollary 2.2]. By [**NT08**, Lemma 2.1 and Theorem 2.3], we know that there exists a unique real extension $\widetilde{\chi}$ such the order of $\det(\widetilde{\chi})$ is the same as the order of $\det(\chi)$. The claim follows directly from the uniqueness. □

We now state an easy consequence of [**Isa76**, (6.28)] that sometimes ensures the existence of suitable character extensions.

Lemma 4.9. [**Joh22**, Lemma 5.2] *Let M be a perfect finite group and $A' \leq A \leq \mathrm{Aut}(M)$ such that A' contains representatives for all outer automorphisms in A. Assume that A' is an ℓ-group for $\ell \in \mathbb{P}$. Then, for every $\chi \in \mathrm{Irr}_{\ell'}(M)$ there exists an $(A \times \mathcal{H}_\ell)_\chi$-invariant extension $\widetilde{\chi} \in \mathrm{Irr}(M \rtimes A_\chi)$ of χ such that its central character on $Z(\widetilde{\chi})$ is trivial.*

Proof. The determinant and the central character of χ are trivial because M is perfect. Since A' is an ℓ-group, it is solvable and we have $(|A'|, \chi(1)) = 1$. By [**Isa76**, (6.28)], we find a unique extension $\widehat{\chi} \in \mathrm{Irr}(M \rtimes A'_\chi)$ of χ such that $\widehat{\chi}$ has trivial determinant. Since the action of $(A \times \mathcal{H}_\ell)_\chi$ does not change the determinant of $\widehat{\chi}$, the uniqueness already implies that $\widehat{\chi}$ is invariant under $(A \times \mathcal{H}_\ell)_\chi$. Now, let $g \in Z(\widehat{\chi})$, i.e. $\widehat{\chi}(g) = \varepsilon\chi(1)$ for some $\varepsilon \in \mathbb{Q}^{\mathrm{ab}}$. Then, we have $g^{\mathrm{ord}(g)_\ell} \in M$ and

$$\det(\widehat{\chi}(g)) = \varepsilon^{\chi(1)} = 1, \quad \varepsilon^{\mathrm{ord}(g)_\ell} = 1.$$

Since $\mathrm{ord}(g)_\ell$ and $\chi(1)$ are coprime, it follows $\varepsilon = 1$ and we get the claim by extending the characters canonically to the inner automorphisms in A. □

As already mentioned in the beginning of the section, there is a canonical extension of characters to inner automorphisms. The following Proposition generalizes this observation and shows us that it suffices to replace $G \rtimes \Gamma$ with a group inducing all automorphisms in Γ.

Proposition 4.10. [**NSV20**, Theorem 2.9] *Let $(H, G, \chi)_\mathcal{H}$ and $(M, N, \psi)_\mathcal{H}$ be $\mathcal{H}$-triples with*

$$(H, G, \chi)_\mathcal{H} \geq_c (M, N, \psi)_\mathcal{H}.$$

Let $\widehat{G}, \widehat{N}$ be finite groups with $G \lhd \widehat{G}$ and $NC_{\widehat{G}}(G) \subseteq \widehat{N} \lhd \widehat{G}$. We denote the group homomorphisms induced by conjugation with elements of H and $\widehat{G}$ by $\varepsilon : H \to \mathrm{Aut}(G)$ and $\widehat{\varepsilon} : \widehat{G} \to \mathrm{Aut}(G)$, respectively. If $\varepsilon(H) = \widehat{\varepsilon}(\widehat{G})$ and $\varepsilon(M) = \widehat{\varepsilon}(\widehat{N})$, then we have

$$(\widehat{G}, G, \chi)_\mathcal{H} \geq_c (\widehat{N}, N, \psi)_\mathcal{H}.$$

In the case of finite groups of Lie type, there is a canonical choice for a group that induces all group automorphisms. Let $\mathbf{G}$ be a simply connected group and $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ a regular embedding as in Section 2.6. Let F be a Frobenius endomorphism of $\mathbf{G}$ and write $G := \mathbf{G}^F$, $\widetilde{G} := \widetilde{\mathbf{G}}^F$. We consider a group $D \leq \mathrm{Aut}(\widetilde{G})$ generated by graph and field endomorphisms of $\widetilde{G}$ that commute with F. Then, conjugation with $\widetilde{G} \rtimes D$ induces

Aut(G) by Proposition 2.56 and [**CE04**, Section 15.1]. Let $N \leq G$ be a D-stable subgroup and $\widetilde{N} \leq \widetilde{G}$ such that $N = \widetilde{N} \cap G$. If we have $\mathcal{H}$-triples

$$((\widetilde{G} \rtimes D)_\chi, G, \chi)_{\mathcal{H}} \geq_c ((\widetilde{N} \rtimes D)_\chi, N, \psi)_{\mathcal{H}}$$

for a D-stable subgroup $N \leq G$, we can therefore apply Proposition 4.10 to conclude

$$(G \rtimes \Gamma_\chi, G, \chi)_{\mathcal{H}} \geq_c (N \rtimes \Gamma_\chi, N, \psi)_{\mathcal{H}}.$$

The following proposition summarizes constructions from [**Ruh21**, Section 7.1, Lemma 7.2] that are helpful if G has non-trivial diagonal automorphisms.

Proposition 4.11. *Let $\widetilde{X} \in \{\widetilde{G}, \widetilde{N}\}$ and $X = G \cap \widetilde{X} \in \{G, N\}$. Assume that $\chi \in \mathrm{Irr}(X)$ such that $(\widetilde{X} \rtimes D)_\chi = \widetilde{X}_\chi \rtimes D_\chi$ and χ has an extension to $X \rtimes D_\chi$ afforded by a representation $\mathcal{D}_2$.*

(a) *The character χ also extends to $\widetilde{X}_\chi$ and we denote a representation affording this extension by $\mathcal{D}_1$.*

(b) *A projective representation of $(\widetilde{X} \rtimes D)_\chi$ is given by*

$$\mathcal{P}(\widetilde{x}d) = \mathcal{D}_1(\widetilde{x})\mathcal{D}_2(d)$$

for $\widetilde{x} \in \widetilde{X}_\chi$ and $d \in D_\chi$.

(c) *Let $a \in (N_{\widetilde{N} \rtimes D}(X \rtimes D_\psi) \times \mathcal{H})_\chi$ and suppose*

$$\mathcal{D}_1^a \sim \mu_1 \mathcal{D}_1, \quad \mathcal{D}_2^a \sim \mu_2 \mathcal{D}_2$$

where $\mu_1 \in \mathrm{Irr}(\widetilde{X}_\chi / X)$ and $\mu_2 \in \mathrm{Irr}((X \rtimes D_\chi)/X)$. Then, we have $\mathcal{P}^a \sim \mu_1 \mu_2 \mathcal{P}$.

Therefore, in this situation we can consider extensions to the diagonal automorphisms and to the field automorphisms separately.

In the following, we often need a good way of extending characters of the F-fixed points of a torus to their stabilizers in the normalizer of the torus. This will be helpful for the parametrization of the local characters in the inductive condition as well as for the construction of character extensions. In order to formalize this, we define so-called extension maps.

Definition 4.12. Let $X \trianglelefteq M$ be normal. Assume that every $\chi \in \mathrm{Irr}(X)$ extends to an irreducible character $\widetilde{\chi}$ of its inertia group M_χ. Then, we say that we have an *extension map for $X \trianglelefteq M$* given by

$$\Lambda : \mathrm{Irr}(X) \to \bigcup_{\chi \in \mathrm{Irr}(X)} \mathrm{Irr}(M_\chi), \quad \chi \mapsto \widetilde{\chi}.$$

An extension map is in general not unique and depends on the choice of the different character extensions.

4.3. Descent of scalars

Next, we present some methods to construct suitable character extensions of some Deligne–Lusztig characters using results from Section 3.4. Let $\mathbf{G}$ be a connected reductive group with connected center and F a Steinberg endomorphism of $\mathbf{G}$.

In order to verify the inductive McKay–Navarro condition, we have to consider extensions of characters to the group $\mathbf{G}^F \rtimes \Gamma$. Since we have a canonical way of extending characters to the inner automorphisms, we often consider characters of the group $\mathbf{G}^F \rtimes \langle F_0 \rangle$ where F_0 is some automorphism of $\mathbf{G}^F$. This group arises from a disconnected algebraic group and we therefore cannot apply classical Deligne–Lusztig theory. However, we can use the Deligne–Lusztig theory for disconnected groups from Section 3.4 that has been developed in [**DM94**]. We first have to find a way to consider $\mathbf{G}^F \rtimes \langle F_0 \rangle$ such that it arises from the semidirect product of a connected reductive group with a quasi-central automorphism.

4.3.1. Constructing isomorphisms. We use the constructions called *Descent of scalars* from [**Ruh21**, Section 6.3], see also [**Dig99**] for more details. Let F_0 be a Steinberg endomorphisms of $\mathbf{G}$ and k a positive integer. For any F_0^k-stable closed subgroup $\mathbf{H}$ of $\mathbf{G}$, we define

$$\underline{\mathbf{H}} = \mathbf{H} \times F_0^{k-1}(\mathbf{H}) \times \ldots \times F_0(\mathbf{H}) \leq \mathbf{G}^k = \underline{\mathbf{G}}.$$

We set

$$\tau : \underline{\mathbf{G}} \to \underline{\mathbf{G}}, \quad \tau(g_1, \ldots, g_k) = (g_2, \ldots, g_k, g_1).$$

As in [**Ruh21**, Section 6.3], τ is a quasi-central automorphism of $\underline{\mathbf{G}}$ and the projection onto the first coordinate pr induces isomorphisms

$$\underline{\mathbf{H}}^{\tau F_0} \cong \mathbf{H}^{F_0^k}, \quad \underline{\mathbf{G}}^{\tau F_0} \rtimes \langle \tau \rangle \cong \mathbf{G}^{F_0^k} \rtimes \langle F_0 \rangle.$$

Since the groups are isomorphic, we have bijections between the respective irreducible characters that can be linearly extended to

$$\mathrm{pr}^\vee_{\mathbf{G} \rtimes \langle \tau \rangle} : \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^{F_0^k} \rtimes \langle F_0 \rangle) \to \mathbb{Z}\,\mathrm{Irr}(\underline{\mathbf{G}}^{\tau F_0} \rtimes \langle \tau \rangle),$$

$$\mathrm{pr}^\vee_{\mathbf{G}} : \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^{F_0^k}) \to \mathbb{Z}\,\mathrm{Irr}(\underline{\mathbf{G}}^{\tau F_0}), \quad \mathrm{pr}^\vee_{\mathbf{H}} : \mathbb{Z}\,\mathrm{Irr}(\mathbf{H}^{F_0^k}) \to \mathbb{Z}\,\mathrm{Irr}(\underline{\mathbf{H}}^{\tau F_0}).$$

The group $\underline{\mathbf{G}} \rtimes \langle \tau \rangle$ is a disconnected group with Steinberg endomorphism τF_0 as considered in [**DM94**]. As described in Section 3.4, Digne and Michel generalized the concept of Deligne–Lusztig characters to disconnected groups of this form. Thus, for $F = F_0^k$ we can use these isomorphisms to construct Deligne–Lusztig characters of the disconnected group $\mathbf{G}^F \rtimes \langle F_0 \rangle$. Note that this is the case if F_0 is the standard field automorphism and $\mathbf{G}^F$ is untwisted as well as if F_0 is an exceptional graph endomorphism of $\mathbf{G}$ and $\mathbf{G}^F$ is either untwisted or a Suzuki or Ree group.

With the required relation of F_0 and F, we cannot apply the previous construction to twisted groups and field automorphisms F_0. However, we can generalize the construction by considering an additional graph endomorphism. We follow [**Ruh21**, Section 6.4].

We consider a graph endomorphism $\rho : \mathbf{G} \to \mathbf{G}$ commuting with F_0 such that we have $F := F_0 \circ \rho$. Let l be the order of ρ and $\mathbf{H}$ an F_0^{kl}-stable subgroup of $\mathbf{G}$. Then, we set

$$\underline{\mathbf{H}} := \mathbf{H} \times F_0^{kl-1}(\mathbf{H}) \times \cdots \times F_0(\mathbf{H})$$

and consider the automorphism

$$\tau : \underline{\mathbf{G}} \to \underline{\mathbf{G}}, \quad (g_1, \ldots, g_{kl}) \mapsto (g_2, \ldots, g_{kl}, g_1).$$

We know from the constructions for untwisted groups that projection onto the first coordinate pr yields an isomorphism $\underline{\mathbf{H}}^{\tau F_0} \cong \mathbf{H}^{F_0^{kl}}$. Thus, we have to consider a suitable subset of $\underline{\mathbf{H}}$ to obtain an isomorphism to $\mathbf{H}^F$. We define the closed subset

$$\underline{\mathbf{G}}_\rho := \{(g_1, \ldots, g_{kl}) \in \underline{\mathbf{G}} \mid g_i = \rho(g_{k+i}) \text{ for all } 1 \leq i \leq k(l-1)\} \subseteq \underline{\mathbf{G}}$$

and set $M_\rho := M \cap \underline{\mathbf{G}}_\rho$ for any subgroup $M \leq \underline{\mathbf{G}}$. Then, we know by [**Ruh21**, Lemma 6.5] that pr induces isomorphisms

$$\underline{\mathbf{G}}_\rho^{\tau F_0} \rtimes \langle \tau \rangle \cong \mathbf{G}^F \rtimes \langle F_0 \rangle, \quad \underline{\mathbf{G}}_\rho^{\tau F_0} \cong \mathbf{G}^F.$$

As above, the group $\underline{\mathbf{G}}_\rho^{\tau F_0} \rtimes \langle \tau \rangle$ with Steinberg endomorphism τF_0 is as considered in [**DM94**] and we have bijections between the (almost) characters of the isomorphic groups.

4.3.2. Application to Gelfand–Graev characters. As a first application of the isomorphisms above, we use Gelfand–Graev characters of $\mathbf{G}^F \rtimes \langle F_0 \rangle$ to obtain suitable extensions of semisimple and regular characters of $\mathbf{G}^F$. Since we only need the results in this section for untwisted groups with exceptional graph automorphisms and the Suzuki and Ree groups, we state them for the case $\rho = \mathrm{id}$ and $Z(\mathbf{G}) = 1$ in order to simplify the notation. However, the proof also holds for twisted groups and groups with non-trivial center, see [**Ruh21**, Proposition 6.7.].

Assume that $\mathbf{G}$ has connected center and F is again a Steinberg endomorphism of $\mathbf{G}$. As in Section 3.3, let $\mathbf{B}$ be an F-stable Borel subgroup of $\mathbf{G}$ containing an F-stable maximal torus $\mathbf{T}$ and $\mathbf{U} := R_u(\mathbf{B})$. Let ξ be a character of $\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F$ as defined there. To simplify notation, we also write ξ for its inflation to $\mathbf{U}^F$. Then,

$$\Gamma_1 = \mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{G}^F}(\xi)$$

is the Gelfand–Graev character of $\mathbf{G}^F$. Again, we denote the Alvis–Curtis duality map for $\mathbf{G}$ by $D_{\mathbf{G}} : \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F) \to \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F)$.

We are now able to extend [**Ruh21**, Proposition 6.7] to Steinberg endomorphisms. Although Ruhstorfer considers only Galois automorphisms in $\mathcal{H}_p$ where p is the defining characteristic, the proof given there also holds for all Galois automorphisms that satisfy [**Ruh21**, Assumption 6.6] and act like some field automorphism of the group. Thus, the changes we make are only due to the generalization to Steinberg automorphisms F_0 and the limited knowledge about duality for disconnected groups in this setting.

Proposition 4.13. [**Joh22**, Proposition 3.6] *Let $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ be a semisimple or regular character and $\sigma \in \mathcal{G}$ such that we have $\chi^\sigma = \chi^{F_p^e}$ for some $e \in \mathbb{N}$. Let F_0 be a Steinberg endomorphism of $\mathbf{G}$ that fixes χ such that there is a $k \in \mathbb{N}$ with $F_0^k = F$. Assume that there exists a $t \in \mathbf{T}$ with $\xi^t = \xi^\sigma$ that is invariant under all graph and field endomorphisms of $\mathbf{G}$. Then, there exists an extension $\widehat{\chi} \in \mathrm{Irr}(\mathbf{G}^F \rtimes \langle F_0 \rangle)$ of χ such that $\widehat{\psi}^{F_p^e t\sigma^{-1}} = \widehat{\chi}$.*

Proof. We first assume that χ is semisimple.

Step 1: Translating the characters to $\underline{\mathbf{G}}$. We keep the notation of the maps $\mathrm{pr}^\vee_{\mathbf{G}\rtimes\langle\tau\rangle}$, $\mathrm{pr}^\vee_{\mathbf{G}}$, $\mathrm{pr}^\vee_{\mathbf{U}}$ from above. We set

$$\underline{\chi} := \mathrm{pr}^\vee_{\mathbf{G}}(\chi), \quad \underline{\xi} := \mathrm{pr}^\vee_{\mathbf{U}}(\xi), \quad \underline{\Gamma_1} = \mathrm{Ind}_{\underline{\mathbf{U}}^{\tau F_0}}^{\underline{\mathbf{G}}^{\tau F_0}}(\underline{\xi}).$$

Using the character formula for induction, we see that

$$\mathrm{pr}^\vee_{\mathbf{G}} \circ \mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{G}^F} = \mathrm{Ind}_{\underline{\mathbf{U}}^{\tau F_0}}^{\underline{\mathbf{G}}^{\tau F_0}} \circ \mathrm{pr}^\vee_{\mathbf{U}}$$

and it follows $\underline{\Gamma_1} = \mathrm{pr}^\vee_{\mathbf{G}}(\Gamma_1)$.

Since χ is semisimple, χ is a constituent of $D_{\mathbf{G}}(\Gamma_1)$ by Proposition 3.40. From [**GM20**, Proposition 3.4.3] we know that Alvis-Curtis duality and $\mathrm{pr}^\vee_{\mathbf{G}}$ also commute in this setting. Consequently, we have

$$D_{\underline{\mathbf{G}}} \circ \mathrm{pr}^\vee_{\mathbf{G}} = \mathrm{pr}^\vee_{\mathbf{G}} \circ D_{\mathbf{G}}$$

and $\underline{\chi}$ is a constituent of $D_{\underline{\mathbf{G}}}(\underline{\Gamma_1})$. Let $\phi : \mathbf{G} \to \mathbf{G}$ be the morphism given by the action of $F_p^e t$ and set

$$\underline{\phi} : \underline{\mathbf{G}} \to \underline{\mathbf{G}}, \quad (g_1, \ldots g_k) \mapsto (\phi(g_1), \ldots \phi(g_k)).$$

Then, we have $\underline{\chi}^{\underline{\phi}\sigma^{-1}} = \underline{\chi}$. By the choice of t and since ξ is invariant under field automorphisms, it follows that

$$\xi^{\phi\sigma^{-1}} = \xi^{t\sigma^{-1}} = \xi.$$

This implies

$$\underline{\xi}^{\underline{\phi}\sigma^{-1}} = \underline{\xi}, \quad \underline{\Gamma_1}^{\underline{\phi}\sigma^{-1}} = \underline{\Gamma_1}.$$

Step: 2 Extending characters to the disconnected group. Since ξ is fixed by F_0, the character $\underline{\xi} \in \mathrm{Irr}(\underline{\mathbf{U}}^{\tau F_0})$ is τ-invariant and we can extend $\underline{\xi}$ to a character $\widehat{\underline{\xi}} \in \mathrm{Irr}(\underline{\mathbf{U}}^{\tau F_0} \rtimes \langle\tau\rangle)$ by setting $\widehat{\underline{\xi}}(\tau) = 1$. Let

$$\widehat{\underline{\Gamma_1}} := \mathrm{Ind}_{\underline{\mathbf{U}}^{\tau F_0} \rtimes \langle\tau\rangle}^{\underline{\mathbf{G}}^{\tau F_0} \rtimes \langle\tau\rangle}(\widehat{\underline{\xi}}).$$

Then, $\widehat{\underline{\Gamma_1}}$ extends $\underline{\Gamma_1}$ by the Mackey formula for characters in Theorem 1.13.

We need the construction of Deligne–Lusztig characters and duality for disconnected groups from Definition 3.45 and 3.48. Digne and Michel have only considered finite groups of Lie type that arise via Frobenius endomorphisms. As we have mentioned in Section 3.4, we checked that all results that we need in the following still hold for Steinberg endomorphisms τF_0.

Using Lemma 3.46, it is easy to see that

$$\begin{aligned}
\mathrm{Res}_{\underline{\mathbf{G}}^{\tau F_0}}^{(\underline{\mathbf{G}} \rtimes \langle\tau\rangle)^{\tau F_0}} \circ D_{\underline{\mathbf{G}} \rtimes \langle\tau\rangle} &= \mathrm{Res}_{\underline{\mathbf{G}}^{\tau F_0}}^{(\underline{\mathbf{G}} \rtimes \langle\tau\rangle)^{\tau F_0}} \circ \sum_{\widetilde{\mathbf{P}} \supset \widetilde{\mathbf{B}}} (-1)^{r_{ss}(\widetilde{\mathbf{P}})} \cdot R_{\underline{\mathbf{L}} \rtimes \langle\tau\rangle}^{\underline{\mathbf{G}} \rtimes \langle\tau\rangle} \circ {}^*R_{\underline{\mathbf{L}} \rtimes \langle\tau\rangle}^{\underline{\mathbf{G}} \rtimes \langle\tau\rangle} \\
&= \sum_{\widetilde{\mathbf{P}} \supset \widetilde{\mathbf{B}}} (-1)^{r_{ss}(\widetilde{\mathbf{P}})} R_{\underline{\mathbf{L}}}^{\underline{\mathbf{G}}} \circ \mathrm{Res}_{\underline{\mathbf{L}}^{\tau F_0}}^{(\underline{\mathbf{L}} \rtimes \langle\tau\rangle)^{\tau F_0}} \circ {}^*R_{\underline{\mathbf{L}} \rtimes \langle\tau\rangle}^{\underline{\mathbf{G}} \rtimes \langle\tau\rangle} \\
&= \sum_{\underline{\mathbf{P}} \supset \underline{\mathbf{B}}} (-1)^{r_{ss}(\underline{\mathbf{P}})} R_{\underline{\mathbf{L}}}^{\underline{\mathbf{G}}} \circ {}^*R_{\underline{\mathbf{L}}}^{\underline{\mathbf{G}}} \circ \mathrm{Res}_{\underline{\mathbf{G}}^{\tau F_0}}^{(\underline{\mathbf{G}} \rtimes \langle\tau\rangle)^{\tau F_0}} \\
&= D_{\underline{\mathbf{G}}} \circ \mathrm{Res}_{\underline{\mathbf{G}}^{\tau F_0}}^{(\underline{\mathbf{G}} \rtimes \langle\tau\rangle)^{\tau F_0}}
\end{aligned}$$

where

- $\widetilde{\mathbf{B}}$ is a fixed τF_0-stable Borel subgroup of $\underline{\mathbf{G}} \rtimes \langle\tau\rangle$ containing τ, the sum runs over all τF_0-stable parabolic subgroups $\widetilde{\mathbf{P}}$ of $\underline{\mathbf{G}} \rtimes \langle\tau\rangle$ containing $\widetilde{\mathbf{B}}$, and $\underline{\mathbf{L}} \rtimes \langle\tau\rangle$ denotes the τF_0-stable Levi subgroup corresponding to $\widetilde{\mathbf{P}}$ containing τ;
- $\underline{\mathbf{B}}$ is an F-stable fixed Borel subgroup of $\underline{\mathbf{G}}$, the sum runs over all τF_0-stable parabolic subgroups $\underline{\mathbf{P}}$ of $\underline{\mathbf{G}}$ containing $\underline{\mathbf{B}}$, and $\underline{\mathbf{L}}$ denotes the τF_0-stable Levi subgroup corresponding to $\underline{\mathbf{P}}$.

A bijection between the summation indices as implied by the notation exists by the remarks after [**DM94**, Definition 3.10] and the considerations about Levi subgroups and parabolic subgroups in [**DM94**, Section 1.1].

Thus, $D_{\underline{\mathbf{G}} \rtimes \langle\tau\rangle}(\widehat{\underline{\Gamma_1}})$ extends $D_{\underline{\mathbf{G}}}(\underline{\Gamma_1})$.

Step 3: Conclusion. In our setting, Harish-Chandra induction and restriction still commute with Galois automorphisms and field automorphisms in the generalized case. Thus, we have

$$D_{\underline{\mathbf{G}} \rtimes \langle\tau\rangle}\left(\widehat{\underline{\Gamma_1}}\right)^{\phi\sigma^{-1}} = D_{\underline{\mathbf{G}} \rtimes \langle\tau\rangle}\left(\widehat{\underline{\Gamma_1}}^{\phi\sigma^{-1}}\right) = D_{\underline{\mathbf{G}} \rtimes \langle\tau\rangle}\left(\widehat{\underline{\Gamma_1}}\right).$$

As mentioned before, the character $\underline{\chi}$ is a constituent of $D_{\underline{\mathbf{G}}}(\underline{\Gamma_1})$. Since $D_{\underline{\mathbf{G}}}(\underline{\Gamma_1})$ and $D_{\underline{\mathbf{G}} \rtimes \langle\tau\rangle}(\widehat{\underline{\Gamma_1}})$ are both multiplicity free, there is only one constituent $\widehat{\underline{\chi}}$ of $D_{\underline{\mathbf{G}} \rtimes \langle\tau\rangle}(\widehat{\underline{\Gamma_1}})$ that extends $\underline{\chi}$. Thus, it is $\underline{\phi}\sigma^{-1}$-stable and it follows that

$$(\mathrm{pr}_{\underline{\mathbf{G}} \rtimes \langle\tau\rangle}^{\vee})^{-1}(\widehat{\underline{\chi}}) \in \mathrm{Irr}(\mathbf{G}^F \rtimes \langle F_0\rangle)$$

is a $\phi\sigma^{-1}$-stable character extending χ.

In the case that χ is a regular character, it is a constituent of Γ_1. Since $\widehat{\underline{\Gamma_1}}$ is multiplicity free, there is a unique extension $\widehat{\underline{\chi}}$ of $\underline{\chi}$ that is a constituent of $\widehat{\underline{\Gamma_1}}$ and we can argue as before. □

4.4. Primes and the universal covering group

In the inductive McKay–Navarro condition, we have to consider the Schur cover of a simple group S to verify the condition for S and ℓ. In this section, we show that we do not always have to consider the full Schur cover.

We first define a smaller covering group that depends on the prime ℓ.

Definition 4.14. Let S be a finite group with covering group G. The *ℓ'-part of the covering group* G is a covering group H with $G \twoheadrightarrow H \twoheadrightarrow S$ such that $|G|/|H|$ is an ℓ-power and $|H|/|S|$ is prime to ℓ.

As we see in the following lemma, it is enough to consider the ℓ'-part of the Schur cover of S.

Lemma 4.15. [**Joh22**, Lemma 5.1] *Let S be a finite simple non-abelian group, ℓ a prime, G the Schur cover of S and H the ℓ'-part of the Schur cover. Let $M \leq H$ such that it contains the normalizer of a Sylow ℓ-subgroup R of H and set $\Gamma = \mathrm{Aut}(H)_R$. Assume that there exists a $\Gamma \times \mathcal{H}_\ell$-equivariant bijection $\Omega' : \mathrm{Irr}_{\ell'}(H) \to \mathrm{Irr}_{\ell'}(M)$ such that for all $\chi \in \mathrm{Irr}_{\ell'}(H)$ we have*

$$(H \rtimes \Gamma_{\chi^{\mathcal{H}}}, H, \chi) \geq_c (M \rtimes \Gamma_{\chi^{\mathcal{H}}}, M, \Omega'(\chi)).$$

Then, the inductive Galois–McKay condition holds for S and ℓ.

Proof. We have $G/Z(G) \cong S$ and $G/Z(G)_\ell \cong H$; thus G is a central ℓ-extension of H. Consider an irreducible character of G with degree d such that its restriction to $Z(G)_\ell$ is not trivial. Then we find a $z \in Z(G)_\ell$ such that an affording representation $\mathcal{R}$ is of the form $\mathcal{R}(z) = \zeta I_d$ for some $\zeta \in \mathbb{C}^\times$ with $\zeta^{|Z(G)_\ell|} = 1$ and $\zeta \neq 1$. Since S is perfect, G is also perfect and its only linear character is the trivial one. Thus, the determinant of $\mathcal{R}(g)$ has to be 1 for all elements $g \in G$ and it follows

$$\det(\mathcal{R}(z)) = \zeta^d = 1.$$

Since $|Z(G)_\ell|$ is an ℓ-power and $\zeta \neq 1$, d and ℓ cannot be coprime and it follows $\ell \mid d$. Therefore, all irreducible ℓ'-characters of G have $Z(G)_\ell$ in their kernel and can be obtained by inflating irreducible ℓ'-characters of H. Thus, deflation yields a bijection Def_H^G between ℓ'-characters of G and H.

Let P and N be the preimages of R and M in G, respectively. One can easily see that P is a Sylow ℓ-subgroup of G and that N contains the normalizer of P in G. Since $Z(G)_\ell \leq N$ by definition and Sylow theory, N is a central ℓ-extension of M. Thus, inflation yields a bijection Inf_M^N between the ℓ'-characters of M and N. It follows that Ω' induces a bijection

$$\Omega := \mathrm{Inf}_M^N \circ \Omega' \circ \mathrm{Def}_H^G : \mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(N).$$

Since H is perfect, we know from [**Asc00**, (33.8)] that G is a universal covering group of H. Thus, the automorphism groups of S and H are both isomorphic to the automorphism group of G by Lemma 1.29. Consequently, every automorphism of H stabilizing R extends uniquely to an automorphism of G stabilizing P and we can consider Γ as a subset of $\mathrm{Aut}(G)$. Since every $\chi \in \mathrm{Irr}_{\ell'}(G)$ is uniquely determined by the values of χ on H, the groups Γ and $\mathcal{H}$ act on $\mathrm{Irr}_{\ell'}(G)$ in the same way as on $\mathrm{Irr}_{\ell'}(H)$. Thus, Def_H^G and Inf_M^N are $\Gamma \times \mathcal{H}$-equivariant. It follows that Ω is $\Gamma \times \mathcal{H}$-equivariant.

It remains to show that the extension conditions (2A) and (2B) of the inductive McKay–Navarro condition are satisfied. Given projective representations of $H \rtimes \Gamma_{\mathrm{Def}_H^G(\chi)}$ and $M \rtimes \Gamma_{\Omega'(\mathrm{Def}_H^G(\chi))}$ for $\chi \in \mathrm{Irr}_{\ell'}(G)$ such that

$$(H \rtimes \Gamma_{\mathrm{Def}_H^G(\chi)^{\mathcal{H}}}, H, \mathrm{Def}_H^G(\chi)) \geq_c (M \rtimes \Gamma_{\mathrm{Def}_H^G(\chi)^{\mathcal{H}}}, M, \Omega'(\mathrm{Def}_H^G(\chi))),$$

we can extend them trivially to $Z(G)_\ell$. These extended projective representations still satisfy (2A) and (2B) and we thereby know that the inductive McKay–Navarro condition holds for S and ℓ. □

This observation will be very useful for the computational validation of the inductive condition for some finite groups that have to be considered separately.

