Wreath combinatorics in the context of restricted rational Cherednik algebras

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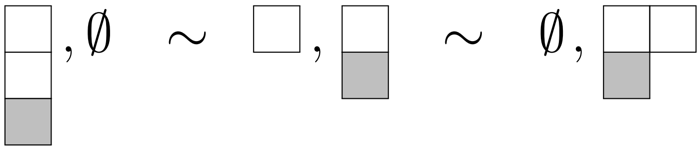

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Abstract

Wreath product groups $C_\ell \wr \mathfrak{S}_n$ have a rich combinatorial representation theory coming from the symmetric group case and involving partitions, Young tableaux, and Specht modules. To such a wreath product group W, one can associate various algebras and geometric objects: Hecke algebras, quantum groups, Hilbert schemes, Calogero–Moser spaces, and (restricted) rational Cherednik algebras. Over the years, surprising connections have been made between a lot of these objects, with many of these connections having been traced back to combinatorial constructions and properties of the group W itself.

In this thesis, we have studied one of the algebras, namely the *restricted rational Cherednik algebra* $\overline{\mathsf{H}}_{\mathbf{c}}(W)$, in order to find combinatorial models which describe certain representation theoretical phenomena around $\overline{\mathsf{H}}_{\mathbf{c}}(W)$. In particular, we generalize a result by Gordon and describe the graded W-characters of the simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}(W)$ for *generic parameter* $\mathbf{c}$ using Haiman's wreath Macdonald polynomials. These graded W-characters turn out to be specializations of Haiman's wreath Macdonald polynomials. In the *non-generic parameter* case, we use recent results by Maksimau to combinatorially express an inductive rule of $\overline{\mathsf{H}}_{\mathbf{c}}(W)$-modules first described by Bellamy. We use our results in type B to describe the (ungraded) B_n-character of simple $\overline{\mathsf{H}}_{\mathbf{c}}(B_n)$-modules associated to bipartitions with one empty part. Afterwards, we relate this combinatorial induction to various other algebras and families of W-characters found in the literature such as Lusztig's constructible characters, as well as detail some connections between generic and non-generic parameter using wreath Macdonald polynomials.

Zusammenfassung

Kranzproduktgruppen $C_\ell \wr \mathfrak{S}_n$ besitzen eine umfangreiche kombinatorische Darstellungstheorie, welche sich von dem Spezialfall der symmetrischen Gruppe herleitet, und stark mit Partitionen, Young Tableaux, und Specht-Moduln zusammenhängt. Zu solch einer Kranzproduktgruppe W können wir verschiedene Algebren und geometrische Objekte assoziieren: Hecke-Algebren, Quantengruppen, Hilbert-Schemata, Calogero–Moser-Räume, und (eingeschränkte) rationale Cherednik-Algebren. Über die Jahre wurden überraschende Zusammenhänge zwischen vielen dieser Konzepte hergestellt, und einige dieser Zusammenhänge finden ihren Ursprung in kombinatorischen Konstruktionen und Eigenschaften der Gruppe W selbst.

In dieser Arbeit haben wir eine dieser Algebren, die eingeschränkte rationale Cherednik-Algebra $\overline{\mathsf{H}}_{\mathbf{c}}(W)$, mit dem Ziel untersucht, kombinatorische Modelle für bestimmte darstellungstheoretische Phänomene rundum $\overline{\mathsf{H}}_{\mathbf{c}}(W)$ zu finden. Im Speziellen haben wir ein Resultat von Gordon verallgemeinert und die graduierten W-Charaktere von einfachen $\overline{\mathsf{H}}_{\mathbf{c}}(W)$-Moduln für *generischen Parameter* $\mathbf{c}$ beschrieben. Diese W-Charaktere sind durch eine Spezialisierung von Haimans Kranzprodukt-Macdonald-Polynomen gegeben. In *nicht-generischem* Parameter verwenden wir kürzlich erschienene Resultate von Maksimau, um eine Induktionsregel von $\overline{\mathsf{H}}_{\mathbf{c}}$-Moduln, welches ursprünglich von Bellamy beschrieben worden ist, kombinatorisch auszudrücken. Wir benutzen unsere Resultate im Typ B, um die (ungraduierten) B_n-Charaktere von einfachen $\overline{\mathsf{H}}_{\mathbf{c}}(B_n)$-Moduln, welche zu Bipartitionen mit einem leeren Teil assoziiert sind, zu beschreiben. Im Anschluss vergleichen wir unsere kombinatorische Induktion mit anderen Algebren und Familien von W-Charakteren aus der Literatur wie beispielsweise Lusztigs konstruierbare Charaktere, und detaillieren eine Verbindung zwischen generischem und nicht-generischem Parameter mithilfe von Kranzprodukt-Macdonald-Polynomen.

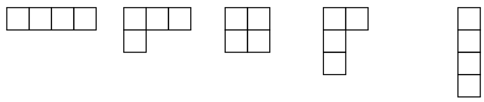
Introduction

The representation theory of the symmetric group $\mathfrak{S}_n$ has been the subject of research for more than a century [Fro68, Sec. 60]. In [Spe35], Specht first used the combinatorics of *partitions* to construct and parametrize a complete set of irreducible representations of $\mathfrak{S}_n$. A *partition* λ of $n \in \mathbb{N}$ is a weakly decreasing sequence

$$(\lambda_1, ..., \lambda_k)$$

of nonnegative integers summing up to n. The partition λ can be visualized using a *Young diagram* which consists of left-justified rows of λ_i boxes for $1 \leq i \leq k$ (cf. Ch. 2). For example when $n = 4$, we get an irreducible representation of $\mathfrak{S}_n$ for each of the following partitions and Young diagrams:

$$(4) \qquad (3,1) \qquad (2,2) \qquad (2,1,1) \qquad (1,1,1,1)$$



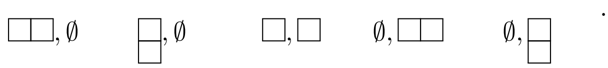
Over the decades, many interesting algebras arising from different mathematical worlds have been associated to the symmetric group. The representation theory of these algebras can often be understood in terms of representation theory of $\mathfrak{S}_n$. The algebras of particular interest to us are: Hecke algebras, quantum groups, and—most importantly for us—rational Cherednik algebras. There exist geometric constructions as well, e.g. the Hilbert scheme, quiver varieties, and—most importantly for us—the Calogero–Moser space, whose geometric properties all have similarly close ties to the combinatorics of $\mathfrak{S}_n$. In a lot of cases, surprising connections have been made between the seemingly disparate worlds of all these algebras and geometric objects (see for example [GGOR03], [EG02], [Prz20]).

One can generalize these algebras and geometric objects to larger classes of groups, of which the symmetric group is only one infinite sub-series. A larger series is given by *wreath product groups* of the form

$$C_\ell \wr \mathfrak{S}_n$$

where C_ℓ denotes the cyclic group of order $\ell \in \mathbb{N}$. The irreducible representations of $C_\ell \wr \mathfrak{S}_n$ have again been described by Specht in [Spe33] and the combinatorics generalize beautifully from partitions to *ℓ-multipartitions*, i.e. ℓ-tuples of partitions which in total sum up to n. For example when $\ell = n = 2$, we get an irreducible representation of $C_\ell \wr \mathfrak{S}_n$ for each of the following multipartitions and multi-Young diagrams:

$$((2), \emptyset) \quad ((1,1), \emptyset) \quad ((1),(1)) \quad (\emptyset, (2)) \quad (\emptyset, (1,1))$$



The group $C_\ell \wr \mathfrak{S}_n$ lies in the intersection of the family of complex reflection groups, which have long been believed to generalize Coxeter groups in some aspects, and groups of the form $\Gamma \wr \mathfrak{S}_n$ for a finite group $\Gamma \leq \mathrm{SL}_2(\mathbb{C})$, which have strong connections to quiver varieties and the geometry of certain quotient singularities (cf. [Nak98]). Therefore, it is not unreasonable to assume that some of these connections can be traced back to properties of the group itself.

We now want to explore the combinatorial representation theory of $C_\ell \wr \mathfrak{S}_n$ in order to derive new connections between the algebras and spaces associated to wreath product groups. To do this, we will use *restricted rational Cherednik algebras (RRCAs)* as our entry point, and let them guide us on the path of wreath product combinatorics in order to see how RRCAs relate to the other theories attached to the group $C_\ell \wr \mathfrak{S}_n$.

The *rational Cherednik algebra (RCA)* $\mathsf{H}_{\mathbf{c}}(W)$ is an infinite-dimensional $\mathbb{C}$-algebra associated to a complex reflection group W—this includes $C_\ell \wr \mathfrak{S}_n$—and a *parameter* $\mathbf{c}$ (cf. Ch. 1). Rational Cherednik algebras have been defined first by Etingof and Ginzburg in [EG02] in order to study the Calogero–Moser space associated to the pair $(W, \mathbf{c})$. There is a more general version of the RCA denoted by $\mathsf{H}_{t,\mathbf{c}}(W)$ that depends on an additional parameter $t \in \mathbb{C}$. In our setting, this parameter t is equal to 0.

The algebra $\mathsf{H}_{\mathbf{c}}(W)$ is a finite-dimensional module over its center $\mathsf{Z}_{\mathbf{c}}(W)$. The *restricted rational Cherednik algebra (RRCA)* $\overline{\mathsf{H}}_{\mathbf{c}}(W)$ is a finite-dimensional quotient of $\mathsf{H}_{\mathbf{c}}(W)$ taken by a centrally generated ideal of $\mathsf{H}_{\mathbf{c}}(W)$. It was first studied by Gordon in [Gor03] (cf. Sec. 1.2). The RRCA is $\mathbb{Z}$-graded and admits a triangular decomposition which can be used to define standard modules $M_{\mathbf{c}}(\lambda)$ of $\overline{\mathsf{H}}_{\mathbf{c}}(W)$ indexed by the complex, irreducible W-characters $\operatorname{Irr} W$. For $\lambda \in \operatorname{Irr} W$, each standard module $M_{\mathbf{c}}(\lambda)$ has a unique simple head $L_{\mathbf{c}}(\lambda)$, and the modules

$$\{L_{\mathbf{c}}(\lambda) \mid \lambda \in \operatorname{Irr} W\}$$

form a complete set of pairwise nonisomorphic, graded, simple $\overline{\mathsf{H}}_{\mathbf{c}}(W)$-modules up to a shift in grading. The aforementioned triangular decomposition of $\overline{\mathsf{H}}_{\mathbf{c}}(W)$ contains the group algebra $\mathbb{C}W$ in the degree zero piece. We can therefore restrict the action of $\overline{\mathsf{H}}_{\mathbf{c}}(W)$ on $L_{\mathbf{c}}(\lambda)$ to $\mathbb{C}W$ and associate to $L_{\mathbf{c}}(\lambda)$ a graded W-character for $\lambda \in \operatorname{Irr} W$. This graded W-character of $L_{\mathbf{c}}(\lambda)$ turns out to be highly dependent on the parameter $\mathbf{c}$ and thus the following natural question emerges which will function as our starting point for diving into the combinatorics of $W = C_\ell \wr \mathfrak{S}_n$.

Question. For a parameter $\mathbf{c}$ and $\lambda \in \operatorname{Irr} W$, what is the graded W-character of the simple $\overline{\mathsf{H}}_{\mathbf{c}}(W)$-module $L_{\mathbf{c}}(\lambda)$?

This question has first been posed by Gordon in [Gor03, Prob. 1] and was partially answered in *loc. cit.* as well. We will generalize his results in this thesis.

We begin in Chapter 1 by introducing complex reflection groups and restricted rational Cherednik algebras. After following [Gor03] for the construction of standard and simple modules associated to an RRCA, we arrive at the question above. In Section 1.4, we review the construction of a "generic" version of the rational Cherednik algebra and observe that the representation theory of RRCAs splits in two parts: one where $\mathbf{c}$ is a *generic parameter* and one where $\mathbf{c}$ lies on a finite union of hyperplanes, in which case $\mathbf{c}$ is called a *special parameter*. We begin with the generic parameter case.

First of all, any generic parameter $\mathbf{c}$ gives rise to the same set of graded W-characters of simple $\overline{\mathsf{H}}_{\mathbf{c}}(W)$-modules. We have furthermore for generic $\mathbf{c}$ that

$$L_{\mathbf{c}}(\lambda) \cong \mathbb{C}W$$

as ungraded W-modules for all $\lambda \in \operatorname{Irr} W$ (cf. [EG02, Thm. 1.7(*iv*)]).

For the group $W = C_\ell \wr \mathfrak{S}_n$, the parameter space may be identified with $\mathbb{C}^\ell$, which becomes just $\mathbb{C}$ when specializing to the symmetric group $\mathfrak{S}_n$ for $\ell = 1$. Since the special parameters for $\mathfrak{S}_n$ must then only be the origin, we differentiate the cases $\mathbf{c} = 0$ and $\mathbf{c} \neq 0$ for $\mathfrak{S}_n$ with the latter comprising the generic case. It is this generic $\mathfrak{S}_n$-case for which the above question was first answered completely by Gordon [Gor03] using the theory of symmetric functions (cf. Ch. 3).

The theory of symmetric functions has for a long time been an important tool in the study of representations of the symmetric group. The *Frobenius character map* ch (cf. Defn. 3.2.9) allows us to associate to any $\mathfrak{S}_n$-module V a symmetric function $\operatorname{ch} V$ in order to study V combinatorially. In [Gor03], Gordon used the Frobenius character map to show that the graded $\mathfrak{S}_n$-characters of simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}(\mathfrak{S}_n)$ are given by specializations of *Macdonald symmetric functions*. These are symmetric functions that depend on two parameters q, t and they have been shown to be connected to Hilbert schemes, where they famously appear as characters of certain vector bundles [Hai01], as well as to various other theories [Che92], [GH96].

To be able to understand Gordon's result and generalize it to the generic parameter case of $C_\ell \wr \mathfrak{S}_n$, we will survey the preliminary combinatorial theories in Chapter 2 and Chapter 3. The *wreath Macdonald symmetric function* for $\boldsymbol{\lambda} \in W$ is denoted by $H_{\boldsymbol{\lambda}}(x; q, t)$. It was first defined and conjectured to exist in [Hai03] with the proof of existence being given in [BF14]. We prove the following.

Theorem (Cor. 3.5.3). *Let $W = C_\ell \wr \mathfrak{S}_n$. For $\boldsymbol{\lambda} \in \operatorname{Irr} W$, let $H_{\boldsymbol{\lambda}}(x; t, t^{-1})$ be the specialized wreath Macdonald symmetric function associated to $\boldsymbol{\lambda}$. Let furthermore $\mathbf{c}$ be a generic Calogero–Moser parameter. We then have*

$$\operatorname{ch} L_{\mathbf{c}}(\boldsymbol{\lambda}) = t^{b(\boldsymbol{\lambda}^*)} H_{\boldsymbol{\lambda}}(x; t, t^{-1})$$

where $b(\boldsymbol{\lambda}^*) \in \mathbb{N}$ is an invariant associated to multipartitions (cf. Sec. 3.3).

Having now treated the generic case, we begin exploring the special parameter case in Chapter 4. In Section 4.1, we encounter *Calogero–Moser partitions* which are set partitions of $\operatorname{Irr} W$ that are obtained by associating the simple $\overline{\mathsf{H}}_{\mathbf{c}}(W)$-module $L_{\mathbf{c}}(\boldsymbol{\lambda})$ to the corresponding block of $\overline{\mathsf{H}}_{\mathbf{c}}(W)$ for $\boldsymbol{\lambda} \in \operatorname{Irr} W$. An element of the Calogero–Moser partition is called *Calogero–Moser family*. For generic $\mathbf{c}$ and $W = C_\ell \wr \mathfrak{S}_n$, the elements of $\operatorname{Irr} W$ all form singleton Calogero–Moser families and the set partition is trivial. For special $\mathbf{c}$, however, we see non-singleton families emerge. The structure of the Calogero–Moser partition has representation theoretic consequences, namely we have

$$\dim L_{\mathbf{c}}(\lambda) < |W|$$

whenever $\lambda \in \operatorname{Irr} W$ belongs to a non-singleton Calogero–Moser family. When $\{\lambda\}$ is a singleton family of a special parameter Calogero–Moser partition, the simple module $L_{\mathbf{c}}(\lambda)$ is isomorphic to the one in generic parameter (cf. Sec. 1.4).

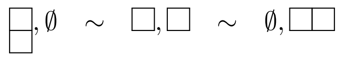
Next up, we describe the geometrical side of rational Cherednik algebras: the *Calogero–Moser space* $\mathsf{X}_{\mathbf{c}}(W)$ [EG02]. This space is given by the spectrum of the center of the rational Cherednik algebra, i.e.

$$\mathsf{X}_{\mathbf{c}}(W) = \operatorname{Spec} \mathsf{Z}_{\mathbf{c}}(W) \ .$$

One can define a central polynomial subalgebra $\mathsf{P}_{\mathbf{c}}(W) \subseteq \mathsf{Z}_{\mathbf{c}}(W)$ such that $\mathsf{Z}_{\mathbf{c}}(W)$ is a free $\mathsf{P}_{\mathbf{c}}(W)$-module of rank $|W|$ (cf. [EG02, Prop. 4.15]). Using the associated scheme-theoretic morphism

$$\Upsilon_{\mathbf{c}} : \mathsf{X}_{\mathbf{c}}(W) \twoheadrightarrow \operatorname{Spec} \mathsf{P}_{\mathbf{c}}(W) \ ,$$

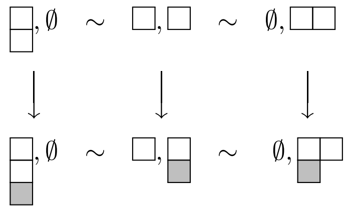
one is able to the study the simple $\overline{\mathsf{H}}_{\mathbf{c}}(W)$-modules in a Calogero–Moser family $\mathcal{F}$ by associating to $\mathcal{F}$ a point in a specific fiber of $\Upsilon_{\mathbf{c}}$ (cf. Sec. 4.3, [Thi17, Sec. 1.9]). This geometric representation theory culminates in a phenomenon we call *cuspidal module induction* which was first described by Bellamy in [Bel11]. In essence, the (ungraded) W-characters of the members of a given Calogero–Moser family $\mathcal{F}$ are induced from the (ungraded) W'-characters of a *cuspidal family* $\mathcal{F}'$ where W' is a parabolic subgroup of W. A family is called *cuspidal* if it lies on a zero-dimensional *symplectic leaf* of a Calogero–Moser space. For $\ell = n = 2$ and $\mathbf{c} = (1, -1) \in \mathbb{C}^2$, we have a cuspidal family given by



where $\sim$ denotes the Calogero–Moser family relation on $\operatorname{Irr} W$. The second part of this thesis is devoted to finding a combinatorial description for cuspidal module induction and using it to describe the ungraded W-characters of $\overline{\mathsf{H}}_{\mathbf{c}}(W)$-modules.

On the one side, this description is obtained via recent results by Maksimau [Mak22] who augmented a known connection between Calogero–Moser spaces and cyclic quiver varieties (cf. Sec. 4.6) by parametrizing $\mathbb{C}^*$-fixed points on symplectic leaves of $\mathsf{X}_{\mathbf{c}}(W)$ (cf. Sec. 5.5). On the other side, these parametrization results rely heavily on the combinatorics of $C_\ell \wr \mathfrak{S}_n$ such as the ℓ-quotient map which is best understood using the theory of *abaci* (cf. Sec. 5.1). An *abacus* (or *bead diagram)* is a certain binary marking of the grid positions $\mathbb{Z} \times \{0, ..., \ell - 1\}$ with which we are able to encode partitions and multipartitions. Bead diagrams have for a long time been a tool in understanding the representation theory of $\mathfrak{S}_n$ (cf. [JK81]).

We use abaci to describe a combinatorial counterpart to cuspidal module induction, which we call *cuspidal family induction.* On the level of partitions, cuspidal family induction consists of adding specific boxes to Young diagrams of multipartitions which preserve the Calogero–Moser family structure. For $\ell = 2$ and $\mathbf{c} = (1, -1)$, we can induce a cuspidal family for $n = 2$ to $n = 3$ as such:



The description of cuspidal family induction is very straightforward for $W = C_2 \wr \mathfrak{S}_n$, i.e. Coxeter type B_n. It relies on a classification of *rigid modules* found in [BT16] which are simple modules $\overline{\mathsf{H}}_{\mathbf{c}}(W)$ that are already irreducible as W-modules. We use these rigid modules to compute the ungraded character of $L_{\mathbf{c}}(\boldsymbol{\lambda})$ whenever $\boldsymbol{\lambda}$ is given by a bipartition $(\lambda^{(0)}, \lambda^{(1)})$ with one empty part.

Theorem (Cor. 5.8.4). *Let $\mathbf{c} = (1, m)$ be a Calogero–Moser parameter for B_n and let $\boldsymbol{\lambda}$ be a bipartition $(\lambda, \emptyset)$ such that $\mu = (k^{k+m}) \subseteq \lambda$ is the largest rectangular sub-diagram of λ having m more rows than columns. We then have*

$$L_{\mathbf{c}}(\boldsymbol{\lambda}) \cong \operatorname{Ind}_{B_{n'}}^{B_n} (\mu, \emptyset)$$

where $n' = k \cdot (k + m)$. The same holds for $\boldsymbol{\lambda} = (\emptyset, \lambda)$ and $\mu = ((k + m)^k)$.

We call this construction method *Durfee induction* and it works in particular for the trivial representation given by the bipartition $((n), \emptyset)$.

Cuspidal family induction for $C_\ell \wr \mathfrak{S}_n$ and $\ell > 2$ is a bit more difficult to describe: for a non-cuspidal family $\mathcal{F}$ and associated cuspidal family $\mathcal{F}'$, there seems to be a nonobvious choice of a bijective map $\mathcal{F} \to \mathcal{F}'$ involved. To remedy this, we propose a formulation of cuspidal family induction using crystal operators coming from the theory of quantum groups, crystal bases, and level ℓ Fock spaces (cf. Sec. 5.9). We then consider possible generalizations of the type B result to higher wreath products by discussing both cuspidal families and rigid modules (cf. Sec. 5.10). The main problem lies in the fact that, unlike in the type B case, not all cuspidal families contain rigid modules (cf. Exm. 5.10.7) which makes generalizations of the theorem above much more difficult.

Having now defined cuspidal family induction, we want to see if we can replicate this inductive phenomenon in other theories related to wreath product groups. In Section 5.11, we begin with Lusztig's $\mathbf{j}$-induction for type B [Lus03, Ch. 22]. Here, we see that the combinatorics developed by Lusztig essentially replicate in our setting of cuspidal family induction of type B, and we get the following result.

PROPOSITION (Prop. 5.11.7). *In type B, Lusztig's $\mathbf{j}$-induction and cuspidal family induction agree.*

Next up, in Section 5.12, we study connections between cuspidal family induction and various W-characters that come from Hecke algebras [GJ11], quantum groups [LM04], and a recently developed cell theory for rational Cherednik algebras [BR17].

In the closing section of Chapter 5, we focus our attention again on wreath Macdonald polynomials. Although these polynomials are only defined for generic parameters and the specialization we investigated in Chapter 3 are equal for all generic parameters, the unspecialized version $H_{\boldsymbol{\lambda}}(x;q,t)$ might also admit a relationship to the combinatorial structures emerging in the *special* parameter case. More precisely, we prove the following result.

THEOREM (Cor. 5.13.8). *For $\boldsymbol{\lambda} \in \operatorname{Irr} B_n$ and Calogero–Moser parameter $\mathbf{c} = (1, n-1)$, let $H_{\boldsymbol{\lambda}}(x;q,t)$ and $H'_{\boldsymbol{\lambda}}(x;q,t)$ be the two wreath Macdonald polynomials of $\boldsymbol{\lambda}$ associated to the two neighboring chambers of $\mathbf{c}$. We then have*

$$H_{\boldsymbol{\lambda}}(x;q,t) = H'_{\boldsymbol{\lambda}}(x;q,t)$$

whenever $\boldsymbol{\lambda}$ lies in a singleton Calogero–Moser family of parameter $\mathbf{c}$.

The above statement holds true for all hyperplanes in the cases $W = B_2, B_3$ and we conjecture that it holds true for all parameters $\mathbf{c}$ and all wreath product groups.

The final chapter briefly recapitulates the contents of this thesis with an emphasis on possible future work augmenting the results presented here.

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Contents

CHAPTER 1

Restricted rational Cherednik algebras I

In this first chapter, we introduce the main topics of this thesis: complex reflection groups and restricted rational Cherednik algebras. We also state our main problem, namely for a complex reflection group W of type $G(\ell, 1, n)$ we want to determine the graded W-characters of simple modules of the restricted rational Cherednik algebra $\overline{\mathsf{H}}_{\mathbf{c}}(W)$.

For the remainder of this thesis, all vector spaces are complex and finite-dimensional and all groups are finite unless stated otherwise.

1.1. Complex reflection groups

Since restricted rational Cherednik algebras are associated to complex reflection groups, we will start there. We refer the reader to [LT09] as general reference for the contents of this section. Let W be a nontrivial finite group and $\mathfrak{h}$ a faithful complex representation of W with $\dim(\mathfrak{h}) = n \in \mathbb{N}$.

Definition 1.1.1. A *complex reflection* of W (with respect to $\mathfrak{h}$) is a nontrivial element $s \in W$ that fixes pointwise some hyperplane in $\mathfrak{h}$. By identifying W with its image in $\mathrm{GL}(\mathfrak{h})$, we have that $s \in W \subseteq \mathrm{GL}(\mathfrak{h})$ is a complex reflection if and only if

$$\dim(\mathrm{Ker}(s - \mathrm{id}_{\mathfrak{h}})) = n - 1 \tag{1.1}$$

where $\mathrm{id}_{\mathfrak{h}} \in \mathrm{GL}(\mathfrak{h})$ denotes the identity map on $\mathfrak{h}$. We write $\mathcal{S} \subseteq W$ for the set of complex reflections of W.

Definition 1.1.2. The tuple $(W, \mathfrak{h})$ is called *complex reflection group* if W is generated by the set of complex reflections $\mathcal{S} \subseteq W$. The vector space $\mathfrak{h}$ is called *complex reflection representation* and its *rank* is $\dim(\mathfrak{h})$. Furthermore, we say $(W, \mathfrak{h})$ is *irreducible* as a complex reflection group whenever $\mathfrak{h}$ is an irreducible representation of W.

Let from now on $(W, \mathfrak{h})$ denote a complex reflection group. After a choice of basis we identify $\mathfrak{h} = \mathbb{C}^n$ and $W \subseteq \mathrm{GL}_n(\mathbb{C})$.

Definition 1.1.3. The *type* of a complex reflection group $(W, \mathfrak{h})$ is the $\mathrm{GL}_n(\mathbb{C})$-conjugacy class of $W \subseteq \mathrm{GL}_n(\mathbb{C})$.

We will give a large classes of examples next which turn out to give the types of all irreducible complex reflection groups up to a finite number of exceptions.

Example 1.1.4. We list some examples of complex reflection groups here.

(*i*) The symmetric group $\mathfrak{S}_n$ acts naturally on $\mathfrak{h} = \mathbb{C}^n$ by permuting the elementary basis vectors $e_1, ..., e_n$. Furthermore, the neighboring transposition

$$s_{i,i+1} \quad \text{for} \quad 1 \leq i \leq n-1 \tag{1.2}$$

generate $\mathfrak{S}_n$ and act as complex reflections on $\mathfrak{h}$. This turns $(\mathfrak{S}_n, \mathfrak{h})$ into a complex reflection group. However, this reflection representation is not irreducible since all reflections fix pointwise the complex line spanned by $e_1 + \cdots + e_n$. If we take the quotient of $\mathfrak{h}$ by that line we obtain an irreducible complex reflection group.

(*ii*) For $\ell \geq 2$ let C_ℓ denote the cyclic group of order ℓ generated by $g \in C_\ell$. Let $\zeta \in \mathbb{C}$ be a primitive ℓ^{th} root of unity. We define a 1-dimensional reflection representation by

$$C_\ell \to \mathrm{GL}_1(\mathbb{C}),\ g^k \mapsto (x \mapsto \zeta^k \cdot x)\ . \tag{1.3}$$

This turns C_ℓ into a complex reflection group by condition (1.1).

(*iii*) As in (*ii*), let $\ell \geq 2$ and $\zeta \in \mathbb{C}$ a primitive ℓ^{th} root of unity. For $n \geq 2$, we define the complex matrices

$$\gamma_j := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \zeta & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \quad s_{i,i+1} := \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \tag{1.4}$$

for $1 \leq j \leq n$, $1 \leq i \leq n-1$, where γ_j has ζ in the j^{th} row and $s_{i,i+1}$ transposes rows i and $i+1$ of the identity matrix. We define the matrix group

$$G = \langle \gamma_j, s_{i,i+1} \mid 1 \leq j \leq n, 1 \leq i \leq n-1 \rangle\ . \tag{1.5}$$

One can check easily that the matrices in (1.4) are complex reflections by equation (1.1), which means that G is already a complex reflection group. As an abstract group, G is isomorphic to the wreath product

$$C_\ell \wr \mathfrak{S}_n \cong (\mathbb{Z}/\ell\mathbb{Z})^n \rtimes \mathfrak{S}_n\ . \tag{1.6}$$

(*iv*) We can construct more examples from the group G in (*iii*). Take any positive integer d dividing ℓ and define $p = \ell/d$. Let $H \subseteq G$ be the subset of all elements $g \in G$ such that the product of the nonzero matrix entries of g is a p^{th} root of unity. One can show that H is a normal subgroup of G, and also that H is a complex reflection group. When we choose $d = 1$, we have $H = G$.

Definition 1.1.5. Let A_{n-1} denote the type of the irreducible complex reflection group $\mathfrak{S}_n$ in Example 1.1.4(*i*). The type of H as described in Example 1.1.4(*iv*) is denoted $G(\ell, d, n)$. The type $G(2, 1, n)$ is also referred to as type B_n.

Theorem 1.1.6 ([ST54]). *Every irreducible complex reflection group is either*

(*i*) *of type A_{n-1} for $n \geq 5$,*
(*ii*) *of the same type as $(C_\ell, \mathbb{C})$ for $\ell \geq 2$ as described in Example 1.1.4(ii),*
(*iii*) *of type $G(\ell, d, n)$ for $\ell, n \geq 2$, $d \mid \ell$, and $(\ell, d, n) \notin \{(2,2,2),(4,4,2)\}$,*
(*iv*) *of one of 34 exceptional types $G_4, ..., G_{37}$.*

Remark 1.1.7. We extend the types of Definition 1.1.5(*iii*) to include the symmetric group $\mathfrak{S}_n$ as type $G(1, 1, n)$ and the cyclic group C_ℓ as type $G(\ell, 1, 1)$. We will not need the irreducible reflection representation of type A.

Additionally, we will identify the class type $G(\ell, 1, n)$ with the representative in Example 1.1.4(*iii*).

Complex reflection groups have a lot of nice properties. We are going to highlight the ones that are most important to us, namely concerning invariants and coinvariants. Throughout, let V denote some finite dimensional complex vector space.

Definition 1.1.8. Let G be a finite subgroup of $\mathrm{GL}(V)$. We denote by $\mathbb{C}[V]$ the symmetric algebra of V^* to which the action of G extends. We denote the fixed point set of the action of G on $\mathbb{C}[V]$ by

$$\mathbb{C}[V]^G = \{f \in \mathbb{C}[V] \mid g.f = f \text{ for all } g \in G\} \tag{1.7}$$

and call it the *ring of invariants* of G. After a choice of basis of V^*, we may identify $\mathbb{C}[V]^G$ with $\mathbb{C}[x_1, ..., x_n]^G$.

Since we can embed any group into $\mathrm{GL}_n(\mathbb{C})$, we can extend the definition of the invariant ring to all groups. (Note that this definition depends on the chosen embedding.)

Theorem 1.1.9 ([ST54],[Che55]). *A finite group $G \leq \mathrm{GL}(V)$ is a complex reflection group if and only if ring of invariants of G is isomorphic to a polynomial algebra.*

The ring of invariants is only one half of the story. To get a good understanding of the algebras we want to discuss, we need to define the coinvariant ring as well.

Definition 1.1.10. Let V be a finite dimensional vector space with symmetric algebra $\mathbb{C}[V]$ and let $G \leq \mathrm{GL}(V)$ be a finite group. Furthermore, let

$$\mathbb{C}[V]^G_+ = \{f \in \mathbb{C}[V]^G \mid f(0) = 0\} \tag{1.8}$$

denote the set of invariants without a constant term. We define the quotient of $\mathbb{C}[V]$ with respect to the respective ideal generated by $\mathbb{C}[V]^G_+$ by

$$\mathbb{C}[V]_G = \mathbb{C}[V] \Big/ \mathbb{C}[V] \cdot \mathbb{C}[V]^G_+ \tag{1.9}$$

and call it the *coinvariant ring* of G. Again, we may write $\mathbb{C}[x_1, ..., x_n]_G$ when fixing a basis of V^*.

The action of G descends to the coinvariant ring which makes $\mathbb{C}[V]_G$ into a G-module. We again get a very nice property of the coinvariant ring from the theory of complex reflection groups and Theorem 1.1.9.

Theorem 1.1.11 ([CG97],[Ste75]). *For a complex reflection group $(W, \mathfrak{h})$ we have*

$$\mathbb{C}[\mathfrak{h}]_W \cong \mathbb{C}W \tag{1.10}$$

as W-modules where $\mathbb{C}W$ is the complex group ring of W affording the regular representation.

This means that for an irreducible representation λ of W, the multiplicity of λ inside $\mathbb{C}[\mathfrak{h}]_W$ is equal to $\dim(\lambda)$.

The algebra $\mathbb{C}[V]$ affords a grading by setting $\deg f = 1$ for $f \in V^*$. Since the action of W on $\mathbb{C}[V]$ preserves this grading, the ideal in (1.9) by which we take the quotient has a homogeneous generating set. This means the ring $\mathbb{C}[\mathfrak{h}]_W$ is graded as well and carries a W-action. We want to fill in the details on $\mathbb{Z}$-graded representations now. Let for that G be an arbitrary finite group.

Definition 1.1.12. A *graded* module of a group G is a $\mathbb{Z}$-graded vector space

$$V = \bigoplus_{i \in \mathbb{Z}} V_i \tag{1.11}$$

with a G-action such that for all $g \in G$ we have

$$g.V_i \subseteq V_i\ , \tag{1.12}$$

i.e. the action of G preserves the homogeneous degree pieces of V.

REMARK 1.1.13. Instead of using Definition 1.1.12, one could also define graded modules of G by viewing the group ring $\mathbb{C}G$ as a graded ring concentrated in degree 0. Both definitions are equivalent.

Now we want to describe the graded irreducible modules of G.

DEFINITION 1.1.14. The *i-shift* $V[i]$ of a $\mathbb{Z}$-graded vector space V is defined to be the $\mathbb{Z}$-graded vector space for which

$$V[i]_j = V_{j-i} \tag{1.13}$$

holds.

Let $\operatorname{Irr} G$ denote a set of pairwise nonisomorphic irreducible complex representations of G. We view the elements of $\operatorname{Irr} G$ as concentrated in degree 0 and the graded irreducible modules of G are now given by all $\mathbb{Z}$-graded shifts of elements of $\operatorname{Irr} G$. We denote the set of graded irreducible G-modules obtained this way by $\operatorname{Irr}^{\mathrm{gr}} G$. Now, we define graded multiplicities.

DEFINITION 1.1.15. Let λ be an (ungraded) complex simple G-module and V be an (ungraded) finite-dimensional complex G-module. We denote by

$$[V : \lambda] \in \mathbb{N} \tag{1.14}$$

the multiplicity of λ inside V. Let V be a graded G-module with homogeneous degree piece V_i of degree $i \in \mathbb{Z}$. We define the *graded multiplicity* of λ in V by

$$[V : \lambda]^{\mathrm{gr}} = \sum_{i \in \mathbb{Z}} [V_i : \lambda] \cdot t^i \in \mathbb{N}[t, t^{-1}] \tag{1.15}$$

for a complex indeterminate t.

By extending Definition 1.1.15, we can define the *graded character* of a graded G-module V by

$$\sum_{\lambda \in \operatorname{Irr} G} [V : \lambda]^{\mathrm{gr}} \cdot \lambda \tag{1.16}$$

where we identify the complex irreducible G-module λ with its character.

To better work with graded characters, we use the *graded Grothendieck ring $\mathcal{G}^{\mathrm{gr}}(W)$ of* W. It is defined as the commutative $\mathbb{Z}[t, t^{-1}]$-algebra generated by the set of symbols

$$\{[\lambda] \mid \lambda \in \operatorname{Irr} W\} \tag{1.17}$$

admitting to the relations $[\lambda \oplus \mu] - [\lambda] - [\mu]$ and $[\lambda \otimes \mu] - [\lambda] \cdot [\mu]$. For a graded W-representation V, we denote by $[V]^{\mathrm{gr}}$ its isomorphism class inside $\mathcal{G}^{\mathrm{gr}}(W)$.

Now, since $\mathbb{C}[\mathfrak{h}]_W$ is a graded version of $\mathbb{C}W$, it is natural to ask the question in which degree pieces the copies of λ appear inside $\mathbb{C}[\mathfrak{h}]_W$. This information is captured in the fake degree of λ.

DEFINITION 1.1.16. Let λ be an irreducible representation of a finite complex reflection group W. We define the *fake degree* of λ by

$$f_\lambda(t) = [\mathbb{C}[\mathfrak{h}]_W : \lambda]^{\mathrm{gr}} . \tag{1.18}$$

We furthermore define

$$\bar{f}_\lambda(t) := t^{-b} \cdot f_\lambda(t) \tag{1.19}$$

where b is the smallest degree appearing with nonzero coefficient in $f_\lambda(t)$. The number b is called *trailing degree* of (the fake degree of) λ.

We will mostly be interested in wreath products $C_\ell \wr \mathfrak{S}_n$ for $\ell, n \geq 1$ and their combinatorial representation theory as outlined in Chapter 2. Furthermore, we will use the (not necessarily irreducible) reflection representation detailed in Example 1.1.4(*iii*).

1.2. Restricted rational Cherednik algebras

In this section we want to state the main problem which this thesis is tackling, i.e. the description of representations of the restricted rational Cherednik algebra $\overline{\mathsf{H}}_{\mathbf{c}}$. The rational Cherednik algebra was introduced in [EG02] and a certain finite-dimensional quotient, the restricted rational Cherednik algebra, was first studied in [Gor03]. We begin with some definitions.

Let $(W, \mathfrak{h})$ be a complex reflection group and let $\mathcal{S} \subseteq W$ be its set of complex reflections. Let $\mathfrak{h}^*$ denote the dual space of $\mathfrak{h}$. The space $\mathfrak{h}^*$ is a W-module via the dual action

$$w.f(y) = f(w^{-1}.y) \tag{1.20}$$

for $w \in W$, $f \in \mathfrak{h}^*$, and $y \in \mathfrak{h}$. We then define the W-module $V := \mathfrak{h} \oplus \mathfrak{h}^*$. Let now

$$T_{\mathbb{C}}(V) := \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus ... \tag{1.21}$$

be the (complex) tensor algebra of V.

Definition 1.2.1. We define the *(complex) skew tensor algebra* $T_{\mathbb{C}}(V) \rtimes W$ as the complex vector space

$$T_{\mathbb{C}}(V) \otimes \mathbb{C}W \tag{1.22}$$

with multiplication given by

$$(f \otimes w) \cdot (f' \otimes w') = f \cdot w.f' \otimes w \cdot w' \tag{1.23}$$

for $w, w' \in W$, $f, f' \in T(V)$.

