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DECOUPLING NORMALIZING TRANSFORMATIONS AND LOCAL STABILIZATION OF NONLINEAR SYSTEMS

S. Nikitin

UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Arbeitsgruppe Technomathematik
Postfach 3049
W-6750 Kaiserslautern

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Abstract

A method of decoupling normalizing transformations has been developed. According to the method only the part of differential equations corresponding to the dynamic on a center manifold has to be modifyed by means of the normalizing transformations of a Poincare type. The existence of the normalizing transformation completely decoupling the stable dynamic from the center manifold dynamic has been proved. A numerical procedure for the calculation of asymptotic series for the decoupling normalizing transformation has been proposed. The developed method is especially important for the perturbation theory of center manifold and, in particular, for the local stabilization theory. In the paper some sufficient conditions for local stabilization have been given.

1 Introduction

Consider the system

$$\dot{\bar{x}} = A\bar{x} + \bar{\Phi}(\bar{x}, \bar{y}),$$
$$\dot{\bar{y}} = B\bar{y} + \bar{\Psi}(\bar{x}, \bar{y}), \tag{1}$$

where $(\bar{x}, \bar{y}) \in \mathrm{R^m} \times \mathrm{R^n}$, $\mathrm{R^n}$ is used for n-dimensional Euclidean space, the eigenvalues of $A \in \mathrm{R^{m\times m}}$ have zero real parts, the eigenvalues of $B \in \mathrm{R^{n\times n}}$ have negative real parts, $\bar{\Phi}$ and $\bar{\Psi}$, are at least C^2 functions which vanish together with their derivatives at the origin, i.e.,

$$\bar{\Phi} \in C^k(\mathrm{R^m} \times \mathrm{R^n}, \mathrm{R^m}), \qquad \bar{\Phi}(0,0) = 0, \qquad \mathrm{d}\bar{\Phi}(0,0) = 0,$$
$$\bar{\Psi} \in C^k(\mathrm{R^m} \times \mathrm{R^n}, \mathrm{R^n}), \qquad \bar{\Psi}(0,0) = 0, \qquad \mathrm{d}\bar{\Psi}(0,0) = 0, \tag{2}$$

where $k \geq 2$, $d\Phi = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}\right)$ and $C^k(\mathrm{R^m} \times \mathrm{R^n}, \mathrm{R}^\ell)$ is the class of all functons

$$\varphi : \mathrm{R^m} \times \mathrm{R^n} \to \mathrm{R}^\ell,$$

which have the continuous derivatives of order k.

To investigate the dynamic of the system (1) in a neighborhood of the origin we apply the center manifold theory which mainly consists of the following three theorems.

Theorem 1.1 [4,8] Given the conditions (2), then there exists a center manifold

$$M_c = \{(\bar{x}, \bar{y}) \in B_\delta(0) \times \mathrm{R^n}; \bar{\mathrm{y}} = \mathrm{h}(\bar{\mathrm{x}})\},$$

where $B_\delta(0) = \{x \in \mathrm{R^m}; |\,\bar{\mathrm{x}}\,| < \delta\}$, $|\,\bar{\mathrm{x}}\,|^2 = <\bar{\mathrm{x}}, \bar{\mathrm{x}}>$ and $<\mathrm{x}, \mathrm{z}> = \sum_{i=1}^{\ell} \mathrm{x_i z_i}$ for $\mathrm{x, z} \in \mathrm{R}^\ell$, $\mathrm{h} \in \mathrm{C^k(R^m, R^n)}$ and δ is a sufficiently small real positive number.

It is convenient to use the following notations:

$$\bar{f}(\bar{x}, \bar{y}) = \left(A\bar{x} + \bar{\Phi}(\bar{x}, \bar{y}), B\bar{y} + \bar{\Psi}(\bar{x}, \bar{y})\right)^T,$$

$e^{t\bar{f}}$ denotes the flow generated by the vector field $\bar{f}$. $e^{t\bar{f}}(x, y)$ is the point drifted by the flow $e^{t\bar{f}}$ in time t from the point (x, y). The zero solution is said to be stable, iff for every neighborhood W there exists a neighborhood V, such that

$$e^{t\bar{f}}V \subset W \qquad \forall t \geq 0,$$

where $e^{t\bar{f}}V = \{e^{t\bar{f}}(x,y);\ (x,y) \in V\}$. The zero solution is asymptotically stable, iff it is stable and there exists a neighborhood Ξ, such that

$$\lim_{t\to+\infty} e^{t\bar{f}}(x,y) = 0$$

for all $(x,y) \in \Xi$. The flow on the center manifold M_c is governed by the system

$$\dot{z} = Az + \bar{\Phi}(z, h(z)). \tag{3}$$

The next theorem tells us that (3) has all the necessary information needed to determine the asymptotic behavior of (1) in a neighborhood of the origin.

Theorem 1.2 [4]

(a) If the zero solution of (3) is stable (asymptotically stable) (unstable), then the zero solution of (1) is stable (asymptotically stable) (unstable).

(b) If the zero solution of (3) is stable, then there exists a neighborhood V of the origin, such that for every $(x_0, y_0) \in V$ one can find z_0, such that

$$e^{t\bar{f}}(x_0, y_0) = (z(t, z_0), h(z(t, z_0))) + O(e^{-\gamma t}),$$

where $\gamma > 0$ is a constant, $z(t, z_0)$ is the solution of(3) with initial condition $z(0, z_0) = z_0$.

The center manifold can be approximated to any degree of accuracy. For C^1 functions $\varphi : \mathrm{R}^n \to \mathrm{R}^m$ define the nonlinear operator

$$(M\varphi)(\bar{x}) = d\varphi(\bar{x})[A\bar{x} + \Phi(\bar{x}, \varphi(\bar{x}))] - B\varphi(\bar{x}) - \Psi(\bar{x}, \varphi(\bar{x})).$$

For the function $h(\bar{x})$ defining the center manifold M_c we have $(Mh)(\bar{x}) = 0$.

Theorem 1.3 [4] Let φ be a C^1 mapping of a neighborhood of the origin in R^n into R^m with $\varphi(0) = 0, d\varphi(0) = 0$. Suppose that as $x \to 0$, $(M\varphi)(x) = O(|\ x\ |^q)$ where $q > 1$. Then as $x \to 0$, $|\ h(x) - \varphi(x)\ | = O(|\ x\ |^q)$.

The main results of this paper occupy the place of Theorem 1.2 among these three theorems. In fact, Theorem 1.2 can be replaced by two stronger theorems (Theorem 2.2

and Theorem 3.1), which are the core of the theory proposed here. At the same time, the method developed here together with the theorems 1.1, 1.3 give us a powerful tool for the investigation of stability and stabilizability of nonlinear systems.