CHAPTER 5

Groups of Lie type in their defining characteristic

We now verify the inductive McKay–Navarro condition for some infinite families of finite groups of Lie type. The observations in Section 2.7 already imply that we have to distinguish between the case that the prime ℓ in the inductive condition equals the characteristic p of the group of Lie type and the case that they are different.

In this chapter, we consider the case $p = \ell$ that is also called the *defining characteristic* case. We already know that the inductive McKay condition holds for all finite simple groups of Lie type in their defining characteristic by [**Spä12**]. In [**Ruh21**], Ruhstorfer extended this and verified the inductive McKay–Navarro condition for most finite groups of Lie type in their defining characteristic. He excluded the Suzuki and Ree groups, the groups with exceptional graph automorphisms, $\mathsf{B}_n(2)$ $(n \geq 2)$, and the groups with non-generic Schur multiplier. We verify the inductive McKay–Navarro condition for these remaining cases and thereby complete the treatment of the defining characteristic case.

With some small changes, this chapter has been published in [**Joh22**]. Section 5.1 and some additional details have been added and the explanation of some general theory has been modified such that we can use the constructions and statements that have already been introduced in earlier chapters of this thesis.

5.1. About the bijection

Before we start with the verification of the inductive McKay–Navarro condition for the remaining groups, we give an idea of the bijection from the inductive McKay condition in defining characteristic as constructed in [**Mas10**]. With this, we can explain why the groups that we consider in the following have been excluded in [**Ruh21**].

Ruhstorfer used the bijection that has been constructed by Maslowski and showed that this bijection also satisfies the inductive McKay–Navarro condition. Since Maslowski restricted his work to Frobenius endomorphisms, Ruhstorfer excluded the Suzuki and Ree groups from the beginning. Especially for the parametrization of the p'-characters in the local case, Maslowski did some case by case distinctions and we cannot transfer the results to the Suzuki and Ree groups without additional work. However, the construction of the characters ξ and ϕ_S below also works for Steinberg endomorphisms.

Moreover, the simple groups with exceptional Schur multiplier have been excluded. This is due to the fact that we consider the simple group $\mathbf{G}^F/Z(\mathbf{G}^F)$ where $\mathbf{G}$ is a simply connected group such that $\mathbf{G}^F$ is perfect and is equal to the universal covering group of $\mathbf{G}^F/Z(\mathbf{G}^F)$. As we have seen in Section 2.5.3, this is the case unless $\mathbf{G}^F$ is one of the groups in Proposition 2.53 or Table 2.3.

5.1.1. Describing the local characters. Let $\mathbf{G}$ be of simply connected type and F a Steinberg endomorphism of $\mathbf{G}$. Let $\mathbf{T} \subseteq \mathbf{B}$ be an F-stable maximal torus contained in an F-stable Borel subgroup of $\mathbf{G}$ and $\mathbf{U} = R_u(\mathbf{B})$. We know from Proposition 2.67 that we can choose $\mathbf{B}^F$ as the local subgroup in the inductive McKay–Navarro condition.

First, we consider the local characters in $\mathrm{Irr}_{p'}(\mathbf{B}^F)$. We recall the construction of linear characters of $\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F$ in Section 3.3 and use the notation introduced there. As we have seen in Lemma 3.34, there is a bijection

$$\delta : \mathrm{Irr}(\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F) \to \prod_{i=1}^{r} \mathbb{F}_{q^{|\delta_i|}}.$$

We know that inflation yields a bijection between the irreducible characters of the group $\mathbf{U}^F/[\mathbf{U}^F,\mathbf{U}^F]$ and the linear characters of $\mathbf{U}^F$. By [**How74**] we have $[\mathbf{U}^F,\mathbf{U}^F] = [\mathbf{U},\mathbf{U}]^F$ unless $\mathbf{G}^F$ is one of the groups

$$\mathsf{B}_n(2) \cong \mathsf{C}_n(2) \text{ for some } n \geq 2, \quad \mathsf{G}_2(2), \mathsf{F}_4(2), \mathsf{G}_2(3), {}^2\mathsf{B}_2(2), {}^2\mathsf{F}_4(2), {}^2\mathsf{G}_2(3).$$

Therefore, these groups have not been studied in [**Mas10**] and [**Ruh21**] and have to be considered separately.

Assuming $[\mathbf{U}^F,\mathbf{U}^F] = [\mathbf{U},\mathbf{U}]^F$, we can even say more about the linear characters of $\mathbf{U}^F$. We consider the restriction

$$\delta^{-1} : \prod_{i=1}^{r}\{0,1\} \to \mathrm{Irr}(\mathbf{U}^F/[\mathbf{U}^F,\mathbf{U}^F]).$$

Every character in its image can be described by an r-tuple $(a_1,\dots,a_r)$ with $a_i \in \{0,1\}$. Thus, we can also parametrize them by a set $S \subseteq \{1,\dots,r\}$ that contains all $1 \leq i \leq r$ with $a_i = 1$. We denote the corresponding character by ϕ_S and use the same notation for the inflated character of $\mathbf{U}^F$.

From now, we assume that F is a Frobenius endomorphism. Let $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding and $\widetilde{\mathbf{T}}$, $\widetilde{\mathbf{B}}$ as defined in Section 2.6. As already mentioned, this definition slightly differs from the definition of the universal group in [**Mas10**]. We can still translate most constructions to our setting and state the following result from [**Mas10**, Section 8 and 10].

Proposition 5.1. *A complete set of representatives for the orbits of the action of $\widetilde{\mathbf{B}}^F$ on the linear characters of $\mathbf{U}^F$ is given by*

$$\{\phi_S \mid S \subset \{1,\dots,r\}\} \subseteq \mathrm{Lin}(\mathbf{U}^F).$$

Further, all $\phi_S \in \mathrm{Irr}(\mathbf{U}^F)$ extend to their stabilizers in $\widetilde{\mathbf{B}}^F$.

We know that $\mathbf{U}^F \leq \widetilde{\mathbf{B}}^F$ is a p-subgroup. By Clifford theory, every $\psi \in \mathrm{Irr}_{p'}(\widetilde{\mathbf{B}}^F)$ lies over a linear character of $\mathbf{U}^F$. Thus, there exists a subset $S \subseteq \{1,\dots,r\}$ such that $\psi \in \mathrm{Irr}_{p'}(\widetilde{\mathbf{B}}^F \mid \phi_S)$. The character ϕ_S extends to $\widetilde{\mathbf{B}}^F_{\phi_S}$ and the unique Clifford correspondent $\lambda \in \mathrm{Irr}_{p'}(\widetilde{\mathbf{B}}^F_{\phi_S} \mid \phi_S)$ with

$$\mathrm{Ind}_{\widetilde{\mathbf{B}}^F_{\phi_S}}^{\widetilde{\mathbf{B}}^F}(\lambda) = \psi$$

is an extension of ϕ_S.

Therefore, $\psi \in \mathrm{Irr}_{p'}(\widetilde{\mathbf{B}}^F)$ is determined by the set S and the possible extensions of ϕ_S to $\widetilde{\mathbf{B}}^F_{\phi_S}$. Maslowski determined the structure of $\widetilde{\mathbf{B}}^F_{\phi_S}$ and used this to attach a label to each $\psi \in \mathrm{Irr}_{p'}(\widetilde{\mathbf{B}}^F)$. With some additional considerations, one can also find labels for the elements of $\mathrm{Irr}_{p'}(\mathbf{B}^F)$, see [**Mas10**, Section 11].

5.1.2. Global characters and equivariance. We already know that every character $\chi \in \mathrm{Irr}_{p'}(\widetilde{\mathbf{G}}^F)$ is contained in a Deligne–Lusztig series $\mathcal{E}(\widetilde{\mathbf{G}}^F, s)$ for some semisimple $s \in (\widetilde{\mathbf{G}}^*)^{F^*}$ where $(\widetilde{\mathbf{G}}^*, F^*)$ is in duality with $(\widetilde{\mathbf{G}}, F)$.

From Corollary 3.22 we also know that the p-part of the degree of χ equals the p-part of the degree of the corresponding unipotent character of $C_{\widetilde{\mathbf{G}}^*}(s)^{F^*}$. It has been shown in [**Mal07**, Theorem 6.8] that the trivial character is the only p'-character of $C_{\widetilde{\mathbf{G}}^*}(s)^{F^*}$ if $\widetilde{\mathbf{G}}^F$ is none of the groups that have been excluded before. The corresponding unique p'-character in $\mathcal{E}(\widetilde{\mathbf{G}}^F, s)$ is a semisimple character, see e.g. [**Bru09**, Lemma 5].

Thus, the parametrization of the p'-characters of $\widetilde{\mathbf{G}}^F$ corresponds to the parametrization of the $(\widetilde{\mathbf{G}}^*)^{F^*}$-conjugacy classes of semisimple elements of $(\widetilde{\mathbf{G}}^*)^{F^*}$. This can be used to determine labels of $\mathrm{Irr}_{p'}(\widetilde{\mathbf{G}}^F)$ [**Mas10**, Chapter 14]. Note that although the groups $\mathsf{D}_n(2)$ and ${}^2\mathsf{D}_n(2)$ for $n \geq 4$ have also been excluded in [**Mas10**, Proposition 13.1], they still satisfy all needed assumptions by the results mentioned above. Therefore, we can treat them together with the other groups.

In order to verify the inductive McKay condition, Maslowski showed that there is a bijection between the labels of $\mathrm{Irr}_{p'}(\widetilde{\mathbf{G}}^F)$ and $\mathrm{Irr}_{p'}(\widetilde{\mathbf{B}}^F)$ that is equivariant under field and graph automorphisms [**Mas10**, Theorem 15.1]. However, Maslowski did not consider the action of exceptional graph automorphisms and therefore excluded the groups $\mathsf{B}_2(2^i)$, $\mathsf{G}_2(3^i)$, and $\mathsf{F}_4(2^i)$ for $i \geq 1$.

For all other groups, this yields a Γ-equivariant bijection between $\mathrm{Irr}_{p'}(\mathbf{G}^F)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F)$ that preserves central characters [**Mas10**, Theorem 15.4]. In [**Ruh21**, Theorem 5.1 and Theorem 5.7], Ruhstorfer shows that the bijection is additionally equivariant under the action of the Galois automorphisms in $\mathcal{H}$.

This finally completes the explanation why the groups in question have been excluded in [**Ruh21**].

5.2. Groups with exceptional graph automorphisms and the Suzuki and Ree groups

In this section, we consider the groups with exceptional graph automorphisms $\mathsf{B}_2(2^i)$, $\mathsf{G}_2(3^i)$, $\mathsf{F}_4(2^i)$ for $i \geq 2$, and the Suzuki and Ree groups ${}^2\mathsf{B}_2(2^{2f+1})$ for $f \geq 2$, ${}^2\mathsf{G}_2(3^{2f+1})$, ${}^2\mathsf{F}_4(2^{2f+1})$ for $f \geq 1$. We prove that these groups satisfy the inductive McKay–Navarro condition in their defining characteristic. The groups $\mathsf{B}_2(2)$, $\mathsf{G}_2(3)$, $\mathsf{F}_4(2)$, ${}^2\mathsf{B}_2(8)$, ${}^2\mathsf{G}_2(3)$, and ${}^2\mathsf{F}_4(2)'$ will be studied separately in Section 5.4. The group ${}^2\mathsf{B}_2(2)$ is solvable and thus we do not have to consider it further. We follow [**Mas10**] and [**Ruh21**] and extend the results from there.

We now fix the notation we will use throughout this section. Let

$$G \in \left\{\mathsf{B}_2(2^i), \mathsf{G}_2(3^i), \mathsf{F}_4(2^i), {}^2\mathsf{B}_2(2^{2f+1}), {}^2\mathsf{G}_2(3^{2f+1}), {}^2\mathsf{F}_4(2^{2f+1})\right\}$$

for $i \geq 2$ and $f \geq 1$. If $G = {}^2\mathsf{B}_2(2^{2f+1})$, we assume $f \geq 2$. Let p be the defining characteristic 2 or 3, respectively. For the Suzuki and Ree groups, we write $q^2 := p^{2f+1}$. The group G is simple, non-abelian, has trivial Schur multiplier, and trivial center by Proposition 2.54 and [**MT11**, Table 24.2].

Let $\mathbf{G}$ be the corresponding simple algebraic group of type B_2, G_2, or F_4, respectively, defined over an algebraic closure $\mathbf{k}$ of $\mathbb{F}_p$. We denote by F_p the standard field automorphism of $\mathbf{G}$ and by γ the exceptional graph endomorphism of $\mathbf{G}$ as in Proposition 2.43(c). If G is a Suzuki or Ree group, we set $F = F_p^f \circ \gamma$. Otherwise, we set $F = F_p^i$. Then, F is a Steinberg endomorphism such that $\mathbf{G}^F = G$.

We fix a maximally split torus $\mathbf{T} \subseteq \mathbf{G}$ and an F-stable Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. As mentioned in Proposition 2.67, $U = \mathbf{U}^F$ is a Sylow p-subgroup of G with normalizer $B = \mathbf{B}^F$. We denote by Φ the root system of $\mathbf{G}$ with respect to $\mathbf{T}$ and by $\Phi^\vee$ the set of coroots. Let n be the rank of Φ and denote the set of simple roots with respect to $\mathbf{B}$ by $\Delta = \langle \alpha_j \mid 1 \leq j \leq n \rangle$.

5.2.1. Dual fundamental weights. Let $X(\mathbf{T})$ be the character group of $\mathbf{T}$ and $Y(\mathbf{T})$ the group of cocharacters of $\mathbf{T}$. For $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ be the root subgroup of $\mathbf{G}$ associated to α. We fix an isomorphism $u_\alpha : \mathbf{k}^+ \to \mathbf{U}_\alpha$, and let $h_\alpha \in Y(\mathbf{T})$ be as in Theorem 2.36.

For $1 \le j \le n$, let $\omega_j^\vee \in Y(\mathbf{T})$ be the dual fundamental weights as in Definition 2.63. Note that the dual fundamental weights only depend on the type of the root system and the characteristic of $\mathbf{G}$ and not on F.

By definition, we have $\gamma^2 = F_p$ and

$$\gamma(h_\alpha(c)) = \begin{cases} h_{\rho(\alpha)}(c) & \text{if } \alpha \text{ is long,} \\ h_{\rho(\alpha)}(c^p) & \text{if } \alpha \text{ is short,} \end{cases}$$

for all $c \in \mathbf{k}^\times$ where ρ is the permutation of roots in Φ induced by γ.

We follow the ideas in [**Ruh21**] and imitate the considerations that were already made there for Frobenius endomorphisms and groups without exceptional graph automorphisms.

Lemma 5.2. *The action of γ on the dual fundamental weights is given by*

$$\gamma(\omega_j^\vee(c)) = \begin{cases} \omega_{\rho(j)}^\vee(c) & \text{if } \alpha \text{ is long,} \\ \omega_{\rho(j)}^\vee(c^p) & \text{if } \alpha \text{ is short,} \end{cases}$$

for all $c \in \mathbf{k}^\times$. Here, we also use ρ to denote the permutation of the indices of the α_i induced by ρ.

Proof. Since the center of $\mathbf{G}$ is trivial, we can write

$$\omega_j^\vee : \mathbf{k}^\times \to \mathbf{T}, \quad \omega_j^\vee(c) = \prod_{k=1}^{n} h_{\alpha_k}(c^{D_{kj}})$$

with $(D_{kj})_{k,j} \in \mathbb{Q}^{n\times n}$ the inverse of the Cartan matrix of $\mathbf{G}$. For type G_2, the fundamental weights have already been determined in Example 2.65(a). Since h_{α_1} and h_{α_2} commute by Theorem 2.36, we have

$$\gamma(\omega_1^\vee(c)) = \gamma(h_{\alpha_1}(c^2)h_{\alpha_2}(c^3)) = h_{\alpha_2}(c^{2p})h_{\alpha_1}(c^3) = h_{\alpha_1}(c^3)h_{\alpha_2}(c^6) = \omega_2^\vee(c^3),$$
$$\gamma(\omega_2^\vee(c)) = \gamma(h_{\alpha_1}(c)h_{\alpha_2}(c^2)) = h_{\alpha_2}(c^{p})h_{\alpha_1}(c^2) = h_{\alpha_1}(c^2)h_{\alpha_2}(c^3) = \omega_1^\vee(c).$$

For the types B_2 and F_4, this follows by a similar computation using the inverses of the Cartan matrices as in [**Mas10**, Appendix]. □

5.2.2. An equivariant bijection. As already pointed out, we know that the inductive McKay condition holds in defining characteristic. More precisely, we know by [**Bru09**, Theorem 5] that there exists a Γ-equivariant bijection

$$\Omega_{\text{iMcK}} : \mathrm{Irr}_{p'}(\mathbf{G}^F) \to \mathrm{Irr}_{p'}(\mathbf{B}^F)$$

such that (2A) of the inductive McKay–Navarro condition holds. We want to show that this bijection is also $\mathcal{H}$-equivariant.

Now, $(\mathbf{G}, F)$ is self-dual by Example 2.59 except for type B_2 where we have a bijection between rational semisimple elements of $\mathbf{G}$ and its dual with an isomorphism of centralizers [**Lus77**, p. 164]. In the following, we will use this bijection without further notice. The center of $\mathbf{G}$ is trivial by Proposition 2.29. We can consider Deligne–Lusztig characters as introduced in Section 3.1.

As described in Example 2.57, the outer automorphism group of $\mathbf{G}^F$ is generated by γ. If $\mathbf{G}^F$ is a Suzuki or Ree group, it has order $2f+1$ and is equal to $\langle F_p \rangle$; otherwise, it has order $2i$. We already know how the outer automorphisms and Galois automorphisms act on the character sets $\mathcal{E}(\mathbf{G}^F, s)$ by Section 4.1. For the characters in $\mathrm{Irr}_{p'}(\mathbf{G}^F)$, this implies the following statement.

Lemma 5.3. *Let $s \in \mathbf{G}^F$ be semisimple and $\chi \in \mathcal{E}(\mathbf{G}^F, s)$ a character of p'-degree. Let $\sigma \in \mathcal{H}$ such that every p'-root of unity is mapped to its p^k-th power where k is an integer. Then, χ^σ lies in $\mathcal{E}(\mathbf{G}^F, s^{p^k}) = \mathcal{E}(\mathbf{G}^F, F_p^k(s))$ and we have $\chi^\sigma = \chi^{F_p^k}$.*

PROOF. Similar to Lemma 4.2, the first claim follows from the character formula in Proposition 3.12 and the fact that all maximal tori have an order prime to p. As already mentioned in the previous section, we know from [**Bru09**, Lemma 5] that in every Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$ there is only one character of p'-degree. Together with Lemma 4.1, this yields the equality of the characters $\chi^\sigma, \chi^{F_p^k} \in \mathcal{E}(\mathbf{G}^F, s^{p^k})$. □

We show an analogous result for the local case.

Lemma 5.4. *Let $\sigma \in \mathcal{H}$ such that every p'-root of unity is mapped to its p^k-th power where k is an integer. Then, σ acts on $\mathrm{Irr}_{p'}(\mathbf{B}^F)$ in the same way as F_p^k.*

PROOF. If F is a Frobenius map, the claim is included in [**Ruh21**, Remark 5.2]. We consider the Suzuki and Ree groups case by case. First, let $\mathbf{G}^F$ be of type ${}^2\mathsf{B}_2$ or ${}^2\mathsf{G}_2$. Then, $\mathbf{B}^F$ is a Frobenius group with

$$\mathbf{B}^F = \mathbf{U}^F \cdot \mathbf{T}^F \quad \text{ with } \mathbf{U}^F \in \mathrm{Syl}_p(\mathbf{G}^F) \text{ and } \mathbf{T}^F \cong \mathbb{F}_{q^2}^\times$$

by [**HB82**, Sect. XI.3] and [**Eat00**, Proof of Lemma 5]. Thus, the irreducible characters of $\mathbf{B}^F$ consist of the inflations of characters in $\mathrm{Irr}(\mathbf{T}^F)$ and the characters induced by the linear characters of $\mathbf{U}^F$ that are non-trivial.

As in [**IMN07**, (16C)] and [**Eat00**, Proof of Lemma 5], we see that the non-trivial linear characters of $\mathbf{U}^F$ are all in the same orbit under $\mathbf{T}^F$ and therefore induce the same unique irreducible character of $\mathbf{B}^F$ of degree $q^2 - 1$. Thus, we have

$$\mathrm{Irr}_{p'}(\mathbf{B}^F) = \{\mathrm{Inf}_{\mathbf{T}^F}^{\mathbf{B}^F}(\tau) \mid \tau \in \mathrm{Lin}(\mathbf{T}^F)\} \cup \{\psi_0\}$$

for the character $\psi_0 = \mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{B}^F}(\xi)$ with $\xi \in \mathrm{Lin}(\mathbf{U}^F)$ as in Section 5.1.

Since F_p^k acts on the elements of $\mathbf{T}^F$ by mapping them to their p^k-th power and $\mathbf{T}^F$ has order prime to p, F_p^k and σ act in the same way on $\mathrm{Lin}(\mathbf{T}^F)$. This is inherited by the inflated characters. The character ψ_0 is the only character of $\mathbf{B}^F$ of degree $q^2 - 1$ and therefore fixed by F_p^k and σ. Thus, we have $\psi^{F_p^k} = \psi^\sigma$ for all $\psi \in \mathrm{Irr}_{p'}(\mathbf{B}^F)$.

Finally, let $\mathbf{G}^F$ be of type ${}^2\mathsf{F}_4$. With the notation and index sets as in [**HH09**], we have as in [**HH09**, Proof of Lemma 6.1]

$$\mathrm{Irr}_{2'}(\mathbf{B}^F) = \{{}_B\chi_1(k,l)\} \cup \{{}_B\chi_2(k)\} \cup \{{}_B\chi_5(k)\} \cup \{{}_B\chi_8\}.$$

As described there, F_2 acts on these characters by doubling the character parameters. By looking at the explicit character values given in [**HH09**, Table A.6] we see that σ acts on the characters in the same way as F_2^k. □

Proposition 5.5. *The Γ-equivariant bijection Ω_{iMcK} from* [**Bru09**] *is also $\mathcal{H}$-equivariant.*

PROOF. Let $\sigma \in \mathcal{H}$. As we have seen in Lemma 5.3 and Lemma 5.4, every σ acts on $\mathrm{Irr}_{p'}(\mathbf{G}^F)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F)$ in the same way as $F_p^k \in \Gamma$ for some $k \in \mathbb{Z}$. Since Ω_{iMcK} is Γ-equivariant, it follows for all $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}^F)$

$$\Omega_{\mathrm{iMcK}}(\chi^\sigma) = \Omega_{\mathrm{iMcK}}(\chi^{F_p^k}) = \Omega_{\mathrm{iMcK}}(\chi)^{F_p^k} = \Omega_{\mathrm{iMcK}}(\chi)^\sigma.$$

This shows the claim. □

5.2.3. Character extensions. We use the same constructions and notation for the characters ϕ_S and ξ of $\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F$ and $\mathbf{U}^F$ as in Section 5.1. Since $\mathbf{G}$ has trivial center, we have $\mathbf{G} = \widetilde{\mathbf{G}}$. Here, S is a subset of $\{1, \ldots, r\}$ with $r = n$ if F is a Frobenius map and $r = n/2$ if we are in the case of Suzuki and Ree groups.

Recall from Section 3.3 that $\Gamma_1 = \mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{G}^F}(\xi)$ is the Gelfand–Graev character of $\mathbf{G}^F$ and that

$$D_{\mathbf{G}} : \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F) \to \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F)$$

is the Alvis–Curtis duality map for $\mathbf{G}$ as in Definition 3.38.

Lemma 5.6. *For every $\sigma \in \mathcal{G}$ there exists a $t \in \mathbf{T}^\gamma$ such that $\phi_S^\sigma = \phi_S^t$ for all subsets $S \subseteq \{1, \dots, r\}$.*

PROOF. As in the proof of [**Ruh21**, Lemma 6.3(a)], choose an element $b \in \mathbb{F}_p^\times$ such that $\phi_0^\sigma(a) = \phi_0(ba)$ for all $a \in \mathbb{F}_q$. It is shown there that for

$$s_i := \omega_i^\vee(b) \in \mathbf{T}^{F_p} \text{ for } 1 \leq i \leq n, \quad t := \prod_{i=1}^n s_i \in \mathbf{T}^{F_p}$$

we have $\phi_S^t = \phi_S^\sigma$. Since $b^p = b$, it follows with Lemma 5.2 that

$$\gamma(t) = \prod_{i=1}^n \gamma(s_i) = \prod_{i=1,\ \alpha_i \text{ long}}^n \omega_{\rho(j)}^\vee(b) \prod_{i=1,\ \alpha_i \text{ short}}^n \omega_{\rho(j)}^\vee(b^p) = \prod_{i=1}^n \omega_j^\vee(b) = t.$$

Thus, t is γ-invariant. □

In particular, for every $\sigma \in \mathcal{G}$ there exists a $t \in \mathbf{T}^\gamma$ with $\xi^t = \xi^\sigma$. We now show that the global and local characters can be extended to their stabilizers in Γ such that the corresponding μ_a are trivial.

Proposition 5.7. *Let $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}^F)$ and $\sigma \in \mathcal{H}$ mapping every p'-root of unity to its p^k-th power where k is an integer. Then, there exists an extension $\widehat{\chi} \in \mathrm{Irr}(\mathbf{G} \rtimes \langle\gamma\rangle_\chi)$ of χ such that $\widehat{\chi}^{F_p^k t\sigma^{-1}} = \widehat{\chi}$ where $t \in \mathbf{T}^\gamma$ with $\xi^t = \xi^\sigma$.*

PROOF. We know that all characters in $\mathrm{Irr}_{p'}(\mathbf{G}^F)$ are semisimple and by Lemma 5.3 we have $\chi^\sigma = \chi^{F_p^k}$. Since F is a power of γ and γ generates the outer automorphism group of $\mathbf{G}^F$, we can now apply Proposition 4.13 to obtain the claim. □

It remains to show the existence of convenient extensions of the local characters.

Proposition 5.8. *Let $\psi \in \mathrm{Irr}_{p'}(\mathbf{B}^F)$ and $\sigma \in \mathcal{G}$ a Galois automorphism such that we have $\psi^\sigma = \psi^{F_p^e}$ for some $e \in \mathbb{N}$. Let t be an element in $\mathbf{T}^\gamma$ as is Lemma 5.6 and set $x_\sigma := F_p^e \sigma^{-1} t$. Then, there exists an extension $\widehat{\psi} \in \mathrm{Irr}(\mathbf{B}^F \rtimes \langle\gamma\rangle_\psi)$ of ψ such that $\widehat{\psi}^{x_\sigma} = \widehat{\psi}$.*

PROOF. For the groups $\mathsf{B}_2(2^i), \mathsf{G}_2(3^i)$, and $\mathsf{F}_4(2^i)$, this can be shown in the same way as in [**Ruh21**, Prop. 6.10]. For the Suzuki and Ree groups, we give a different proof. Let $\mathbf{G}^F$ be ${}^2\mathsf{B}_2(q^2)$ or ${}^2\mathsf{G}_2(q^2)$. We recall the p'-characters of $\mathbf{B}^F$ from the proof of Lemma 5.4:

$$\mathrm{Irr}_{p'}(\mathbf{B}^F) = \{\mathrm{Inf}_{\mathbf{T}^F}^{\mathbf{B}^F}(\tau) \mid \tau \in \mathrm{Lin}(\mathbf{T}^F)\} \cup \{\mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{B}^F}(\xi)\}.$$

Since the characters $\psi = \mathrm{Inf}_{\mathbf{T}^F}^{\mathbf{B}^F}(\tau)$ are linear, we obtain an extension to $\mathbf{B}^F \rtimes \langle\gamma\rangle_\psi$ that is invariant under $F_p^e \sigma^{-1}$ by Lemma 4.7. These extensions are also invariant under x_σ since $t \in \mathbf{B}^F$.

We now consider the linear character $\xi \in \mathrm{Irr}(\mathbf{U}^F)$. As noted before, ξ is invariant under x_σ and can be extended to $\hat{\xi} \in \mathrm{Irr}(\mathbf{U}^F \rtimes \langle\gamma\rangle)$ such that it is x_σ-invariant. This also gives us an x_σ-invariant extension $\mathrm{Ind}_{\mathbf{U}^F \rtimes \langle\gamma\rangle}^{\mathbf{B}^F \rtimes \langle\gamma\rangle}(\hat{\xi})$ of the character $\mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{B}^F}(\xi)$.

For $\mathbf{G}^F = {}^2\mathsf{F}_4(q^2)$, the characters of $\mathbf{B}^F$ were explicitly constructed in [**HH09**]. As in the proof of Lemma 5.4, we use the notation and index sets from there. Since ${}_B\chi_1(k,l)$ is linear, we can extend it as claimed. With the roots labeled as in [**HH09**], we write

$$H := C_{\mathbf{T}^F}(u_{\alpha_3}(1)u_{\alpha_2}(1)u_{\alpha_2+\alpha_3}(1))\mathbf{U}^F$$

and denote by $\lambda_2(k) \in \mathrm{Irr}(H)$ the character inducing ${}_B\chi_2(k)$ as given in [**HH09**, p. 9]. Then, we can easily see $\lambda_2(k)^{F_2} = \lambda_2(2k)$ and it follows

$$D := \langle F_2 \rangle_{{}_B\chi_2(k)} = \langle F_2 \rangle_{\lambda_2(k)}.$$

The character $\lambda_2(k)$ restricts to a linear character $\phi_{\{1\}} \in \mathrm{Irr}(\mathbf{U}^F)$. The action of $\mathbf{T}^F$ on the root subgroups is described in [**HH09**, Table 2] and H is the inertia subgroup of $\phi_{\{1\}}$ in $\mathbf{B}^F$. By Clifford correspondence, $\lambda_2(k)$ is the unique character of H inducing ${}_B\chi_2(k)$ such that $\lambda_2(k)|_{\mathbf{U}^F}$ has $\phi_{\{1\}}$ as a constituent. Since $\phi_{\{1\}}$ and ${}_B\chi_2(k)$ are x_σ-invariant, the Clifford correspondent $\lambda_2(k)$ is also x_σ-invariant. Thus, we can extend the linear character $\lambda_2(k)$ to an x_σ-invariant character $\hat{\lambda}_2(k) \in \mathrm{Irr}(H \rtimes D)$. It follows that $\mathrm{Ind}_{H \rtimes D}^{\mathbf{B}^F \rtimes D}(\hat{\lambda}_2(k))$ is an x_σ-invariant extension of ${}_B\chi_2(k)$. In the same way, this can be shown for the other characters. □

5.2.4. Verification of the inductive condition. We are now able to verify the inductive Galois–McKay condition for all considered groups in their defining characteristic.