The vector space V becomes a symplectic vector space via the symplectic form

$$\omega : V \times V \to \mathbb{C}, ((y, x), (y', x')) \mapsto x(y') - x'(y) \ . \tag{1.24}$$

For $s \in \mathcal{S} \subseteq \mathrm{GL}(\mathfrak{h})$, we use the action of s on $V = \mathfrak{h} \oplus \mathfrak{h}^*$ to write

$$V = \mathrm{Im}(s - \mathrm{id}_V) \oplus \mathrm{Ker}(s - \mathrm{id}_V) \tag{1.25}$$

as symplectic subspaces and define the form ω_s to be equal to ω on $\mathrm{Im}(s - \mathrm{id}_V)$ and 0 on $\mathrm{Ker}(s - \mathrm{id}_V)$. Alternatively, we can define for each $s \in \mathcal{S}$ a *root* $\alpha_s \in \mathfrak{h}^*$ and *coroot* $\alpha_s^\vee \in \mathfrak{h}$ by

$$\langle \alpha_s \rangle = \mathrm{Im}(s - \mathrm{id}_V)|_{\mathfrak{h}^*}, \ \langle \alpha_s^\vee \rangle = \mathrm{Im}(s - \mathrm{id}_V)|_{\mathfrak{h}}, \text{ and } \alpha_s(\alpha_s^\vee) \neq 0 \ . \tag{1.26}$$

We then get for $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ the expression

$$\omega_s(x, y) = \frac{x(\alpha_s^\vee) \cdot \alpha_s(y)}{\alpha_s(\alpha_s^\vee)} \ . \tag{1.27}$$

Definition 1.2.2. A function

$$\mathbf{c} : \mathcal{S} \to \mathbb{C} \tag{1.28}$$

is called *Calogero–Moser parameter* for W if $\mathbf{c}(s) = \mathbf{c}(s')$ for s and s' conjugate in W, i.e. the function $\mathbf{c}$ is *W-equivariant* with respect to the conjugacy action of W on $\mathcal{S}$. We denote the space of parameters by $\mathscr{C}$.

Definition 1.2.3 ([EG02]). The *rational Cherednik algebra (RCA)* of W at parameter $\mathbf{c}$ and "at $t = 0$" is defined as

$$\mathsf{H}_{\mathbf{c}} := T_{\mathbb{C}}(V) \rtimes W \big/ I_{\mathbf{c}} \tag{1.29}$$

where $I_{\mathbf{c}}$ is the ideal generated by the relations

$$0 = [x, x'] = [y, y'] = [x, y] - \sum_{s \in \mathcal{S}} \mathbf{c}(s) \omega_s(x, y) s \tag{1.30}$$

for $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$.

REMARK 1.2.4. There is a more general version of this algebra that includes an additional parameter $t \in \mathbb{C}$ which in our case is equal to 0. To differentiate these two cases, one writes "at $t = 0$" when talking about this specific class of Cherednik algebras. That t is *not to be confused* with the parameter t of our symmetric functions and graded modules. We will restrict to the $t = 0$ case in this work.

The rational Cherednik algebra admits a $\mathbb{Z}$-grading given by

$$\deg(\mathfrak{h}^*) = 1 \ , \ \deg(\mathfrak{h}) = -1 \ , \text{ and } \deg(W) = 0 \ . \tag{1.31}$$

We also have a triangular decomposition and a *PBW property* by [EG02, Thm 1.3], i.e. $\mathsf{H}_{\mathbf{c}}$ has a graded vector space decomposition

$$\mathsf{H}_{\mathbf{c}} \cong \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*] \ . \tag{1.32}$$

Let $Z(\mathsf{H}_{\mathbf{c}})$ denote the center of $\mathsf{H}_{\mathbf{c}}$. It was first shown in [EG02, Prop. 4.15] and later in [Gor03, Prop. 3.6] using elementary methods that

$$\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \subseteq Z(\mathsf{H}_{\mathbf{c}}) \tag{1.33}$$

This motivates the following definition.

DEFINITION 1.2.5 ([Gor03]). We define a quotient of $\mathsf{H}_{\mathbf{c}}$, called the *restricted rational Cherednik algebra (RRCA)*, by

$$\overline{\mathsf{H}}_{\mathbf{c}} := \mathsf{H}_{\mathbf{c}} \Big/ A_+ \mathsf{H}_{\mathbf{c}} \tag{1.34}$$

where $A_+ \subseteq \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ is the set of elements with no constant term.

LEMMA 1.2.6. *The PBW property of $\mathsf{H}_{\mathbf{c}}$ in [EG02, Thm 1.3] descends to $\overline{\mathsf{H}}_{\mathbf{c}}$ such that we get a graded vector space decomposition*

$$\overline{\mathsf{H}}_{\mathbf{c}} \cong \mathbb{C}[\mathfrak{h}]_W \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*]_W \tag{1.35}$$

where $\mathbb{C}[\mathfrak{h}]_W$ is the coinvariant ring of Definition 1.1.10. Using Theorem 1.1.11 we get

$$\dim(\overline{\mathsf{H}}_{\mathbf{c}}) = |W|^3 \ . \tag{1.36}$$

In particular, the restricted rational Cherednik algebra is finite dimensional.

1.3. Representation theory of RRCAs

We are interested in the representation theory of $\overline{\mathsf{H}}_{\mathbf{c}}$ whose basic notions have first been described in [Gor03].

Using the methods of [HN91] on (1.35), Gordon defined in [Gor03] the subalgebra of $\mathsf{B}_{\mathbf{c}} \subseteq \overline{\mathsf{H}}_{\mathbf{c}}$ generated by the negative degree coinvariants and the elements of W inside $\overline{\mathsf{H}}_{\mathbf{c}}$. Under the isomorphism afforded by (1.32), we have

$$\mathsf{B}_{\mathbf{c}} \cong \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}W \ . \tag{1.37}$$

For a W-module λ, we define an action of $q \otimes w \in \mathsf{B}_{\mathbf{c}}$ on λ by

$$(q \otimes w).v := q(0) \cdot w.v \tag{1.38}$$

for all $v \in \lambda$, which turns λ into a $\mathsf{B}_{\mathbf{c}}$-module. Let $\operatorname{Irr} W$ be a set of representatives of the isomorphism classes of simple W-modules. We identify $\lambda \in \operatorname{Irr} W$ with the graded version concentrated in degree 0.

DEFINITION 1.3.1 ([Gor03, 4.2]). For $\lambda \in \operatorname{Irr} W$ we define the *standard* (or *baby Verma*) *module* of λ by

$$M_{\mathbf{c}}(\lambda) := \overline{\mathsf{H}}_{\mathbf{c}} \otimes_{\mathsf{B}_{\mathbf{c}}} \lambda \tag{1.39}$$

Let V be a module of $\overline{\mathsf{H}}_{\mathbf{c}}$. By Lemma 1.2.6 and because $\mathbb{C}W \subseteq \overline{\mathsf{H}}_{\mathbf{c}}$ is concentrated in degree 0, we can restrict the action of $\overline{\mathsf{H}}_{\mathbf{c}}$ on V to the subalgebra $\mathbb{C}W$. Whenever V is a graded vector space, we obtain a graded W-module and therefore a graded W-character of V. It is these W-characters of $\overline{\mathsf{H}}_{\mathbf{c}}$-modules which we aim to study.

REMARK 1.3.2. We have

$$M_{\mathbf{c}}(\lambda) \cong \mathbb{C}[\mathfrak{h}]_W \otimes \lambda \tag{1.40}$$

as vector spaces by the vector space decomposition of $\overline{\mathsf{H}}_{\mathbf{c}}$ (1.35). In particular, the W-character of $M_{\mathbf{c}}(\lambda)$ is independent of $\mathbf{c} \in \mathscr{C}$. We will therefore often omit the parameter $\mathbf{c}$ from our notation of standard modules.

Recall from Section 1.1 that $\mathcal{G}^{\mathrm{gr}}(W)$ denotes the graded Grothendieck group of W. We have the following identity (see [Gor03, Sec. 4.2] or [Thi17, Cor. 2.6] for a proof).

LEMMA 1.3.3. *When viewing equation* (1.40) *in* $\mathcal{G}^{\mathrm{gr}}(W)$ *we can use the fake degree from Definition 1.1.16 to obtain*

$$[M(\lambda)]^{\mathrm{gr}} = \sum_{\mu \in \operatorname{Irr} W} f_\mu(t) \cdot [\mu \otimes \lambda] \ . \tag{1.41}$$

From the straightforward construction of the standard modules we are actually able to give a complete set of pairwise nonisomorphic simple graded modules of $\overline{\mathsf{H}}_{\mathbf{c}}$.

THEOREM 1.3.4 ([Gor03, Prop. 4.3]). *Each* $\overline{\mathsf{H}}_{\mathbf{c}}$*-module* $M(\lambda)$ *has a unique maximal submodule, hence a unique simple head denoted by* $L_{\mathbf{c}}(\lambda)$*. Furthermore, the set*

$$\{L_{\mathbf{c}}(\lambda)[i] \mid \lambda \in \operatorname{Irr} W, i \in \mathbb{Z}\} \tag{1.42}$$

is a complete set of pairwise nonisomorphic simple graded $\overline{\mathsf{H}}_{\mathbf{c}}$*-modules.*

Note that we have $L_{\mathbf{c}}(\lambda[i]) \cong L_{\mathbf{c}}(\lambda)[i]$ in the construction above. We will focus on the description of $L_{\mathbf{c}}(\lambda)$ with $\lambda \in \operatorname{Irr}^{\mathrm{gr}} W$ concentrated in degree 0. Alternatively, we could characterize $L_{\mathbf{c}}(\lambda)$ by being concentrated in nonnegative degree and having a copy of λ in its degree 0 piece.

When studying the representation theory of restricted rational Cherednik algebras, we will mainly be interested in the relationship of three classes of modules: standards of $\overline{\mathsf{H}}_{\mathbf{c}}$, simples of $\overline{\mathsf{H}}_{\mathbf{c}}$, and simples of W, which are respectively denoted by

$$(M(\lambda))_{\lambda \in \operatorname{Irr} W} \ , \quad (L_{\mathbf{c}}(\lambda))_{\lambda \in \operatorname{Irr} W} \ , \quad (\lambda)_{\lambda \in \operatorname{Irr} W} \ . \tag{1.43}$$

All three of these classes are indexed by the simple modules of W and we can therefore define the three square matrices over the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$

$$D_\Delta = ([M(\lambda) : L_{\mathbf{c}}(\mu)]^{\mathrm{gr}})_{\lambda, \mu \in \operatorname{Irr} W} \ , \tag{1.44}$$

$$C_\Delta = ([M(\lambda) : \mu]^{\mathrm{gr}})_{\lambda, \mu \in \operatorname{Irr} W} \ , \tag{1.45}$$

$$C_L = ([L_{\mathbf{c}}(\lambda) : \mu]^{\mathrm{gr}})_{\lambda, \mu \in \operatorname{Irr} W} \ . \tag{1.46}$$

Here, D_Δ is defined via graded multiplicities in the category of graded $\overline{\mathsf{H}}_{\mathbf{c}}$-modules while C_Δ and C_L use multiplicities in the category of graded W-modules. See [BT18, §3.7] for more details.

When viewing D_Δ in the graded Grothendieck group of W, we can relate the three matrices by

$$C_\Delta = D_\Delta \cdot C_L \ . \tag{1.47}$$

We are mainly interested in the matrix C_L, i.e. the decomposition of simple modules of the restricted rational Cherednik algebra into simple modules of W. Because of Definition 1.3.1 we have that $C_\Delta \in \mathbb{Z}[t,t^{-1}]^{\operatorname{Irr} W \times \operatorname{Irr} W}$ specializes to the identity matrix for $t = 0$, and therefore, C_Δ is invertible over $\mathbb{Q}(t)$. This makes D_Δ and C_L invertible over $\mathbb{Q}(t)$ as well and we can transform (1.47) into

$$C_L = D_\Delta^{-1} \cdot C_\Delta \,. \tag{1.48}$$

Since the matrix C_Δ is completely determined by the fake degrees, which are easy to compute (see Theorem 3.3.13) and the Kronecker coefficients, which we need for both C_L and D_Δ, the matrices C_L and D_Δ are basically equally difficult to attain. We can therefore concentrate our efforts on C_L and pose the following natural question, which goes back to the inception of the restricted rational Cherednik algebra (cf. [Gor03, Prob. 1]).

Question 1.3.5. For given parameter $\mathbf{c}$, what are the entries of the matrix C_L? Or equivalently, what is the graded character of $L_{\mathbf{c}}(\lambda)$ for given parameter $\mathbf{c}$ and $\lambda \in \operatorname{Irr} W$?

Before we can tackle this question and also see which cases have already been answered, namely in [Gor03], we need introduce the notion of generic parameters.

1.4. Generic parameters

The structure of the simple module $L_{\mathbf{c}}(\lambda)$ is highly dependent on the parameter $\mathbf{c}$, for which one needs to differentiate two main cases: *generic* parameters (of which there are again two types) and *special* parameters. The different types of generic parameters have been studied in [Thi16] and [Thi18] which are reviewed in the survey paper [Thi17]. The different notions of parameters are obtained by defining a "generic" version H of the RCA that does not depend on a parameter choice. We can then "specialize" H to a given rational Cherednik algebra $\mathsf{H}_{\mathbf{c}}$. The notions of generic and special parameters will be given by the properties of these specializations $\mathsf{H} \rightsquigarrow \mathsf{H}_{\mathbf{c}}$.

For a reflection $s \in \mathcal{S}$ we denote by $[s] \in \mathcal{S}/W$ its W-conjugacy class. We have a basis for the space of parameters $\mathscr{C}$ given by the set of maps

$$\begin{aligned} \mathbf{c}_{[s]} &: \mathcal{S} \to \mathbb{C} \\ r &\mapsto \mathbf{c}_{[s]}(r) = \begin{cases} 1, & r \in [s] \\ 0 & \text{else} \end{cases} \end{aligned} \tag{1.49}$$

for all $[s] \in \mathcal{S}/W$. The corresponding dual basis of $\mathscr{C}^*$ is given by the evaluation maps

$$\begin{aligned} \mathbf{c}^*_{[s]} &: \mathscr{C} \to \mathbb{C} \\ \mathbf{c} &\mapsto \mathbf{c}(s) \end{aligned} \tag{1.50}$$

for all $[s] \in \mathcal{S}/W$. We denote the coordinate ring of $\mathscr{C}$ by

$$\mathsf{C} := \mathbb{C}[\mathscr{C}] = \mathbb{C}[\mathbf{c}^*_{[s]} \mid [s] \in \mathcal{S}/W\} \,. \tag{1.51}$$

Furthermore, we denote the tensor algebra of $V = \mathfrak{h} \oplus \mathfrak{h}^*$ over C by T_{C} which gets tensored with the group ring $\mathsf{C}W$ to generate the skew tensor algebra of W over C analogously to Definition 1.2.1.

Definition 1.4.1 ([EG02],[Thi17]). The *generic rational Cherednik algebra* of W "at $t = 0$" is defined as

$$\mathsf{H} := \left. T_{\mathsf{C}}(V) \rtimes W \middle/ I_{\mathsf{C}} \right. \tag{1.52}$$

where I_{C} is the ideal generated by the relations

$$0 = [x, x'] = [y, y'] = [x, y] - \sum_{s \in \mathcal{S}} \mathbf{c}^*_{[s]} \omega_s(x, y) s \tag{1.53}$$

for $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$.

Note that in the generic case the commutator relations are now inside $\mathsf{C}W$. The generic rational Cherednik algebra behaves similarly its complex version from the previous section: H is a C-algebra with a PBW property analogous to (1.32). Furthermore, one can define the restriction of the generic RCA in the same way as before.

Definition 1.4.2. Let H be a generic rational Cherednik algebra. We define the *generic restricted rational Cherednik algebra* by

$$\overline{\mathsf{H}} := \mathsf{H} \Big/ A_+ \mathsf{H} \tag{1.54}$$

where $A_+ \subseteq \mathsf{C}[\mathfrak{h}]^W \otimes \mathsf{C}[\mathfrak{h}^*]^W$ is the set of elements with no constant term.

When we view $\overline{\mathsf{H}}$ as a sheaf of algebras over $\operatorname{Spec} \mathsf{C}$, we can follow the discussion in [Thi17] and identify the set of closed points of $\operatorname{Spec} \mathsf{C}$ with $\mathscr{C}$. For such a closed point $\mathbf{c}$, we obtain the $\overline{\mathsf{H}}_{\mathbf{c}}$-relations in (1.29) and (1.30) from the $\overline{\mathsf{H}}$-relations in (1.52) and (1.53) via

$$\overline{\mathsf{H}}_{\mathbf{c}} = \mathsf{C}/\mathbf{c} \otimes_{\mathsf{C}} \overline{\mathsf{H}} \tag{1.55}$$

which replaces in (1.52) the algebra C by $\mathsf{C}/\mathbf{c} = \mathbb{C} = \operatorname{Frac}(\mathsf{C}/\mathbf{c})$ and in (1.53) the $\mathbf{c}^*_{[s]}$ by their images in $\mathsf{C}/\mathbf{c}$ given by $\mathbf{c}^*_{[s]}(\mathbf{c}) = \mathbf{c}(s)$. This construction can be extended to nonclosed points of $\operatorname{Spec} \mathsf{C}$ as detailed in [Thi17].

Definition 1.4.3. For a (not necessarily closed) point $\mathbf{c} \in \operatorname{Spec} \mathsf{C}$, we define

$$\overline{\mathsf{H}}|_{\mathbf{c}} := \mathsf{C}/\,\mathbf{c} \otimes_{\mathsf{C}} \overline{\mathsf{H}}\ . \tag{1.56}$$

Furthermore, we denote the *specialization* (or *fiber*) of $\overline{\mathsf{H}}$ in $\mathbf{c}$ by

$$\overline{\mathsf{H}}_{\mathbf{c}} := \operatorname{Frac}(\mathsf{C}/\,\mathbf{c}) \otimes_{\mathsf{C}} \overline{\mathsf{H}}\ . \tag{1.57}$$

Remark 1.4.4. Specialization and restriction commute, i.e. when we start in the specialized setting of Section 1.2 and restrict as in Definition 1.2.5, we could also start with the generic restricted RCA of Definition 1.4.2 and then specialize.

Let $\mathbf{c} \in \operatorname{Spec} \mathsf{C}$ be a prime ideal. As has been done in [Thi17], we assume from now on that the quotient $\mathsf{C}/\mathbf{c}$ is normal. We can identify the spectrum of $\mathsf{C}/\mathbf{c}$ with the zero locus $\mathsf{V}(\mathbf{c})$ of $\mathbf{c}$ in $\operatorname{Spec} \mathsf{C}$ and a closed point inside $\mathsf{V}(\mathbf{c})$ then gives rise to the specialization from before. We will be interested in the case when $\mathbf{c} = \bullet$ is the generic point of $\operatorname{Spec} \mathsf{C}$ defined by the zero ideal. We then have

$$\overline{\mathsf{H}}|_{\bullet} = \overline{\mathsf{H}} \tag{1.58}$$

and

$$\overline{\mathsf{H}}_{\bullet} = \operatorname{Frac}(\mathsf{C}) \otimes_{\mathsf{C}} \overline{\mathsf{H}}\ . \tag{1.59}$$

Using the theory found in [GP00; CJ16], Thiel described in [Thi17, Sec. 3.1] a decomposition map between graded Grothendieck groups of restricted rational Cherednik algebras denoted by

$$\mathsf{d}^{\mathbf{c}}_{\bullet} : \mathcal{G}^{\mathrm{gr}}(\overline{\mathsf{H}}_{\bullet}) \to \mathcal{G}^{\mathrm{gr}}(\overline{\mathsf{H}}_{\mathbf{c}}) \tag{1.60}$$

where $\mathbf{c}$ is a (not necessarily closed) point in $V(\bullet) = \operatorname{Spec} \mathsf{C}$. We will review the construction for the ungraded version here, following [GP00, Sec. 7]. The graded version follows a similar approach and is detailed in [CJ16].

Let $\mathbf{c}$ be a prime ideal of C and denote by $\theta : \mathsf{C} \to \mathrm{Frac}(\mathsf{C}/\mathbf{c})$ the composition of the quotient map and the embedding map. Furthermore, we have an embedding $\mathsf{C} \subseteq \mathrm{Frac}(\mathsf{C})$. This can be illustrated by

$$\begin{tikzcd} \mathsf{C} \arrow[d, "\theta"] & \subseteq & \mathrm{Frac}(\mathsf{C}) \\ \mathrm{Frac}(\mathsf{C}/\mathbf{c}) & & \end{tikzcd} \tag{1.61}$$

Now, the fields $\mathrm{Frac}(\mathsf{C})$ and $\mathrm{Frac}(\mathsf{C}/\mathbf{c})$ correspond to restricted rational Cherednik algebras $\overline{\mathsf{H}}_\bullet$ and $\overline{\mathsf{H}}_\mathbf{c}$, respectively. We construct the map (1.60) by going through a valuation ring: as summarized in [GP00, Sec. 7.3.5], there exists a valuation ring $\mathscr{O}_\mathbf{c}$ with maximal ideal $\mathfrak{m}$ for each $\mathbf{c} \in \operatorname{Spec} \mathsf{C}$ such that

$$\mathsf{C} \subseteq \mathscr{O}_\mathbf{c} \subseteq \mathrm{Frac}(\mathsf{C}/\mathbf{c}) \quad \text{and} \quad \mathfrak{m} \supseteq \mathbf{c}\ . \tag{1.62}$$

The situation then becomes

$$\begin{tikzcd} \mathsf{C} \arrow[d, "\theta"] & \subseteq & \mathscr{O}_\mathbf{c} \arrow[d, "\pi"] & \subseteq & \mathrm{Frac}(\mathsf{C}) \\ \mathrm{Frac}(\mathsf{C}/\mathbf{c}) & \subseteq & \mathscr{O}_\mathbf{c}/\mathfrak{m} & & \end{tikzcd} \tag{1.63}$$

where π denotes the canonical quotient map. This diagram appears in [GP00, Sec. 7.4.1] in a more general setting.

Let V be a $\overline{\mathsf{H}}_\bullet$-module, i.e. we are working over the field $K := \mathrm{Frac}(\mathsf{C})$. In [GP00, Sec. 7.3.7], there is now a construction of a K-basis $\mathcal{B}$ of V such that the matrices associated of $\overline{\mathsf{H}}_\bullet$ with respect to $\mathcal{B}$ contain only entries in $\mathscr{O}_\mathbf{c}$. The conditions for this construction is met since K is the field of fractions of $\mathscr{O}_\mathbf{c}$ (cf. [GP00, Sec. 7.3.7]). Now, the basis $\mathcal{B}$ of V generates an $\mathscr{O}_\mathbf{c}\overline{\mathsf{H}}$-lattice $\tilde{V}$ where we denote $\mathscr{O}_\mathbf{c}\overline{\mathsf{H}} := \mathscr{O}_\mathbf{c} \otimes_\mathsf{C} \overline{\mathsf{H}}$. We now have two modules associated to $\tilde{V}$, namely

$$K \otimes_{\mathscr{O}_\mathbf{c}} \tilde{V} \quad \text{and} \quad \mathscr{O}_\mathbf{c}/\mathfrak{m} \otimes_{\mathscr{O}_\mathbf{c}} \tilde{V} \tag{1.64}$$

and we have $K \otimes_{\mathscr{O}_\mathbf{c}} \tilde{V} \cong V$. By [GP00, Thm. 7.4.3], there now exists a map

$$\mathcal{G}(\overline{\mathsf{H}}_\bullet) \to \mathcal{G}(\mathscr{O}_\mathbf{c}\overline{\mathsf{H}}) \tag{1.65}$$

uniquely determined by

$$[K \otimes_{\mathscr{O}_\mathbf{c}} \tilde{V}] \mapsto [\mathscr{O}_\mathbf{c}/\mathfrak{m} \otimes_{\mathscr{O}_\mathbf{c}} \tilde{V}]\ . \tag{1.66}$$

Finally, we use that $\overline{\mathsf{H}}_\mathbf{c}$ is split by [BR17, Prop. 9.1.3], which gives us

$$\mathcal{G}(\overline{\mathsf{H}}_\mathbf{c}) \cong \mathcal{G}(\mathscr{O}_\mathbf{c}\overline{\mathsf{H}}) \tag{1.67}$$

induced by the field extension $\mathrm{Frac}(\mathsf{C}/\mathbf{c}) \subseteq \mathscr{O}_\mathbf{c}/\mathfrak{m}$ in (1.63). Combining (1.66) and (1.67), we obtain the ungraded version of (1.60), which is furthermore independent of the choice of valuation ring by [GP00, Thm. 7.4.3]. As mentioned above, the methods described here extend to the graded setting (cf. [CJ16]).

We collect some of the properties of $\mathsf{d}_\bullet^\mathbf{c}$ from [Thi17, Sec. 3.1] in the following lemma. Note that the construction of standard and simple modules outlined in Section 1.2 extends to specializations at nonclosed points.

Lemma 1.4.5 ([Thi17, Sec. 3.1]). *Let* $\mathbf{c}$ *be a (not necessarily closed) point of* $\operatorname{Spec} \mathsf{C}$. *The decomposition map* $\mathsf{d}_\bullet^\mathbf{c} : \mathcal{G}^{\mathrm{gr}}(\overline{\mathsf{H}}_\bullet) \to \mathcal{G}^{\mathrm{gr}}(\overline{\mathsf{H}}_\mathbf{c})$ *has the following properties:*

(*i*) $\mathsf{d}_\bullet^\mathbf{c}$ *is stable with respect to grade-shifts, i.e.*

$$\mathsf{d}_\bullet^\mathbf{c}([V[m]]^{\mathrm{gr}}) = \mathsf{d}_\bullet^\mathbf{c}([V]^{\mathrm{gr}})[m] \tag{1.68}$$

for $[V]^{\mathrm{gr}} \in \mathcal{G}^{\mathrm{gr}}(\overline{\mathsf{H}}_\bullet)$, (*ii*) $\mathsf{d}^{\mathbf{c}}_{\bullet}$ *sends standard modules to standard modules, i.e.*

$$\mathsf{d}^{\mathbf{c}}_{\bullet}([M_{\bullet}(\lambda)]^{\mathrm{gr}}) = [M_{\mathbf{c}}(\lambda)]^{\mathrm{gr}} \tag{1.69}$$

for $\lambda \in \mathrm{Irr}\, W$*,*

(*iii*) *the simple* $\overline{\mathsf{H}}_{\mathbf{c}}$*-module* $L_{\mathbf{c}}(\lambda)$ *is a constituent of the* $\overline{\mathsf{H}}_{\mathbf{c}}$*-module corresponding to* $\mathsf{d}^{\mathbf{c}}_{\bullet}([L_{\bullet}(\lambda)]^{\mathrm{gr}})$.

Now that there is a way to relate the modules of the generic RRCA with its various specializations, one can define the first notion of genericity.

Definition 1.4.6. We define the set of *decomposition generic* points of $\mathrm{Spec}\, \mathsf{C}$ by

$$\mathsf{DecGen}(\overline{\mathsf{H}}) := \{\mathbf{c} \in \mathrm{Spec}\, \mathsf{C} \mid \mathsf{d}^{\mathbf{c}}_{\bullet}([L_{\bullet}(\lambda)]^{\mathrm{gr}}) = [L_{\mathbf{c}}(\lambda)]^{\mathrm{gr}} \text{ for all } \lambda \in \mathrm{Irr}\, W\} \tag{1.70}$$

and analogously the set of *decomposition exceptional* (or *decomposition special*) points by

$$\mathsf{DecEx}(\overline{\mathsf{H}}) := \mathrm{Spec}\, \mathsf{C} \ \setminus \mathsf{DecGen}(\overline{\mathsf{H}}) \tag{1.71}$$

The set $\mathsf{DecGen}(\overline{\mathsf{H}})$ is one of the two notions of genericity we are going to need. The second one comes from the block structure of $\overline{\mathsf{H}}_{\mathbf{c}}$.

The algebra $\overline{\mathsf{H}}_{\mathbf{c}}$ is noetherian and therefore admits a *block decomposition*, i.e. there exist $e_1, ..., e_k \in Z(\overline{\mathsf{H}}_{\mathbf{c}})$ such that

$$e_1 + \cdots + e_k = 1 \in \overline{\mathsf{H}}_{\mathbf{c}}$$

with $e_i e_j = \delta_{ij} e_i$ for $1 \leq i, j \leq k$. Furthermore, the ideal

$$B_i := \overline{\mathsf{H}}_{\mathbf{c}} e_i \tag{1.72}$$

is an indecomposable (left) $\overline{\mathsf{H}}_{\mathbf{c}}$-module called *block* of $\overline{\mathsf{H}}_{\mathbf{c}}$ for each $1 \leq i \leq k$. We can then decompose $\overline{\mathsf{H}}_{\mathbf{c}}$ as

$$\overline{\mathsf{H}}_{\mathbf{c}} = \bigoplus_{i=1}^{k} B_i \tag{1.73}$$

and furthermore any (left) $\overline{\mathsf{H}}_{\mathbf{c}}$-module V as

$$V = \bigoplus_{i=1}^{k} e_i V\ .$$

Therefore, if V is an indecomposable $\overline{\mathsf{H}}_{\mathbf{c}}$-module, we have $e_i V = V$ for some i and $e_j V = 0$ for $j \neq i$. We then say V *belongs to* B_i. In this manner, we can associate to any simple $\overline{\mathsf{H}}_{\mathbf{c}}$-module $L_{\mathbf{c}}(\lambda)$ a block of $\overline{\mathsf{H}}_{\mathbf{c}}$ and this process partitions the set $\{L_{\mathbf{c}}(\lambda) \mid \lambda \in \mathrm{Irr}^{\mathrm{gr}}\, W\}$. Because this association is independent of grade shifts of the $L_{\mathbf{c}}(\lambda)$, the partition descends to $\mathrm{Irr}\, W$.

Definition 1.4.7 ([Gor03]). For $\mathbf{c} \in \mathrm{Spec}\, \mathsf{C}$ the partition of $\mathrm{Irr}\, W$ induced by the block structure of $\overline{\mathsf{H}}_{\mathbf{c}}$ is called *Calogero–Moser (*$\mathbf{c}$*-)partition* denoted by $\mathrm{CM}_{\mathbf{c}}$. The classes of $\mathrm{CM}_{\mathbf{c}}$ are then called *Calogero–Moser (*$\mathbf{c}$*-)families.*

Note that Calogero–Moser partitions are certain subsets of the power set of $\mathrm{Irr}\, W$ called *set partitions*. They are not to be confused with the partitions described in Chapter 2.

Definition 1.4.8. The set of *block generic* points of $\mathrm{Spec}\, \mathsf{C}$ is defined as

$$\mathsf{BlGen}(\overline{\mathsf{H}}) := \{\mathbf{c} \in \mathrm{Spec}\, \mathsf{C} \mid \mathrm{CM}_{\bullet} = \mathrm{CM}_{\mathbf{c}}\} \tag{1.74}$$

and analogously the set of *block exceptional* (or *block special*) points is given by

$$\mathsf{BlEx}(\overline{\mathsf{H}}) := \mathrm{Spec}\, \mathsf{C} \ \setminus \mathsf{BlGen}(\overline{\mathsf{H}}) \tag{1.75}$$

THEOREM 1.4.9 ([Thi18, Thm. 4.3], [BST18, Thm. 5.1]). *We have*

$$\mathsf{DecGen}(\overline{\mathsf{H}}) \subseteq \mathsf{BlGen}(\overline{\mathsf{H}}) \ . \tag{1.76}$$

Furthermore, the set $\mathsf{BlEx}(\overline{\mathsf{H}}) \subseteq \operatorname{Spec} \mathsf{C}$ *is a finite union of hyperplanes.*

We also call $\mathsf{BlEx}(\overline{\mathsf{H}})$ the *Calogero–Moser hyperplanes* and the hyperplane chambers of $\mathsf{BlGen}(\overline{\mathsf{H}})$ we call *Calogero–Moser chambers* of $\operatorname{Spec} \mathsf{C}$. The above theorem also tells us that "most" parameters are decomposition and block generic.

Everything up to this point generalizes to any finite complex reflection group W. To be able to give explicit character formulae in the chapters to come, we will set now W to be of type $G(\ell, 1, n)$ for some fixed $\ell, n \in \mathbb{N}$.

THEOREM 1.4.10. *For W of type $G(\ell, 1, n)$ the following statements are equivalent:*

(*i*) $\mathbf{c} \in \mathsf{BlGen}(\overline{\mathsf{H}})$,
(*ii*) $\mathbf{c} \in \mathsf{DecGen}(\overline{\mathsf{H}})$,
(*iii*) $\dim L_{\mathbf{c}}(\lambda) = |W|$ *for all* $\lambda \in \operatorname{Irr} W$,
(*iv*) $L_{\mathbf{c}}(\lambda) \cong \mathbb{C}W$ *for all* $\lambda \in \operatorname{Irr} W$ *as ungraded W-modules,*
(*v*) *the partition* $\mathrm{CM}_{\mathbf{c}}$ *contains only singleton families,*
(*vi*) D_Δ *is a diagonal matrix with entries* $\bar{f}_\lambda(t)$.

PROOF. Since the Calogero-Moser space of $G(\ell, 1, n)$ is generically smooth by [EG02, 1.12–14], we can use [Thi17, Lem. 3.21], [Thi17, Cor. 3.22] [Gor03, 5.2+5.5], and the proof of [Gor03, 5.6]. ∎

DEFINITION 1.4.11. For W of type $G(\ell, 1, n)$ a point $\mathbf{c} \in \mathsf{DecGen}(\overline{\mathsf{H}}) = \mathsf{BlGen}(\overline{\mathsf{H}})$ is simply called *generic*. A nongeneric point is called *special*.

It is conjectured in [Thi17, Conj. 6.1] that $\mathsf{DecGen}(\overline{\mathsf{H}}) = \mathsf{BlGen}(\overline{\mathsf{H}})$ holds for all complex reflection groups.

As we can see in Theorem 1.4.10, the theory becomes much simpler for W of type $G(\ell, 1, n)$ and $\mathbf{c}$ generic. Now that both the matrices C_Δ and D_Δ are controlled by the fake degrees of elements of $\operatorname{Irr} W$ (cf. Lem. 1.3.3), the matrix C_L is as well. For the same reasons, all three matrices are constant across the generic locus of $\operatorname{Spec} \mathsf{C}$. This leads us to the following question which is the generic $G(\ell, 1, n)$-case of Question 1.3.5 and forms the starting point for that question as well.

QUESTION 1.4.12. For generic parameter $\mathbf{c}$ and W of type $G(\ell, 1, n)$, what are the entries of the matrix C_L? Or equivalently, what is the graded character of $L_{\mathbf{c}}(\lambda)$ for generic $\mathbf{c}$ and $\lambda \in \operatorname{Irr} W$?

Now, there is an extensive combinatorial toolbox for this case which we can use to fully describe the graded character of $L_{\mathbf{c}}(\lambda)$ for all $\lambda \in \operatorname{Irr} W$. We will therefore start by reviewing the relevant theories in Chapter 2 and Chapter 3 where everything up to Section 3.5 are well-known preliminaries.

After having solved the generic case, we are going to dive into the more difficult, but much more enticing special parameter case in Chapter 4.

CHAPTER 2

Partition combinatorics

In this chapter we are going to review the combinatorial representation theory of $W = C_\ell \wr \mathfrak{S}_n$ in order to answer Question 1.4.12, i.e. determining the structure of C_L for W of type $G(\ell, 1, n)$ and generic $\mathbf{c}$.

We begin by introducing first the theory of partitions and Young tableaux following the classic book by Fulton [Ful97]. Fix once and for all some nonnegative integers ℓ and n.

2.1. Partitions and Young tableaux

Definition 2.1.1. A *partition* of n is a weakly decreasing sequence of nonnegative integers

$$\lambda = (\lambda_1, ..., \lambda_k) \tag{2.1}$$

such that its *size* (or *rank*) is $|\lambda| := \sum_{i=1}^{k} \lambda_k = n$. Its *length* is k, denoted by $l(\lambda)$. The set of all partitions of n is denoted by $\mathscr{P}(n)$ and we also write $\mu \vdash n$ for $\mu \in \mathscr{P}(n)$. We denote by $\emptyset \in \mathscr{P}(0)$ the empty partition.

For ease of notation, we will sometimes extend a partition by a (possibly infinite) number of trailing 0's.

Remark 2.1.2. There is also a "multiplicative notation" for partitions we will use as well. For $\mu \in \mathscr{P}(n)$, we write

$$(\mu_1^{a_1}, \mu_2^{a_2}, ..., \mu_k^{a_k}) \tag{2.2}$$

where $\mu_1 > \mu_2 > ... > \mu_k > 0$ and $a_i > 0$ denotes the number of times μ_i appears in the partition μ for $1 \leq i \leq k$, i.e. $|\mu| = \sum_{i=1}^{k} a_i \cdot \mu_i = n$. We also omit all exponents which are equal to 1.

When working with partitions, one identifies them with their respective Young diagrams, which we define next.

Definition 2.1.3. The *Young diagram* of a partition $\lambda = (\lambda_1, ..., \lambda_k)$ is a collection of boxes arranged in left-justified rows where row i has λ_i boxes. We identify its boxes by their matrix coordinates given by the set

$$\{(i, j) \in \mathbb{Z}_+^2 \mid j \leq \lambda_i\} \tag{2.3}$$

and we will write $(i, j) \in \lambda$ whenever the coordinate (i, j) belongs to a box in the Young diagram of λ.

Example 2.1.4. There are five partitions of the number 4. They are given in Figure 2.1 together with their Young diagrams.

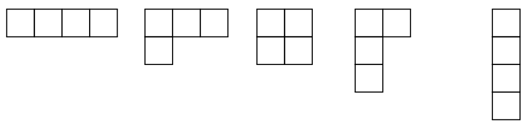
Definition 2.1.5. Let $\lambda \in \mathscr{P}(n)$ and let $(i, j) \in \lambda$ be the coordinate of one of the boxes of λ.

The *arm* of (i, j) in λ is the set of boxes of λ strictly to the right of (i, j), i.e. the set of all coordinates of the form $(i, j + k)$ for some $k \geq 1$. We write $a((i, j))$ for the size of the arm of (i, j).

The *leg* of (i, j) in λ is the set of boxes of λ strictly to the bottom of (i, j), i.e. the set of all coordinates of the form $(i+k, j)$ for some $k \geq 1$. We write $l((i, j))$ for the size of the leg of (i, j).

The *hook* of (i, j) in λ is the set of boxes in the arm and leg of (i, j) including (i, j) itself. We write $h((i, j))$ for the *hook-length* of (i, j), i.e. the number of boxes in the hook of (i, j).

EXAMPLE 2.1.6. For $\lambda = (4, 2, 2, 1)$ and $(i, j) = (2, 1)$ we have illustrated its arm, leg, and hook in Figure 2.2.



(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
(4)	$(3,1)$	(2^2)	$(2,1^2)$	(1^4)

FIGURE 2.1. The five Young diagrams and partitions of the number 4 in both notations.

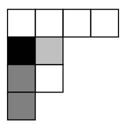


FIGURE 2.2. The Young diagram of the partition $\lambda = (4, 2, 2, 1)$ with light gray arm, dark gray leg, and hook all nonwhite boxes at the black coordinate $(2, 1)$.

There are some standard notations and operations associated to partitions that will be needed throughout this discussion.

DEFINITION 2.1.7. For $n' \geq n$, $\lambda \in \mathscr{P}(n), \mu \in \mathscr{P}(n')$ we write $\lambda \subseteq \mu$ when for each box $(i, j) \in \mu$ we have $(i, j) \in \lambda$ as well, i.e. we have $\lambda_i \leq \mu_i$ for all i.