We prove for small $(\bar{x}, \bar{y})$ the existence of the decoupling normalizing transformation

$$\tilde{x} = \bar{x} + \nu(\bar{x}, \bar{y} - h(\bar{x})),\ \nu(\bar{x}, 0) = 0,\ d\nu(0,0) = 0, \qquad (4)$$

$$\tilde{y} = \bar{y} - h(\bar{x}),$$

under which the system (1) has the form

$$\dot{\tilde{x}} = A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})), \qquad (5)$$

$$\dot{\tilde{y}} = B\tilde{y} + \tilde{\Psi}(\tilde{x}, \tilde{y}),$$

where $h(x)$ is the function from Theorem 1.1, $\bar{\Phi}(\tilde{x}, h(\tilde{x}))$ is from (3), $\tilde{\Psi}(\tilde{x}, 0) = 0$ for all $\tilde{x}$ sufficiently small and $d\tilde{\Psi}(0,0) = 0$. If $\bar{\Phi}, \bar{\Psi}$ are C^k functions, then $\nu(\bar{x}, \bar{y} - h(\bar{x}))$ is also C^k function. $\nu(x, y)$ can be approximated to any degree of accuracy. We will show that the theorem analogous to Theorem 1.3 holds. To know $\nu(x, y,)$ is important, both for the investigation of the stabilization and for the design of a stabilizing feedback. To illustrate that, we will prove several sufficient conditions for local stabilizability of nonlinear systems with noncontrollable linearizations and propose a stabilizer design procedure for a bilinear system.

2 Existence of decoupling normalizing transformation

Here we prove the existence of the decoupling normalizing transformation (4). The proof is analogous to the proof of Theorem 1.1 [8].

The system (1) is more convenient to rewrite in the new coordinates

$$x = \bar{x},$$

$$y = \bar{y} - h(\bar{x}),$$

where $h(\bar{x})$ is from Theorem 1.1. Under the coordinate transformation the system (1) has the form

$$\dot{x} = Ax + \hat{\Phi}(x, y), \qquad (6)$$

$$\dot{y} = By + \hat{\Psi}(x, y),$$

where

$$\hat{\Phi}(x,y) = \bar{\Phi}(x, y + h(x)),$$

$$\hat{\Psi}(x,y) = dh(x)(\bar{\Phi}(x,h(x)) - \bar{\Phi}(x,y+h(x))) + \bar{\Psi}(x,y+h(x)) - \bar{\Psi}(x,h(x)).$$

Now for the system (5) we prove the existence of the function $\nu(x,y)$, such that under the transformation

$$\tilde{x} = x + \nu(x,y) \qquad (7)$$

$$\tilde{y} = y$$

the system (6) has the form (5).

Theorem 2.1. Let $\hat{\Phi}(x,y)$, $\hat{\Psi}(x,y)$ be C^k functions ($k \geq 3$) which vanish together with their derivatives at the origin, i.e., $\hat{\Phi}(0,0) = 0$, $d\hat{\Phi}(0,0) = 0$, $d\hat{\Psi}(0,0) = 0$ and in addition $\hat{\Psi}(x,0) = 0$ for all $(x,0) \in Q$, where Q is a neighborhood of the origin. Then there exists in a neighborhood $\grave{Q} \subseteq Q$ of the origin a C^{k-2} function $\nu(x,y)$, such that $\nu(x,0) = 0 \quad \forall\, (x,0) \in \grave{Q}$, $d\nu(0,0) = 0$ and under the normalizing transformation (7) the system (6) has the form (5).

Proof. Introducing the scalar change of variables $(x,y) \longrightarrow (\lambda x, \lambda y)$ and multiplying $\hat{\Phi}$, $\hat{\Psi}$ by $\omega(\mid x \mid^2 + \mid y \mid^2 + K\lambda^2)$ where K is a sufficiently large positive constant and $\omega(r)$ is a C^∞ real valued function satisfying

$$0 \leq \omega(r) \geq 1 \quad \forall\, r \geq 0,$$

$$\omega(r) \equiv 1 \quad \forall\, 0 \leq r \leq \frac{1}{2},$$

$$\omega(r) \equiv 0 \quad \forall\, 1 \leq r < \infty,$$

we obtain

$$\dot{x} = Ax + \Phi(x,y,\lambda), \qquad (8)$$

$$\dot{y} = By + \Psi(x,y,\lambda),$$

where

$$\Phi(x,y,\lambda) = \frac{1}{\lambda}\omega(\mid x \mid^2 + \mid y \mid^2 + K\lambda^2)\hat{\Phi}(\lambda x, \lambda y),$$

$$\Psi(x,y,\lambda) = \frac{1}{\lambda}\omega(\mid x \mid^2 + \mid y \mid^2 + K\lambda^2)\hat{\Psi}(\lambda x, \lambda y)$$

and the following conditions hold:

(ai) $\Phi(x,y,\lambda)$, $\Psi(x,y,\lambda)$ exist and are continuous for all (x,y,λ) and for each fixed λ are C^k functions in (x,y).

(aii) $\Phi(0,0,\lambda) = 0$, for any fixed λ $d\Phi(0,0,\lambda) = 0$, $d\Psi(0,0,\lambda) = 0$. There exists a real positive value $\delta > 0$, such that $\Psi(x,0,\lambda) = 0 \quad \forall\, x \in \mathrm{R}^m, \ |\lambda| < \delta$.

(aiii) $\Phi, \Psi \equiv 0 \quad \forall \quad |x|^2 + |y|^2 \geq 1$, where $|\cdot|$ represents the Euclidean norm corresponding to the usual scalar product $< \cdot, \cdot >$ on pairs of vectors.

(aiv) $\left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j (\Phi, \Psi) \longrightarrow 0$ uniformly in $(x,y) \in \mathrm{R}^m \times \mathrm{R}^n$ as $\lambda \to 0$ for $|i| + |j| \leq k$;

$$\left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j = \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{i_m} \left(\frac{\partial}{\partial y_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial y_n}\right)^{j_n},$$

where $i = (i_1, \dots, i_m)$, $j = (j_1, \dots, j_n)$ are an m - tuple and an n - tuple of nonnegative integers respectivly, $|i| = i_1 + \cdots + i_m$, $|j| = j_1 + \cdots + j_n$.

If $\lambda \neq 0$, then system (6) and (8) are locally (near the origin) related by a scalar change of variables. Therefore it is sufficient to prove Theorem 2.1 only for system (8).

The function $\nu(x,y)$ is a solution of the following equation in partial derivatives.

$$A\nu - \frac{\partial \nu}{\partial x} Ax - \frac{\partial \nu}{\partial y} By = \frac{\partial \nu}{\partial x} \Phi(x,y,\lambda) + \frac{\partial \nu}{\partial y} \Psi(x,y,\lambda) + \Phi(x,y,\lambda) - \Phi(x+\nu,0,\lambda), \quad (9)$$

$$\nu(x,0) = 0 \quad \forall\, x \in \mathrm{R}^m,$$

$$d\nu(0,0) = 0.$$

To solve the equation (9) we take into account that

$$\frac{d}{dt}[e^{At}(e^{-tf})^* \nu(x,y)] = e^{At}(e^{-tf})^*[\Phi(x,y,\lambda) - \Phi(x+\nu,0,\lambda)], \qquad (10)$$

where $f = \left(Ax + \Phi(x,y,\lambda), \ By + \Psi(x,y,\lambda)\right)^T$, $\frac{d}{dt}e^{At} = Ae^{At}$, $e^{At}|_{t=0} = I$, I is the identity matrix,

$$(e^{tf})^* \varphi(x,y) = \varphi(e^{tf}(x,y)) \ \ \forall t \in \mathrm{R}.$$

After integrating (10) with respect to t we obtain

$$e^{tA}(e^{-tf})^* \nu(x,y) - \nu(x,y) = \int_0^t e^{A\tau}(e^{-\tau f})^*[\Phi(x,y,\lambda) - \Phi(x+\nu,0,\lambda)]d\tau. \qquad (11)$$