Theorem 5.9. *The groups* $\mathsf{B}_2(2^i)$, $\mathsf{G}_2(3^i)$, $\mathsf{F}_4(2^i)$, ${}^2\mathsf{B}_2(2^{2f+1})$, ${}^2\mathsf{G}_2(3^{2f+1})$ *, and* ${}^2\mathsf{F}_4(2^{2f+1})$ *($i \geq 2$, $f \geq 1$, and $f \geq 2$ in the case of ${}^2\mathsf{B}_2$) satisfy the inductive McKay–Navarro condition in their defining characteristic.*

Proof. Let S be one of the groups above. As already mentioned, the group S is simple, non-abelian, and has trivial Schur multiplier; thus we can consider $G = S$. We want to verify the condition for $N = \mathbf{B}^F$.

Let Ω be the $\Gamma \times \mathcal{H}$-equivariant bijection from Proposition 5.5 and $D = \langle \gamma \rangle$. Let $\chi \in \mathrm{Irr}_{p'}(G)$. We know by Lemma 5.3 that every $\sigma \in \mathcal{H}$ acts on χ in the same way as F_p^k for some $k \in \mathbb{Z}$. Therefore, every $a \in (D \times \mathcal{H})_\chi$ is of the form (dF_p^k, σ^{-1}) for some $d \in D_\chi$. As before, let $x_\sigma := F_p^k t \sigma^{-1}$ with $t \in \mathbf{T}^\gamma$ as in Lemma 5.6. We know by Proposition 4.13 that we find an extension $\widehat{\chi} \in \mathrm{Irr}(G \rtimes D_\chi)$ such that $\widehat{\chi}^{x_\sigma} = \widehat{\chi}$ for all $\sigma \in \mathcal{H}$. As we have $t \in \mathbf{T}^D \subseteq B \subseteq G$ and $d \in D_\chi$, both act trivially on $\widehat{\chi}$ and we have

$$\widehat{\chi} = \widehat{\chi}^{x_\sigma} = \widehat{\chi}^{F_p^k \sigma^{-1}} = \widehat{\chi}^{dF_p^k \sigma^{-1}}.$$

We can now extend $\widehat{\chi}$ to the inner automorphisms in Γ as described in Lemma 4.4. If $\mathcal{P}$ is a representation affording this extended character, then $\mathcal{P}$ and $\mathcal{P}^a$ afford the same character for all $a \in ((B \rtimes \Gamma) \times \mathcal{H})_\chi$ and are thereby similar. Therefore, the μ_a are trivial. Using Proposition 5.8, the same can be done for the local character $\Omega(\chi)$ and the claim follows. □

5.3. The groups $\mathsf{B}_n(2)$ in defining characteristic

In this section, we verify the inductive McKay–Navarro condition for $\mathsf{B}_n(2)$ with an integer $n \geq 4$ and $p = 2$. For $n = 2$ and $n = 3$, the groups have an exceptional Schur multiplier and are treated separately in Proposition 5.13.

The group $S = \mathsf{B}_n(2)$ is the symplectic group $\mathrm{Sp}_{2n}(2)$ and can be defined by

$$\mathrm{Sp}_{2n}(2) := \left\{ A \in \mathrm{GL}_{2n}(2) \mid A^T J_{2n} A = J_{2n} \right\} \text{ where } J_m := \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \in \mathrm{GL}_m(2)$$

for $m \in \mathbb{Z}_{>0}$. Here, A^T is the transpose of A.

5.3.1. Action of Galois automorphisms in the global case. The group S is simple, non-abelian, and has trivial Schur multiplier [**MT11**, Remark 24.19]. Hence, we can take $G = S$ in the inductive McKay–Navarro condition. We already know that the inductive McKay condition holds for S by [**Cab11**, Theorem 5].

We first consider the action of the Galois automorphisms in $\mathcal{H}$ on the characters in $\mathrm{Irr}_{2'}(G)$. Parts of the next proof have been corrected following the suggestions of the anonymous referee of [**Joh22**].

Lemma 5.10. *The Galois automorphisms in $\mathcal{H}$ act trivially on* $\mathrm{Irr}_{2'}(G)$.

PROOF. By [**Lus77**, p. 164], there exists a bijection between the rational semisimple elements of $\mathrm{Sp}_{2n}(2)$ and of its dual $\mathrm{SO}_{2n+1}(2)$ with an isomorphism of centralizers. Thus, the Jordan decomposition of $\chi \in \mathrm{Irr}_{2'}(G)$ can be written as (s, ν) with $s \in G$ semisimple and ν a unipotent character of $C_G(s)$.

Let $\sigma \in \mathcal{H}$ be such that every $2'$-root of unity is mapped to its 2^k-th power for some $k \in \mathbb{Z}$. Since the semisimple elements of G have odd order, χ^σ has Jordan decomposition (s^{2^k}, ν^σ) by Theorem 4.3. As in the proof of [**Cab11**, Proposition 2], we have

$$C_G(s) \cong \mathrm{Sp}_{2j}(2) \times C$$

for some $0 \leq j \leq n$ and C a product of finitely many general linear or general unitary groups. By [**Lus02**, Corollary 1.12], every unipotent character of a group of type A or B is rational-valued and it follows that $\nu^\sigma = \nu$.

It remains to show that s and s^{2^k} are conjugate for all $k \in \mathbb{Z}$. The conjugacy classes of G are uniquely determined by the characteristic polynomials of their elements, see the proof of [**Cab11**, Proposition 2]. Let

$$\pi = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{F}_2[X]$$

be the characteristic polynomial of s with roots $\lambda_1, \ldots, \lambda_n$ over $\overline{\mathbb{F}_2}$. By Vieta's formula, we have $a_{n-i} = e_i(\lambda_1, \ldots, \lambda_n)$ where e_i is the i-th elementary symmetric polynomial in n variables. Applying the standard Frobenius map F_2 yields

$$F_2(a_{n-i}) = a_{n-i} = e_i(\lambda_1^2, \ldots, \lambda_n^2)$$

since all coefficients of the e_i lie in $\mathbb{F}_2$.

We know that the eigenvalues of s^2 over $\overline{\mathbb{F}_2}$ are given by $\lambda_1^2, \ldots, \lambda_n^2$. Again by Vieta's formula, the coefficients of the characteristic polynomial of s^2 coincide with the coefficients of π. Therefore, s and s^2 are conjugate which implies that s and s^{2^k} are conjugate for all $k \in \mathbb{Z}$. This shows that $\chi^\sigma = \chi$. □

5.3.2. Action of Galois automorphisms in the local case. Let $\mathrm{SymM}_n(2)$ be the additive group of symmetric $n \times n$ -matrices over $\mathbb{F}_2$, $U_n(2) \leq \mathrm{GL}_n(2)$ the group of upper triangular unipotent matrices over $\mathbb{F}_2$ and

$$R := \left\{ \begin{pmatrix} x & xsJ_n \\ 0 & J_n(x^{-1})^T J_n \end{pmatrix} \mid x \in U_n(2),\ s \in \mathrm{SymM}_n(2) \right\}.$$

By [**Cab11**, Proposition 3], R is a self-normalizing Sylow-2-subgroup of G. We want to show that the inductive McKay–Navarro condition is satisfied for $N = R$.

Lemma 5.11. *The Galois automorphisms in $\mathcal{H}$ act trivially on* $\mathrm{Irr}_{2'}(R)$.

PROOF. We want to show that all linear characters of R have integer character values. Since all linear characters of R can be obtained as inflated characters from its abelianization R/R', it suffices to show that all linear characters of R/R' have integer character values.

As in [**Cab11**], we see $R \cong \mathrm{SymM}_n(2) \rtimes U_n(2)$ with $U_n(2)$ acting on $\mathrm{SymM}_n(2)$ by $x.s = xsx^T$ for $x \in U_n(2)$ and $s \in \mathrm{SymM}_n(2)$. With the same considerations as in [**Cab11**, proof of Proposition 3], it follows

$$R/R' \cong (\mathrm{SymM}_n(2)/[\mathrm{SymM}_n(2), U_n(2)]) \times (U_n(2)/U_n(2)') \cong (C_2)^{n+1}.$$

Thus, every value of a linear character of R/R' is either 1 or -1 and thereby an integer. Now the claim follows. □

Proposition 5.12. *The inductive McKay–Navarro condition holds for the groups $\mathsf{B}_n(2)$ with $n \geq 4$ in their defining characteristic $p = 2$.*

PROOF. The group $G = \mathsf{B}_n(2)$ has trivial outer automorphism group [**GLS98**, Section 2.5]. Thus, Γ consists of inner automorphisms and acts trivially on all characters in $\mathrm{Irr}_{2'}(G)$ and $\mathrm{Irr}_{2'}(R)$. By Lemma 5.10 and Lemma 5.11, the group $\mathcal{H}$ also acts trivially on these characters. We know $|\mathrm{Irr}_{2'}(G)| = |\mathrm{Irr}_{2'}(R)| = 2^{n+1}$ by [**Cab11**, Section 2]. Thus, there obviously exists a $\Gamma \times \mathcal{H}$-equivariant bijection between $\mathrm{Irr}_{2'}(G)$ and $\mathrm{Irr}_{2'}(R)$. We can canonically extend all characters to the inner automorphisms as in Lemma 4.4. Thereby, the extension part of the inductive McKay–Navarro condition also holds. □

5.4. Groups with non-generic Schur multiplier

In this section, we show that the inductive McKay–Navarro condition is satisfied for the groups $\mathsf{B}_2(2)'$, $\mathsf{G}_2(2)'$, ${}^2\mathsf{G}_2(3)'$, ${}^2\mathsf{F}_4(2)'$, and the finite groups of Lie type with non-generic Schur multiplier in their defining characteristic, see Section 2.5.3. Since these are just finitely many groups, we can do this by explicit computations.

In the following proof, all computations were made with GAP [**GAP19**].

Proposition 5.13. *The inductive McKay–Navarro condition holds for the groups* $\mathsf{B}_2(2)'$, $\mathsf{G}_2(2)'$, ${}^2\mathsf{G}_2(3)'$, ${}^2\mathsf{F}_4(2)'$, *and the simple groups of Lie type with non-generic Schur multiplier in their defining characteristic.*

PROOF. Let S be one of the groups $\mathsf{B}_2(2)'$, $\mathsf{G}_2(2)'$, ${}^2\mathsf{G}_2(3)'$, ${}^2\mathsf{F}_4(2)'$, or a simple group of Lie type with non-generic Schur multiplier and p its defining characteristic. By Lemma 4.15, it suffices to let G be the p'-part of the Schur cover of S. Note that the proof of [**Ruh21**, Theorem 7.3] does not use the fact that G is the universal covering group of S. For all groups except $\mathsf{B}_2(2)'$, the exceptional part of the Schur multiplier is a p-group, see Table 2.3. Therefore, by Lemma 4.15 and [**Ruh21**, Theorem 7.3], the inductive McKay–Navarro condition is satisfied in defining characteristic for S being one of the groups $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_4(2)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(3)$, ${}^2\mathsf{E}_6(2)$, $\mathrm{PSL}_2(9)$, $\mathrm{O}_7(3)$, $\mathrm{PSU}_6(2)$, or $\mathsf{D}_4(2) = \mathrm{O}_8^+(2)$. The remaining groups are $\mathsf{G}_2(2)'$, ${}^2\mathsf{G}_2(3)'$, $\mathrm{Sp}_6(2)$, $\mathsf{G}_2(3)$, ${}^2\mathsf{B}_2(8)$, $\mathsf{F}_4(2)$, ${}^2\mathsf{F}_4(2)'$, and $\mathsf{B}_2(2)'$.

Note that $\mathsf{G}_2(2)'$ has trivial Schur multiplier. If S is one of the groups $\mathrm{Sp}_6(2)$, $\mathsf{G}_2(3)$, ${}^2\mathsf{B}_2(8)$, $\mathsf{G}_2(2)'$, or ${}^2\mathsf{G}_2(3)'$, then the outer automorphism group of S and thereby also of G is cyclic by [**CCN⁺85**]. Let $R \in \mathrm{Syl}_p(G)$, $N = N_G(R)$, and $\Gamma = \mathrm{Aut}(G)_R$. We can explicitly compute the actions of $\Gamma \times \mathcal{H}$ on $\mathrm{Irr}_{p'}(G)$ and on $\mathrm{Irr}_{p'}(N)$ and see that they are permutation isomorphic. Thus, there exists a $\Gamma \times \mathcal{H}$-equivariant bijection Ω between the sets. We find an outer automorphism $\gamma \in \mathrm{Aut}(G)$ such that $\langle\gamma\rangle \cong \mathrm{Out}(G)$ and can compute the character tables of $G \rtimes \langle\gamma\rangle_\chi$ and $N \rtimes \langle\gamma\rangle_\chi$ for every $\chi \in \mathrm{Irr}_{p'}(G)$. If G is not $\mathsf{G}_2(3)$, we always find $(\Gamma \times \mathcal{H})_\chi$-invariant extensions of χ and $\Omega(\chi)$. These characters can be canonically extended to the inner automorphisms in Γ such that they are still $(\Gamma \times \mathcal{H})_\chi$-invariant. Thus, the inductive McKay–Navarro condition is true for these groups.

If $G = \mathsf{G}_2(3)$, we can construct suitable $(\Gamma \times \mathcal{H})_\chi$-invariant extensions for all characters $\chi \in \mathrm{Irr}_{3'}(G)$ and $\Omega(\chi) \in \mathrm{Irr}_{3'}(N)$ except for $\chi_2 \in \mathrm{Irr}_{3'}(G)$ of degree 14 as in [**CCN⁺85**, p.60] and a character $\psi \in \mathrm{Irr}_{3'}(N)$ of degree 2. The actions of both $\mathcal{H}$ and $\langle\gamma\rangle \cong C_2$ are trivial on these characters. Thus, we can assume that Ω maps the characters onto another. We can compute the character tables of $G \rtimes \langle\gamma\rangle$ and $N \rtimes \langle\gamma\rangle$. Since $N \rtimes \Gamma$ acts trivially on all extensions, we see for both χ_2 and ψ

$$\mu_{y\sigma} = \begin{cases} \lambda & \text{if } {\zeta_3}^\sigma = \zeta_3^2, \\ 1 & \text{if } {\zeta_3}^\sigma = \zeta_3, \end{cases}$$

with ζ_3 a primitive third root of unity and $(y, \sigma) \in (N \rtimes \Gamma) \times \mathcal{H}$. Here, λ is the character of $G \rtimes \langle\gamma\rangle$ or $N \rtimes \langle\gamma\rangle$, respectively, given by inflation from the non-trivial character of $\langle\gamma\rangle \cong C_2$, i.e.

$$\lambda(g, \tau) = \begin{cases} 1 & \text{if } \tau = \mathrm{id}, \\ -1 & \text{else}, \end{cases}$$

for all $(g, \tau) \in N \rtimes \langle\gamma\rangle$. Since we can extend the characters canonically to the inner automorphisms in Γ, the inductive McKay–Navarro condition holds.

For $S = \mathsf{F}_4(2)$, the Schur multiplier has order 2 and we can consider $G = S$. We see that $R \in \mathrm{Syl}_2(G)$ is self-normalizing and that the character values of the linear characters of R are integers. The values of the characters in $\mathrm{Irr}_{2'}(G)$ are given in [**CCN⁺85**] and we see that $\mathcal{H}$ acts trivially on both $\mathrm{Irr}_{2'}(G)$ and $\mathrm{Irr}_{2'}(R)$. The outer automorphism group of G is generated by a graph automorphism of order 2 stabilizing R [**GLS98**, Section 2.5]. We can compute the actions of γ on the conjugacy classes of G and N and we see that the actions of γ on $\mathrm{Irr}_{2'}(G)$ and $\mathrm{Irr}_{2'}(R)$ are permutation isomorphic. Thus, for $\chi \in \mathrm{Irr}_{2'}(G)$ we now have either $\Gamma_\chi = \mathrm{Inn}(G \mid R)$ or we can read off the character values of $G \rtimes \langle\gamma\rangle$ from the character table of the split extension $\mathsf{F}_4(2) : 2$ given in [**CCN⁺85**]. As before, we conclude that the group satisfies the inductive McKay–Navarro condition.

We now consider $S = \mathsf{B}_2(2)' \cong \mathrm{PSL}_2(9)$ which has cyclic Schur multiplier of order 6. For $p = 3$, this was already treated above; thus let $p = 2$. Let G be the 3-cover of S, R a Sylow 2-subgroup of G, and $N = N_G(R)$. Note first $\Gamma = \langle \mathrm{Inn}(G \mid N_G(R)), \gamma_1, \gamma_2\rangle$ with $\gamma_1, \gamma_2 \in \mathrm{Aut}(G)_R$ of order 2 but there is no subgroup of Γ that is isomorphic to the outer automorphism group. Thus, we choose a subgroup $\Gamma' \subseteq \Gamma$ of order 8 containing representatives of all outer automorphisms of G. We can explicitly compute the actions of Γ' and $\mathcal{H}$ on $\mathrm{Irr}_{2'}(G)$ and $\mathrm{Irr}_{2'}(N)$ and construct a $\Gamma \times \mathcal{H}$-equivariant bijection. For all $\chi \in \mathrm{Irr}_{2'}(G)$, we can apply Lemma 4.9 and find $(\Gamma \times \mathcal{H})_\chi$-invariant extensions that are trivial on the centralizer of N in $G \rtimes \Gamma'_\chi$. The local characters $\psi \in \mathrm{Irr}_{2'}(N)$ are linear and can be extended trivially by setting $\widehat{\psi}(x, \gamma) = \psi(x)$ for all $(x, \gamma) \in N \rtimes \Gamma'$. As always, these extensions are $(\Gamma \times \mathcal{H})_\psi$-invariant but since Γ' also contains inner automorphisms, we have to make sure that they are trivial on $C_{G \rtimes \Gamma'}(G)$. We have $\widehat{\psi}(x, \gamma) = \psi(x)$ and can explicitly compute $\psi(x) = 1$ for all $x \in N$ inducing inner automorphisms in Γ'. Thus, the corresponding central character is trivial and it follows that the inductive McKay–Navarro condition holds.

For the Tits group $S = G = {}^2\mathsf{F}_4(2)'$, the Schur multiplier is trivial and the extension by the outer automorphism group of order 2 is again not split. However, we find a representative $\gamma \in \Gamma$ of the outer automorphism with $\mathrm{ord}(\gamma) = 4$. As before, we can construct the $2'$-characters of G and R and see that γ acts trivially on them. If $\sigma \in \mathcal{H}$ fixes the fourth root of unity ζ_4, it fixes all considered characters. Otherwise, it acts by interchanging two character pairs of G and R each. Thus, we can find a $\Gamma \times \mathcal{H}$-equivariant bijection. We can apply Lemma 4.9 to get suitable character extensions of the global characters that are invariant under the action of their stabilizers in $\Gamma \times \mathcal{H}$ and have a trivial central character on $C_{G \rtimes \langle\gamma\rangle}(G)$. The linear local characters ψ can again be extended trivially as above and by Lemma 4.6 we also find $(\Gamma \times \mathcal{H})_\psi$-invariant extensions that have a trivial central character on $C_{G \rtimes \langle\gamma\rangle}(G)$. Thus, we have constructed extensions of the local and global characters such that the inductive McKay–Navarro condition is satisfied. □

CHAPTER 6

Suzuki and Ree groups

In this chapter, we verify the inductive McKay–Navarro condition for the Suzuki and Ree groups and all primes. As already mentioned in previous chapters, the Suzuki and Ree groups are the only simple groups of Lie type that do not arise from Frobenius endomorphisms. Since some results about finite groups of Lie type and their characters are only available for Frobenius endomorphism, it is often convenient to restrict our considerations to these. Thereby, the Suzuki and Ree groups often have to be treated separately which we want to do here.

In the first section, we introduce a general parametrization of the irreducible ℓ'-characters of the global and local groups in non-defining characteristic. It has been introduced in [**Mal07**] and works for most groups of Lie type and in particular also for the Suzuki and Ree groups. We use this parametrization to verify the equivariance part of the inductive McKay–Navarro condition and show that there is a Galois-equivariant Jordan decomposition for the Suzuki and Ree groups. In the last section, we construct character extensions that satisfy the extension condition of the inductive McKay–Navarro condition.

With minor changes, this chapter has been published in [**Joh21**, Section 3 and 4].

6.1. Parametrization of ℓ'-characters

Before we concentrate on the Suzuki and Ree groups, we give a short overview of the parametrizations of the local and global characters that we will use in the following. In [**Mal07**], Malle showed that, with some exceptions, we can use normalizers of Sylow d-tori to obtain a good choice of the local subgroup in the inductive McKay condition. Additionally, he used Jordan decomposition of characters and d-Harish-Chandra theory to determine a parametrization of the global and local irreducible ℓ'-characters. This immediately showed that there is a bijection between these character sets. In later works, this bijection has often been used to verify the inductive McKay condition.

In this section, we aim to give the reader an idea how these bijections can be obtained from the theory we have seen so far. We thereby present the basic steps of most proofs.

Let $\mathbf{G}$ be a connected reductive group over an algebraically closed field of characteristic p and F a Steinberg endomorphism of $\mathbf{G}$.

Following [**Mal07**], we first give a parametrization of the irreducible ℓ'-characters of $\mathbf{G}^F$ where ℓ is a prime different from p. We assume that $Z(\mathbf{G})$ is connected since we only stated the Jordan decomposition of characters for this case. However, note that there is a generalization of the following statements to groups with disconnected center that has already been considered in [**Mal07**].

Using the Jordan decomposition of characters with the correspondence of character degrees, we observe the following description of ℓ'-characters of $\mathbf{G}^F$. Let $(\mathbf{G}^*, F^*)$ be in duality with $(\mathbf{G}, F)$.

Proposition 6.1. [**Mal07**, Proposition 7.2] *The irreducible ℓ'-characters of $\mathbf{G}^F$ are parametrized by pairs (s, ν) where*

(a) *$s \in C_{\mathbf{G}^{*F^*}}(R)$ is semisimple up to $\mathbf{G}^{*F^*}$-conjugation for a fixed $R \in \mathrm{Syl}_\ell(\mathbf{G}^{*F^*})$, and*

(b) *$\nu \in \mathcal{E}(C_{\mathbf{G}^{*F^*}}(s), 1)$ such that $\ell \nmid \nu(1)$.*

Proof. We know from the Jordan decomposition of characters that $\mathrm{Irr}(\mathbf{G}^F)$ can be parametrized by pairs (s,ν) where $s \in \mathbf{G}^{*F^*}$ is semisimple up to $\mathbf{G}^{*F^*}$-conjugacy and ν is a unipotent character of $C_{\mathbf{G}^{*F^*}}(s)$. The degree of the character $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ corresponding to (s,ν) is by Corollary 3.22 given by

$$\chi(1) = |\mathbf{G}^{*F^*} : C_{\mathbf{G}^{*F^*}}(s)|_{\ell'} \cdot \nu(1).$$

Thus, χ is an ℓ'-character if and only if $\ell \nmid \nu(1)$ and $\ell \nmid |\mathbf{G}^{*F^*} : C_{\mathbf{G}^{*F^*}}(s)|$. The latter is true if and only if $C_{\mathbf{G}^{*F^*}}(s)$ contains the whole ℓ-part of $\mathbf{G}^{*F^*}$ and thereby a Sylow ℓ-subgroup of $\mathbf{G}^{*F^*}$. This shows the claim. □

To refine this statement, we consider unipotent ℓ'-characters of reductive groups. Let $\mathbf{H} \leq \mathbf{G}$ be an F-stable connected reductive subgroup and let q be the positive real number attached to F. We define $d := d_\ell(q)$ as the order of q modulo ℓ if ℓ is odd and $d := d_2(q)$ as the order of q modulo 4 if $\ell = 2$.

To simplify notation, we state the following results in terms of Sylow d-tori and d-Harish-Chandra series and give the references for the case that F is a Frobenius map. However, all statements still hold for Steinberg maps F and Sylow ϕ-tori, see [**Mal07**, Section 8].

Proposition 6.2. [**Mal07**, Corollary 6.6] *Let $\chi \in \mathrm{Irr}(\mathbf{H}^F)$ be a unipotent character that lies in the d-Harish-Chandra series of the d-cuspidal pair $(\mathbf{L}^F, \lambda)$. Then $\ell \nmid \chi(1)$ if and only if the following are true:*

(a) $\mathbf{L} = C_{\mathbf{H}}(\mathbf{S}_d)$ *for some Sylow d-torus $\mathbf{S}_d$ of $\mathbf{H}$,*

(b) $\lambda \in \mathcal{E}(\mathbf{L}^F, 1)$ *is unipotent, and*

(c) *the degrees of λ and $\mathcal{I}^{-1}(\chi)$ are not divisible by ℓ where*

$$\mathcal{I} : \mathrm{Irr}(\mathbf{W}_{\mathbf{H}^F}(\mathbf{L}, \lambda)) \to \mathcal{E}(\mathbf{H}^F, \mathbf{L}^F, \lambda)$$

is a bijection as in Theorem 3.33.

With this in mind, we can now refine the above parametrization. We assume that $\mathbf{G}$ is simple and fix a Sylow d-torus $\mathbf{S}_d$ of $\mathbf{G}$. Let $\mathbf{C} := C_{\mathbf{G}}(\mathbf{S}_d)$ and $\mathbf{N} := N_{\mathbf{G}}(\mathbf{S}_d)$. We can find a group $\mathbf{C}^* \subseteq \mathbf{G}^*$ that is in duality with $\mathbf{C} \subseteq \mathbf{G}$ and let $\mathbf{S}_d^*$ be the Sylow d-torus of $\mathbf{G}^*$ such that $\mathbf{C}^* = C_{\mathbf{G}^*}(\mathbf{S}_d^*)$. We write $\mathbf{N}^* := N_{\mathbf{G}^*}(\mathbf{S}_d^*)$.

Theorem 6.3. [**Mal07**, Theorem 7.5] *The irreducible ℓ'-characters of $\mathbf{G}^F$ are in bijection with triples (s, λ, η) where*

(a) $s \in \mathbf{C}^{*F^*}$ *is a semisimple element centralizing a Sylow ℓ-subgroup of $\mathbf{N}^{*F^*}$ up to $\mathbf{N}^{*F^*}$-conjugation,*

(b) $\lambda \in \mathcal{E}(C_{\mathbf{C}^{*F^*}}(s), 1)$ *such that $\ell \nmid \lambda(1)$ up to $C_{\mathbf{N}^{*F^*}}(s)$-conjugation, and*

(c) $\eta \in \mathrm{Irr}_{\ell'}(W_{C_{\mathbf{G}^*}(s)^{F^*}}(\mathbf{C}^*, \lambda))$.

Proof (Sketch). We modify the statement from Proposition 6.1. Every semisimple element that centralizes a Sylow ℓ-subgroup also centralizes a Sylow d-torus of $\mathbf{G}$ by [**Mal07**, Theorem 5.9]. From [**Mal07**, Proposition 5.11 and 5.21], we know that it is enough to consider Sylow subgroups and conjugacy in $\mathbf{N}^{*F^*}$. Thus, (a) is equivalent to Proposition 6.1(a).

We now apply Proposition 6.2 to a unipotent ℓ'-character ν of $\mathbf{H} := C_{\mathbf{G}^*}(s)$. Then, ν lies in the d-Harish Chandra series $\mathcal{E}(\mathbf{H}^{F^*}, \mathbf{L}^{F^*}, \lambda)$ where $\mathbf{L}$ is the centralizer of some Sylow d-torus of $\mathbf{H}$ and λ is a unipotent ℓ'-character. Using the Sylow theorems and the fact that conjugate d-cuspidal pairs give rise to the same d-Harish-Chandra series, we can assume $\mathbf{L}^{F^*} = C_{\mathbf{H}^{F^*}}(\mathbf{S}_d^*) = C_{\mathbf{C}^{*F^*}}(s)$. Thus, ν lies in $\mathcal{E}(\mathbf{H}^{F^*}, C_{\mathbf{C}^{*F^*}}(s), \lambda)$ where λ is any unipotent ℓ'-character of $C_{\mathbf{C}^{*F^*}}(s)$. Together with Proposition 6.2(c) and some more identifications of conjugacy actions, we can use the bijection $\mathcal{I}$ to obtain the claim. □

Following [**CS13**], we denote the set of triples (s, λ, η) as in the previous theorem up to $\mathbf{N}^{*F^*}$-conjugacy by $\mathcal{M}$. Then, we have a bijection

$$\varphi^{(G)} : \mathcal{M} \to \mathrm{Irr}_{\ell'}(\mathbf{G}^F).$$

We have a similar correspondence for the local characters that we can even state more concretely. Remember from Section 2.7 that we can choose $N := \mathbf{N}^F$ as the local subgroup in the inductive McKay–Navarro condition. To determine the irreducible ℓ'-characters of N, we first consider the irreducible characters of $\mathbf{C}^F$.

Proposition 6.4. [**Mal07**, Proposition 7.7] *There is a bijection between* $\mathrm{Irr}_{\ell'}(\mathbf{C}^F)$ *and pairs* (s, λ) *where*

(a) $s \in \mathbf{C}^{*F^*}$ *is semisimple and centralizes a Sylow* ℓ*-subgroup of* $\mathbf{C}^{*F^*}$ *up to conjugation in* $\mathbf{C}^{*F^*}$*, and*

(b) $\lambda \in \mathcal{E}(C_{\mathbf{C}^{*F^*}}(s), 1)$ *such that* $\ell \nmid \lambda(1)$.

If $\chi \in \mathrm{Irr}_{\ell'}(\mathbf{C}^F)$ *corresponds to the pair* (s, λ)*, then there is an isomorphism*

$$i_{s,\lambda} : N/N_\chi \to W_{C_{\mathbf{G}^*}(s)^{F^*}}(\mathbf{C}^*, \lambda).$$

Proof. If $\mathbf{C}$ has connected center, we can apply Proposition 6.1 to the group $\mathbf{C}$ to obtain the first part. In particular, this is the case if $\mathbf{C}$ is a torus, i.e. d is a so-called regular number as in [**Spr74**]. Otherwise, we can apply the more general statement in [**Mal07**, Proposition 7.2]. The second part is shown in [**Mal07**, Proposition 7.7] or [**CS13**, Corollary 3.3]. □

By the main result of [**Spä06**], any irreducible character of $\mathbf{C}^F$ extends to its inertia subgroup in N. Therefore, there exists an extension map for $\mathbf{C}^F \trianglelefteq N$ as in Definition 4.12. We can use this to complete the parametrization of the local characters.

Corollary 6.5. [**CS13**, Section 4.1] *Let* Λ *be an extension map for* $\mathbf{C}^F \trianglelefteq N$*. There is a bijection given by*

$$\varphi^{(N)} : \mathcal{M} \to \mathrm{Irr}_{\ell'}(N), \quad (s, \lambda, \eta) \mapsto \mathrm{Ind}_{N_{\chi^{\mathbf{C}}_{s,\lambda}}}^{N} (\Lambda(\chi^{\mathbf{C}}_{s,\lambda})(\eta \circ i_{s,\lambda}))$$

where we denote the character of $\mathbf{C}^F$ *corresponding to* (s, λ) *as in Proposition 6.4 by* $\chi^{\mathbf{C}}_{s,\lambda}$.