DEFINITION 2.1.8. For $\lambda \in \mathscr{P}(n)$ we define the *transpose* ${}^t\lambda \in \mathscr{P}(n)$ as the partition

$$ {}^t\lambda_i = |\{k \mid \lambda_k \geq i\}| \,, \tag{2.4} $$

i.e. the column lengths of the Young diagram of λ become the row lengths of the Young diagram of ${}^t\lambda$.

For $\lambda \in \mathscr{P}(n)$, the Young diagram of ${}^t\lambda$ is given by transposing the Young diagram of λ as one would transpose a matrix.

DEFINITION 2.1.9. We have a partial order $\trianglelefteq$ on $\mathscr{P}(n)$ defined by

$$ \lambda \trianglelefteq \mu \;:\Longleftrightarrow\; \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \quad \text{for all } k \geq 1 \tag{2.5} $$

for $\lambda, \mu \in \mathscr{P}(n)$. We extend the sequence of partitions by trailing 0's here.

EXAMPLE 2.1.10. Figure 2.1 shows the five partitions of 4 in descending order with respect to $\trianglelefteq$.

REMARK 2.1.11. For $n \leq 5$, the order $\trianglelefteq$ is total. Starting with $n = 6$, we get incomparable partitions, e.g. $(4,1,1)$ and $(3,3)$.

The proof of the following lemma is straightforward when visualizing it in terms of Young diagrams.

LEMMA 2.1.12. *For $\lambda, \mu \in \mathscr{P}(n)$ we have*

$$\lambda \trianglelefteq \mu \Longleftrightarrow {}^t\lambda \trianglerighteq {}^t\mu \ . \tag{2.6}$$

There is one important integer attached to each partition that will play a role in the representation theory later, see also [GP00, Sec. 5.5].

DEFINITION 2.1.13. The *b-invariant* of a partition λ is defined as

$$b(\lambda) = \sum_{i=1}^{l(\lambda)} (i-1) \cdot \lambda_i \ . \tag{2.7}$$

REMARK 2.1.14. We will extend $b(\alpha)$ to arbitrary vectors $\alpha \in \mathbb{N}^k$ for some k via

$$b(\alpha) = \sum_{i=1}^{k} (i-1) \cdot \alpha_i \ . \tag{2.8}$$

Also note that $b(\alpha)$ is often denoted by $n(\alpha)$ by many authors, see for example [Mac95].

Partitions and Young diagrams are only one part of the story. Now, we want to fill the boxes of a Young diagram with positive integers.

DEFINITION 2.1.15. A *filling* of the Young diagram associated to a partition λ is a map

$$T : \{(i,j) \in \mathbb{Z}_+^2 \mid j \leq \lambda_i\} \to \mathbb{Z}_+ \tag{2.9}$$

which associates to each box in the Young diagram a positive integer. The *shape* of T is λ, denoted $\mathrm{sh}(T)$. If T is injective, it is called a *numbering*. A filling is called a *Young tableau*, or simply *tableau* if two conditions are met:

(*i*) $T(i,j) \leq T(i,j+1)$ (weakly increasing along rows),
(*ii*) $T(i,j) < T(i+1,j)$ (strictly increasing along columns).

A filling is called *semistandard* if T maps onto $\{1, ..., n\}$ for $\lambda \in \mathscr{P}(n)$. If in addition T is a bijection, we say it is *standard*.

The *weight* of a tableau T is the vector $\mathrm{wt}(T) = (\mu_1, \mu_2, ...)$ where μ_i denotes the number of times the letter $i \in \mathbb{Z}_+$ appears in T. We identify an infinite weight vector with its finite version obtained by removing trailing 0's.

REMARK 2.1.16. Let λ, μ be partitions of n. Then μ is a weight vector of a semistandard Young tableau T of shape λ if and only if $\mu \trianglelefteq \lambda$. All permutations of μ are then weight vectors as well.

EXAMPLE 2.1.17. There are six semistandard Young tableaux of shape $(2,1)$. They are given in Figure 2.3 together with their respective weights. The two middle ones are standard.

The number of standard Young tableaux of a given shape can be computed very easily using the *hook-length formula*.

PROPOSITION 2.1.18 ([FRT54, Thm. 1]). *For $\lambda \in \mathscr{P}(n)$ the number of standard Young tableaux of shape λ is given by*

$$f_\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)} \ . \tag{2.11}$$

$$\begin{array}{cccccc} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} & \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \\ \\ (2,1) & (1,2) & (1,1,1) & (1,1,1) & (0,2,1) & (0,1,2) \end{array} \tag{2.10}$$

FIGURE 2.3. The six semistandard Young tableaux of shape $(2, 1)$ together with their weights.

The setting of partitions and Young diagrams, we will be use to work with the group $\mathfrak{S}_n$. For the wreath product groups $C_\ell \wr \mathfrak{S}_n$, we need to extend our combinatorial model to multipartitions and multitableaux.

DEFINITION 2.1.19. An *ℓ-multipartition* of n is an ℓ-tuple

$$\boldsymbol{\lambda} = (\lambda^{(0)}, ..., \lambda^{(\ell-1)}) \tag{2.12}$$

of (possibly empty) partitions such that their sizes add up to n, i.e.

$$|\boldsymbol{\lambda}| := \sum_{i=0}^{\ell-1} |\lambda^{(i)}| = n\ , \tag{2.13}$$

which is called the *size* (or *rank*) of $\boldsymbol{\lambda}$. Denote the set of all ℓ-multipartitions of n by $\mathscr{P}(\ell, n)$ and we also write $\boldsymbol{\mu} \vdash n$ for $\boldsymbol{\mu} \in \mathscr{P}(\ell, n)$. We write $\boldsymbol{\emptyset} \in \mathscr{P}(\ell, 0)$ for the empty ℓ-multipartition. The Young diagram of $\boldsymbol{\lambda}$ is obtained by listing the Young diagrams of the $\lambda^{(i)}$.

EXAMPLE 2.1.20. For $\ell = n = 2$ there are five 2-multipartitions (or *bipartitions*) displayed in Figure 2.4.

$$\begin{array}{ccccc} \begin{array}{|c|c|}\hline & \\ \hline \end{array}, \emptyset & \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}, \emptyset & \begin{array}{|c|}\hline \\ \hline \end{array}, \begin{array}{|c|}\hline \\ \hline \end{array} & \emptyset, \begin{array}{|c|c|}\hline & \\ \hline \end{array} & \emptyset, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \\ \\ ((2), \emptyset) & ((1,1), \emptyset) & ((1), (1)) & (\emptyset, (1,1)) & (\emptyset, (2)) \end{array}$$

FIGURE 2.4. The five partitions and Young diagrams for $\ell = n = 2$.

We extend the definition of fillings, numberings, and tableaux to Young diagrams of multipartitions by only demanding that the conditions hold *within* each partition in the tuple, and having no condition on the relationship *between* the partitions. We give all standard *multitableaux* (or *bitableaux*) of shape $\boldsymbol{\lambda} = ((2), \emptyset, (1))$ in Figure 2.5

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \ \emptyset, \ \begin{array}{|c|}\hline 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array}, \ \emptyset, \ \begin{array}{|c|}\hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}, \ \emptyset, \ \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

FIGURE 2.5. The three standard Young multitableaux of shape $\boldsymbol{\lambda} = ((2), \emptyset, (1))$.

2.2. Combinatorial representation theory of $\mathfrak{S}_n$

We now want to illustrate the well-known connection of the combinatorial structures that we have seen thus far to representation theory. Let G be a finite group and denote by $\mathrm{Irr}(G)$ a set of pairwise nonisomorphic irreducible complex representations of G. We will identify the elements of $\mathrm{Irr}(G)$ with their respective complex characters as well.

It is a standard fact from representation theory that

$$|\mathrm{Irr}(G)| = |\mathrm{Cl}(G)| \tag{2.14}$$

where $\mathrm{Cl}(G)$ denotes the set of conjugacy classes of G. This means that the number of irreducible characters of a finite group G is the same as the number of its conjugacy classes. For a general group G, the identity in (2.14) cannot be extended to a "meaningful" bijection. By this we mean a parametrization of the elements of $\mathrm{Irr}(G)$ by conjugacy classes of G such that we can retrieve representation theoretic information of $\mathrm{Irr}(G)$ from $\mathrm{Cl}(G)$, e.g. the dimension of an irreducible representation. This kind of parametrization however *is* very much possible in the case $G = C_\ell \wr \mathfrak{S}_n$.

Let us start with $G = \mathfrak{S}_n$. The conjugacy class of a given permutation $\pi \in \mathfrak{S}_n$ is given by its *cycle type*, i.e. the multiplicities of lengths of cycles in the permutation, wherein fixed elements become cycles of length 1. Because two permutations with different cycle types are never conjugate in $\mathfrak{S}_n$, we have that the set $\mathrm{Cl}(\mathfrak{S}_n)$ is parametrized by all possible cycle types.

To each cycle type we can associate a unique partition $\lambda \in \mathscr{P}(n)$ by listing the lengths of cycles in descending order. It is easy to see that this is in fact a bijection between cycle types and partitions of n which proves the following lemma.

LEMMA 2.2.1. *There is a bijection*

$$\mathrm{Cl}(\mathfrak{S}_n) \leftrightarrow \mathscr{P}(n) \tag{2.15}$$

given by the cycle types of elements of $\mathfrak{S}_n$.

Now that we have parametrized the right-hand side of (2.14) using partitions, we take a look at the set $\mathrm{Irr}(\mathfrak{S}_n)$. The irreducible representations of $\mathfrak{S}_n$ indexed by partitions of n were first constructed by Specht in [Spe35] and are thus named *Specht modules*. There exist more recent versions of this construction, e.g. [JK81] and [Ful97, Ch. 7], the latter of which we will give here in an abbreviated form.

For this construction we will work with fillings of Young diagrams with n boxes that are standard numberings, i.e. the fillings contain all numbers $1, ..., n$ exactly once and have no restriction on their order. We let the group $\mathfrak{S}_n$ act on the set of standard numberings by permuting the entries of the boxes.

DEFINITION 2.2.2. For a given standard numbering T let $R(T) \leq \mathfrak{S}_n$ denote the subgroup of permutations which permute the entries of T only within their respective rows. It is called the *row group* of T. Let analogously denote $C(T) \leq \mathfrak{S}_n$ the *column group* of T.

DEFINITION 2.2.3. For a given standard numbering T, its *(row) tabloid* $\{T\}$ is given by the set $R(T) \cdot T$. The *shape* of a tabloid is given by the shape of its elements.

The set of tabloids defines an equivalence relation on the set of standard numberings. For $\sigma \in \mathfrak{S}_n$, we get an action on the set of tabloids by

$$\sigma \cdot \{T\} = \{\sigma \cdot T\} \; . \tag{2.16}$$

DEFINITION 2.2.4. For $\lambda \in \mathscr{P}(n)$ denote by M^λ the vector space generated by all tabloids of shape λ with action of $\mathfrak{S}_n$ given by (2.16).

The irreducible representations of $\mathfrak{S}_n$ can be realized as submodules of M^λ.

DEFINITION 2.2.5. For a standard numbering T of shape $\lambda \in \mathscr{P}(n)$ we define

$$v_T := \sum_{q \in C(T)} \operatorname{sgn}(q) \cdot \{q \cdot T\} \in M^\lambda \ . \tag{2.17}$$

DEFINITION 2.2.6. For $\lambda \in \mathscr{P}(n)$ the *Specht module* S^λ is given by the $\mathbb{C}$-vector space

$$S^\lambda = \langle v_T \mid T \text{ standard numbering of shape } \lambda \rangle_{\mathbb{C}} \leq M^\lambda \ . \tag{2.18}$$

together with the action of $\mathfrak{S}_n$ on tabloids.

THEOREM 2.2.7 ([Ful97, Prop. 7.2.1]). *The Specht modules form a complete set of pairwise nonisomorphic complex irreducible representations of* $\mathfrak{S}_n$.

Before we enrich this parametrization further, we want to take a look at some examples first.

EXAMPLE 2.2.8. Any partition has $n!$ standard numberings.

For $\lambda = (n)$ and standard numbering T of shape λ we have $R(T) = \mathfrak{S}_n$. This means there is only one tabloid class $\{T_0\}$ with $n!$ elements and $M^\lambda = \langle\{T_0\}\rangle$ is a 1-dimensional complex vector space on which $\mathfrak{S}_n$ acts trivially. Since $C(T)$ is trivial, we have $v_T = \{T_0\}$ and $S^\lambda = M^\lambda$ affords the trivial representation of $\mathfrak{S}_n$.

For $\lambda = (1^n)$ and standard numbering T of shape λ we have that $R(T)$ is trivial. This gives us $n!$ singleton tabloid classes, one for every standard numbering. We can identify each standard numbering T with a permutation $\pi_T \in \mathfrak{S}_n$ sending the number i to $T(i,1)$, i.e. to the content of the i^{th} box. When we act with $\sigma \in \mathfrak{S}_n$ on a standard numbering T corresponding to $\pi_T \in \mathfrak{S}_n$ we obtain the standard numbering corresponding to $\sigma \circ \pi_T$. This makes M^λ into an $n!$-dimensional vector space affording the regular representation of $\mathfrak{S}_n$. Using the identification $\{T\} \leftrightarrow T \leftrightarrow \pi_T$ we have that

$$v_T = \sum_{\pi \in \mathfrak{S}_n} \operatorname{sgn}(\pi) \cdot \pi \tag{2.19}$$

which is the same for all T. This makes $S^\lambda = \langle v_T \rangle$ into a 1-dimensional vector space carrying the sign representation of $\mathfrak{S}_n$ since

$$\begin{aligned} \sigma \cdot v_T &= \sum_{\pi \in \mathfrak{S}_n} \operatorname{sgn}(\pi) \cdot \sigma \circ \pi \\ &= \sum_{\pi \in \mathfrak{S}_n} \underbrace{\operatorname{sgn}(\sigma)^2}_{=1} \operatorname{sgn}(\pi) \cdot \sigma \circ \pi \\ &= \operatorname{sgn}(\sigma) \sum_{\pi \in \mathfrak{S}_n} \operatorname{sgn}(\sigma \circ \pi) \cdot \sigma \circ \pi \\ &= \operatorname{sgn}(\sigma) \cdot v_T \end{aligned} \tag{2.20}$$

for all $\sigma \in \mathfrak{S}_n$.

We want to record some representation theoretic properties that can be extracted from the combinatorial data of the partitions.

COROLLARY 2.2.9. *There is a bijection*

$$\mathscr{P}(n) \leftrightarrow \operatorname{Irr}(\mathfrak{S}_n) \tag{2.21}$$

given by sending a partition $\lambda \in \mathscr{P}(n)$ *to the Specht modules* S^λ.

PROPOSITION 2.2.10 ([Ful97, Prop. 7.2.2]). *Let $\lambda \in \mathscr{P}(n)$ and S^λ the Specht module of λ. Then the set*

$$\{v_T \mid T \text{ is a standard Young tableau of shape } \lambda\} \tag{2.22}$$

is a basis of S^λ. In particular, the dimension of S^λ is equal to the number of standard Young tableaux of shape λ.

DEFINITION 2.2.11. For $\lambda, \mu \in \mathscr{P}(n)$ we define the *Kostka number* $K_{\mu\lambda}$ by

$$K_{\mu\lambda} = [M^\lambda : S^\mu] \tag{2.23}$$

where $[M^\lambda : S^\mu]$ denotes the multiplicity of S^μ inside M^λ.

PROPOSITION 2.2.12 ([Ful97, Cor. 7.3.1]). *For $\lambda, \mu \in \mathscr{P}(n)$ we have*

$$K_{\mu\lambda} = |\{T \mid T \text{ is a semistandard tableau of shape } \mu \text{ and weight } \lambda\}| \tag{2.24}$$

The next application of Theorem 2.2.7 is known as the *branching rule*. It relates representations of $\mathfrak{S}_n$ with $\mathfrak{S}_{n+1}$, where we view $\mathfrak{S}_n$ as the subgroup of $\mathfrak{S}_{n+1}$ which fixes the letter $n+1$.

PROPOSITION 2.2.13 ([Ful97, Cor. 7.3.3]). *For $\lambda \in \mathscr{P}(n)$ we have*

$$\operatorname{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} S^\lambda = \bigoplus_{\substack{\mu \in \mathscr{P}(n+1) \\ \lambda \subseteq \mu}} S^\mu\ , \tag{2.25}$$

i.e. the induction of S^λ to $\mathfrak{S}_{n+1}$ is the direct sum over all S^μ where $\mu \in \mathscr{P}(n+1)$ is obtained from λ by adding one box.

Recall Definition 1.1.16 of the fake degree $f_\lambda(t)$ for an irreducible representation of a finite complex reflection group W. The proof of the following proposition is a combination of [Sta79, Prop. 4.11] and [Sta99, Cor. 7.21.5].

PROPOSITION 2.2.14. *For $\lambda \in \mathscr{P}(n)$ the fake degree $f_\lambda(t) = f_{S^\lambda}(t)$ is given by*

$$f_\lambda(t) = t^{b(\lambda)} \cdot \frac{\prod_{i=1}^n (1-t^i)}{\prod_{u\in\lambda}(1-t^{h(u)})}\ . \tag{2.26}$$

The fake degree formula from the above proposition can be viewed as a "quantization" of Proposition 2.1.18, wherein we replace each factor $i \in \mathbb{Z}_{\geq 1}$ by $(1-t^i)$.

As we can see, the combinatorial structure afforded by Theorem 2.2.7 is very rich and versatile. Furthermore, it allows us to recover important invariants of Specht modules without much effort. From here on out we will identify $\mathscr{P}(n)$ and $\operatorname{Irr}(\mathfrak{S}_n)$, in particular we will denote the Specht module S^λ simply by λ.

Let us see how Theorem 2.2.7 and its corollaries can be extended to wreath product groups.

2.3. Generalizing to $C_\ell \wr \mathfrak{S}_n$

We have seen in Chapter 1 that the theory of Cherednik algebras encompasses not only the symmetric group but any complex reflection group, in particular the wreath product groups $C_\ell \wr \mathfrak{S}_n$ of type $G(\ell, 1, n)$ for some positive integers ℓ, n. We are now going to give the generalization of the theory of $\mathfrak{S}_n$ laid out in the previous section to these wreath products, starting again with their conjugacy classes. The construction of conjugacy classes of $C_\ell \wr \mathfrak{S}_n$ is well-known and can for example be found in [Poi98, Sec. 2].

Fix some positive integer ℓ together with a primitive ℓ^{th} root of unity $\zeta \in \mathbb{C}$. When working with the reflection representation of $C_\ell \wr \mathfrak{S}_n$ given in Example 1.1.4 we can define the set of vectors

$$\mathcal{N} := \{\zeta^j e_i \mid 0 \leq j \leq \ell - 1,\ 1 \leq i \leq n\} \tag{2.27}$$

where e_i is the i^{th} standard basis vector of $\mathbb{C}^n$. Because of the faithful monomial matrix representation Example 1.1.4(*iii*), we are then able to view $C_\ell \wr \mathfrak{S}_n$ as the group permutations $\pi \in \mathfrak{S}_{\mathcal{N}}$ that are linear with respect to the root power ζ^j, i.e. we demand

$$\pi(\zeta^j e_i) = \zeta^j \cdot \pi(e_i) \tag{2.28}$$

for $0 \leq j \leq \ell - 1,\ 1 \leq i \leq n$.

As in the matrix case, each of these permutations is uniquely determined by its images of the e_i. By identifying $e_i \leftrightarrow i$, we can write any element $\pi \in C_\ell \wr \mathfrak{S}_n$ as

$$\begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ \pi(e_1) & \pi(e_2) & \cdots & \pi(e_n) \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n \\ \zeta^{k_1} a_1 & \zeta^{k_2} a_2 & \cdots & \zeta^{k_n} a_n \end{pmatrix} \tag{2.29}$$

for $k_1, ..., k_n \in \{0, ..., \ell - 1\}$ and $a_1, ..., a_n \in \{1, ..., n\}$. When we now ignore the power of ζ in the bottom row, we obtain an element of $\mathfrak{S}_n$

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} . \tag{2.30}$$

This process is equivalent to isolating the permutation matrix of the monomial matrix in Example 1.1.4(*iii*). The permutation (2.30) has a cycle type given by a partition $\lambda \in \mathscr{P}(n)$. To each part of λ, i.e. each cycle of (2.30), we associate a power of ζ by taking the product over the coefficients of the $\zeta^{k_i} a_i$ appearing in the cycle. This splits the parts of λ into ℓ distinct sets indexed by $\zeta^0, ..., \zeta^{\ell-1}$. The set of parts indexed by ζ^j can be sorted into a partition denoted $\lambda^{(j)}$ which combined for all j form an ℓ-multipartition $\boldsymbol{\lambda}$ of n. We call $\boldsymbol{\lambda}$ the *cycle type* of $\pi \in C_\ell \wr \mathfrak{S}_n$. It is not difficult to see that the multipartition $\boldsymbol{\lambda}$ forms a separating invariant for the conjugacy class of σ. We record this fact in the lemma below.

Lemma 2.3.1. *There is a bijection*

$$\mathrm{Cl}(C_\ell \wr \mathfrak{S}_n) \leftrightarrow \mathscr{P}(\ell, n) \tag{2.31}$$

given by the cycle types of elements of $C_\ell \wr \mathfrak{S}_n$.

Let us give a small example of this parametrization.

Example 2.3.2. Let $\ell = 2$, $n = 4$ and let $\zeta \in \mathbb{C}$ be a 3^{rd} root of unity. Denote by $\pi \in C_3 \wr \mathfrak{S}_4$ the element corresponding to the matrix

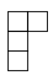
$$\begin{pmatrix} 0 & 0 & 0 & \zeta \\ 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta & 0 \\ \zeta & 0 & 0 & 0 \end{pmatrix} \tag{2.32}$$

given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \zeta 4 & \zeta^2 2 & \zeta 3 & \zeta 4 \end{pmatrix} . \tag{2.33}$$

Ignoring the powers of ζ in (2.33), we obtain the permutation $(1,4)(2)(3) \in \mathfrak{S}_n$ with cycle type

(2.34)



$$(2, 1, 1) \in \mathscr{P}(4) .$$

Taking the product of the ζ^j within every cycle, we get

$$(1, 4) \mapsto \zeta^2 \ ,\ (2) \mapsto \zeta^2 \ ,\ (3) \mapsto \zeta \tag{2.35}$$

and therefore obtain the 3-multipartition

$$\left(\emptyset, \ydiagram{1}, \ydiagram{2,1}\right) \tag{2.36}$$

$$(\emptyset, (1), (2, 1)) \in \mathscr{P}(3, 4)$$

which describes the cycle type of $\pi \in C_3 \wr \mathfrak{S}_4$.

As is the case with $\mathfrak{S}_n$, we can extend the parametrization of conjugacy classes by partitions to simple modules of $C_\ell \wr \mathfrak{S}_n$. We follow Stembridge [Ste89, Ch. 5] for the construction but the results go back to Specht [Spe33].

Let $g \in C_\ell$ be such that $\langle g \rangle = C_\ell$. For $0 \leq j \leq \ell - 1$, we can define the C_ℓ-representation

$$U_j : C_\ell \to \mathrm{End}(\mathbb{C}),\ g^k \mapsto \left(x \mapsto \left(\zeta^j\right)^k \cdot x\right) . \tag{2.37}$$

It is well known that the U_j exhaust $\mathrm{Irr}(C_\ell)$ for $0 \leq j \leq \ell - 1$ and we have therefore described $\mathrm{Irr}(C_\ell)$ completely.

Definition 2.3.3 ([Ste89, 4.A]). Let U be a representation of C_ℓ and let V be a representation of $\mathfrak{S}_n$. We define a representation of $C_\ell \wr \mathfrak{S}_n \cong (C_\ell)^n \rtimes \mathfrak{S}_n$ on the vector space

$$U \wr V := U^{\otimes n} \otimes_{\mathbb{C}} V \tag{2.38}$$

by the action on $u_1 \otimes ... \otimes u_n \otimes v \in U \wr V$ given by

$$(g_1, ..., g_n).(u_1 \otimes ... \otimes u_n \otimes v) = g_1.u_1 \otimes ... \otimes g_n.u_n \otimes v \tag{2.39}$$

for $(g_1, ..., g_n) \in (C_\ell)^n$, and

$$\sigma.(u_1 \otimes ... \otimes u_n \otimes v) = u_{\sigma^{-1}(1)} \otimes ... \otimes u_{\sigma^{-1}(n)} \otimes \sigma.v \tag{2.40}$$

for $\sigma \in \mathfrak{S}_n$.

Equations (2.39) and (2.40) give a well-defined representation on the whole group $C_\ell \wr \mathfrak{S}_n$ and we have that $U \wr V$ is irreducible if U and V both are. To obtain a full list of irreducible representations of the wreath product group, we need to induce modules from *Young subgroups*. Let $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{Z}^k_{\geq 0}$ for some positive integer k such that

$$\alpha_1 + \cdots + \alpha_k = n , \tag{2.41}$$

i.e. α is an unsorted partition with possible internal zeroes. We define the *Young subgroup*

$$\mathfrak{S}_\alpha := \mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k} \leq \mathfrak{S}_n \tag{2.42}$$

by splitting the set $\{1, ..., n\}$ into

$$\{1, ..., \alpha_1\}, \{\alpha_1 + 1, ..., \alpha_1 + \alpha_2\}, ..., \{\alpha_1 + \cdots \alpha_{k-1} + 1, ..., n\} \tag{2.43}$$

and having S_{α_i} permute the set $\{\alpha_1 + \cdots + \alpha_{i-1} + 1, ..., \alpha_1 + \cdots + \alpha_i\}$.

Lastly, we define analogously

$$C_\ell \wr \mathfrak{S}_\alpha := C_\ell \wr \mathfrak{S}_{\alpha_1} \times \cdots \times C_\ell \wr \mathfrak{S}_{\alpha_k} \leq C_\ell \wr \mathfrak{S}_n . \tag{2.44}$$

We can now construct $C_\ell \wr \mathfrak{S}_n$ modules by inducing modules from $C_\ell \wr \mathfrak{S}_\alpha$.

Definition 2.3.4. Let $\boldsymbol{\lambda} = (\lambda^{(0)}, ..., \lambda^{(\ell-1)}) \in \mathscr{P}(\ell, n)$ be an ℓ-multipartition. Let $U_0, ..., U_{\ell-1}$ be the complete set of irreducible C_ℓ-modules given in (2.37). Using Definition 2.3.3 we get a representation of $C_\ell \wr \mathfrak{S}_{|\lambda^{(i)}|}$ given by

$$U_i \wr \lambda^{(i)} \text{ for } 0 \leq i \leq \ell - 1 \tag{2.45}$$

where $\lambda^{(i)}$ is identified with the Specht module of $\mathfrak{S}_n$. From (2.45) we construct

$$U_0 \wr \lambda^{(0)} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} U_{\ell-1} \wr \lambda^{(\ell-1)} \tag{2.46}$$

and define

$$\alpha := \left(|\lambda^{(0)}|, ..., |\lambda^{(\ell-1)}|\right) . \tag{2.47}$$

The vector space (2.46) becomes a $C_\ell \wr \mathfrak{S}_\alpha$-module by (2.44) and a diagonal action on the modules given by Definition 2.3.3. From (2.46) we induce a representation of $C_\ell \wr \mathfrak{S}_n$ by

$$S^{\boldsymbol{\lambda}} := \operatorname{Ind}_{C_\ell \wr \mathfrak{S}_\alpha}^{C_\ell \wr \mathfrak{S}_n} \left(U_0 \wr \lambda^{(0)} \otimes \cdots \otimes U_{\ell-1} \wr \lambda^{(\ell-1)}\right) . \tag{2.48}$$

THEOREM 2.3.5 ([Spe33]). *The modules $S^{\boldsymbol{\lambda}}$ from Definition 2.3.4 form a complete set of pairwise nonisomorphic complex irreducible representations of $C_\ell \wr \mathfrak{S}_n$.*

We again record some representation theoretic properties of this combinatorial parametrization.

COROLLARY 2.3.6. *There is a bijection*

$$\operatorname{Irr}(C_\ell \wr \mathfrak{S}_n) \leftrightarrow \mathscr{P}(\ell, n) \tag{2.49}$$

given by the construction of $S^{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)$.

PROPOSITION 2.3.7. *For $\boldsymbol{\lambda} \in \mathscr{P}(n)$ the dimension of $S^{\boldsymbol{\lambda}}$ is equal to the number of standard Young tableaux of shape $\boldsymbol{\lambda}$.*

PROOF. Following the notation of Definition 2.3.4, the index of $C_\ell \wr \mathfrak{S}_\alpha$ inside $C_\ell \wr \mathfrak{S}_n$ is the multinomial coefficient $\binom{n}{\alpha}$ which is equal to the number of ways to distribute the numbers $1, ..., n$ between the different partitions of the multipartition. The claim then follows from the $\mathfrak{S}_n$ case given by Proposition 2.2.10.

■

PROPOSITION 2.3.8 ([AK94, Cor. 3.12]). *For $\boldsymbol{\lambda} \in \mathscr{P}(n)$ we have*

$$\operatorname{Ind}_{C_\ell \wr \mathfrak{S}_n}^{C_\ell \wr \mathfrak{S}_{n+1}} S^{\boldsymbol{\lambda}} = \bigoplus_{\substack{\boldsymbol{\mu} \in \mathscr{P}(\ell, n+1) \\ \boldsymbol{\lambda} \subseteq \boldsymbol{\mu}}} S^{\boldsymbol{\mu}} , \tag{2.50}$$

i.e. the induction of $S^{\boldsymbol{\lambda}}$ to $\mathfrak{S}_{n+1}$ is the direct sum over all $S^{\boldsymbol{\mu}}$ where $\boldsymbol{\mu} \in \mathscr{P}(n+1)$ is obtained from $\boldsymbol{\lambda}$ by adding one box.

We now know enough about the representation theory of $C_\ell \wr \mathfrak{S}_n$ to understand and utilize another combinatorial tool outlined in the next chapter: the theory of symmetric functions.

CHAPTER 3

Symmetric functions

This chapter is devoted to the theory of symmetric functions. The goal is to state the definition of the Macdonald polynomial [Mac88] and its generalization to wreath product groups [Hai03] which we will use to provide an answer to Question 1.4.12 in Section 3.5, i.e. the description of the graded W-character of simple $\overline{\mathsf{H}}_{\mathbf{c}}$-modules in generic parameter $\mathbf{c}$ for $W = C_\ell \wr \mathfrak{S}_n$.

As with the partitions and representation theory in Chapter 2, we will start with the $\mathfrak{S}_n$-case and work ourselves up from there. A general reference for this chapter will be the classic book by Macdonald [Mac95], which also contains in Chapter 1 of Appendix B most of what we are going to need for the wreath product case. We again fix some positive integer n.

3.1. Symmetric functions and plethysms

We will introduce the ring of symmetric functions by looking at the polynomial ring first, following to the approach found [LR11].

Let $x_1, ..., x_N$ be indeterminates over $\mathbb{C}$ for some fixed integer $N \geq n$. We let the symmetric group $\mathfrak{S}_N$ act on these variables by permuting their indices and extend this action to the polynomial ring $\mathbb{C}[x_1, ..., x_N]$.

Definition 3.1.1. We define the *ring* Λ_N *of symmetric polynomials in* N *variables* as the set of polynomials in $\mathbb{C}[x_1, ..., x_N]$ which are fixed by the action of $\mathfrak{S}_N$, and we write

$$\Lambda_N = \mathbb{C}[x_1, ..., x_N]^{\mathfrak{S}_N} \ . \tag{3.1}$$

For a nonnegative integer k we denote by Λ_N^k the vector space of homogeneous symmetric polynomials of degree k.

Definition 3.1.2. For $r \in \{1, ..., N\}$, the *power-sum symmetric polynomial* is given by

$$p_{r,N} = x_1^r + \cdots + x_N^r \tag{3.2}$$

and for a partition of $\lambda \in \mathscr{P}(n)$ define

$$p_{\lambda,N} = \prod_{i=1}^{l(\lambda)} p_{\lambda_i,N} \tag{3.3}$$

Proposition 3.1.3 ([Mac95, II.2]). *The ring* Λ_N *is isomorphic to a polynomial ring, more precisely we have*

$$\Lambda_N = \mathbb{C}[p_{1,N}, ..., p_{N,N}] \ . \tag{3.4}$$

Furthermore, for $k \leq N$ *we have*

$$\Lambda_N^k = \langle p_{\lambda,N} \mid \lambda \in \mathscr{P}(k) \rangle_{\mathbb{C}} \tag{3.5}$$

as a $\mathbb{C}$*-vector space.*

In our approach, the power-sum symmetric functions will serve as our entry and reference point for studying the symmetric polynomials. Often times other bases are preferred because they generate the symmetric polynomials with integer coefficients as a free $\mathbb{Z}$-algebra while the power-sums do not. But this is not needed for the purpose of our discussion.

Because the identity (3.5) only holds for the degree pieces k with $k \in \mathbb{N}$ smaller or equal to N, it is very natural to try to extend the set of variables to a countably infinite set. In [Mac95], this is done via an "inverse limit". We instead follow [LR11] and use an abstract polynomial ring which we can map onto Λ_N for various N. So, let now p_r for $r \in \mathbb{Z}_+$ denote indeterminates over $\mathbb{C}$.

Definition 3.1.4. The *ring of (abstract) symmetric functions* is the polynomial ring

$$\Lambda = \mathbb{C}[p_1, p_2, ...] \ . \tag{3.6}$$

We set $\deg(p_r) = r$ and refer to the p_r as the *(abstract) power-sum symmetric function of degree* r for $r \in \mathbb{Z}_+$. The degree-k pieced of Λ is denoted Λ^k.

The following proposition gives the connection between Λ and Λ_N.

Proposition 3.1.5 ([LR11, 2.1]). *There exists a $\mathbb{C}$-algebra evaluation homomorphism*

$$\begin{aligned} \mathrm{ev}_N : \Lambda \quad &\rightarrow \quad \Lambda_N \\ p_r \quad &\mapsto \quad p_{r,N} \end{aligned} \tag{3.7}$$

that restricts to a vector space isomorphism

$$\mathrm{ev}_N^k : \Lambda^k \xrightarrow{\sim} \Lambda_N^k \tag{3.8}$$

for all $k \leq N$. We also write $\mathrm{ev}_N(f)$ as $f(x_1, ..., x_N)$ for $f \in \Lambda$.

Remark 3.1.6. Note that we refer to the elements of Λ as symmetric *functions* and to the elements of Λ_N as symmetric *polynomials*. Nonetheless, we will often use ev_N implicitly to identify a symmetric function with its polynomial image.

Using Proposition 3.1.5, we can now work in Λ and restrict to a fixed set of variables whenever we want to look at a specific ring of symmetric polynomials. This has the advantage that we need not worry about the relationship between the number of variables and the degree piece we are interested in.

Next, we will define the plethysm operation on Λ. This is a certain technical tool which has been used for a long time to study representation-theoretic phenomena around the symmetric group $\mathfrak{S}_n$ (see [LR11] for concise introduction). The plethysm has proved incredibly useful in the study of Macdonald polynomials which is what we will use it for as well. We start with a proposition followed by the definition.

Proposition 3.1.7 ([LR11, Thm. 1]). *There exists a unique binary operation*

$$\cdot[\cdot] : \Lambda \times \Lambda \rightarrow \Lambda \tag{3.9}$$

that satisfies the following properties:

(*i*) *for all $r_1, r_2 \in \mathbb{Z}_+$ we have $p_{r_1}[p_{r_2}] = p_{r_2}[p_{r_1}] = p_{r_1 \cdot r_2}$,*

(*ii*) *for all $m \geq 1$, the map*

$$L_m : \Lambda \rightarrow \Lambda, \ g \mapsto p_m[g]$$

is a homomorphism of $\mathbb{C}$-algebras,

(*iii*) *for all $g \in \Lambda$, the map*

$$R_g : \Lambda \rightarrow \Lambda, \ f \mapsto f[g]$$

is a homomorphism of $\mathbb{C}$-algebras.

DEFINITION 3.1.8. The binary operation given by Proposition 3.1.7 is called *plethysm* or *plethystic substitution.*

We record some properties of the plethysm in the next lemma.

LEMMA 3.1.9 ([LR11, 2.3]). *For* $f, g, h \in \Lambda$, $r \geq 1$ *we have*
(*i*) $f[g[h]] = (f[g])[h]$,
(*ii*) $p_1[f] = f[p_1] = f$
(*iii*) $(p_r[g])(x_1, ..., x_N) = g(x_1^r, ..., x_N^r)$,
(*iv*) $p_r[0] = 0$ *and* $p_r[1] = 1$ *for* $0, 1 \in \Lambda$.

When thinking about the plethysm operation, it usually suffices to look at Proposition 3.1.7(*i*) and Lemma 3.1.9(*iii*). These two identities give us the possibility to operate on each variable in a symmetric polynomial by raising its power. Or, in the context of symmetric functions, replace p_r by $p_{r \cdot s}$ in a given expression in Λ for $r, s \in \mathbb{Z}_+$. When working out the plethystic substitution $f[g]$ for some $f, g \in \Lambda$, one can express the elements f and g in terms of the p_r and then use the algebra homomorphism properties from Proposition 3.1.7.

As stated at the beginning of this chapter, we want to define the Macdonald polynomial $P_\lambda(x; q, t)$, which is a symmetric polynomial introduced in [Mac88] whose coefficients are rational functions in the two complex indeterminates q, t. Thus, we want to extend our field of scalars.

DEFINITION 3.1.10. We define the *ring of symmetric polynomials in* N *variables with parameters* q, t as

$$\Lambda_N(q, t) = \mathbb{C}(q, t) \otimes_{\mathbb{C}} \Lambda_N \tag{3.10}$$

and analogously the *ring of symmetric functions with parameters* q, t as

$$\Lambda(q, t) = \mathbb{C}(q, t) \otimes_{\mathbb{C}} \Lambda\ . \tag{3.11}$$

REMARK 3.1.11. We also have a generalization of the map ev_N in the setting of Definition 3.1.10 where the proofs in [LR11, 2.1] work verbatim.

Now, when defining the plethysm on $\Lambda(q, t)$, it is very tempting to extend the scalars of the plethistic operation to $\mathbb{C}(q, t)$ and make the maps L_m, R_m in Proposition 3.1.7 into $\mathbb{C}(q, t)$-algebra homomorphisms. This, however, is *not* the behavior we desire. Instead, we would like the plethysm to treat q, t as *variables*, i.e. we want for $r \in \mathbb{Z}_+$ that

$$p_r[q] = q^r,\ p_r[t] = t^r \tag{3.12}$$

holds, analogously to the $x_1, ..., x_N$ in Lemma 3.1.9(*iii*). We can accomplish this by following [LR11, Exm. 1].

PROPOSITION 3.1.12. *There exists a unique* $\mathbb{C}$*-algebra homomorphism*

$$L_m^{q,t} : \Lambda(q, t) \to \Lambda(q, t) \tag{3.13}$$

that satisfies the following properties:
(*i*) *for all* $r \in \mathbb{Z}_+$ *we have* $L_m^{q,t}(p_r) = p_{m \cdot r}$,
(*ii*) $L_m^{q,t}(q) = q^m$,
(*iii*) $L_m^{q,t}(t) = t^m$.
The map $L_m^{q,t}$ *restricts to* $L_m : \Lambda \to \Lambda$ *from Proposition 3.1.7.*

The plethysm $\cdot[\cdot] : \Lambda(q, t) \to \Lambda(q, t)$ is now given by taking the maps L_m and R_m of Proposition 3.1.7, replacing L_m by $L_m^{q,t}$, and making R_m into a $\mathbb{C}(q, t)$-linear map. When applying $L_m^{q,t}$ to $a(q, t) \in \mathbb{C}(q, t)$, we identify the rational function $a(q, t)$ with its respective formal power series. This way, we can use conditions (*ii*) and (iii) of Proposition 3.1.12 to operate on rational functions as well. To reiterate, L_m is *not* a homomorphism of $\mathbb{C}(q,t)$-algebras as it is not linear with respect to the indeterminates q, t. We will give an example now.