In according with condition (aii) we have

$$\Psi(x,y,\lambda) = \int_0^1 \frac{\partial}{\partial y} \Psi(x,sy,\lambda) ds \cdot y$$

and (aiv) yields

$$\int_0^1 \frac{\partial}{\partial y} \Psi(x,sy,\lambda) ds \longrightarrow 0$$

uniformly in $(x,y) \in \mathrm{R}^m \times \mathrm{R}^n$ as $\lambda \to 0$. Therefore we can choose the positive real value δ from (aii), such that

$$\lim_{t\to\infty} P_y(e^{tf}(x,y)) = 0 \quad \forall\, (x,y) \in \mathrm{R}^m \times \mathrm{R}^n,$$

where $P_y : \mathrm{R}^m \times \mathrm{R}^n \to \mathrm{R}^n$, $\quad \mathrm{P_y(x,y) = y}$. Thus it follows from boundary condition $\nu(x,0) = 0 \quad \forall x \in \mathrm{R}^m$ and (11) that

$$\nu(x,y) = \int_{-\infty}^{0} e^{A\tau}(e^{-\tau f})^*[\Phi(x,y,\lambda) - \Phi(x+\nu(x,y),0,\lambda)]d\tau.$$

Consider the nonlinear operator

$$T_\lambda \nu(x,y) = \int_{-\infty}^{0} e^{A\tau}(e^{-\tau f})^*[\Phi(x,y,\lambda) - \Phi(x+\nu(x,y),0,\lambda)]d\tau$$

which is defined on the following Banach space Υ^{k-2}.

$$\Upsilon^l = \{\nu = \nu(x,y) \quad \text{satisfying (bi - biv)}\}.$$

(bi) ν is a real vector - valued C^l function defined on $\mathrm{R}^m \times \mathrm{R}^n$ and $\nu : \mathrm{R}^m \times \mathrm{R}^n \to \mathrm{R}^m$.

(bii) $\nu(x,0) = 0 \quad \forall\, x \in \mathrm{R}^m,\ \ \mathrm{d}\nu(0,0) = 0.$

(biii) $\|\nu\| = \max_{|i|+|j|\le l} \sup \mid \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^{j+1} \nu(x,y) \mid < \infty.$

It follows from the theorem of differentiation of improper integrals with respect to parameter (see [9]) that $T_\lambda \nu \in \Upsilon^{k-2} \ \forall\, \nu \in \Upsilon^{k-2}$.

Introduce the notations

$$X^{i,j}_{x,y}(t) = \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j P_x(e^{tf}(x,y)),$$

$$Y^{i,j}_{x,y}(t) = \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j P_y(e^{tf}(x,y)).$$

Then $\{(X^{i,j}_{x,y}(t), Y^{i,j}_{x,y}(t))\}_{|i|+|j|\le k-1}$ is the solution of the following system

$$\dot{x}(t) = Ax(t) + \Phi(x(t),y(t),\lambda),$$

$$\dot{y}(t) = By(t) + \Psi(x(t),y(t),\lambda),$$

$$\frac{d}{dt}X^{i,j}_{x,y}(t) = AX^{i,j}_{x,y}(t) + \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j \Phi(x(t),y(t),\lambda),$$

$$\frac{d}{dt}Y^{i,j}_{x,y}(t) = BY^{i,j}_{x,y}(t) + \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j \Psi(x(t),y(t),\lambda),$$

where $|i|+|j|\leq k-1$, $x(t)=P_x(e^{tf}(x,y))$, $y(t)=P_y(e^{tf}(x,y))$ and $X^{i,j}_{x,y}(0)=0$, $Y^{i,j}_{x,y}(0)=0$ for $|i|+|j|\geq 2$, $\frac{\partial}{\partial y}P_x(e^{tf}(x,y))|_{t=0}=0$, $\frac{\partial}{\partial x}P_y(e^{tf}(x,y))|_{t=0}=0$,

$$[\frac{\partial}{\partial x_1}P_x(e^{tf}(x,y)),\dots,\frac{\partial}{\partial x_m}P_x(e^{tf}(x,y))]|_{t=0}=I_m,$$

$$[\frac{\partial}{\partial y_1}P_y(e^{tf}(x,y)),\dots,\frac{\partial}{\partial y_n}P_y(e^{tf}(x,y))]|_{t=0}=I_n,$$

where $I_m\in \mathrm{R}^{m\times m}$, $\mathrm{I_n}\in \mathrm{R}^{n\times n}$ are identity matrices. Using the method of the induction with respect to $|i|+|j|=l$ we can prove the existence of $\delta>0$, such that for $|\lambda|<\delta$

$$\sup_{(x,y)}\left|\left(\frac{\partial}{\partial x}\right)^i\left(\frac{\partial}{\partial y}\right)^j P_y(e^{tf}(x,y))\right|\leq \tilde{\alpha}(t)\cdot e^{(-\mu+\tilde{\beta}(\lambda))t},\quad |i|+|j|\geq 1 \tag{12}$$

where $\tilde{\alpha}(t)$ is a polynomial in t with positive coefficients, $\tilde{\beta}(\lambda)\geq 0$ is continuous in λ and $\tilde{\beta}(\lambda)\to 0$ as $\lambda\to 0$, $\mu=\max\{|Re\ z|;\ z\in\sigma(B)\}$, $\sigma(B)$ is the set of eigenvalues of B.

Step 1. Let $|i|+|j|=0$. Then

$$\dot{y}(t)=By(t)+\int_0^1\frac{\partial}{\partial y}\Psi(x(t),sy(t),\lambda)ds\cdot y(t)$$

and the eigenvalues of B have negative real parts . Therefore there exists positive real value $\delta>0$, such that for $|\lambda|<\delta$ and $|i|+|j|=1$ the inequality (12) holds.

Step 2. Let inequality hold for all $|i|+|j|<l$. Consider the case $|i|+|j|=l$.

$$\frac{d}{dt}Y^{i,j}_{x,y}(t)=BY^{i,j}_{x,y}(t)+\frac{\partial}{\partial y}\Psi(x(t),y(t),\lambda)\cdot Y^{i,j}_{x,y}(t)+\Xi(\{X^{i,j}_{x,y}(t)\}_{|i|+|j|<l},\{Y^{i,j}_{x,y}(t)\}_{|i|+|j|<l},\lambda), \tag{13}$$

where $X^{0,0}_{x,y}(t)=x(t)$, $Y^{0,0}_{x,y}(t)=y(t)$ and the function $\Xi(\cdot,\cdot,\lambda)$ satisfyes the following conditions

$$\Xi(\{X^{i,j}_{x,y}(t)\}_{|i|+|j|<l},0,\lambda)=0,$$

$$\Xi(\{X^{i,j}_{x,y}(t)\}_{|i|+|j|<l},\{Y^{i,j}_{x,y}(t)\}_{|i|+|j|<l},0)=0.$$

Due to the conjecture of the induction

$$\sup_{(x,y)}|\Xi(\{X^{i,j}_{x,y}(t)\}_{|i|+|j|<l},\{Y^{i,j}_{x,y}(t)\}_{|i|+|j|<l},\lambda)|\leq\bar{\alpha}(t)\cdot e^{(-\mu+\bar{\beta}(\lambda))t}\quad\forall\, t\geq 0\quad |\lambda|<\delta, \tag{14}$$

where $\delta>0$ is small enough, $\bar{\alpha}(t)$ is polinomial in t with positive coefficients, $\bar{\beta}(\lambda)\geq 0$ is continuous in λ and $\bar{\beta}(\lambda)\to 0$ as $\lambda\to 0$.
Thus (13) and (14) imply (12).

