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STABILIZABILITY OF NONLINEAR SYSTEMS

S. Nikitin

UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Arbeitsgruppe Technomathematik
Postfach 3049
W-6750 Kaiserslautern

April 1992

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Abstract

Elements of the differential topology are used to prove nesessary conditions for stabilizability in large by a smooth feedback. Criteria for the smooth feedback stabilizing a smooth nonlinear system locally to have the smooth piecewise smooth extention, which stabilizes the system over a given compact set, have been obtained.

1 Introduction

An investigation of the stabilizability of dynamic system is one of the basic problem in the system theory. A solution of this problem is especially complicater for nonlinear systems. It was proved in [9], that "if a real analytic control system is completely controllable, then for every point p in the state space there exist a piecewise analytic feedback control that steers every state into p". An examples of the analytic controllable system being not smoothly stabilized (i.e. stabilized by a smooth feedback) has been given also in [1,3,9,11]. The local smooth stabilizability (i.e. the stabilizability by smooth feedback in some neighbourhood of equilibrium point) has been investigated in [3] and with the help of the Lyaponov's second method the necessary topological type condition of the smooth local stabilizability has been obtained. In [1,4] the local stabilizability of the real analytic control system having the linearization with noncontrollable imaginary modes has been considered.

The purpose of this paper is twofold. Firstly we prove the necessary conditions of a topological type for a smooth control system to be smoothly stabilized in large. Secondly we establish criteria for the smooth feedback $w = w(x)$ stabilizing a smooth nonlinear system locally in equilibrium x^* to have the smooth (piecewise smooth) extention $u = \overline{u}(x)$ which steers every point from a given compact K into the equilibrium x^* and $\overline{u}(x) = w(x)$ in some neighbourhood of x^*. The feedback $u = \overline{u}(x)$ is called the smoothly stabilizing extention of $w = w(x)$ over K.

The paper is organized as follows: In Section 2 we give necessary definitions and notations. In Section 3 we present some facts about the transversality, the degree of a function, the intersection numbers, the Euler characteristic and the linking numbers. In Section 4 we prove the necessary conditions of smooth stabilizability. In Section 5 the criteria for the smooth feedback locally stabilizing a smooth control system to have a smoothly (piecewise smoothly) stabilizing extention over a given compact K are proved. Finally, Section 6 contains concluding remarks.

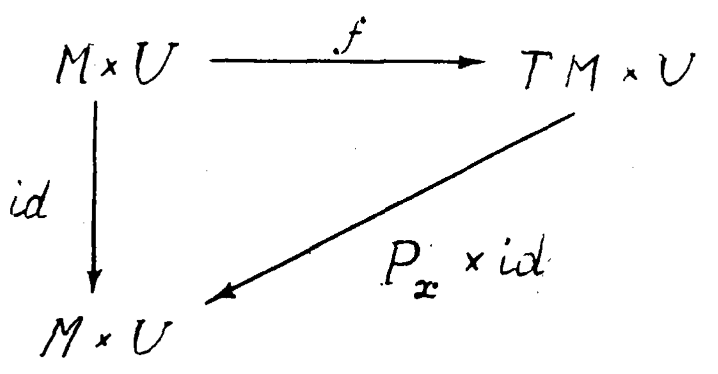
2 Preliminaries

In this paper we consider control systems whose state space is a real smooth manifold (C^∞-manifold) $M \in C^\infty$, i.e., M Hausdorff, a finite dimensional, paracompact topological space with the real smooth differential structure. The term "submainfold" is used in the sense of "regulary embedded submanifold". N is a submainfold of M, when immersion $\zeta : N \to M$ is an embedding, i.e., ζ maps N homeomorphically into its image $\zeta(M) \subset M$. The standard notations $\mathbb{R}^n, \mathbb{R}^n_+$ are used for n-dimensional Euclidean space, and for its nonnegative orthant, respectively.
A control system Σ_f is defined as a 3-tuple (M, U, f), where

(a) M is a C^∞- manifold, called the state space of the system,

(b) U is a C^∞-manifold, called the control space,

(c) f is a mapping, such that the following diagram is commutativ:



where TM is a tangent bundle of the manifold M, $P_x : TM \to M$ is the standard natural projection TM onto M, id is an identity mapping.

Such a treatment of a control system is close to that of used in [6]. Throughout the paper, it is tacitly assumed that $\dim U < \dim M$ and $f(x,u)$ denotes simultaneously both the element in $T_x \times M$ and the map $f : M \times U \to TM \times U$. We call the system Σ_f smooth, if $f \in C^\infty(M \times U, TM \times U)$- the set of all C^∞-mappings from $M \times U$ to $TM \times U$, i.e., the mappings having continuous derivatives of any order. The feedback from only two classes $C^\infty(M,U)$ and $PS(M,U)$ are considered in the paper. $PS(M,U)$ is the set of all piecewise smooth mappings form M into U. A function $u = u(x)$ is called piecewise smooth on some set Q, iff there exist a covering $Q \subseteq \bigcup_i V_i$, such that $V_i \cap V_j = \emptyset$ when $i \neq j$, for all i $Int\, V_i \neq \emptyset$, the closure $\overline{Int\, V_i}$ of the interior $Int\, V_i$ coincides with $\overline{V_i}$ the restriction $u|_{Int\, V_i}$ of u to the $Int\, V_i$ is smooth and all derivatives of u are continuous functions on $\overline{V_i}$.

Having fixed the feedback $u = u(x)$ we obtain the flow e^{tf} generated by the closed loop system

$$\dot{x} = f(x, u(x))$$

evolving over the manifold M. $e^{tf}x_0$ denotes the point into which the flow e^{tf} steers x and $e^{tf}(V) = \{e^{tf}x_0; x_0 \in V\}$. V is called an invariant set of the system, iff $e^{tf}V \subseteq V$ for all $t \geq 0$.

To define the smooth or piecewise smooth stabilizability we need the notions of an asymptotically stable equilibrium and the domain of its attraction.

Definition 2.1 A point $x^* \in M$ is called an equilibrium or a singular point of a vector field ξ on M, if $\xi(x^*) = 0$, where 0 is the zero in the tangent space $T_{x^*}M$ at x^* to the manifold M. The singular point x^* is stable, iff for every neighbourhood V of x^* ("neighbourhood V" means: V is a connected open subset of M) there exists a neighbourhood Ξ of x^*, such that $e^{tf}(\Xi) \subset V$ for all $t \geq 0$. The singlar point x^* is asymptotically stable, iff x^* is stable and there exists a neighbourhood W of x^*, such that

$$\lim_{t\to\infty} e^{tf} x_0 = x^*$$

for all $x_0 \in W$ ■

Remark: If a singular point x^* is stable, then for each x_0 in some neighbourhood of x^* the solution $e^{t\xi}x_0$ can be prolonged onto the infinit time inverval $[0,\infty)$
■

Definition 2.2 Let x^* be an asymptotically stable singular point of a vector field ξ. $D(x^*)$ is said to be the domain of x^*- attraction, iff for every $x_0 \in D(x^*)$ the solution $e^{t\xi}x_0$ exists for all $t \geq 0$ and $\lim_{t\to\infty} e^{t\xi}x_0 = x^*$ ■

Remark: It follows from the definition 1, that $D(x^*)$ is not empty and open for an asymptotically stable singular point x^* ■

The set

$$E(\Sigma_f) = \{(x,u) \in M \times U; f(x,u) = 0 \in T_x M\}$$

is called an equilibria set of the control system Σ_f.
As we shall see later, $E(\Sigma_f)$ plays a very important role in the investigation of stabilizability.

Definition 2.3. A system Σ_f is said to be smoothly (piecewise smootly) stabilized in $(x^*, u^*) \in E(\Sigma_f)$ over a domain $\mathbb{S} \subseteq M$, if there exist a smooth (piecewise smooth) feedback $u = u(x)$, such that $u(x^*) = u^*, x^*$ is an asymptotically stable singular point of the field $f(x, u(x))$ and $\mathbb{S} \subseteq D(x^*)$ ■

If $\mathbb{S} = M$, then Σ_f is called (completely) smoothly (piecewise smoothly) stabilized in $(x^*, u^*) \in E(\Sigma_f)$ (over M or in large). If there exists a neighbourhood $O(x^*)$ and Σ_f is smoothly stabilized in $(x^*, u^*) \in E(\Sigma_f)$ over $O(x^*)$, then Σ_f is said to be locally smoothly (piecewise smoothly) stabilized in (x^*, u^*).

3 Transversality, degree of function, intersection numbers, Euler characteristic, linking numbers.

3.1 Transversality

Let $C^\infty_w(M,N)$ be the set of C^∞-maps from M to N with C^∞-Whitney topology on it (see [7]). If $\varphi \in C^\infty(M,N)$ and $A \subset N$ a submainfold, then $\varphi \vdash A$ denote transversality φ to A, i.e., T_yN is spanned by T_yA and the image $d_x\varphi(T_xM)$, wherever $\varphi(x) = y \in A$, where $d_x\varphi : T_xM \to T_{\varphi(x)}N$ is the derivative of $\varphi(x)$.$\varphi(M) = \{\varphi(x); x \in M\} \cap A = \emptyset$ implies $\varphi \vdash A$. If $B, A \subset N$ are submainfolds, then $B \vdash A$ denotes $i_B \vdash A$, where i_B is an embedding corresponding to the submanifold $B \subset N$.

Theorem 3.1 [7] Let M, N be C^∞-manifolds and $A \subset N$ a C^∞-submanifold. Then the set $\{\varphi \in C^\infty(M, N) : \varphi \vdash A\}$ is residual (and therefore dense) in $C^\infty_w(M, N)$. ■

If follows from the Theorem 3.1, that for almost every control system with $f \in C^\infty_w(M \times U, TM \times U)$ the equilibira set $E(\Sigma_f)$ is a C^∞-submainfold in MxU.Let $V(x)$ be a set in T_xM. Then $rank_x V$ denotes the dimension of the vector space spanned by $V(x)$.

Theorem 3.2 The set

$$\{f \in C^\infty_w(M \times U, TM \times U) : rank_{(x,u)}\{d_x f, d_u f\} = dim M$$

for every $(x, u) \in E(\Sigma_f)\}$ is dense in $C^\infty_w(M \times U, TM \times U)$.