EXAMPLE 3.1.13. Let $c \in \mathbb{C}$, $r, s \in \mathbb{Z}_+$. We can calculate in $\Lambda(q,t)$ that

$$p_r\left[c \cdot t^2 \cdot p_1 - \frac{1}{1-q} \cdot p_s\right] = c \cdot t^{2r} \cdot p_r - \frac{1}{1-q^r} \cdot p_{sr} \ . \tag{3.14}$$

When using the evaluation homomorphism we have

$$p_r[c \cdot t \cdot p_1](x_1, ..., x_N) = c \cdot t^r \cdot p_r(x_1, ..., x_N) = c \cdot t^r(x_1^r + \cdots + x_N^r) \tag{3.15}$$

in $\Lambda_N(q,t)$.

We close this section by introducing standard notation in the context of plethystic calculus.

DEFINITION 3.1.14. We define

$$X_N = x_1 + \cdots + x_N \in \Lambda_N \ , \quad X = p_1 \in \Lambda \tag{3.16}$$

viewing p_1 as a formal sum over an infinite set of indeterminates, and for $f \in \Lambda$ we write

$$f[X_N] = \mathrm{ev}_N(f) \tag{3.17}$$

analogous to $f[X] = f[p_1] = f$.

3.2. Macdonald polynomials

In this section, we want to give a brief review of Schur and Macdonald polynomials and how they relate to the representation theory of chapters 1 and 2. There are many ways of defining Schur polynomials, e.g. using semistandard Young tableaux [Ful97, Ch. 2] or matrix determinants [Mac95, II.2]. We will stick to the combinatorial approach of the former but use their relation to the power-sums as our definition.

For $\lambda \in \mathscr{P}(n)$, we denote by χ^λ the character of the Specht module S^λ. Furthermore, for the evaluation of χ^λ at an element $\sigma \in \mathfrak{S}_n$ of cycle type μ, we write χ^λ_μ (as is standard notation).

DEFINITION 3.2.1. For $\lambda \in \mathscr{P}(n)$, we define the *Schur (symmetric) function* s_λ of λ by

$$s_\lambda = \sum_{\mu \in \mathscr{P}(n)} z_\mu^{-1} \cdot \chi^\lambda_\mu \cdot p_\mu \in \Lambda \ , \tag{3.18}$$

where we have $z_\mu = \prod_{i \in \mathbb{N}} i^{a_i} \cdot a_i!$ with a_i denoting the multiplicity of the number i inside μ.

Using the evaluation homomorphism, we can define $s_{\lambda,N} := \mathrm{ev}_N(s_\lambda) \in \Lambda_N$ whenever $N \geq n$ and call it the *Schur (symmetric) polynomial of* λ *in* N *variables.* We have

$$s_{\lambda,N} = s_\lambda[X_N] = \sum_{\mu \in \mathscr{P}(n)} z_\mu^{-1} \cdot \chi^\lambda_\mu \cdot p_\mu[X_N] \in \Lambda_N \ . \tag{3.19}$$

The definition above is incredibly useful and seems very natural but it fails to convey an intuition about what the polynomials actually look like. Let us give a more explicit construction of the s_λ next (see [Ful97, Sec. 2.2+7.3] for a proof). For a vector $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$, we write

$$x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N} \in \mathbb{C}[x_1, ..., x_N] \ . \tag{3.20}$$

PROPOSITION 3.2.2. *For $\lambda \in \mathscr{P}(n)$ we have*

$$s_{\lambda,N} = \sum_T x^{\mathrm{wt}(T)} \tag{3.21}$$

where the sum runs over the set of all semistandard Young tableaux T of shape λ filled with numbers $1, ..., N$ and $\mathrm{wt}(T)$ denotes the weight of T.

EXAMPLE 3.2.3. For $N = 3$, $n = 2$ we have

$$\begin{aligned} s_{(2),3} &= x_1^2 + x_1x_2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 \ , \\ s_{(1,1),3} &= x_1x_2 + x_1x_3 + x_2x_3 \ . \end{aligned} \tag{3.22}$$

For $N = n$, the latter expression simplifies and we have

$$s_{(1^n),n} = x_1 \cdot x_2 \cdots x_n \ . \tag{3.23}$$

We continue with our study of the Schur polynomials. With a simple counting argument, one can see that for any partition $\mu \in \mathscr{P}(n)$ and permutation $\sigma \in \mathfrak{S}_n$ of cycle type μ we have

$$z_\mu = |\mathrm{Stab}(\sigma)| = \frac{|\mathfrak{S}_n|}{|\sigma^{\mathfrak{S}_n}|} \tag{3.24}$$

where z_μ is the number given in Definition 3.2.1 and $\mathrm{Stab}(\sigma)$ (resp. $\sigma^{\mathfrak{S}_n}$) denotes the stabilizer (resp. orbit) of σ inside $\mathfrak{S}_n$ with respect to conjugation. Next, we can apply the orthogonality relations of character tables of finite groups to obtain the inverse decomposition of (3.18) which we record next.

PROPOSITION 3.2.4. *For $\mu \in \mathscr{P}(n)$ we have*

$$p_\mu = \sum_{\lambda \in \mathscr{P}(n)} \chi_\mu^\lambda \cdot s_\lambda \ . \tag{3.25}$$

We then get the following corollary.

COROLLARY 3.2.5. *The Schur functions and polynomials are linearly independent, in particular we have*

$$\Lambda^k = \langle s_\lambda \mid \lambda \vdash k \rangle_{\mathbb{C}} \, , \ \Lambda_N^k = \langle s_\lambda[X_N] \mid \lambda \vdash k \rangle_{\mathbb{C}} \, . \tag{3.26}$$

for $k \leq N$. In particular, we have

$$\Lambda = \langle s_\lambda \mid \lambda \vdash \mathscr{P} \rangle_{\mathbb{C}} \ . \tag{3.27}$$

We see now that the equations (3.18) and (3.25) are simply base changes between the two vector space bases of Λ^k for some $k \leq N$. Base changes like these and their associated base change matrices are what Gordon used to answer Question 1.4.12 in the symmetric group case (see Theorem 3.5.1) and how we are going to generalize his result to type $G(\ell, 1, n)$.

DEFINITION 3.2.6. For $k \in \mathbb{N}$, we define a scalar product $\langle \cdot, \cdot \rangle$ on Λ^k by demanding that the set of all Schur functions $\{s_\lambda \mid \lambda \vdash k\}$ forms an orthonormal basis with respect to $\langle \cdot, \cdot \rangle$.

REMARK 3.2.7. With respect to the scalar product defined in 3.2.6, we have

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \ , \tag{3.28}$$

where $\delta_{\lambda\mu}$ is the Kronecker delta on $\mathscr{P}(n)$, which means the p_λ form an orthogonal but not an orthonormal basis.

Let G be a finite group.

Definition 3.2.8. Denote by $\mathrm{CF}(G)$ the complex vector space of complex-valued class functions on G, i.e. the vector space of maps $\chi : G \to \mathbb{C}$ such that $\chi(g) = \chi(h)$ whenever g and h are in the same G-conjugacy class.

Recall that we have $|\mathrm{Irr}(G)| = |\mathrm{Cl}(G)|$. Because the character table of a finite group has full rank over $\mathbb{C}$, we know that the irreducible characters form a basis of $\mathrm{CF}(G)$. For $\mathfrak{S}_n$, we can thus extend equation (3.18) to a vector space homomorphism by allowing any class function in place of χ^λ.

Definition 3.2.9 ([Mac95, I.7.2]). For a class function $\chi \in \mathrm{CF}(\mathfrak{S}_n)$ we again write χ_μ for the evaluation of χ at the class of cycle type μ. We define the *Frobenius character map* as

$$\begin{array}{rcl} \mathrm{ch} : \mathrm{CF}(\mathfrak{S}_n) & \to & \Lambda^n \\ \chi & \mapsto & \sum_{\mu \in \mathscr{P}(n)} z_\mu^{-1} \cdot \chi_\mu \cdot p_\mu . \end{array} \tag{3.29}$$

As with the Schur functions, one obtains the polynomial version $\mathrm{ch}_N : \mathrm{CF}(\mathfrak{S}_n) \to \Lambda^n_N$ by evaluating the p_μ, i.e. replacing them with $p_\mu[X_N]$.

The Frobenius character map will be our main tool when studying representations of $\mathfrak{S}_n$ via the symmetric functions because it has the following isometry property (see for example [Ful97, Thm 7.3] for a proof).

Theorem 3.2.10. *The Frobenius character map is an isometry of $\mathbb{C}$-vector spaces with the scalar product on $\mathrm{CF}(\mathfrak{S}_n)$ being the inner product of characters and the scalar product on Λ^n being the scalar product from Definition 3.2.6.*

Before we begin to discuss the Macdonald symmetric functions, we want to briefly mention Littlewood-Richardson coefficients as they will become important later on.

Definition 3.2.11. For partitions μ, λ, ν the *Littlewood–Richardson coefficient* $c^\nu_{\mu\lambda}$ is defined by

$$s_\mu s_\lambda = \sum_\nu c^\nu_{\mu\lambda} s_\nu \; . \tag{3.30}$$

They are the structure coefficients of Λ with respect to the Schur functions (cf. Cor. 3.2.5).

Lemma 3.2.12 ([Ful97, Ch. 5]). *The numbers $c^\nu_{\mu\lambda}$ are nonzero only if*

$$|\lambda| + |\mu| = |\nu| \; . \tag{3.31}$$

The Macdonald symmetric functions are defined in $\Lambda(q,t)$, so we are again in the setting of extended scalars $\mathbb{C}(q,t)$. Our discussion of Schur functions, base change matrices, and even the Frobenius character map generalize to $\Lambda(q,t)$. For the latter, we also need to replace $\mathrm{CF}(\mathfrak{S}_n)$ by

$$\mathrm{CF}(\mathfrak{S}_n)(q,t) := \mathbb{C}(q,t) \otimes_{\mathbb{C}} \mathrm{CF}(\mathfrak{S}_n) \; . \tag{3.32}$$

Whenever we have an element $\chi \in \mathrm{CF}(\mathfrak{S}_n)(q,t)$ that is an $\mathbb{N}[q,t,q^{-1},t^{-1}]$-linear combination of irreducible characters, we can (and will) interpret χ as a character of some $\mathbb{Z}$-bigraded $\mathfrak{S}_n$-representation. The Macdonald symmetric functions are given by a unitriangularity condition with respect to the following symmetric functions.

Definition 3.2.13. For $N \geq n$ we fill the sequence $\lambda \in \mathscr{P}(n)$ with $N - l(\lambda)$ many 0's and define the *monomial symmetric polynomials* as

$$m_{\lambda,N} := \sum_\alpha x^\alpha \in \Lambda_N(q,t) \tag{3.33}$$

where the sum runs over all distinct permutations of the vector $(\lambda_1, ..., \lambda_{l(\lambda)}, 0, ..., 0)$, i.e. each monomial appears in the sum only once. We define the *monomial symmetric functions* m_λ as the preimage of $m_{\lambda,N}$ under ev_N^n.

PROPOSITION 3.2.14 ([Mac95, I.2]). *For $k \in \mathbb{N}$, set $\{m_\lambda \mid \lambda \vdash k\}$ is linearly independent and thus forms a vector space basis of $\Lambda^k(q,t)$ for all $k \leq N$.*

In addition to a unitriangularity condition, the Macdonald symmetric functions have an orthonomality property given by the following scalar product.

DEFINITION 3.2.15 ([Mac95, VI.2.2]). We define a q,t-version of the scalar product on Λ^n from Definition 3.2.6 by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \tag{3.34}$$

for $\lambda, \mu \in \mathscr{P}(n)$.

The previous scalar product can be seen as a specialization of this one by $\langle \cdot, \cdot \rangle_{0,0} = \langle \cdot, \cdot \rangle$. Recall the partial order $\trianglelefteq$ on $\mathscr{P}(n)$ given in Definition 2.1.9 by

$$\lambda \trianglelefteq \mu \;:\Longleftrightarrow\; \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \quad \text{for all } k \geq 1 \;. \tag{3.35}$$

THEOREM 3.2.16 ([Mac95, VI.4.7]). *For each partition $\lambda \in \mathscr{P}(n)$ there is a unique symmetric function $P_\lambda = P_\lambda(x;q,t) \in \Lambda^n(q,t)$ such that*

(i) $P_\lambda = \sum_{\mu \trianglelefteq \lambda} u_{\lambda\mu} \cdot m_\mu$ *where* $u_{\lambda\mu} \in \mathbb{C}(q,t)$ *and* $u_{\lambda\lambda} = 1$,
(ii) $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ *if* $\lambda \neq \mu$.

DEFINITION 3.2.17. For $\lambda \in \mathscr{P}(n)$, the symmetric function $P_\lambda(x;q,t)$ given by Theorem 3.2.16 is called the *Macdonald symmetric function* of λ.

Because of their unitriangular nature and by Proposition 3.2.14, the Macdonald functions also form a vector space basis of $\Lambda^n(q,t)$. To define q,t-Kostka–Macdonald coefficients as a certain base change matrix, we will first need to define the *integral form* of the Macdonald symmetric functions. We begin with a q,t-version of the hook polynomial. Recall Definition 2.1.5 where we introduced the concepts of arms, legs, and hooks in diagrams.

DEFINITION 3.2.18 ([Mac95, VI.8.1]). For $\lambda \in \mathscr{P}(n)$ we define $c_\lambda(q,t) \in \mathbb{C}(q,t)$ by

$$c_\lambda(q,t) = \sum_{u \in \lambda} \left(1 - q^{a(u)} t^{l(u)+1}\right) . \tag{3.36}$$

DEFINITION 3.2.19 ([Mac95, VI.8.3]). For $\lambda \in \mathscr{P}(n)$ with Macdonald symmetric function $P_\lambda(x;q,t)$ we define

$$J_\lambda(x;q,t) := c_\lambda(q,t) \cdot P_\lambda(x;q,t) \tag{3.37}$$

which will be called the *integral Macdonald symmetric function* of λ.

The integral Macdonald symmetric functions are differently scaled versions of the ordinary Macdonald symmetric functions and therefore form a vector space basis of $\Lambda^n(q,t)$ as well. From this observation, one can define the q,t-Kostka–Macdonald coefficients.

DEFINITION 3.2.20 ([Mac95, VI.8.11 + VI.8 Exm. 1]). For $\lambda, \mu \in \mathscr{P}(n)$ we define the *q,t-Kostka–Macdonald coefficient* $K_{\mu\lambda}(q,t) \in \mathbb{C}(q,t)$ via the base change

$$J_\lambda(x;q,t) = \sum_{\mu \in \mathscr{P}(n)} K_{\mu\lambda}(q,t) \cdot s_\mu[(1-t) \cdot X] \;. \tag{3.38}$$

For $n \leq 6$, the tables of $K_{\mu\lambda}(q,t)$ can be found in [Mac95, VI.8]. It has been shown by Haiman in 2001 that the q,t-Kostka–Macdonald coefficients are in fact Laurent polynomials with nonnegative integer coefficients, which is why the $J_\lambda(x;q,t)$ are called *integral* Macdonald symmetric functions. This statement is known as *Macdonald positivity* and it is recorded in the following theorem.

THEOREM 3.2.21 ([Hai01, Thm. 2]). *We have*

$$K_{\mu\lambda}(q,t) \in \mathbb{N}[q,t,q^{-1},t^{-1}] \tag{3.39}$$

for all $\lambda, \mu \in \mathscr{P}(n)$.

We want to use the Frobenius character map to find representations and characters of $\mathfrak{S}_n$ that afford the Kostka–Macdonald coefficients as their graded multiplicities. To this end, it is easier for us to define $K_{\mu\lambda}(q,t)$ via a base change into the Schur functions instead of the plethystically transformed $s_\lambda[(1-t)\cdot X]$ in (3.38). For that, we can "move" the factor $(1-t)$ to the left side by performing the plethystic substitution

$$p_r \mapsto p_r\left[\frac{X}{1-t}\right] \tag{3.40}$$

for $r \geq 1$ and obtain

$$J_\lambda(x;q,t)\left[\frac{X}{1-t}\right] = \sum_{\mu\in\mathscr{P}(n)} K_{\mu\lambda}(q,t)\cdot s_\mu\ . \tag{3.41}$$

The symmetric function on the left-hand side of (3.41) is very similar to the normalized version $H_\lambda(x;q,t)$ which Haiman introduced in [Hai99] and used in his proof of Theorem 3.2.21. It is given by

$$H_\lambda(x;q,t) := t^{b(\lambda)}\cdot J_\lambda(x;q,t^{-1})\left[\frac{X}{1-t^{-1}}\right]\ . \tag{3.42}$$

The idea behind Macdonald symmetric functions is that they generalize a wide range of families of symmetric functions. To examine this generalization property, we will need to specialize the coefficients $u_{\mu\lambda} \in \mathbb{C}(q,t)$ from Theorem 3.2.16 by replacing one or both of the variables q,t by some elements of $\mathbb{C}(q,t)$. One specialization that will be central to our discussion is $q = t$.

We note here that this *could* cause problems for general elements of $\mathbb{C}(q,t)$ but in all cases we will discuss we have that the rational functions in q and t do not have poles at $q-t$. Most of the time, we will specialize rational functions whose coefficients are already inside $\mathbb{N}[q,t,q^{-1},t^{-1}]$. We will also often specialize elements of $\Lambda(q,t)$ that are also plethystically transformed. In those cases, we will *first* apply the specialization and *afterwards* perform the plethysm. We give an example to illustrate this.

EXAMPLE 3.2.22. We define $R(q,t) := t\cdot p_4 + qt\cdot p_2^2 \in \Lambda(q,t)$ and compute

$$\begin{aligned} R(t^2,t^3)[(1-t)X] &= (t^3\cdot p_4 + t^2t^3\cdot p_2^2)[(1-t)X] \\ &= t^3\cdot p_4[(1-t)X] + t^5\cdot p_2^2[(1-t)X] \\ &= t^3\left(1-t^4\right)\cdot p_4[X] + t^5\left(1-t^2\right)^2\cdot p_2^2[X] \end{aligned}$$

REMARK 3.2.23. Note that by performing the computations in the described order, we would be able to transform a $q = t$ specialized symmetric function that only contains t into one containing both variables again. This could make the notation (t,t) a bit confusing but we will not encounter this behavior in our discussion.

Let us now focus on the $q = t$ specialization by examining the integral Macdonald symmetric functions first. Using Definition 3.2.19, we specialize the two factors one after the other.

DEFINITION 3.2.24. When we specialize $c_\lambda(q,t)$ at $q = t$, we obtain

$$c_\lambda(t,t) = \prod_{u\in\lambda}\left(1 - t^{a(u)+l(u)+1}\right) = \prod_{u\in\lambda}\left(1 - t^{h(u)}\right) =: H_\lambda(t) \in \mathbb{C}(t) \tag{3.43}$$

which is called the *hook polynomial of* λ.

REMARK 3.2.25. The hook polynomial is *not to be confused* with Haiman's normalized Macdonald symmetric function $H_\lambda(x;q,t)$ in (3.42). To help with this, we have denoted Haiman's symmetric function with a dependency on x.

The hook polynomial has already appeared implicitly in the context of fake degrees of the symmetric group, namely in Proposition 2.2.14. One important fact is captured in the following lemma. Recall the b-invariant of a partition given in Definition 2.1.13.

LEMMA 3.2.26 ([Mac95, Exm. I.3.2]). *For a partiton λ we have*

$$s_\lambda\left[\frac{1}{1-t}\right] = t^{b(\lambda)} \cdot H_\lambda^{-1}(t)\ . \tag{3.44}$$

Combining the above lemma with Proposition 2.2.14 gives us the following description of the fake degree.

PROPOSITION 3.2.27. *For $\lambda \in \mathscr{P}(n)$, the fake degree $f_\lambda(t)$ can be written as*

$$f_\lambda(t) = (1-t)(1-t^2)\cdots(1-t^n)\cdot s_\lambda\left[\frac{1}{1-t}\right] \tag{3.45}$$

The $q = t$-specialization of the second factor of Definition 3.2.19 is given by the following proposition.

PROPOSITION 3.2.28 ([Mac95, IV.4.14(i)]). *We have*

$$P_\lambda(x;t,t) = s_\lambda\ . \tag{3.46}$$

COROLLARY 3.2.29. *The $q = t$ specialized integral Macdonald polynomials are given by*

$$J_\lambda(x;t,t) = H_\lambda(t)\cdot s_\lambda\ . \tag{3.47}$$

If we use the plethysm of (3.41)*, we obtain*

$$J_\lambda(x;t,t)\left[\frac{X}{1-t}\right] = H_\lambda(t)\cdot s_\lambda\left[\frac{X}{1-t}\right]\ . \tag{3.48}$$

DEFINITION 3.2.30. For $\lambda \in \mathscr{P}(n)$, we define $G_\lambda(x;t,t) \in \Lambda^n(q,t)$ by

$$G_\lambda(x;t,t) := t^{b(\lambda)}\cdot H_\lambda(x;t,t^{-1}) = J_\lambda(x;t,t)\left[\frac{X}{1-t}\right] = H_\lambda(t)\cdot s_\lambda\left[\frac{X}{1-t}\right]\ . \tag{3.49}$$

We obtain a specialized version of the q,t-Kostka–Macdonald coefficients by

$$G_\lambda(x;t,t) = \sum_{\mu\in\mathscr{P}(n)} K_{\mu\lambda}(q,t)\cdot s_\mu\ , \tag{3.50}$$

which we call the *t,t-Kostka–Macdonald coefficients.*

The symmetric function $G_\lambda(x;t,t)$ turns out to be the image under the Frobenius character map ch of the graded $\mathfrak{S}_n$-character afforded by the irreducible $\overline{\mathsf{H}}_{\mathbf{c}}$-representation $L_{\mathbf{c}}(\lambda)$ in generic parameter $\mathbf{c}$, which is a result due to Gordon.

To close our discussion on specializations, we want to quickly mention some of the other constants and polynomials that are generalized by $K_{\mu\lambda}(q,t)$.

Proposition 3.2.31 ([Mac95, IV.8.12–17]). *For $\lambda, \mu \in \mathscr{P}(n)$ we have*

(i) $K_{\mu\lambda}(0,t) = K_{\mu\lambda}(t)$ *(Kostka polynomial),*
(ii) $K_{\mu\lambda}(0,1) = K_{\mu\lambda}$ *(Kostka number),*
(iii) $K_{\mu\lambda}(0,0) = \delta_{\mu\lambda}$ *(Kronecker delta),*
(iv) $K_{\mu\lambda}(1,1) = \chi^{\lambda}_{(1^n)}$ *(character degree),*
(v) $K_{\mu,(n)}(t,t) = f_\mu(t)$ *(fake degree, see 1.1.16).*

Note that (n) corresponds to the trivial character and $(1)^n$ to the trivial conjugacy class of $\mathfrak{S}_n$.

The first two specializations do not "factor through" $q = t$ but the rest do. When generalizing the t,t-Kostka–Macdonald coefficients to multipartitions, we will see that an analogous version of Proposition 3.2.31 holds there as well.

3.3. Multisymmetric functions

As with partitions and multipartitions, we want to generalize the theory of symmetric functions to the wreath product $C_\ell \wr \mathfrak{S}_n$ as well. We will use [Mac95, App. B], where the theory has been described for $G \wr \mathfrak{S}_n$ where G is an arbitrary finite group. This adaptation is common in the literature, see for example [Poi98], from which we will additionally use the plethystic formulation of the Frobenius character map.

Fix some positive integers ℓ and N, as well as some primitive ℓ^{th} root of unity $\zeta \in \mathbb{C}$. Let

$$\{x_i^{(j)} \mid 1 \le i \le N,\ 0 \le j \le \ell - 1\} \tag{3.51}$$

be a set of indeterminates over $\mathbb{C}$ and let $\Lambda_N^{(j)}$ denote the ring of symmetric polynomials in the variables $x_i^{(j)}$ for $0 \le j \le \ell - 1$.

Definition 3.3.1. We define the *ring of ℓ-multisymmetric* (or *ℓ-wreath symmetric*) *polynomials in N variables* $\mathbf{\Lambda}_N$ by

$$\mathbf{\Lambda}_N = \bigotimes_{j=0}^{\ell-1} \Lambda_N^{(j)} \tag{3.52}$$

For a nonnegative integer k we denote by $\mathbf{\Lambda}_N^k$ the homogeneous multisymmetric polynomials of degree k.

Remark 3.3.2. In the literature, the term "multisymmetric" is often used to describe the ring of diagonal invariants of the symmetric group. We chose this term for the wreath product case to mirror the combinatorial theory of partitions and multipartitions.

We again have a symmetric function version of $\mathbf{\Lambda}_N$.

Definition 3.3.3. We define the *ring of ℓ-multisymmetric* (or *ℓ-wreath symmetric*) *functions* by

$$\mathbf{\Lambda} = \Lambda^{\otimes \ell}\ . \tag{3.53}$$

Its grading is given by the sum of the degrees of the elementary tensors, i.e.

$$\deg\left(\bigotimes_{i=0}^{\ell-1} p_{r_i}\right) = \sum_{i=0}^{\ell-1} r_i\ . \tag{3.54}$$

We extend our convention and write

$$X_N^{(j)} := x_1^{(j)} + \cdots + x_N^{(j)}\ . \tag{3.55}$$

for $0 \leq j \leq \ell - 1$. Furthermore, we denote by $X^{(j)}$ the power-sum symmetric function of degree 1 inside the j^{th} factor of $\mathbf{\Lambda} = \Lambda^{\otimes \ell}$ such that we define

$$p_r[X^{(j)}] := 1 \otimes \cdots \otimes 1 \otimes p_r \otimes 1 \otimes \cdots \otimes 1 \in \Lambda^{\otimes \ell} \tag{3.56}$$

where p_r is in the j^{th} position for $r \geq 1$.

We define the generalization of the evaluation homomorphism next, given directly in plethystic notation. The linear transformation of the variables $X_1, ..., X_N$ is there to ensure the evaluation is well-behaved with respect to Schur functions.

Proposition 3.3.4. *There exists an evaluation homomorphism*

$$\begin{array}{rcl} \mathbf{ev}_N : \mathbf{\Lambda} & \to & \mathbf{\Lambda}_N \\ \bigotimes_{j=0}^{\ell-1} p_{r_j} & \mapsto & \prod_{j=0}^{\ell-1} p_{r_j}\Big[\sum_{i=0}^{\ell-1} \zeta^{i \cdot j} X_N^{(i)}\Big] \end{array} \tag{3.57}$$

that restricts to a vector space isomorphism

$$\mathbf{ev}_N^k : \mathbf{\Lambda}^k \xrightarrow{\sim} \mathbf{\Lambda}_N^k \tag{3.58}$$

for all $k \leq N$.

The proof works very similarly to the symmetric group case of [LR11, 2.1] and we again write $f[X_N]$ for the image of $f \in \mathbf{\Lambda}$ under $\mathbf{ev}_N$. As was the case for $\ell = 1$, we also want to extend our scalars from $\mathbb{C}$ to $\mathbb{C}(q,t)$ for some complex indeterminates q, t. We define

$$\mathbf{\Lambda}(q,t) := \mathbb{C}(q,t) \otimes_{\mathbb{C}} \mathbf{\Lambda}\ , \quad \mathbf{\Lambda}_N(q,t) := \mathbb{C}(q,t) \otimes_{\mathbb{C}} \mathbf{\Lambda}_N\ . \tag{3.59}$$

On our road to the multipartition analogue of the t,t-Kostka–Macdonald coefficients, we again start with the Schur functions.

Definition 3.3.5. For $\boldsymbol{\lambda} = (\lambda^{(0)}, ..., \lambda^{(\ell-1)}) \in \mathscr{P}(\ell, n)$ we define the *ℓ-multi-Schur* (or *ℓ-wreath Schur*) *function* of $\boldsymbol{\lambda}$ as

$$s_{\boldsymbol{\lambda}} = \bigotimes_{j=0}^{\ell-1} s_{\lambda^{(j)}} \in \mathbf{\Lambda} \tag{3.60}$$

The above definition behaves well with respect to the evaluation homomorphism $\mathbf{ev}_N$ as can be seen in the following proposition.

Proposition 3.3.6 ([Poi98, Cor. 3]). *For $\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)$ we have*

$$\mathbf{ev}_N(s_{\boldsymbol{\lambda}}) = \prod_{j=0}^{\ell-1} s_{\lambda^{(j)}}\Big[X_N^{(j)}\Big]\ . \tag{3.61}$$

Since we have $\mathbf{\Lambda} = \Lambda^{\otimes \ell}$ and we know that the Schur functions form a vector space basis of Λ^k for all k, we also have that the ℓ-multi-Schur functions form a vector space basis of $\mathbf{\Lambda}^k$ as well.

Definition 3.3.7. For $k \in \mathbb{N}$, we define a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbf{\Lambda}^k$ by demanding that the set of all ℓ-multi-Schur functions $\{s_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \vdash k\}$ forms an orthonormal basis with respect to $\langle \cdot, \cdot \rangle$.

Next, we want to give the generalization of the Frobenius character map. Recall that $\mathrm{CF}(G)$ denotes the set of complex-valued class functions of a group G. For $\boldsymbol{\mu} \in \mathscr{P}(\ell, n)$, denote by

$$z_{\boldsymbol{\mu}} = |\mathrm{Stab}_{C_\ell \wr \mathfrak{S}_n}(\sigma_{\boldsymbol{\mu}})| \tag{3.62}$$

the size of the stabilizer of an element $\sigma_{\boldsymbol{\mu}} \in C_\ell \wr \mathfrak{S}_n$ with cycle type $\boldsymbol{\mu}$. For a class function $\chi \in \mathrm{CF}(C_\ell \wr \mathfrak{S}_n)$, we again write $\chi_{\boldsymbol{\mu}}$ for the evaluation of χ at the class of cycle type $\boldsymbol{\mu}$.

Definition 3.3.8 ([Poi98]). We define the *Frobenius character map* of $C_\ell \wr \mathfrak{S}_n$ as

$$\begin{array}{rcl} \mathrm{ch} : \mathrm{CF}(C_\ell \wr \mathfrak{S}_n) & \to & \mathbf{\Lambda} \\ \chi & \mapsto & \sum_{\boldsymbol{\mu} \in \mathscr{P}(\ell,n)} z_{\boldsymbol{\mu}}^{-1} \cdot \chi_{\boldsymbol{\mu}} \cdot \prod_{j=0}^{\ell-1} p_{\mu^{(j)}}\Big[\sum_{i=0}^{\ell-1} \overline{\zeta^{i \cdot j}} X^{(i)}\Big] \ . \end{array} \tag{3.63}$$

Theorem 3.3.9 ([Poi98, Thm. 2]). *The Frobenius character map* ch *is an isometry of* $\mathbb{C}$*-vector spaces with the scalar product on* $\mathrm{CF}(C_\ell \wr \mathfrak{S}_n)$ *being given by the inner product of characters and the scalar product on* $\mathbf{\Lambda}$ *being given by Definition 3.3.7.*

As is the case with $\mathfrak{S}_n$, we identify a $C_\ell \wr \mathfrak{S}_n$-module V with its character $\chi_V \in \mathrm{CF}(C_\ell \wr \mathfrak{S}_n)$ when applying ch.

The classical Littlewood–Richardson coefficients $c^{\nu}_{\mu\lambda}$ also generalize to the multipartition setting. Because $c^{\nu}_{\mu\lambda}$ is only nonzero whenever $|\mu| + |\lambda| = |\nu|$ holds, we can think of (μ, λ) as a bipartition of $|\nu|$ and generalize the Littlewood–Richardson coefficients to products of Schur polynomials with ℓ factors.

Definition 3.3.10. For an ℓ-multipartition $\boldsymbol{\rho} \in \mathscr{P}(\ell, n)$ we define

$$\prod_{i=0}^{\ell-1} s_{\rho^{(i)}} = \sum_{\nu \in \mathscr{P}(n)} c^{\nu}_{\boldsymbol{\rho}} s_\nu \in \Lambda \ . \tag{3.64}$$

There are two very important properties of the Schur functions when it comes to plethystic substitutions. The first relates to sums of variables.

Proposition 3.3.11 ([LR11, 3.2]). *For a partition* $\lambda \in \mathscr{P}(n)$ *and indices* i, j *such that* $0 \leq i \neq j \leq \ell - 1$ *we have*

$$s_\lambda\Big[X^{(i)} + X^{(j)}\Big] = \sum_{(\rho^{(0)},\rho^{(1)}) \in \mathscr{P}(2,n)} c^{\lambda}_{\rho^{(0)}\rho^{(1)}} \cdot s_{\rho^{(0)}}\Big[X^{(i)}\Big] \cdot s_{\rho^{(1)}}\Big[X^{(j)}\Big] \ . \tag{3.65}$$

We can expand Proposition 3.3.11 iteratively to arbitrary sums of variables.

Corollary 3.3.12. *For a partition* $\lambda \in \mathscr{P}(n)$ *we have*

$$s_\lambda\left[\sum_{i=0}^{\ell-1} X^{(i)}\right] = \sum_{\boldsymbol{\rho} \in \mathscr{P}(\ell,n)} c^{\lambda}_{\boldsymbol{\rho}} \cdot \prod_{i=0}^{\ell-1} s_{\rho^{(i)}}\Big[X^{(i)}\Big] \ . \tag{3.66}$$

The second important property is a generalization of the fake degree formula from Proposition 3.2.27.

Theorem 3.3.13 ([Ste89, Thm 5.3]). *For a multipartition* $\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)$ *with fake degree* $f_{\boldsymbol{\lambda}}(t)$ *we have*

$$f_{\boldsymbol{\lambda}}(t) = t^{b(\alpha(\boldsymbol{\lambda}))} \cdot (1 - t^\ell) \cdot (1 - t^{2 \cdot \ell}) \cdots (1 - t^{n \cdot \ell}) \cdot \prod_{i=0}^{\ell-1} t^{\ell \cdot b(\lambda^{(i)})} \cdot H_{\lambda}^{-1}(t^\ell) \tag{3.67}$$

where $b(\alpha(\boldsymbol{\lambda}))$ *is the* b*-invariant of the vector* $\alpha(\boldsymbol{\lambda}) = (|\lambda^{(0)}|, \ldots, |\lambda^{(\ell-1)}|)$*. When we express* (3.67) *in Schur functions using Lemma 3.2.26, we obtain*

$$f_{\boldsymbol{\lambda}}(t) = t^{b(\alpha(\boldsymbol{\lambda}))} \cdot (1 - t^\ell) \cdot (1 - t^{2 \cdot \ell}) \cdots (1 - t^{n \cdot \ell}) \cdot \prod_{i=0}^{\ell-1} s_{\lambda^{(i)}}\left[\frac{1}{1 - t^\ell}\right] \ . \tag{3.68}$$

Definition 3.3.14. We define for $\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)$ the number

$$b(\boldsymbol{\lambda}) = b(\alpha(\boldsymbol{\lambda})) + \ell \cdot \sum_{i=0}^{\ell-1} b(\lambda^{(i)}) \ . \tag{3.69}$$

REMARK 3.3.15. Note when $\ell = 1$ the definition of $b(\boldsymbol{\lambda})$ agrees with $b(\lambda)$. Also, using Theorem 3.3.13 we can see that the lowest power appearing with nonzero coefficient in $f_{\boldsymbol{\lambda}}$ is in degree $b(\boldsymbol{\lambda})$. This means for the grade-shifted version $\bar{f}_{\boldsymbol{\lambda}}(t)$ of the fake degree from Definition 1.1.16 we have

$$\bar{f}_{\boldsymbol{\lambda}}(t) = t^{-b(\boldsymbol{\lambda})} \cdot f_{\boldsymbol{\lambda}}(t) \ . \tag{3.70}$$

3.4. Wreath Macdonald polynomials at $q = t$

Our goal in this section is to define the wreath product analogue of Macdonald's symmetric function in the plethystically transformed and normalized version $H_\lambda(x; q, t)$ (3.42) given first by Haiman in [Hai03]. For this, it will be useful to look at a different characterization of $H_\lambda(x; q, t)$ than the one given by Theorem 3.2.16. Recall the partial order $\trianglelefteq$ on $\mathscr{P}(n)$ given by

$$\lambda \trianglelefteq \mu \iff \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \quad \text{for all } k \geq 1 \tag{3.71}$$

for $\lambda, \mu \in \mathscr{P}(n)$.

PROPOSITION 3.4.1 ([Hai99, Prop. 2.6]). *For $\lambda \in \mathscr{P}(n)$ the symmetric function $H_\lambda(x; q, t) \in \mathbf{\Lambda}(q, t)$ is uniquely determined by*

(*i*) $H_\lambda(x; q, t) \in \langle s_\mu [X/(1 - q)] \mid \mu \trianglerighteq \lambda \rangle_{\mathbb{Q}(q,t)}$,
(*ii*) $H_\lambda(x; q, t) \in \langle s_\mu [X/(1 - t)] \mid \mu \trianglelefteq \lambda \rangle_{\mathbb{Q}(q,t)}$,
(*iii*) *the coefficient of $s_{(n)}$ in $H_\lambda(x; q, t)$ is 1.*

There is a family of partial orders on the set of ℓ-multipartitions $\mathscr{P}(\ell, n)$ derived from the dominance order (3.71), which can be used to generalize the definition obtained from Proposition 3.4.1. This family of partial orders is associated to the chamber decomposition $\mathsf{BlGen}(\overline{\mathsf{H}})$ such that each chamber gives rise to a partial order. The precise definition of these partial orders is not important here, it will be given later in Section 5.13. Fix now one such partial order on $\mathscr{P}(\ell, n)$ and denote it by $\preceq$. We then have the following theorem first conjectured by Haiman in [Hai03] and later proven by Bezrukavnikov–Finkelberg in [BF14].

THEOREM 3.4.2 (Bezrukavnikov–Finkelberg). *Let $\mathfrak{h}$ be the reflection representation of $G(\ell, 1, n)$. There exists a basis of $\mathbf{\Lambda}^n(q, t)$ indexed by partitions $\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)$ uniquely determined by*

(*i*) $H^{\preceq}_{\boldsymbol{\lambda}}(x; q, t) \cdot \sum_i (-q)^i \operatorname{ch}(\wedge^i \mathfrak{h}) \in \langle s_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \succeq \boldsymbol{\lambda} \rangle_{\mathbb{Q}(q,t)}$,
(*ii*) $H^{\preceq}_{\boldsymbol{\lambda}}(x; q, t) \cdot \sum_i (-t)^{-i} \operatorname{ch}(\wedge^i \mathfrak{h}) \in \langle s_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \preceq \boldsymbol{\lambda} \rangle_{\mathbb{Q}(q,t)}$,
(*iii*) *the coefficient of $s_{((n),\emptyset,\ldots,\emptyset)}$ in $H^{\preceq}_{\boldsymbol{\lambda}}(x; q, t)$ is 1,*

where $\operatorname{ch}(\wedge^i \mathfrak{h}) \in \mathbf{\Lambda}$ *is the image of the character of the i-fold alternating product of the reflection representation $\mathfrak{h}$ under the Frobenius character map* ch.

DEFINITION 3.4.3. For $\lambda \in \mathscr{P}(n)$, the symmetric function $H^{\preceq}_{\boldsymbol{\lambda}}(x; q, t)$ given by Theorem 3.4.2 is called the *wreath Macdonald symmetric function* of λ.