Proof. Every $\psi \in \mathrm{Irr}(N)$ lies over an N-conjugacy class of irreducible characters of $C := \mathbf{C}^F$ by Proposition 1.18. Using Clifford correspondence, we obtain a bijection

$$\bigsqcup_{\chi \in \mathrm{Irr}(C)/\sim_N} \mathrm{Irr}(N_\chi \mid \chi) \longrightarrow \bigsqcup_{\chi \in \mathrm{Irr}(C)/\sim_N} \mathrm{Irr}(N \mid \chi) = \mathrm{Irr}(N).$$

Let $\psi \in \mathrm{Irr}(N)$ correspond to $\theta \in \mathrm{Irr}(N_\chi \mid \chi)$ for some $\chi \in \mathrm{Irr}(C)$ via this bijection. Since Clifford correspondence is given by induction, we then have

$$\psi(1) = |N : N_\chi| \cdot \theta(1).$$

Thereby, the above bijection restricts to

$$\bigsqcup_{\substack{\chi \in \mathrm{Irr}(C)/\sim_N \\ |N:N_\chi|_\ell = 1}} \mathrm{Irr}_{\ell'}(N_\chi \mid \chi) \longrightarrow \mathrm{Irr}_{\ell'}(N).$$

Since any irreducible character of C extends to its stabilizer in N, we know by Proposition 1.22 that we have

$$\mathrm{Irr}(N_\chi \mid \chi) = \{\eta\Lambda(\chi) \mid \eta \in \mathrm{Irr}(N_\chi/C)\}$$

for all $\chi \in \mathrm{Irr}(C)$. Therefore, $\mathrm{Irr}_{\ell'}(N)$ can be parametrized by pairs (χ, η) with $\chi \in \mathrm{Irr}_{\ell'}(C)$ where $|N : N_\chi|_\ell = 1$ and $\eta \in \mathrm{Irr}_{\ell'}(N_\chi/C)$. We can now apply Proposition 6.4 to obtain the claim. □

We can now clearly use the maps $\varphi^{(G)}$ and $\varphi^{(N)}$ to obtain a bijection between $\mathrm{Irr}_{\ell'}(G)$ and $\mathrm{Irr}_{\ell'}(N)$. This bijection has the additional property that it preserves the central character over $Z(\mathbf{G}^F)$ by [**Mal07**, Theorem 7.8(c)]. In the remaining part of this chapter, we will apply these constructions to the special case of Suzuki and Ree groups. As already mentioned above, all constructions still hold for this setting with a slightly adjusted terminology.

6.2. Equivariant bijections

In this section, we verify the equivariance part of the inductive McKay–Navarro condition for the Suzuki and Ree groups and all primes. Since the defining characteristic has already been considered in Section 5.2, we only consider other primes.

We first establish some notation that we use throughout the chapter. Let G be one of the Suzuki and Ree groups ${}^2\mathsf{B}_2(2^{2f+1}), {}^2\mathsf{G}_2(3^{2f+1})$, or ${}^2\mathsf{F}_4(2^{2f+1})$ for $f \geq 0$. We denote its defining characteristic by p and let ℓ be a prime dividing $|G|$ different from p. As mentioned in Section 2.5, the Suzuki and Ree groups are simple and have trivial Schur multiplier except for ${}^2\mathsf{B}_2(2), {}^2\mathsf{B}_2(8), {}^2\mathsf{G}_2(3)$, and ${}^2\mathsf{F}_4(2)$ that will be treated separately in Section 7.4.

Let $\mathbf{G}$ be the simple algebraic group of type B_2, G_2, or F_4, respectively, defined over an algebraic closure of $\mathbb{F}_p$. We denote by F_p the standard field endomorphism of $\mathbf{G}$ and by γ the exceptional graph endomorphism of $\mathbf{G}$. Setting $F = F_p^f \circ \gamma$, we obtain $\mathbf{G}^F \cong G$.

We know that $\mathbf{G}^F$ is self-dual, see Example 2.59, and the center of $\mathbf{G}$ is trivial by Proposition 2.29. Thus, we can consider the Jordan decomposition of characters as stated in Theorem 3.21 and identify $(\mathbf{G}, F)$ with its dual group. The outer automorphism group of $\mathbf{G}^F$ is $\langle \gamma \rangle = \langle F_p \rangle$ and has order $2f+1$, see Example 2.57. We already know that we have $\mathcal{E}(\mathbf{G}^F, s)^\kappa = \mathcal{E}(\mathbf{G}^F, \kappa(s))$ for every outer automorphism κ of $\mathbf{G}^F$ and every semisimple element $s \in \mathbf{G}$ by Lemma 5.3.

We first prove the existence of a $\Gamma \times \mathcal{H}$-equivariant bijection as in (1) of the inductive McKay–Navarro condition. We already know that $\mathbf{G}^F$ satisfies the inductive McKay condition for all $\ell \in \mathbb{P}$ by [**IMN07**] and [**CS13**]. More precisely, we know that we can choose N as the normalizer of a Sylow ϕ-torus and have a Γ-equivariant bijection

$$\Omega_{\mathrm{iMcK}} : \mathrm{Irr}_{\ell'}(\mathbf{G}^F) \to \mathrm{Irr}_{\ell'}(N), \quad \varphi^{(G)}(s, \lambda, \eta) \mapsto \varphi^{(N)}(s, \lambda, \eta)$$

for all $(s, \lambda, \eta) \in \mathcal{M}$ as described in the previous section. The Γ-equivariance has been shown in [**CS13**, Theorem 6.1] for type ${}^2\mathsf{F}_4$. For the other types, this can be shown in the same way or by looking at the explicit bijections in [**IMN07**, Section 16 and 17] that coincide with Malle's parametrization.

We want to show that this bijection can also be chosen $\mathcal{H}$-equivariant. In order to do this, we will show that there is a choice for both bijections $\varphi^{(G)}$ and $\varphi^{(N)}$ that is $\Gamma \times \mathcal{H}$-equivariant.

6.2.1. Galois-equivariant Jordan decomposition. In the following, we need a $\Gamma \times \mathcal{H}$-equivariant Jordan decomposition for the irreducible characters of $\mathbf{G}^F$. Therefore, we first prove that Theorem 4.3 continues to hold for the Suzuki and Ree groups.

Proposition 6.6. *Let $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ with Jordan decomposition (s, ν) for $s \in \mathbf{G}^F$ semisimple and ν a unipotent character of $C_{\mathbf{G}^F}(s)$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ be described by $b \in \mathbb{Z}$. Then, χ^σ has Jordan decomposition (s^b, ν^σ).*

Proof. We know by Lemma 4.2 that we have

$$\mathcal{E}(\mathbf{G}^F, s)^\sigma = \mathcal{E}(\mathbf{G}^F, s^b).$$

If χ is a unipotent character, its Jordan decomposition is $(1, \chi)$. Then, χ^σ is also unipotent and its Jordan decomposition is $(1, \chi^\sigma)$ as claimed.

Assume that the characters in $\mathcal{E}(\mathbf{G}^F, s)$ have pairwise distinct degrees. The degrees of the characters in $\mathcal{E}(\mathbf{G}^F, s)$ correspond uniquely to the degrees of the unipotent characters of $C_{\mathbf{G}^F}(s)$ by Corollary 3.22. It follows that ν is fixed by σ and, if $\chi \in \mathcal{E}(\mathbf{G}^F, s)$ has Jordan decomposition (s, ν), then the character χ^σ has Jordan decomposition (s^b, ν).

It remains to consider the characters that are not unipotent and cannot be distinguished in their rational series by their character degrees. We identify and study these cases by looking at the known generic character tables (see e.g. [**GHL**$^+$**96**]).

If $\mathbf{G}^F$ is of type ${}^2\mathsf{B}_2$ or ${}^2\mathsf{G}_2$, no character like this exists. For Ree groups of type ${}^2\mathsf{F}_4$, we use the notation of characters as in [**GHL**$^+$**96**] and denote the characters of type j by $\chi_j(k)$. We do not specify the parameter k here but assume it to be as described in [**GHL**$^+$**96**]. The only characters that still have to be considered are

$$\mathcal{E}(\mathbf{G}^F, s_2(k)) = \{\chi_{22}(k), \chi_{23}(k), \chi_{24}(k), \chi_{25}(k)\}$$

where we denote the semisimple elements $s_j(k)$ as labeled in [**GHL**$^+$**96**]. As before, it is clear that the claim holds for $\chi_{22}(k)$ and $\chi_{25}(k)$ since they are determined by their degrees. By looking at the explicit character values given in [**GHL**$^+$**96**] we see

$$\chi_{23}(k)^\sigma = \begin{cases} \chi_{23}(bk) & \text{if } \zeta_4^\sigma = \zeta_4, \\ \chi_{24}(bk) & \text{if } \zeta_4^\sigma = -\zeta_4, \end{cases} \qquad \chi_{24}(k)^\sigma = \begin{cases} \chi_{24}(bk) & \text{if } \zeta_4^\sigma = \zeta_4, \\ \chi_{23}(bk) & \text{if } \zeta_4^\sigma = -\zeta_4, \end{cases}$$

where ζ_4 denotes a primitive fourth root of unity.

In the notation of [**SI75**] and using the Steinberg presentation, the semisimple elements $s_2(k)$ correspond to elements that are $\mathbf{G}^F$-conjugate to

$$t_1(k) = \left(1, 1, \zeta_{q^2-1}^{k}, \zeta_{q^2-1}^{k(2^{f+1}-1)}\right)$$

with ζ_{q^2-1} a primitive (q^2-1)-th root of unity. We see that $s_2(k)^b = s_2(bk)$.

Now, the centralizers $\mathbf{H} := C_{\mathbf{G}}(s_2(k))$ are of type ${}^2\mathsf{B}_2$ and thus have four unipotent characters $1_{\mathbf{H}^F}, \chi_1, \chi_2, \chi_{\mathrm{St}}$ as described in Example 3.23. Their values are given in [**Gec03**, Theorem 4.6.9] and we have

$$1_{\mathbf{H}^F}^\sigma = 1_{\mathbf{H}^F}, \quad \chi_1^\sigma = \begin{cases} \chi_1 & \text{if } \zeta_4^\sigma = \zeta_4, \\ \chi_2 & \text{if } \zeta_4^\sigma = -\zeta_4, \end{cases} \quad \chi_2^\sigma = \begin{cases} \chi_2 & \text{if } \zeta_4^\sigma = \zeta_4, \\ \chi_1 & \text{if } \zeta_4^\sigma = -\zeta_4, \end{cases} \quad {\chi_{\mathrm{St}}}^\sigma = \chi_{\mathrm{St}}.$$

This shows the claim for groups of type ${}^2\mathsf{F}_4$. □

6.2.2. Galois equivariance of Malle's bijection. We now want to show that the bijections $\varphi^{(N)}$ and $\varphi^{(G)}$ can be chosen $\Gamma \times \mathcal{H}$-equivariant.

First, we prove a number-theoretical lemma about the prime divisors of some cyclotomic polynomials.

Lemma 6.7. *Assume $q^2 = 2^{2f+1}$ for an integer $f \geq 1$ and that ℓ is an odd prime.*

(a) *If $\ell \mid q^2 - 1$, then $\ell \equiv 1, 7 \mod 8$.*
(b) *If $\ell \mid q^4 - q^2 + 1$ and $\ell \neq 3$, then $\ell \equiv 1 \mod 3$.*
(c) *If $\ell \mid q^4 \pm \sqrt{2}q^3 + q^2 \pm \sqrt{2}q + 1$ and $\ell \neq 3$, then $\ell \equiv 1, 11 \mod 12$.*
(d) *If $\ell \mid q^2 \pm \sqrt{2}q + 1$, then $\ell \equiv 1 \mod 4$.*

Proof. We know by Fermat's little theorem that we have $2^{\ell-1} \equiv 1 \mod \ell$.

(a) We have $q^2 - 1 = 2^{2f+1} - 1 \equiv 0 \mod \ell$, thus $2^{2f+1} \equiv 1 \mod \ell$. It follows

$$2^{\gcd(2f+1, \ell-1)} = 2^{\gcd(2f+1, \frac{\ell-1}{2})} \equiv 1 \mod \ell$$

and since ℓ is odd further $2^{(\ell-1)/2} \equiv 1 \mod \ell$. The claim follows from the second supplement to quadratic reciprocity.

(b) Assume that we have additionally $\ell \mid q^4 - 1$. Then it follows $\ell \mid q^2 + 1$ or $\ell \mid q^2 - 1$. Now $\ell \mid q^4 - q^2 + 1$ implies $\ell \mid q^2 - 2$ which is not possible at the same time for $\ell \neq 3$. Thus, we know $\ell \nmid q^4 - 1$. Since $\ell \mid q^4 - q^2 + 1$, it follows

$$\ell \mid q^6 + 1 = (q^4 - q^2 + 1)(q^2 + 1)$$

and $2^{3(2f+1)} \equiv -1 \mod \ell$. Together with Fermat's little theorem, we have

$$2^{\gcd(\ell-1, 6(2f+1))} \equiv 1 \mod \ell.$$

If $\ell \equiv 2 \mod 3$, it follows $\gcd(\ell - 1, 6(2f + 1)) = \gcd(\ell - 1, 2(2f + 1))$ which implies $2^{2(2f+1)} = q^4 \equiv 1 \mod \ell$ and $\ell \mid q^4 - 1$. This is a contradiction and since $\ell \neq 3$, the claim holds.

(c) We know

$$\ell \mid (q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1)(q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1)(q^4 + 1) = q^{12} + 1$$

and as before it follows

$$(*) \quad 2^{\gcd(6(2f+1),\ell-1)} \equiv -1 \mod \ell, \quad 2^{\gcd(12(2f+1),\ell-1)} \equiv 1 \mod \ell.$$

Since $\ell \in \mathbb{P}$, only $\ell \equiv 1, 5, 7, 11 \mod 12$ are possible. If we have $\ell \equiv 5 \mod 12$, then it follows $2^{4\gcd(2f+1, \frac{\ell-1}{4})} \equiv 1 \mod \ell$. If we have $\ell \equiv 7 \mod 12$, then it follows $2^{6\gcd(2f+1, \frac{\ell-1}{6})} \equiv 1 \mod \ell$. These equations contradict $(*)$, thus we have shown the claim.

(d) Let $\ell \mid (q^2 + \sqrt{2}q + 1)(q^2 - \sqrt{2}q + 1) = q^4 + 1$. Then it follows $2^{2(2f+1)} \equiv -1 \mod \ell$. If we have $\ell \equiv 3 \mod 4$, then we get as before $2^{2\gcd(2f+1, \frac{\ell-1}{2})} \equiv 1 \mod \ell$ which leads again to a contradiction. The claim follows. □

In the following, we write $\phi^{(\ell)}$ for the cyclotomic polynomial over $\mathbb{Z}[\sqrt{p}]$ as defined in Theorem 2.71 (2). We assume $\ell \neq 3$ if $q^2 \equiv 2, 5 \mod 9$ and $\mathbf{G}^F = {}^2\mathsf{F}_4(q^2)$, and $\ell \neq 2$ if $\mathbf{G}^F = {}^2\mathsf{G}_2(q^2)$. The excluded groups and primes will be considered separately in Proposition 7.1.

Let Q be a Sylow ℓ-subgroup of $\mathbf{G}^F$. By Theorem 2.71, there exists a Sylow $\phi^{(\ell)}$-torus $\mathbf{S}_0$ of $\mathbf{G}$ such that $N_{\mathbf{G}^F}(Q) \leq N_{\mathbf{G}^F}(\mathbf{S}_0)$. Let $\mathbf{T} = C_{\mathbf{G}}(\mathbf{S}_0)$ be its centralizer in $\mathbf{G}$. We know from the proof of [**Spä09**, Lemma 6.5] that $\mathbf{T}$ is a maximal torus.

We first consider the map $\mathcal{I}$ that has been introduced in Theorem 3.33. As we can already presume from the construction of $\varphi^{(G)}$ in the previous section, this will be important in order to show that the parametrization of the global characters is $\Gamma \times \mathcal{H}$-equivariant.

Proposition 6.8. *Let* $\mathbf{T}$ *be as above and* $s \in \mathbf{T}^F$ *semisimple. For* $\mathbf{H} = C_{\mathbf{G}}(s)$*, there exists a* $\Gamma \times \mathcal{H}$*-equivariant bijection*

$$\mathcal{I} : \mathrm{Irr}(\mathbf{W}_{\mathbf{H}^F}(\mathbf{T})) \to \mathcal{E}(\mathbf{H}^F, \mathbf{T}^F, 1).$$

Proof. We first consider ${}^2\mathsf{F}_4(q^2)$. As in [**CS13**, Proof of Theorem 6.1], we know that $(\mathbf{H}, F)$ is of type ${}^2\mathsf{F}_4$, ${}^2\mathsf{B}_2$, ${}^2\mathsf{A}_2$, A_1, or A_0. From there we also know that the action of Γ on all relevant characters is trivial and that there exists a bijection between the character sets.

As already mentioned in the proof of Proposition 6.6, the unipotent characters of $\mathbf{H}^F$ can be distinguished by their degrees if $(\mathbf{H}, F)$ is of type ${}^2\mathsf{A}_2$, A_1, or A_0. Thus, $\mathcal{H}$ acts trivially on these unipotent characters.

If $(\mathbf{H}, F)$ is of type A_0, the corresponding relative Weyl group is trivial and $\mathcal{H}$ acts trivially on its unique irreducible character.

If $(\mathbf{H}, F)$ is of type A_1, all F-stable maximal tori $\mathbf{T}$ are minimal d-split Levi subgroups for a $d \in \{1, 2\}$. The structure of $\mathbf{W}_{\mathbf{H}}(\mathbf{T})^F$ is described in [**GM20**, Example 3.5.14(b)] and we see $|\,\mathbf{W}_{\mathbf{H}}(\mathbf{T})^F| = 2$. Thus, its characters have integer character values and $\mathcal{H}$ acts trivially.

If $\mathbf{H}^F$ is the group ${}^2\mathsf{A}_2(q^2)$, then all F-stable maximal tori $\mathbf{T}$ are minimal d-split Levi subgroups for a $d \in \{1, 2, 6\}$. The structure of $\mathbf{W_H}(\mathbf{T})^F$ is described in [**GM20**, Example 3.5.14 (c)]. For $d = 1$ and $d = 2$, we have $|\mathbf{W_H}(\mathbf{T})^F| = 2$ and $\mathbf{W_H}(\mathbf{T})^F \cong \mathrm{Sym}(3)$, respectively. Thus, their characters have integer values and $\mathcal{H}$ acts trivially. If $d = 6$, then $\mathbf{W_H}(\mathbf{T})^F \cong C_3$ and $\mathbf{S}_0$ is by definition a Sylow 6-torus of $\mathbf{H}$ with

$$|\mathbf{S}_0^F| = q^4 - q^2 + 1.$$

Thus, we have $\ell \mid q^4 - q^2 + 1$ and by Lemma 6.7 we know $\ell \equiv 1 \mod 3$. It follows that $\mathcal{H}$ acts trivially on all characters of $\mathbf{W_H}(\mathbf{T})^F$.

If $(\mathbf{H}, F)$ is of type ${}^2\mathsf{B}_2$, the structure of all possible maximal tori $\mathbf{T}$ is given in Example 2.51 or [**Gec03**, Table 4.4]. We use the notation introduced in Example 2.51. If $\mathbf{T} = \mathbf{T}_0$, then we have

$$\mathbf{W_H}(\mathbf{T})^F \cong C_2, \quad \mathcal{E}(\mathbf{H}^F, \mathbf{T}^F, 1) = \{1_{\mathbf{H}^F}, \chi_{\mathrm{St}}\}$$

by Example 3.23. We see that $\mathcal{H}$ acts trivially on all occurring characters. If $\mathbf{T}$ is of type s_1 or $s_1 s_2 s_2$, then

$$\mathbf{W_H}(\mathbf{T})^F \cong C_4, \quad \mathcal{E}(\mathbf{H}^F, \mathbf{T}, 1) = \{1_{\mathbf{H}^F}, \chi_1, \chi_2, \chi_{\mathrm{St}}\}.$$

As in the proof of Proposition 6.6 we see that $\sigma \in \mathcal{H}$ acts trivially on both sets if a primitive fourth root of unity is fixed and interchanges two characters otherwise. Therefore, we find a bijection as claimed.

Assume that $(\mathbf{H}, F)$ is of type ${}^2\mathsf{F}_4$. The F-stable maximal tori of $\mathbf{G}$ are described in [**SI75**]. We use the same notation and denote the corresponding fixed point sets by $T(1), \dots, T(11)$. Looking at the order of $\mathbf{G}^F$ in Table 2.2, we see that we have the following Sylow $\phi^{(\ell)}$-tori:

$$\begin{gathered}
\ell \mid q^2 - 1 \text{ and } \mathbf{T}^F \cong T(1),\\
\ell \mid q^2 - \sqrt{2}q + 1 \text{ and } \mathbf{T}^F \cong T(6),\\
\ell \mid q^2 + \sqrt{2}q + 1 \text{ and } \mathbf{T}^F \cong T(7),\\
\ell \mid q^2 + 1 \text{ and } \mathbf{T}^F \cong T(8),\\
\ell \mid q^4 - q^2 + 1 \text{ and } \mathbf{T}^F \cong T(9),\\
\ell \neq 3,\ \ell \mid q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1 \text{ and } \mathbf{T}^F \cong T(10), \text{ or}\\
\ell \neq 3,\ \ell \mid q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1 \text{ and } \mathbf{T}^F \cong T(11).
\end{gathered}$$

The 21 unipotent characters of $\mathbf{G}^F$ are described in [**Mal90a**]. We denote them by $\chi_1, \dots, \chi_{21}$ and enumerate them as in the database of [**GHL$^+$96**]. By explicitly studying the character values, we see that σ acts on the characters by permuting the indices as

$$\begin{array}{rl}
(15,16) & \text{if } \zeta_{12}^\sigma = \zeta_{12}^5,\\
(2,3)(11,12)(13,14)(19,20) & \text{if } \zeta_{12}^\sigma = \zeta_{12}^7,\\
(2,3)(11,12)(13,14)(15,16)(19,20) & \text{if } \zeta_{12}^\sigma = \zeta_{12}^{11}.
\end{array}$$

Here, ζ_{12} is a primitive 12-th root of unity. For every possible Sylow torus $\mathbf{T}$ of $\mathbf{G}$, the elements of $\mathcal{E}(\mathbf{H}^F, \mathbf{T}^F, 1)$ are identified in [**Lus84**, p.374]. The corresponding relative Weyl group can be determined using the data in [**SI75**] or is given in [**BMM93**, Table 3].

First, let $\ell \mid q^2 - 1$. Then, the relative Weyl group is isomorphic to the dihedral group of order 16. Constructing its characters, we see by using Lemma 6.7(a) that $\mathcal{H}$ acts trivially on them. The same is true for the characters in

$$\mathcal{E}(\mathbf{H}^F, \mathbf{T}^F, 1) = \{\chi_1, \chi_4, \chi_5, \chi_6, \chi_7, \chi_{18}, \chi_{21}\}.$$

For $\ell \mid q^2 \pm \sqrt{2}q + 1$, the relative Weyl group is isomorphic to the complex reflection group with Shephard–Todd number 8 and has the group structure $C_4.\mathrm{Sym}(4)$. We can compute its character table with GAP and see with Lemma 6.7(d) that $\mathcal{H}$ acts trivially on its characters. Since $\ell \equiv 1 \mod 4$, it is easy to see that the same is true for the characters in $\mathcal{E}(\mathbf{H}^F, \mathbf{T}^F, 1)$.

For $\ell \mid q^2 + 1$, the relative Weyl group is isomorphic to $\mathrm{GL}_2(3)$. Constructing its characters we see that $\mathcal{H}$ acts trivially on them and the same is true for $\mathcal{E}(\mathbf{H}^F, \mathbf{T}^F, 1)$.

For $\ell \mid q^4 - q^2 + 1$, the relative Weyl group is isomorphic to C_6. Since $\ell \equiv 1 \mod 3$ by Lemma 6.7(b), we see that $\mathcal{H}$ acts trivially on the characters of the Weyl group and on $\mathcal{E}(\mathbf{H}^F, \mathbf{T}^F, 1)$.

For $\ell \mid q^4 \pm \sqrt{2}q^3 + q^2 \pm \sqrt{2}q + 1$, the relative Weyl group is isomorphic to C_{12}. Since $\ell \equiv 1, 11 \mod 12$ by Lemma 6.7(c), it is clear that $\mathcal{H}$ acts trivially on the characters of the Weyl group and on $\mathcal{E}(\mathbf{H}^F, \mathbf{T}^F, 1)$. With this, we have shown the claim for ${}^2\mathsf{F}_4(q^2)$.

We now study the Suzuki groups ${}^2\mathsf{B}_2(q^2)$. Then, we know from [**DL85**, p. 52] that $(\mathbf{H}, F)$ is of type A_0 or ${}^2\mathsf{B}_2$. Thus, we have already shown the existence of a suitable bijection before.

For ${}^2\mathsf{G}_2(q^2)$, $(\mathbf{H}, F)$ is of type A_0, A_1 or ${}^2\mathsf{G}_2$ [**DL85**, p. 52]. The cases A_0 and A_1 can be treated as above. If $(\mathbf{H}, F)$ is of type ${}^2\mathsf{G}_2(q^2)$, there are four different types of tori $\mathbf{T}$. The respective sets $\mathcal{E}(\mathbf{H}^F, \mathbf{T}^F, 1)$ are given in [**Lus84**, p. 376]. If $\mathbf{T}^F$ is of order $q^2 - 1$, there are two characters in $\mathcal{E}(\mathbf{H}^F, \mathbf{T}^F, 1)$ and they are both $\mathcal{H}$-invariant. For all other tori, there are six characters in $\mathcal{E}(\mathbf{H}^F, \mathbf{T}^F, 1)$ and $\sigma \in \mathcal{H}$ induces a double transposition if a third root of unity is not preserved by σ and fixes them otherwise.

The relative Weyl group is cyclic by [**BMM93**, Table 3] and thus isomorphic to C_2 or C_6. Thus, it is easy to see that the action of $\mathcal{H}$ on the characters of the Weyl group is the same as on the respective unipotent characters. This proves the claim. □

We now consider the "ingredients" of the parametrization of the local characters $\varphi^{(N)}$. First, we choose a suitable extension map for $\mathbf{T}^F \lhd N_{\mathbf{G}}(\mathbf{T})^F$.

Lemma 6.9. *For* $\mathbf{T}$ *as defined above, there exists a* $\Gamma \times \mathcal{H}$*-equivariant extension map for* $\mathbf{T}^F \lhd N_{\mathbf{G}}(\mathbf{T})^F$.

Proof. If q^2 is even, this follows directly from the construction in [**Spä09**, Lemma 4.2] together with the existence of a very good Sylow twist [**Spä09**, Theorem D] and the fact that H as in [**Spä09**, Setting 2.1] is trivial.

For type ${}^2\mathsf{G}_2$, we have $N_{\mathbf{G}^F}(\mathbf{T}) = \mathbf{T}^F \rtimes C$ for some cyclic group C (see the proof of Proposition 6.8, [**Kle88**, Theorem C], and [**LN85**, p. 20]). Every irreducible character of $\mathbf{T}^F$ is linear and we can extend it trivially to its inertia subgroup in $N_{\mathbf{G}^F}(\mathbf{T})$. This extension is clearly $\Gamma \times \mathcal{H}$-equivariant. □

We also have to consider the isomorphisms $i_{s,\lambda}$ from Proposition 6.4 in the special situation that $\lambda \in \mathrm{Irr}(\mathbf{T}^F, 1)$ is the trivial character.

Remark 6.10. Note that, for $s \in \mathbf{T}^F$ semisimple and $k \in \mathbb{Z}$ coprime to $\mathrm{ord}(s)$, the maps $i_{s,1}$ and $i_{s^k,1}$ constructed in [**CS13**, Proof of Corollary 3.3] are the same. The equality of domain and codomain follows directly from the coprimeness. We see that the maps are equal since for $t \in \mathbf{T}$ with $tst^{-1} = s'$ we also have $t(s^k)t^{-1} = (s')^k$.

Finally, we are able to show that the bijection Ω_{iMcK} from above is also equivariant under the action of Galois automorphisms in $\mathcal{H}$.

Proposition 6.11. *With* $\mathbf{T}$ *defined as above, there exists a* $\Gamma \times \mathcal{H}$*-equivariant bijection*

$$\Omega : \mathrm{Irr}_{\ell'}(\mathbf{G}^F) \to \mathrm{Irr}_{\ell'}(N_{\mathbf{G}^F}(\mathbf{T})).$$

Proof. We follow the proofs of [**CS13**, Theorem 4.5 and Theorem 6.1]. As described above, the ℓ'-characters of $\mathbf{G}^F$ and of $N := N_{\mathbf{G}^F}(\mathbf{T}) = N_{\mathbf{G}^F}(\mathbf{S}_0)$ can be parametrized by triples (s, λ, η) where $s \in \mathbf{T}^F$ is semisimple, $\lambda \in \mathcal{E}(\mathbf{T}^F, 1)$ is an ℓ'-character, and

$\eta \in \mathrm{Irr}_{\ell'}(\mathbf{W_H}(\mathbf{T},\lambda)^F)$ for $\mathbf{H} = C_{\mathbf{G}}(s)$. Since $\mathbf{T}^F$ is a torus, λ is always the trivial character and we set

$$\mathcal{M}_0 = \{(s,\eta) \mid s \in \mathbf{T}^F \text{ semisimple up to } N\text{-conjugacy}, \eta \in \mathrm{Irr}_{\ell'}(\mathbf{W_H}(\mathbf{T})^F)\}.$$

We consider the bijections

$$\varphi^{(N)} : \mathcal{M}_0 \to \mathrm{Irr}_{\ell'}(N_{\mathbf{G}^F}(\mathbf{T})), \quad \varphi^{(G)} : \mathcal{M}_0 \to \mathrm{Irr}_{\ell'}(\mathbf{G}^F)$$

as described in Section 6.1. As in the proof of [**CS13**, Theorem 4.5], we want to show that these bijections can be chosen such that

$$\left(\varphi^{(N)}(s,\eta)^{\gamma'}\right)^\sigma = \varphi^{(N)}\left(\gamma'(s^k),(\eta^{\gamma'})^\sigma\right), \quad \left(\varphi^{(G)}(s,\eta)^{\gamma'}\right)^\sigma = \varphi^{(G)}\left(\gamma'(s^k),(\eta^{\gamma'})^\sigma\right)$$

for any $\gamma' \in \Gamma$ and $\sigma \in \mathcal{H}$ described by k. We already know that the maps $\varphi^{(N)}$ and $\varphi^{(G)}$ as we consider them here are Γ-equivariant from [**CS13**, Theorem 6.2]. Thus, we only have to verify the $\mathcal{H}$-equivariance.