The inequality (12) yields

$$\|(e^{tf})^*\|\leq\alpha(t)e^{(-\mu+\beta(\lambda))t}\ \ \forall\, t\geq 0, \tag{15}$$

where $\|(e^{tf})^*\|$ is the norm of the operator

$$(e^{tf})^* \ : \ \Upsilon^{k-1} \to \Upsilon^{k-1}$$

and $\alpha(t)$, $\beta(\lambda)$ are of the same type as $\tilde{\alpha}(t), \tilde{\beta}(\lambda)$ and $\bar{\alpha}(t), \bar{\beta}(\lambda)$.
The condition (aii - aiv) imply

$$\Phi(x, y, \lambda) - \Phi(x + \nu(x, y), 0, \lambda) \in \Upsilon^{k-2},$$

whenever Φ is C^k function and $\nu \in \Upsilon^{k-2}$. Moreover,

$$\|\Phi(x, y, \lambda) - \Phi(x + \nu(x, y), 0, \lambda)\| \leq \|\Phi(x, y, \lambda) - \Phi(x, 0, \lambda)\| +$$
$$\|\Phi(x, 0, \lambda) - \Phi(x + \nu(x, y), 0, \lambda)\| \leq$$
$$\|\Phi(x, y, \lambda) - \Phi(x, 0, \lambda)\| + D_k \cdot \|\Phi(x, 0, \lambda)\|_{C^{k-1}} \cdot (\|\nu\| + 1)^{k-1},$$

where $k \geq 3$, constant D_k dependes only on k and

$$\|\Phi(x, 0, \lambda)\|_{C^{k-1}} = \max_{|i| \leq k-1} \sup_x \mid \Big(\frac{\partial}{\partial x}\Big)^i \Phi(x, 0, \lambda) \mid$$

Thus, taking into account (15), we obtain

$$\|T_\lambda \nu\| \leq \int_{-\infty}^{0} \hat{\alpha}(\tau) e^{(-\mu+\beta(\lambda))\tau} d\tau \cdot (\|\Phi(x, y, \lambda) - \Phi(x, 0, \lambda)\| + D_k \cdot \|\Phi(x, 0, \lambda)\|_{C^{k-1}} \cdot (1 + \|\nu\|)^{k-1})$$

$$\forall\, \nu \in \Upsilon^{k-2},$$

where $\hat{\alpha}(t)$ is polynomial in t with positive coefficients.
Therefore there exists $\delta > 0$, such that for $\mid \lambda \mid < \delta$

$$T_\lambda \ : \ \Upsilon^{k-2} \to \Upsilon^{k-2}.$$

(aiv) implies

$$\lim_{\lambda \to 0} \{ \int_{-\infty}^{0} \hat{\alpha}(\tau) e^{(-\mu+\beta(\lambda))\tau} d\tau \cdot (\|\Phi(x, y, \lambda) - \Phi(x, 0, \lambda)\| + D_k \cdot \|\Phi(x, 0, \lambda)\|_{C^{k-1}} \cdot (1 + r)^{k-1})\} = 0$$

for any positive real value r. Hence for any $r > 0$ there exists $\delta(r) > 0$, such that

$$T_\lambda \ : \ B_r \to B_r$$

for $\mid \lambda \mid < \delta(r)$, where $B_r = \{\nu \in \Upsilon^{k-2} \ ; \ \|\nu\| \leq r\}$.
We now prove the existence $r > 0$, such that for all $\nu_1, \nu_2 \in B_r$

$$\|T_\lambda \nu_1 - T_\lambda \nu_2\| \ \leq \ \frac{1}{2} \cdot \|\nu_1 - \nu_2\| \tag{16}$$

and $\mid \lambda \mid < \delta(r)$. It follows from (15) and the definition of T_λ that

$$\|T_\lambda \nu_1 - T_\lambda \nu_2\| \ \leq \ \int_{-\infty}^{0} \hat{\alpha}(\tau) e^{(-\mu+\beta(\lambda))\tau} d\tau \cdot \|\Phi(x+\nu_1, 0, \lambda) - \Phi(x+\nu_2, 0, \lambda)\|. \qquad (17)$$

It is easy to see that

$$\Phi(x+\nu_1(x,y), 0, \lambda) - \Phi(x+\nu_2(x,y), 0, \lambda) = \int_0^1 \frac{\partial}{\partial x}\Phi(x + s\nu_1 + (1-s)\nu_2, 0, \lambda) ds \times \qquad (18)$$

$$\int_0^y \left(\frac{\partial}{\partial y}\nu_1(x,\theta) - \frac{\partial}{\partial y}\nu_2(x,\theta)\right) d\theta.$$

Due to (aiii) we obtain from (18)

$$\|\Phi(x+\nu_1, 0, \lambda) - \Phi(x+\nu_2, 0, \lambda)\| \leq \|\Phi(x, 0, \lambda)\|_{C^{k-1}} \cdot C(r) \cdot \|\nu_1 - \nu_2\|, \qquad (19)$$

where $C(r)$ is a constant depending only on r.

Thus (19) together with (17) and (aiv) yield (16). We have proved the existence of $r > 0$ and $\delta > 0$, such that for $\mid \lambda \mid < \delta$ $\; T_\lambda$ is a contraction mapping on $B_r \subset \Upsilon^{k-2}$. Therefore according to the Banach's contraction principle [6] there exists the function $\nu(x,y) \in \Upsilon^{k-2}$, which we are looking for.

Theorem 2.1 can be reformulated in terms of the original system (1).

Theorem 2.2. Let $\bar{\Phi}(\bar{x}, \bar{y}), \bar{\Psi}(\bar{x}, \bar{y})$ be C^k functions ($k \geq 3$) which vanish together with their derivatives at the origin, i.e., $\bar{\Phi}(0,0) = 0, \;\; \bar{\Psi}(0,0) = 0, \;\; d\bar{\Phi}(0,0) = 0, \;\; d\bar{\Psi}(0,0) = 0$. Then there exist in a nieghborhood Q of the origin a C^{k-2} function $\nu(x,y)$ and a C^k function $h(x)$, such that $\nu(x,0) = 0 \;\; \forall\, (x,0) \in Q, \;\; d\nu(0,0) = 0, \;\; h(0) = 0, \;\; dh(0) = 0$ and under the normalizing transformation

$$\tilde{x} = \bar{x} + \nu(\bar{x}, \bar{y} - h(\bar{x})),$$

$$\tilde{y} = \bar{y} - h(\bar{x}),$$

the system (1) has the form

$$\dot{\tilde{x}} = A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})),$$

$$\dot{\tilde{y}} = B\tilde{y} + \tilde{\Psi}(\tilde{x}, \tilde{y}),$$

where $\tilde{\Psi}(\tilde{x}, 0) = 0 \;\; \forall\, (\tilde{x}, 0) \in Q, \;\; d\tilde{\Psi}(0,0) = 0.$

Remark. Decoupling normalizing transformation is not unique becouse of non - uniqueness of the center manifold.