There is a way to describe the multiplication by the virtual character

$$\sum_i (-q)^i \operatorname{ch}(\wedge^i \mathfrak{h}) \tag{3.72}$$

of Theorem 3.4.2 by a plethysm on the Schur symmetric functions resembling the conditions in Proposition 3.4.1. This has been done by Wen in [Wen19] by defining plethystic variables for $0 \leq j \leq \ell-1$ by

$$
\begin{aligned}
Z_q^{(j)} &= \sum_{i=0}^{\ell-1} q^{\overline{(i-j)}} X^{(i)} \in \left\langle X^{(i)} \;\middle|\; 0 \leq i \leq \ell-1 \right\rangle_{\mathbb{Q}(t)} \\
Z_{t^{-1}}^{(j)} &= \sum_{i=0}^{\ell-1} t^{-\overline{(i-j)}} X^{(i)} \in \left\langle X^{(i)} \;\middle|\; 0 \leq i \leq \ell-1 \right\rangle_{\mathbb{Q}(t)}
\end{aligned}
\tag{3.73}
$$

where $\overline{(i-j)}$ denotes the representative of the class of $i-j \mod \ell$ in $\{0,...,\ell-1\}$.

Lemma 3.4.4 ([Wen19, Lem. 2.1]). *Using the plethystic variables in* (3.73) *the first two conditions in Theorem 3.4.2 transform to*

(*i*) $H_{\boldsymbol{\lambda}}^{\preceq}(x;q,t) \in \left\langle \prod_{j=0}^{\ell-1} s_{\mu^{(j)}}\left[Z_q^{(j)}/(1-q^{\ell})\right] \;\middle|\; \boldsymbol{\mu} \succeq \boldsymbol{\lambda} \right\rangle_{\mathbb{Q}(q,t)}$,

(*ii*) $H_{\boldsymbol{\lambda}}^{\preceq}(x;q,t) \in \left\langle \prod_{j=0}^{\ell-1} s_{\mu^{(j)}}\left[Z_{t^{-1}}^{(j)}/(1-t^{-\ell})\right] \;\middle|\; \boldsymbol{\mu} \preceq \boldsymbol{\lambda} \right\rangle_{\mathbb{Q}(q,t)}$.

If we want to generalize the symmetric function $G_\lambda(x;t,t)$ from Definition 3.2.30, we need to specialize $H_{\boldsymbol{\lambda}}$ in the same way, i.e. simultaneously by $q \mapsto t$ and $t \mapsto t^{-1}$. This gives us

$$
Z_q^{(j)} \mapsto Z_t^{(j)} \text{ and } Z_{t^{-1}}^{(j)} \mapsto Z_t^{(j)} \tag{3.74}
$$

for all $0 \leq j \leq \ell-1$. This means the two plethysms in Lemma 3.4.4 agree and the dominance sets intersect to $\{\boldsymbol{\mu} \mid \boldsymbol{\mu} = \boldsymbol{\lambda}\}$ for all $\boldsymbol{\lambda}$. We obtain the following lemma.

Lemma 3.4.5. *For $\boldsymbol{\lambda} \in \mathscr{P}(\ell,n)$ and any partial order $\preceq$ we have*

$$
H_{\boldsymbol{\lambda}}^{\preceq}(x;t,t^{-1}) = H_{\boldsymbol{\lambda}}(x;t,t^{-1}) = a_{\boldsymbol{\lambda}}(t) \cdot \prod_{j=0}^{\ell-1} s_{\lambda^{(j)}}\left[\frac{Z_t^{(j)}}{(1-t^{\ell})}\right] \tag{3.75}
$$

for some scalar $a_{\boldsymbol{\lambda}}(t) \in \mathbb{C}(t)$.

The scalar $a_{\boldsymbol{\lambda}}(t) \in \mathbb{C}(t)$ from the previous lemma is uniquely determined by Theorem 3.4.2(*iii*) and we it will be given later.

Definition 3.4.6. We denote

$$
Z^{(j)} := Z_t^{(j)} \tag{3.76}
$$

and for $\boldsymbol{\lambda} \in \mathscr{P}(\ell,n)$ let

$$
G_{\boldsymbol{\lambda}}(x;t,t) := \prod_{j=0}^{\ell-1} G_{\lambda^{(j)}}(x;t^{\ell},t^{\ell})\left[Z^{(j)}\right] = \prod_{j=0}^{\ell-1} H_{\lambda^{(j)}}(t) s_{\lambda^{(j)}}\left[\frac{Z^{(j)}}{(1-t^{\ell})}\right] . \tag{3.77}
$$

Definition 3.4.7. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}(\ell,n)$ we define the *t,t-Kostka–Macdonald coefficient* $K_{\boldsymbol{\mu}\boldsymbol{\lambda}}(t,t) \in \mathbb{C}(t)$ via the decomposition

$$
G_{\boldsymbol{\lambda}}(x;t,t) = \sum_{\boldsymbol{\mu} \in \mathscr{P}(n)} K_{\boldsymbol{\mu}\boldsymbol{\lambda}}(t,t) \cdot s_{\boldsymbol{\mu}} . \tag{3.78}
$$

The $G_{\boldsymbol{\lambda}}(x;t,t)$ will turn out to describe the characters of simple $\overline{\mathsf{H}}_{\mathbf{c}}$-modules for generic $\mathbf{c}$ in type $G(\ell,1,n)$ and they will be equal to $H_{\boldsymbol{\lambda}}(x;t,t^{-1})$ up to a shift in grading (see Corollary 3.5.3). In particular, the entries of C_L will be given by the $K_{\boldsymbol{\mu}\boldsymbol{\lambda}}(t,t)$.

3.5. Generic character formulae for $\overline{\mathsf{H}}_{\mathbf{c}}$-modules

We want to return to the study of the restricted rational Cherednik algebra $\overline{\mathsf{H}}_{\mathbf{c}}$ by answering Question 1.4.12, i.e. describing the graded W-character of simple $\overline{\mathsf{H}}_{\mathbf{c}}$-modules in generic parameter for W of type $G(\ell,1,n)$. The case $W=\mathfrak{S}_n$ has been dealt with by Gordon in [Gor03, Thm. 6.4(ii)] and we generalize his methods using the theories in Section 3.3 and Section 3.4. The results of this section have appeared in [MT22].

Fix again some nonnegative integers ℓ, n and let W denote the complex reflection group of type $G(\ell,1,n)$ together with its restricted rational Cherednik algebra $\overline{\mathsf{H}}_{\mathbf{c}}(W)$ for some generic parameter $\mathbf{c}$ according to Definition 1.4.11. Recall that we denote by $[V]$ the class of a graded W-module V inside the graded Grothendieck group $\mathcal{G}^{\mathrm{gr}}(W)$ of W.

By Theorem 1.4.10, the characters $[L_{\mathbf{c}}(\boldsymbol{\lambda})]^{\mathrm{gr}}$ are generically equal, i.e. they are constant for all generic parameters $\mathbf{c}$. To emphasize this, we will omit the parameter $\mathbf{c}$ in this section and write

$$L(\boldsymbol{\lambda}) := L_{\mathbf{c}}(\boldsymbol{\lambda}) \tag{3.79}$$

for $\boldsymbol{\lambda} \in \operatorname{Irr} W$ and $\mathbf{c}$ generic.

We will begin with Gordon's character formula for the $\mathfrak{S}_n$-case. When W is the symmetric group, we have by Theorem 1.4.9 that $\mathbf{c}$ is generic if and only if $\mathbf{c} \in \mathbb{C} \setminus \{0\}$. Gordon then proved the following.

THEOREM 3.5.1 ([Gor03, Thm. 6.4(ii)]). *Let $W=\mathfrak{S}_n$ and let $\mathbf{c} \neq 0$. Then the decomposition of the simple $\overline{\mathsf{H}}_{\mathbf{c}}(W)$-module $L(\lambda)$ for $\lambda \in \mathscr{P}(n)$ in the graded Grothendieck group of W is given by*

$$[L(\lambda)]^{\mathrm{gr}} = \sum_{\mu \in \mathscr{P}(n)} K_{\mu\lambda}(t,t) \cdot [\mu] \tag{3.80}$$

where $K_{\mu\lambda}(t,t)$ is the specialized q,t-Kostka–Macdonald coefficient given by Definition 3.2.20.

In the language of decomposition matrices from Section 1.2, we can rewrite Theorem 3.5.1 as

$$C_L = \left(K_{\mu\lambda}(t,t)\right)_{\mu,\lambda \in \operatorname{Irr}(\mathfrak{S}_n)} \tag{3.81}$$

with the q,t-Kostka–Macdonald coefficients as given in Definition 3.2.20. Using $G_\lambda(x;t,t)$ given by Definition 3.2.30 we can also write

$$\operatorname{ch} L(\lambda) = G_\lambda(x;t,t) \tag{3.82}$$

where $\operatorname{ch} L(\lambda)$ denotes the image of the W-character of $L(\lambda)$ under the Frobenius character map from Definition 3.2.9.

We can generalize Theorem 3.5.1 to W of type $G(\ell,1,n)$ and obtain the following result.

THEOREM 3.5.2 ([MT22, Thm. 3.11]). *Let W be of type $G(\ell,1,n)$ and let $\mathbf{c}$ be generic. Then the decomposition of the simple $\overline{\mathsf{H}}_{\mathbf{c}}(W)$-module $L(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathscr{P}(\ell,n)$ in the graded Grothendieck group of W is given by*

$$[L(\boldsymbol{\lambda})]^{\mathrm{gr}} = \sum_{\boldsymbol{\mu} \in \mathscr{P}(\ell,n)} K_{\boldsymbol{\mu}\boldsymbol{\lambda}}(t,t) \cdot [\boldsymbol{\mu}] \tag{3.83}$$

where $K_{\boldsymbol{\mu}\boldsymbol{\lambda}}(t,t)$ is the t,t-Kostka–Macdonald coefficient given by Definition 3.4.7.

Before we give a proof of this theorem, we want to record a corollary relating the symmetric function $G_{\boldsymbol{\lambda}}(x;t,t)$ to Haiman's wreath Macdonald symmetric function $H_{\boldsymbol{\lambda}}(x;q,t)$. By Definition 3.4.7 and Theorem 3.5.2, we have

$$\operatorname{ch} L(\boldsymbol{\lambda}) = G_{\boldsymbol{\lambda}}(x;t,t) \ . \tag{3.84}$$

We then get the following connection to wreath Macdonald symmetric functions which combined with (3.84) generalizes Definition 3.2.30 .

Corollary 3.5.3 ([MT22, Thm. 3.15]). *For $\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)$ let $H_{\boldsymbol{\lambda}}(x;t,t^{-1})$ be the specialized wreath Macdonald polynomial of $\boldsymbol{\lambda}$. We have*

$$\operatorname{ch} L(\boldsymbol{\lambda}) = t^{b(\boldsymbol{\lambda}^*)} H_{\boldsymbol{\lambda}}(x;t,t^{-1}) \tag{3.85}$$

where $\boldsymbol{\lambda}^*$ parametrizes the dual of the representation parametrized by $\boldsymbol{\lambda}$.

Proof. By Lemma 3.4.5 we have

$$H_{\boldsymbol{\lambda}}(x;t,t^{-1}) \in \mathbb{C}(t) \cdot \prod_{j=0}^{\ell-1} s_{\lambda^{(j)}} \left[\frac{Z^{(j)}}{1-t^{\ell}} \right] \tag{3.86}$$

where the $Z^{(j)}$ are the ones defined in (3.76). By comparing now (3.86) to Definition 3.4.7, we now know that the polynomials $H_{\boldsymbol{\lambda}}(x;t,t^{-1})$ and $G_{\boldsymbol{\lambda}}(x;t,t)$ differ by a scalar $u(t)\mathbb{C}(t)$. We thus need to show that $u(t) = t^{b(\boldsymbol{\lambda}^*)}$.

The third characteristic of wreath Macdonald polynomials in Theorem 3.4.2 says that the coefficient of the Schur function $s_{((n),\emptyset,\dots,\emptyset)}$ corresponding to the trivial representation in $H_{\boldsymbol{\lambda}}(x;q,t)$ is 1. Therefore we need to show that the coefficient of $s_{((n),\emptyset,\dots,\emptyset)}$ in $\operatorname{ch} L(\boldsymbol{\lambda})$ is $t^{b(\boldsymbol{\lambda}^*)}$.

The trivial representation appears in $M(\boldsymbol{\lambda})$ only in the degrees in which the coinvariant ring contains the dual representation $\boldsymbol{\lambda}^*$. These degrees are given by $f_{\boldsymbol{\lambda}^*}(t)$. Using [Ste89, Thm. 4.3], we see that dualizing

$$\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell-1)}) \tag{3.87}$$

yields the ℓ-multipartition

$$\boldsymbol{\lambda}^* = (\lambda^{(0)}, \lambda^{(\ell-1)}, \lambda^{(\ell-2)} \dots, \lambda^{(1)}) \ . \tag{3.88}$$

This means we have $\bar{f}_{\boldsymbol{\lambda}}(t) = \bar{f}_{\boldsymbol{\lambda}^*}(t)$ by Theorem 3.3.13. Therefore by Theorem 1.4.10, the trivial representation appears in $L(\boldsymbol{\lambda})$ only once, namely in degree $b(\boldsymbol{\lambda}^*)$. Using now (3.86) we have determined $H_{\boldsymbol{\lambda}}(x;t,t^{-1})$ uniquely and thus shown that $u(t) = t^{b(\boldsymbol{\lambda}^*)}$ proving the claim. ■

We will now prove Theorem 3.5.2 by starting with a lemma.

Lemma 3.5.4. *Let $\left(Z^{(j)}\right)_{0 \le j \le \ell-1}$ be the variables defined in (3.76). We define two vectors*

$$\mathbf{z} = \left(Z^{(0)} \ , \ \dots \ , \ Z^{(\ell-1)}\right)^{\mathrm{tr}} \ , \quad \mathbf{x} = \left(X^{(0)} \ , \ \dots \ , \ X^{(\ell-1)}\right)^{\mathrm{tr}} \tag{3.89}$$

and two matrices

$$T = (\zeta^{i \cdot j})_{0 \le i,j \le \ell-1} \ , \quad D = \operatorname{diag}\left(\frac{1-t^{\ell}}{1-\zeta^0 t} \ , \ \dots \ , \ \frac{1-t^{\ell}}{1-\zeta^{\ell-1} t} \right) \ . \tag{3.90}$$

We then have

$$\overline{T} \cdot \mathbf{z} = D \cdot \overline{T} \cdot \mathbf{x} \tag{3.91}$$

where $\overline{T}$ is obtained by complex conjugation of the entries of T.

PROOF. We show this by solving the system of linear equations for $\mathbf{z}$. We first see that

$$T \cdot \overline{T} = \operatorname{diag}(\ell, ..., \ell) \tag{3.92}$$

by row orthogonality of the character table of the cyclic group C_ℓ (which is equal to T). This means we have

$$T \cdot \overline{T} \cdot \mathbf{z} = \ell \cdot \mathbf{z}\,. \tag{3.93}$$

On the right side of equation (3.91), we obtain after multiplication with T for the p^{th} entry

$$(T \cdot D \cdot \overline{T} \cdot \mathbf{x})_p = \sum_{j=0}^{\ell-1} \zeta^{p\cdot j} \left(\frac{1-t^\ell}{1-\zeta^j t} \sum_{i=0}^{\ell-1} \overline{\zeta^{i\cdot j}} X^{(i)} \right)\,. \tag{3.94}$$

Combining (3.93) and (3.94), we get for $Z^{(p)}$

$$\begin{aligned}
Z^{(p)} &= \frac{1}{\ell} \cdot \sum_{j=0}^{\ell-1} \zeta^{p\cdot j} \left(\frac{1-t^\ell}{1-\zeta^j t} \sum_{i=0}^{\ell-1} \overline{\zeta^{i\cdot j}} X^{(i)} \right) \\
&= \frac{1}{\ell}(1-t^\ell) \cdot \sum_{j=0}^{\ell-1} \zeta^{p\cdot j} \left(\frac{1}{1-\zeta^j t} \sum_{i=0}^{\ell-1} \overline{\zeta^{i\cdot j}} X^{(i)} \right) \\
&= \frac{1}{\ell}(1-t^\ell) \cdot \sum_{i=0}^{\ell-1} \left(\sum_{j=0}^{\ell-1} \frac{1}{1-\zeta^j t} \zeta^{p\cdot j - i\cdot j} \right) X^{(i)} \\
&= \frac{1}{\ell}(1-t^\ell) \cdot \sum_{i=0}^{\ell-1} \left(\sum_{j=0}^{\ell-1} \left(\zeta^{-(i-p)} \right)^j \sum_{k=0}^{\infty} (\zeta^j t)^k \right) X^{(i)} \\
&= \frac{1}{\ell}(1-t^\ell) \cdot \sum_{i=0}^{\ell-1} \left(\sum_{k=0}^{\infty} t^k \sum_{j=0}^{\ell-1} \left(\zeta^{k-(i-p)} \right)^j \right) X^{(i)}\,.
\end{aligned}$$

The inner sum over j is equal to ℓ if $k-(i-p) \equiv 0 \bmod \ell$, and else 0. Therefore, only every ℓ^{th} k-summand remains. Note that $i-p$ is in $\{-(\ell-1), ..., \ell-1\}$. If $i-p$ is nonnegative, because $k \geq 0$, all k of the form $k = \ell \cdot k' + (i-p)$ for $k' \geq 0$ remain. If $i-p$ is negative, all k of the form $k = \ell \cdot k' + \ell + (i-p)$ for $k' \geq 0$ remain. That is equivalent to saying all k of the form $k = \ell \cdot k' + \overline{(i-p)}$ for some $k' \geq 0$ remain where $\overline{(i-p)}$ denotes the unique element in $\{0, ..., \ell-1\}$ that is equivalent to $i-p \bmod \ell$. We therefore obtain

$$\begin{aligned}
Z^{(p)} &= \frac{1}{\ell}(1-t^\ell) \cdot \sum_{i=0}^{\ell-1} \left(\sum_{k'=0}^{\infty} \ell \cdot t^{\ell k' + \overline{(i-p)}} \right) X^{(i)} \\
&= \frac{1}{\ell}(1-t^\ell) \cdot \sum_{i=0}^{\ell-1} \frac{\ell \cdot t^{\overline{(i-p)}}}{1-t^\ell} X^{(i)} \\
&= \frac{1}{\ell} \cdot \sum_{i=0}^{\ell-1} \ell \cdot t^{\overline{(i-p)}} \cdot X^{(i)} \\
&= \sum_{i=0}^{\ell-1} t^{\overline{(i-p)}} X^{(i)}\,.
\end{aligned}$$

That means that $Z^{(p)}$ is a cyclic permutation of $Z^{(0)} = \sum_{i=0}^{\ell-1} t^i X^{(i)}$ which is in accordance with (3.76).

■

The next proposition is a wreath analogue of the classical case (see the proof of [Mac95, IV.8.16]). Denote by $\mathbf{1}$ (resp. 1) the ℓ-multipartition (resp. partition) corresponding to the trivial representation, i.e. $\mathbf{1} = ((n), \emptyset, ..., \emptyset)$ (resp. $1 = (n)$). Let $\zeta \in \mathbb{C}$ again be a primitive ℓ^{th} root of unity. Recall that

$$H_\lambda(t) = \prod_{u \in \lambda} \left(1 - t^{h(u)}\right) \tag{3.95}$$

denotes the hook polynomial of a partition λ given in Definition 3.2.24 and that $l(\lambda)$ denotes the length of λ.

PROPOSITION 3.5.5. *Let $\boldsymbol{\rho} = (\rho^{(0)}, ..., \rho^{(\ell-1)}) \in \mathscr{P}(\ell, n)$. We then have the following properties of the t, t-Kostka–Macdonald coefficients:*

(i) $\sum_{\boldsymbol{\mu}} K_{\boldsymbol{\mu}\boldsymbol{\lambda}}(t,t)\ \chi^{\boldsymbol{\mu}}_{\boldsymbol{\rho}} = \prod_{j=0}^{\ell-1} \left(H_{\lambda^{(j)}}(t^\ell) \prod_{i=0}^{l(\rho^{(j)})} \left(1 - \zeta^j t^{\rho_i^{(j)}}\right)^{-1} \right) \chi^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}}$,

(ii) $K_{\boldsymbol{\mu}\mathbf{1}}(t,t) = f_{\boldsymbol{\mu}}(t)$.

PROOF. (i) : We look at the preimages of $G_{\boldsymbol{\lambda}}(x;t,t)$ and $\sum_{\boldsymbol{\mu}} K_{\boldsymbol{\mu}\boldsymbol{\lambda}}(t,t) s_{\boldsymbol{\mu}}$ under the Frobenius character map ch (see Definition 3.3.8) and use Lemma 3.5.4 to obtain

$$\begin{aligned}
G_{\boldsymbol{\lambda}}(x;t,t) &= \prod_{j=0}^{\ell-1} G_{\lambda^{(j)}}(x;t^\ell,t^\ell)\Big[Z^{(j)}\Big] = \prod_{j=0}^{\ell-1} H_{\lambda^{(j)}}(t^\ell) s_{\lambda^{(j)}}\left[\frac{Z^{(j)}}{1-t^\ell}\right] \\
&\overset{\text{ch}}{=} \sum_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}}^{-1} \cdot \left(\prod_{j=0}^{\ell-1} H_{\lambda^{(j)}}(t^\ell) \chi^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}}\right) \cdot \prod_{j=0}^{\ell-1} p_{\rho^{(j)}}\left[\frac{\sum_{i=0}^{\ell-1} \overline{\zeta^{i\cdot j}} Z^{(i)}}{1-t^\ell}\right] \\
&\overset{3.5.4}{=} \sum_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}}^{-1} \cdot \left(\prod_{j=0}^{\ell-1} H_{\lambda^{(j)}}(t^\ell) \chi^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}}\right) \cdot \prod_{j=0}^{\ell-1} p_{\rho^{(j)}}\left[\frac{\frac{1-t^\ell}{1-\zeta^j t} \cdot \sum_{i=0}^{\ell-1} \overline{\zeta^{i\cdot j}} X^{(i)}}{1-t^\ell}\right] \\
&= \sum_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}}^{-1} \cdot \left(\prod_{j=0}^{\ell-1} H_{\lambda^{(j)}}(t^\ell) \prod_{i=1}^{l(\rho^{(j)})} \left(1 - \zeta^j t^{\rho_i^{(j)}}\right)^{-1} \chi^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}}\right) \cdot \prod_{j=0}^{\ell-1} p_{\rho^{(j)}}\left[\sum_{i=0}^{\ell-1} \overline{\zeta^{i\cdot j}} X^{(i)}\right]
\end{aligned}$$

and

$$\sum_{\boldsymbol{\mu}} K_{\boldsymbol{\mu}\boldsymbol{\lambda}}(t,t) s_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}}^{-1} \cdot \left(\sum_{\boldsymbol{\mu}} K_{\boldsymbol{\mu}\boldsymbol{\lambda}}(t,t) \chi^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}}\right) \cdot \prod_{j=0}^{\ell-1} p_{\rho^{(j)}}\left[\sum_{i=0}^{\ell-1} \overline{\zeta^{i\cdot j}} X^{(i)}\right]. \tag{3.96}$$

The claim now follows from the bijectivity of ch.

(ii) : Using (i) we now show that

$$G_{\mathbf{1}}(x;t,t) = \sum_{\boldsymbol{\mu}} f_{\boldsymbol{\mu}}(t) \cdot s_{\boldsymbol{\mu}}\ . \tag{3.97}$$

By Definition 3.2.30 and Lemma 3.5.4 for $\boldsymbol{\lambda} = \mathbf{1} = ((n), \emptyset, ..., \emptyset)$ we get

$$\prod_{j=0}^{\ell-1} G_{\lambda^{(j)}}(x;t^\ell,t^\ell)\left[Z^{(j)}\right] = G_1(x,t^\ell,t^\ell)\left[Z^{(0)}\right] \tag{3.98}$$

which transforms (3.97) into

$$G_1(x;t^\ell,t^\ell)\left[Z^{(0)}\right] = \sum_{\boldsymbol{\mu}} f_{\boldsymbol{\mu}}(t) \cdot s_{\boldsymbol{\mu}}\left[Z^{(0)}\right]\ . \tag{3.99}$$

We can solve (3.99) by using the definition of the classical q, t-Kostka–Macdonald coefficients together with (3.76) and Corollary 3.3.12. Recall that for $\boldsymbol{\rho} \in \mathscr{P}(\ell, n)$ we use the notation $\alpha(\boldsymbol{\rho})$ for the vector $(|\rho^{(0)}|, ..., |\rho^{(0)}|)$. We obtain

$$
\begin{aligned}
G_1(x;t^\ell,t^\ell)\left[Z^{(0)}\right] &= \sum_\mu K_{\mu\,1}(t^\ell,t^\ell)\cdot s_\mu\left[Z^{(0)}\right] \\
&= \sum_\mu K_{\mu\,1}(t^\ell,t^\ell)\cdot s_\mu\left[\sum_{i=1}^{\ell-1} t^i X^{(i)}\right] \\
&= \sum_\mu K_{\mu\,1}(t^\ell,t^\ell)\cdot \sum_{\boldsymbol{\rho}} c^\mu_{\boldsymbol{\rho}} \prod_{i=1}^{\ell-1} s_{\rho^{(i)}}\left[t^i X^{(i)}\right] \\
&= \sum_\mu K_{\mu\,1}(t^\ell,t^\ell)\cdot \sum_{\boldsymbol{\rho}} c^\mu_{\boldsymbol{\rho}}\ t^{b(\alpha(\boldsymbol{\rho}))}\cdot \prod_{i=1}^{\ell-1} s_{\rho^{(i)}}\left[X^{(i)}\right] \\
&= \sum_\mu K_{\mu\,1}(t^\ell,t^\ell)\cdot \sum_{\boldsymbol{\rho}} c^\mu_{\boldsymbol{\rho}}\ t^{b(\alpha(\boldsymbol{\rho}))} s_{\boldsymbol{\rho}} \\
&= \sum_{\boldsymbol{\rho}} \left(\sum_\mu K_{\mu\,1}(t^\ell,t^\ell)\ c^\mu_{\boldsymbol{\rho}}\ t^{b(\alpha(\boldsymbol{\rho}))}\right)\cdot s_{\boldsymbol{\rho}}\ .
\end{aligned}
$$

We now calculate the coefficient of $s_{\boldsymbol{\rho}}$. We have $K_{\mu\,1}(t,t) = f_\mu(t)$ by Proposition 3.2.27. Using this, we get

$$
\begin{aligned}
&\sum_\mu K_{\mu\,1}(t^\ell,t^\ell)\ c^\mu_{\boldsymbol{\rho}}\ t^{b(\alpha(\boldsymbol{\rho}))} \\
&\quad = t^{b(\alpha(\boldsymbol{\rho}))}\sum_\mu K_{\mu\,1}(t^\ell,t^\ell)\ c^\mu_{\boldsymbol{\rho}} \\
&\quad = t^{b(\alpha(\boldsymbol{\rho}))}\sum_\mu f_\mu(t^\ell)\ c^\mu_{\boldsymbol{\rho}} \\
&\quad = t^{b(\alpha(\boldsymbol{\rho}))}\cdot(1-t^\ell)(1-t^{2\cdot\ell})\cdots(1-t^{n\cdot\ell})\cdot\sum_\mu s_\mu\left[\frac{1}{1-t^\ell}\right]\cdot c^\mu_{\boldsymbol{\rho}} \\
&\quad = t^{b(\alpha(\boldsymbol{\rho}))}\cdot(1-t^\ell)(1-t^{2\cdot\ell})\cdots(1-t^{n\cdot\ell})\cdot\prod_{i=1}^{\ell-1} s_{\rho^{(i)}}\left[\frac{1}{1-t^\ell}\right] \\
&\quad = f_{\boldsymbol{\rho}}(t)\ .
\end{aligned}
$$

■

We now have all the tools to prove Theorem 3.5.2.

Proof of Theorem 3.5.2. As in [Gor03, Thm. 6.4(*ii*)], we compare the corresponding characters of

$$[M(\boldsymbol{\lambda})]^{\mathrm{gr}} \quad \text{and} \quad t^{-b(\boldsymbol{\lambda})} f_{\boldsymbol{\lambda}}(t)\cdot\sum_{\boldsymbol{\mu}} K_{\boldsymbol{\mu}\,\boldsymbol{\lambda}}(t,t)[\boldsymbol{\lambda}]\ . \tag{3.100}$$

The second expression is obtained by first decomposing $M(\boldsymbol{\lambda})$ into $L(\boldsymbol{\lambda})$ using Theorem 1.4.10 and Remark 3.3.15, and then decomposing the $L(\boldsymbol{\lambda})$ into $\boldsymbol{\mu}$ by inserting the character formula we want to prove. The character of $[M(\boldsymbol{\lambda})]^{\mathrm{gr}}$ is determined by Lemma 1.3.3. All in all, we must show that for $\boldsymbol{\rho}\in\mathscr{P}(\ell,n)$ the equation

$$\sum_{\boldsymbol{\mu}} f_{\boldsymbol{\mu}}(t)\cdot\chi^{\boldsymbol{\mu}}_{\boldsymbol{\rho}}\chi^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}} = t^{-b(\boldsymbol{\lambda})} f_{\boldsymbol{\lambda}}(t)\sum_{\boldsymbol{\mu}} K_{\boldsymbol{\mu}\,\boldsymbol{\lambda}}(t,t)\chi^{\boldsymbol{\mu}}_{\boldsymbol{\rho}} \tag{3.101}$$

holds. We will only manipulate the right-hand side of (3.101). Using Proposition 3.5.5(i), we obtain

$$t^{-b(\boldsymbol{\lambda})} f_{\boldsymbol{\lambda}}(t) \prod_{j=0}^{\ell-1} \left(H_{\lambda^{(j)}}(t^\ell) \prod_{i=0}^{l(\rho^{(j)})} \left(1-\zeta^j t^{\rho_i^{(j)}}\right)^{-1} \right) \chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}} \,. \tag{3.102}$$

We plug in the fake degree formula from Theorem 3.3.13 and get

$$\prod_{k=1}^{n} (1-t^{k\cdot\ell}) \prod_{j=0}^{\ell-1} \prod_{i=0}^{l(\rho^{(j)})} \left(1-\zeta^j t^{\rho_i^{(j)}}\right)^{-1} \chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}}. \tag{3.103}$$

We multiply with $\chi_{\boldsymbol{\rho}}^{\mathbf{1}} = 1$, and write the first product as the hook polynomial of $\mathbf{1} = ((n), \emptyset, ..., \emptyset)$. Expression (3.103) now reads

$$\prod_{j=0}^{\ell-1} \left(H_{\mathbf{1}^{(j)}}(t^\ell) \prod_{i=0}^{l(\rho^{(j)})} \left(1-\zeta^j t^{\rho_i^{(j)}}\right)^{-1} \right) \chi_{\boldsymbol{\rho}}^{\mathbf{1}} \cdot \chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}}. \tag{3.104}$$

where we denote the partitions of $\mathbf{1}$ as $\mathbf{1}^{(j)}$. We use Proposition 3.5.5(i) for $\boldsymbol{\lambda} = \mathbf{1}$ and get

$$\sum_{\boldsymbol{\mu}} K_{\boldsymbol{\mu}\,\mathbf{1}}(t,t) \cdot \chi_{\boldsymbol{\rho}}^{\boldsymbol{\mu}} \chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}} \,. \tag{3.105}$$

Equation (3.101) from the beginning now reduces to

$$\sum_{\boldsymbol{\mu}} f_{\boldsymbol{\mu}}(t) \cdot \chi_{\boldsymbol{\rho}}^{\boldsymbol{\mu}} \chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu}} K_{\boldsymbol{\mu}\,\mathbf{1}}(t,t) \cdot \chi_{\boldsymbol{\rho}}^{\boldsymbol{\mu}} \chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}} \,, \tag{3.106}$$

which follows from Proposition 3.5.5(ii).

■

REMARK 3.5.6. The first statement of Proposition 3.5.5 completely determines our t,t-Kostka–Macdonald coefficients (similar to the classical case, see [Mac95, VI.8.17]). Namely, if we take the scalar product with an irreducible character $\chi^{\boldsymbol{\mu}}$ on both sides of Proposition 3.5.5(i), we obtain

$$K_{\boldsymbol{\mu}\,\boldsymbol{\lambda}}(t,t) \;= \sum_{\boldsymbol{\rho}} z_{\boldsymbol{\rho}}^{-1} \cdot \prod_{j=0}^{\ell-1} \left(H_{\lambda^{(j)}}(t^\ell) \prod_{i=0}^{\ell-1} \left(1-\zeta^j t^{\rho_i^{(j)}}\right)^{-1} \right) \cdot \chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}}\, \chi_{\boldsymbol{\rho}}^{\boldsymbol{\mu}} \,. \tag{3.107}$$

REMARK 3.5.7. From Proposition 3.5.5 and the discussions in Section 1.3, we get analogous results to Proposition 3.2.31(iii)-(v).

Computing the matrix C_L in the generic parameter case in practice is not difficult. We start with a character table of $W = C_\ell \wr \mathfrak{S}_n$ whose columns and rows are indexed by ℓ-multipartitions. From the ℓ-multipartitions, we can straightforwardly compute the fake degrees using Theorem 3.3.13. The decomposition numbers of tensor products $\boldsymbol{\lambda} \otimes \boldsymbol{\mu}$ are attained by solving a system of linear equations given by the character table of W. Together, we have determined

$$[M(\boldsymbol{\lambda})]^{\mathrm{gr}} = \sum_{\boldsymbol{\mu} \in \operatorname{Irr} W} f_{\boldsymbol{\mu}}(t) \cdot [\boldsymbol{\mu} \otimes \boldsymbol{\lambda}] \,. \tag{3.108}$$

Now, we need to divide by the grade shifted fake degrees $\bar{f}_{\boldsymbol{\lambda}}(t)$ and obtain C_L.

To compute the matrix C_L via the unspecialized wreath Macdonald symmetric functions $H_{\boldsymbol{\lambda}}^{\preceq}(x;q,t)$ is also straight-forward (cf. Sec. 5.13 for the precise definition), although it is computationally more expensive.

We close this section by giving a small example of the characters of simple $\overline{\mathsf{H}}_{\mathbf{c}}$-modules. Let for that $W = B_2 = G(2,1,2) = C_2 \wr \mathfrak{S}_2$. Denote the variables in $\boldsymbol{\Lambda}$ by

$X = X^{(0)}$ and $Y = X^{(1)}$. The irreducible representations of B_2 are parametrized by the bipartitions

$$\begin{array}{ccccc} \ydiagram{2},\emptyset & \ydiagram{1,1},\emptyset & \ydiagram{1},\ydiagram{1} & \emptyset,\ydiagram{1,1} & \emptyset,\ydiagram{2} \\ ((2),\emptyset) & ((1,1),\emptyset) & ((1),(1)) & (\emptyset,(1,1)) & (\emptyset,(2)) \end{array} \tag{3.109}$$

with grade-shifted fake degrees

$$\bar{f}_{\boldsymbol{\lambda}}(t) = \begin{cases} 1+t^2 & \text{if } \boldsymbol{\lambda} = ((1),(1))\ , \\ 1 & \text{else}\ , \end{cases} \tag{3.110}$$

and hook polynomials

$$H_{\lambda^{(0)}}(t^2)\cdot H_{\lambda^{(1)}}(t^2) = \begin{cases} (1-t^2)^2 & \text{if } \boldsymbol{\lambda} = ((1),(1))\ , \\ (1-t^2)(1-t^4) & \text{else}\ . \end{cases} \tag{3.111}$$

The character table of $C_2 = \langle g \mid g^2 = 1\rangle$ is equal to

$$\begin{array}{c|cc} & 1 & g \\ \hline \chi_1 & 1 & 1 \\ \chi_2 & 1 & -1 \end{array} \tag{3.112}$$

which means the Frobenius character map is defined as

$$\chi \mapsto \sum_{\boldsymbol{\mu}\in\mathscr{P}(2,n)} z_{\boldsymbol{\mu}}^{-1}\cdot\chi_{\boldsymbol{\mu}}\cdot p_{\mu^{(0)}}[X+Y]\cdot p_{\mu^{(1)}}[X-Y]\ . \tag{3.113}$$

Since the character table of $\mathfrak{S}_2$ is also given by (3.112), one can work out the multi-Schur polynomials and $G_{\boldsymbol{\lambda}}(x;t,t)$ by hand, the latter of which is given by the formula

$$H_{\lambda^{(0)}}(t^2)\, s_{\lambda^{(0)}}\left[\frac{X+tY}{1-t^2}\right]\cdot H_{\lambda^{(1)}}(t^2)\, s_{\lambda^{(1)}}\left[\frac{tX+Y}{1-t^2}\right]\ . \tag{3.114}$$

From the grade-shifted fake degrees (3.110) we obtain the diagonal decomposition matrix D_Δ. The matrices C_Δ and C_L are given in Figure 3.1 and Figure 3.2, respectively.

	$((2),\emptyset)$	$((1,1),\emptyset)$	$((1),(1))$	$(\emptyset,(2))$	$(\emptyset,(1,1))$
$((2),\emptyset)$	1	t^2	t^3+t	t^2	t^4
$((1,1),\emptyset))$	t^2	1	t^3+t	t^4	t^2
$((1),(1))$	t^3+t	t^3+t	t^4+2t^2+1	t^3+t	t^3+t
$(\emptyset,(2))$	t^2	t^4	t^3+t	1	t^2
$(\emptyset,(1,1))$	t^4	t^2	t^3+t	t^2	1

FIGURE 3.1. The decomposition matrix $C_\Delta = ([M(\boldsymbol{\lambda}) : \boldsymbol{\mu}]^{\mathrm{gr}})_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in type B_2 for $\mathbf{c}$ generic.

	$((2),\emptyset)$	$((1,1),\emptyset)$	$((1),(1))$	$(\emptyset,(2))$	$(\emptyset,(1,1))$
$((2),\emptyset)$	1	t^2	t^3+t	t^2	t^4
$(1,1),\emptyset)$	t^2	1	t^3+t	t^4	t^2
$((1),(1))$	t	t	t^2+1	t	t
$(\emptyset,(2))$	t^2	t^4	t^3+t	1	t^2
$(\emptyset,(1,1)$	t^4	t^2	t^3+t	t^2	1

FIGURE 3.2. The decomposition matrix $C_L = ([L(\boldsymbol{\lambda}) : \boldsymbol{\mu}]^{\mathrm{gr}})_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in type B_2 for $\mathbf{c}$ generic.

CHAPTER 4

Restricted rational Cherednik algebras II

In this chapter, we review the preliminary theories associated to the special parameter case, which turns out to be much more geometric in nature. Our target will be this special parameter version of Question 1.4.12.

QUESTION 4.0.1. For a special parameter $\mathbf{c}$ and W of type $G(\ell, 1, n)$, what are the entries of the matrix C_L? Or equivalently, what is the graded character of $L_{\mathbf{c}}(\lambda)$ for a special $\mathbf{c}$ and $\lambda \in \operatorname{Irr} W$?

This question will lead us first to the explicit description of the two generic and special loci of Section 1.4. Afterwards, we will review the symplectic structure of Calogero–Moser spaces in the classic and the quiver variety setting with our main aim being the description of what we call "cuspidal module induction".

4.1. Special parameters

The special parameter case forms the complement of the generic parameter case as outlined in Section 1.4. Special parameters are much richer combinatorially but also a lot more complicated to understand at the same time. To gather intuition for this problem, we will continue the example we gave at the end of Chapter 3.