Proof Let $P : M \times U \to TM \times U$ is the standard embedding of $M \times U$ in $TM \times U$, such that for every $(x, u) \in M \times U$ $P(x, u) \cap T_x M = 0$. $P(M \times U)$ is C^∞-submanifold in $TM \times U$. It follows from Theorem 3.1, that the set $\{f \in C^\infty_w(M \times U, TM \times U) : f \vdash P(M \times U)\}$ is dense in $C^\infty_w(M \times U, TM \times U)$. $f \vdash P(M \times U)$ implies $rank_{(x,u)}\{d_x f, d_u f\} = \dim M$, whenever $f(x, u) \in P(M \times U)$, i.e., whenever $(x, u) \in E(\Sigma_f)$. This completes the proof. ■

Corollary 1. The set of all systems Σ_f, whose equilibria set $E(\Sigma_f)$ is C^∞-manifold of the dimention equal to $\dim U$, is dense in $C^\infty_w(M \times U, TM \times U)$ ■

3.2 Degree of function

Let M, N be compact manifolds with fixed orientations, $\dim M = \dim N$ and N is connected. Let $\varphi : M \to N$ be a C^1-map (i.e., φ has at least first continuous derivative) and $y \in N$ is a regular point of φ, i.e., $rank_x\{d_x\varphi\} = \dim N$ for all $x \in \varphi^{-1}(y) = \{\varphi(x) = y\}$. Due to compactness of M, the set $\varphi^{-1}(y)$ is finite. Thus the degree of φ can be defined in the following way.

Definition 3.1 Let M, N be oriented manifolds. Suppose $y \in N$ is any regular value for $\varphi \in C^\infty(M, N)$. Φ and Ξ are fixed atlases on M and N, respectivly. The degree of φ over y is defined as

$$\deg(\varphi, y) = \sum_{x \in \varphi^{-1}(y)} sgn(\det\left[\frac{\partial \varphi(x)}{\partial z}\right]),$$

where $\frac{\partial \varphi}{\partial z} = \{\frac{\partial \varphi_i}{\partial z_j}\}_{i,j=1}^{n}$ ($n = \dim M = \dim N$) and an n-tuple $(z_1, ..., z_n)$ is local coordinates in a neighbourhood of x. If $\varphi^{-1}(y) = \emptyset$, then $deg(\varphi, y) = 0$. For M, N which are not oriented, the degree is defined as

$$deg_2(\varphi, y) = \{\sum_{x \in \varphi^{-1}(y)} sgn(\det[\frac{\partial \varphi(x)}{\partial z}])\} mod(2)$$

■

Two maps $\varphi, \psi : M \rightarrow N$ are called C^r-homotopic ($0 \leq r \leq \infty$), as long as there exist C^r-homotopy

$$F : M \times [0,1] \rightarrow N,$$

such that $F \in C^r(M \times [0,1], N)$,

$$F \mid_{M \times 0} = \varphi \text{ and } F \mid_{M \times 1} = \psi.$$

The following Theorem is important to define the degree of a continuous map $\varphi : M \rightarrow N$.

Theorem 3.3 [7]. Let M, N be a compact oriented n-manifolds without boundaries, $n \geq 1$, with N connected. Then homotopic C^∞-maps $\varphi, \psi : M \rightarrow N$ have the same degrees, i.e., $deg(\psi, y) = deg(\varphi, z)$ for all $(y, z) \in N$.

If $\varphi : M \rightarrow N$ is continuous map, then φ can be approximated by C^∞-maps homotopic to φ ■

This theorem allows us to define the degree of a continuous map $\varphi : M \rightarrow N$ by the following way. $deg\varphi$ is defined to be $deg(g, z)$ where $g : M \rightarrow N$ is a C^∞-map homotopic to φ and $z \in N$ is a regular value for g. By Theorem 3.3 such a g exists, and $deg\varphi$ is independent of g and z. If M, N are not orientable, then $deg_2(\varphi)$ is defined to be $deg_2(g, z)$.

3.3 Linking and intersection numbers, Euler Characteristic

Let $M, N \subset \mathbb{R}^q$ be compact oriented submanifolds without boundaries, of dimensions m, n respectively. Assume that $M \cap N = \emptyset$ and $\dim M + \dim N = g - 1$. The linking number $Lk(M, N)$ is degree of the map

$$\begin{aligned} L &: M \times N \rightarrow S^{q-1}, \\ L(x,y) &= (x-y)/_{|x-y|}, \end{aligned}$$

where $S^{q-1} - (q-1)$ - dimensional sphere, $| x-y |^2 = < x-y, x-y >$ and $< x, z > = \sum_{i=1}^{n} x_i z_i$.
As a corollary of Theorem 3.3 we obtain the proposition.

Theorem 3.4 [7]. Let $F : M \times [0,1] \rightarrow \mathbb{R}^q$ be a homotopy of manifold M, such that $F(X \times 0) = M$ and $F(M \times t) \cap N = \emptyset$ for all $t \in [0,1]$. Then $Lk(F(M \times 1), N) = Lk(M, N)$ ■

By the Theorem 3.4 we have $Lk(M, N) = 0$, when tere exists a homotopy $F : M \times [0,1] \rightarrow \mathbb{R}^{q-1}$, $F(M \times 0) = M$, $F(M \times t) \cap N = \emptyset$ for all $t \in [0,1]$ and $\dim N + \dim(F(M \times 1)) < q-1$.

Now we recall the notion of intersection number. Let W be an oriented mainfold of dimension $m+n$ and $N \subset W$ a closed n-dimensional submanifold with oriented normal bundle. Let M be a compact oriented m-dimensional manifold, $\partial M = \partial N = \emptyset$. Let

$\varphi : M \to W$ be a C^∞-map transverse to N. We assign to point $x \in \varphi^{-1}(N)$ a positive or negative type according as the composite linear isomorphism

$$T_x M \xrightarrow{d\varphi} T_y W \to T_y W / T_y N, \; y = \varphi(x)$$

preserves or reverses orientation; we write $\sharp_x(\varphi, N) = 1$ or -1, respectively.
The intersection number of (φ, N) is the integer

$$\sharp(\varphi, N) = \sum_{x \in \varphi^{-1}(N)} \sharp_x(\varphi, N).$$

$\sharp(\varphi, N) = 0$, whenever $\varphi(M) \cap N = \emptyset$. It is known (see[7]), that if $\varphi, g : M \to W$ are homotopic C^∞-maps, then $\sharp(\varphi, N) = \sharp(g, N)$. Thus for any continuous map $g : M \to W$ $\sharp(g, N)$ is defined as $\sharp(g, N) = \sharp(\varphi, N)$ where φ is a C^∞-map, which is transverse to N and homotopic to g. If M is also a submainfold of W, then $\sharp(M, N)$ denotes $\sharp(i, N)$, where $i : M \to N$ is the inclusion.
For M, N or W which are not oriented, intersection numbers $\sharp_2(f, N)$ and $\sharp_2(M, N)$ are defined in the same way as $\deg_2(\varphi)$.
Let $\xi(M)$ be a zero section of a tangent bundle TM and φ be a vector field on M. Then Euler characteristic of M is

$$\chi(M) = \sharp(\varphi, \xi(M)).$$

Every vector field φ is homotopic to zero section ξ. Therefore $\chi(M)$ is independent from φ.
If z is an asymptotically stable singular point of a vector field φ, then $|\sharp_2(\varphi, \xi(M))| = 1$. Thus $D(z) = M$ implies $|\chi(M)| = 1$, where $D(z)$ is the domain of z-attraction.

4 Necessary conditions for smooth stabilizability

Firstly we consider the system Σ_f with $E(\Sigma_f)$ being a smooth submanifold in $M \times U$ without boundary and $P_x(\Sigma_f)$ being a compact subset in M. P_x, P_u denote the natural (projectors) from $M \times U$ onto M and U, respectivly, i.e., $P_x(x, u) = x$ and $P_u(x, u) = u$. We write $u_1 \simeq u_2$, whenever $u_1, u_2 : M \to U$ are homotopic continuous maps. The compactness of $P_x(E(\Sigma_f))$ implies the compactness of the graph

$$\Gamma_u(P_x(F(\Sigma_f))) = \{(x, u(x)) \in M \times U : x \in P_x(E(\Sigma_f))\}$$

for any continuous mapping $u : M \to U$.
Therefore the intersection number

$$\sharp(\Gamma_u(M), E(\Sigma_f))$$

where $\Gamma_u(M) = \{(x, u(x)) : x \in M\}$ is well defined and

$$\sharp(\Gamma_u(M), E(\Sigma_f)) = \sharp(\Gamma_g(M), E(\Sigma_f))$$

for $g \simeq u$.

Theorem 4.1. Let M, U be connected C^∞-manifolds, Σ_f be a smooth control system, $E(\Sigma_f)$ be submanifold in $M \times U$ without boundary, and $P_x(E(\Sigma_f))$ be a compact subset in M. Then if Σ_f is completly smoothly stabilized over M in a point (x^*, u^*) by the feedback $u = u(x)$, then

$$|\sharp(\Gamma_u(M), E(\Sigma_f))| = 1$$

Proof: If the smooth feedback $u = u(x)$ stabilizes Σ_f in (x^*, u^*) over M, then $\Gamma_u(M) \cap E(\Sigma_f) = (x^*, u^*)$ and

$$|\sharp_{(x^*,u^*)}(\Gamma_u(M), E(\Sigma_f))| = 1$$

■

A function $u : M \to U$ is said to be homotopic to zero, if $u \simeq v$, where v is a constant-value map, i.e., $v(x) = \overline{v} \in U$ for all $x \in M$.

Corollary 1. Let all the conditions of Theorem 4.1 be fulfiled. Then if Σ_f is completly smoothly stabilized over M in a point (x^*, u^*) by the feedback $u = u(x)$ being homotopic to zero, then

$$P_u(E(\Sigma_f)) = U. \qquad (1)$$

Proof: If $U \backslash P_u(E(\Sigma_f)) \neq \emptyset$ and $u = u(x)$ is homotopic to zero, then it follows from the connectness of U, that $u \simeq \overline{u}$, where $\overline{u} \in U \backslash P_u(E(\Sigma_f))$. Therefore

$$\sharp(\Gamma_u(M), E(\Sigma_f)) = \sharp(\Gamma_{\overline{u}}(M), E(\Sigma_f)) = 0$$

and according to Theorem 4.1 $u = u(x)$ does not stabilize Σ_f in (x^*, u^*) over M. Thus (1) is a necessary condition for the smooth stabilizability in large. ■

Corollary 2 Let $U = \mathbb{R}^m$ and

$$rank_{(x,u)}\{d_u f(x,u)\} = m$$

for all $(x, u) \in E(\Sigma_f)$; the set

$$P_x^{-1}(z) = \{(z, u) \in E(\Sigma_f); P_x(z, u) = z\}$$

be finite for every $z \in P_x(E(\Sigma_f))$. Then the smooth system Σ_f with compact $P_x(E(\Sigma_f)) \subset M$ is not smoothly stabilized over M.