3 Approximation of the decoupling normalizing transformation

The function $\zeta(\bar{x},\bar{y}) = \nu(\bar{x},\bar{y}-h(\bar{x}))$ can be approximated to any degree of accuracy. To show that we introduce the following nonlinear operator

$$\Im(\mu) = A\mu - L_{\bar{f}}\mu + \bar{\Phi}(\bar{x}+\mu, h(\bar{x}+\mu)) - \bar{\Phi}(\bar{x},\bar{y}),$$

where $L_{\bar{f}}\mu$ is Lie derivativ, i.e.,

$$L_{\bar{f}}\mu = \frac{d}{dt}(e^{t\bar{f}})^{*}\mu|_{t=0},$$

$h(x)$ is the function from Theorem 1.1. We remind that

$$g(x,y) = O((\mid \bar{x} \mid + \mid \bar{y} \mid)^{q} \cdot \mid \bar{y} - h(\bar{x}) \mid) \quad \text{as} \quad (\bar{x},\bar{y}) \to 0,$$

iff there exists a neighborhood of the origin W, such that

$$\mid g(\bar{x},\bar{y}) \mid \leq C \cdot (\mid \bar{x} \mid + \mid \bar{y} \mid)^{q} \cdot \mid \bar{y} - h(\bar{x}) \mid \qquad \forall\, (\bar{x},\bar{y}) \in W,$$

where C is a positive real constant.

Theorem 3.1. Suppose that μ is C^1 function with $d\mu(0,0) = 0$ and there exists $\rho > 0$ such that $\mu(\bar{x}, h(\bar{x})) = 0 \quad \forall \quad \mid \bar{x} \mid < \rho$ and that

$$\Im(\mu) = O((\mid \bar{x} \mid + \mid \bar{y} \mid)^{q} \cdot \mid \bar{y} - h(\bar{x}) \mid) \quad \text{as} \quad (\bar{x},\bar{y}) \to 0$$

where $q \geq 1$. Then

$$\zeta(\bar{x},\bar{y}) - \mu(\bar{x},\bar{y}) = O((\mid \bar{x} \mid + \mid \bar{y} \mid)^{q} \cdot \mid \bar{y} - h(\bar{x}) \mid) \quad \text{as} \quad (\bar{x},\bar{y}) \to 0. \tag{20}$$

Proof. Following the proof of Theorem 2.1, it is sufficient to prove (20) only for the system (8) with λ sufficiently small. Take the function

$$\theta_\lambda(x,y) = \frac{1}{\lambda}\mu(\lambda x, \lambda y) \cdot \omega(\mid x \mid^2 + \mid y \mid^2 + K\lambda^2), \tag{21}$$

where $x = \bar{x},\ y = \bar{y} - h(\bar{x})$ and $\omega(r)$ is truncated function introduced in the proof of Theorem 2.1. Then $\theta_\lambda \in \Upsilon^0$ and there exists $\bar{\lambda} > 0$, such that

$$\theta_\lambda \in \mathrm{Int}B_r = \{\nu \in \Upsilon^0 \ ;\ \|\nu\| < r\} \ \forall \mid \lambda \mid < \bar{\lambda}.$$

Define a mapping $S_\lambda : \Upsilon^0 \to \Upsilon^0$ by

$$S_\lambda z = T_\lambda(z + \theta_\lambda) - \theta_\lambda.$$

Since T_λ is a contraction mapping on B_r for $|\lambda| < \delta(r)$, S_λ is a contraction mapping on

$$\Xi(\lambda, q) = \{\varphi \in \Upsilon^0 \ ; \ \|\varphi + \theta_\lambda\| \leq r, \ |\varphi(x,y)| \leq \tilde{K} \cdot ((|x| + |y|)^q \cdot |y|) \ \forall (x,y) \in \mathrm{R^m} \times \mathrm{R^n}\},$$

where $\tilde{K}$ is a positive real constant. Indeed, it is only sufficient to show that

$$S_\lambda : \Xi(\lambda, q) \to \Xi(\lambda, q).$$

If $\varphi \in \Xi(\lambda, q)$, then

$$\|S_\lambda \varphi + \theta_\lambda\| = \|T_\lambda(\varphi + \theta_\lambda)\| \leq r,$$

where the last inequality follows from

$$T_\lambda : B_r \to B_r.$$

Thus it remains to prove that for all $(x,y) \in \mathrm{R^m} \times \mathrm{R^n}$

$$|\varphi(x,y)| \leq \tilde{K} \cdot ((|x| + |y|)^q \cdot |y|)$$

yields

$$|(S_\lambda \varphi)(x,y)| \leq \tilde{K} \cdot ((|x| + |y|)^q |y|)$$

for some positive $\tilde{K}$.

The function $\theta_\lambda(x)$ can be represented as

$$-\theta_\lambda(x) = -\int_{-\infty}^{0} \frac{d}{d\tau}(e^{A\tau}(e^{-\tau f})^* \theta(x)) d\tau = -\int_{-\infty}^{0} e^{A\tau}(e^{-\tau f})^*(A\theta - L_f \theta) d\tau.$$

Since $\Im(\mu) = O((|x| + |y|)^q \cdot |y|)$ and hence $\Im(\theta_\lambda) = O((|x| + |y|)^q \cdot |y|)$ we obtain

$$-\theta_\lambda(x,y) = -\int_{-\infty}^{0} e^{A\tau}(e^{-\tau f})^* \{[\Phi(x,y,\lambda) - \Phi(x+\theta, 0, \lambda)] + N(x,y)\} d\tau,$$

where

$$N(x,y) = A\theta - L_f\theta + \Phi(x+\theta, 0, \lambda) - \Phi(x,y,\lambda).$$

Thus

$$(S_\lambda \varphi)(x,y) = \int_{-\infty}^{0} e^{A\tau}(e^{-\tau f})^* [\Phi(x+\theta, 0, \lambda) - \Phi(x+\theta+\varphi, 0, \lambda) - N(x,y)] d\tau.$$

Hence having applied (15), (19) we obtain the existence of $\tilde{\delta} > 0$, such that

$$|\varphi(x,y)| \leq \tilde{K} \cdot (|x| + |y|)^q \cdot |y|, \quad \varphi \in \Xi(\lambda, q)$$

which yields

$$|(S_\lambda \varphi)(x,y)| \leq \tilde{K} \cdot (|x| + |y|)^q |y|$$

for all (x,y) and $|\lambda| \leq \tilde{\delta}$. The proof is completed.

Now using Theorem 1.3 and Theorem 3.1 we can approximate the decoupling normalizing transformation

$$\tilde{x} = \bar{x} + \nu(\bar{x}, \bar{y} - h(\bar{x}))$$
$$\tilde{y} = \bar{y} - h(\bar{x}),$$

to any degree of accuracy, where $\nu(\bar{x}, 0) = 0, \; h(0) = 0, \; d\nu(0,0) = 0, \; dh(0) = 0.$