Let W be of type B_2 and let $\mathbf{c} \equiv 1$, i.e. $\mathbf{c}(s) = 1$ for all conjugacy classes. We will see later in Section 4.2 that $\mathbf{c}$ is indeed a special parameter in this case (without having to consult the explicit decomposition of simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$). Recall that D_Δ denotes the decomposition matrix of standard $\overline{\mathsf{H}}_{\mathbf{c}}$-modules into simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$ and that in the generic B_2-case the matrix D_Δ diagonal and consists of grade-shifted fake degrees by Theorem 1.4.10. For $\mathbf{c} \equiv 1$, the matrix D_Δ can be computed using CHAMP [Thi15] and it is displayed in Figure 4.1.

	$((2), \emptyset)$	$((1,1), \emptyset)$	$((1),(1)))$	$(\emptyset, (2))$	$(\emptyset, (1,1)$
$((2), \emptyset)$	1	0	0	0	0
$((1,1), \emptyset)$	0	1	t	t^4	0
$((1),(1))$	0	t^3+t	t^2+1	t^3+t	0
$(\emptyset, (2))$	0	t^4	t	1	0
$(\emptyset, (1,1))$	0	0	0	0	1

FIGURE 4.1. The matrix $D_\Delta = ([M(\boldsymbol{\lambda}) : L(\boldsymbol{\mu})]^{\mathrm{gr}})_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in type B_2 for $\mathbf{c} \equiv 1$.

This is a block-diagonal matrix whose blocks form exactly the Calogero–Moser families for $\mathbf{c} \equiv 1$ (cf. [BR17, Rem. 12.4.3]). Note that although it might look as if the entries of D_Δ always consist of grade-shifted fake degrees, this is false already for type B_3, which can again be computed using CHAMP [Thi15]. From D_Δ it is not difficult to determine C_L, which is given in Figure 4.2.

Let us now continue with the more general discussion of special parameters. Let W be a complex reflection group. We recall some notation from Section 1.4.

	$((2),\emptyset)$	$((1,1),\emptyset)$	$((1),(1))$	$(\emptyset,(2))$	$(\emptyset,(1,1))$
$((2),\emptyset)$	1	t^2	t^3+t	t^2	t^4
$(1,1),\emptyset)$	0	1	0	0	0
$((1),(1)))$	t	0	t^2+1	0	t
$(\emptyset,(2))$	0	0	0	1	0
$(\emptyset,(1,1)$	t^4	t^2	t^3+t	t^2	1

FIGURE 4.2. The decomposition matrix $C_L = ([L(\boldsymbol{\lambda}):\boldsymbol{\mu}]^{\mathrm{gr}})_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in type B_2 for $\mathbf{c} \equiv 1$.

Denote by $\mathscr{C}$ the space of complex-valued W-equivariant functions on the set of complex reflections $\mathcal{S} \subseteq W$. Furthermore, let $\mathsf{C} = \mathbb{C}[\mathscr{C}]$ be the complex algebra of polynomial functions on $\mathscr{C}$ and let $\bullet$ denote the generic point in $\operatorname{Spec}\mathsf{C}$ given by the zero ideal, i.e. $V(\bullet) = \operatorname{Spec}\mathsf{C}$. We have discussed the decomposition map $\mathsf{d}_\bullet^{\mathbf{c}}$ and the two genericity loci $\mathsf{DecGen}(\overline{\mathsf{H}})$ and $\mathsf{BlGen}(\overline{\mathsf{H}})$ with respect to the generic algebras $\overline{\mathsf{H}} = \overline{\mathsf{H}}|_\bullet$ and $\overline{\mathsf{H}}_\bullet$ (cf. Sec. 1.4).

The whole construction of $\mathsf{d}_\bullet^{\mathbf{c}}$ and associated results hold in more generality by replacing $\bullet$ with any prime ideal $\mathfrak{p} \subseteq \mathsf{C}$ and replacing $\operatorname{Spec}\mathsf{C} = V(\bullet)$ by $V(\mathfrak{p})$ as well as $\overline{\mathsf{H}}$ by $\overline{\mathsf{H}}/\mathfrak{p}\overline{\mathsf{H}}$. We then demand that for the prime ideal $\mathbf{c}$ we have $\mathbf{c} \in V(\mathfrak{p})$, i.e. $\mathbf{c} \supseteq \mathfrak{p}$. The diagram (1.63) in Section 1.4 then becomes

$$\begin{tikzcd} \mathsf{C}/\mathfrak{p} \arrow[d, "\theta"] & \subseteq & \mathscr{O}_{\mathbf{c}} \arrow[d, "\pi"] & \subseteq & \operatorname{Frac}(\mathsf{C}/\mathfrak{p}) \\ \operatorname{Frac}(\mathsf{C}/\mathbf{c}) & \subseteq & \mathscr{O}_{\mathbf{c}}/\mathfrak{m} & & \end{tikzcd} \tag{4.1}$$

Doing this, we obtain for each prime ideal $\mathfrak{p}$ and $\mathbf{c} \in V(\mathfrak{p})$ a decomposition map $\mathsf{d}_{\mathfrak{p}}^{\mathbf{c}}$ and genericity loci

$$\mathsf{DecGen}(\overline{\mathsf{H}}|_{\mathfrak{p}})\ ,\ \mathsf{BlGen}(\overline{\mathsf{H}}|_{\mathfrak{p}}) \subseteq V(\mathfrak{p})\ . \tag{4.2}$$

Because $\mathbf{c}$ need not be a closed point in $V(\mathfrak{p})$, we can iterate this process and construct isotone sequences of prime ideals of C ending in a maximal ideal corresponding to a closed point in $\operatorname{Spec}\mathsf{C}$. It is more natural to view generic and special parameters this way since there is no choice of some "generic enough" function $\mathbf{c} : \mathcal{S}/W \to \mathbb{C}$ to be made at any point.

We want to record some important properties of this specialization with respect to Calogero–Moser families. Recall from Definition 1.4.7 that Calogero–Moser partitions are set partitions of $\operatorname{Irr}W$ obtained by associating the simple $\overline{\mathsf{H}}_{\mathbf{c}}$-module $L_{\mathbf{c}}(\lambda)$ to the corresponding block of $\overline{\mathsf{H}}_{\mathbf{c}}$ for $\lambda \in \operatorname{Irr}W$. The set of Calogero–Moser families associated to $\mathbf{c}$ is denoted by $\mathrm{CM}_{\mathbf{c}}$ for $\mathbf{c} \in \operatorname{Spec}\mathsf{C}$.

THEOREM 4.1.1 ([Thi18], [Thi17, Thm. 3.14]). *Let $\mathfrak{p} \in \operatorname{Spec}\mathsf{C}$ and $\mathfrak{q} \in V(\mathfrak{p})$.*

(*i*) *We have*

$$|\mathrm{CM}_{\mathfrak{p}}| \geq |\mathrm{CM}_{\mathfrak{q}}|\ , \tag{4.3}$$

i.e. the number of Calogero–Moser families decreases when specializing.

(*ii*) *We have*

$$|\mathrm{CM}_{\mathfrak{p}}| = |\mathrm{CM}_{\mathfrak{q}}| \Longleftrightarrow \mathrm{CM}_{\mathfrak{p}} = \mathrm{CM}_{\mathfrak{q}}\ , \tag{4.4}$$

i.e. the Calogero–Moser partition is equal if the number of families is the same. Furthermore, the families of $\mathrm{CM}_{\mathfrak{q}}$ are obtained by gluing certain families of $\mathrm{CM}_{\mathfrak{p}}$.

(*iii*) *The map*

$$V(\mathfrak{p}) \to \mathbb{N}, \ \mathfrak{r} \mapsto |\mathrm{CM}_{\mathfrak{r}}| \tag{4.5}$$

is lower semicontinuous, i.e. the subsets

$$\{\mathfrak{r} \in V(\mathfrak{p}) \mid |\mathrm{CM}_{\mathfrak{r}}| \leq k\} \tag{4.6}$$

are closed in $V(\mathfrak{p})$ *for all* $k \in \mathbb{N}$.

The most interesting part for us in Theorem 4.1.1 is (*ii*), especially the gluing process. This gluing is fairly well understood and it can be described using a standard operation on set partitions: the join of two elements.

Definition 4.1.2. For $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} \mathsf{C}$, we define a graph $G = (V, E)$ where $V = \mathrm{CM}_{\mathfrak{p}} \cup \mathrm{CM}_{\mathfrak{q}}$ and edge $\mathcal{F} \leftrightarrow \mathcal{F}'$ whenever $\mathcal{F} \cap \mathcal{F}' \neq \emptyset$. Denote by $\mathcal{C}_1, ..., \mathcal{C}_k$ the connected components of G. The *join* of $\mathrm{CM}_{\mathfrak{p}}$ and $\mathrm{CM}_{\mathfrak{q}}$ is then given by

$$\mathrm{CM}_{\mathfrak{p}} \vee \mathrm{CM}_{\mathfrak{q}} := \left\{ \bigcup_{\mathcal{F} \in \mathcal{C}_i} \mathcal{F} \ \middle| \ 1 \leq i \leq k \right\}, \tag{4.7}$$

i.e. we unite all overlapping families.

Remark 4.1.3. In [Thi18] the operation of Definition 4.1.2 is called the *meet* operation and it is denoted by $\wedge$. This is because of a different choice of partial order on set partitions. We revert back to the more common notation.

Let us review how the join operation was used in [Thi18] to describe the gluing of Calogero–Moser families. Recall that by Theorem 1.4.9 the set $\mathsf{BlEx}(\overline{\mathsf{H}}) \subseteq \operatorname{Spec} \mathsf{C}$ is a finite union of hyperplanes. We denote the set of irreducible components or *atoms* of $\mathsf{BlEx}(\overline{\mathsf{H}})$ by $\mathsf{At}(\overline{\mathsf{H}})$. We write $\mathrm{CM}_{\mathcal{H}}$ for the Calogero–Moser partition corresponding to $\mathcal{H} \in \mathsf{At}(\overline{\mathsf{H}})$ by identifying $\mathcal{H}$ with the corresponding prime ideal $\mathfrak{p} \in \operatorname{Spec} \mathsf{C}$. The set partition $\mathrm{CM}_{\mathcal{H}}$ is also the unique maximal Calogero–Moser partition associated to points of $\mathcal{H}$. Here, we mean that a partition is maximal with respect to the number of families, which is well-defined by Theorem 4.1.1(*ii*). We can now state the description of $\mathrm{CM}_{\mathbf{c}}$ for a given point $\mathbf{c} \in \operatorname{Spec} \mathsf{C}$.

Theorem 4.1.4 ([Thi18, Thm. 1.4]). *For* $\mathbf{c}$ *a (not necessarily closed) point in* $\operatorname{Spec} \mathsf{C}$ *we have*

$$\mathrm{CM}_{\mathbf{c}} = \bigvee_{\substack{\mathcal{H} \in \mathsf{At}(\overline{\mathsf{H}}): \\ \mathbf{c} \in \mathcal{H}}} \mathrm{CM}_{\mathcal{H}}, \tag{4.8}$$

i.e. we join the partitions of all irreducible components that contain $\mathbf{c}$. *This means* $\mathrm{CM}_{\mathbf{c}}$ *is the finest Calogero–Moser partition that is refined by all the* $\mathrm{CM}_{\mathcal{H}}$ *for* $\mathbf{c} \in \mathcal{H}$.

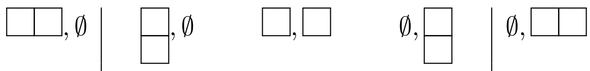
Because there are only finitely many irreducible components of $\mathsf{BlEx}(\overline{\mathsf{H}})$, there are at most $2^{|\mathsf{At}(\overline{\mathsf{H}})|}$ ways for $\mathbf{c}$ to be related to the atoms. This changes the problem of determining Calogero–Moser families to a finite problem governed by the Calogero–Moser partitions $\mathrm{CM}_{\bullet}$, $\mathrm{CM}_{\mathcal{H}}$ for $\mathcal{H} \in \mathsf{At}(\overline{\mathsf{H}})$. Furthermore, we have that $\mathsf{DecGen}(\overline{\mathsf{H}}\,|_{\mathfrak{p}}) = \mathsf{BlGen}(\overline{\mathsf{H}}\,|_{\mathfrak{p}})$ by Theorem 1.4.10 and that Calogero–Moser partitions uniquely determine the decomposition matrices C_L and D_Δ by [Thi17, Lem. 3.6]. This means, we have transformed Question 4.0.1 into a finite problem as well, which we aim to describe combinatorially. We begin with the explicit description of $\mathsf{BlGen}(\overline{\mathsf{H}})$ for $G(\ell, 1, n)$ following [BST18].

4.2. Parameters and hyperplanes

For this subsection, let W denote a complex reflection group of type $G(\ell, 1, n)$ for some $\ell, n \in \mathbb{N}$ and let $\zeta \in \mathbb{C}$ be some primitive ℓ^{th} root of unity.

We have already seen in Figure 4.2 the first example of special Calogero–Moser families for the group B_2, namely the partition for $\mathbf{c} \equiv 1$ displayed in Figure 4.3. The results and discussions found in [Gor08a] and [BST18] complete this

(4.9)



$$((2), \emptyset) \;\Big|\; ((1,1), \emptyset) \quad ((1),(1)) \quad (\emptyset, (1,1)) \;\Big|\; (\emptyset, (2))$$

FIGURE 4.3. The Calogero–Moser partition in type B_2 for $\mathbf{c} \equiv 1$.

example for the other hyperplanes of B_2, as well as generalize it to any group of type $G(\ell, 1, n)$.

We start by reiterating the discussion on wreath product complex reflection groups found in [BST18, Sec. 3.3]. Recall that we gave a complex reflection representation for W in Example 1.1.4, where W is generated by

$$\gamma_j := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \zeta & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \quad s_{i,i+1} := \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \tag{4.10}$$

for $1 \leq j \leq n$, $1 \leq i < n$. It is easy to see that the matrices in (4.10) are complex reflections. We obtain the remaining complex reflections as products

$$\gamma_j^{-k} s_{ij} \gamma_j^k = s_{ij} \gamma_i^{-k} \gamma_j^k \ , \qquad \gamma_j^k, \tag{4.11}$$

where $1 \leq k \leq \ell - 1,\ \ 1 \leq i < j \leq n$, and

$$s_{ij} = s_{i,i+1} \cdot s_{i+1,i+2} \cdot \cdots \cdot s_{j-2,j-1} \cdot s_{j-1,j} \cdot s_{j-2,j-1} \cdots s_{i+1,i+2} \cdot s_{i,i+1} \tag{4.12}$$

is the transposition matrix swapping the i^{th} and j^{th} elementary basis vector. The matrix of $s_{ij}\gamma_i^{-k}\gamma_j^k$ contains ζ^k in its i^{th} row and ζ^{-k} in its j^{th} row. We have therefore ℓ different conjugacy classes of reflections, namely

$$\begin{aligned} \mathcal{S}_0 &:= \{s_{ij}\gamma_i^k\gamma_j^{-k} \mid 1 \leq i \neq j \leq n, 0 \leq k \leq \ell - 1\} \ , \\ \mathcal{S}_k &:= \{\gamma_j^k \mid 1 \leq j \leq n\} \ , \qquad 1 \leq k \leq \ell - 1 \ , \end{aligned} \tag{4.13}$$

with the product of their nonzero matrix entries forming a separating invariant.

To each reflection $s \in \mathcal{S}$ we associate the hyperplane $\mathcal{H}_s \subseteq \mathfrak{h}$ of the fixed space of s, where $\mathfrak{h}$ again denotes the complex reflection representation of W. There are two hyperplanes up to W-conjugation, namely

$$\begin{aligned} \mathcal{H}_0 &:= \operatorname{Ker}(1 - s_{12}) = (1, -1, 0, ..., 0)^{\perp} \ , \\ \mathcal{H}_1 &:= \operatorname{Ker}(1 - \gamma_1) = (1, 0, ..., 0)^{\perp} \ , \end{aligned} \tag{4.14}$$

where $\mathcal{H}_0$ is conjugate to the hyperplanes of elements of $\mathcal{S}_0$, and $\mathcal{H}_1$ is conjugate to the hyperplanes of elements of $\mathcal{S}_1, ..., \mathcal{S}_{\ell-1}$. We denote their respective W-orbits by $[\mathcal{H}_0]$ and $[\mathcal{H}_1]$. Their respective stabilizer subgroups are

$$\begin{aligned} W_{\mathcal{H}_0} &:= W_0 := \{1, s_{12}\} \cong C_2 \ , \\ W_{\mathcal{H}_1} &:= W_1 := \{\gamma_1^k \mid 0 \leq k \leq \ell - 1\} \cong C_\ell \ . \end{aligned} \tag{4.15}$$

Let $y_1, ..., y_n \in \mathfrak{h}$ be the standard basis and let $x_1, ..., x_n \in \mathfrak{h}^*$ be the dual basis with respect to the natural pairing on $\mathfrak{h} \times \mathfrak{h}^*$ given by $(x, y) \mapsto y(x)$. For the roots and coroots (1.26) of the complex reflections in (4.11), we then have

$$\begin{gathered} \alpha^\vee_{s_{ij}\gamma_i^{-k}\gamma_j^k} = y_i - \zeta^{-k} y_j \in \mathfrak{h} \ , \quad \alpha_{s_{ij}\gamma_i^{-k}\gamma_j^k} = x_i - \zeta^k x_j \in \mathfrak{h}^* \ , \\ \alpha_{s_{ij}\gamma_i^{-k}\gamma_j^k}(\alpha^\vee_{s_{ij}\gamma_i^{-k}\gamma_j^k}) = 2 \ , \\ \\ \alpha^\vee_{\gamma_i^k} = y_i \in \mathfrak{h} \ , \quad \alpha_{\gamma_i^k} = x_i \in \mathfrak{h}^*, \\ \alpha_{\gamma_i^k}(\alpha^\vee_{\gamma_i^k}) = 1 \ . \end{gathered} \tag{4.16}$$

When we use the root and coroot identity (1.27) for the relations of the (restricted) rational Cherednik algebra (1.30), we get

$$[x, y] - \sum_{s \in \mathcal{S}} \mathbf{c}(s) \frac{x(\alpha_s^\vee) \cdot \alpha_s(y)}{\alpha_s(\alpha_s^\vee)} s \ , \quad x \in \mathfrak{h}^*, y \in \mathfrak{h} \ . \tag{4.17}$$

For $1 \leq i \neq j \leq n$, we can then calculate

$$[x_i, y_j] = -\sum_{k=0}^{\ell-1} \mathbf{c}(s_{12}) \frac{1}{2} \zeta^k s_{ij} \gamma_i^{-k} \gamma_j^k \ , \tag{4.18}$$

$$[x_i, y_i] = \sum_{k=1}^{\ell-1} \mathbf{c}(\gamma_1^k) \gamma_i^k + \sum_{k=1}^{\ell-1} \sum_{i \neq j} \mathbf{c}(s_{12}) \frac{1}{2} s_{ij} \gamma_i^{-k} \gamma_j^k \ . \tag{4.19}$$

When working with special parameters, it is often useful to introduce another basis of the space of class functions $\mathscr{C}$. The space $\mathscr{C}$ is ℓ-dimensional with one basis defined by

$$\mathbf{c}_{[s]} : \mathcal{S} \to \mathbb{C} \ , \ r \mapsto \mathbf{c}_{[s]}(r) = \begin{cases} 1, & r \in [s] \\ 0 & \text{else} \end{cases} \tag{4.20}$$

for all $[s] \in \mathcal{S}/W$. We will denote the $\mathbf{c}_{[s]}$ in accordance with (4.13) by

$$\begin{aligned} \boldsymbol{\kappa} &:= \mathbf{c}_{[s]}, \quad s \in \mathcal{S}_0 \ , \\ \mathbf{c}_k &:= \mathbf{c}_{[s]}, \quad s \in \mathcal{S}_k \ , \quad 1 \leq k \leq \ell - 1 \ . \end{aligned} \tag{4.21}$$

The basis of $\mathscr{C}$ given by the vectors in (4.21) has first been used in [EG02] and we call it *Etingof–Ginzburg basis* of $\mathscr{C}$. For an element $\mathbf{c} \in \mathscr{C}$, we define the *Etingof–Ginzburg coordinates* $\kappa, c_1, ..., c_{\ell-1}$ of $\mathbf{c}$ by the expression of $\mathbf{c}$ in the Etingof–Ginzburg basis, i.e.

$$\mathbf{c} = \kappa \boldsymbol{\kappa} + \sum_{k=1}^{\ell-1} c_k \mathbf{c}_k \ . \tag{4.22}$$

Using the Etingof–Ginzburg coordinates we have for $s \in \mathcal{S}_k$ that

$$\mathbf{c}(s) = c_k \mathbf{c}_k(s) = c_k \ . \tag{4.23}$$

It is useful to introduce the vector $\mathbf{c}_0 = 0$ with coordinate c_0 from which we demand that $c_0 = 0$ to guarantee a unique expression in the $\mathbf{c}_k$ for $0 \leq k \leq \ell - 1$. It follows from the commutator relations (1.30), that the algebras $\mathsf{H}_\mathbf{c}$ and $\mathsf{H}_{a \cdot \mathbf{c}}$ are isomorphic for $a \in \mathbb{C} \setminus \{0\}$. Therefore, one usually normalizes the Etingof–Ginzburg parameter by setting $\kappa = 1$ whenever κ is nonzero.

The *Gordon basis* of $\mathscr{C}$ was first used in [Gor08a] and it is given by the vectors

$$\begin{aligned} \mathbf{h} &:= 2 \cdot \boldsymbol{\kappa} \ , \\ \mathbf{H}_i &:= \sum_{k=0}^{\ell-1} \zeta_\ell^{-ik} \, \mathbf{c}_k \ , \ 1 \le i \le \ell - 1 \ , \end{aligned} \tag{4.24}$$

with *Gordon coordinates* $h, H_1, ..., H_{\ell-1}$, respectively. We also define another vector $\mathbf{H}_0$ with coordinate H_0 by $\mathbf{H}_0 := -\sum_{i=1}^{\ell-1} \mathbf{H}_i$ and restrict to the coordinates satisfying $\sum_{i=0}^{\ell-1} H_i = 0$ to guarantee a unique expression.

For a general element of $\mathscr{C}$ we have

$$\begin{aligned} h\mathbf{h} &= 2h\boldsymbol{\kappa} \ , \\ \sum_{i=0}^{\ell-1} H_i \mathbf{H}_i &= \sum_{k=1}^{\ell-1} \left(\sum_{i=0}^{\ell-1} H_i \zeta_\ell^{-ik} \right) \mathbf{c}_k \ , \end{aligned} \tag{4.25}$$

from which we can read off the coordinate relation

$$\begin{aligned} h &= \frac{1}{2}\kappa \ , \\ c_k &= \sum_{i=0}^{\ell-1} \zeta_\ell^{-ki} H_i \ , \quad 1 \le k \le \ell - 1 \end{aligned} \tag{4.26}$$

which can be extended to $c_0 = H_0 + \cdots + H_{\ell-1} = 0$. When we now rewrite the Cherednik algebra relations in the Gordon basis, we obtain

$$[x_i, y_j] = -\sum_{k=0}^{\ell-1} h \cdot \zeta^k \cdot s_{ij} \gamma_i^{-k} \gamma_j^k \ , \tag{4.27}$$

$$[x_i, y_i] = \sum_{k=1}^{\ell-1} \left(\sum_{i=0}^{\ell-1} \zeta_\ell^{-ki} H_i \right) \gamma_i^k + \sum_{k=1}^{\ell-1} \sum_{j \neq i} h \cdot s_{ij} \gamma_i^{-k} \gamma_j^k \ . \tag{4.28}$$

The Calogero–Moser hyperplanes for W of type $G(\ell, 1, n)$ have already been determined. The statement is a combination of [EG02, 1.12–14], [Thi17, Lem. 3.21], [Thi17, Cor. 3.22] and—most importantly—[Gor08a, Thm. 3.10, Proof of Lem. 4.3].

THEOREM 4.2.1. *For W of type $G(\ell, 1, n)$ we have that $\mathsf{BlEx}(\overline{\mathsf{H}})$ is a union of hyperplanes given by the equations*

$$\begin{gathered} h = 0 \ , \ \textit{and} \ \ (H_i + \cdots + H_j) + mh = 0 \ , \\ 1 \le i \le j \le \ell - 1, \quad -n < m < n \ . \end{gathered} \tag{4.29}$$

Coming back to our example of B_2 and $\mathbf{c} \equiv 1$, we see that the Etingof–Ginzburg coordinates $(\kappa, c_1) = (1, 1)$ correspond to the Gordon coordinates

$$(h, H_0, H_1) = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \tag{4.30}$$

which lie on exactly one of the hyperplanes in (4.29). The other nontrivial finest Calogero–Moser partitions are given in Figure 4.4, where the second column consists of generic points (h, H_1) on the hyperplane given by the first column. Figure 4.5 shows the Calogero–Moser hyperplane arrangement of type B. There exists a very nice combinatorial rule to calculate the families of Figure 4.4 by [Mar10], which we will discuss in Section 5.5. It is obtained via the geometry of quiver varieties which we will reiterate in Section 4.6. We close this section by introducing a restriction on our parameter space.

DEFINITION 4.2.2. Let $\mathbf{c} = (\kappa, c_1, ..., c_{\ell-1})$ be a Calogero–Moser parameter for $G(\ell, 1, n)$. If $\kappa = 0$, we say $\mathbf{c}$ is a *degenerate parameter*. We call $\mathbf{c}$ *nondegenerate* otherwise.

The degenerate case of wreath products can be reduced to the symmetric group case as has been done for type B in [BT16, Sec. 6.5]. In Section 6.9 of *loc. cit.*, this reduction is used to compute the ungraded B_n-characters of simple $\overline{\mathsf{H}}_{\mathbf{c}}$-modules. Because of these results, we will restrict to nondegenerate parameters.

> From now on, we assume that all Calogero–Moser parameters are nondegenerate, i.e. $\kappa \neq 0$ in Etingof–Ginzburg coordinates.

It is possible to compute the decomposition matrix C_L for certain cases using the action of $\overline{\mathsf{H}}_{\mathbf{c}}$ on standard modules. In [Thi17, App. B], Thiel constructed the graded W-characters of simple $\overline{\mathsf{H}}_{\mathbf{c}}$-modules when W is a dihedral group of order 2ℓ, i.e. W is of type $G(\ell, \ell, 2)$. However, a lot of the methods he used are specific to groups of that type. In order to give a more general approach to the wreath product case, we review the geometric representation theory of rational Cherednik algebras next.

Hyperplane	(h, H_1)	Calogero–Moser Partition		
$H_1 + h = 0$	$(1, -1)$	$\square\square, \emptyset$ $((2), \emptyset)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}, \emptyset$ $\quad$ $\square, \square$ $\quad$ $\emptyset, \square\square$ $((1,1), \emptyset)$ $\quad$ $((1), (1))$ $\quad$ $(\emptyset, (2))$	$\emptyset, \begin{smallmatrix}\square\\\square\end{smallmatrix}$ $(\emptyset, (1,1))$
$H_1 = 0$	$(1, 0)$	$\square\square, \emptyset$ $\quad$ $\emptyset, \square\square$ $((2), \emptyset)$ $\quad$ $(\emptyset, (2))$	$\square, \square$ $((1), (1))$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}, \emptyset$ $\quad$ $\emptyset, \begin{smallmatrix}\square\\\square\end{smallmatrix}$ $((1,1), \emptyset)$ $\quad$ $(\emptyset, (1,1))$
$H_1 - h = 0$	$(1, 1)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}, \emptyset$ $((1,1), \emptyset)$	$\square\square, \emptyset$ $\quad$ $\square, \square$ $\quad$ $\emptyset, \begin{smallmatrix}\square\\\square\end{smallmatrix}$ $((2), \emptyset)$ $\quad$ $((1), (1))$ $\quad$ $(\emptyset, (1,1))$	$\emptyset, \square\square$ $(\emptyset, (2))$
$h = 0$	$(0, 1)$	$\square\square, \emptyset$ $\quad$ $\begin{smallmatrix}\square\\\square\end{smallmatrix}, \emptyset$ $((2), \emptyset)$ $\quad$ $((1,1), \emptyset)$	$\square, \square$ $((1), (1))$	$\emptyset, \square\square$ $\quad$ $\emptyset, \begin{smallmatrix}\square\\\square\end{smallmatrix}$ $(\emptyset, (2))$ $\quad$ $(\emptyset, (1,1))$

FIGURE 4.4. The four nontrivial finest Calogero–Moser partitions for B_2.

4.3. Geometric representation theory

The rational Cherednik algebra and its geometric context were studied in [EG02] because of its relation to the *symplectic singularity* (in the sense of [Bea00])

$$\operatorname{Spec} \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W, \tag{4.31}$$

i.e. the spectrum of the invariants of the doubled action of W on $\mathfrak{h} \oplus \mathfrak{h}^*$. We will reiterate the connection between this invariant ring and the theory of rational Cherednik algebras. We refer the reader to [Thi17; Gor08b] for a more detailed account.

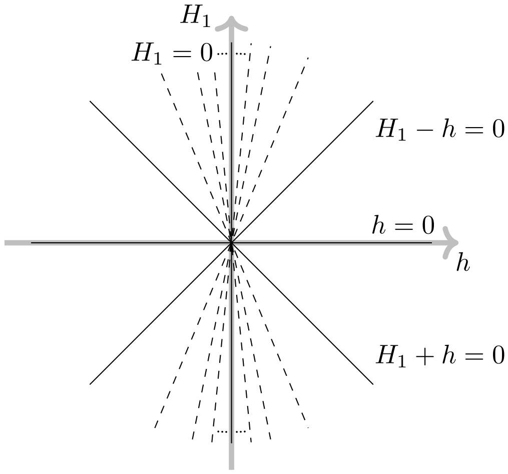


FIGURE 4.5. B_n-hyperplanes for $n = 2$ (dashed lines are hyperplanes for $n > 2$).

Denote by $\mathsf{Z}_\mathbf{c} = Z(\mathsf{H}_\mathbf{c})$ the center of a rational Cherednik algebra $\mathsf{H}_\mathbf{c}$ with parameter $\mathbf{c} \in \operatorname{Spec} \mathsf{C}$ and by H_0 the Cherednik algebra associated to the zero-function $\mathbf{c} \equiv 0$. It is easy to see that

$$\mathsf{H}_0 \cong \mathbb{C}[V] \rtimes W \tag{4.32}$$

as algebras. Furthermore, for the center we then have

$$\mathsf{Z}_0 \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W . \tag{4.33}$$

Thus, one can view (1.30) as a "deformation" of the commutator relations of $\mathbb{C}[V] \rtimes W$ and $\mathsf{Z}_\mathbf{c}$ as a "deformation" of the invariant ring $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ (cf. [Thi17, Thm. 1.19(1)]). For a nonclosed point $\mathbf{c} \in \operatorname{Spec} \mathsf{C}$, we change the scalars and replace the invariant ring $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ by

$$(\mathsf{C}/\mathbf{c}) \otimes_\mathbb{C} \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W = (\mathsf{C}/\mathbf{c})[\mathfrak{h} \oplus \mathfrak{h}^*]^W \tag{4.34}$$

The rings $\mathsf{Z}_\mathbf{c}$ and $(\mathsf{C}/\mathbf{c})[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ are studied via their relationship to another algebra, namely

$$\mathsf{P}_\mathbf{c} := (\mathsf{C}/\mathbf{c})[\mathfrak{h}]^W \otimes (\mathsf{C}/\mathbf{c})[\mathfrak{h}^*]^W . \tag{4.35}$$

By diagonally embedding $W \subseteq W \times W \subseteq \mathrm{GL}(\mathfrak{h} \oplus \mathfrak{h}^*)$ we readily see that

$$\mathsf{P}_\mathbf{c} \subseteq (\mathsf{C}/\mathbf{c})[\mathfrak{h} \oplus \mathfrak{h}^*]^W \tag{4.36}$$

holds and by [Thi17, Lem. 1.18(3)] this extension is free of degree $|W|$. There is a similar result for $\mathsf{Z}_\mathbf{c}$.

PROPOSITION 4.3.1 ([EG02, Prop. 4.15]). *For* $\mathbf{c} \in \operatorname{Spec} \mathsf{C}$ *we have*

(*i*) $\mathsf{P}_\mathbf{c} \subseteq \mathsf{Z}_\mathbf{c}$,
(*ii*) $\mathsf{Z}_\mathbf{c}$ *is a free* $\mathsf{P}_\mathbf{c}$*-module of rank* $|W|$.

The above proposition gives us the ability to pass onto the geometric viewpoint. We start by defining the corresponding space.

DEFINITION 4.3.2 ([EG02]). For $\mathbf{c} \in \operatorname{Spec} \mathsf{C}$ and $\mathsf{Z}_\mathbf{c}$ the center of the rational Cherednik algebra $\mathsf{H}_\mathbf{c}$ we define the *Calogero–Moser space* of W at parameter $\mathbf{c}$ by

$$\mathsf{X}_\mathbf{c} := \operatorname{Spec} \mathsf{Z}_\mathbf{c} . \tag{4.37}$$

As discussed in [EG02], the space $\mathsf{X}_{\mathbf{c}}$ is an irreducible affine variety. We will reiterate its relationship to the affine space $\operatorname{Spec} \mathsf{P}_{\mathbf{c}}$ next. The following construction goes back to [EG02, Sec. 4].

Definition 4.3.3. We denote the finite, thus surjective, morphism associated to $\mathsf{P}_{\mathbf{c}} \subseteq \mathsf{Z}_{\mathbf{c}}$ by

$$\Upsilon_{\mathbf{c}} : \mathsf{X}_{\mathbf{c}} \twoheadrightarrow \operatorname{Spec} \mathsf{P}_{\mathbf{c}} \ . \tag{4.38}$$

Note that whenever $\mathsf{C}/\mathbf{c}$ is a field, i.e. equal to $\mathbb{C}$, we have $\operatorname{Spec} \mathsf{P}_{\mathbf{c}} = \mathfrak{h}/W \times \mathfrak{h}^*/W$.

The representation theory of $\overline{\mathsf{H}}_{\mathbf{c}}$ is now studied by looking at the set-theoretic fiber $\Upsilon_{\mathbf{c}}^{-1}(p)$ for a closed point $p \in \operatorname{Spec} \mathsf{P}_{\mathbf{c}}$. Because $\Upsilon_{\mathbf{c}}$ is finite by Proposition 4.3.1(ii) and thus closed, we have that $\Upsilon_{\mathbf{c}}^{-1}(p)$ is finite set of closed points in $\operatorname{Spec} \mathsf{Z}_{\mathbf{c}}$. By Proposition 4.3.1$(i)$ and [Thi17, Thm 1.19(2)] we have that the extension $\mathsf{Z}_{\mathbf{c}} \subseteq \mathsf{H}_{\mathbf{c}}$ is finite as well from which one can prove the following statements. Denote by $\operatorname{Max} A$ the set of maximal ideals of a ring A.

Lemma 4.3.4 ([Thi17, Lem. 1.31]). *Let* $\mathbf{c} \in \operatorname{Spec} \mathsf{C}$*,* $S \in \operatorname{Irr} \mathsf{H}_{\mathbf{c}}$*, and* $A \in \{\mathsf{P}_{\mathbf{c}}, \mathsf{Z}_{\mathbf{c}}, \mathsf{H}_{\mathbf{c}}\}$. *We then have*

$$\operatorname{Ann}_A(S) \in \operatorname{Max} A \ . \tag{4.39}$$

This means we can decompose $\operatorname{Irr} \mathsf{H}_{\mathbf{c}}$ *as*

$$\operatorname{Irr} \mathsf{H}_{\mathbf{c}} = \coprod_{p \in \operatorname{Max} \mathsf{P}_{\mathbf{c}}} \operatorname{Irr} \mathsf{H}_{\mathbf{c}}/p\mathsf{H}_{\mathbf{c}} = \coprod_{p \in \operatorname{Max} \mathsf{P}_{\mathbf{c}}} \coprod_{\mathfrak{m} \in \Upsilon_{\mathbf{c}}^{-1}(p)} \operatorname{Irr} \mathsf{H}_{\mathbf{c}}/\mathfrak{m}\mathsf{H}_{\mathbf{c}} \ . \tag{4.40}$$

Theorem 4.3.5 ([Thi18, Thm. 1.32]). *For* $\mathbf{c} \in \operatorname{Spec} \mathsf{C}$ *and* $p \in \operatorname{Max} \mathsf{P}_{\mathbf{c}}$ *there is a canonical bijection*

$$\mathsf{Bl}(\mathsf{H}_{\mathbf{c}}/p\mathsf{H}_{\mathbf{c}}) \cong \mathsf{Bl}(\mathsf{Z}_{\mathbf{c}}/p\mathsf{Z}_{\mathbf{c}}) \cong \Upsilon_{\mathbf{c}}^{-1}(p) \tag{4.41}$$

where $\mathsf{Bl}(\cdot)$ *denotes the set of blocks of a finite dimensional algebra. Furthermore, the simple* $\mathsf{H}_{\mathbf{c}}$*-modules annihilated by* $\mathfrak{m} \in \Upsilon_{\mathbf{c}}^{-1}(p)$ *are precisely the simple modules of the corresponding block of* $\mathsf{H}_{\mathbf{c}}/p\mathsf{H}_{\mathbf{c}}$.

For the latter result, see also the discussion in [Gor03, Ch. 5]. From Theorem 4.3.5, we know that we can naturally identify the set $\Upsilon_{\mathbf{c}}^{-1}(p)$ with the set of blocks $\mathsf{Bl}(\mathsf{H}_{\mathbf{c}}/p\mathsf{H}_{\mathbf{c}})$, i.e. we are able to study the representation theoretic properties of $\mathsf{H}_{\mathbf{c}}$ by looking at its restrictions to maximal ideals of $\mathsf{P}_{\mathbf{c}}$ and using the geometric properties of the points in $\Upsilon_{\mathbf{c}}^{-1}(p)$. The following theorem makes this more explicit. The result is an application of [EG02, Thm. 1.7(iv)], [BG97, Lem 3.3], [BG02, Thm. III.1.6], and [Gor03, Lem. 7.2].

Theorem 4.3.6 ([Thi17, Thm. 1.33]). *For* $\mathbf{c} \in \mathscr{C}$ *and* $S \in \operatorname{Irr} \mathsf{H}_{\mathbf{c}}$ *denote by* $\mathfrak{m}_S$ *(resp.* p_S*) the annihilator of* S *inside* $\mathsf{Z}_{\mathbf{c}}$ *(resp.* $\mathsf{P}_{\mathbf{c}}$*). We then have* $\dim S \leq |W|$ *and the following statements are equivalent:*

(i) $\mathfrak{m}_S$ *is a smooth point of* $\mathsf{X}_{\mathbf{c}}$*,*
(ii) $\dim S = |W|$*,*
(iii) $S \cong \mathbb{C}W$ *as (ungraded)* W*-representations,*
(iv) *the block of* $\mathsf{H}_{\mathbf{c}}/p_S\mathsf{H}_{\mathbf{c}}$ *containing* S *contains up to isomorphism no further simple modules.*

As we can see from the theorem above, the points of $\mathsf{X}_{\mathbf{c}}$ corresponding to the blocks of simple modules of the $\overline{\mathsf{H}}_{\mathbf{c}}$ lie in the smooth locus of $\mathsf{X}_{\mathbf{c}}$ whenever $\mathbf{c}$ is a generic parameter and W is of type $G(\ell, 1, n)$. In this case in fact, the variety $\mathsf{X}_{\mathbf{c}}$ is already smooth which is a result we have used in the proof of Theorem 1.4.10. There is a classification result due to [EG02] and [Bel09] (see [Thi17, Sec. 5.2] for more details).