Proof: Since $rank\{d_u f(x,u)\} = m$ we can choose for each point $z \in P_x(E(\Sigma_f))$ a neighbourhood $O(z)$, such that

$$P_x^{-1}(O(z)) = \bigcup_{i=1}^{l} \{(x, u_i(x)) : x \in O(z)\},$$

where l is a finite natural number, and $\{u_i(x)\}_{i=1}^{l}$ are the functions satisfying

$$f(x, u_i(x)) = 0 \in T_x M$$

for all $x \in O(z)$. $P_x(E(\Sigma_f))$ is compact. That means the existence of the finite covering $P_x(E(\Sigma_f)) \subset \bigcup_{j=1}^{\nu} O(z_j)$. Thus

$$E(\Sigma_f) \subset P_x^{-1}(\bigcup_{j=1}^{\nu} O(z_j)) = \bigcup_{j=1}^{\nu} P_x^{-1}(O(z_j)) = \bigcup_{j=1}^{\nu}\bigcup_{i=1}^{l_j}\{(x, u_i(x)) : x \in O(z_j)\}.$$

Therefore $P_u(E(\Sigma_f))$ is bounded. An application of the Corollary 1 of Theorem 4.1 completes the proof. ■

Corollary 3: Given $U = \mathbb{R}^m$ and $M = \mathbb{R}^n$. Then under the condition

$$\liminf_{|x|^2+|u|^2 \to \infty} < f(x,u), f(x,u) > > 0 \tag{2}$$

the system Σ_f is not completely smoothly stabilized.

Proof: it follows from (2), that $E(\Sigma_f)$ is compact. Therefore $P_u(E(\Sigma_f)) \neq \mathbb{R}^m$. Thus by the Corollary 1 we complete the proof. ■

The following two Theorems 4.2, 4.3 give necessary conditions for the smooth stabilizability of the system

$$\Sigma_{f,h} : \begin{cases} \dot{x} &= f(x,u), \\ y &= h(x) \end{cases}$$

by output-feedback controller $u = u(y)$ and by dynamic-feedback

$$D_g : \begin{cases} u &= u(z,y), \\ \dot{z} &= g(z,y). \end{cases}$$

h, Σ_f are supposed to be a smooth mapping and a smooth system respectivly. That means Σ_g corresponding to 3-tuple (Z, Y, g) with Z being C^∞-manifold, and the conditions $(a), (b), (c)$ from the Section 2 are fulfiled. $h \in C^\infty(M, Y)$, Y is a C^∞-manifold.
Consider

$$\mathcal{H}(E(\Sigma_f)) = \{(h(x), u) \in Y \times U : (x,u) \in E(\Sigma_f)\}.$$

If $\mathcal{H}(E(\Sigma_f))$ is a submanifold in $Y \times U$, and $P_y(\mathcal{H}(E(\Sigma_f)))$ is compact, then

$$\sharp(\Gamma_u(Y), \mathcal{H}(E(\Sigma_f)))$$

is well defined, where

$$\Gamma_u(Y) = \{(y, u(y)) \in Y \times U : y \in Y\}$$

and $u \in C^\infty(Y, U)$ is an output feedback controller.

Theorem 4.2. Let M, U, Y be connected C^∞-manifolds, $\Sigma_{f,h}$ be a smooth control system with output $y = h(x), h(x) \in C^\infty(M, Y), \mathcal{H}(E(\Sigma_f))$ be submanifold in $Y \times U$, and $P_y(\mathcal{H}(E(\Sigma_f)))$ be a compact subset in Y. Then if $\Sigma_{f,h}$ is completely smoothly stabilized over M in a point $(x^*, u^*) \in \mathcal{H}((E(\Sigma_f)))$ by the output-feedback $u = u(y)$, then:

(i) $\sharp(\Gamma_u(Y), \mathcal{H}(E(\Sigma_f)) = 1$;

(ii) if $u = u(y)$ homotopic to zero, then $P_u(E(\Sigma_f)) = U$

∎

The proofs of the conditions (i), (ii) are exactly the same as the proof of the Theorem 4.1 and the Corollary 1.
Let $\Sigma_{f,h,g}$ denote the system $\Sigma_{f,h}$ closed by the dynamic feedback D_g, i.e.,

$$\Sigma_{f,h,g}: \begin{cases} \dot{x} &= f(x,u), \\ \dot{z} &= g(z,y), \\ u &= u(z,y) \\ y &= h(x). \end{cases}$$

$E(\Sigma_{f,h,g})$ is the equilibria set of the system $\Sigma_{f,h,g}$. Thus $E(\Sigma_{f,h,g}) \subset X \times Z \times Y \times U$ and

$$E(\Sigma_{f,h,g}) = \{(x,z,y,u) \in X \times Z \times Y \times U : f(x,u) = 0, y = h(x), g(z,y) = 0\}.$$

Having applied the Theorem 4.2 to the system $\Sigma_{f,h,g}$ with the output-manifold $Y \times Z$ we obtain the following proposition.

Theorem 4.3 Let X, U, Y, Z be connected C^∞-manifolds, $\Sigma_{f,h,g}$ be a smooth control system closed by a dynamic smooth feedback D_g, $P_{z,y,u}(E(\Sigma_{f,h,g}))$ be a submanifold in $Z \times Y \times U$, and $P_{z,y}(E(\Sigma_{f,h,g}))$ be a compact subset in $Z \times Y$, where $P_{z,y}(\nu) = (\nu_z, \nu_y)$ for all $\nu = (\nu_x, \nu_z, \nu_y, \nu_u) \in X \times Z \times Y \times U$. Then if $\Sigma_{f,h,g}$ is completely smoothly stabilized over $X \times Z \times Y \times U$ in a point $(x^*, z^*, y^*, u^*) \in E(\Sigma_{f,h,g})$ by the feedback $u = u(z,y)$, then:

(i) $\sharp(\Gamma_u(Z \times Y), P_{z,y,u}(E(\Sigma_{f,h,g})) = 1$;

(ii) if $u = u(z,y)$ homotopic to zero, then $P_u(E(\Sigma_{f,h,g})) = U$

∎

Corollary 1 Given $U = \mathbb{R}^m$ and $M = \mathbb{R}^n, \mathbb{R}^n = \mathbb{R}^{n_1}_{x_1} \times \mathbb{R}^{n_2}_{x_2}$; Then under the condition

$$\liminf_{|x_2|^2 + |u|^2 \to \infty} < f(x,u), f(x,u) >> 0$$

the system

$$\begin{cases} \dot{x}_1 &= Ax_1 + \varphi(x_1, x_2), \\ \dot{x}_2 &= f(x_2, u), \\ y &= x_2 \end{cases}$$

with $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n_1}_{x_1}), \varphi(x_1, 0) = 0$ for all $x_1 \in \mathbb{R}^{n_1}_{x_1}, A \in \mathbb{R}^{n \times n}, \sigma(A) \subset \mathbb{C}_-(\sigma(A)$-eigenvalues of A, $\mathbb{C}_- = \{V \in \mathbb{C}; ReV < 0\})$ is not completely stabilized by the dynamic feedback of the form

$$\begin{aligned} u &= u(z, y), \\ \dot{z} &= Rz + ly, z \in \mathbb{R}^r_z \end{aligned}$$

where u is C^∞-mapping, $R \in \mathbb{R}^{r \times r}, l \in \mathbb{R}^r$ ■

Now we consider the smooth stabilizability of a system Σ_f over a compact set in $\mathbb{R}^n$ by a smooth feedback taking values in $\mathbb{R}^m$.
Let $u : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth controller and $\mathbb{S}$ be an (n-1)-dimensional surface homotopic to the sphere $\mathcal{S}^{n-1}$ and $f(x, u(x)) \neq 0$ forall $x \in \mathbb{S}$. Then $(u = u(x); \mathbb{S})$ generates the following submainfold in $\mathbb{R}^n \times \mathbb{R}^m$

$$\Gamma_u(\mathbb{S}) = \{(x, u(x)) \in \mathbb{R}^n \times \mathbb{R}^m : x \in \mathbb{S}\}.$$

It is easily seen, that $\dim \Gamma_u(\mathbb{S}) = n - 1$. As long as $\dim E(\Sigma_f) = m$ the linking numbers $Lk(\Gamma_u(\mathbb{S}), E_i(\Sigma_f))$ $(i = 1, 2, ..., \nu(\nu = \infty$ also possible$))$,
where $\{E_i(\Sigma_f)\}_{i=1}^{\nu}$ are connected components of $E(\Sigma_f)$, are correctly defined.

Definition 4.1. Let G be a connected closed subset in $\mathbb{R}^n$ with the smooth ∂G being homotopic to the sphere $\mathcal{S}^{n-1}$ let $u : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth feedback, such that $\Gamma_u(\partial G) \cap E(\Sigma_f) = \emptyset$. Then $\{\kappa_i(G, u, \Sigma_f) = Lk(\Gamma_u(\partial G), E_i(\Sigma_f))\}_{j=1}^{\nu}$ are called equilibria numbers of the closed loop system Σ_f in G. ■

The equilibria numbers provide the information about equilibria of the closed loop system Σ_f in $G \subset \mathbb{R}^n$.

Theorem 4.4. Let Σ_f be closed by the smooth feedback $u = u(x) : \mathbb{R}^n \to \mathbb{R}^m$ and μ_i be the number of equilibria in $E_i(\Sigma_f) \cap (G \times \mathbb{R}^m)(i = 1, 2, ..., \nu)$, where G is as in the definition 4.1. Then

$$\mu_i \geq | \kappa_i(G, u, \Sigma_f) | \quad (i = 1, ...2, \nu).$$

Proof: The Theorem follows from

$$|\kappa_i(G, u, \Sigma_f)| = | \sum_{x \in \Gamma_u(G) \cap E(\Sigma_f)} \sharp_x(\Gamma_u(\overline{G}), E(\Sigma_f))|.$$

This equality is quite wellknown in the theory of linking and intersection numbers [2] ■

The following necessary condition of smooth stabilizability ona compact set follows immediately from the Theorem 4.4.