We denote by $\chi_s^{\mathbf{T}}$ the irreducible character of $\mathbf{T}^F$ with Jordan decomposition $(s,1)$ and by $\chi_{s,\nu}^{\mathbf{G}}$ the one of $\mathbf{G}^F$ with Jordan decomposition (s,ν). First, note that the Jordan decomposition can be chosen $\mathcal{H}$-equivariant by Proposition 6.6. Denote by

$$\Lambda : \mathrm{Irr}(\mathbf{T}^F) \to \bigcup_{\mathbf{T}^F \leq I \leq N_{\mathbf{G}^F}(\mathbf{T})} \mathrm{Irr}(I)$$

the extension map from Lemma 6.9. Then, we have

$$\begin{aligned}
\varphi^{(N)}(s,\eta)^\sigma &= \mathrm{Ind}_{N_{\chi_s^{\mathbf{T}}}}^N \left(\Lambda(\chi_s^{\mathbf{T}})(\eta \circ i_{s,1})\right)^\sigma = \mathrm{Ind}_{N_{\chi_s^{\mathbf{T}}}}^N \left(\Lambda(\chi_s^{\mathbf{T}})^\sigma(\eta \circ i_{s,1})^\sigma\right) \\
&= \mathrm{Ind}_{N_{\chi_s^{\mathbf{T}}}}^N \left(\Lambda((\chi_s^{\mathbf{T}})^\sigma)(\eta^\sigma \circ i_{s,1})\right) = \mathrm{Ind}_{N_{\chi_{s^k}^{\mathbf{T}}}}^N \left(\Lambda(\chi_{s^k}^{\mathbf{T}})(\eta^\sigma \circ i_{s^k,1})\right) \\
&= \varphi^{(N)}(s^k,\eta^\sigma).
\end{aligned}$$

For the characters of $\mathbf{G}^F$, let $\mathcal{I}$ be the map from Proposition 6.8. Then it follows

$$\varphi^{(G)}(s,\eta)^\sigma = (\chi_{s,\mathcal{I}(\eta)}^{\mathbf{G}})^\sigma = \chi_{s^k,\mathcal{I}(\eta)^\sigma}^{\mathbf{G}} = \chi_{s^k,\mathcal{I}(\eta^\sigma)}^{\mathbf{G}} = \varphi^{(G)}(s^k,\eta^\sigma).$$

This shows the claim. □

6.3. Character extensions

In this section, we verify the extension condition (2B) of the inductive McKay–Navarro condition. We first use Deligne–Lusztig theory for disconnected groups as in Section 3.4 to construct suitable extensions of the global characters. After this, we consider extensions of the local characters case by case and finally verify the inductive McKay–Navarro condition for the Suzuki and Ree groups and all primes.

We continue to use the notation that was introduced in the previous section and let $N = N_{\mathbf{G}^F}(\mathbf{S}_0)$ for a Sylow $\phi^{(\ell)}$-torus $\mathbf{S}_0$ of $\mathbf{G}$. For the construction of suitable character extensions, it will be helpful to specify a small group of outer automorphisms in Γ. Since γ does not necessarily stabilize a chosen Sylow ℓ-subgroup of $\mathbf{G}^F$ contained in N, it might not be contained in Γ and we have to consider a conjugate outer automorphism. Let $D \leq \Gamma$ be a cyclic subgroup of Γ containing representatives for all outer automorphisms in Γ and let γ' be a generator of D. We can assume that D has odd order since we can otherwise consider $\langle \gamma'^{|D|_2} \rangle$.

6.3.1. Global characters. In order to find suitable extensions of the global characters, we construct character extensions of some Deligne–Lusztig characters. Again, we use the method of the descent of scalars as described in Section 4.3.

Proposition 6.12. *Let χ be a unipotent character of $\mathbf{G}^F$. There exists an $\mathcal{H}_\chi$-invariant extension of χ to $\mathbf{G}^F \rtimes \Gamma$ such that all associated scalars on $C_{G\rtimes\Gamma}(G)$ are trivial.*

PROOF. We know that Γ acts trivially on all unipotent characters of $\mathbf{G}^F$ by [**Mal08a**, Theorem 2.5]. If χ is the only character of its degree, it is real and we can extend it to D as in Proposition 4.8 and then to the inner automorphisms in Γ. By Lemma 4.6, we can assume that the associated scalars on $C_{G\rtimes\Gamma}(G)$ are trivial.

Otherwise, we can always find an F-stable Sylow torus $\mathbf{S}$ of $\mathbf{G}$ such that χ is a constituent of the Deligne–Lusztig character $\beta := R_{\mathbf{S}}^{\mathbf{G}}(1_{\mathbf{S}^F})$ by the tables in [**Lus84**, pp. 373–376]. We can choose $\mathbf{S}$ such that all constituents of β have multiplicity ± 1. We want to use the construction of Deligne–Lusztig characters for disconnected groups to obtain an extension of β and identify a constituent that extends χ.

Step 1: Twisting the setting. First, we have to adjust the setting such that we can use the descent of scalars from Section 4.3. Since all Sylow tori corresponding to the same cyclotomic polynomial are conjugate in $\mathbf{G}^F$, we find a $g \in \mathbf{G}^F$ such that $(\gamma \circ c_g)(\mathbf{S}) = \mathbf{S}$ where c_g denotes conjugation with g. We set $F_0 := \gamma \circ c_g$ and $k := 2f + 1$. We know $\gamma^k = F$ and

$$F_0^k = c_{g^*} \circ F = F \circ c_{g^*}$$

with

$$g^* := \gamma(g) \cdots \gamma^{k-1}(g)\gamma^k(g) \in \mathbf{G}^F.$$

By the Lang–Steinberg theorem, we find an $h \in \mathbf{G}$ such that $g^* = h^{-1}F(h)$. Conjugation with h can be extended to an isomorphism

$$\mathbf{G} \rtimes \langle F_0 \rangle \to \mathbf{G} \rtimes \langle c_h \circ F_0 \circ c_{h^{-1}} \rangle, \quad (x, F_0^i) \mapsto (hxh^{-1}, c_h \circ F_0^i \circ c_{h^{-1}})$$

that maps F_0^k-stable elements to F-stable elements. Thus, it restricts to an isomorphism $\mathbf{G}^{F_0^k} \cong \mathbf{G}^F$ and $\mathbf{S}_1 := c_{h^{-1}}(\mathbf{S})$ is an F_0^k-stable torus of $\mathbf{G}$.

Step 2: Descent of scalars. We can now use the construction of the descent of scalars to obtain isomorphisms

$$\underline{\mathbf{G}}^{\tau F_0} \cong \mathbf{G}^{F_0^k}, \quad \underline{\mathbf{S}_1}^{\tau F_0} \cong \mathbf{S}_1^{F_0^k}$$

and maps

$$\mathrm{pr}^\vee_{\mathbf{G}\rtimes\langle F_0\rangle} : \mathbb{Z}\operatorname{Irr}(\mathbf{G}^{F_0^k} \rtimes \langle F_0 \rangle) \to \mathbb{Z}\operatorname{Irr}(\underline{\mathbf{G}}^{\tau F_0} \rtimes \langle \tau \rangle), \quad \mathrm{pr}^\vee_{\mathbf{G}} : \mathbb{Z}\operatorname{Irr}(\mathbf{G}^{F_0^k}) \to \mathbb{Z}\operatorname{Irr}(\underline{\mathbf{G}}^{\tau F_0})$$

induced by pr that restrict to bijections between the respective irreducible characters.

Let $\underline{\chi} := \mathrm{pr}^\vee_{\mathbf{G}}\left(\chi^{h^{-1}}\right)$ be the character of $\underline{\mathbf{G}}^{\tau F_0}$ corresponding to χ. Setting

$$\underline{\beta} := R_{\underline{\mathbf{S}_1}}^{\underline{\mathbf{G}}}\left(1_{\underline{\mathbf{S}_1}^{\tau F_0}}\right) = \mathrm{pr}^\vee_{\mathbf{G}}\left(\beta^{h^{-1}}\right),$$

we have $\langle \underline{\beta}, \underline{\chi} \rangle = \langle \beta, \chi \rangle$. We use the construction of Deligne–Lusztig characters for disconnected groups from [**DM94**, Definition 2.2] as presented in Section 3.4. As already mentioned before, most of the constructions and results generalize to our setting of Steinberg endomorphisms.

Let $\mathbf{B}_1$ be a Borel subgroup of the disconnected group $\underline{\mathbf{G}} \rtimes \langle \tau \rangle$ containing $\underline{\mathbf{S}_1} \rtimes \langle \tau \rangle$. By Proposition 3.44, we find a τ-stable pair $\widetilde{\mathbf{S}} \subseteq \widetilde{\mathbf{B}}$ that is conjugate to $\underline{\mathbf{S}_1} \rtimes \langle \tau \rangle \subseteq \mathbf{B}_1$ by some element $x \in \underline{\mathbf{G}}^{\tau F_0} \rtimes \langle \tau \rangle$. Then we can define

$$\widehat{\underline{\beta}} := R_{\widetilde{\mathbf{S}}}^{\underline{\mathbf{G}}\rtimes\langle\tau\rangle}\left(1_{\widetilde{\mathbf{S}}^{\tau F_0}}\right) = R_{\underline{\mathbf{S}_1}\rtimes\langle\tau\rangle}^{\underline{\mathbf{G}}\rtimes\langle\tau\rangle}\left(1_{\underline{\mathbf{S}_1}^{\tau F_0}\rtimes\langle\tau\rangle}\right) \in \mathbb{Z}\operatorname{Irr}(\underline{\mathbf{G}}^{\tau F_0} \rtimes \langle \tau \rangle).$$

This character extends $\underline{\beta}$ by Lemma 3.46(a).

Step 3: Unique extending constituents. We now determine the number of irreducible constituents of $\widehat{\underline{\beta}}$ in order to show that only one of them extends $\underline{\chi}$.

We have

$$((\underline{\mathbf{G}} \rtimes \langle\tau\rangle)^\tau)^\circ = (\{(g,\dots,g) \mid g \in \mathbf{G}\} \rtimes \langle\tau\rangle)^\circ \cong \mathbf{G}$$

where the isomorphism is given by projection to the first coordinate pr. Note that τF_0 acts on the elements of $((\underline{\mathbf{G}} \rtimes \langle\tau\rangle)^\tau)^\circ$ as F_0. Analogously, we have $\mathrm{pr}((\widetilde{\mathbf{S}}^\tau)^\circ) = c_x(\mathbf{S}_1)$. It follows

$$R^{((\underline{\mathbf{G}}\rtimes\langle\tau\rangle)^\tau)^\circ}_{(\widetilde{\mathbf{S}}^\tau)^\circ}\left(1_{((\widetilde{\mathbf{S}}^\tau)^\circ)^{\tau F_0}}\right) \circ \mathrm{pr} = R^{\mathbf{G}}_{c_x(\mathbf{S}_1)}\left(1_{c_x(\mathbf{S}_1)^{F_0}}\right).$$

By Proposition 3.49, we have thereby

$$\begin{aligned}\langle\widehat{\underline{\beta}}, \widehat{\underline{\beta}}\rangle_{(\underline{\mathbf{G}}\rtimes\langle\tau\rangle)^{\tau F_0}} &= \left\langle R^{((\underline{\mathbf{G}}\rtimes\langle\tau\rangle)^\tau)^\circ}_{(\widetilde{\mathbf{S}}^\tau)^\circ}\left(1_{((\widetilde{\mathbf{S}}^\tau)^\circ)^{\tau F_0}}\right), R^{((\underline{\mathbf{G}}\rtimes\langle\tau\rangle)^\tau)^\circ}_{(\widetilde{\mathbf{S}}^\tau)^\circ}\left(1_{((\widetilde{\mathbf{S}}^\tau)^\circ)^{\tau F_0}}\right)\right\rangle_{(((\underline{\mathbf{G}}\rtimes\langle\tau\rangle)^\tau)^\circ)^{\tau F_0}} \\ &= \left\langle R^{\mathbf{G}}_{c_x(\mathbf{S}_1)}\left(1_{c_x(\mathbf{S}_1)^{F_0}}\right), R^{\mathbf{G}}_{c_x(\mathbf{S}_1)}\left(1_{c_x(\mathbf{S}_1)^{F_0}}\right)\right\rangle_{\mathbf{G}^{F_0}} = \langle\underline{\beta}, \underline{\beta}\rangle_{\underline{\mathbf{G}}^{\tau F_0}}.\end{aligned}$$

The last equality holds since both scalar products equal $|\mathbf{W}^{F_0}| = |\mathbf{W}^{F_0^k}|$ by Proposition 3.6. Since all constituents of $\underline{\beta}$ have multiplicity ± 1, its norm gives us the number of irreducible constituents of $\underline{\beta}$. Now, all constituents of $\underline{\beta}$ extend to $\underline{\mathbf{G}}^{\tau F_0} \rtimes \langle\tau\rangle$ and we have the same number of irreducible constituents in $\widehat{\underline{\beta}}$. Thus, every constituent of $\underline{\beta}$ has a unique extension that is a constituent of $\widehat{\underline{\beta}}$. For $\underline{\chi}$, we denote this character by $\widehat{\underline{\chi}}$.

The Deligne–Lusztig character β is $\mathcal{H}$-invariant because it arises from the trivial character of $\mathbf{S}^F$. Analogously, $\underline{\beta}$ and $\widehat{\underline{\beta}}$ are also $\mathcal{H}$-invariant and from the uniqueness we know that $\widehat{\underline{\chi}}$ is $\mathcal{H}_\chi$-invariant. It follows that

$$\widehat{\chi} := \left(c_h \circ (\mathrm{pr}^\vee_{\underline{\mathbf{G}}\rtimes\langle\tau\rangle})^{-1}\right)\left(\widehat{\underline{\chi}}\right) \in \mathrm{Irr}\left(\mathbf{G}^F \rtimes \langle c_h \circ F_0 \circ c_{h^{-1}}\rangle\right)$$

is an $\mathcal{H}_\chi$-invariant extension of χ.

Step 4: Twisting back and conclusion. It remains to convince ourselves that $\langle c_h \circ F_0 \circ c_{h^{-1}}\rangle$ forms a subgroup of $\mathrm{Aut}(\mathbf{G}^F)$ that contains no non-trivial inner automorphisms and has order k. By definition, we have

$$c_h \circ F_0 \circ c_{h^{-1}} = c_{h\gamma(gh^{-1})} \circ \gamma$$

and $h\gamma(gh^{-1})$ lies in $\mathbf{G}^F$ since

$$\begin{aligned}F(h\gamma(gh^{-1})) &= F(h)\gamma(g)\gamma(F(h^{-1})) = hg^*\gamma(g)\gamma(g^*)^{-1}\gamma(h^{-1}) \\ &= hg^*\gamma(g)\gamma(g^{-1})\gamma^k(g^{-1})\cdots\gamma^2(g^{-1})\gamma(h^{-1}) = hg^*(g^*)^{-1}\gamma(g)\gamma(h^{-1}).\end{aligned}$$

We further have

$$h\gamma(gh^{-1})\gamma(h\gamma(gh^{-1}))\cdots\gamma^{k-1}(h\gamma(gh^{-1})) = h\gamma(g)\cdots\gamma^k(g)F(h^{-1}) = 1$$

which yields

$$(c_{h\gamma(gh^{-1})} \circ \gamma)^k = F.$$

Thus, the only inner automorphism in $\langle c_h \circ F_0 \circ c_{h^{-1}}\rangle$ is the identity and we can extend $\widehat{\chi}$ canonically to all inner automorphisms in $\mathrm{Aut}(\mathbf{G}^F)$ such that all scalars on $C_{G\rtimes\Gamma}(G)$ are trivial, see Remark 4.5. Restricting this character to $\mathbf{G}^F \rtimes \Gamma$ gives us a character with the required properties. □

6.3.2. Local characters. We now prove an analogous result for the local characters. As in the proof of Proposition 6.11, we know that the irreducible ℓ'-characters of N can be parametrized by $(s,\eta) \in \mathcal{M}_0$ via

$$\varphi^{(N)}(s,\eta) = \mathrm{Ind}^N_{N_{\chi_s^{\mathbf{T}}}}(\Lambda(\chi_s^{\mathbf{T}})(\eta \circ i_{s,1})).$$

We use this to consider the irreducible characters of N case by case.

Proposition 6.13. *Let ψ be an irreducible ℓ'-character of N. Then, there exists an extension $\widehat{\psi} \in \mathrm{Irr}(N \rtimes D_\psi)$ of ψ that is invariant under the action of $(\Gamma \times \mathcal{H})_\psi$.*

Proof. We can use the map $\varphi^{(N)}$ and available results about the Sylow tori of $\mathbf{G}$ to determine the irreducible ℓ'-characters of N. An overview is given in Table 6.1, 6.2, and 6.3. Let $(s, \eta) \in \mathcal{M}_0$.

If $s = 1$, then the corresponding character $\psi := \varphi^{(N)}(1, \eta)$ is the character inflated from $\eta \circ i_{1,1}$ where η is a character of the corresponding relative Weyl group $\mathbf{W}_{\mathbf{G}^F}(\mathbf{T})$. The relative Weyl group is, depending on ℓ and the type of $\mathbf{G}$, either cyclic or isomorphic

$\ell \mid q^2 - 1$, $\mathbf{T}^F \cong C_{q^2-1}$, $N = \mathbf{T}^F \rtimes C_2$

s	$\lvert s^N \rvert$	Associated characters $\varphi^{(N)}(s, \cdot)$
1	1	$\mathrm{Inf}_{C_2}^N(\eta)$ for $\eta \in \mathrm{Irr}(C_2)$
$\neq 1$	2	$\mathrm{Ind}_{\mathbf{T}^F}^N(\chi_s^{\mathbf{T}})$

$\ell \mid q^2 \pm \sqrt{2}q + 1$, $\mathbf{T}^F \cong C_{q^2 \pm \sqrt{2}q+1}$, $N = \mathbf{T}^F \rtimes C_4$

s	$\lvert s^N \rvert$	Associated characters $\varphi^{(N)}(s, \cdot)$
1	1	$\mathrm{Inf}_{C_4}^N(\eta)$ for $\eta \in \mathrm{Irr}(C_4)$
$\neq 1$	4	$\mathrm{Ind}_{\mathbf{T}^F}^N(\chi_s^{\mathbf{T}})$

Table 6.1. Description of Sylow $\phi^{(\ell)}$-tori for type ${}^2\mathsf{B}_2$, their normalizers N, the size of their N-conjugacy classes and the associated irreducible characters [**IMN07**, Section 16].

$\ell \mid q^2 - 1$, $\mathbf{T}^F \cong C_{q^2-1}$, $N = \mathbf{T}^F \rtimes C_2$

s	$\lvert s^N \rvert$	Associated characters $\varphi^{(N)}(s, \cdot)$
1	1	$\mathrm{Inf}_{C_2}^N(\eta)$ for $\eta \in \mathrm{Irr}(C_2)$
$\neq 1$	2	$\mathrm{Ind}_{\mathbf{T}^F}^N(\chi_s^{\mathbf{T}})$

$\ell \mid q^2 \pm \sqrt{3}q + 1$, $\mathbf{T}^F \cong C_{q^2 \pm \sqrt{3}q+1}$, $N = \mathbf{T}^F \rtimes C_6$

1	1	$\mathrm{Inf}_{C_6}^N(\eta)$ for $\eta \in \mathrm{Irr}(C_6)$
$\neq 1$	6	$\mathrm{Ind}_{\mathbf{T}^F}^N(\chi_s^{\mathbf{T}})$

$\ell \mid q^2 + 1$, $\mathbf{T}^F \cong C_{(q^2+1)/2} \times C_2$, $N = \mathbf{T}^F \rtimes C_6$

1	1	$\mathrm{Inf}_{C_6}^N(\eta)$ for $\eta \in \mathrm{Irr}(C_6)$
$\mathrm{ord}(s) = 2$	3	$\mathrm{Ind}_{\mathbf{T}^F \rtimes C_2}^N(\Lambda(\chi_s^{\mathbf{T}})(\eta \circ i_{s,1}))$, $\eta \in \mathrm{Irr}(C_2)$
$\mathrm{ord}(s) > 2$	6	$\mathrm{Ind}_{\mathbf{T}^F}^N(\chi_s^{\mathbf{T}})$

Table 6.2. Description of Sylow $\phi^{(\ell)}$-tori for type ${}^2\mathsf{G}_2$, their normalizers N, the size of their N-conjugacy classes and the associated irreducible characters [**IMN07**, Section 17].

$\ell \mid q^2-1$, $\mathbf{T}^F \cong C_{q^2-1} \times C_{q^2-1} \cong \langle\alpha\rangle \times \langle\alpha'\rangle$, $N = \mathbf{T}^F \rtimes D_{16}$

s	Type in [**SI75**]	$\lvert s^N\rvert$	Associated characters $\varphi^{(N)}(s,\cdot)$
$(1,1)$	t_0	1	$\mathrm{Inf}_{D_{16}}^N(\eta)$ for $\eta \in \mathrm{Irr}(D_{16})$
$(1,\alpha'^i)$	t_1	8	$\mathrm{Ind}_{\mathbf{T}^F \rtimes C_2}^N(\Lambda(\chi_s^{\mathbf{T}})(\eta \circ i_{s,1}))$ for $\eta \in \mathrm{Irr}(C_2)$
$(\alpha^i, \alpha'^{i(\sqrt{2}q+1)})$	t_2	8	$\mathrm{Ind}_{\mathbf{T}^F \rtimes C_2}^N(\Lambda(\chi_s^{\mathbf{T}})(\eta \circ i_{s,1}))$ for $\eta \in \mathrm{Irr}(C_2)$
(α^i, α'^j)	t_3	16	$\mathrm{Ind}_{\mathbf{T}^F}^N(\chi_s^{\mathbf{T}})$

$\ell \mid q^2+1$, $\mathbf{T}^F \cong C_{q^2+1} \times C_{q^2+1} \cong \langle\alpha\rangle \times \langle\alpha'\rangle$, $N = \mathbf{T}^F \rtimes \mathrm{GL}_2(3)$

$(1,1)$	t_0	1	$\mathrm{Inf}_{\mathrm{GL}_2(3)}^N(\eta)$ for $\eta \in \mathrm{Irr}(\mathrm{GL}_2(3))$
$(\alpha^{i\frac{q^2+1}{3}}, \alpha'^{j\frac{q^2+1}{3}})$	t_4	8	$\mathrm{Ind}_{\mathbf{T}^F \rtimes H}^N(\Lambda(\chi_s^{\mathbf{T}})(\eta \circ i_{s,1}))$ for $H \le \mathrm{GL}_2(3)$ of order 6, $\eta \in \mathrm{Irr}(H)$
(α^i, α'^i)	t_5	24	$\mathrm{Ind}_{\mathbf{T}^F \rtimes C_2}^N(\Lambda(\chi_s^{\mathbf{T}})(\eta \circ i_{s,1}))$ for $\eta \in \mathrm{Irr}(C_2)$
(α^i, α'^j)	t_{14}	48	$\mathrm{Ind}_{\mathbf{T}^F}^N(\chi_s^{\mathbf{T}})$

$\ell \mid q^2 \pm \sqrt{2}q+1$, $\mathbf{T}^F \cong C_{q^2\pm\sqrt{2}q+1} \times C_{q^2\pm\sqrt{2}q+1} \cong \langle\alpha\rangle \times \langle\alpha'\rangle$, $N = \mathbf{T}^F \rtimes G_8$

$(1,1)$	t_0	1	$\mathrm{Inf}_{G_8}^N(\eta)$ for $\eta \in \mathrm{Irr}(G_8)$
$(1,\alpha'^i)$	t_7 or t_9	24	$\mathrm{Ind}_{\mathbf{T}^F \rtimes H}^N(\Lambda(\chi_s^{\mathbf{T}})(\eta \circ i_{s,1}))$ for $H \le G_8$ of size 4, $\eta \in \mathrm{Irr}(H)$
(α^i, α'^j)	t_{12} or t_{13}	96	$\mathrm{Ind}_{\mathbf{T}^F}^N(\chi_s^{\mathbf{T}})$

$\ell \mid q^4-q^2+1$, $\mathbf{T}^F \cong C_{q^4-q^2+1} \cong \langle\alpha\rangle$, $N = \mathbf{T}^F \rtimes C_6$

1	t_0	1	$\mathrm{Inf}_{C_6}^N(\eta)$ for $\eta \in \mathrm{Irr}(C_6)$
$\alpha^{\frac{q^4-q^2+1}{3}}$	t_4	2	$\mathrm{Ind}_{\mathbf{T}^F \rtimes C_3}^N(\Lambda(\chi_s^{\mathbf{T}})(\eta \circ i_{s,1}))$ for $\eta \in \mathrm{Irr}(C_3)$
α^i	t_{15}	6	$\mathrm{Ind}_{\mathbf{T}^F}^N(\chi_s^{\mathbf{T}})$

$\ell \mid q^4 \pm \sqrt{2}q^3 + q^2 \pm \sqrt{2}q+1$, $\mathbf{T}^F \cong C_{q^4\pm\sqrt{2}q^3+q^2\pm\sqrt{2}q+1} \cong \langle\alpha\rangle$, $N = \mathbf{T}^F \rtimes C_{12}$

1	t_0	1	$\mathrm{Inf}_{C_{12}}^N(\eta)$ for $\eta \in \mathrm{Irr}(C_{12})$
α^i	t_{16} or t_{17}	12	$\mathrm{Ind}_{\mathbf{T}^F}^N(\chi_s^{\mathbf{T}})$

TABLE 6.3. Description of Sylow $\phi^{(\ell)}$-tori for type ${}^2\mathsf{F}_4$, their normalizers N given in [**Mal91b**], representatives of their semisimple N-conjugacy classes from [**SI75**], and the associated irreducible characters. Here, α and α' denote generators of the respective cyclic groups and G_8 is the complex reflection group with Shephard–Todd number 8.

to D_{16}, $\mathrm{GL}_2(3)$, or the complex reflection group with Shephard–Todd number 8. We see that every character of these groups is linear, rational, or a constituent of $\mathrm{Ind}_H^{\mathbf{W}_{\mathbf{G}^F}(\mathbf{T})}(\tau)$ of multiplicity 1 for some subgroup $H \le \mathbf{W}_{\mathbf{G}^F}(\mathbf{T})$ and a D_ψ-invariant linear character $\tau \in \mathrm{Irr}(H)$. Thus, we find a $(\Gamma \times \mathcal{H})_\psi$-invariant extension of η to $\mathbf{W}_{\mathbf{G}^F}(\mathbf{T}) \rtimes D_\psi$ by Lemma 4.7 and Proposition 4.8. Inflation yields a $(\Gamma \times \mathcal{H})_\psi$-invariant extension of ψ to $N \rtimes D_\psi$.

We assume that s is not trivial and consider

$$\psi_0 := \Lambda(\chi_s^{\mathbf{T}})(\eta \circ i_{s,1}), \quad \psi := \varphi^{(N)}(s,\eta) = \mathrm{Ind}_{N_{\chi_s^{\mathbf{T}}}}^N(\psi_0).$$

Since $\mathbf{T}^F$ is abelian and Λ preserves the character degrees, $\Lambda(\chi_s^{\mathbf{T}})$ is always linear. Let κ be a generator of D_ψ. First, assume that η is linear. Because κ fixes ψ, there exists an $n \in N$ such that $\psi_0^{\kappa n} = \psi_0$. Thus, we find a $(\langle \kappa n\rangle \times \mathcal{H})_\psi$-invariant extension $\widehat{\psi}$ of ψ to $N \rtimes \langle \kappa n\rangle$ by Lemma 4.7. Since $\mathrm{Inn}(N)$ acts trivially on ψ, the extension $\widehat{\psi}$ is also $(\Gamma\times\mathcal{H})_\psi$-invariant. The claim follows by extending $\widehat{\psi}$ to the remaining inner automorphisms of N and then restricting it to $N \rtimes D_\psi$.

For type ${}^2\mathsf{F}_4$ it can happen that $\eta \in \mathrm{Irr}(H)$ for $H := \mathbf{W}_{C_{\mathbf{G}}(s)}(\mathbf{T})^F$ is not linear. In this case, we can see that it has integer character values. Let $n \in N$ such that

$$(\chi_s^{\mathbf{T}})^{\kappa n} = \chi_s^{\mathbf{T}}$$

and κn has odd order. We have a natural isomorphism $(\mathbf{T}^F \rtimes H) \rtimes \langle \kappa n\rangle \cong \mathbf{T}^F \rtimes (H \rtimes \langle \kappa n\rangle)$. By [**CS13**, Proposition 2.3], $\eta \circ i_{s,1}$ is κn -stable and with Proposition 4.8 we can find a $(\Gamma \times \mathcal{H})_\psi$-invariant extension of $\eta \circ i_{s,1}$ to $\widehat{\eta} \in \mathrm{Irr}(H \rtimes \langle \kappa n\rangle)$. Inflating this character and using the construction of irreducible characters of semidirect products from [**Ser77**, Section 8.2], we obtain a $(\Gamma \times \mathcal{H})_\psi$-invariant extension of $\Lambda(\chi_s^{\mathbf{T}})(\eta \circ i_{s,1})$ to the group $\mathbf{T}^F \rtimes (H \rtimes \langle \kappa n\rangle)$. With Lemma 4.7 and as before, this yields a $(\Gamma \times \mathcal{H})_\psi$-invariant extension of ψ to $N \rtimes D_\psi$. □

6.3.3. Verification of the inductive condition. We can now verify the inductive McKay–Navarro condition for the Suzuki and Ree groups and the prime ℓ except for the groups that have been excluded before.

Theorem 6.14. *The inductive McKay–Navarro condition holds for*

(a) *the Suzuki groups* ${}^2\mathsf{B}_2(2^{2f+1})$ *for* $f \ge 2$ *and any odd prime* ℓ*,*
(b) *the Ree groups* ${}^2\mathsf{G}_2(3^{2f+1})$ *for* $f \ge 1$ *and any prime* $\ell \ge 5$*, and*
(c) *the Ree groups* ${}^2\mathsf{F}_4(2^{2f+1})$ *for* $f \ge 1$ *and any odd prime* ℓ *except possibly* $\ell = 3$ *if* $2^{2f+1} \equiv 2, 5 \mod 9$.

Proof. We use the notation established in the previous sections. We can choose $N := N_{\mathbf{G}^F}(\mathbf{T})$ by Theorem 2.71 and we know from Proposition 6.11 that there exists a $\Gamma \times \mathcal{H}$-equivariant bijection

$$\Omega : \mathrm{Irr}_{\ell'}(\mathbf{G}^F) \to \mathrm{Irr}_{\ell'}(N).$$

Looking at the generic character table of $\mathbf{G}^F$ as given in [**GHL**$^+$**96**], we see that every irreducible global ℓ'-character is either unipotent, regular, real, or semisimple.