Consider more thoroughly the numerical procedure for the calculation of asymptotic series for ν. For simplicity we suppose that the coordinate transformation

$$x = \bar{x}$$

$$y = \bar{y} - h(\bar{x})$$

has been already applyed. Thus we deal with the system (6). Then the function $\nu(x, y)$ satisfies the equation

$$\Lambda\nu = -d\nu\Omega - \{\hat{\Phi}(x,y) - \hat{\Phi}(x+\nu,0)\},$$

where

$$\Lambda\nu = ad_A\nu + \frac{\partial\nu}{\partial y}By, \quad ad_A\nu = \frac{\partial\nu}{\partial x}Ax - A\nu$$

and

$$\Omega(x,y) = \left(\hat{\Phi}(x,y), \hat{\Psi}(x,y)\right)^T.$$

Let $y \cdot \wp^i$ be a linear space of vector fields whose coefficients are homogeneous polynomials of degree $i+1$ and for every $g \in y \cdot \wp^i \quad g(x,0) = 0 \quad \forall x \in \mathrm{R}^m$ holds. Suppose further we have the asymptotic series

$$\nu = \sum_{i=1}^{\infty} \nu_i,$$

$$\Omega = \sum_{i \geq 2} \Omega_i,$$

$$\hat{\Phi}(x,y) - \hat{\Phi}(x+\nu,0) = \sum_{i=1}^{\infty} [\Phi(x,y) - \Phi(x+\nu,0)]_{i+1},$$

where $\nu_i, \;\; [\Phi(x,y) - \Phi(x+\nu,0)]_{i+1} \in y \cdot \wp^i$ and $\Omega_i \in \wp^i$, $\wp^i$ is a linear space of vector fields whose coefficients are homogeneous polynomials of degree i. Then we have to solve for $\{\nu_i\}_{i=1}^{\infty}$ the following linear equations in the linear spaces $\{y \cdot \wp^i\}_{i=1}^{\infty}$.

$$\Lambda\nu_l = - \sum_{i+j=l+1\,,\, l\geq 1,\, j \geq 2} d\nu_i\Omega_j - [\Phi(x,y) - \Phi(x+\nu,0)]_{l+1} \;\; (l = 1, 2, \ldots) \qquad (21)$$

The solution $\{\nu_l\}_{l=1}^{\infty}$ exists and is unique. Namely the following statement is true.

Proposition 3.1. There exists $\Lambda^{-1} \; : \; y \cdot \wp^i \to y \cdot \wp^i$ and

$$\Lambda^{-1}h = -\int_0^{\infty} e^{-A\tau} h(e^{A\tau}x, e^{B\tau}y) d\tau$$

for $h \in y \cdot \wp^i \;\; (i = 1, 2, \ldots)$.

Proof. Suppose there exists $g \neq 0$, $g \in y \cdot \wp^i$, such that $\Lambda g = 0$. Then

$$\frac{d}{dt}\{e^{-At}g(e^{At}x, e^{Bt}y)\} = 0.$$

Thus

$$e^{-At}g(e^{At}x, e^{Bt}y)\} = g(x,y).$$

for $t \geq 0$. But $g \in y \cdot \wp^i$ and consequently

$$\lim_{t\to\infty} e^{-At}g(e^{At}x, e^{Bt}y) = 0.$$

Hence $g(x,y) = 0$. Thus $\Lambda g = 0$ implyes $g = 0$. That means the existence of Λ^{-1}.

Example 3.1. Consider the polynomial system

$$\dot{x} = Ax + (V_{11}x + V_{12}y) \cdot < k, y >,$$

$$\dot{y} = By + (V_{21}x + V_{22}y) \cdot < k, y >,$$

where the eigenvalues of $A \in \mathrm{R}^{m\times m}$ have zero real parts, the eigenvalues of $B \in \mathrm{R}^{n\times n}$ have negative real parts, $V_{11} \in \mathrm{R}^{m\times m}$, $V_{12} \in \mathrm{R}^{m\times n}, V_{21} \in \mathrm{R}^{n\times m}$, $V_{22} \in \mathrm{R}^{n\times n}$ and $k \in \mathrm{R}^n$. Then for $l = 1$ the equation (21) has the form

$$\Lambda\nu_1 = -(V_{11}x + V_{12}y) \cdot < k, y > .$$

Using Proposition 3.1, we obtain

$$\nu_1 = \int_0^\infty e^{-A\tau}(V_{11}e^{A\tau}x + V_{12}e^{B\tau}y) \cdot < k, e^{B\tau}y > d\tau$$

and

$$\nu = \nu_1 + O((| x | + | y |)^2 | y |).$$

4 Additional smoothness

Smoothness and/or real analyticity of a decoupling normalizing transformation is complety determined by smoothness and/or real analiticity of a center manifold. Consider the sequence

$$\xi_0 = -\int_{-\infty}^0 e^{A\tau}(e^{-\tau f})^*[\Phi(x,y,\lambda) - \Phi(x,0,\lambda)]d\tau, \;\; \xi_1 = T_\lambda\xi_0, \; \ldots, \xi_j = T_\lambda\xi_{j-1}, \; \ldots,$$

where $\Phi(x,y,\lambda)$ and T_λ are defined in the proof of Theorem 2.1. Then $\{\xi_j\}_{j=0}^\infty$ are C^k functions whenever f is C^k vector field and $| \lambda | < \delta$, where δ is a sufficiently small positive real value. It has been proved in Section 2, that $\lim_{i\to\infty} \xi_i = \nu$ in the Υ^{k-2} topology. Thus a restriction of ν to any closed ball in $\mathrm{R}^m \times \mathrm{R}^n$ is the limit of $\{\xi_i\}_{i=0}^\infty$ in the C^{k-2} topology. Moreover for sufficiently small δ and $| \lambda | < \delta$ the $(k-2)$th derivatives of ν are uniformly Lipschitzian. Using this fact and the proof method of Theorem 4.2

from [5], one can show that, for λ sufficiently small, ν is a C^k function on a closed ball in $R^m \times R^n$.

In general real analiticity of the vector field f does not imply the existence of a real analytic center manifold [8]. But if the function $h(x)$ from Theorem 1.1, the vector field $\bar{f}$ are real analytic and moreover

$$A = -A^T,$$

then the decoupling normalizing transformation is also real analytic. To prove that one define the norm

$$\|g\|_i = \sup_{|x|+|y|=1} \left| \frac{\partial}{\partial y} g(x,y) \right| \quad \text{on} \quad y \cdot \wp^i.$$

If $A = -A^T$, then there exists a constant $K > 0$, such that

$$\|\Lambda^{-1}\|_i \leq K^i \quad \forall\, i = 1, 2, \ldots. \tag{22}$$

Thus using (21) one can show that

$$\|\nu_i\|_i \leq M^i, \tag{23}$$

where constant $M > 0$. (23) means real analyticity of ν. The details of this scenario are quite laborious so we do not present them here. It is necessary only to note, that the condition $A = -A^T$ is quite important. In general, for arbitrary matrix A, whose eigenvalues have zero real parts, there not exists any constant $K > 0$ for which (22) holds.

5 Local stabilization of nonlinear system with non-controllable linearization

Here we continue the work begun in [1,3]. Namely we apply the results obtained above in order to investigate the local stabilization of the single - input nonlinear system

$$\dot{\bar{x}} = A\bar{x} + \bar{\Phi}(\bar{x}, \bar{y}) + \bar{G}(\bar{x}, \bar{y}) \cdot u,$$

$$\dot{\bar{y}} = B\bar{y} + \bar{\Psi}(\bar{x}, \bar{y}) + (q + \bar{Q}(\bar{x}, \bar{y})) \cdot u, \tag{24}$$

where control value $u \in R$ and A, B, $\bar{\Phi}$, $\bar{\Psi}$ have been defined in (1),

$$\bar{G} : R^m \times R^n \to R^m,$$

$$\bar{Q} : R^m \times R^n \to R^n$$

are C^∞ function which vanish at the origin, i.e., $\bar{G}(0,0) = 0$, $\bar{Q}(0,0) = 0$.