Corollary 1 Let G be such a compact subset in $\mathbb{R}^n$ as states in the definition 4.1. Then if a system Σ_f is stabilized in $(x^*, u^*) \in IntG \times \mathbb{R}^m$ over $\overline{G}$ by a smooth feedback $u = u(x)$, then $\sum_{i=1}^{\nu} |\kappa_i(G, u, \Sigma_f)| = 1$, where ν is the number of connected components of $E(\Sigma_f)$ ■

The Corollary 1 has an especial simple form in the case of single-input systems (i.e., $m = 1$).

Corollary 2. Let Σ_f be a single-input system, $w_i = \{(x(\tau), u(\tau)); a_i \leq \tau \leq b_i\}(i = 1, 2, ..., \nu)$ connected components of the intersection $E(\Sigma_f) \cap (G \times \mathbb{R})$ and $x(a_i), x(b_i) \in \partial G$, i.e., $E(\Sigma_f) \cap (G \times \mathbb{R}) = \bigcup_i w_i$. Then if a system Σ_f is stabilized in some point $(x^*, u^*) \in IntG \times \mathbb{R}$ over $\overline{G}$ by a smooth feedback $u = w(x)$ then there exists such a connected component w_i, that

$$(w(x(a_i)) - u(a_i))(w(x(b_i)) - u(b_i)) < 0$$

and for any other $j \neq i$

$$(w(x(a_j)) - u(a_j))(w(x(b_j)) - u(b_j)) > 0$$

■

The corollary 2 provides a simple answer to the question: "When shall the feedback $u = w(x)$ stabilizing a system Σ_f locally in $(x^*, u^*) \in G \times \mathbb{R}$ not stabilize also the Σ_f over $\overline{G}$?"

5 Stabilizing extention

Consider the system Σ_f being locally stabilized in $(x^*, u^*) \in E(\Sigma_f)$, i.e., there exists the feedback $u = u(x)$, which stabilizes the Σ_f in x^* over a neighbourhood $O(x^*)$ of the point x^*. Let K be a compact set in $\mathbb{R}^n$. Then it is natural to ask whether or not we can extend $u = u(x)$ to $\mathbb{R}^n$, such that the extention $\overline{u}(x)$ is smooth (piecewise smooth), $\bar{u}(x)$ stabilizes the Σ_f in (x^*, u^*) over K and $\overline{u}(x) = u(x)$ in some neighbourhood $O'(x^*) \subset O(x^*)$. If the extention $\overline{u}(x)$ exists then $u(x)$ is said to have a (piecewiese smooth) smooth stabilizing extention over K.

5.1 Piecewise smooth stabilizing extention

A control $u : [0, T] \to U$ is said to be piecewise constant, iff there exist time points $0 = t_0 < t_1 < t_2 < ... < t_N = T$ and $u_1, ..., u_N \in U$, such that $u(t) = u_i$ for $t_{i-1} \leq t < t_i (i < N)$ and $u(t) = u_N$ for $t_{N-1} \leq t \leq t_N$.

Definition 5.1 A point p is called piecewise constantly steered into a point q, if there exist $0 < T < \infty$ and pieceweise constant control $u : [0, T] \to U$, such that the solution $x_u(t, P)$ of the initial value problem

$$\begin{aligned} \dot{x} &= f(x, u(t)) \\ x(0) &= p \end{aligned}$$

exists on the time interval $[0,T]$, is unique, $x_u(T,p) = q$. If every point $p \in K \subset M$ is piecewise constantly steered into some point $q \in V \subset M$, then the set K is called piecewise constantly accessible from the set V in the sense of the system Σ_f ■

The piecewise constant accessibility has for us a very important property contained in the following Lemma.

Lemma Let K, V be a compact and an open subset in C^∞-manifold M respectively. Then if K is piecewisely constant accessible from V in the sense of the system Σ_{-f}, then there exist natural number $N \in \mathbf{N}$($\mathbf{N}$ is the set of natural numbers), real number $T > 0$ and

$$\{u_1, ..., u_N\} \in U^N = \underbrace{U \times ... \times U}_{N},$$

where U is the control space, such that for each point $p \in K$ there exist a point $q \in V$ and $t_1 \geq 0, ..., t_N \geq 0, \sum_{i=1}^{N} t_i \leq T$, such that

$$e^{-t_1 f(u_1)} \circ e^{-t_2 f(u_2)} \circ ... \circ e^{-t_N f(u_N)} q = p,$$

where $f(u_i)$ denotes the vector field $f(x, u_i)(i = 1, 2, ..., N)$.

Proof: The set
$V(N,u,T) = \{e^{-t_1 f(u_1)} \circ e^{-t_2 f(u_2)} \circ ... \circ e^{-t_N f(u_N)} z : t_i \geq 0 (i = 1, 2, ..., N) \sum_{i=1}^{N} t_i \leq T, z \in V\}$
is open for all $N \in \mathbf{N}, T > 0, u \in U^N$. The piecewiese constant accessibility K from V in the sense of Σ_{-f} implies

$$K \subset \bigcup_{n=1}^{\infty} \bigcup_{\substack{u \in U^n \\ T>0}} V(n, u, T).$$

Thus from the compactness of K it follows, that there exists $\mu \in \mathbf{N}$, such that

$$K \subset \bigcup_{i=1}^{\mu} V(n_i, u_i, T_i),$$

where $u_i = (u_{i1}, u_{i2}, ..., u_{in_i})(i = 1, 2, ..., \mu)$. Therefore we can take $N = \sum_{i=1}^{\mu} n_i$.

$$\{u_1, ..., u_N\} = \{u_{11}, u_{12}, ..., u_{1n}, u_{21}, u_{22}, ..., u_{\mu n_\mu}\}$$

and $T = \sum_{i=1}^{\mu} T_i$ ■

Theorem 5.1 Let $M = \mathbf{R}^n, U$ be C^∞-manifold, $w = w(x)$ a smooth feedback stabilize locally the Σ_f in (x^*, w^*), K a compactset in M. Then $w = w(x)$ has a piecewise smoothly stabilizing extention over K, iff K is pieceweise constantly accessible in the sense of the system Σ_{-f} from such an open neighbourhood $O(x^*)$ of x^*, that $\overline{O}(x^*)$ is an invariant set of the closed loop system

$$\dot{x} = f(x, w(x))$$

and $w = w(x)$ stabilizes the Σ_f in (x^*, u^*) over $\overline{O(x^*)}$.

Proof. Necessity. If $w(x)$ has a piecewise smoothly stabilizing extention over K, then K is evidently piecewise constantly accessible in the sense of the system Σ_{-f} from the neighbourhood $O(x^*)$.

Sufficiency. Let $w = w(x)$ stabilize the Σ_f over $\overline{O(x^*)}$. Then according to the article [10] there exists a Liapunov's function $v(x) > 0$ in some neighbourhood $\overline{O'(x^*)} \subseteq O(x^*), v(x^*) = 0$, $\overline{O'(x^*)} = \{x \in \mathbb{R}^n; v(x) \leq \epsilon\}$ and $L_f v(x) < 0$ for $x \in \overline{O'(x^*)} \backslash x^*$, where $L_f v(x)$ is the Lie derivative of the function $v(x)$ with respect to the vector field $f(x, w(x))$, i.e., $L_f v(x) = \frac{d}{dt} v(e^{tf} x) \mid_{t=0}$. Under the conditions of the Theorem K is also piecewise constantly accessible form the $O'(x^*)$ in the sense of the system Σ_{-f}. Let $\overline{v}(x)$ be a smooth extention of $v(x)$ to $\mathbb{R}^n$, such that $\overline{v}(x) \mid_{\overline{O'(x^*)}} = v(x)$ and $\overline{v}(x) > \epsilon$ for all $x \notin \overline{O'(x^*)}$. Then the following function is correctly defined.

$$\tilde{v}(t_1, t_2, ..., t_N, x) = (e^{t_1 f(u_1)})^* \circ (e^{t_2 f(u_2)})^* \circ ... \circ (e^{t_N f(u_N)})^* \overline{v}(x),$$

where $N \in \mathbb{N}$, $\{u_1, ..., u_N\}$ are the same as in the Lemma and $(e^{t_i f(u_i)})^* \varphi(x) = \varphi(e^{t_i f(u_i)} x)$ for $\varphi : \mathbb{R}^n \to \mathbb{R}$. Due to the Lemma $T \in \mathbb{R}_+, N \in \mathbb{N}, \{u_1, ..., u_N\}$ can be chosen in such a way, that

$$K \subset \bigcup_{i=1}^{\nu} v(t_{1i}, t_{2i}, ..., t_{Ni}), \tag{3}$$

where

$$v(t_1, t_2, ..., t_N) = \{x \in \mathbb{R}^n : \tilde{v}(t_1, t_2, ..., t_N, x) < \epsilon\}.$$

$(t_{1i}, t_{2i}, ..., t_{Ni}) \in \mathbb{R}_+^N (i = 1, ..., \nu)$ are fixed N-tupels and $\sum_{j=1}^{N} t_{ji} \leq T$ for $1 \leq i \leq \nu$. The piecewise smoothly stabilizing extention $\overline{u}(x)$ of $w = w(x)$ over K is defined in the following way

$$\begin{aligned}
\overline{u}(x) &= w(x) \text{ for } x \in \overline{v(0, ..., 0)}, \\
\overline{u}(x) &= u_N \text{ for } x \in \overline{\bigcup_{0<\tau\leq t_{N1}} v(0, ..., \tau)} \backslash \overline{v(0, ..., 0)}, \\
\overline{u}(x) &= u_{N-1} \text{ for } x \in \overline{\bigcup_{0<\tau\leq t_{N-11}} v(0, ..., \tau, t_{N1})} \backslash \{ \overline{\bigcup_{0<\tau\leq t_{N1}} v(0, ..., \tau)} \bigcup \overline{v(0, ..., 0)} \}
\end{aligned}$$

and so on until

$$u(x) = u_1 \text{ for } x \in \overline{\bigcup_{0<\tau\leq t_{11}} v(\tau, t_{21}, ..., t_{N1})} \backslash \{ \overline{\bigcup_{0<\tau\leq t_{21}} v(0, \tau, ..., t_{N1})} \bigcup$$

$$(\overline{\bigcup_{0<\tau\le t_{31}} v(0,0,\tau,...,t_{N1})}) \bigcup \dots \bigcup \overline{v(0,...,0)}\}.$$