Every semisimple or regular $\chi \in \mathrm{Irr}_{\ell'}(\mathbf{G}^F)$ can be extended to a $(\Gamma \times \mathcal{H})_\chi$-invariant character of $\mathbf{G}^F \rtimes \langle \gamma\rangle_\chi$ by Proposition 4.13. We can therefore extend it canonically to all inner automorphisms of $\mathbf{G}^F$ and then restrict it to $\mathbf{G}^F \rtimes \Gamma_\chi$ to obtain a $(\Gamma\times\mathcal{H})_\chi$-invariant extension such that all associated scalars on $C_{G\rtimes\Gamma_{\chi^{\mathcal{H}}}}(G)$ are trivial.

For all real and unipotent characters in $\mathrm{Irr}_{\ell'}(\mathbf{G}^F)$, this holds analogously by Proposition 4.8 and Proposition 6.12. By Proposition 6.13, the local characters $\psi \in \mathrm{Irr}_{\ell'}(N)$

have a $(\Gamma \times \mathcal{H})_\psi$-invariant extension to $\widehat{\psi} \in \mathrm{Irr}(N \rtimes D_\psi)$ and by Lemma 4.6 we can assume that the associated scalars on $C_{G \rtimes \Gamma_{\chi^{\mathcal{H}}}}(G)$ are trivial. This shows that the inductive McKay–Navarro condition holds for $\mathbf{G}^F$ and ℓ. □

We already know that the inductive McKay–Navarro condition holds for the Suzuki and Ree groups in their defining characteristic by Chapter 5. The remaining cases of Suzuki and Ree groups will be considered in Proposition 7.1 and Proposition 7.11 in the next chapter.

CHAPTER 7

Groups with non-generic Sylow normalizers

In this chapter, we consider some groups and primes for which the normalizer of a Sylow subgroup does not necessarily lie in the normalizer of the corresponding Sylow torus, i.e. the cases that have been excluded in Theorem 2.71. This happens for the primes and Suzuki and Ree groups that have been excluded in the previous chapter as well as for some groups of type A_2, ${}^2\mathsf{A}_2$, and G_2 for the prime $\ell = 3$ and of type C_n for the prime $\ell = 2$. The latter case has already been settled in [**RSF22**, Theorem A]. We consider the remaining groups since we want to complete the case of Suzuki and Ree groups and it seems natural to look at the other groups of Lie type where we do not have a generic choice of the local subgroup.

With some minor changes, this chapter has been published in [**Joh21**, Section 5].

7.1. Suzuki and Ree groups

In this section, we consider the inductive McKay–Navarro condition for the Ree groups ${}^2\mathsf{G}_2(3^{2f+1})$ for $\ell = 2$ and ${}^2\mathsf{F}_4(2^{2f+1})$ for $\ell = 3$ in the case that we have $2^{2f+1} \equiv 2, 5 \mod 9$. We show that the condition holds by looking at the global and local characters case by case. We use the notation that was introduced in the previous chapter.

Proposition 7.1. *The inductive McKay–Navarro condition is satisfied for $f \geq 1$ and*
(a) ${}^2\mathsf{G}_2(3^{2f+1})$ *and* $\ell = 2$*;*
(b) ${}^2\mathsf{F}_4(2^{2f+1})$ *with* $2^{2f+1} \equiv 2, 5 \mod 9$ *and* $\ell = 3$.

Proof. (a) First, let $G = {}^2\mathsf{G}_2(3^{2f+1})$ and $\ell = 2$. We start by considering the local characters. As in [**IMN07**, Proof of Theorem 17.1], $R \in \mathrm{Syl}_2(G)$ is elementary abelian of order 8 and we have $N = N_G(R) = R \cdot H$ where $H \cong C_7 \rtimes C_3$ is a Frobenius group. Therefore, the irreducible characters of N are as listed in Table 7.1.

$\mathrm{Irr}_{2'}(N)$	Parameters and construction of characters	Degree
$\theta_i = \mathrm{Inf}_{C_3}^N(\tilde{\theta}_i)$	$1 \leq i \leq 3$, $\tilde{\theta}_i \in \mathrm{Irr}(C_3)$	1
$\psi_i = \mathrm{Inf}_H^N(\mathrm{Ind}_{C_7}^H(\tau_i))$	$i \in \{1, 2\}$, $1 \neq \tau_i \in \mathrm{Irr}(C_7)$ with $\tau_1 \not\sim_H \tau_2$	3
$\lambda_1, \lambda_2, \lambda_3$	$\mathrm{Ind}_R^N(\chi) = \lambda_1 + \lambda_2 + \lambda_3$, $1 \neq \chi \in \mathrm{Irr}(R)$	7

Table 7.1. Irreducible characters of N with odd degree.

From [**IMN07**, Proof of Theorem 17.1], we know that F_3 acts trivially on N and thus on all of these characters. The explicit construction of the character values shows us, after choosing a suitable labeling of the characters, that

$$\theta_2^\sigma = \begin{cases} \theta_2 & \text{if } \zeta_3^\sigma = \zeta_3, \\ \theta_3 & \text{if } \zeta_3^\sigma = \zeta_3^2, \end{cases} \quad , \quad \lambda_2^\sigma = \begin{cases} \lambda_2 & \text{if } \zeta_3^\sigma = \zeta_3, \\ \lambda_3 & \text{if } \zeta_3^\sigma = \zeta_3^2, \end{cases}$$

for all $\sigma \in \mathcal{H}$ where $\zeta_3 \in \mathbb{C}^\times$ is a third root of unity. The characters θ_3 and λ_3 are mapped analogously. All other characters of N are fixed by $\mathcal{H}$.

It is clear that the linear characters θ_i can be extended to $N \rtimes \langle F_3 \rangle$ such that they are $\mathcal{H}_{\theta_i}$-invariant. For the characters ψ_1 and ψ_2, an extension can be found as in Lemma 4.7

and by inflating the obtained character. Similarly, the linear character $\chi \in \mathrm{Irr}(R)$ can be extended to an $\mathcal{H}_{\lambda_i}$-invariant character of $R \rtimes \langle F_3 \rangle$. With Lemma 4.7, we obtain $\mathcal{H}_{\lambda_i}$-invariant extensions of λ_1, λ_2, and λ_3 to $N \rtimes \langle F_3 \rangle$.

In the global case, we have

$$\mathrm{Irr}_{2'}(G) = \{\chi_1, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8, \chi_a, \chi'_a\}$$

in the notation of [**IMN07**, Table 4]. The automorphism F_3 acts trivially on them since they are unipotent or uniquely determined by their degree. The Galois automorphism $\sigma \in \mathcal{H}$ acts by permuting the indices as $(4,5)(6,7)$ if we have $\xi_3^\sigma = \xi_3^2$ and trivially otherwise. Thus, there exists a $\Gamma \times \mathcal{H}$-equivariant bijection.

We find $\mathcal{H}_\chi$-invariant character extensions to $G \rtimes \langle F_3 \rangle$ for the real characters χ_1, χ_8, χ_a, χ'_a by Proposition 4.8 and for the unipotent characters χ_4, χ_5, χ_6, χ_7 in the same way as in Proposition 6.12. Thus, the inductive McKay–Navarro condition is satisfied.

(b) Let $G = {}^2\mathsf{F}_4(2^{2f+1})$ for $f \geq 1$, assume $2^{2f+1} \equiv 2, 5 \mod 9$, and set $\ell = 3$. In the global case, we see that G has six unipotent characters of $3'$-degree and three other $3'$-characters that all have distinct degrees [**GHL⁺96**]. The characters are all Γ-invariant, $\mathcal{H}$-invariant, and real. Since F_2 has odd order, there exists a unique $\mathcal{H}$-invariant extension to $G \rtimes \Gamma$ for every character in $\mathrm{Irr}_{3'}(G)$ by Proposition 4.8.

As mentioned in [**Mal07**, Proof of Theorem 8.4], the normalizer of a Sylow 3-subgroup of G is $N \cong \mathrm{SU}_3(2).2$ and we can explicitly compute its irreducible $3'$-characters. They are given in Table 7.2.

N	1a	2a	4a	4b	2b	3a	3b	8a	6a	12a	12b	6b	8b	12c
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	1	1	-1	1	-1	1	1	-1
χ_3	1	-1	1	1	1	1	1	-1	-1	1	1	1	-1	1
χ_4	1	1	1	-1	1	1	1	-1	1	1	-1	1	-1	-1
χ_5	2	.	-2	.	2	2	2	.	.	-2	.	2	.	.
χ_6	2	.	.	.	-2	2	2	$-\zeta_8 - \zeta_8^3$	.	.	.	-2	$\zeta_8 + \zeta_8^3$	.
χ_7	2	.	.	.	-2	2	2	$\zeta_8 + \zeta_8^3$	.	.	.	-2	$-\zeta_8 - \zeta_8^3$	.
χ_8	8	-2	.	.	.	-1	8	.	1	.	.	.	.	.
χ_9	8	2	.	.	.	-1	8	.	-1	.	.	.	.	.

TABLE 7.2. Values of the irreducible $3'$-characters of $N \cong \mathrm{SU}_3(2).2$ computed with [**GAP19**]. The conjugacy classes are labeled as it is done there. A dot "." represents the character value 0 and ζ_8 denotes a primitive eighth root of unity.

We know by [**Mal08a**, Proof of Proposition 3.16] that Γ acts trivially on $\mathrm{Irr}_{3'}(N)$ and we see that $\mathcal{H}$ also acts trivially.

For the rational characters, we obtain $\mathcal{H}$-invariant extensions to $N \rtimes \Gamma$ again by Proposition 4.8. We can show with GAP that the characters χ_6, χ_7 are each induced by two linear characters λ_6, λ'_6 and λ_7, λ'_7 of $((C_3 \times C_3) : C_3) : C_8 \leq N$, respectively. Since F_2 fixes χ_6 and χ_7, both λ_6 and λ_7 are fixed or mapped to λ'_6 and λ'_7, respectively. The orbit cannot have length 2 because F_2 has odd order, thus λ_6, λ_7 are fixed by F_2. Since χ_6 is $\mathcal{H}$-invariant, λ_6 and λ_6^σ are conjugate by some element of N for every $\sigma \in \mathcal{H}$. It follows as before that there is an $\mathcal{H}$-invariant extension of χ_6 to $N \rtimes \langle F_2 \rangle$. The same can be done for χ_7. Thereby, the inductive McKay–Navarro condition holds. □

7.2. Special linear and special unitary groups

In this section, we consider the twisted and untwisted groups of type A_2 that have been excluded in Theorem 2.71 and show that the inductive McKay–Navarro condition holds for the prime $\ell = 3$.

Let $X = \mathrm{PSL}_3(q)$ with $q \equiv 4, 7 \mod 9$ or $X = \mathrm{PSU}_3(q)$ with $q \neq 2$ and $q \equiv 2, 5 \mod 9$. The inductive McKay condition has been verified for X and ℓ in [**Mal08a**, Section 3.1 and 3.2]. We verify the inductive McKay–Navarro condition for these groups by recalling the considerations from there and extending them to the stronger condition.

Note that we do not consider $\mathrm{PSU}_3(2)$ since it is solvable. The group $\mathrm{PSL}_3(4)$ has exceptional Schur multiplier and is considered separately in Proposition 7.11. For all other groups, the Schur multiplier of X has order 3 and we can consider X itself (instead of its universal covering group) by Lemma 4.15. We follow [**Mal08a**, Section 3.1 and 3.2] and use the notation from there.

7.2.1. Global characters. Let $S := \mathrm{SL}_3(q)$ or $S := \mathrm{SU}_3(q)$, respectively. Since the irreducible characters of S with $Z(S)$ in their kernel are in bijection with the irreducible characters of $X \cong S/Z(S)$, we can consider the irreducible $3'$-characters of S with $Z(S)$ in their kernel. We use this bijection without further notice. The group S has six irreducible $3'$-characters that are listed in Table 7.3. They all have $Z(S)$ in their kernel.

$\mathrm{Irr}_{3'}(S)$	Degree	Property
$\rho_1 = 1_S$	1	unipotent
ρ_2	$q(q+\varepsilon)$	unipotent
ρ_3	q^3	unipotent
φ_1	$(q+\varepsilon)(q^2+\varepsilon q+1)/3$	semisimple
φ_2	$(q+\varepsilon)(q^2+\varepsilon q+1)/3$	semisimple
φ_3	$(q+\varepsilon)(q^2+\varepsilon q+1)/3$	semisimple

Table 7.3. Irreducible $3'$-characters of S as determined in [**Mal08a**, Lemma 3.3 and 3.10] where $\varepsilon = 1$ if $S = \mathrm{SL}_3(q)$ and $\varepsilon = -1$ if $S = \mathrm{SU}_3(q)$.

Lemma 7.2. *The irreducible $3'$-characters ρ_1, ρ_2, ρ_3, φ_1, φ_2, φ_3 of S are invariant under the action of $\mathcal{H}$.*

Proof. The unipotent characters ρ_1, ρ_2, ρ_3 have distinct degrees and are thus $\mathcal{H}$-invariant. The character values of φ_1, φ_2, φ_3 that are different for the three characters are given in [**Mal08a**, Section 3.1 and 3.2] and we see that they are rational. □

Let p be the prime with $q = p^f$ and note that the condition on q ensures that f is odd in the case of unitary groups. The structure of the outer automorphism group of S is described in [**Mal08a**, Lemma 3.4 and Section 3.2]. We write

$$D := \langle \gamma, \delta, \psi \rangle \cong \mathrm{Out}(S) \cong \mathrm{Sym}(3) \times C_f$$

where $\psi := F_p$ is the standard field automorphism of S, γ is the transpose-inverse automorphism, and δ is a diagonal automorphism of order 3.

Lemma 7.3. *For $X = \mathrm{PSL}_3(q)$, every $\chi \in \mathrm{Irr}_{3'}(X)$ has an $\mathcal{H}_\chi$-invariant extension to $X \rtimes D_\chi$.*

Proof. Let $G := \mathrm{PGL}_3(q)$ and B be a Borel subgroup of G. The unipotent characters ρ_1, ρ_2, ρ_3 can be extended to $\widetilde{\rho}_1, \widetilde{\rho}_2, \widetilde{\rho}_3 \in \mathrm{Irr}(G \rtimes \langle \psi \rangle)$ as in the proof of [**Mal08a**, Theorem 2.4] and we have

$$\mathrm{Ind}_{B \rtimes \langle \psi \rangle}^{G \rtimes \langle \psi \rangle}(1) = \widetilde{\rho}_1 + 2\widetilde{\rho}_2 + \widetilde{\rho}_3.$$

Similarly, we get

$$\mathrm{Ind}_{B\rtimes\langle\gamma\rangle}^{G\rtimes\langle\gamma\rangle}(1) = \widehat{\rho}_1 + 2\widehat{\rho}_2 + \widehat{\rho}_3$$

with $\widehat{\rho}_1, \widehat{\rho}_2, \widehat{\rho}_3 \in \mathrm{Irr}(G \rtimes \langle\gamma\rangle)$ extending ρ_1, ρ_2, ρ_3 by [**Mal91a**, pp. 447f].

Since we know that all unipotent characters extend to $G \rtimes \langle\gamma, \psi\rangle$, at least one of these extensions is a constituent of $\mathrm{Ind}_{B\rtimes\langle\gamma,\psi\rangle}^{G\rtimes\langle\gamma,\psi\rangle}(1)$. These constituents also extend $\widehat{\rho}_i$ and $\widetilde{\rho}_i$. Thus, there is only one constituent of $\mathrm{Ind}_{B\rtimes\langle\gamma,\psi\rangle}^{G\rtimes\langle\gamma,\psi\rangle}(1)$ extending ρ_i and it follows that it is $\mathcal{H}$-invariant.

The semisimple character φ_1 is a constituent of the dual of the Gelfand-Graev character Γ_1 of S and satisfies

$$((S \rtimes \langle\delta\rangle)\langle\psi, \gamma\rangle)_{\varphi_1} = (S \rtimes \langle\delta\rangle)_{\varphi_1}\langle\psi, \gamma\rangle_{\varphi_1} = S\langle\psi, \gamma\rangle.$$

As in [**Ruh21**, Corollary 6.9], we can use Proposition 4.13 to construct an $\mathcal{H}$-invariant extension $\widehat{\varphi}_1 \in \mathrm{Irr}(S\langle\psi, \gamma\rangle)$ of φ_1. Note that we can use this result since Ruhstorfer does not use any properties coming from the prime corresponding to $\mathcal{H}$.

The characters

$$\widehat{\varphi}_1^{\delta} \in \mathrm{Irr}(S\langle\psi\delta, \gamma\delta\rangle), \quad \widehat{\varphi}_1^{\delta^2} \in \mathrm{Irr}(S\langle\psi\delta^2, \gamma\delta^2\rangle)$$

are extensions of φ_2 and φ_3, respectively. Since they are $\mathcal{H}$-invariant, this shows the claim. □

Lemma 7.4. *For $X = \mathrm{PSU}_3(q)$, every $\chi \in \mathrm{Irr}_{3'}(X)$ has an $\mathcal{H}$-invariant extension to $X \rtimes D_\chi$.*

Proof. For the semisimple characters as well as for ρ_1 and ρ_3, this can be shown in the same way as for $\mathrm{PSL}_3(q)$. However, the unipotent character ρ_2 does not lie in the principal series of X and has to be treated differently. We first show that there is an $\mathcal{H}$-invariant extension of ρ_2 to $X \rtimes \langle\gamma\rangle$.

Step 1: Odd characteristic. If p is odd, then the two extensions of ρ_2 to $X \rtimes \langle\gamma\rangle$ have already been considered in [**Mal90b**, Proposition 2.1] for $p \equiv 3 \mod 4$. In the same way as in the proof given there, one can show for all odd p that the only non-rational character value of an extension $\widehat{\rho}_2 \in \mathrm{Irr}(X \rtimes \langle\gamma\rangle)$ is $\sqrt{-q}$. By the law of quadratic reciprocity, 3 is a square modulo p if $p \equiv 1 \mod 4$ and not a square otherwise. We can use the quadratic Gauss sum to write

$$\zeta_4\sqrt{q} = \begin{cases} p^{(f-1)/2}\sum_{k=0}^{p-1}\zeta_p^{k^2}\cdot\zeta_4 & \text{if } p \equiv 1 \mod 4, \\ p^{(f-1)/2}\sum_{k=0}^{p-1}\zeta_p^{k^2} & \text{if } p \equiv 3 \mod 4, \end{cases}$$

with ζ_p and ζ_4 primitive p-th and fourth roots of unity, see [**Gau65**, Section 14]. With this, it is easy to see that $\sigma \in \mathcal{H}$ fixes $\sqrt{-q}$. Thus, $\widehat{\rho}_2$ is $\mathcal{H}$-invariant.

Step 2: Outer conjugacy classes in even characteristic. If $p = 2$, we first have to study the character values for the extensions of ρ_2 to $\widetilde{X} := X \rtimes \langle\gamma\rangle$. We first determine the outer conjugacy classes of $\widetilde{X}$, i.e. the conjugacy classes that do not lie in $X \trianglelefteq \widetilde{X}$. We can parametrize the outer conjugacy classes as described in [**Bru07**, Method 3.1] by

$$\left\{ (g,1)x_i \;\middle|\; \begin{array}{l} g \in X \text{ representative of a } \gamma\text{-stable conjugacy class of odd order,} \\ x_i \text{ a class representative of an outer class of } 2'\text{-elements in } C_{\widetilde{X}}(g,1) \end{array} \right\}.$$

With the notation of [**SF73**], the γ-stable conjugacy classes of X are

$$C_1,\ C_2,\ C_3^{(0)},\ C_5',\ C_6^{(k,l,m)},\ C_7^{(rk)}.$$

The classes C_2 and $C_3^{(0)}$ consist of 2-elements of X. For an element $x \in X$ in a γ-stable conjugacy class, we have

$$2|C_X(x)| = |C_{\widetilde{X}}(x,1)|.$$

Thus, the centralizer orders in $\widetilde{X}$ for the elements of odd order in C_5', $C_6^{(k,l,m)}$, $C_7^{(rk)}$ can be deduced from [**SF73**, Table 2]. Since the Sylow 2-subgroups of the centralizers have order 2, the 2-elements of the centralizers are all conjugate. Thus, the γ-stable classes C_5', $C_6^{(k,l,m)}$, $C_7^{(rk)}$ each correspond to one outer conjugacy class with representatives $c_5'(1,\gamma)$, $c_6^{(k,l,m)}(1,\gamma)$, $c_7^{(rk)}(1,\gamma)$ each where we denote a representative of C_5' by c_5' and so on.

The number of outer conjugacy classes of $\widetilde{X}$ is the number of γ-stable conjugacy classes of X. Therefore, there are 3 conjugacy classes corresponding to the trivial class C_1. They consist of 2-elements and will be denoted by $(1,\gamma), A, B$.

Step 3: Character values and centralizer orders. Let $\widehat{\rho}_2$ be an extension of ρ_2 to $\widetilde{X}$. Then, the other extension of ρ_2 to $\widetilde{X}$ is given by sign $\cdot\widehat{\rho}_2$ where sign denotes the non-trivial extension of the trivial character of X to $\widetilde{X}$.

X	C_1	C_2	$C_3^{(0)}$	C_5'	$C_6^{(k,l,m)}$	$C_7^{(rk)}$
ρ_2	$q(q-1)$	$-q$	0	2	2	0

$\widetilde{X}$	C_1	$(1,\gamma)$	A	$B=A^{-1}$	$C_5'(1,\gamma)$	$C_6^{(k,l,m)}(1,\gamma)$	$C_7^{(rk)}(1,\gamma)$
$\widehat{\rho}_2$	$q(q-1)$	0	α	$\overline{\alpha}$	0	0	0
sign $\cdot\widehat{\rho}_2$	$q(q-1)$	0	$\overline{\alpha}$	α	0	0	0

Table 7.4. Some character values of ρ_2 and its extensions. Here, α denotes the value of $\widehat{\rho}_2$ at the outer class A.

The conjugacy classes $C_6^{(k,l,m)}(1,\gamma)$ and $C_7^{(rk)}(1,\gamma)$ are each algebraically conjugate to an odd number of classes of the same type with different parameters. Here, two conjugacy classes are called algebraically conjugate if their elements generate conjugate subgroups. One can easily see that the values of a character on algebraically conjugate classes are also algebraically conjugate, i.e. in one orbit under the action of Galois automorphisms in $\mathcal{G}$. Since we only have two extensions of ρ_2, these vanish on all $c_6^{(k,l,m)}(1,\gamma), c_7^{(rk)}(1,\gamma)$ and obviously also on $c_5'(1,\gamma)$.

Now, $(1,\gamma)$ is a rational class and since the extensions of ρ_2 have to be different on at least one class, A and B cannot be rational and it follows $B = A^{-1}$. Note that sign $\cdot\widehat{\rho}_2$ is also the complex conjugate of $\widehat{\rho}_2$.

We now determine $|C_{\widetilde{X}}(a)|$ for $a \in A$. From [**Bru07**, Lemma 3.1 2.], we know

$$C_{\widetilde{X}}(c_i(1,\gamma)) = C_{\widetilde{X}}(c_i) \cap C_{\widetilde{X}}((1,\gamma)).$$

We have $C_{\widetilde{X}}((1,\gamma)) \cong \mathrm{PSL}_2(q) \times \langle\gamma\rangle$. Since $a \in A$ has order 4 or 8, we have $a^2 \in C_2$ or $a^4 \in C_2$ and it follows with [**SF73**, Table 2] that $|C_{\widetilde{X}}(a)|$ divides $2|C_X(c_2)| = 2q^2\frac{q+1}{3}$. We know that half of the elements of $\widetilde{X}$ lie in an outer class, i.e. the sum

$$\frac{\widetilde{X}}{|C_{\widetilde{X}}((1,\gamma))|} + \frac{\widetilde{X}}{|C_{\widetilde{X}}(c_5'(1,\gamma))|} + \sum_{k,l,m} \frac{\widetilde{X}}{|C_{\widetilde{X}}(c_6^{(k,l,m)}(1,\gamma))|} + \sum_{k} \frac{\widetilde{X}}{|C_{\widetilde{X}}(c_7^{(rk)}(1,\gamma))|} + 2\cdot\frac{\widetilde{X}}{|C_{\widetilde{X}}(a)|}$$

adds up to

$$|X| = \frac{1}{3}q^3(q+1)^2(q-1)(q^2-q+1).$$

Looking at the centralizer orders in [**SF73**, Table 2] we see that q divides all summands except possibly the last one. It follows that $|C_{\widetilde{X}}(a)| = 4qd$ with d odd.

We now consider the unknown character value $\alpha := \widehat{\rho}_2(a)$. We can write

$$\alpha = \sum_{j=0}^{3} n_j \zeta_8^j, \quad n_j \in \mathbb{Z}$$

where ζ_8 is a primitive eighth root of unity. Since we have $\langle \widehat{\rho}_2, 1_{\widetilde{X}} \rangle_{\widetilde{X}} = 0$, it follows that $\overline{\alpha} = -\alpha$ and thereby $n_0 = 0$.

We also know

$$1 = \langle \widehat{\rho}_2, \widehat{\rho}_2 \rangle_{\widetilde{X}} = \frac{1}{2} \langle \rho_2, \rho_2 \rangle_X + \frac{1}{2|X|} \cdot \frac{2|X|}{|C_{\widetilde{X}}(a)|} \cdot \alpha\overline{\alpha} \cdot 2 = \frac{1}{2} + \frac{2\alpha\overline{\alpha}}{|C_{\widetilde{X}}(a)|}$$

and thereby

$$\alpha\overline{\alpha} = \frac{1}{4}|C_{\widetilde{X}}(a)| = qd.$$

Step 4: Action of Galois automorphisms. Let $\sigma_3, \sigma_5 \in \mathcal{G}$ such that $\zeta_8^{\sigma_3} = \zeta_8^3$ and $\zeta_8^{\sigma_5} = \zeta_8^5$. The characters $\widehat{\rho}_2^{\sigma_3}$ and $\widehat{\rho}_2^{\sigma_5}$ are either $\widehat{\rho}_2$ or $\mathrm{sign} \cdot \widehat{\rho}_2$, respectively.

First assume that $\widehat{\rho}_2^{\sigma_3} = \mathrm{sign} \cdot \widehat{\rho}_2$ and $\widehat{\rho}_2^{\sigma_5} = \widehat{\rho}_2$. Then, we have $n_1 = n_3 = 0$ and

$$\alpha = (n_2 - n_6)\zeta_8^2.$$

Thereby, we have

$$qd = \alpha\overline{\alpha} = (n_2 - n_6)^2.$$

This is not possible since d is odd and q is an odd power of 2. Therefore, we know that we have

$$\widehat{\rho}_2^{\sigma_3} = \widehat{\rho}_2, \quad \widehat{\rho}_2^{\sigma_5} = \mathrm{sign} \cdot \widehat{\rho}_2$$

which implies that $\widehat{\rho}_2$ is invariant under $\mathcal{H}$.

By Proposition 4.8 and the results of Lusztig as stated in [**Mal08a**, Proposition 2.1], we find a unique $\mathcal{H}$-invariant extension of ρ_2 to $\widetilde{\rho}_2 \in \mathrm{Irr}(X \rtimes \langle \delta, \psi^2 \rangle)$. We know from [**Mal08a**, Lemma 3.11] that ρ_2 extends to all outer automorphisms. Thus, we find an extension $\widetilde{\widetilde{\rho}}_2$ extending both $\widetilde{\rho}_2$ and $\widehat{\rho}_2$. For $\sigma \in \mathcal{H}$, we have $\widetilde{\widetilde{\rho}}_2^{\,\sigma} = \beta\widetilde{\widetilde{\rho}}_2$ for some $\beta \in \mathrm{Irr}(D)$. By construction we have $\beta|_{\langle \delta, \psi^2 \rangle} = 1$ and $\beta|_{\langle \gamma \rangle} = 1$ and we thereby have found an $\mathcal{H}$-invariant extension of ρ_2 to $X \rtimes D$. □

7.2.2. Local characters. Now we turn to the local characters. As in the proof of [**Mal08a**, Theorem 3.12], there is a natural embedding

$$\mathrm{SU}_3(q) \hookrightarrow \mathrm{SL}_3(q^2)$$

that maps the normalizers of Sylow 3-subgroups N onto another. We further know that $N \cong 3^{1+2}.Q_8$ where Q_8 is the quaternion group. Since every automorphism of $\mathrm{SU}_3(q)$ naturally extends to $\mathrm{SL}_3(q^2)$, it suffices to study the characters and character extensions for N as a subgroup of $\mathrm{SL}_3(q^2)$. The group N is D-stable and with

$$\widehat{H} := N \rtimes \langle \gamma, \delta \rangle \cong 3^{1+2}.\,\mathrm{SL}_2(3).2$$

we have

$$\Gamma \cong \langle \mathrm{Inn}(G \mid N), \psi, \gamma, \delta \rangle \cong \widehat{H} \times \langle \psi \rangle$$

in the inductive condition.

The irreducible $3'$-characters of N and their extensions to $\widehat{H}$ are displayed in Table 7.5. Note that each irreducible $3'$-character of $\widehat{H}$ lies in $\mathrm{Irr}(\widehat{H} \mid \rho_i')$ for some $1 \leq i \leq 3$.

Lemma 7.5. *Every $\chi \in \mathrm{Irr}_{3'}(N)$ has an $\mathcal{H}_\chi$-invariant extension to $N \rtimes D_\chi$.*

Proof. We only have to consider the non-linear characters ρ_2' and ρ_3'. We already know from [**Mal08a**, Proof of Lemma 3.7] that ρ_2' and ρ_3' both extend to $\widehat{H} \times \langle \psi \rangle$ since we can compute the irreducible characters of $\widehat{H}$ explicitly and extend them trivially to the direct product. Table 7.5 shows that the character extensions are all $\mathcal{H}$-invariant. □

N	1a	12a	4a	3a	4b	6a	12b	12c	2a	6b	12d	12e	4c	12f	3b	3c
ρ'_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ'_2	1	-1	-1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1
φ'_3	1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	1	1	1	1
φ'_1	1	1	1	1	-1	1	-1	-1	1	1	1	-1	-1	-1	1	1
ρ'_2	2	.	.	2	.	-2	.	.	-2	-2	.	.	.	.	2	2
ρ'_3	8	.	.	-1	.	.	.	.	.	.	.	.	.	.	8	8

$\widehat{H}$	1a	3a	3b	6a	2a	12a	4a	3c	3d	3e	9a	6b	6c	6d	2b	6e	8a	8b
(ρ'_1)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
(ρ'_1)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
(ρ'_2)	2	2	2	-2	-2	.	.	-1	-1	-1	-1	1	1	1	.	.	$-\alpha$	α
(ρ'_2)	2	2	2	-2	-2	.	.	-1	-1	-1	-1	1	1	1	.	.	α	$-\alpha$
(ρ'_3)	8	8	-1	.	.	.	.	2	2	2	-1	.	.	.	-2	1	.	.
(ρ'_3)	8	8	-1	.	.	.	.	2	2	2	-1	.	.	.	2	-1	.	.