Definition 5.1 The system (24) is said to be locally stabilizable in the origin iff there exists the C^2 feedback $u = w(x, y)$ which vanish together with their derivatives at the origin (i.e., $w(0,0) = 0, dw(0,0) = 0$), such that the zero solution of the closed loop system (the system (24) with $u = w(x, y)$) is asymptotically stable.

Due to Theorem 2.2 there exists a decoupling normalizing transformation (4) under which the system (24) has the form

$$\dot{\tilde{x}} = A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})) + \tilde{G}(\tilde{x}, \tilde{y}) \cdot u,$$
$$\dot{\tilde{y}} = B\tilde{y} + \tilde{\Psi}(\tilde{x}, \tilde{y}) + (q + \tilde{Q}(\tilde{x}, \tilde{y})) \cdot u, \tag{25}$$

where

$$\tilde{G}(\bar{x}, \bar{y}) = \bar{G}(\bar{x}, \bar{y}) + \frac{\partial}{\partial z}\nu(z, \bar{y} - h(\bar{x}))|_{z=\bar{x}}\bar{G}(\bar{x}, \bar{y}) + \frac{\partial}{\partial \bar{y}}\nu(\bar{x}, \bar{y} - h(\bar{x}))(q + \bar{Q}(\bar{x}, \bar{y}) - \frac{\partial}{\partial \bar{x}}h(\bar{x})\bar{G}(\bar{x}, \bar{y})$$

$$\tilde{Q}(\bar{x}, \bar{y}) = \bar{Q}(\bar{x}, \bar{y}) - \frac{\partial}{\partial \bar{x}}h(\bar{x})\bar{G}(\bar{x}, \bar{y})$$

and $(\tilde{x}, \tilde{y})$, $(\bar{x}, \bar{y})$ are connected by the decoupling normalizing transformation (4). It is easy to see that $\tilde{y} = 0$ yields $\tilde{x} = \bar{x}$ and $\bar{y} = h(\tilde{x})$. Thus

$$\tilde{G}(\tilde{x}, 0) = \bar{G}(\tilde{x}, h(\tilde{x})) + \frac{\partial}{\partial \bar{y}}\nu(\tilde{x}, 0)(q + \bar{Q}(\tilde{x}, h(\tilde{x})) - \frac{\partial}{\partial \bar{x}}h(\tilde{x})\bar{G}(\tilde{x}, h(\tilde{x})))$$

The next theorem gives us some sufficient conditions for local stabilizability of nonlinear system (24).

Theorem 5.1. Let the system

$$\dot{\tilde{x}} = A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})) \tag{26}$$

be stable, $V(\tilde{x})$ be its C^∞ weak Liapunov's function, i.e, there exists $\delta > 0$, such that $V(\tilde{x}) > 0$ for all $0 <| \tilde{x} |< \delta$, $V(0) = 0$ and $< dV(\tilde{x}), A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})) > \leq 0 \ \forall \ | \tilde{x} |< \delta$. Suppose further that for every complete trajectory $\tilde{x}(t, \tilde{x}(0)) = \{\tilde{x}(t); \ | \tilde{x}(0) |< \delta, \ 0 \leq t < \infty\}$ of (26) which satisfyes

$$< dV(\tilde{x}(t)), \tilde{G}(\tilde{x}(t), 0) >= 0 \quad \forall t \geq 0 \tag{27}$$

it follows that $x(t) = 0$. Then the system (25) is locally stabilizable in the origin by the feedback $u = - < dV(\tilde{x}), \tilde{G}(\tilde{x}, \tilde{y}) >$.

Proof. According to Theorem 1.1 the system (25) with $u = - < dV(\tilde{x}), \tilde{G}(\tilde{x}, \tilde{y}) >$ has a center manifold $\tilde{y} = H(\tilde{x})$. Then due to Theorem 1.2 (and/or Theorem 2.2) the zero solution of the closed loop system is asymptotically stable iff the zero solution of the system

$$\dot{\tilde{x}} = A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})) - \tilde{G}(\tilde{x}, H(\tilde{x})) < dV(\tilde{x}), \tilde{G}(\tilde{x}, H\tilde{x}) > \tag{28}$$

is asymptotically stable. If there exists $\delta > 0$, such that $\lim_{t\to\infty} \tilde{x}(t, x^*) = 0 \quad \forall \quad | x^* | < \delta$, where $\tilde{x}(t, x^*)$ is the solution of (28) generated by the initial conditions $\tilde{x}(0, x^*) = x^*$, then the proof is completed. Otherwise for every $\delta > 0$ one can find $0 < | x^* | < \delta$, such that $\lim_{t\to\infty} \tilde{x}(t, x^*) \neq 0$ and $\tilde{x}(t, x^*)$ saticfyes

$$< dV(\tilde{x}(t, x^*)), \tilde{G}(\tilde{x}(t, x^*), H(\tilde{x}(t, x^*)) >= 0 \quad \forall \ t \geq 0.$$

But $\big(\tilde{x}(t, x^*), H(\tilde{x}(t, x^*))\big)$ is a solution of the system (25) with $u = 0$. Hence, due to the stability of the zero solution of (26) $\lim_{t\to\infty} H(\tilde{x}(t, x^*)) = 0$. Thus there exists nontrivial trajectory of (26) which satisfyes (27). That contradictes the conditions of the theorem. The proof is completed.

Using the sufficient conditions of stabilization obtained in [7] we can formulate the following corollary of Theorem 5.1.

Corollary 1. Let $\bar{\Phi}(\tilde{x}, h(\tilde{x})) = 0, \quad A^T = -A, \quad \tilde{G}(\tilde{x}, 0)$ be C^∞ function and for δ sufficiently small

$$rank\{ad_A^i \tilde{G}(\tilde{x}, 0)\}_{i=0}^{\infty} = m, \quad \forall \ 0 < | \tilde{x} | < \delta$$

where $ad_A^0 \tilde{G}(\tilde{x}, 0) = \tilde{G}(\tilde{x}, 0), \quad ad_A \tilde{G}(\tilde{x}, 0) = \frac{\partial}{\partial \tilde{x}} \tilde{G}(\tilde{x}, 0) A\tilde{x} - A\tilde{G}(\tilde{x}, 0)$ and $ad_A^i \tilde{G}(\tilde{x}, 0) = ad_A(ad_A^{i-1} \tilde{G}(\tilde{x}, 0))$. Then the system (25) is locally stabilizable in the origin by the feedback $u = - < \tilde{x}, \tilde{G}(\tilde{x}, \tilde{y}) >$.

The next theorem follows from the sufficient conditions of the stability of homogeneous polynomial systems [2].

Theorem 5.2. Let $A = -A^T$,

$$\bar{\Phi}(\tilde{x}, h(\tilde{x})) = \bar{\Phi}_\theta(\tilde{x}) + O(| \tilde{x} |^{\theta+1}),$$

$$\tilde{G}(\tilde{x}, \tilde{y}) = \tilde{G}_\eta(\tilde{x}, \tilde{y}) + O((| \tilde{x} | + | \tilde{y} |)^{\eta+1}),$$

where $\bar{\Phi}_\theta \in \wp^\theta, \quad \tilde{G}_\eta \in \wp^\eta$ and $\wp^\theta, \quad \wp^\eta$ are defined in Section 3. Suppose further $\theta \geq 2\eta + 1$ and

$$\{\tilde{x} \in S^{m-1} \ ; < \tilde{x}, \tilde{G}_\eta(\tilde{x}, 0) >= 0\} \subset \{x \in S^{m-1} \ ; < x, \bar{\Phi}_\theta(\tilde{x}) >< 0\},$$

where S^{m-1} is the (m-1)- dimensional unit sphere. Then there exists $\gamma > 0$, such that the feedback

$$u(\tilde{x}) = -\gamma < \tilde{x}, \tilde{G}_\eta(\tilde{x}, 0) > | \tilde{x} |^{\theta-2\eta-1}$$

stabilizes the system (25).