Thus $\bar{u}(x)$ is defined on the set

$$\Xi_1 = (\overline{\bigcup_{0<\tau\le t_{11}} v(\tau, t_{21},...,t_{N1})}) \bigcup (\overline{\bigcup_{0<\tau\le t_{21}} v(0,\tau,...,t_{N1})}) \bigcup \dots \bigcup \overline{v(0,...,0)}.$$

If $K \subseteq \Xi_1$, then the proof is completed. If $K\backslash\Xi_1 \neq \emptyset$, then there exist $1 \le i \le \nu$ for which

$$v(t_{1i}, t_{2i}, ..., t_{Ni}) \not\subset \Xi_1.$$

We put

$$\begin{aligned}
\bar{u}(x) &= u_N \text{ for } x \in \overline{\bigcup_{0<\tau\le t_{Ni}} v(0,...,\tau,)}\backslash\Xi_1, \\
\bar{u}(x) &= u_{N-1} \text{ for } x \in \overline{\bigcup_{0<\tau\le t_{N-1i}} v(0,0,...,\tau,t_{Ni})}\backslash\{\Xi_1 \cup (\overline{\bigcup_{0<\tau\le t_{Ni}} v(0,...,\tau,)})\} \\
&\vdots \\
\bar{u}(x) &= u_1 \text{ for } x \in \overline{\bigcup_{0<\tau\le t_{1i}} v(\tau, t_{2i},...,t_{Ni})}\backslash\{\Xi_1 \cup \\
&\quad (\overline{\bigcup_{0<\tau\le t_{Ni}} v(0,...,\tau,)}) \cup \dots \cup (\overline{\bigcup_{0<\tau\le t_{2i}} v(0,\tau,...,t_{Ni})})\}
\end{aligned}$$

Thus we have defined $\bar{u}(x)$ on the set

$$\Xi_2 = (\overline{\bigcup_{0<\tau\le t_{1i}} v(\tau,...,t_{Ni})}) \cup (\overline{\bigcup_{0<\tau\le t_{2i}} v(0,\tau,...,t_{Ni})}) \cup \dots \cup \Xi_1$$

If $K \subseteq \Xi_2$, then the proof is finished. Otherwise, if $K\backslash\Xi_2 \neq \emptyset$, then there exist $j \neq 1$ and $v(t_{1j},...,t_{Nj}) \not\subset \Xi_2$. We can define the feedback $\bar{u}(x)$ on the set

$$\Xi_3 = (\overline{\bigcup_{0<\tau\le t_{1j}} v(\tau,...,t_{Nj})}) \cup (\overline{\bigcup_{0<\tau\le t_{2j}} v(0,\tau,...,t_{Nj})}) \cup \dots \cup \Xi_2$$

in the same way as it has been done on the set Ξ_2.
The existence of a natural number μ, such that $K \subseteq \Xi_\mu$ follows from the (3) ∎

Many important consequences for applications follow from the above prooved Theorem. To formulate some of them we introduce the notion of approximate controllability.

Definition 5.2 The system Σ_f is called completely approximately piecewise constantly controllable (CAPC controllable), if for every $\epsilon > 0$ and every $p, q \in \mathbf{R}^n$ there exist $p' \in B_\epsilon(p) = \{x \in \mathbf{R}^n; |x - p| < \epsilon\}, q' \in B_\epsilon(q)$, such that p' is piecewise constantly steered

into q', where $| x |= \sqrt{\sum_{i=1}^{n} x_i^2}$.
It is easily seen, that CAPC controllability of the system Σ_f implies the piecewise constant accessiblity of every compact $K \subset \mathbb{R}^n$ from any open set $O \subset \mathbb{R}^n$ in the sense of the system Σ_{-f}. At the same time there are relatively simple verifiable sufficient conditions of CAPC controllability (see for example [5,6]). All these allow us to formulate the following corollaries of the Theorem 5.1.

Corollary 1 Let a system Σ_f be CAPC controllable. Then the Σ_f is piecewise smoothly stabilizable over $\mathbb{R}^n$ in every point $(x^*, u^*) \in E(\Sigma_f)$, such that Σ_f is locally stabilizable in (x^*, u^*).

Proof. $\mathbb{R}^n$ is a locally compact separable space. Therefore $\mathbb{R}^n \subseteq \sum_{i=1}^{\infty} K_i$, where K_i is compact for all $i = 1, 2, \ldots$. After application of the Theorem 5.1 we have the piecewise smooth feedbacks $u_1(x), u_2(x), \ldots$, such that $u_i(x)$ stabilizes the Σ_f in (x^*, u^*) over $K_i (i = 1, 2, \ldots)$. Thus $\mathbb{R}^n \subseteq \sum_{i=1}^{\infty} \tilde{K}_i$, where $\tilde{K}_i$ is an invariant compact set of the system $\dot{x} = f(x, u_i(x))$, $K_i \subseteq \tilde{K}_i$ and $u_i(x)$ stabilizes the Σ_f in (x^*, u^*) over $\tilde{K}_i$. We define the piecewise smooth feedback

$$
\begin{aligned}
\bar{u}(x) = u_1(x) \quad & \text{for} \quad x \in \tilde{K}_1, \\
\bar{u}(x) = u_2(x) \quad & \text{for} \quad x \in \tilde{K}_2 \backslash \tilde{K}_1, \\
\bar{u}(x) = u_3(x) \quad & \text{for} \quad x \in \tilde{K}_3, \backslash (\tilde{K}_2 \cup \tilde{K}_1)
\end{aligned}
$$

and so on, $\bar{u}(x) = u_i(x)$ when $x \in \tilde{K}_i \backslash (\bigcup_{j=1}^{i-1} \tilde{K}_j)$. On account of the design procedure the feedback $\bar{u}(x)$ stabilizes the system Σ_f in (x^*, u^*) over $\mathbb{R}^n$ ■

Let $[X, Y]$ denote the lie bracket of the vector fields X, Y i.e., $[X, Y]$ is such a vector field, that $L_{[X,Y]}\varphi = L_X L_Y \varphi - L_Y L_X \varphi$ for every smooth real function φ. $ad_X^i Y = ad_X(ad_X^{i-1} Y)$ and $ad_X^0 Y = Y, ad_X Y = [X, Y]$.

Corollary 2 Let $U = \mathbb{R}^m, M = \mathbb{R}^n, \Sigma_f$ be CAPC controllable. Then Σ_f is piecewise smoothly stabilizable over $\mathbb{R}^n$ in every point $(x^*, u^*) \in E(\Sigma_f)$, such that

$$
rank_{x^*} \{ ad_{f(u^*)}^i \frac{\partial f}{\partial u}(u^*) \}_{i=0}^{n-1} = n,
$$

where $f(u^*), \frac{\partial f}{\partial u}(u^*)$ denote $f(x, u^*)$, $\{\frac{\partial f}{\partial u_1}(x, u^*), \ldots, \frac{\partial f}{\partial u_m}(x, u^*)\}$, respectively ■

Let $\mathcal{L}$ be a family of smooth vector fields. Then $ad_{\mathcal{L}}^i f$ denotes the following vector fields set $ad_{\mathcal{L}}^i f = \{ad_{\xi}^i f; \xi \in \mathcal{L}\}$. $Lie(\mathcal{L})$ is lie algebra generated by $\mathcal{L}$, i.e., $Lie(\mathcal{L})$ is the minimum vector fields module over the ring $C^\infty(\mathbb{R})$ satisfying the conditions:

a) $\mathcal{L} \subset Lie(\mathcal{L})$,

b) $X, Y \in Lie(\mathcal{L})$ implies $[X, Y] \in Lie(\mathcal{L})$.

Corollary 3
Let $M = \mathbb{R}^n, U = \mathbb{R}^m, \Sigma_{f,B}$ be of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i b_i(x), \exists k \in \mathbb{N}\ rank\ \mathcal{L}_k = n,$$

$B(x) = \{b_1(x), ..., b_m(x)\}, \mathcal{L}_0 = Lie B$ and

$$\mathcal{L}_i = Lie(ad_{Q_{i-1}}^{2k-1} f \cup \mathcal{L}_{i-1}), \text{ where}$$

Q_{i-1} is a maximal subset of $\mathcal{L}_{i-1}$, which fulfils the condition $ad_{Q_{i-1}}^{2k} f \subseteq \mathcal{L}_{i-1}$ for some natural number $k \geq 1$. Then the system $\Sigma_{f,B}$ is piecewise smoothly stabilizable over $\mathbb{R}^n$ in every point $(x^*, u^*) \in E(\Sigma_{f,B})$, such that

$$rank_{x^*}\{(ad_f + \sum_{j=1}^{m} u_j^* ad_{b_j})^i\ b_\nu; 0 \leq i \leq n-1, 1 \leq \nu \leq m)\} = n$$

■

The Corollary 3 follows immediately from the sufficent conditions of CAPC controllability obtained in [6] and from the corollary 1 of the Theorem 5.1.

Corollary 4 Let $\Sigma_{f,b}$ be the system of the form

$$\dot{x} = f(x) + ub(x),$$

$x \in \mathbb{R}^2, f(x), b(x)$ be smooth vector field and $\varphi(x) = det(f(x), b(x))$, where $det(\cdot)$ is determinant of $(\cdot)$. Then if $L_b\varphi(x) \neq 0$ for $x \in \varphi^{-1}(0) = \{x \in \mathbb{R}^2; \varphi(x) = 0\}$ and for every point $x \in \mathbb{R}^2$ there exists $t \in \mathbb{R}$, such that $\varphi(e^{tb}x) = 0$, then $\Sigma_{f,b}$ is piecewise smoothly stabilizable over $\mathbb{R}^2$ in any point $x^* \in \varphi^{-1}(0)$,

$$u^* = \frac{-1}{|\ b(x^*)\ |^2} < f(x^*), b(x^*) >,$$

where $< z, y >= \sum_{i=1}^{2} z_i y_i$ and $|\ z\ |^2 = < z, z >$ ■

This corollary is a consequence of the Theorem 5.1 and the criterion of CAPC controllability prooved in [5].