TABLE 7.5. Values of the irreducible $3'$-characters of $N \cong 3^{1+2}.Q_8$ and their extensions to $\widehat{H}$ computed with [**GAP19**]. We label the conjugacy classes as it is done there and write $\alpha := \zeta_8 + \zeta_8^3$. For $\widehat{H}$, we do not label the characters but write (ρ'_i) to indicate that the respective irreducible character extends $\rho'_i \in \mathrm{Irr}(N)$.

Theorem 7.6. *The groups* $\mathrm{PSL}_3(q)$ *for* $q \neq 4$, $q \equiv 4, 7 \mod 9$ *and* $\mathrm{PSU}_3(q)$ *for* $q \neq 2$, $q \equiv 2, 5 \mod 9$ *satisfy the inductive McKay–Navarro condition for* $\ell = 3$.

PROOF. As described before, $\mathrm{Irr}_{3'}(\mathrm{PSL}_3(q))$ is in bijection with $\mathrm{Irr}_{3'}(\mathrm{SL}_3(q))$ via inflation. We verify the condition with respect to the local subgroup N. Since $\mathcal{H}$ acts trivially on the local and global characters by Lemma 7.2, the bijection from [**Mal08a**, Proposition 3.8] is also $\mathcal{H}$-equivariant and for all $\chi \in \mathrm{Irr}_{3'}(\mathrm{SL}_3(q))$ we have $(\Gamma \times \mathcal{H})_\chi = \Gamma_\chi \times \mathcal{H}$. Thus, it suffices to extend the $\mathcal{H}$-invariant character extensions in $\mathrm{Irr}(G \rtimes D_\chi)$ and $\mathrm{Irr}(N \rtimes D_\chi)$ from Lemma 7.3 and 7.5 to the remaining inner automorphisms in Γ. Thus, the inductive McKay–Navarro condition is satisfied. Using Lemma 7.4, we can argue analogously for $\mathrm{PSU}_3(q)$. □

The group $\mathrm{PSL}_3(4)$ is considered in Proposition 7.11.

7.3. The groups $\mathsf{G}_2(q)$

In this section, we consider the groups $G := \mathsf{G}_2(q)$ for $q = 2, 4, 5, 7 \mod 9$ and $\ell = 3$. We assume $q \neq 2$ since the group $\mathsf{G}_2(2)$ is considered in Proposition 7.11. Again, we follow [**Mal08a**, Section 3.3].

From there we know that the subgroup

$$M := \begin{cases} \mathrm{SL}_3(q).2 \text{ (extension with the graph automorphism)} & \text{if } q \equiv 4, 7 \mod 9, \\ \mathrm{SU}_3(q).2 \text{ (extension with the graph-field automorphism)} & \text{if } q \equiv 2, 5 \mod 9 \end{cases}$$

contains the normalizer of a Sylow 3-subgroup of G. Therefore, we can choose M as the local subgroup in the inductive condition. Since we have $M' = \mathrm{SL}_3(q)$ and $M' = \mathrm{SU}_3(q)$, respectively, we can use the results from the previous section about the irreducible $3'$-characters of M'.

The outer automorphism group of G is generated by a field automorphism ψ and we can assume that M is ψ-stable. We know from [**Mal08a**, Section 3.3] that there are each nine global and local $3'$-characters that are all ψ-invariant.

Lemma 7.7. *The characters in* $\mathrm{Irr}_{3'}(G)$ *and* $\mathrm{Irr}_{3'}(M)$ *are invariant under* Γ *and* $\mathcal{H}$.

Proof. Looking at the character values of the global characters in [**CR74**] and [**EY86**] we see that $\mathcal{H}$ acts trivially on them. We already obtained the irreducible local characters of M' in the previous section. The irreducible characters of M are $\mathrm{Ind}_{M'}^{M}(\varphi_2)$ and the extensions of ρ_1, ρ_2, ρ_3, and φ_1 to M. We know by Lemma 7.3 and Lemma 7.4 that one of these extensions is $\mathcal{H}$-invariant. Thus, the other extension also has to be $\mathcal{H}$-invariant. Since $\mathrm{Ind}_{M'}^{M}(\varphi_2)$ is the only irreducible character of its degree, it is also $\mathcal{H}$-invariant. As described in [**Mal08a**, Proposition 3.14], Γ acts trivially on all occurring characters. □

Lemma 7.8. *The irreducible* $3'$*-characters of* M *have* $\mathcal{H}$*-invariant extensions to* $M \rtimes \langle\psi\rangle$.

Proof. For the extensions of ρ_1, ρ_2, ρ_3, and φ_1 this is clear from the previous section. We also know that there is an $\mathcal{H}$-invariant extension of φ_2 to $\mathrm{SL}_3(q) \rtimes \langle\psi\rangle$, $\mathrm{SL}_3(q) \rtimes \langle\psi\gamma\rangle$, and $\mathrm{SU}_3(q) \rtimes \langle\psi^2\rangle$, respectively. Inducing this to $M \rtimes \langle\psi\rangle$ leads to an $\mathcal{H}$-invariant extension of $\mathrm{Ind}_{M'}^{M}(\varphi_2)$. □

Let $\mathbf{G}$ be the underlying algebraic group of type G_2 and $F := F_q$ a Frobenius endomorphism such that we have $\mathbf{G}^F = G$.

Lemma 7.9. *The irreducible* $3'$*-characters of* G *have* $\mathcal{H}$*-invariant extensions to* $G \rtimes \langle\psi\rangle$.

Proof. First, assume that q is odd. Then, all unipotent $3'$-characters are in the principal series of G by [**CR74**, p.402]. They can be extended to $G \rtimes \langle\psi\rangle$ as described before in the proof of Lemma 7.3. In the notation of [**CR74**], the remaining three irreducible $3'$-characters are denoted by X_{31}, X_{32}, X_{33}. We know

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta) = X_{31} + X_{32} - X_{33}$$

for some $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ and an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ with

$$\mathbf{T}^F = \begin{cases} \mathfrak{H}_3 & \text{if } q \equiv 1 \mod 3 \\ \mathfrak{H}_6 & \text{if } q \equiv -1 \mod 3 \end{cases}$$

in the notation of [**CR74**]. Since we know that all irreducible $3'$-characters are invariant under the actions of Γ and $\mathcal{H}$, $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is also invariant and we thereby find a $g \in \mathbf{G}^F$ such that $\psi \circ c_g$ fixes the pair $(\mathbf{T}, \theta)$. Now we can argue as in the proof of Proposition 6.12 to obtain $\mathcal{H}$-invariant character extensions of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, X_{31}, X_{32}, and X_{33} to $G \rtimes \langle\psi\rangle$. Note that this still works for non-trivial θ since we can trivially extend the linear character corresponding to θ to the considered semidirect product.

Now let q be even. The irreducible characters of G are described in [**EY86**] and the irreducible $3'$-characters are denoted by θ_0, θ_1, θ_2, θ_3, θ_4, θ_6, θ_7, θ_8. Let P, Q be subgroups of G as defined there. The decompositions of some characters induced to G are given in [**EY86**, Table A] and we can read off

$$\mathrm{Ind}_P^G(1) = \theta_0 + \theta_1 + \theta_2 + \theta_3, \quad \mathrm{Ind}_Q^G(1) = \theta_0 + \theta_1 + \theta_2 + \theta_4.$$

Since P and Q are ψ-invariant, Lemma 4.7 already shows the claim for $\theta_0, \theta_1, \theta_2, \theta_3$, and θ_4. We also know that θ_6 is semisimple and θ_7 is regular. Therefore they can be extended as described in Proposition 4.13. For the character θ_8, we have

$$\langle \mathrm{Ind}_P^G(\chi_1((q-1)/3)), \theta_8 \rangle = 1$$

for some linear character $\chi_1((q-1)/3)$ of P as given in [**EY86**, p. 335]. We have

$$\chi_1((q-1)/3)^\psi = \chi_1(2(q-1)/3)$$

and we know from [**EY86**, p. 339] that both characters induce the same character of G. Thus, we find a $g \in G$ such that

$$\chi_1((q-1)/3)^{\psi g} = \chi_1((q-1)/3).$$

Since it is linear, we find an $\mathcal{H}$-invariant extension of $\chi_1((q-1)/3)$ to $P \rtimes \langle \psi g \rangle$. It follows that θ_8 also has an $\mathcal{H}$-invariant extension to $G \rtimes \langle \psi g \rangle$ and as before we can conclude that we have an $\mathcal{H}$-invariant extension of θ_8 to $G \rtimes \langle \psi \rangle$. ☐

The previous considerations directly give us the following result.

Theorem 7.10. *The groups $\mathsf{G}_2(q)$ with $q \notin \{2,4\}$, $q \equiv 2,4,5,7 \mod 9$ satisfy the inductive McKay–Navarro condition for $\ell = 3$.*

7.4. Some individual groups

We now study the groups that we excluded before. The group ${}^2\mathsf{B}_2(2)$ is the Frobenius group of order 20, hence solvable, and we do not have to consider it here.

Proposition 7.11. *The groups ${}^2\mathsf{B}_2(8), {}^2\mathsf{G}_2(3)', \mathsf{G}_2(2)'$, and ${}^2\mathsf{F}_4(2)'$ satisfy the inductive McKay–Navarro condition for every prime ℓ. The groups $\mathrm{PSL}_3(4)$ and $\mathsf{G}_2(4)$ satisfy the inductive McKay–Navarro condition for $\ell = 3$.*

Proof. The computations were all made with [**GAP19**]. If ℓ is the defining characteristic, the inductive McKay–Navarro condition has already been verified in Chapter 5. Thus, we only consider primes ℓ that are different from the defining characteristic.

First, let $S = {}^2\mathsf{B}_2(8)$. The universal covering group of S is $G = 2^2.{}^2\mathsf{B}_2(8)$. For $\ell = 5$, let N be the normalizer of a Sylow 5-subgroup of G. Then, $\mathcal{H}$ acts trivially on all characters of N and since we know by [**Mal08b**, Theorem 4.1] that the inductive McKay-condition holds, it remains to consider the character extensions to their stabilizer in $G \rtimes \Gamma$ or $N \rtimes \Gamma$. Here, we see by explicit constructions that there is an $\mathcal{H}$-invariant extension for every ℓ'-character of N and G. This is sufficient since the outer automorphism group of G is cyclic. The remaining primes $\ell = 7$ and $\ell = 13$ can be treated in the same way.

Similarly, the actions on the irreducible ℓ'-characters and the extended characters can be computed straightforwardly for ${}^2\mathsf{G}_2(3)' \cong \mathrm{PSL}_2(8)$ and $\mathsf{G}_2(2)'$ for all primes.

The Tits group ${}^2\mathsf{F}_4(2)'$ has trivial Schur multiplier and again we can explicitly determine the actions of $\mathcal{H}$ and the group automorphisms on the global and local characters. The extension ${}^2\mathsf{F}_4(2)'.2 \cong {}^2\mathsf{F}_4(2)$ contained in the ATLAS is not split but we can find an outer automorphism $\kappa \in \mathrm{Aut}(G)$ of order 4. Since ${}^2\mathsf{F}_4(2)' \rtimes \langle \kappa \rangle$ is isoclinic to ${}^2\mathsf{F}_4(2) \times C_2$, we can obtain its character table from the ATLAS [**CCN$^+$85**]. As before, the extensions of the local characters can be directly computed. We see that all characters have $(\Gamma \times \mathcal{H})$-invariant extensions to their stabilizers in the automorphism group.

For $S = \mathrm{PSL}_3(4)$ and $\ell = 3$, it suffices to consider the $3'$-part of the Schur multiplier. Thus, let $G = 4^2.\mathrm{PSL}_3(4)$ and let N be the normalizer of a Sylow 3-group. As before, we can explicitly construct the extensions of the irreducible $3'$-characters of G and N to their inertia groups in $G \rtimes \Gamma$ or $N \rtimes \Gamma$. We see that the action of $(\Gamma \times \mathcal{H})_\psi$ on these extensions is trivial.

For $\mathsf{G}_2(4)$, the character table of its universal covering group $G = 2.\mathsf{G}_2(4)$ and its split extension $2.\mathsf{G}_2(4).2$ with its outer automorphism group of order 2 are included in the ATLAS [**CCN$^+$85**]. Again, we let N be the normalizer of a Sylow 3-group and see that we find a $\Gamma \times \mathcal{H}$-equivariant bijection between $\mathrm{Irr}_{3'}(G)$ and $\mathrm{Irr}_{3'}(N)$. Looking at the character table of $2.\mathsf{G}_2(4).2$ and the corresponding extension of N, we see that the extensions of the Γ-invariant characters in $\mathrm{Irr}_{3'}(G)$ and $\mathrm{Irr}_{3'}(N)$ are not all $\mathcal{H}$-invariant. However, the actions of $\mathcal{H}$ on the extended characters are permutation isomorphic. Thus, the inductive McKay–Navarro condition holds for $\ell = 3$. ☐

CHAPTER 8

Extension condition for some groups of Lie type for $\ell = 2$

In this chapter, we consider the extension part of the inductive McKay–Navarro condition for some finite groups of Lie type and $\ell = 2$.

As already mentioned before, the inductive McKay condition has been verified for all finite simple groups and the prime $\ell = 2$ in [**MS15**]. The situation is especially nice for characters of odd degree since they lie in the principal series for most finite groups of Lie type. This suggests considering the inductive McKay–Navarro condition for $\ell = 2$.

Indeed, the equivariance condition has been verified for $\ell = 2$ and all groups of Lie type not of type A and with some restrictions for type D in [**SF22**]. Moreover, the extension condition has been verified for the untwisted groups of Lie type without graph automorphisms and $\ell = 2$ in [**RSF22**].

In this chapter, we first verify the extension part of the inductive McKay–Navarro condition for $\ell = 2$ and groups of type G_2 with exceptional graph automorphisms. After this, we consider the extension condition for twisted groups of Lie type and show that it holds for groups of type ${}^3\mathsf{D}_4$ and ${}^2\mathsf{E}_6$.

8.1. The groups $\mathsf{G}_2(3^f)$ for $\ell = 2$

In this section, we verify the extension part of the inductive McKay–Navarro condition for the groups $\mathsf{G}_2(3^f)$ ($f \geq 1$) and $\ell = 2$. We consider this family of groups since they are the only groups in odd characteristic that have an exceptional graph automorphism. Therefore, they were excluded in [**RSF22**] and have to be studied separately.

Let $f \geq 2$ be an integer and set $G = \mathsf{G}_2(q)$ for $q := 3^f$. Then G is simple, non-abelian, has trivial Schur multiplier, and trivial center by Proposition 2.54 and [**MT11**, Table 24.2]. Let $\mathbf{G}$ be the corresponding algebraic group of type G_2 defined over an algebraic closure of $\mathbb{F}_3$ and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism such that $\mathbf{G}^F \cong G$.

Let γ be an exceptional graph endomorphism of $\mathbf{G}$ as in Proposition 2.43(c). Then, the outer automorphism group of G is $D := \langle \gamma \rangle$ and has order $2f$.

We consider the inductive McKay–Navarro condition for the prime $\ell = 2$. Although this case has been excluded there, the arguments in [**RSF22**, Proof of Theorem A] still apply to G if we have $q \equiv 1 \mod 4$. Otherwise, the proof given there does not work anymore since the order of γ, considered as an automorphism of G, is even. Therefore, we give an alternative proof that works for all q.

Let $d := d_2(q)$ be the order of q modulo 4, $\mathbf{S}$ a Sylow d-torus of $\mathbf{G}$, and $N := N_G(\mathbf{S})$. Then, we have a $\Gamma \times \mathcal{H}$-equivariant bijection

$$\Omega : \mathrm{Irr}_{2'}(G) \to \mathrm{Irr}_{2'}(N)$$

by [**SF22**, Theorem A]. Further, all characters in $\mathrm{Irr}_{2'}(G)$ have rational character values [**SF22**, Proposition 8.1]. We want to show that all characters in $\mathrm{Irr}_{2'}(G)$ and $\mathrm{Irr}_{2'}(N)$ possess $\Gamma \times \mathcal{H}$-invariant extensions to their stabilizers in $G \rtimes D$ and $N \rtimes D$, respectively.

Proposition 8.1. *Every $\chi \in \mathrm{Irr}_{2'}(G)$ has a $\Gamma \times \mathcal{H}$-invariant extension to $G \rtimes D_\chi$.*

Proof. All irreducible characters of G can be found in the generic character table of $\mathsf{G}_2(q)$ in [**GHL+96**]. Since G is simple, the determinants of the characters are all trivial. For irreducible characters χ of odd degree, we can apply Proposition 4.8 to obtain a $\Gamma \times \mathcal{H}$-invariant extension of χ to $G \rtimes D_\chi$. □

Proposition 8.2. *Every $\psi \in \mathrm{Irr}_{2'}(N)$ has a $\Gamma \times \mathcal{H}$-invariant extension to $N \rtimes D_\psi$.*

Proof. From [**Kle88**, Table 1] we know that there are six classes of F-stable maximal tori in $\mathbf{G}$. With the notation from there, we set

$$T := \begin{cases} T_+ \cong C_{q-1} \times C_{q-1} & \text{if } d = 1, \\ T_- \cong C_{q+1} \times C_{q+1} & \text{if } d = 2. \end{cases}$$

Then, T consists of the F-fixed points of a Sylow d-torus $\mathbf{S} \subseteq \mathbf{G}$ and we can assume $N = N_G(T) = T.D_{12}$ where D_{12} is the dihedral group of order 12. If $d = 1$, then $\mathbf{S}$ is maximally split and we can assume that γ has been chosen such that $\mathbf{S}$ is γ-stable. If $d = 2$, we know from [**MS15**, Section 3.A] that we find an automorphism that acts on G in the same way as γ and stabilizes T_- and N. In a slight abuse of notation, we continue to write γ for this automorphism.

We already know from [**SF22**] that there is a $\Gamma \times \mathcal{H}$-equivariant bijection between $\mathrm{Irr}_{2'}(G)$ and $\mathrm{Irr}_{2'}(N)$. The eight characters in $\mathrm{Irr}_{2'}(N)$ can be obtained in the following way: The group D_{12} has four linear characters and two irreducible characters of degree 2. Inflation gives rise to four linear characters of N. Further, T has three characters of order 2 that are conjugate under N. Each of these characters extends to four irreducible characters of $T.(C_2 \times C_2)$. Inducing these extensions to N yields four different irreducible characters of N of degree 3.

Thus, $\mathrm{Irr}_{2'}(N)$ consists of four linear characters and four characters of degree 3. The linear characters of N can be trivially extended to $N \rtimes D_\psi$.

The characters of degree 3 are of the form

$$\psi := \mathrm{Ind}_{T.(C_2 \times C_2)}^{N}(\widehat{\theta})$$

where $\widehat{\theta} \in \mathrm{Irr}(T.(C_2 \times C_2))$ is an extension of a linear character $\theta \in \mathrm{Irr}(T)$ of order 2. Then, θ is invariant under field automorphisms. If ψ is γ-invariant, then at least one of the three N-conjugates of θ has to be γ-invariant since γ^2 is a field automorphism. Thus, we can choose θ such that it is D_ψ-invariant. We have a $\Gamma \times \mathcal{H}$-equivariant extension map from T to $T.(C_2 \times C_2)$ and can thereby choose $\widehat{\theta}$ such that it is $D_\psi \times \mathcal{H}$-invariant. We can now extend $\widehat{\theta}$ trivially to $T.(C_2 \times C_2) \rtimes D_\psi$ and induce the resulting character to $N \rtimes D_\psi$. This gives us a $\Gamma \times \mathcal{H}$-invariant extension of ψ to $N \rtimes D_\psi$. □

With this, we can show that the inductive McKay–Navarro condition holds for G and $\ell = 2$.

Theorem 8.3. *The inductive McKay–Navarro condition is satisfied for $\mathsf{G}_2(3^f)$ for all $f \geq 1$ and $\ell = 2$.*

Proof. The equivariance part of the inductive McKay–Navarro condition has been verified in [**SF22**, Theorem A]. For $f \geq 2$, the extension part is also true by Proposition 8.1 and Proposition 8.2. For $\mathsf{G}_2(3)$, one can easily verify with [**GAP19**] that we can use Proposition 4.8 to find $(\Gamma \times \mathcal{H})_\chi$-invariant extensions for the local and global characters $\chi \in \mathrm{Irr}_{2'}(3.G) \cup \mathrm{Irr}_{2'}(N)$ where N is the normalizer of a Sylow 2-subgroup of $3.G$. □

8.2. Extensions for twisted groups

In this section, we consider extensions of characters of twisted groups of Lie type. We use the descent of scalars from Section 4.3 to study the Harish–Chandra series of certain disconnected groups and extend the constructions in [**RSF22**, Section 2 and 3] to twisted groups of Lie type. This yields the verification of the inductive McKay–Navarro condition for groups of types ${}^3\mathsf{D}_4$ and ${}^2\mathsf{E}_6$.

Since we already know from [**Ruh21**] that the inductive McKay–Navarro condition holds for all twisted groups of Lie type in their defining characteristic, we only consider groups of Lie type that are defined over a field of odd characteristic.

We study irreducible characters of odd degree in order to verify the inductive McKay–Navarro condition for $\ell = 2$. As it has already been shown and used in the verification of the inductive McKay condition, all irreducible characters of the global group with odd degree lie in the principal series.

Theorem 8.4. [**MS15**, Theorem 7.7] *Let* $\mathbf{G}$ *be a simple group of simply connected type and* F *a Frobenius endomorphism of* $\mathbf{G}$*. Assume that* $\mathbf{G}$ *is not of type* A_n *and neither* $\mathbf{G}^F = \mathrm{Sp}_{2n}(q)$ *for some odd* $n \geq 1$ *and* $q \equiv 3 \mod 4$*. Then, every* $\chi \in \mathrm{Irr}_{2'}(\mathbf{G}^F)$ *lies in the principal series of* $\mathbf{G}^F$*.*

This result suggests that we can use Harish-Chandra series to find out more about the irreducible characters of odd degree. This will be very convenient in the study of extensions of these characters. Note that it has already been shown that the groups $\mathbf{G}^F = \mathrm{Sp}_{2n}(q)$ for odd $n \geq 1$ and $q \equiv 3 \mod 4$ that have been excluded in this theorem satisfy the inductive McKay–Navarro condition for $\ell = 2$ in [**RSF22**, Section 5].

8.2.1. Harish-Chandra series of disconnected groups. We first recall two results from [**RSF22**] about the Harish-Chandra series of disconnected groups. They describe how they are related to the Harish-Chandra series of the connected component and how Galois automorphisms act on the corresponding irreducible characters. We use the notion of Harish-Chandra series for disconnected groups as described in Section 3.4.

Proposition 8.5. *Let* $\mathbf{G} = \mathbf{G}^\circ \rtimes \langle\tau\rangle$ *be a reductive group where* τ *is a quasi-central automorphism of* $\mathbf{G}^\circ$ *and* F *a Frobenius endomorphism of* $\mathbf{G}$*. Let* $\mathbf{L} = \mathbf{L}^\circ \rtimes \langle\tau\rangle$ *be an* F*-stable quasi-Levi subgroup of* $\mathbf{G}$ *contained in an* F*-stable quasi-parabolic subgroup of* $\mathbf{G}$ *and* $\delta^\circ \in \mathrm{Irr}((\mathbf{L}^\circ)^F)$*. Assume that*

- δ° *is a* τ*-invariant cuspidal character extending to its stabilizer in* $N_{\mathbf{G}^F}(\mathbf{L}^\circ)$*,*
- τ *centralizes* $N_{(\mathbf{G}^\circ)^F}(\mathbf{L}^\circ)/(\mathbf{L}^\circ)^F$*, and*
- χ° *is an irreducible character of* $(\mathbf{G}^\circ)^F$ *in the Harish-Chandra series of* δ°*.*

Then, for every $\chi \in \mathrm{Irr}(\mathbf{G}^F \mid \chi^\circ)$ *there exists a unique* $\delta \in \mathrm{Irr}(\mathbf{L}^F \mid \delta^\circ)$ *such that* χ *lies in the Harish-Chandra series of* δ*.*

This is a restricted version of [**RSF22**, Proposition 2.3] where we additionally require that τ centralizes the relative Weyl group $N_{(\mathbf{G}^\circ)^F}(\mathbf{L}^\circ)/(\mathbf{L}^\circ)^F$. The authors of [**RSF22**] noted that this is necessary to apply the Mackey formula for Harish-Chandra induction as in the proof of [**RSF22**, Proposition 2.2].

Proposition 8.6. [**RSF22**, Lemma 2.5] *In the setting of Proposition 8.5, let* $\sigma \in \mathcal{G}$ *be a Galois automorphism that stabilizes* χ° *and assume that* τ *acts trivially on* χ°*. Then, we have* $\chi^\sigma = \mu\chi$ *for some* $\mu \in \mathrm{Irr}(\mathbf{G}^F)$*.*

We later want to apply these statements to the disconnected groups occurring in the inductive McKay–Navarro condition. Therefore, we first have to establish a setting that allows us to apply the descent of scalars that has been introduced in Section 4.3.

Let $\mathbf{G}$ be a connected reductive group defined over an algebraically closed field of odd characteristic. Let F, F_0 be Frobenius endomorphisms of $\mathbf{G}$ such that we have $F = F_0^k\rho$ for a positive integer k and a graph endomorphism $\rho : \mathbf{G} \to \mathbf{G}$ of order l commuting with F_0. We use the notation of τ, pr, $\mathrm{pr}^\vee$, $\underline{\mathbf{H}}$, and $\underline{\mathbf{H}}_\rho$ for F_0^{kl}-stable subgroups $\mathbf{H} \subseteq \mathbf{G}$ as defined in Section 4.3.

Let $\mathbf{P}$ be an $\langle F_0, F\rangle$-stable parabolic subgroup of $\mathbf{G}$ with $\langle F_0, F\rangle$-stable Levi subgroup $\mathbf{L}$. We write $G := \mathbf{G}^F$, $L := \mathbf{L}^F$, and $P := \mathbf{P}^F$. From now on, we assume that

$$(N_G(\mathbf{L})/L)^{F_0} = N_G(\mathbf{L})/L.$$

This implies

$$\mathrm{pr}(N_{\underline{\mathbf{G}}_\rho^{\tau F_0}}(\underline{\mathbf{L}} \rtimes \langle\tau\rangle)) = N_G(\mathbf{L}).$$

For every F_0-stable $H \leq G$, we write $\widehat{H} := H \rtimes \langle F_0\rangle$. As in [**RSF22**, Section 2.1], we can define generalized Harish–Chandra inductions

$$R_{\widehat{L}}^{\widehat{G}} := \mathrm{Ind}_{\widehat{P}}^{\widehat{G}} \circ \mathrm{Inf}_{\widehat{L}}^{\widehat{P}}, \quad R_{\underline{\mathbf{L}}_\rho \rtimes\langle\tau\rangle}^{\underline{\mathbf{G}}_\rho \rtimes\langle\tau\rangle} := \mathrm{Ind}_{\underline{\mathbf{P}}_\rho^{\tau F_0} \rtimes\langle\tau\rangle}^{\underline{\mathbf{G}}_\rho^{\tau F_0} \rtimes\langle\tau\rangle} \circ \mathrm{Inf}_{\underline{\mathbf{L}}_\rho^{\tau F_0} \rtimes\langle\tau\rangle}^{\underline{\mathbf{P}}_\rho^{\tau F_0} \rtimes\langle\tau\rangle}$$

that naturally satisfy

$$\mathrm{pr}^\vee \circ R_{\widehat{L}}^{\widehat{G}} = R_{\underline{\mathbf{L}}_\rho \rtimes\langle\tau\rangle}^{\underline{\mathbf{G}}_\rho \rtimes\langle\tau\rangle} \circ \mathrm{pr}^\vee .$$

Note that this is a special case of the constructions in Section 3.4, see [**DM94**, Section 3].

We now state [**RSF22**, Proposition 2.6, Corollary 2.7] in our setting. The proof translates directly from the proof in the untwisted case.

Proposition 8.7. *Let $\delta \in \mathrm{Irr}(L)$ be an F_0-invariant cuspidal character that extends to $N_G(\mathbf{L})_\delta \rtimes \langle F_0\rangle$ and $\chi \in \mathrm{Irr}(G)$ a character in the Harish-Chandra series of δ.*

(a) *For $\widehat{\chi} \in \mathrm{Irr}(\widehat{G} \mid \chi)$, there exists a unique $\widehat{\delta} \in \mathrm{Irr}(\widehat{L} \mid \delta)$ such that $\widehat{\chi}$ is a constituent of $R_{\widehat{L}}^{\widehat{G}}(\widehat{\delta})$.*

(b) *If χ is fixed by F_0, then this induces a bijection $\mathrm{Irr}(\widehat{G} \mid \chi) \to \mathrm{Irr}(\widehat{L} \mid \delta)$.*

(c) *Let χ be fixed by F_0 and under the action of $(\kappa, \sigma) \in \mathrm{Aut}(G) \times \mathcal{H}$. Then, we have $\widehat{\chi}^{\kappa\sigma} = \widehat{\chi}\lambda$ for some $\lambda \in \mathrm{Irr}(\widehat{G}/G)$. Let $x \in N_G(\mathbf{L})$ such that $\delta^{\kappa\sigma x} = \delta$. Then, we have $\widehat{\delta}^{\kappa\sigma x} = \widehat{\delta}\lambda$.*

Proof. The character $\underline{\delta} := \mathrm{pr}^\vee(\delta) \in \mathrm{Irr}(\underline{\mathbf{L}}_\rho^{\tau F_0})$ is cuspidal and τ-invariant. Since Harish-Chandra induction and pr commute, $\underline{\chi} := \mathrm{pr}^\vee(\chi) \in \mathrm{Irr}(\underline{\mathbf{G}}_\rho^{\tau F_0})$ lies in the Harish-Chandra series of $\underline{\delta}$. Since $\underline{\mathbf{L}}_\rho$ is τ-stable, we have

$$\mathrm{pr}(N_{\underline{\mathbf{G}}_\rho^{\tau F_0} \rtimes\langle\tau\rangle}(\underline{\mathbf{L}_\rho})_{\underline{\delta}}) = \mathrm{pr}(N_{\underline{\mathbf{G}}_\rho^{\tau F_0}}(\underline{\mathbf{L}_\rho})_{\underline{\delta}} \rtimes \langle\tau\rangle) = N_G(\mathbf{L})_\delta \rtimes \langle F_0\rangle.$$

Further, τ acts on τF_0-stable points in $\underline{\mathbf{G}}_\rho$ in the same way as F_0^{-1} and we thereby know that τ centralizes $N_{\underline{\mathbf{G}}_\rho^{\tau F_0}}(\underline{\mathbf{L}}_\rho)/\underline{\mathbf{L}}_\rho^{\tau F_0}$.