THEOREM 4.3.7. *Let W be an irreducible complex reflection group. The variety $\mathsf{X}_{\mathbf{c}}$ is smooth for some $\mathbf{c} \in \mathscr{C}$ if and only if W is of type $G(\ell, 1, n)$ or G_4.*

In the above theorem, we again include the irreducible complex reflection group of type A as type $G(1,1,n)$, even though, strictly speaking, type $G(1,1,n)$ as given in Example 1.1.4 is not irreducible.

The results in Theorem 4.3.6 hold for any quotient $\mathsf{H}_{\mathbf{c}}/p_S\mathsf{H}_{\mathbf{c}}$ where $p_S \subseteq \mathsf{P}_{\mathbf{c}}$ annihilates $S \in \operatorname{Irr} \mathsf{H}_{\mathbf{c}}$, but there is of course one particular quotient we are most interested in: the restricted rational Cherednik algebra $\overline{\mathsf{H}}_{\mathbf{c}}$. Following the discussion in [Thi17, Ch. 2], the RRCA is realized in this context by taking the ring

$$\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \subseteq \mathsf{P}_{\mathbf{c}} \,, \tag{4.42}$$

and the maximal ideal of $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ corresponding to the point

$$(0,0) \in \mathfrak{h}/W \times \mathfrak{h}^*/W \,, \tag{4.43}$$

which is given by

$$\mathbb{C}[\mathfrak{h}]^W_+ \mathbb{C}[\mathfrak{h}^*] + \mathbb{C}[\mathfrak{h}^*]^W_+ \mathbb{C}[\mathfrak{h}] \subseteq \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \tag{4.44}$$

and generated by the fundamental invariants. We can lift the above ideal to a prime ideal of $\mathsf{P}_{\mathbf{c}}$ by

$$(\mathsf{C}/\mathbf{c})[\mathfrak{h}]^W_+ (\mathsf{C}/\mathbf{c})[\mathfrak{h}^*] + (\mathsf{C}/\mathbf{c})[\mathfrak{h}^*]^W_+ (\mathsf{C}/\mathbf{c})[\mathfrak{h}] \,. \tag{4.45}$$

The quotient of $\mathsf{H}_{\mathbf{c}}$ by the $\mathsf{H}_{\mathbf{c}}$-ideal which is generated by (4.45) is precisely the restricted rational Cherednik algebra $\overline{\mathsf{H}}_{\mathbf{c}}$.

In summary, we get for a simple $\overline{\mathsf{H}}_{\mathbf{c}}$-module $L_{\mathbf{c}}(\lambda)$ a corresponding annihilator $\mathfrak{m}_{L_{\mathbf{c}}(\lambda)} \in \mathsf{X}_{\mathbf{c}}$ which is a maximal ideal of $\mathsf{Z}_{\mathbf{c}}$ lying over some maximal ideal of $p \in \mathsf{P}_{\mathbf{c}}$, which contains the ideal in (4.45). We then have that the point $\mathfrak{m}_{L_{\mathbf{c}}(\lambda)} \in \mathsf{X}_{\mathbf{c}}$ is smooth if and only if that particular $\overline{\mathsf{H}}_{\mathbf{c}}$-module $L_{\mathbf{c}}(\lambda)$ is isomorphic to the regular representation of W as a W-representation if and only if the Calogero–Moser family of $\lambda \in \operatorname{Irr} W$ is singleton. When we specialize parameters, the radical of the standard module $M(\lambda)$ becomes bigger. Because of this, the simple module $L_{\mathbf{c}}(\lambda)$ can be viewed as a graded W-subrepresentation of its generic version $L_\bullet(\lambda) \in \operatorname{Irr} \overline{\mathsf{H}}_\bullet$. We therefore have that $L_{\mathbf{c}}(\lambda)$ is already isomorphic to $L_\bullet(\lambda)$ whenever λ lies in a singleton family.

For W of type $G(\ell, 1, n)$, we have solved the singleton family case previously with Theorem 3.5.2. This means we are mostly interested in the nonsingleton Calogero–Moser families, i.e. the points $\mathfrak{m} \in \mathsf{X}_{\mathbf{c}}$ that annihilate more than one simple module of $\overline{\mathsf{H}}_{\mathbf{c}}$ or, equivalently, the points $\mathfrak{m} \in \mathsf{X}_{\mathbf{c}}$ that lie in the non-smooth locus of $\mathsf{X}_{\mathbf{c}}$. The remainder of this chapter is devoted to a geometric discussion of these points.

4.4. Fixed points and symplectic leaves

The discussion of the singular locus of $\mathsf{X}_{\mathbf{c}}$ starts with the description of two important geometric features of $\mathsf{X}_{\mathbf{c}}$: fixed points and symplectic leaves. We follow the discussions in [EG02] and [Bel11]. To make things easier, we are going to assume from now on that $\mathbf{c} \in \operatorname{Spec} \mathsf{C}$ is closed, i.e. $\mathbf{c} \in \mathscr{C}$. We can naturally define a $\mathbb{C}^*$-action on $\mathfrak{h} \oplus \mathfrak{h}^*$ by

$$\phi_z(y \oplus x) = zy \oplus z^{-1}x \tag{4.46}$$

for $z \in \mathbb{C}^*$. We can extend this action multiplicatively to $\mathsf{H}_{\mathbf{c}}$, to $\mathsf{Z}_{\mathbf{c}}$ and thus get a $\mathbb{C}^*$-action on $\operatorname{Spec} \mathsf{Z}_{\mathbf{c}} = \mathsf{X}_{\mathbf{c}}$. Since the action on $\mathfrak{h} \oplus \mathfrak{h}^*$ also induces one on $\mathfrak{h}/W \times \mathfrak{h}^*/W$, we get that Υ is $\mathbb{C}^*$-equivariant with respect to that action. Also, since $(0,0) \in \mathfrak{h}/W \times \mathfrak{h}^*/W$ is the only $\mathbb{C}^*$-fixed point, we have that $\Upsilon_{\mathbf{c}}^{-1}(0)$ is exactly the set of $\mathbb{C}^*$-fixed points of $\mathsf{X}_{\mathbf{c}}$, since the fixed points of $\mathsf{X}_{\mathbf{c}}$ must lie over the fixed points of $\operatorname{Spec} \mathsf{P}_{\mathbf{c}}$ by the $\mathbb{C}^*$-equivariance of $\Upsilon_{\mathbf{c}}$ (see [EG02, Proof of Prop. 4.16] for more details). We obtain the following well-known result as a consequence.

LEMMA 4.4.1. *There is a bijection between the Calogero–Moser families and the $\mathbb{C}^*$-fixed points of $\mathsf{X}_{\mathbf{c}}$.*

Recall that the the set $\Upsilon_{\mathbf{c}}^{-1}(0)$ is in bijection with the blocks of $\overline{\mathsf{H}}_{\mathbf{c}}$ by Theorem 4.3.5. To get an even better handle on the properties of the points in $\Upsilon_{\mathbf{c}}^{-1}(0)$, we can try to "locate" these points inside $\mathsf{X}_{\mathbf{c}}$ using the notion of symplectic leaves.

We can define a Poisson bracket $\{-,-\}$ on $\mathsf{X}_{\mathbf{c}}$ as in [EG02], (see also [Bel11, 4.1]) and view $\mathsf{X}_{\mathbf{c}}$ as a complex *analytic* Poisson space, i.e. affording the analytic topology.

DEFINITION 4.4.2. Let $\mathfrak{m} \in \mathsf{X}_{\mathbf{c}}$ be a point. The *symplectic leaf* $\mathcal{L}(\mathfrak{m})$ of $\mathsf{X}_{\mathbf{c}}$ is the maximal, connected, analytic subspace of $\mathsf{X}_{\mathbf{c}}$ containing $\mathfrak{m}$ on which the Poisson bracket $\{-,-\}$ is nondegenerate.

Following [BG03, Sec. 3.5], we can stratify the variety $\mathsf{X}_{\mathbf{c}}$ using symplectic leaves: we begin with the smooth locus of $\mathsf{X}_{\mathbf{c}}$ denoted by $\mathsf{X}_{\mathbf{c}}^{\mathrm{sm}}$. We can then stratify $\mathsf{X}_{\mathbf{c}}^{\mathrm{sm}}$ by a finite number of symplectic leaves. This gives us the remaining space

$$\mathsf{X}_1 = \mathsf{X}_{\mathbf{c}} \setminus \mathsf{X}_{\mathbf{c}}^{\mathrm{sm}} \,. \tag{4.47}$$

This space is again an affine Poisson variety, so we can stratify the smooth locus $\mathsf{X}_1^{\mathrm{sm}}$ of X_1 again by symplectic leaves. Iterating this process yields a stratification for all of $\mathsf{X}_{\mathbf{c}}$. This method has also been described in [Mar06, Sec. 1.6] in more detail. We only get finitely many symplectic leaves this way, which is captured in the following theorem.

THEOREM 4.4.3 ([BG03, Sec. 3.5+7.8]). *There are only finitely many symplectic leaves and they stratify $\mathsf{X}_{\mathbf{c}}$, i.e.*

$$\mathsf{X}_{\mathbf{c}} = \coprod \mathcal{L} \,. \tag{4.48}$$

We will be interested in the interplay between the block of $\overline{\mathsf{H}}_{\mathbf{c}}$ corresponding to a fixed point $\mathfrak{m} \in \Upsilon_{\mathbf{c}}^{-1}(0)$ and its corresponding leaf $\mathcal{L}(\mathfrak{m})$. It turns out that the symplectic leaf $\mathcal{L}(\mathfrak{m})$ controls much of the representation theory of $\mathsf{H}_{\mathbf{c}}/\mathfrak{m}\mathsf{H}_{\mathbf{c}}$. This "control" is captured by the results in [Bel11] which we will summarize the next section.

4.5. Cuspidal module induction

We want to describe an inductive phenomenon discovered by Bellamy in [Bel11] that connects symplectic leaves of $\mathsf{X}_{\mathbf{c}}$ to the representation theory of $\overline{\mathsf{H}}_{\mathbf{c}}$.

Let $(W,\mathfrak{h})$ be any complex reflection group of type and denote by $\mathsf{X}_{\mathbf{c}}(W,\mathfrak{h})$ its Calogero–Moser space associated to a point $\mathbf{c} \in \mathscr{C}$. For a point $\mathfrak{m} \in \mathsf{X}_{\mathbf{c}}$, we also define

$$\mathsf{H}_{\mathbf{c},\mathfrak{m}} := \mathsf{H}_{\mathbf{c}}/\mathfrak{m}\mathsf{H}_{\mathbf{c}} \,. \tag{4.49}$$

We have already seen that the set-theoretic fiber $\Upsilon_{\mathbf{c}}^{-1}(0) \subseteq \mathsf{X}_{\mathbf{c}}$ consists of all $\mathbb{C}^*$-fixed points of the Calogero–Moser space $\mathsf{X}_{\mathbf{c}}$ with respect to the $\mathbb{C}^*$-action in (4.46). Each point $\mathfrak{m} \in \Upsilon_{\mathbf{c}}^{-1}(0)$ lies on exactly one symplectic leaf $\mathcal{L}(\mathfrak{m})$ by Theorem 4.4.3. Since $\mathcal{L}(\mathfrak{m})$ is a symplectic manifold, it has even dimension denoted by $2l$. We will first discuss a special case of leaves.

DEFINITION 4.5.1 ([Bel11]). We say a point $\mathfrak{m} \in \mathsf{X}_{\mathbf{c}}$ is *cuspidal* if $\dim \mathcal{L}(\mathfrak{m}) = 0$, i.e. $\mathfrak{m}$ lies on a zero-dimensional symplectic leaf. We then also refer to the algebra $\mathsf{H}_{\mathbf{c},\mathfrak{m}}$ as a *cuspidal algebra.*

Because symplectic leaves are connected, we already have that $\mathcal{L}(\mathfrak{m}) = \{\mathfrak{m}\}$ for cuspidal $\mathfrak{m} \in \mathsf{X}_{\mathbf{c}}$. Furthermore, by [Bel11, Cor. 4.7] we know that a cuspidal point is already a $\mathbb{C}^*$-fixed point, i.e. $\mathfrak{m} \in \Upsilon_{\mathbf{c}}^{-1}(0)$ for cuspidal $\mathfrak{m}$. This gives us a corresponding property for Calogero–Moser families.

Definition 4.5.2 ([BT16]). The Calogero–Moser families associated to cuspidal points are called *cuspidal families.*

We have already seen two examples of cuspidal families in Figure 4.4, namely the 3-member families associated to the coordinates $(h, H_1) = (1, -1)$ and $(h, H_1) = (1, 1)$. We will see in Section 5.7 using [BT16] that these are indeed cuspidal families.

Cuspidal module induction is, as the name suggests, an inductive phenomenon. In this case, we "descend" to Calogero–Moser spaces of parabolic subgroups of W. A subgroup $W' \leq W$ is called *parabolic* if W' is the stabilizer of some set $S \subseteq \mathfrak{h}$ where $\mathfrak{h}$ is the reflection representation of W. Because the action of W is linear on $\mathfrak{h}$, the group W' already fixes a linear subspace $\mathfrak{h}^{W'}$. By [Ste64, Thm. 1.5] we have that W' is a complex reflection group as well and a reflection representation is given by $(\mathfrak{h}^{W'})^{\perp} =: \mathfrak{h}'$. Any parameter $\mathbf{c} \in \mathscr{C}$ can be restricted to the reflections of W' to obtain a parameter $\mathbf{c}'$ for W'. We can now state this incredibly useful result by Bellamy [Bel11, Sec. 5.5].

Theorem 4.5.3 (Cuspidal module induction). *Let $\mathfrak{m} \in \mathsf{X}_{\mathbf{c}}(W, \mathfrak{h})$ be a point with symplectic leaf $\mathcal{L}(\mathfrak{m}) \subseteq \mathsf{X}_{\mathbf{c}}(W, \mathfrak{h})$ of dimension $2l$. There exists a parabolic subgroup $W' \leq W$ of rank $\dim(\mathfrak{h}) - l$, a cuspidal algebra $\mathsf{H}_{\mathbf{c}',\mathfrak{m}'}$ with $\mathfrak{m}' \in \mathsf{X}_{\mathbf{c}'}(W', \mathfrak{h}')$ such that*

$$\mathsf{H}_{\mathbf{c},\mathfrak{m}} \cong \mathrm{Mat}_{|W/W'|}(\mathsf{H}_{\mathbf{c}',\mathfrak{m}'}) \,. \tag{4.50}$$

Furthermore, we have an equivalence

$$\Phi_{\mathfrak{m},\mathfrak{m}'} : \mathsf{H}_{\mathbf{c}',\mathfrak{m}'}\text{-mod} \xrightarrow{\sim} \mathsf{H}_{\mathbf{c},\mathfrak{m}}\text{-mod} \tag{4.51}$$

of categories of left modules such that for $M \in \mathsf{H}_{\mathbf{c}',\mathfrak{m}'}$-mod we have

$$\Phi_{\mathfrak{m},\mathfrak{m}'}(M) \cong \mathrm{Ind}_{W'}^{W} M \tag{4.52}$$

as ungraded W-modules.

Note that the result holds in particular for points inside $\Upsilon_{\mathbf{c}}^{-1}(0)$ which is where we will use it. Note also that the equivalence of categories given by the above theorem induces a bijection between the simple modules of $\mathsf{H}_{\mathbf{c},\mathfrak{m}}$ and the simple modules of $\mathsf{H}_{\mathbf{c}',\mathfrak{m}'}$ and therefore a bijection between the respective Calogero–Moser families associated to the points $\mathfrak{m}, \mathfrak{m}'$. It is this bijection which we want to study combinatorially.

We now have that the ungraded W-module structure of simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$ is completely controlled by modules of cuspidal families of parabolic subgroups of W. We give our first example of cuspidal module induction.

Example 4.5.4. Consider a point $\mathfrak{m} \in \mathsf{X}_{\mathbf{c}}^{\mathrm{sm}}$ on the smooth locus of the Calogero–Moser space. Cuspidal module induction tells us that the family associated to $\mathfrak{m} \in \mathsf{X}_{\mathbf{c}}(W)$ must be induced from a parabolic subgroup $W' \leq W$ of rank 0 with $\dim \mathsf{X}_{\mathbf{c}}(W') = 0$. This is true for the trivial group having a Calogero–Moser space of a single point (which is cuspidal). This cuspidal point corresponds to a singleton family of the trivial representation of the trivial group. The W-module $\mathrm{Ind}_1^W(1)$ is isomorphic to the regular representation. We will thus view a singleton Calogero–Moser family affording the regular representation of W as being induced from the trivial parabolic group.

As mentioned already at the end of Section 4.2, Bellamy and Thiel have computed the ungraded W-characters of simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$ in [BT16, Sec. 6.9] for degenerate parameters. This is done via cuspidal module induction as well.

For small groups, e.g. B_3, we can use CHAMP [Thi15] to compute the W-characters of simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$ for all special parameter hyperplanes as well as all parabolic subgroups. This way, we can compare the ungraded W-characters of simple modules of the various restricted rational Cherednik algebras and infer the structure of cuspidal module induction. In the example of B_3, all parabolic subgroups used for the induction are of the form $B_{n'}$ for some $n' \leq 3$ where we identify B_0 with the trivial group inducing the smooth locus. The fact that all parabolic subgroups which are used for the induction have the form $B_{n'}$ for some n' will turn out to be true in general (see Proposition 4.5.7). When we use the transitivity of module induction and view each family of B_n as being induced from the maximal parabolic subgroup $B_{n'}$, we obtain a forest graph structure with each cuspidal family forming a root. The inductive forest for the hyperplane given by the equation $h + H_1 = 0$ is given in Figure 4.6. As we can see, we obtain an "inductive pairing" of family members $\lambda \in \mathscr{P}(2, n)$, $\lambda' \in \mathscr{P}(2, n')$ such that $\lambda' \subseteq \lambda$, i.e. there are traces of an inductive phenomenon within the combinatorics. Our aim will be to describe this combinatorial side and link it to Theorem 4.5.3.

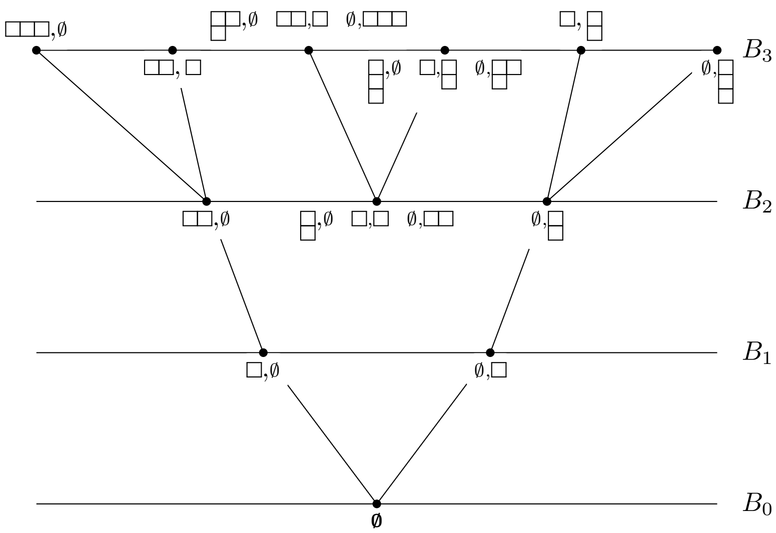


FIGURE 4.6. The inductive forest for B_3 and hyperplane $h + H_1 = 0$.

If we want to use cuspidal module induction effectively, we need to answer the following questions.

QUESTION 4.5.5. Let W be a complex reflection group of type $G(\ell, 1, n)$ and let $\mathbf{c}$ be some Calogero–Moser parameter.

(1) For a given family $\mathcal{F}$ of $\operatorname{Irr} W$ associated to $\mathbf{c}$, which cuspidal family $\mathcal{F}'$ of which parabolic subgroup W' induces $\mathcal{F}$?
(2) For a given family member $\boldsymbol{\lambda} \in \mathcal{F}$, which family member $\boldsymbol{\lambda}' \in \mathcal{F}'$ induces $\boldsymbol{\lambda}$?
(3) What is the ungraded W-module structure of $L_{\mathbf{c}'}(\boldsymbol{\lambda}')$?

We will begin by partially answering part (1) of the above question. The parabolic subgroups of $G(\ell, 1, n)$ have been classified in the following theorem by Taylor.

THEOREM 4.5.6 ([Tay12, Thm. 3.11]). *The parabolic subgroups of $G(\ell, 1, n)$ are (up to $G(\ell, 1, n)$-conjugacy) of the form*

$$G(\ell, 1, \alpha_0) \times \prod_{i \geq 1} \mathfrak{S}_{\alpha_i} \text{ with } \sum_{i \geq 0} \alpha_i \leq n \tag{4.53}$$

for some $\alpha_0, \alpha_1, ... \in \mathbb{Z}_{\geq 0}$.

Now, whenever we have a reducible complex reflection group $W = W_1 \times W_2$, we get a decomposition of the reflection representation $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and thus a decomposition of rational Cherednik algebras

$$\mathsf{H}_{\mathbf{c}}(W) \cong \mathsf{H}_{\mathbf{c}_1}(W_1) \otimes_{\mathbb{C}} \mathsf{H}_{\mathbf{c}_2}(W_2) \tag{4.54}$$

where $\mathbf{c}_i := \mathbf{c}|_{W_i}$ for $i = 1, 2$. The isomorphism is given by the product of two elements. This gives us a decomposition of centers

$$\mathsf{Z}_{\mathbf{c}}(W) \cong \mathsf{Z}_{\mathbf{c}_1}(W_1) \otimes_{\mathbb{C}} \mathsf{Z}_{\mathbf{c}_2}(W_2) \tag{4.55}$$

and then a decomposition of Calogero–Moser spaces

$$\mathsf{X}_{\mathbf{c}}(W) \cong \mathsf{X}_{\mathbf{c}_1}(W_1) \times_{\mathbb{C}} \mathsf{X}_{\mathbf{c}_2}(W_2) \ . \tag{4.56}$$

Now, let W be a parabolic subgroup of $G(\ell, 1, n)$ containing S_α as a factor for $\alpha \geq 2$. Since $\mathsf{X}_{\mathbf{c}}(\mathfrak{S}_k)$ is smooth of dimension $2k$ for $\mathbf{c} \neq 0$ (cf. [Gor03, Sec. 6.1]), there do not exist any zero-dimensional symplectic leaves in $\mathsf{X}_{\mathbf{c}}(W)$ whenever $\mathbf{c}$ is nondegenerate. So, only parabolic subgroups of the form $G(\ell, 1, n - l)$ admit zero-dimensional symplectic leaves. Since the rank of $G(\ell, 1, n - l)$ is $n - l$, we have proven the following proposition.

PROPOSITION 4.5.7. *Let $\mathbf{c}$ be a nondegenerate Calogero–Moser parameter for the group $G(\ell, 1, n)$. Cuspidal module induction associates to a symplectic leaf $\mathcal{L}$ of dimension $2l$ a zero-dimensional symplectic leaf of $\mathsf{X}_{\mathbf{c}}(W')$ where W' is of type $G(\ell, 1, n - l)$.*

We now know where to look for cuspidal Calogero–Moser families and have therefore answered the part of Question 4.5.5(1) concerning the parabolic subgroup. The problem is that we still do not know *which* cuspidal family is the correct one whenever there are multiple—not to mention the specific family *member*. Also, we have not seen a way to describe properties of $\mathfrak{m}$ or $\mathcal{L}(\mathfrak{m})$ using the geometry of $\mathsf{X}_{\mathbf{c}}$. To accomplish the latter, we will review the theory quiver varieties next and detail their connections to Calogero–Moser spaces given in [EG02], [Gor08a].

4.6. Quiver varieties

The main tool that has been used to study fixed points and symplectic leaves of Calogero–Moser spaces are *quiver varieties*, or rather Marsden–Weinstein reductions for quivers via moment maps. They are a special case of Nakajima quiver varieties [Nak98] studied in [CB01]. Let us start with the notion of a quiver. We will mainly follow the survey [Gin12] for this introduction.

DEFINITION 4.6.1. A *quiver* is a tuple $Q = (I, A)$ where I is a finite set whose elements are called *vertices*, and A is the set of *arrows* together with two maps

$$h, t : A \to I \tag{4.57}$$

giving the *head* and *tail* of an arrow $a \in A$.

We write Q^{op} for the *opposite* quiver obtained from Q by reversing the orientation of each arrow $a \in A$ and denote the reversed arrow of a by a^*. Furthermore write $\overline{Q}$ for the *double quiver* containing both the arrows of Q and Q^{op}.

EXAMPLE 4.6.2.

(*i*) Let $Q = (I, A)$ be given by $I = \{i\}$ and $A = \{a_1, ..., a_k\}$ with

$$h(a_s) = i = t(a_s) \tag{4.58}$$

for all $1 \leq s \leq k$, i.e. Q has a single vertex with $k \in \mathbb{N}$ loops. This quiver is visualized by Figure 4.7.

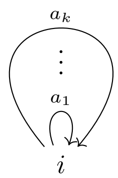


FIGURE 4.7. The quiver with one vertex and k loops.

(*ii*) Let $k \in \mathbb{N}$ and $Q = (I, A)$ with $I = \{i_1, ..., i_k\}$ and $A = \{a_1, ..., a_{k-1}\}$ such that $h(a_j) = j$, $t(a_j) = j + 1$ for all $1 \leq j \leq k - 1$. This quiver is visualized by the picture below.

$$i_1 \xleftarrow{a_1} i_2 \xleftarrow{a_2} \cdots \xleftarrow{a_{k-1}} i_k$$

(*iii*) Let $\ell \in \mathbb{N}$ and $I = \{0, ..., \ell - 1\}$ where we identify I with the classes of $\mathbb{Z}/\ell\mathbb{Z}$. The *cyclic quiver* on ℓ vertices is given by the arrows $a_j : j - 1 \to j$ inside $\mathbb{Z}/\ell\mathbb{Z}$ for all $j \in I$. This quiver is displayed in Figure 4.8. The cyclic quiver will be the most interesting example for us.

The representation theoretic story of quivers starts with the *path algebra* which we will define now. Let for that $Q = (I, A)$ be a quiver. Denote by $\mathbb{C}I$ the algebra of complex-valued functions on the set I generated as an algebra by the characteristic functions

$$1_i : I \to \mathbb{C}, j \mapsto \delta_{ij} \tag{4.59}$$

for $i \in I$ where δ_{ij} is the Kronecker delta. Furthermore we write $\mathbb{C}A$ for the vector space with basis the arrows of Q. The vector space $\mathbb{C}A$ becomes a $\mathbb{C}I$-bimodule with the action

$$1_i \cdot a \cdot 1_j = \begin{cases} a & \text{if } h(a) = j \text{ and } t(a) = i \ , \\ 0 & \text{else} \ . \end{cases} \tag{4.60}$$

DEFINITION 4.6.3. For a quiver $Q = (I, A)$ the *path algebra* of Q is

$$\mathbb{C}Q := T_{\mathbb{C}I}(\mathbb{C}A) \ , \tag{4.61}$$

i.e. $\mathbb{C}Q$ is the tensor algebra of the $\mathbb{C}I$ bimodule $\mathbb{C}A$. We identify $\mathbb{C}I \subseteq \mathbb{C}Q$ with the trivial paths at the respective vertices.

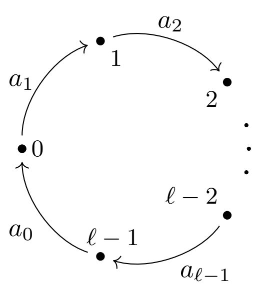


FIGURE 4.8. The cyclic quiver on ℓ vertices.

Let us give some examples for path algebras.

Example 4.6.4.

(*i*) Let Q be the quiver given in Example 4.6.2(*i*) and Figure 4.7. Its path algebra $\mathbb{C}Q$ is the free noncommutative algebra generated by the symbols $a_1, ..., a_k$.

(*ii*) Let Q be the quiver given in Example 4.6.2(*ii*). Then, $\mathbb{C}Q$ is isomorphic to the space of upper triangular $n \times n$ matrices over $\mathbb{C}$ by mapping 1_{i_k} to the matrix E_{ii} and a_k to the matrix $E_{i,i+1}$.

The algebra $\mathbb{C}I$ is generated by the complete set of orthogonal idempotents $\{1_i \mid i \in I\}$. This makes any left $\mathbb{C}I$-module V into an I-graded vector space. For a finite-dimensional I-graded vector space

$$V = \bigoplus_{i \in I} V_i \quad \text{with} \quad \dim V_i = d_i \ , \tag{4.62}$$

denote by $\mathrm{Rep}(\mathbb{C}Q, V)$ the vector space of algebra homomorphisms

$$\varphi : \mathbb{C}Q \to \mathrm{End}_{\mathbb{C}}(V) \tag{4.63}$$

on which $\prod_{i \in I} \mathrm{GL}(V_i)$ acts via base changes. We call $\mathbf{d} := \dim_I V := (d_i)_{i \in I} \in \mathbb{Z}^I_{\geq 0}$ the *dimension vector* of V and write

$$\mathrm{Rep}(Q, \mathbf{d}) := \mathrm{Rep}(\mathbb{C}Q, V) \ , \ G_{\mathbf{d}} := \prod_{i \in I} \mathrm{GL}(V_i) \ . \tag{4.64}$$

We can also choose a basis and identify

$$\mathrm{Rep}(Q, \mathbf{d}) = \bigoplus_{a \in A} \mathbb{C}^{h(a) \times t(a)} \ , \tag{4.65}$$

i.e. we get a matrix for each arrow that affords linear maps between vector spaces of dimensions $h(a)$ and $t(a)$.

The geometric space which can now be associated to a quiver Q and dimension vector $\mathbf{d}$ is the (categorical) quotient

$$\mathrm{Rep}(Q, \mathbf{d}) /\!/ G_{\mathbf{d}} := \mathrm{Spec}\, \mathbb{C}[\mathrm{Rep}(Q, \mathbf{d})]^{G_{\mathbf{d}}} \ , \tag{4.66}$$

i.e. the spectrum of the invariant ring of $\mathrm{Rep}(Q, \mathbf{d})$ as a $G_{\mathbf{d}}$-module. Because $G_{\mathbf{d}}$ is a reductive group, the construction in (4.66) is an affine variety, which we call *quiver variety*. The closed points of $\mathrm{Rep}(Q, \mathbf{d}) /\!/ G_{\mathbf{d}}$ are naturally in bijection with $G_{\mathbf{d}}$-orbits of semisimple $\mathbb{C}Q$-representations with dimension vector $\mathbf{d}$ (cf. [Gin12, Thm. 2.1.3]).

Next, we will replace the algebra $\mathbb{C}Q$ by a quotient of $\mathbb{C}\overline{Q}$ depending on a parameter $\boldsymbol{\theta} = (\theta_i)_{i \in I} \in \mathbb{C}^I$. This parameter will play the role of the Calogero–Moser parameter $\mathbf{c}$ in the quiver setting. It is denoted by λ in [CB01] and subsequent works but to avoid confusion we are using the convention found in [Gor08a]. We note though that this could also cause confusion with the stability parameter associated to GIT which is denoted by $\boldsymbol{\theta}$ as well.

Definition 4.6.5. For $\boldsymbol{\theta} \in \mathbb{C}^I$ the *preprojective algebra* $\Pi_{\boldsymbol{\theta}} = \Pi_{\boldsymbol{\theta}}(Q)$ is the quotient of $\mathbb{C}\overline{Q}$ generated by the relations

$$\sum_{a \in A} (aa^* - a^*a) - \sum_{i \in I} \theta_i \cdot 1_i \in \mathbb{C}\overline{Q} \ . \tag{4.67}$$

For more details on the preprojective algebra $\Pi_{\boldsymbol{\theta}}$, we refer the reader to [Gin12, Sec. 4.3]. We denote again by $\mathrm{Rep}(\Pi_{\boldsymbol{\theta}}, \mathbf{d})$ the vector space of algebra homomorphisms $\varphi : \Pi_{\boldsymbol{\theta}} \to \mathrm{End}_{\mathbb{C}}(V)$ where V is again an I-graded vector space of dimension vector $\mathbf{d}$.

The space $\mathrm{Rep}(\Pi_{\boldsymbol{\theta}}, \mathbf{d})$ can be expressed in terms of the space $\mathrm{Rep}(\overline{Q}, \mathbf{d})$ by using constructions from differential geometry. As detailed in [Gin12, Sec. 4.1-4-3], we can naturally associate

$$\mathrm{Rep}(\overline{Q}, \mathbf{d}) \cong \mathrm{Rep}(Q, \mathbf{d}) \times \mathrm{Rep}(Q^{\mathrm{op}}, \mathbf{d}) \cong \mathrm{Rep}(Q, \mathbf{d}) \times \mathrm{Rep}(Q, \mathbf{d})^* \tag{4.68}$$

which gives us the cotangent bundle $T^*(\mathrm{Rep}(Q, \mathbf{d}))$ on the vector space $\mathrm{Rep}(Q, \mathbf{d})$. Expanding on (4.65), one identifies

$$\mathrm{Rep}(\overline{Q}, \mathbf{d}) = \bigoplus_{a \in A} \mathbb{C}^{h(a) \times t(a)} \oplus \bigoplus_{a \in A} \mathbb{C}^{t(a) \times h(a)} . \tag{4.69}$$

The Lie algebra associated to $G_{\mathbf{d}}$ is

$$\mathrm{Lie}(G_{\mathbf{d}}) = \mathrm{Lie}\left(\prod_{i \in I} \mathrm{GL}_{d_i} \right) = \bigoplus_{i \in I} \mathfrak{gl}_{d_i} =: \mathfrak{gl}_{\mathbf{d}} \tag{4.70}$$

and the *moment map* μ is defined as

$$\mu : \mathrm{Rep}(\overline{Q}, \mathbf{d}) \to \mathfrak{gl}_{\mathbf{d}} \ , \ \ ((X_a)_{a \in A}, (Y_a)_{a \in A}) \mapsto \sum_{a \in A} X_a Y_a - Y_a X_a \tag{4.71}$$

for an element $((X_a)_{a \in A}, (Y_a)_{a \in A}) \in \mathrm{Rep}(\overline{Q}, \mathbf{d})$. We identify a parameter $\boldsymbol{\theta} \in \mathbb{C}^I$ with

$$(\theta_0 \cdot \mathrm{id}_{d_0}, ..., \theta_{\ell-1} \cdot \mathrm{id}_{d_{\ell-1}}) \in \mathfrak{gl}_{\mathbf{d}} \tag{4.72}$$

and therefore with a point in the codomain of μ. To get a better understanding of μ and its properties, we have compiled some of the results of [Gin12, Sec. 4.1-4.5] below.

LEMMA 4.6.6. *For a parameter $\boldsymbol{\theta} \in \mathbb{C}^I$, a quiver $Q = (I, A)$, and associated preprojective algebra $\Pi_{\boldsymbol{\theta}}$ we have*

(*i*) $\mu^{-1}(\boldsymbol{\theta}) = \mathrm{Rep}(\Pi_{\boldsymbol{\theta}}, \mathbf{d})$,
(*ii*) $\mathrm{Rep}(\Pi_{\boldsymbol{\theta}}, \mathbf{d}) = \emptyset$ *whenever* $\sum_{i \in I} \theta_i \cdot d_i \neq 0$,
(*iii*) *the fiber $\mu^{-1}(\boldsymbol{\theta})$ is a $G_{\mathbf{d}}$-stable subvariety of* $\mathrm{Rep}(\overline{Q}, \mathbf{d})$.

To abbreviate notation in the future, we define

$$\boldsymbol{\theta} \cdot \mathbf{d} := \sum_{i \in I} \theta_i \cdot d_i \ . \tag{4.73}$$

The last step in order to model Calogero–Moser space via the geometry associated to quivers is the introduction of framings.

DEFINITION 4.6.7. Given a quiver $Q = (I, A)$ and vector $\mathbf{w} = (w_i)_{i \in I} \in \mathbb{Z}^I_{\geq 0}$ we define the *framed* quiver $Q^{\mathbf{w}} = (I^{\mathbf{w}}, A^{\mathbf{w}})$ by $I^{\mathbf{w}} := I \cup \{\infty\}$ and we obtain $A^{\mathbf{w}}$ by adding w_i edges $i \to \infty$ to A.

EXAMPLE 4.6.8. For the cyclic quiver on ℓ vertices given in Example 4.6.2(*iii*) and $\mathbf{w} = (w_i)_{i \in I} = (1, 0, ..., 0) \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}_{\geq 0}$ we get the framed quiver displayed in Figure 4.9. This is the main quiver whose corresponding quotient variety we will be interested in.

To a dimension vector $\mathbf{d}$ of a quiver Q we can naturally associate a *framed dimension vector* $\hat{\mathbf{d}} = (\hat{d}_i)_{i \in \hat{I}}$ of the framed quiver $Q^{\mathbf{w}}$ with framing vector $\mathbf{w}$ by setting

$$\hat{d}_i = d_i \text{ for all } i \in I \text{ and } \hat{d}_\infty = 1 \ . \tag{4.74}$$

Furthermore, define the *framed parameter* $\hat{\boldsymbol{\theta}} = (\hat{\theta}_i)_{i \in \hat{I}} \in \mathbb{C}^{\hat{I}}$ by

$$\hat{\theta}_i = \theta_i \text{ for all } i \in I \text{ and } \hat{\theta}_\infty = 0 \ . \tag{4.75}$$

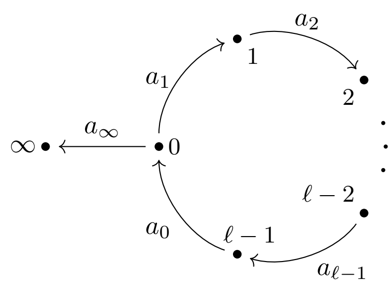


FIGURE 4.9. The framed cyclic quiver on ℓ vertices with $\mathbf{w} = (1, 0, ..., 0)$.

We can still have $G_{\mathbf{d}}$ act on $\mathrm{Rep}(Q^{\mathbf{w}}, \hat{\mathbf{d}})$ by fixing some basis vector inside the 1-dimensional space V_∞ corresponding to the ∞-vertex, and letting $G_{\mathbf{d}}$ act trivially on V_∞.

There is also an analogous construction of the cotangent bundle and the moment map for framed quivers (see [CB01], [Gin12, Sec. 5.1] for more details). We denote by $\overline{Q^{\mathbf{w}}}$ the doubling of the framing of Q and write as before

$$\begin{aligned} \mathrm{Rep}(\overline{Q^{\mathbf{w}}}, \hat{\mathbf{d}}) &= T^*(\mathrm{Rep}(Q^{\mathbf{w}}, \hat{\mathbf{d}})) \\ &= \mathrm{Rep}(Q, \mathbf{d}) \times \mathrm{Rep}(Q, \mathbf{d})^* \\ &\quad \times \prod_{i \in I} (\mathrm{Hom}_{\mathbb{C}I}(V_i, \mathbb{C}))^{w_i} \times \prod_{i \in I} (\mathrm{Hom}_{\mathbb{C}I}(\mathbb{C}, V_i))^{w_i} \end{aligned} \tag{4.76}$$

with an element

$$\mathbf{x} := ((X_a)_{a \in A}, (Y_a)_{a \in A}, (x_i)_{i \in I}, (y_i)_{i \in I}) \in \mathrm{Rep}(\overline{Q^{\mathbf{w}}}, \hat{\mathbf{d}}) \tag{4.77}$$

on which $G_{\mathbf{d}} = \prod_{i \in I} \mathrm{GL}_{d_i}$ acts by

$$\left(\prod_{i \in I} g_i\right) \cdot \mathbf{x} = ((g_{h(a)} X g_{t(a)}^{-1})_{a \in A}, (g_{t(a)} Y g_{h(a)}^{-1})_{a \in A}, (g_i \cdot x_i)_{i \in I}, (y_i \cdot g_i^{-1})_{i \in I}) \ . \tag{4.78}$$

We also get a *framed* moment map

$$\hat{\mu} : \mathrm{Rep}(\overline{Q^{\mathbf{w}}}, \hat{\mathbf{d}}) \to \mathfrak{gl}_{\mathbf{d}} \ , \mathbf{x} \mapsto \sum_{a \in A} X_a Y_a - Y_a X_a + \sum_{i \in I} x_i y_i \ . \tag{4.79}$$

As previously, we have

$$\hat{\mu}^{-1}(\hat{\boldsymbol{\theta}}) = \mathrm{Rep}(\Pi_{\hat{\boldsymbol{\theta}}}, \hat{\mathbf{d}}) \tag{4.80}$$

where $\Pi_{\hat{\boldsymbol{\theta}}}$ is the preprojective algebra of the quiver $Q^{\mathbf{w}}$.