5.2 Smooth stabilizing extention.

Here we describe a sufficient condition (it turns out to be a criterion for a nonlinear affine system) for the feedback $w = w(x)$ stabilizing Σ_f locally to have a smooth extention $u = \bar{u}(x)$, which stabilizes Σ_f over a given compact $K \subset \mathbb{R}^n$. We consider only the case $M = \mathbb{R}^n, U = \mathbb{R}^m$.

Definition 5.3. Let $\mathcal{P}\mathbb{R}^m$ be the class of all subsets in $\mathbb{R}^m$ and $|x| = \sum_{i=1}^{m} x_i^2$ for $x \in \mathbb{R}^m$. The topology on $\mathcal{P}\mathbb{R}^m$ defined by the base of the neighbourhoods

$$\begin{aligned}
&\{\hat{0}_\epsilon(\mathbf{S}); \mathbf{S} \subseteq \mathbb{R}^m, \epsilon > 0\},\\
&\hat{0}_\epsilon(\mathbf{S}) = \{\Xi \in \mathcal{P}\mathbb{R}^m; \Xi \backslash \mathbf{S} \subset 0_\epsilon(\mathbf{S}), \mathbf{S}\backslash \Xi \subset 0_\epsilon(\Xi)\} \text{ for } \mathbf{S} \neq \mathbb{R}^m,\\
&\hat{0}_\epsilon(\mathbb{R}^m) = \{\Xi \in \mathcal{P}\mathbb{R}^m; B_{\frac{1}{\epsilon}}(0) \subset \overline{\Xi}\},
\end{aligned}$$

where $0_\epsilon(\mathbf{S}) = \{x \in \mathbb{R}^m; \inf_{y\in\mathbf{S}} |x-y| < \epsilon\}$, is called Hausdorff's topology on $\mathcal{P}\mathbb{R}^m$ ■

The crucial role in the investigation of the smooth stabilizability is played by the notions of U-accessibility and U-controllability.

Definition 5.4 Let mapping $U : \mathbb{R}^n \to \mathcal{P}\mathbb{R}^m$ be continuous in the sense of Hausdorff's topology. Then a point p is piecewise constantly U-steered into a point q (in the sense of the system Σ_f on a connected set $P \subset \mathbb{R}^n p, q \in P$) if there exist $0 < T < \infty$ and piecewise constant $u : [0,T] \to \mathbb{R}^m$ such that the solution $x_u(t,p)$ of the initial value problem

$$\begin{aligned}
\dot{x} &= f(x, u(t)),\\
x(0) &= p
\end{aligned}$$

exists on the time interval $[0,T]$ is unique, $x_u(T,p) = q$ $(x_u(t,p) \in P$ for all $0 \le t \le T)$ and $\lim_{\tau - t \to +0} u(\tau) \in U(x_u(t,p))$ for all $0 < t < T$ and $\lim_{t\to+0} u(\tau) \in U(p)$, $\lim_{t-T\to-0} u(\tau) \in U(q)$. A set K is called piecewise constantly U-accessible in the sense of the system Σ_f on P from a set $V (V \cup K \subset P)$ if for each point $q \in K$ one can find a point $p \in V$ being piecewise constantly U-steered into q (in the sense of the system Σ_f on P). ■

Generally speaking controllability does not imply the smooth stabilizability. Some examples have been given in [1,3,9]. But if the set mapping $U : \mathbb{R}^n \to \mathcal{P}\mathbb{R}^m$ is of some special type then U-controllability implies the smooth stabilizability of the system Σ_f satisfying the condition

(S): there exist real numbers $\nu_+ > \nu_- > 0$, such that
$f(x, \alpha u_1 + (1-\alpha)u_2) \in \{k \cdot (\beta f(x,u_1) + (1-\beta) f(x,u_2)); \nu_- \le k \le \nu_+, \ 0 \le \beta \le 1\}$,
for all $0 \le \alpha \le 1$ and $u_1, u_2 \in U(x)$.

Namely, we have the following proposition.

Theorem 5.2 Let $w = w(x)$ stabilize locally Σ_f in $(x^*, w^*) \in E(\Sigma_f), K$ be a compact in $\mathbb{R}^n$. Then $w = w(x)$ has smoothly stabilizing extention over K, if there exists a set mapping $U : \mathbb{R}^n \to \mathcal{P}\mathbb{R}^m$ such that:

a) $U(x)$ is continuous on $\mathbb{R}^n \backslash x^*$ in the sense of Hausdorff's topology;

b) $U(x)$ is convex for all $x \in \mathbf{R}^n$, and condition (S) is fullfiled, $U(x^*) = w^*, (x, U(x)) \cap E(\Sigma_f) = \emptyset$ for $x \neq x^*$ and $w(x) \in Int\ U(x)$ for $x \in O(x^*)\backslash x^*$, where $O(x^*)$ is such an open neighbourhood of x^*, that $\overline{O(x^*)}$ is an invariant set of the sysem

$$\dot{x} = f(x, w(x))$$

and $w = w(x)$ stabilizes Σ_f over $\overline{O(x^*)}$;

c) there exists a connected set P, such that $O(x^*) \cup K \subset P, (x, U(x)) \cap E(\Sigma_f) = \emptyset$ for $x \in P\backslash x^*$ and K is piecewise constantly $Int\ U$-accessible from the neighbourhood $O(x^*)$ in the sense of the system Σ_{-f} on P.

Proof. Without loss of generality we can assume that $P = \mathbf{R}^n$. The feedback $w = w(x)$ stabilizes the system locally in (x^*, w^*). Therefore there exists Liapunov's function $V(x)$ (see [10]) in some neighbourhood $\overline{O'(x^*)} \subseteq O(x^*), \overline{O'(x^*)} = \{x \in \mathbf{R}^n; V(x) \leq \epsilon\}, V(x) > 0$ for $x \in \overline{O'(x^*)}\backslash x^*, V(x^*) = 0, L_f V(x) < 0$ for $x \in O'(x^*)\backslash x^*$, where $f = f(x, w(x))$. It is easy to see, that the Lemma from the previous section is true also in the case of U-accessiblity. Therefore, according to the Lemma

$$K \subset \bigcup_{\substack{t_1 \geq 0, \ldots, t_N \geq 0 \\ \sum_{i=1}^{N} t_i \leq T}} v_{\frac{\epsilon}{2}}(t_1, t_2, \ldots, t_N)$$

for some $N \in \mathbf{N}, T \in \mathbf{R}_+$, where

$$v_{\frac{\epsilon}{2}}(t_1, t_2, \ldots, t_N) = \{x \in \mathbf{R}^n; \tilde{V}(t_1, t_2, \ldots, t_N, x) < \frac{\epsilon}{2}\}$$

and $\tilde{V}$ is defined in the proof of the Theorem 5.1. Let us take

$$x \in v_{\frac{\epsilon}{2}}(t_1, t_2, \ldots, t_N).$$

Due to Sard's Theorem [7] we can assume without loss of generality, that

$$L_{f(u_N)}\tilde{V}(e^{\tau f(u_N)}\bar{x}_1) < 0, \ \tilde{V}(e^{\tau\ f(u_N)}\bar{x}_1) = \frac{\epsilon}{2}$$

where $\bar{x}_1 = e^{t_{N-1} f(u_{N-1})} \circ \ldots \circ e^{t_1 f(u_1)} x$. Otherwise we can change a little bit $\epsilon > 0$, in order to get this inequality. Let $w_\delta(x) \in C^\infty(\mathbf{R}^n, \mathbf{R}), 0 \leq w_\delta(x) \leq 1$ for all $x \in \mathbf{R}^n$ and

$$w_\delta(x) = 1 \quad \text{for} \quad x \in \overline{v_{\frac{\epsilon}{2}}(0, \ldots, 0)},$$
$$w_\delta(x) = 0 \quad \text{for} \quad x \in \mathbf{R}^n \backslash O_\delta(\overline{v_{\frac{\epsilon}{2}}(0, \ldots, 0)})$$

(the existence of $w_\delta(x)$) is wellknown [7]). Then the smooth feedback

$$\bar{u}_1^\delta(x) = w(x) w_\delta(x) + (1 - w_\delta(x)) u_N$$

stabilizes the system Σ_f in (x^*, w^*) over the set

$$W_1^\delta = \{x \in \mathbb{R}^n; (e^{tf(\bar{u}_1^\delta)})^* \tilde{V}(x) \leq \frac{\epsilon}{2}\}.$$

Now we prove the existence of such $\bar{\delta} > 0$, that for all $\delta < \bar{\delta}$ $\bar{x}_1 \in Int\ W_1^\delta$. Let $y = e^{\bar{t}f(u_N)}\bar{x}_1$ and $\tilde{V}(y) = \frac{\epsilon}{2}$. Then $\tilde{V}(y) = V(y)$ and for some $\rho > 0$ there exist $\gamma > 0$, such that $L_{f(w(x))}V(x) < -\rho, L_{f(u_N)}V(x) < -\rho$ for $x \in B_\gamma(y) = \{x \in \mathbb{R}^n; |x - y| \leq \gamma\}$. The set mapping $U(x)$ is continuous, $u_N \in Int\ U(x)$ and consequently $\gamma > 0$ can be chosen in such a way that guaranties $\alpha w(x) + (1-\alpha)u_N \in U(x)$ for all $x \in B_\gamma(y)$ and all $0 \leq \alpha \leq 1$. Thus $f(x, \bar{u}_1^\delta(x)) \neq 0$ for $x \in B_\gamma(y)$ and $\max_{x \in B_\gamma(y)} |f(x, \bar{u}_1^\delta(x))| \leq M < \infty$. Moreover, in according to the conditions (S) $f(x, \bar{u}_1^\delta(x)) = k(\beta f(x, w(x)) + (1-\beta) f(x, u_N))$ for $\nu_- \leq k \leq \nu_+, 0 \leq \beta \leq 1$ and $x \in \{O_\delta(v_{\frac{\epsilon}{2}}(0, ..., 0)) \backslash v_{\frac{\epsilon}{2}}(0, ..., 0)\} \cap B_\gamma(y)$.
Therefore

$$\frac{d}{dt}\tilde{V}(e^{tf(\bar{u}_1^\delta)}\bar{x}_1) = \frac{d}{dt}V(e^{tf(\bar{u}_1^\delta)}\bar{x}_1) \leq -\nu_- \cdot \rho$$

and

$$\tilde{V}(e^{(t+\Delta)f(\bar{u}_1^\delta)}\bar{x}_1) \leq \tilde{V}(e^{tf(\bar{u}_1^\delta)}\bar{x}_1) - \nu_- \rho \cdot \Delta \tag{4}$$

where as $e^{(t+\tau)f(\bar{u}_1^\delta)}\bar{x}_1 \in \{O_\delta(\overline{v_{\frac{\epsilon}{2}}(0, ..., 0)}) \backslash \overline{v_{\frac{\epsilon}{2}}(0, ..., 0)}\} \cap B_\gamma(y)$
for $0 \leq \tau \leq \Delta$.