Thus, we can apply Proposition 8.5 and for every $\widehat{\underline{\chi}} \in \mathrm{Irr}(\underline{\mathbf{G}}_\rho^{\tau F_0} \rtimes \langle\tau\rangle \mid \underline{\chi})$ there exists a unique $\widehat{\underline{\delta}} \in \mathrm{Irr}(\underline{\mathbf{L}}_\rho^{\tau F_0} \rtimes \langle\tau\rangle \mid \underline{\delta})$ such that

$$\left\langle \widehat{\underline{\chi}}, R_{\underline{\mathbf{L}}_\rho \rtimes\langle\tau\rangle}^{\underline{\mathbf{G}}_\rho \rtimes\langle\tau\rangle}(\widehat{\underline{\delta}}) \right\rangle_{\underline{\mathbf{G}}_\rho^{\tau F_0} \rtimes\langle\tau\rangle} \neq 0.$$

Using the isomorphisms given by pr, this gives the claim in (a) which implies (b).

The first claim in (c) follows directly from Gallagher's theorem since both $\widehat{\chi}^{\kappa\sigma}$ and $\widehat{\chi}$ extend χ. Since χ^σ lies in the Harish-Chandra series of δ and $\delta^{\kappa\sigma}$, we find an $x \in N_G(\mathbf{L})$ such that $\delta^{\kappa\sigma x} = \delta$. Then, $\widehat{\chi}^{\kappa\sigma} = \widehat{\chi}\lambda$ corresponds to $\widehat{\delta}^{\kappa\sigma x}$ as well as to $\lambda\widehat{\delta}$ under the bijection in (b). This shows the claim. □

8.2.2. Setting and Sylow twists. We now take a closer look at the inductive McKay–Navarro condition for $\ell = 2$. In order to have a good understanding of the local subgroup, it is useful to consider the following setting.

Let $\mathbf{G}$ be a simple simply connected group defined over an algebraically closed field of odd characteristic p. We fix a maximal torus $\mathbf{T}_0$ contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. With respect to $\mathbf{T}_0$, let F_p be the standard field endomorphism of $\mathbf{G}$ and ρ a standard graph endomorphism of order $l \in \{2, 3\}$ commuting with F_p. We consider the Frobenius endomorphism $F := F_p^f \circ \rho$ for a positive integer f and set $q = p^f$. Then, the groups $\mathbf{T}_0$, $\mathbf{B}$, and $N_{\mathbf{G}}(\mathbf{T}_0)$ are fixed by F and F_p and we write $G := \mathbf{G}^F$, $T_0 := \mathbf{T}_0^F$, and $N_0 := N_{\mathbf{G}}(\mathbf{T}_0)^F$.

The outer automorphism group of $\mathbf{G}^F$ consists of the field automorphisms and the diagonal automorphisms of $\mathbf{G}^F$, see Section 2.5.4. We denote the group of field automorphisms of $\mathbf{G}^F$ by $D := \langle F_p \rangle$. Let $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding as in Section 2.6 and $\widetilde{\mathbf{T}}_0$ an F-stable maximally split torus of $\widetilde{\mathbf{G}}$ such that $\mathbf{T}_0 = \widetilde{\mathbf{T}}_0 \cap \mathbf{G}$. We write $\widetilde{G} := \widetilde{\mathbf{G}}^F$. Note that $\widetilde{T}_0 := \widetilde{\mathbf{T}}_0^F$ induces all diagonal automorphisms of $\mathbf{G}^F$.

Let d be the order of q modulo 4, i.e. $d \in \{1, 2\}$. If $d = 1$, then we find a Sylow 1-torus $\mathbf{S}$ of $(\mathbf{G}, F)$ such that we have $\mathbf{T}_0 = C_{\mathbf{G}}(\mathbf{S})$ and $N_0 = N_{\mathbf{G}}(\mathbf{S})^F$. We want to have similar properties for $d = 2$ which can be realized using a so-called Sylow twist.

Let $v \in \mathbf{G}$ be the identity if $d = 1$ and the canonical representative of the longest element in the Weyl group $\mathbf{W} := N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ as in [**Spä10**, Definition 3.2] if $d = 2$. It induces an isomorphism $G \cong \mathbf{G}^{vF}$. Then, we find a Sylow d-torus $\mathbf{S}$ of $(\mathbf{G}, vF)$ such that $\mathbf{T}_0 = C_{\mathbf{G}}(\mathbf{S})$ by [**MS15**, Lemma 3.2], see also [**Spr74**, Section 4 and 6]. In the following, we often work with vF instead of F and consider the groups

$$T_1 := \mathbf{T}_0^{vF}, \quad N_1 := N_{\mathbf{G}^{vF}}(\mathbf{S}), \quad \widetilde{T}_1 := \widetilde{\mathbf{T}}_0^{vF}, \quad \widetilde{N}_1 := N_{\widetilde{\mathbf{G}}^{vF}}(\mathbf{S}).$$

Note that $T_1 = T_0$ and $N_1 = N_0$ if $d = 1$. By [**MS15**, Section 3.A], we find a field automorphism $(F_p)_1$ that acts on $\mathbf{G}^{vF}$ in the same way as F_p and stabilizes T_1 and N_1. To simplify notation, we also write F_p for this automorphism and, in the twisted setting, it will be clear that we actually consider the action of $(F_p)_1$.

Using these isomorphisms, we can view a character $\chi \in \mathrm{Irr}_{2'}(G)$ also as a character of $\mathbf{G}^{vF}$. Thus, χ lies in a Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$ for some semisimple element $s \in \mathbf{G}^{*F^*}$ as well as in $\mathcal{E}(\mathbf{G}^{vF}, s_1)$ for some semisimple element $s_1 \in \mathbf{G}^{*(vF)^*}$.

By construction, we can identify N_1 with the normalizer of a Sylow d-torus of $(\mathbf{G}, F)$. Thus, by Theorem 2.71 we can choose N_1 as the local subgroup in the inductive McKay–Navarro condition.

8.2.3. The groups $^3\mathsf{D}_4(q)$. With this, we can now show analogously to the proof of [**RSF22**, Theorem A] that the inductive McKay–Navarro condition holds for $^3\mathsf{D}_4(q)$ and $\ell = 2$. In fact, in an earlier version of [**RSF22**], it has already been pointed out that all requirements on the irreducible characters of odd degree and the automorphism groups are satisfied for $^3\mathsf{D}_4(q)$. Since the descent of scalars and the corresponding results have only been considered for untwisted groups, we recall the arguments that were given there.

Theorem 8.8. *The group $^3\mathsf{D}_4(q)$, where q is an odd prime power, satisfies the inductive McKay–Navarro condition for $\ell = 2$.*

Proof. We use the notation from the previous section and assume that $\mathbf{G}$ is of type D_4 and ρ is the standard graph endomorphism of order 3. Then, we have $G := \mathbf{G}^F \cong {}^3\mathsf{D}_4(q)$ for $q = p^f$. The group G is simple, has trivial Schur multiplier and outer automorphism group D.

Let $P \in \mathrm{Syl}_2(G)$. In [**SF22**, Theorem A], it has been proven that there exists an $\mathrm{Aut}(G)_P \times \mathcal{H}$-equivariant bijection

$$\Omega : \mathrm{Irr}_{2'}(G) \to \mathrm{Irr}_{2'}(M)$$

where $M := N_G(\mathbf{S}_0)$ for some Sylow d-torus $\mathbf{S}_0$ of $\mathbf{G}$ with respect to F. Every $\chi \in \mathrm{Irr}_{2'}(G)$ is rational-valued and lies in a principal series $\mathcal{E}(G, T_0, \delta)$ with $\delta^2 = 1$ by [**SF22**, Lemma 2.6, Proposition 8.1]. Thus, δ is invariant under all Galois and field automorphisms.

Let $\chi \in \mathrm{Irr}_{2'}(G)$ and $\psi := \Omega(\chi) \in \mathrm{Irr}_{2'}(M)$. Let F_0 be a field endomorphism of $\mathbf{G}$ generating D_χ. Then, F_0 centralizes N_0/T_0.

We identify M with N_1 as described above, see also [**NSV20**, Lemma 2.1]. We have

$$\psi = \mathrm{Ind}_{(N_1)_{\delta_1}}^{N_1}(\Lambda_1(\delta_1)\eta_1)$$

for some $\delta_1 \in \mathrm{Irr}(T_1)$ of order 2 and $\eta_1 \in \mathrm{Irr}_{2'}((N_1)_{\delta_1}/T_1)$. By [**RSF22**, Lemma 3.1(3)], we find an $\mathcal{H}$-invariant extension of ψ to $N_1 \rtimes D_\chi$. Further, we can apply [**RSF22**, Lemma 3.3(1)] to obtain $\mathcal{H}$-invariant extensions of χ to $G \rtimes D_\chi$. These $\mathcal{H}$-invariant extensions of ψ and χ are also invariant under the action of $(D \times \mathcal{H})_\chi = D_\chi \times \mathcal{H}$. Together with Proposition 4.10, this gives the claim. □

8.2.4. The groups $^2\mathsf{E}_6(q)$. We now consider the twisted groups of type $^2\mathsf{E}_6$. We continue to use the notation from Section 8.2.2 and assume that $\mathbf{G}$ is of type E_6 and ρ is a graph endomorphism of order 2.

We know from [**SF22**, Corollary 4.4] that we have an $(N_1 \rtimes D) \times \mathcal{H}$-equivariant extension map Λ_1 with respect to $T_1 \lhd N_1$. Further, we already know that the equivariance part of the inductive McKay–Navarro condition is satisfied, i.e. there is a $\Gamma \times \mathcal{H}$-equivariant bijection

$$\Omega : \mathrm{Irr}_{2'}(G) \to \mathrm{Irr}_{2'}(M)$$

preserving central characters where $M := N_G(\mathbf{S}_0)$ for some Sylow d-torus $\mathbf{S}_0$ of $\mathbf{G}$ with respect to F. As described above, we can identify M with N_1. Then, the bijection above is given by

$$R_{T_0}^G(\delta)_\eta \mapsto \mathrm{Ind}_{(N_1)_{\delta_1}}^{N_1}(\Lambda_1(\delta_1)\eta_1)$$

where $\delta \in \mathrm{Irr}(T_0)$, $\eta \in \mathrm{Irr}((N_0)_\delta/T_0)$, $\delta_1 \in \mathrm{Irr}(T_1)$, and $\eta_1 \in \mathrm{Irr}((N_1)_{\delta_1}/T_1)$. Moreover, if $(\mathbf{T}_0, \delta)$ corresponds to $(\mathbf{T}_0^*, s)$ as described in Proposition 3.15, then $(\mathbf{T}_0, \delta_1)$ corresponds to $(\mathbf{T}_0^*, s_1)$ where χ lies in both Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$ and $\mathcal{E}(\mathbf{G}^{vF}, s_1)$.

We first collect some information about the irreducible characters in $\mathrm{Irr}_{2'}(G)$ from the proof of [**SF22**, Proposition 8.2]. The unipotent characters of G of odd degree are all invariant under $\mathcal{H}$ and also under Γ by [**Mal08a**, Theorem 2.5]. For every non-trivial semisimple $s \in \mathbf{G}^{*F^*}$, the Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$ contains either zero or eight irreducible characters of odd degree. They can be distinguished by their degrees. If there are characters of odd degree in $\mathcal{E}(\mathbf{G}^F, s)$, then $C_{\mathbf{G}^{*F^*}}(s)$ is of type $^2\mathsf{D}_5(q).(q+1)$ and we know that $C_{\mathbf{G}^*}(s)$ is connected by [**Lü**].

Thus, the image of a non-unipotent character $\chi \in \mathcal{E}(\mathbf{G}^F, s)$ of odd degree under the actions of Γ and $\mathcal{H}$ is uniquely determined by the image of $\mathcal{E}(\mathbf{G}^F, s)$ under these actions. Recall from Section 4.1 that we have $\mathcal{E}(\mathbf{G}^F, s)^\kappa = \mathcal{E}(\mathbf{G}^F, \kappa^*(s))$ for every $\kappa \in \Gamma$ and further $\mathcal{E}(\mathbf{G}^F, s)^\sigma = \mathcal{E}(\mathbf{G}^F, s^b)$ for every $\sigma \in \mathcal{H}$ described by b. This also implies that all characters in $\mathrm{Irr}_{2'}(G)$ are fixed by the diagonal automorphisms in Γ. Since η is the character of a Weyl group, it is rational and fixed by all Galois automorphisms.

We now fix a character

$$\chi := R_{T_0}^G(\delta)_\eta \in \mathrm{Irr}_{2'}(G)$$

with $\delta \in \mathrm{Irr}(T_0)$ and $\eta \in \mathrm{Irr}((N_0)_\delta/T_0)$. Let

$$\psi := \Omega(\chi) = \mathrm{Ind}_{(N_1)_{\delta_1}}^{N_1}(\Lambda_1(\delta_1)\eta_1) \in \mathrm{Irr}_{2'}(N_1)$$

with $\delta_1 \in \mathrm{Irr}(T_1)$ and $\eta_1 \in \mathrm{Irr}((N_1)_{\delta_1}/T_1)$. We write F_0 for a generator of $D_\chi = D_\psi$.

Proposition 8.9. *There is a $(D \times \mathcal{H})_\chi$-invariant extension of χ to $\widehat{\chi} \in \mathrm{Irr}(G \rtimes D_\chi)$.*

Proof. First, assume that $\delta^{F_0} = \delta$. We can extend the linear character δ trivially to $\widehat{\delta} \in \mathrm{Irr}(T_0 \rtimes D_\chi)$ by setting

$$\widehat{\delta}(t, F_0^i) = \delta(t)$$

for all $(t, F_0^i) \in T_0 \rtimes D_\chi$. For all $(\kappa, \sigma) \in (D \times \mathcal{H})_\chi$, we find an $x \in N_0$ such that $\delta^{\kappa\sigma x} = \delta$ and we easily see $\widehat{\delta}^{\kappa\sigma x} = \widehat{\delta}$.

Let $\widehat{\chi}$ be the unique character in $\mathrm{Irr}(G \rtimes D_\chi)$ corresponding to $\widehat{\delta}$ as in Proposition 8.7(a). Then, $\widehat{\chi}^{\kappa\sigma x}$ is a constituent of

$$R_{T_0 \rtimes D_\chi}^{G \rtimes D_\chi}(\widehat{\delta}^{\kappa\sigma x})$$

since the actions of group and Galois automorphisms commute with induction and inflation. The uniqueness in Proposition 8.7(a) yields

$$\widehat{\chi}^{\kappa\sigma} = \widehat{\chi}^{\kappa\sigma x} = \widehat{\chi}$$

for all $(\kappa, \sigma) \in (D \times \mathcal{H})_\chi$.

If δ is not F_0-invariant, then we find an $n \in N_0$ such that $\delta^{F_0} = \delta^n$. The group of F^*-fixed points of the Weyl group of type D_5 is isomorphic to $(C_2)^4 \rtimes \mathrm{Sym}(4)$. By [**SF19**, Lemma 4.5], this group is isomorphic to $(N_0)_\delta/T_0$. Since the Weyl group N_0/T_0 is isomorphic to the complex reflection group with Shephard–Todd number 28, it has order $2^7 \cdot 3^2$ and we thereby have

$$[N_0 : (N_0)_\delta] = 3.$$

Thus, we can assume that the order of F_0, as an automorphism of $\mathbf{G}^F$, is a multiple of 3. Further, $F_p^{f_2} \circ \rho$ generates all outer automorphisms of odd order in D and we have $(F_p^{f_2} \circ \rho)^{f_{2'}} = F$. We can thereby assume $F_0^k = F$ for some $k \in \mathbb{N}$ that is divisible by 3.

Let V be the extended Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}_0$ as in [**Spä09**, Setting 2.1]. We have $N_0 = T_0V$ and can choose n such that it lies in V. Since the orbit of δ under the action of N_0 has size 3, we can consider powers of n and assume that n has 3-power order. We can explicitly see that the Weyl group N_0/T_0 has the Sylow 3-subgroup $C_3 \times C_3$. Thus, n^3 is contained in T_0 and we even have $n^3 \in V \cap T_0$ which is an elementary abelian 2-group, see [**Spä09**, Setting 2.1]. We can conclude $n^3 = 1$.

By the theorem of Lang–Steinberg, we find an $h \in \mathbf{G}$ such that $h^{-1}F_0(h) = n^{-1}$. Then, $c_h(\mathbf{T}_0)$ is F_0-stable and δ^h is F_0-invariant. Further, F_0 acts trivially on V and we have

$$h^{-1}F(h) = h^{-1}F_0(h)F_0(h^{-1}) \cdots F_0^k(h) = n^{-1}F_0(n^{-1}) \cdots F_0^{k-1}(n^{-1}) = n^{-k} = 1.$$

We thereby know that $h \in \mathbf{G}^F$ and it follows that $c_h(\mathbf{T}_0)$ is again maximally split. Thus, $(c_h(\mathbf{T}_0), \delta^h)$ and $(\mathbf{T}_0, \delta)$ give rise to the same Deligne–Lusztig character and we can extend δ^h as described above to obtain a $(D \times \mathcal{H})_\chi$-invariant extension of χ to $\widehat{\chi} \in \mathrm{Irr}(G \rtimes D_\chi)$. □

Proposition 8.10. *There is a $(D \times \mathcal{H})_\chi$-invariant extension of ψ to $\widehat{\psi} \in \mathrm{Irr}(N_1 \rtimes D_\chi)$.*

Proof. Let $n \in N_1$ such that $F_0' := c_n \circ F_0$ stabilizes δ_1. We can extend δ_1 trivially to $\widehat{\delta}_1 \in \mathrm{Irr}(T_1 \rtimes \langle F_0' \rangle)$. For any $\lambda \in \mathrm{Irr}(T_1 \rtimes \langle F_0' \rangle)$, let $\widehat{\Lambda}_1(\lambda)$ be the unique common extension of $\Lambda_1(\lambda|_{T_1})$ and λ. Then, $\widehat{\Lambda}_1$ is an extension map for

$$(T_1 \rtimes \langle F_0' \rangle) \lhd (N_1 \rtimes \langle F_0' \rangle).$$

By uniqueness, it is $(N_1 \rtimes D) \times \mathcal{H}$-equivariant. The character

$$\widehat{\psi} := \mathrm{Ind}_{(N_1)_{\delta_1} \rtimes \langle F_0' \rangle}^{N_1 \rtimes \langle F_0' \rangle}(\widehat{\Lambda}_1(\widehat{\delta}_1)\eta_1) \in \mathrm{Irr}(N_1 \rtimes \langle F_0' \rangle)$$

is an extension of ψ. Let $(\kappa, \sigma) \in (D \times \mathcal{H})_\psi$. Then, we have $\delta_1^{\kappa\sigma x} = \delta_1$ and by construction also $\widehat{\delta}_1^{\kappa\sigma x} = \widehat{\delta}_1$. Since η_1 is invariant under the action of $\kappa\sigma$, we can easily see $\widehat{\psi}^{\kappa\sigma x} = \widehat{\psi}$. We can now extend $\widehat{\psi}$ to the remaining inner automorphisms in Γ and restrict it again to $N_1 \rtimes D_\chi$ to obtain the claim. □

We now consider character extensions to $\widetilde{G}$ and $\widetilde{N}_1$. In the following, we naturally identify the characters of the isomorphic groups $\widetilde{G}/G$, $\widetilde{N}_1/N_1$, $\widetilde{T}_0/T_0$, and $\widetilde{T}_1/T_1$ by restriction.

Proposition 8.11. *There are extensions $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G})$ and $\widetilde{\psi} \in \mathrm{Irr}(\widetilde{N}_1)$ of χ and ψ, respectively, lying over the same central character such that for all $(\kappa, \sigma) \in (D \times \mathcal{H})_\chi$ there exists a $\mu \in \mathrm{Irr}(\widetilde{G}/G)$ with $\widetilde{\chi}^{\kappa\sigma} = \mu\widetilde{\chi}$ and $\widetilde{\psi}^{\kappa\sigma} = \mu\widetilde{\psi}$.*

Proof. The characters χ and ψ lie over the same central character of $Z(G)$ and we can choose an extension of this character to $Z(\widetilde{G})$. Let $\chi' \in \mathrm{Irr}(GZ(\widetilde{G}))$ and $\psi' \in \mathrm{Irr}(N_1 Z(\widetilde{G}))$ be the extensions of χ and ψ over this central character.

By [**MS15**, Theorem 6.3], there is a D-equivariant bijection

$$\widetilde{\Omega} : \mathrm{Irr}\left(\widetilde{G} \mid \mathrm{Irr}_{2'}(G)\right) \to \mathrm{Irr}\left(\widetilde{N}_1 \mid \mathrm{Irr}_{2'}(N_1)\right)$$

preserving central characters. For $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G} \mid \chi')$, we find characters $\widetilde{\delta} \in \mathrm{Irr}(\widetilde{T}_0 \mid \delta)$ and $\widetilde{\eta} \in \mathrm{Irr}((\widetilde{N}_0)_{\widetilde{\delta}}/\widetilde{T}_0)$ such that

$$\widetilde{\chi} = R_{\widetilde{T}_0}^{\widetilde{G}}(\widetilde{\delta})_{\widetilde{\eta}}.$$

As mentioned above, we know that $C_{\mathbf{G}^*}(s)$ is connected where $s \in \mathbf{G}^{*F^*}$ is a semisimple element corresponding to δ. Thus, [**SF19**, Lemma 4.5] and [**MS15**, Lemma 5.1] imply that the relative Weyl groups $(\widetilde{N}_0)_{\widetilde{\delta}}/\widetilde{T}_0$ and $(N_0)_\delta/T_0$ are equal. Therefore, $\eta = \widetilde{\eta}$ is fixed by all Galois automorphisms.

The character $\widetilde{\psi} := \widetilde{\Omega}(\widetilde{\chi})$ lies in $\mathrm{Irr}(\widetilde{N}_1 \mid \psi')$ and we find a $\widetilde{\delta}_1 \in \mathrm{Irr}(\widetilde{T}_1 \mid \delta_1)$ and $\widetilde{\eta}_1 \in \mathrm{Irr}((\widetilde{N}_1)_{\widetilde{\delta}_1}/\widetilde{T}_1)$ such that

$$\widetilde{\psi} = \mathrm{Ind}_{(\widetilde{N}_1)_{\widetilde{\delta}_1}}^{\widetilde{N}_1}(\widetilde{\Lambda}_1(\widetilde{\delta}_1)\widetilde{\eta}_1).$$

Here, for all $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{T}_1)$ the character $\widetilde{\Lambda}_1(\widetilde{\lambda})$ is again the unique common extension of $\Lambda_1(\widetilde{\lambda}|_{T_1})$ and $\widetilde{\lambda}$.

Since $(\widetilde{N}_1)_{\widetilde{\delta}_1}/\widetilde{T}_1$ is a Coxeter group, the character $\widetilde{\eta}_1$ is rational-valued and fixed by all Galois automorphisms. We further know that the action of D on $\widetilde{\chi}$ and $\widetilde{\psi}$ is equivariant. Thus, we only have to consider the action of $(\kappa, \sigma) \in (D \times \mathcal{H})_\psi$ on $\widetilde{\delta}$ and $\widetilde{\Lambda}(\widetilde{\delta}_1)$ and show that both characters transform in the same way up to inner automorphisms of G.

Let $x \in N$ and $x_1 \in N_1$ such that we have $\delta^{\kappa\sigma x} = \delta$ and $\delta_1^{\kappa\sigma x_1} = \delta_1$. Then, we have

$$\widetilde{\delta}^{\kappa\sigma x} = \mu\widetilde{\delta}, \quad \widetilde{\delta}_1^{\kappa\sigma x_1} = \mu_1\widetilde{\delta}_1$$

for some $\mu \in \mathrm{Irr}(\widetilde{T}_0/T_0)$ and $\mu_1 \in \mathrm{Irr}(\widetilde{T}_1/T_1)$. It remains to show that μ and μ_1 coincide under the identification $\widetilde{T}_1/T_1 \cong \widetilde{T}_0/T_0$.

To relate these characters, we consider the Lusztig series of $\widetilde{\mathbf{G}}^F$ and $\widetilde{\mathbf{G}}^{vF}$. The character $\widetilde{\chi}$ is a member of $\mathcal{E}(\widetilde{\mathbf{G}}^F, \widetilde{s})$ as well as of $\mathcal{E}(\widetilde{\mathbf{G}}^{vF}, \widetilde{s}_1)$ where $\widetilde{s}$ corresponds to $\widetilde{\delta}$ and $\widetilde{s}_1$ corresponds to $\widetilde{\delta}_1$. Since $\widetilde{\mathbf{G}}$ has connected center, the Jordan decomposition of its irreducible characters is $(D \times \mathcal{H})$-equivariant by Theorem 4.3 and [**CS13**, Theorem 3.1]. Thus, $\widetilde{\chi}^{\kappa\sigma}$ is a member of $\mathcal{E}(\widetilde{\mathbf{G}}^F, (\widetilde{s}^b)^\kappa)$ as well as of $\mathcal{E}(\widetilde{\mathbf{G}}^{vF}, (\widetilde{s}_1^b)^\kappa)$ where $b \in \mathbb{Z}$ such that σ is described by b. Then $(\widetilde{s}^b)^\kappa$ and $(\widetilde{s}_1^b)^\kappa$ correspond to $\mu\widetilde{\delta}$ and $\mu_1\widetilde{\delta}_1$, respectively.

By [**GM20**, Proposition 2.5.21], there are unique elements

$$z_\mu \in Z(\widetilde{\mathbf{G}}^*)^{F^*}, \quad z_{\mu_1} \in Z(\widetilde{\mathbf{G}}^*)^{(vF)^*}$$

such that $(\widetilde{s}^b)^\kappa$ is $\mathbf{G}^{*F^*}$-conjugate to $z_\mu\widetilde{s}$ and $(\widetilde{s}_1^b)^\kappa$ is $(\mathbf{G}^*)^{(vF)^*}$-conjugate to $z_{\mu_1}\widetilde{s}_1$. Thus, $\widetilde{\chi}^{\kappa\sigma}$ is a member of $\mathcal{E}(\widetilde{\mathbf{G}}^F, z_\mu\widetilde{s})$ as well as of $\mathcal{E}(\widetilde{\mathbf{G}}^{vF}, z_{\mu_1}\widetilde{s}_1)$. This shows that z_μ and z_{μ_1} correspond to the same element of $Z(\widetilde{\mathbf{G}}^*)^{F^*}$ under the isomorphism $(\widetilde{\mathbf{G}}^*)^{(vF)^*} \cong (\widetilde{\mathbf{G}}^*)^{F^*}$. Therefore, μ and μ_1 also coincide under the identification of $\widetilde{\mathbf{G}}^F$ and $\widetilde{\mathbf{G}}^{vF}$. □

We can now verify the inductive McKay–Navarro condition. We follow the proof of [**Ruh21**, Theorem 7.3].

Theorem 8.12. *The inductive McKay–Navarro condition holds for $S = G/Z(G)$ and the prime $\ell = 2$.*

Proof. As described above, it remains to verify the extension part of the inductive McKay–Navarro condition. Let $\chi \in \mathrm{Irr}_{2'}(G)$ and $\psi := \Omega(\chi)$ where Ω is as described at the beginning of the section. Let $\widehat{\mathcal{D}}$ and $\widehat{\mathcal{D}}'$ be affording representations of the characters $\widehat{\chi} \in \mathrm{Irr}(G \rtimes D_\chi)$ and $\widehat{\psi} \in \mathrm{Irr}(N_1 \rtimes D_\chi)$ as in Proposition 8.9 and 8.10. Analogously, let $\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{D}}'$ be affording representations of $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G})$ and $\widetilde{\psi} \in \mathrm{Irr}(\widetilde{N}_1)$ as in Proposition 8.11.

We know from Proposition 4.11 that we can define projective representations $\mathcal{P}$ of $\widetilde{G} \rtimes D_\chi$ and $\mathcal{P}'$ of $\widetilde{N}_1 \rtimes D_\chi$ by setting

$$\mathcal{P}(\widetilde{g}d) = \widetilde{\mathcal{D}}(\widetilde{g})\widehat{\mathcal{D}}(d), \quad \mathcal{P}'(\widetilde{n}d) = \widetilde{\mathcal{D}}'(\widetilde{n})\widehat{\mathcal{D}}'(d)$$

for $\widetilde{g} \in \widetilde{G}$, $\widetilde{n} \in \widetilde{N}_1$, and $d \in D_\chi$.

Let $a \in ((\widetilde{N}_1 \rtimes D) \times \mathcal{H})_\chi = \widetilde{N}_1(D \times \mathcal{H})_\chi$. We find $x \in \widetilde{N}_1$ and $y \in (D \times \mathcal{H})_\chi$ such that $a = xy$. Then, we have

$$\mathcal{P}(\widetilde{g}d)^x = \mathcal{P}(x)\mathcal{P}(\widetilde{g}d)\mathcal{P}(x)^{-1}$$

by [**Nav18**, Lemma 10.10(a)] for all $\widetilde{g} \in \widetilde{G}$ and $d \in D_\chi$. Together with Proposition 4.11, we can conclude

$$\mathcal{P}(\widetilde{g}d)^a = \mathcal{P}(x)^y\mathcal{P}(\widetilde{g}d)^y(\mathcal{P}(x)^{-1})^y \sim \mu(\widetilde{g})\mathcal{P}(\widetilde{g}d)$$

where $\mu \in \mathrm{Irr}(\widetilde{G}/G)$ such that $\widetilde{\chi}^y = \mu\widetilde{\chi}$. Since we also have $\widetilde{\psi}^y = \mu\widetilde{\psi}$, we can show analogously that we have

$$\mathcal{P}'(\widetilde{n}d)^a \sim \mu(\widetilde{n})\mathcal{P}'(\widetilde{n}d)$$

for all $\widetilde{n} \in \widetilde{N}$ and $d \in D_\chi$. This shows that S satisfies the inductive McKay–Navarro condition. □

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Curriculum Vitae

Education

since 09/2019	**PhD student**, Supervisor: Prof. Dr. Gunter Malle Technische Universität Kaiserslautern, Germany
10/2017 – 07/2019	**Mathematical Physics (M.Sc.)** Julius-Maximilians-Universität Würzburg, Germany
10/2014 – 08/2017	**Mathematical Physics (B.Sc.)** Julius-Maximilians-Universität Würzburg, Germany
10/2014 – 08/2017	**Mathematics (B.Sc.)** Julius-Maximilians-Universität Würzburg, Germany
03/2014	**Abitur** (general qualification for university entrance) Maria Ward-Gymnasium Mainz, Germany

Work experience

since 02/2021	**Research Assistant** Technische Universität Kaiserslautern, Germany
since 09/2019	**Member of the SFB-TRR 195**

Wissenschaftlicher Werdegang

Ausbildung

seit 09/2019	**Doktorandin**, Betreuer: Prof. Dr. Gunter Malle Technische Universität Kaiserslautern
10/2017 – 07/2019	**Mathematische Physik (M.Sc.)** Julius-Maximilians-Universität Würzburg
10/2014 – 11/2017	**Mathematik (B.Sc.)** Julius-Maximilians-Universität Würzburg
10/2014 – 09/2017	**Mathematische Physik (B.Sc.)** Julius-Maximilians-Universität Würzburg
03/2014	**Abitur** Maria Ward-Gymnasium Mainz

Berufserfahrung

seit 02/2021	**Wissenschaftliche Mitarbeiterin** Technische Universität Kaiserslautern
seit 09/2019	**Mitglied des SFB-TRR 195**