Proof. Consider the system (25) closed by $u(\tilde{x}) = -\gamma < \tilde{x}, \tilde{G}_\eta(\tilde{x}, 0) > | \tilde{x} |^{\theta-2\eta-1}$. Having applied Theorem 1.1 we obtain the existence of the center manifold $y = H(\tilde{x})$ for the closed loop system. Hence the feedback stabilizes the system (25), iff the zero solution of the system

$$\dot{\tilde{x}} = A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})) - \tilde{G}(\tilde{x}, H(\tilde{x})) \cdot \gamma < \tilde{x}, \tilde{G}_\eta(\tilde{x}, 0) > | \tilde{x} |^{\theta-2\eta-1}$$

is asymptotically stable. Take the Liapunov's function $V(\tilde{x}) = \frac{1}{2} \mid \tilde{x} \mid^2$. Then

$$\frac{d}{dt}V(\tilde{x}) =< \tilde{x}, \bar{\Phi}_\theta(\tilde{x}) > -\gamma\Big(< \tilde{x}, \tilde{G}_\eta(\tilde{x}, 0) >\Big)^2 \cdot \mid \tilde{x} \mid^{\theta-2\eta-1} + O(\mid \tilde{x} \mid^{\theta+2}) \tag{29}$$

According to the result of [2], there exists $\gamma > 0$ such that

$$< \tilde{x}, \bar{\Phi}_\theta(\tilde{x}) >< \gamma(< \tilde{x}, \tilde{G}_\eta(\tilde{x}, 0) >)^2 \mid \tilde{x} \mid^{\theta-2\eta-1} \quad \forall \tilde{x} \neq 0.$$

Thus the statement of the theorem follows from (29).

Now we formulate sufficient conditions for local stabilizability of the bilinear system

$$\begin{aligned} \dot{x} &= Ax + (V_{11}x + V_{12}y)v, \\ \dot{y} &= \bar{B}y + (q + V_{21}x + V_{22}y)v, \end{aligned} \tag{30}$$

where control value $v \in \mathrm{R}$, $\mathrm{q} \in \mathrm{R}^n$, the system

$$\dot{y} = \bar{B}y + q \cdot v$$

is stabilizable and A, $\{V_{ij}\}_{i,j=1}^2$ are defined in Example 3.1.
We will design the stabilizing feedback in the form

$$v =< k, y > + u(x, y) \tag{31}$$

with $u(0,0) = 0$, $du(0,0) = 0$ and $k \in \mathrm{R}^n$, such that all eigenvalues of $B = \bar{B} + q \cdot k$ have negative real parts.
After inserting (31) in (30) we obtain

$$\begin{aligned} \dot{x} &= Ax + (V_{11}x + V_{12}y) \cdot < k, y > + (V_{11}x + V_{12}y) \cdot u, \\ \dot{y} &= By + (V_{21}x + V_{22}y) \cdot < k, y > + (q + V_{21}x + V_{22}y) \cdot u. \end{aligned} \tag{32}$$

Theorem 5.3. If $A = -A^T$ and

$$< x, V_{11}x > + \int_0^\infty < e^{A\tau}x, V_{11}e^{A\tau}x >< k, e^{B\tau}q > d\tau = 0 \tag{33}$$

implies $x = 0$, then the system (30) is stabilized by the feedback

$$v =< k, y > - < x, V_{11}x > - \int_0^\infty < e^{A\tau}x, V_{11}e^{A\tau}x >< k, e^{B\tau}q > d\tau. \tag{34}$$

Proof. It is easy to see that for the system (32) with $u = 0$ we have $h(x) = 0$ and $\bar{\Phi}(x, h(x)) = 0$. The decoupling normalizing transformation is of the form

$$\tilde{x} = x + \nu(x, y), \qquad\qquad (35)$$
$$\tilde{y} = y,$$

where

$$\nu = \int_0^\infty e^{-A\tau}(V_{11}e^{A\tau}x + V_{12}e^{b\tau}y) < k, e^{B\tau}y > d\tau + O((| x | + | y |^2)\cdot | y |)$$

that was calculated in Example 3.1.

Under the normalizing transformation (35) the system (32) has the form

$$\dot{\tilde{x}} = A\tilde{x} + \tilde{G}(\tilde{x}, \tilde{y}) \cdot u, \qquad\qquad (36)$$
$$\dot{\tilde{y}} = B\tilde{y} + \tilde{\Psi}(\tilde{x}, \tilde{y}) + (q + \tilde{Q}(\tilde{x}, \tilde{y})) \cdot u,$$

where $\tilde{\Psi}, \tilde{Q}$ are analogous to the corresponding functions in (25).

Consider the system (36) closed by

$$u(x) = - < x, V_{11}x > - \int_0^\infty < e^{A\tau}x, V_{11}e^{A\tau}x >< k, e^{B\tau}q > d\tau \qquad (37)$$

where (x, y) and $(\tilde{x}, \tilde{y})$ are connected by the transformation (35). Then using Theorem 1.1 we obtain for the system (36) closed by (37) the center manifold $\tilde{y} = H(\tilde{x})$. Hence to prove the theorem we need to investigate a local behaviour of the system

$$\dot{\tilde{x}} = A\tilde{x} + \tilde{G}(\tilde{x}, H(\tilde{x})) \cdot u(x), \qquad (38)$$

where $x = \tilde{x} - \nu(x, H(\tilde{x}))$. Take the Liapunov's function $V(\tilde{x}) = \frac{1}{2} | \tilde{x} |^2$. Then

$$\frac{d}{dt}V(\tilde{x}) =< \tilde{x}, \tilde{G}(\tilde{x}, H(\tilde{x})) > \cdot u(x).$$

But

$$< \tilde{x}, \tilde{G}(\tilde{x}, H(\tilde{x})) >=< \tilde{x}, V_{11}\tilde{x} > + \int_0^\infty < e^{A\tau}\tilde{x}, V_{11}e^{A\tau}\tilde{x} > \cdot < k, e^{B\tau}q > d\tau + O(| \tilde{x} |^3),$$

$$u(x) = - < \tilde{x}, V_{11}\tilde{x} > - \int_0^\infty < e^{A\tau}\tilde{x}, V_{11}e^{A\tau}\tilde{x} >< k, e^{B\tau}q > d\tau + O(| \tilde{x} |^3).$$

Therefore

$$\frac{d}{dt}V(\tilde{x}) = -\Big(< \tilde{x}, V_{11}\tilde{x} > + \int_0^\infty < e^{A\tau}\tilde{x}, V_{11}e^{A\tau}\tilde{x} >< k, e^{B\tau}q > d\tau\Big)^2 + O(| \tilde{x} |^5).$$

and due to the condition (33) that means asymptotic stability of the zero solution of (38). Hence the zero solution of the system (30) which is closed by the feedback (34) is also asymptotically stable.

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