Let $y_\delta = e^{t_\delta f(u_N)}\bar{x}_1$, where $e^{(t_\delta+\Delta)f(u_N)}\bar{x}_1 \in \overline{O_\delta(v_{\frac{\epsilon}{2}}(0, ..., 0))} \backslash \overline{v_{\frac{\epsilon}{2}}(0, ..., 0)}$ and $e^{(t_\delta-\Delta)f(u_N)}\bar{x}_1 \notin \overline{O_\delta(v_{\frac{\epsilon}{2}}(0, ..., 0))}$ for all sufficiently small $\Delta > 0$. It is easy to see, that $\lim_{\delta \to 0} y_\delta = y$ and consequently $\lim_{\delta \to 0} V(y_\delta) = \frac{\epsilon}{2}$. We take such $\bar{\delta} > 0$, that for all $0 < \delta \leq \bar{\delta}$

$$M(V(y_\delta) - \frac{\epsilon}{2}) < \nu_- \rho \cdot (\gamma - |y - y_\delta|)$$

and

$$\tilde{V}(x) = V(x) \text{ for } x \in \overline{O_\delta(\overline{v_{\frac{\epsilon}{2}}(0, ..., 0)})}.$$

t_δ is defined such, that $e^{t_\delta f(\bar{u}_1^\delta)}\bar{x}_1 = y_\delta$.
Inequality (4) implies

$$V(e^{(t_\delta+\Delta)f(\bar{u}_1^\delta)}\bar{x}_1) \leq \frac{\epsilon}{2}$$

when $\Delta = \frac{V(y_\delta) - \frac{\epsilon}{2}}{\nu_- \cdot \rho}$. At the same time

$$|e^{(t_\delta+\tau)f(\bar{u}_1^\delta)}\bar{x}_1 - y| \leq M\tau + |y - y_\delta| < \gamma$$

for $0 \leq \tau \leq \Delta$. Thus $\bar{x}_1 \in Int\ W_1^\delta$.
Let us take the following set $\Xi_1 = \{x \in \mathbb{R}^n; (e^{t_N f(\bar{u}_1^{\delta_1})})^* \tilde{V}(x) \leq \frac{\epsilon}{2}, \frac{d}{dt}(e^{tf(\bar{u}_1^{\delta_1})})^* \tilde{V}(x)|_{t=\bar{t}} < -\rho_1$ for such $0 \leq \bar{t} \leq t_N$, that $(e^{\bar{t}f(\bar{u}_1^{\delta_1})})^* \tilde{V}(x) = \frac{\epsilon}{2}\}$, where $\delta_1 > 0, \rho_1 > 0$. It was proved,

that $\bar{x}_1 \in Int\ \Xi_1$.
Let $\bar{x}_2 = e^{t_{N-2}f(u_{N-2})} \circ \ldots \circ e^{t_1 f(u_1)}x$. Without loss of generality it is supposed

$$\frac{d}{dt}\{(e^{\tau f(u_{N-1})})^*(e^{t_N f(\bar{u}_1^{\delta})})^*\tilde{V}(\bar{x}_2)\} \neq 0$$

where $\tau > 0$ such, that $(e^{\tau f(u_{N-1})})^*(e^{t_N f(\bar{u}_1^{\delta_1})})^*\tilde{V}(\bar{x}_2) = \frac{\epsilon}{2}$.
We take the feedback

$$\bar{u}_2^{\delta_1,\delta_2}(x) = \bar{u}_1^{\delta_1}(x)w_{\delta_2}(x) + (1 - w_{\delta_2}(x))u_{N-1},$$

where $w_{\delta_2}(x) \in C^\infty(\mathbf{R}^n,\ \mathbf{R}), w_{\delta_2}(x) = 1$ for $x \in \bar{\Xi}_1, w_{\delta_2}(x) = 0$ for $x \in \mathbf{R}^2\backslash O_{\delta_2}(\Xi_1)$ and $0 < w_{\delta_2}(x) < 1$ for $x \in O_\delta(\Xi_1)\backslash\bar{\Xi}_1$.
We constract Ξ_2 in the way being analogous to that used for the constraction of Ξ_1 and choose $\delta_2 > 0$ such, that $\bar{x}_2 \in \Xi_2, \Xi_1 \subset \Xi_2$. Then the following feedback

$$\bar{u}_N^{\delta_1,\delta_2,\ldots\delta_N}(x) = \bar{u}_{N-1}^{\delta_1,\delta_2,\ldots,\delta_{N-1}}(x)w_{\delta_N}\ (x) + (1 + w_{\delta_N}(x))u_{N_2}$$

stabilizing the system Σ_f over the set $\overline{\Xi_N}$ is obtained, where
$\Xi_N = \{z \in \mathbf{R}^n; (e^{Tf(\bar{u}_N^{\delta_1,\ldots,\delta_N})})^*\tilde{V}(z) \leq \frac{\epsilon}{2}, \frac{d}{dt}(e^{tf(\bar{u}_N^{\delta_1,\ldots,\delta_N})})^*\tilde{V}(z)\ |_{t=\bar{t}} < -\rho_n$ for such $0 \leq \bar{t} \leq T$, that $(e^{\bar{t}f(\bar{u}_N^{\delta_1,\ldots,\delta_N})})^*\tilde{V}(z) = \frac{\epsilon}{2}\}$, where $\rho_N > 0$ and $x \in Int\ \Xi_N$. If $K \subset \Xi_N$, then $\bar{u}(x) = \bar{u}_N^{\delta_1,\ldots,\delta_N}(x)$ is the smooth feedback, which we are looking for. Otherwise, if $K\backslash\Xi_N \neq \emptyset$, then we can take $z \in K\backslash\Xi_N$ and $\{t_i \geq 0\}_{i=1}^N, \sum_{i=1}^N t_i \leq T$, such that $e^{t_N f(u_N)} \circ \ldots \circ e^{t_1 f(u_1)}z \in Int\Xi_N$. Therefore by repeating the procedure described above, we obtain the smooth feedback $\bar{u}_{2N}^{\delta_1,\ldots,\delta_{2N}}(x)$ stabilizing the system Σ_f over the set Ξ_{2N} and $\Xi_N \subset Int\ \Xi_{2N}$. If $K \subseteq \Xi_{2N}$, then the smooth feedback is designed. If $K\backslash\Xi_{2N} \neq \emptyset$, then we continue the described procedure. Due to compactness of K and piecewise constant $Int\ U$-accessibility of K from the neighbourhood $O(x^*)$ in the sense of Σ_{-f} this procedure leads us to the smooth feedback stabilizing Σ_f over K and it is necessary to make only finite number of iteration. ■

The Theorem 5.2 turns out to be the criterion for the existence of a smooth stabilizing extention in the case of the nonlinear affine system of the form

$$\Sigma_{f,B} : \dot{x} = f(x) + \sum_{i=1}^m b_i(x)u_i,$$

where $f(x), B(x) = \{b_i(x)\}_{i=1}^m$ are smooth vector fields.

Theorem 5.3 Let $w = w(x)$ stabilize locally the nonlinear affine system $\Sigma_{f,B}$ in $(x^*, w^*) \in E(\Sigma_{f,B})$, $E(\Sigma_{f,B})$ be a submainfold in $\mathbf{R}_x^n \times \mathbf{R}_u^m, K$ be some compact set in $\mathbf{R}^n$. Then $w = w(x)$ has smootly stabilizing extention over K, iff there exists a set mapping $U : \mathbf{R}^m \to \mathcal{P}\mathbf{R}^m$, such that:

a) $U(x)$ is continuous on $\mathbf{R}^n\backslash x^*$ in the sense of Hausdorff topology;

b) $U(x)$ is convex for all $x \in \mathbb{R}^n, U(x^*) = w^*$ and $w(x) \in Int\ U(x)$ in some open neighbourhood $O(x^*)\backslash x^*$ of x^*, such that $\overline{O(x^*)}$ is an invariant set of the system

$$\dot{x} = f(x) + \sum_{i=1}^{m} w_i(x) b_i(x)$$

and $w = w(x)$ stabilizes $\Sigma_{f,B}$ over $\overline{O(x^*)}$;

c) there exist a connected set P, such that $O(x^*) \cup K \subset P$, $(x, U(x)) \cap E(\Sigma_f) = \emptyset$ for $x \in \mathcal{P}\backslash x^*$ and K is piecewise constant $Int\ U$-accessible from the neighbourhood $O(x^*)$ in the sense of the system $\Sigma_{-f,-B}$ on P.

Proof. Sufficiency follows immediately from the Theorem 5.2. We should only note, that $f(x) + \sum_{i=1}^{m}(\alpha w_i + (1-\alpha)u_i)b_i(x) = \alpha(f(x) + \sum_{i=1}^{m} w_i b_i(x)) + (1-\alpha)(f(x) + \sum_{i=1}^{m} u_i b_i(x))$ i.e., the conditions (S) is fulfiled.

Necessity. Let $u = \bar{u}(x)$ be a smoothly stabilizing extention of $w = w(x)$ over the compact K. $\bar{v}(x)$ is the extention of Liapunov's function $v(x)$ to $\mathbb{R}^n$, which was constructed to prove the Theorem 5.1. Consider the function

$$\rho(x) = \inf_{(y,u)\in E(\Sigma_{f,B})} (|x-y| + |\bar{u}(x) - u|).$$

There exist $\delta > 0$, such that $\bar{v}|_{\Xi_\delta} = v$, the restriction $\rho(x)$ to $\Xi_\delta = \{x \in \mathbb{R}^n; \bar{v}(x) \leq \delta\}$ is continuous, $\rho(x) > 0$ for $x \neq x^*$ and $\rho(x^*) = 0$. Introduce the set

$$\mathbf{S} = \{e^{t\tilde{f}(\bar{u}(x))}z; z \in K, t \geq 0\}.$$

where

$$\tilde{f}(\bar{u}(x)) = f(x) + \sum_{i=1}^{m} \bar{u}_i(x) b_i(x).$$

$\bar{\mathbf{S}}$ is a compact and $(x, \bar{u}(x)) \notin E(\Sigma_{f,B})$ for $x \in \overline{\bar{\mathbf{S}}\backslash(\Xi_{\frac{\delta}{2}})}$. Therefore we can find such $\epsilon > 0$, that $\rho(x) > \epsilon$ for all $x \in \overline{\bar{S}\backslash(\Xi_{\frac{\delta}{2}})}$. Let $w(x) \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and

$$w(x) = \begin{cases} 1 & \text{for} \quad x \in \overline{\Xi}_{\frac{\delta}{2}}, \\ 0 & \text{for} \quad x \notin \overline{\Xi}_\delta. \end{cases}$$

The set mapping

$$U(x) = \{u \in \mathbb{R}^m; |u - \bar{u}(x)| \leq \frac{1}{2}(w(x) \cdot \rho(x) + \epsilon(1 - w(x)))\}$$

satisfies all the conditions of the Theorem. ∎

The equilibira set $E(\Sigma_{f,B})$ of a nonlinear affine system $\Sigma_{f,B}$ with $B(x)$ being of complete rank is described by the following way

$$E(\Sigma_{f,B}) = \{(x,u) \in \mathbf{R}^n \times \mathbf{R}^m; rank\{f(x), B(x)\} = m,$$
$$u = -(B^T(x)B(x))^{-1}B^T(x)f(x)\},$$

where $B^T(x)$ is transpose of $B(x) = \{b_1(x), ..., b_m(x)\}$ and $\{b_i(x)\}_{i=1}^m$ are smooth vector fields.

This description of $E(\Sigma_{f,B})$ allows us to formulate the following Corollary of the Theorem 5.3 in the case of a single-input system $\Sigma_{f,B}$.

Corollary 1 Let $P_x(E(\Sigma_{f,B}))$ be submanifold in $\mathbf{R}^n$ and $P_x(E(\Sigma_{f,B})) = \bigcup_i w_i$, $\{w_i\}_i$ regular curves, i.e.

$$w_i = \{x(\tau) \in \mathbf{R}^n; \tau \in \mathbf{R}\} \text{ and } |\frac{dx}{d\tau}| \neq 0 \text{ for all } \tau \in \mathbf{R},$$

K be a compact subset of $\mathbf{R}^n, \Sigma_{f,b}$ be the nonlinear affine single-input system

$$\dot{x} = f(x) + b(x)u$$

with $b(x) \neq 0$ for all $x \in \mathbf{R}^n$ and

$$\{e^{tb}x_0; t \in \mathbf{R}\} \vdash P_x(E(\Sigma_{f,b})), \text{for all } x_0 \in \mathbf{R}^n,$$

$w = w(x)$ stabilize $\Sigma_{f,b}$ locally in

$$(x^*, w^*) = (x(0), u(0)) \in w_1 = \{(x(\tau), u(\tau)); -\infty < \tau < +\infty\},$$

$$U_\zeta(x) = \begin{cases} \mathbf{R} & \text{for } x \notin P_x(E(\Sigma_{f,b})), \\ \zeta_i(u - u^e(x)) > 0 & \text{for } x \in w_i (i \neq 1) \\ \zeta_1^+(u - u^e(x(\tau))) > 0 & \text{for } x(\tau) \in w_1 \ \tau > 0 \\ \zeta_1^-(u - u^e(x(\tau))) > 0 & \text{for } x(\tau) \in w_1 \ \tau < 0 \\ w^* & \text{for } x = x^*, \end{cases}$$

where $u^e(x) = -\frac{<f(x),b(x)>}{<b(x),b(x)>}, \zeta = \{\zeta_i\}_{i\geq 2} \bigcup \{\zeta_1^+, \zeta_1^-\}$ and either $\zeta_i = +1$ or $\zeta_i = -1$,

$$\zeta_1^\pm = \lim_{\tau \to \pm 0} sgn(w(x(\tau)) - u^e(x(\tau))).$$

Then $w = w(x)$ has smoothly stabilizing extention over K, iff there exists $\{\zeta_i\}_{i\geq 2}$, such that K is piecewise constantly U_ζ-accessible from the neighbourhood $O(x^*)$ in the sense of the system $\Sigma_{-f,-b}$ where $\overline{O(x*)}$ is an invariant set of the closed loop system

$$\dot{x} = f(x) + b(x)w(x)$$

and $w = w(x)$ stabilizes $\Sigma_{f,b}$ over $\overline{O(x^*)}$.

Proof. Let $\bar{\delta} > 0, B_{\bar{\delta}}(x^*) \cap w_i = \emptyset$ for $i \neq 1$ and $w_1^\pm = \{x(\tau) \in w_1; \pm\tau > 0\}$. Then there exists for every $R > 0$ such $0 < \delta < \bar{\delta}$, that

$$O_\delta(B_R(x^*) \cap w_i) \cap O_\delta(B_R(x^*) \cap w_j) = \emptyset \text{ for } i \neq j.$$

We introduce the following smooth function

$$\{\xi_1^\pm(x;\delta,R), \{\xi_i(x;\delta,R)\}_{i\geq 2}\},$$

$$\begin{aligned}
0 &\leq \xi_i(x;\delta,R) \leq 1,\\
0 &\leq \xi_1^\pm(x;\delta,R) \leq 1, \text{ for all } x \in \mathbf{R}^n,\\
&\xi_1^\pm(x;\delta,R) = 1 \text{ for } x \in w_1^\pm \backslash B_\delta(x^*),\\
&\xi_i(x;\delta,R) = 1 \text{ for } x \in w_i,\\
&\xi_1^\pm(x;\delta,R) > 0 \text{ for } x \in w_1^\pm \backslash x^*,
\end{aligned}$$

$$\begin{aligned}
\text{supp}\{\xi_1^+(x;\delta,R)\} &\cap \text{supp}\{\xi_1^-(x;\delta,R)\} = \emptyset,\\
\text{supp}\{\xi_i(x;\delta,R)\} &\cap \text{supp}\{\xi_j(x;\delta,R)\} = \emptyset \text{ for } i \neq j,\\
\text{supp}\{\xi_i(x;\delta,R)\} &\cap \text{supp}\{\xi_1^\pm(x;\delta,R)\} = \emptyset \text{ for } i \geq 2,\\
\text{supp}\{\xi_i(x;\delta,R)\} &\cap B_R(x^*) \subset O_\delta(B_R(x^*) \cap w_i),\\
\text{supp}\{\xi_1^\pm(x;\delta,R)\} &\cap B_R(x^*) \subset O_\delta(B_R(x^*) \cap w_1^\pm \backslash B_\delta(x^*)),
\end{aligned}$$

where $\text{supp}\varphi(x) = \{x \in \mathbf{R}; \varphi(x) \neq 0\}$. The functions $\{\xi_1^\pm(x;\delta,R), \{\xi_i(x;\delta,R)\}_{i\geq 2}\}$ evidently exist [7].

Let

$$U_\zeta(x;\delta,R) = \begin{cases}
\mathbf{R} \text{ for } x \bar{\in} \bigcup_i \text{supp}\{\xi_i(x;\delta,R)\} \cup x^*\\
\zeta_i(u - (-\frac{\zeta_i}{\xi_i(x;\delta,R)} + \zeta_i + u^e(x))) > 0\\
\quad \text{for } x \in \text{supp}\{\xi_i(x;\delta,R)\},\\
\zeta_1^\pm(u - (-\frac{\zeta_1^\pm}{\xi_1^\pm(x;\delta,R)} + \zeta_1^\pm + u^q(x))) > 0\\
\quad \text{for } x \in \text{supp}\{\xi_1^\pm(x,\delta,R)\},\\
w^* \text{ for } x = x^*.
\end{cases}$$

$U_\zeta(x;\delta,R)$ is continuous on $\mathbf{R}^n \backslash x^*$ in the sense of Hausdorff's topology and there exists a neighbourhood $O(x^*)$, such that $w(x) \in Int\ U_\zeta(x;\delta,R)$ for $x \in O(x^*)\backslash x^*$. Moreover it follows from the transversality $\{e^{tb}x_0; t \in \mathbf{R}\} \vdash P_x(\Sigma_{f,b})$ for all $x_0 \in \mathbf{R}^n$, that the compact K is piecewise constantly U_ζ-accessible from $O(x^*)$ for some $\{\zeta_i\}_{i\geq 2}$, iff there exist $R > 0, 0 < \delta < \bar{\delta}$, such that $K \subset B_R(x^*)$ and K is $Int\ U_\zeta(x;\delta,R)$-accessible from $O(x^*)$. Thus it is easy to see, that all the conditions of the Theorem 5.3 are fullfilled. Therefore the application of the Theorem 5.3 completes the proof. ∎

Corollary 2 Let $\Sigma_{f,b}$ be the system of the form

$$\dot{x} = f(x) + u \cdot b(x),$$

$x \in \mathbb{R}^2, f(x), b(x)$ be smooth vector fields, $L_b\varphi(x) \neq 0$ for all $x \in \varphi^{-1}(0)$, where $\varphi(x)$ and $\varphi^{-1}(0)$ have been defined in the Corollary 4 of Theorem 5.1, and for every point $x \in \mathbb{R}^2$ there exist $t \in \mathbb{R}$, such that $\varphi(e^{tb}x) = 0$. Then $\Sigma_{f,b}$ is smoothly stabilizable over every compact $K \subset \mathbb{R}^2$ in any given point $x^* \in \varphi^{-1}(0)$,

$$u^* = -\frac{1}{|b(x^*)|^2} < f(x^*), b(x^*) >$$

■

This proposition immediately follows from the Corollary 1 and the results on the controllability of the system $\Sigma_{f,b}$ obtained in [5].

6 Conclusion

We have assumed in the paper, that the system Σ_f is smooth. But all the main results obtained here can be generalized to the system Σ_f with $f(x, u)$ having at least continuous derivatives of the first order. The Theorem 5.1 remains to be true also for an abstract nonlinear system in Banach space under the natural suppositions of the existence and uniqueness of the solution. If the Liapunov's function used to prove the Theorem 5.1, the feedback $w = w(x)$, vector fields $f(x, u), f(x, w(x))$ are real analytic, then the designed stabilizing extention $u = \bar{u}(x)$ is piecewise analytic in the sense of the paper [9].