DEFINITION 4.6.9. Let $\ell, n \in \mathbb{N}$. Denote by Q the cyclic quiver on ℓ vertices. For $\mathbf{d} \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$ and $\mathbf{w} = (1, 0, ..., 0) \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$, we get a framed moment map $\hat{\mu}$. We then define the *Marsden–Weinstein reduction*

$$\mathcal{X}_{\boldsymbol{\theta}}(\mathbf{d}) := \hat{\mu}^{-1}(\hat{\boldsymbol{\theta}}) // G_{\mathbf{d}} \ . \tag{4.81}$$

Fix some $\ell, n \in \mathbb{N}$ and let $\boldsymbol{\theta} \in \mathbb{C}^{\mathbb{Z}/\ell\mathbb{Z}}$ be a parameter. Let from now on $\mathbf{w}$ denote the framing vector $(1, 0, ..., 0) \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$ and let Q denote the cyclic quiver on ℓ vertices. We have a symplectic structure on $\mathrm{Rep}(\overline{Q^{\mathbf{w}}}, \hat{\mathbf{d}})$ through its identification with a cotangent bundle and then $G_{\mathbf{d}}$ acts via symplectic automorphisms on $\mathrm{Rep}(\overline{Q^{\mathbf{w}}}, \hat{\mathbf{d}})$. This way, we also have a Poisson structure on $\mathcal{X}_{\boldsymbol{\theta}}(n)$. For the details, we refer the reader to [Mar06, Ch. 2 Prop. 2.20].

Another important property of $\mathcal{X}_{\boldsymbol{\theta}}(\mathbf{d})$ is that it admits a $\mathbb{C}^*$-action by letting $z \in \mathbb{C}^*$ act on $\mathbf{x} \in \mathrm{Rep}(\overline{Q^{\mathbf{w}}}, \hat{\mathbf{d}})$ via

$$z \cdot \mathbf{x} = ((z^{-1} \cdot X_a)_{a \in A}, (z \cdot Y_a)_{a \in A}, (x_i)_{i \in I}, (y_i)_{i \in I}) \tag{4.82}$$

which commutes with the action of $G_{\mathbf{d}}$. Because the moment map is constant on $\mathbb{C}^*$-orbits, we get an induced action on $\mathcal{X}_{\boldsymbol{\theta}}(\mathbf{d})$.

Let $\mathsf{X}_{\mathbf{H}}$ denote the Calogero–Moser space associated to the complex reflection group of type $G(\ell, 1, n)$ at parameter $\mathbf{H} = (h, H_1, ..., H_{\ell-1})$ given in the Gordon basis. We define

$$\boldsymbol{\theta}_{\mathbf{H}} := (-h + \sum_{i=1}^{\ell-1} H_i,\ H_1, ..., H_{\ell-1}) \tag{4.83}$$

and let $\mathcal{X}_{\boldsymbol{\theta}_{\mathbf{H}}}(n)$ denote the quiver variety given by Definition 4.6.9 associated to ℓ, n and the dimension vector $(n, ..., n) \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}/\ell\mathbb{Z}}$. We are now able to state the remarkable result by Gordon which links quiver varieties and Calogero–Moser spaces compiling results of [EG02] and [Mar08].

THEOREM 4.6.10 ([Gor08a, Thm. 3.10]). *There exists a $\mathbb{C}^*$-equivariant isomorphism of Poisson varieties*

$$\mathsf{X}_{\mathbf{H}} \xrightarrow{\sim} \mathcal{X}_{\boldsymbol{\theta}_{\mathbf{H}}}(n)\ . \tag{4.84}$$

This isomorphic property allows us to describe the points and leaves of the Calogero–Moser space by using the explicit nature of quiver representations. We will start by reiterating some general theory on quotients of moment map fibers. For more details, we refer to the concise account found in [BS21].

Let for this paragraph $\mathcal{X}$ be any variety of the form $\hat{\mu}^{-1}(\hat{\boldsymbol{\theta}})/\!/G_{\mathbf{d}}$ for some quiver Q, dimension vector $\mathbf{d}$, framing $\mathbf{w}$, and parameter $\boldsymbol{\theta}$. A closed point $\mathbf{y} \in \mathcal{X}$ corresponds to an isomorphism class of semisimple representations of $\Pi_{\hat{\boldsymbol{\theta}}}$ with dimension vector $\hat{\mathbf{d}}$. The fiber of $\mathbf{y}$ with respect to the quotient map

$$\Omega : \hat{\mu}^{-1}(\hat{\boldsymbol{\theta}}) \to \mathcal{X} \tag{4.85}$$

consists of a single closed $G_{\mathbf{d}}$-orbit in $\mathrm{Rep}(\Pi_{\hat{\boldsymbol{\theta}}}, \hat{\mathbf{d}})$. For a non-closed point $\mathbf{y}' \in \mathcal{X}$, we still have a unique closed $G_{\mathbf{d}}$-orbit in $\Omega^{-1}(\mathbf{y}')$ and any point in a closed $G_{\mathbf{d}}$-orbit of $\mathrm{Rep}(\Pi_{\hat{\boldsymbol{\theta}}}, \hat{\mathbf{d}})$ is semisimple. This means we can associate to a point $\mathbf{y} \in \mathcal{X}$ a semisimple representation

$$\mathbf{x}(\mathbf{y}) = \bigoplus_{i=1}^{k} \mathbf{x}_i^{\oplus e_i} \in \mathrm{Rep}(\Pi_{\boldsymbol{\theta}}, \mathbf{d}) \tag{4.86}$$

where the $\mathbf{x}_i$ are simple and $\mathbf{x}_i \ncong \mathbf{x}_j$ for $i \neq j$. Using (4.86), we can now define the *dimension type* $\tau(\mathbf{y})$ of $\mathbf{y}$ by

$$\tau(\mathbf{y}) := (e_1, \dim_I \mathbf{x}_1; e_2, \dim_I \mathbf{x}_2; ...; e_k \dim_I \mathbf{x}_k)\ . \tag{4.87}$$

The dimension type affords a stratification of $\mathcal{X}$, i.e. we have

$$\mathcal{X} = \coprod_{\tau} \mathcal{C}_\tau \tag{4.88}$$

where $\mathcal{C}_\tau$ denotes the set of points of $\mathcal{X}$ with dimension type τ. Bellamy and Schedler then proved this very useful fact.

PROPOSITION 4.6.11 ([BS21, Prop. 3.6]). *The connected components of the dimension type strata of $\mathcal{X}$ are precisely the symplectic leaves of $\mathcal{X}$.*

In [Mak22, Sec. 3.F.], this result has been applied to the case $\mathcal{X}_{\boldsymbol{\theta}}(n)$ associated to the double framed cyclic quiver $\overline{Q^{\mathbf{w}}}$ in Definition 4.6.9: each representation

$$M \in \hat{\mu}^{-1}(\hat{\boldsymbol{\theta}}) \subseteq \mathrm{Rep}(\overline{Q^{\mathbf{w}}}, \mathbf{d}) \tag{4.89}$$

has a unique decomposition $M = M_\infty \oplus M_1$ such that

$$M_\infty \in \hat{\mu}^{-1}(\hat{\boldsymbol{\theta}}) \subseteq \mathrm{Rep}(\overline{Q^{\mathbf{w}}}, \hat{\mathbf{d}}') \ , \ M_1 \in \mu^{-1}(\boldsymbol{\theta}) \subseteq \mathrm{Rep}(\overline{Q}, \mathbf{d}'') \tag{4.90}$$

for some dimension vectors $\mathbf{d}', \mathbf{d}'' \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$, and M_∞ indecomposable. We define

$$\dim_I^{\mathrm{reg}}(M) := \mathbf{d}' \ . \tag{4.91}$$

The dimension vector $\dim_I^{\mathrm{reg}}$ affords a stratification of $\mathcal{X}_{\boldsymbol{\theta}}(n)$ as well. Maksimau used Proposition 4.6.11 to prove the following.

LEMMA 4.6.12 ([Mak22, Lem. 3.14]). *The stratification of $\mathsf{X}_{\mathbf{H}}$ by symplectic leaves is equal to the stratification of $\mathcal{X}_{\boldsymbol{\theta}_{\mathbf{H}}}(n)$ by dimension vectors, i.e. two points $M_1, M_2 \in \mathcal{X}_{\boldsymbol{\theta}_{\mathbf{H}}}(n)$ are on the same symplectic leaf if and only if $\dim_I^{\mathrm{reg}}(M_1) = \dim_I^{\mathrm{reg}}(M_2)$.*

With the above statement and the $\mathbb{C}^*$-action on $\mathcal{X}_{\boldsymbol{\theta}}(n)$ from (4.82) we will be able to review in the next chapter the combinatorial constructions [Gor08a], [Mak22] used to parametrize both the fixed points and the symplectic leaves of the Calogero–Moser space $\mathsf{X}_{\mathbf{H}}$.

CHAPTER 5

Abacus combinatorics

Let W be a complex reflection group of type $G(\ell, 1, n)$ for some positive integers ℓ, n, and let $\mathbf{c}$ be some Calogero–Moser parameter. Whenever the Calogero–Moser space $\mathsf{X}_{\mathbf{c}} := \mathsf{X}_{\mathbf{c}}(W)$ is *smooth*, we have seen in Section 4.4 that the $\mathbb{C}^*$-fixed points of $\mathsf{X}_{\mathbf{c}}$ are in bijection with ℓ-multipartitions since the Calogero–Moser families are all singleton. By the action given (4.82), we obtain another parametrization of the $\mathbb{C}^*$-fixed points by certain representations of a framed double cyclic quiver. Since this space of quiver representations is dependent on some parameter $\boldsymbol{\theta} \in \mathbb{C}^{\mathbb{Z}/\ell\mathbb{Z}}$, we would like the bijective map between these quiver representations and the ℓ-multipartitions to be $\boldsymbol{\theta}$-dependent as well. Lastly, we would like to be able to "degenerate" this map to *singular* Calogero–Moser spaces to model the join of Calogero–Moser families combinatorially (cf. Sec. 4.1). All of this has been achieved in [Gor08a; GM09; Mar10; Mak22] using the *ℓ-quotient map* which is best visualized using the combinatorics of *abaci*.

This chapter is devoted to the review and study of the ℓ-quotient map which we will use in conjunction with recent results by Maksimau [Mak22] to give a partial answer to Questions 4.5.5 and thus to its more general versions Question 4.0.1 and Question 1.3.5. The Sections 5.1–5.6 are comprised of reviews of preliminary works, while Sections 5.7–5.10 consist of original work which applies the combinatorics developed in the preliminary sections to define cuspidal family induction and discuss cuspidal families and rigid modules in type $G(\ell, 1, n)$. Sections 5.11–5.13 review some combinatorial theories in the case of Weyl groups, Coxeter groups, and complex reflection groups such as quantum groups, Fock spaces, and cellular characters. The aim is to compare these theories with the constructions of this chapter.

5.1. Introduction to abaci

The two combinatorial structures we will describe in this section are called *β-numbers* and *abaci*. They have long been used to study the character theory of the symmetric group, see for example [JK81] for an in-depth discussion. Both β-numbers and abaci appeared again in the study of the quiver varieties [Hai03; Gor08a; Prz20] which is why we are interested in them now.

Fix some nonnegative integers ℓ, n and denote by $\mathscr{P}(\ell, n)$ the set of ℓ-multipartitions of n (cf. Sec. 2.1). Furthermore, denote by

$$\mathscr{P} = \bigcup_{n \geq 0} \mathscr{P}(n) \tag{5.1}$$

the set of all partitions. For $\lambda \in \mathscr{P}$ and a position $u = (i, j) \in \lambda$, the *ℓ-residue* of u is given by $j - i \mod \ell$. This gives us a map

$$\mathrm{Res}_\ell : \mathscr{P} \to \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \tag{5.2}$$

where the entry of $\mathrm{Res}_\ell(\lambda)$ associated to $j \in \mathbb{Z}/\ell\mathbb{Z}$ is equal to the number of positions of λ with ℓ-residue j. We have given the 3-residues of the partition $\lambda = (4, 2, 2, 1)$ in Figure 5.1. To get better access to the ℓ-residue information of a partition λ,

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 \\ \cline{1-2} 1 & 2 \\ \cline{1-2} 0 \\ \cline{1-1}\end{array}$$

$$\mathrm{Res}_3((4,2,2,1)) = (4,2,3)$$

FIGURE 5.1. The 3-residues of $\lambda = (4,2,2,1)$.

one uses β-numbers (cf. [JK81, Sec. 2.7]).

DEFINITION 5.1.1. A strictly decreasing, infinite sequence of integers

$$\beta = (\beta_1, \beta_2, ...) \tag{5.3}$$

is called *sequence of β-numbers* if the entries of β only decrease by 1 after a certain point, i.e. there exists an index K such that

$$\beta_i - \beta_{i+1} = 1 \text{ for all } i \geq K \ . \tag{5.4}$$

Condition (5.4) basically says that the infinite sequence β only contains finite information. Equivalently, we can write the entries of β as

$$\beta_i = r + 1 - i \quad \text{for } i \geq K \tag{5.5}$$

for some $r \in \mathbb{Z}$, $K \in \mathbb{Z}_{\geq 1}$. We then say *$\beta$ stabilizes with respect to r*. This makes it possible to "isolate" for each sequence of β-numbers an underlying sequence $\underline{r} = (r, r-1, r-2, ...)$ for some $r \in \mathbb{Z}$, and decompose β into a component-wise sum of two sequences

$$\beta = \underline{r} + \lambda \ . \tag{5.6}$$

It is not difficult to see that the sequence λ is now a unique partition with trailing 0's. We summarize this result in the following proposition.

PROPOSITION 5.1.2. *Denote by $\mathscr{B}$ the set of all sequences of β-numbers. We then have a bijection*

$$\mathscr{P} \times \mathbb{Z} \to \mathscr{B} \tag{5.7}$$

by defining for $\lambda \in \mathscr{P}$, $r \in \mathbb{Z}$ the sequence

$$\beta_i = r + 1 - i + \lambda_i \tag{5.8}$$

for $i \geq 1$.

We call two sequences of β-numbers β, β' *equivalent* if they afford the same partition, i.e. the sequence $(\beta_i - \beta'_i)_{i \in \mathbb{Z}}$ is constant. We give some examples of sequences of β-numbers and partitions next.

EXAMPLE 5.1.3.

(1) The sequence of β-numbers of $\lambda = (4,2,1,1)$ stabilizing with respect to 0 is equal to $(4, 1, 0, -2, -4, -5, ...)$.
(2) For $\lambda = (\lambda_1, ..., \lambda_k)$ denote by $h(i,j)$ the hook-length of the box (i,j) (cf. Defn. 2.1.5). The sequence

$$\beta = (h(1,1), h(2,1), ..., h(k,1), -1, -2, ...) \tag{5.9}$$

is the sequence of β-numbers of λ stabilizing with respect to $k-1$. This follows from

$$\beta_i = h(i,1) = (k-1) + 1 - i + \lambda_i \ . \tag{5.10}$$

The following lemma connects sequences of β-numbers and ℓ-residues (cf. [JK81, Lem. 2.7.38]).

LEMMA 5.1.4. *Let β be the sequence of β-numbers of $\lambda \in \mathscr{P}$ stabilizing with respect to $r \in \mathbb{Z}$. Then, we have that the rightmost box of λ_i has ℓ-residue $\beta_i - 1 - r \mod \ell$.*

PROOF. We can see this inductively. For $i = 1$, the rightmost box of λ_1 has ℓ-residue $\lambda_1 - 1 \mod \ell$ which is equal to

$$\beta_1 - r - 1 \mod \ell\ . \tag{5.11}$$

Now, if the rightmost box of λ_i has ℓ-residue k, then the rightmost box of λ_{i+1} has ℓ-residue

$$k - (\lambda_i - \lambda_{i+1}) - 1 \mod \ell\ . \tag{5.12}$$

Expressing (5.12) with β, we get that the ℓ-residue of λ_{i+1} is equal to

$$(\beta_i - 1 - r) - \Big((\beta_i - r + i - 1) - (\beta_{i+1} - r + (i+1) - 1)\Big) - 1 \equiv \beta_{i+1} - r - 1\ . \tag{5.13}$$

■

EXAMPLE 5.1.5. Take $\lambda = (4, 2, 2, 1)$ with sequence of β-numbers

$$\beta = (3, 0, -1, -3, -5, -6, ...) \tag{5.14}$$

stabilizing with respect to -1. We obtain

$$(\beta_i \mod \ell)_{i \geq 1} = (0, 0, 2, 0, 1, 0, ...) \tag{5.15}$$

which are the ℓ-residues of the rightmost boxes in Figure 5.1.

In order to work with partitions, we use Young diagrams to display them. For sequences of β-numbers, one uses *abaci*. We will review their construction next (cf. [JK81, Sec. 2.7]). Start by listing the elements of $\mathbb{Z}$ into ℓ columns of infinite length labelled $0, ..., \ell - 1$ such that column k contains the integers congruent to $k + 1$ modulo ℓ. We identify the column labels with the elements of $\mathbb{Z}/\ell\mathbb{Z}$ as well. For $\ell = 3$, we get

$$\begin{array}{ccc} 0 & 1 & 2 \\ \hline \vdots & \vdots & \vdots \\ -8 & -7 & -6 \\ -5 & -4 & -3 \\ -2 & -1 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ \vdots & \vdots & \vdots \end{array}\ . \tag{5.16}$$

For a sequence of β-numbers β we can now mark the entries of β in this list of integers to obtain the *ℓ-abacus* (or *ℓ-bead diagram*) of β. Take as an example the sequence of β-numbers

$$\beta = (4, 1, 0, -2, -4, -5, ...) \tag{5.17}$$

which corresponds to $\lambda = (4,2,2,1)$ and stabilizes with respect to 0. The corresponding 3-abacus is then given by

$$\begin{array}{ccc} 0 & 1 & 2 \\ \hline \vdots & \vdots & \vdots \\ -8 & -7 & -6 \\ -5 & -4 & -3 \\ -2 & -1 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ \vdots & \vdots & \vdots \end{array} \tag{5.18}$$

where the dots indicate fully marked rows on top and fully unmarked rows on the bottom.

The marked integers are called *beads*. A bead is called *active* if there exists a smaller integer that is unmarked. For marked integers a, b and an unmarked integer c with $a < c < b$, the position of the unmarked integer c is called a *gap* of the abacus. In our example (5.18), the beads of $4, 1, 0$ and -2 are active while $3, 2, -1$ and -3 are gap positions.

Fix a partition λ with sequence of β-numbers β stabilizing with respect to $r \in \mathbb{Z}$. For each entry of the partition λ, there exists a corresponding active bead in any abacus of β. Because the column k of an ℓ-abacus contains all integers congruent to $k+1 \mod \ell$, we have that the active bead of a row λ_i is in column $k + r \mod \ell$ where k is the ℓ-residue of the rightmost box of λ_i. This follows from Lemma 5.1.4.

To reobtain the partition λ from an abacus, we count for each active bead the number of gap positions that come before it, and arrange those counts into a weakly decreasing sequence. This process yields the same partition no matter the stabilizing number r.

We call two ℓ-abaci *ℓ-equivalent* if their respective sequences of β-numbers β, β' are equivalent and the stabilizing numbers of β and β' are congruent modulo ℓ. Because we will be interested in ℓ-residues of boxes of λ, we will work with equivalence classes of ℓ-abaci and replace marked integers with black circles and unmarked integers with white circles. This means, our stabilizing numbers can only be determined up to ℓ-congruence from such an abacus. The 3-abacus in (5.18) transforms into

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ \\ \bullet & \circ & \bullet \\ \bullet & \circ & \circ \\ \bullet & \circ & \circ \end{array} \tag{5.19}$$

where we left out the dots and column labels. We will often identify an abacus with its respective sequence of β-numbers. In the sections to come, Abaci will help us visualize a number of operations which we need to perform on partitions.

5.2. ℓ-core and ℓ-quotient map

To define the *ℓ-core* of a partition, we must first introduce skew shapes. Let $\lambda, \mu \in \mathscr{P}$ be two partitions with $\mu \subseteq \lambda$. We can now construct the *skew shape* λ/μ which is the collection of coordinates $(i,j) \in \lambda$ such that $(i,j) \notin \mu$. The *skew diagram* of λ/μ is obtained from the Young diagram of λ by removing the boxes of μ. We will identify shapes and diagrams the same way we did in the classical case. For $\lambda = (4,2,2,1)$ and $\mu = (2,2)$, we have displayed the skew diagram of λ/μ in Figure 5.2. We call a skew shape λ/μ *connected* if λ/μ cannot be obtained

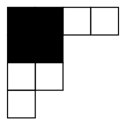


FIGURE 5.2. The skew diagram of $(4,2,2,1)/(2,2)$.

as a union of two disjoint nonempty skew shapes. Equivalently, any two boxes in a connected skew diagram can be connected by a path crossing only the sides of boxes in the skew shape. As a non-example, the skew diagram displayed in Figure 5.2 is not connected. Finally, a *rim ℓ-hook* λ/μ is a connected skew shape of size ℓ that contains no 2×2-square. For example, the bottom three boxes of the skew diagram in Figure 5.2 form the rim 3-hook $(4,2,2,1)/(4,2)$.

DEFINITION 5.2.1 ([JK81, Sec. 2.7]). The *ℓ-core* of a partition $\lambda \in \mathscr{P}$ is the partition obtained from λ by successively removing as many rim ℓ-hooks as possible. We denote this new partition by $\mathrm{Core}_\ell(\lambda)$ and we say $\nu \in \mathscr{P}$ is an ℓ-core whenever $\mathrm{Core}_\ell(\nu) = \nu$.

At first glance, one should wonder if the map $\mathrm{Core}_\ell : \mathscr{P} \to \mathscr{P}$ is well-defined for any choice of sequence of rim ℓ-hooks. However, this becomes easier to answer when translating the definition of ℓ-cores to the language of abaci.

PROPOSITION 5.2.2. *Let $\lambda \in \mathscr{P}$. We obtain an ℓ-abacus of* $\mathrm{Core}_\ell(\lambda)$ *from an ℓ-abacus of λ by iteratively moving active beads upwards as high as possible. As a consequence, the map* Core_ℓ *is well-defined.*

PROOF. First, we need to see that moving an active bead upwards to a gap position is equivalent to the removal of a rim ℓ-hook. We can view the process of upwards movement also in terms of the active beads corresponding to integers between the active bead and the gap position. These beads in between are moved successively to the position that was previously left vacant. We can now view the one-step operation of moving beads upwards as displayed in Figure 5.3 instead as the multi-step operation displayed in Figure 5.4 where we move beads to the left looping around to the row above.

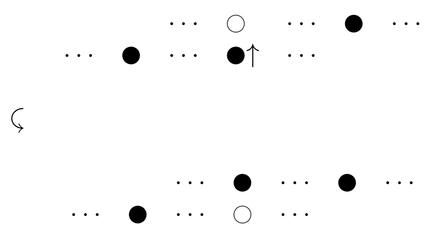


FIGURE 5.3. The one-step ℓ-core operation.

The operation in Figure 5.4 removes a skew shape α. It is not difficult to see that α is a rim ℓ-hook: we moved past all of the columns exactly once, meaning we have removed one box for each ℓ-residue $0, ..., \ell - 1$. Therefore, α does not contain a 2×2-square. The skew shape α is furthermore connected because we move each active bead to a position previously occupied by an active bead. Also, any removal of a rim ℓ-hook is given by such an operation.

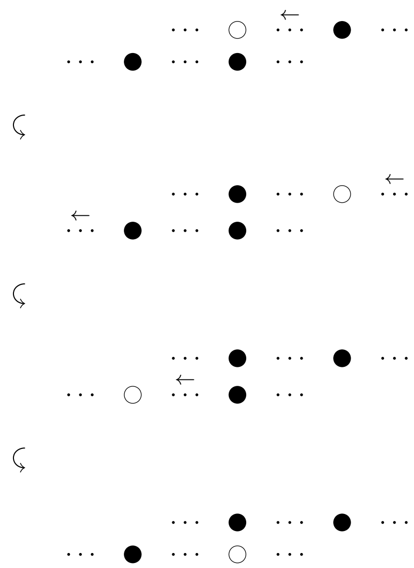


FIGURE 5.4. The multi-step ℓ-core operation.

Now, since the ℓ-core is obtained by doing all possible upward movements of beads and the resulting abacus is independent of the order of these movements, the partition corresponding to the abacus is independent of the order as well. Thus, we have that Core_ℓ is well-defined.

■

We refer the reader to [JK81, Sec. 2.7] for a different proof of the above proposition. Next, we want to give an example of the operation in the proof of Proposition 5.2.2.

EXAMPLE 5.2.3. The following construction is illustrated in Figure 5.5. We take the 3-abacus of $(4,2,2,1)$ with sequence of β-numbers

$$\beta = (4, 1, 0, -2, -4, -5, ...) \tag{5.20}$$

stabilizing with respect to 0. This abacus has an active bead corresponding to 0 that can be moved upwards to position -3. Instead of doing that, we can move the -2 bead to position -3, then the 0 bead to position -2, removing first a box of residue 0 from the row corresponding to $-2 \in \beta$, and then two boxes of residues $1, 2$ from the row corresponding to $0 \in \beta$. The resulting ℓ-core is $(4, 2)$ with sequence of β-numbers

$$\beta' = (4, 1, -2, -3, -4, -5, ...)\ . \tag{5.21}$$

Note that β' still stabilizes with respect to 0 because taking ℓ-cores does not change the stabilization number.

As a result of Proposition 5.2.2, we get a useful abacus characterization of ℓ-cores.

COROLLARY 5.2.4. *A partition λ is an ℓ-core if and only if the ℓ-abaci of λ do not have gaps above active beads.*

We want to use Corollary 5.2.4 to give a different well-known parametrization of the set of ℓ-cores that we will use for the ℓ-quotient map.

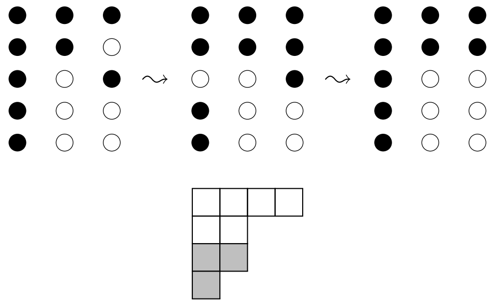


FIGURE 5.5. A visualization of $\mathrm{Core}_3((4, 2, 2, 1))$ with grey boxes for the rim 3-hook.

Take an ℓ-core $\nu \in \mathscr{P}$ with sequence of β-numbers β stabilizing with respect to $r \in \mathbb{Z}/\ell\mathbb{Z}$. We can move the beads in the ℓ-abacus of β by reducing the entries of β as much as possible, ending up with the abacus of the empty partition $\emptyset \in \mathscr{P}$ (see Figure 5.6 for an example). The sequence of β-numbers of that abacus still stabilizes with respect to r, i.e. the biggest marked integer has residue r and is therefore in column $r-1$. The resulting abacus is displayed in Figure 5.7.

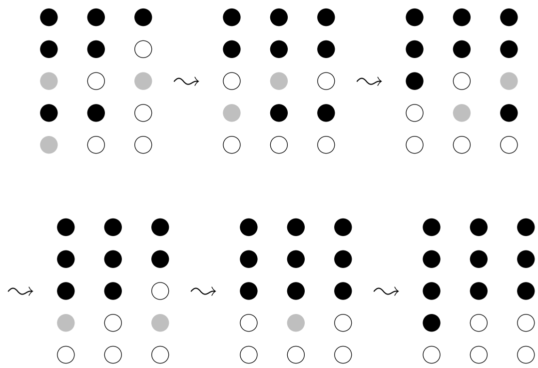


FIGURE 5.6. Shifting beads to construct the abacus of the empty partition $\emptyset$ (moving beads are highlighted).

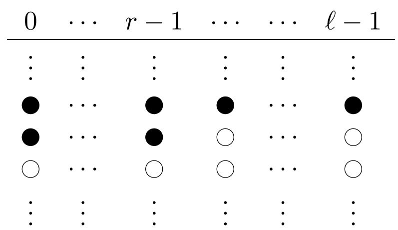


FIGURE 5.7. The ℓ-abacus of $\emptyset$ stabilizing with respect to r.

Next, we label the rows of an abacus by $\mathbb{Z}$ using the integer grid underlying the abacus. The row containing 0 is labeled with $0 \in \mathbb{Z}$ and we count labels in the direction of the integers in the abacus, i.e. the row containing $\ell \cdot i$ is labeled by

$i \in \mathbb{Z}$. This means, we can associate to an ℓ-abacus of an ℓ-core ν a list of integers

$$\mathbf{s} := (s_0, ..., s_{\ell-1}) \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \tag{5.22}$$

called *ℓ-charge* (or simply *charge*) where s_k is the row index of the biggest marked integer in column k. Because $\mathbf{s}$ depends on the stabilizing number r and we have chosen r to be a class modulo $\ell\mathbb{Z}$, the vector $\mathbf{s}$ is only determined modulo $\delta\mathbb{Z}$ where

$$\delta := (1, ..., 1) \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \ . \tag{5.23}$$

Therefore, we will choose the unique representative such that

$$\sum_{k=0}^{\ell-1} s_k \in \{0, ..., \ell-1\} \ , \tag{5.24}$$

where we identify the set $\{0, ..., \ell-1\}$ with $\mathbb{Z}/\ell\mathbb{Z}$. When we now reduce the ℓ-core ν to $\emptyset \in \mathscr{P}$ as before, we see that the active beads are simply repositioned and $\sum_{k=0}^{\ell-1} s_k$ stays invariant. This can be observed from the ℓ-core displayed in Figure 5.5. For the ℓ-core $\emptyset \in \mathscr{P}$ stabilizing with respect to $r \in \mathbb{Z}/\ell\mathbb{Z}$, we have

$$\sum_{k=0}^{\ell-1} s_k = r \ . \tag{5.25}$$

Since both sides of (5.25) are the same for the abaci $\emptyset$ and ν, we have proven the following statement. Denote by $\mathbb{Z}_0^{\mathbb{Z}/\ell\mathbb{Z}}$ all elements of $\mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$ that sum up to $0 \in \mathbb{Z}$.

Lemma 5.2.5. *There is a bijection*

$$\{\nu \mid \nu \textit{ is an } \ell\textit{-core}\} \longleftrightarrow \mathbb{Z}_0^{\mathbb{Z}/\ell\mathbb{Z}} \tag{5.26}$$

given by the charge of the abacus of an ℓ-core stabilizing with respect to 0.

Before we continue with a discussion of the ℓ-quotient map, we want to briefly record a useful characterization of the abaci of a transposed partition.

Lemma 5.2.6. *Let $\lambda \in \mathscr{P}$ be a partition with transpose ${}^t\lambda$. We obtain an ℓ-abacus of ${}^t\lambda$ from the ℓ-abacus of λ by*

1. *inverting the color of the positions, i.e. marked positions of the abacus become unmarked and vice versa,*
2. *reversing the order of the columns,*
3. *mirroring the abacus horizontally such that the marked integers end up on top again.*

The ℓ-abacus of ${}^t\lambda$ constructed this way has ℓ-core ${}^t\nu$ where ν is the ℓ-core of the ℓ-abacus of λ. The charge of ${}^t\nu$ is given by $\mathrm{rev}(-\mathbf{s})$ *where* $\mathbf{s}$ *is the charge of ν and* rev *denotes the reverse of a vector.*

The proof of the above lemma is straight-forward and we have illustrated an example of the operation in Figure 5.8.

Now that we have gotten familiar with some the behavior of abaci, we can construct the ℓ-quotient map (cf. [JK81, Sec. 2.7], [LM02]). Start by taking an ℓ-abacus and viewing it as a tuple of ℓ columns each forming a 1-abacus. The k^{th} column then gives us a partition $\lambda^{(k)}$ and we therefore obtain a map

$$\mathscr{A}_\ell \to \bigcup_{k \geq 0} \mathscr{P}(\ell, k) \tag{5.27}$$

where $\mathscr{A}_\ell$ denotes the set of all ℓ-abaci modulo ℓ-equivalence. For our previous example, we get

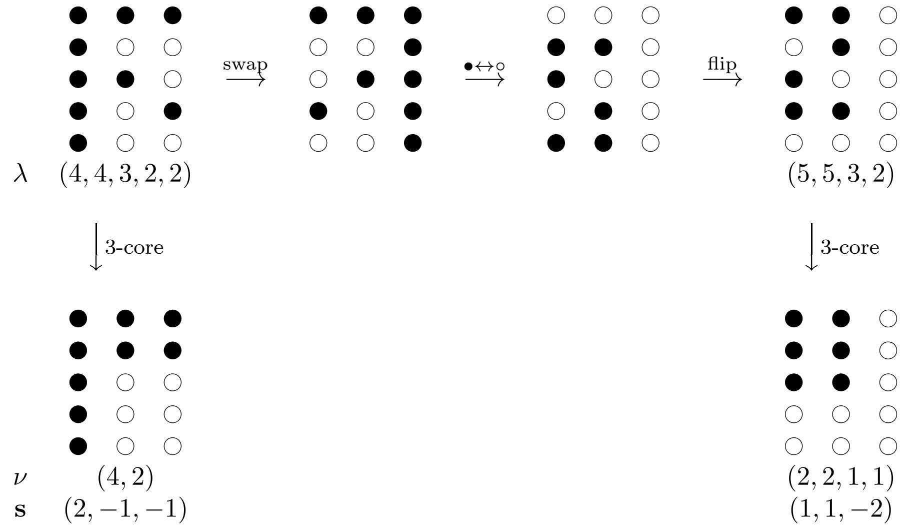


FIGURE 5.8. Abacus operations for transposing a partition λ with 3-core ν and charge $\mathbf{s}$.

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ \\ \bullet & \circ & \bullet \\ \bullet & \circ & \circ \\ \bullet & \circ & \circ \end{array} \rightarrow \begin{array}{c|c|c} \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ \\ \bullet & \circ & \bullet \\ \bullet & \circ & \circ \\ \bullet & \circ & \circ \end{array} \rightarrow (\emptyset, \emptyset, (1)) \ . \tag{5.28}$$

We want to define this map not on abaci but on the partitions of the underlying sequences of β-numbers. Because two equivalent sequences of β-numbers can give two different multipartitions, we restrict ourselves to sequences of β-numbers of ℓ-abaci stabilizing with respect to 0 for the remainder of this section. This restriction gives us a map

$$\mathscr{P} \rightarrow \bigcup_{k \geq 0} \mathscr{P}(\ell, k) \ . \tag{5.29}$$

The map in (5.29) is obviously surjective but it is not injective since the columns of the 1-abacus columns can be shifted downwards or upwards without changing the multipartition image. Let now $\lambda, \lambda' \in \mathscr{P}$ be two partitions that map onto the same ℓ-multipartition. When comparing the respective ℓ-abaci of λ, λ', we can readily see that λ, λ' are uniquely determined by their respective ℓ-core. This is because the ℓ-cores encode the downward and upward shift of 1-abacus columns. Thus, when we fix an ℓ-core ν, the map (5.29) becomes bijective. Denote by $\mathscr{P}_\nu$ the set of partitions which have ℓ-core ν. We thus have a bijective map

$$\mathscr{P}_\nu \leftrightarrow \bigcup_{k \geq 0} \mathscr{P}(\ell, k) \tag{5.30}$$

for any ℓ-core ν. If we let ν vary between all possible ℓ-cores, we can use Lemma 5.2.5 to get the bijection

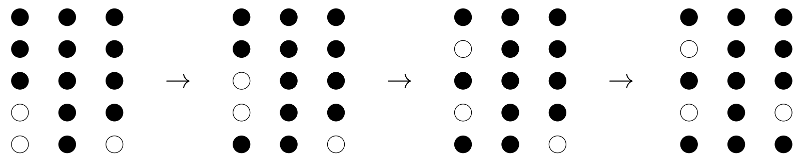
$$\mathscr{P} \leftrightarrow \mathbb{Z}_0^\ell \times \bigcup_{k \geq 0} \mathscr{P}(\ell, k) \ . \tag{5.31}$$

This map is called the *ℓ-quotient map* and the ℓ-multipartition in the image of $\lambda \in \mathscr{P}$ is called the *ℓ-quotient* $\mathrm{Quot}_\ell(\lambda)$ of λ. The ℓ-quotient map associates to each partition λ an ℓ-core $\mathrm{Core}_\ell(\lambda)$ and an ℓ-quotient $\mathrm{Quot}_\ell(\lambda)$ which together determine λ uniquely (cf. [JK81, Thm. 2.7.30]).

When constructing the abacus of a partition whose ℓ-core ν and ℓ-quotient

$$\boldsymbol{\lambda} = (\lambda^{(0)}, ..., \lambda^{(\ell-1)}) \tag{5.32}$$

are known, one can start with the ℓ-abacus of the empty ℓ-multipartition $\boldsymbol{\emptyset}$ with the ℓ-charge associated to ν. Then, move the bottommost bead in column k down $\lambda_1^{(k)}$ rows. The next bead in column k is moved down $\lambda_2^{(k)}$ rows and so on for all $\lambda_i^{(k)}$ and all $k \in \{1, ..., \ell-1\}$. We have given an example of this operation for $\nu = (-1, 1, 0)$ and $\boldsymbol{\lambda} = ((2,1), \emptyset, (1))$ in Figure 5.9.



$(\emptyset, \emptyset, \emptyset)$ $((2), \emptyset, \emptyset)$ $((2,1), \emptyset, \emptyset)$ $((2,1), \emptyset, (1))$

FIGURE 5.9. Constructing the abacus of the partition λ with 3-core $(-1, 1, 0)$ and 3-quotient $((2,1), \emptyset, (1))$.

The ℓ-quotient map is often used in a more specialized form. For a fixed ℓ-core ν, we again have the map (5.30). When we now look at the preimage of one "rank level" $\mathscr{P}(\ell, n)$ for some positive integer n, we obtain on the left side of (5.30) all elements $\lambda \in \mathscr{P}_\nu$ whose ℓ-abaci are obtained from the ℓ-abacus of ν by moving beads down n increments in total. By the arguments in the proof of Proposition 5.2.2, each time we move a bead down by one row we add ℓ new boxes to ν. This means the partition λ has $|\nu| + n \cdot \ell$ boxes. This gives us a new bijection

$$\tau_{\mathbf{s}} : \mathscr{P}_\nu(|\nu| + n \cdot \ell) \leftrightarrow \mathscr{P}(\ell, n) \ . \tag{5.33}$$

where $\mathbf{s} \in \mathbb{Z}_0^\ell$ denotes the charge of the ℓ-core ν. It is this map $\tau_{\mathbf{s}}$ which will be used to parametrize the fixed points of Calogero–Moser spaces where the Calogero–Moser parameter $\mathbf{c}$ will be determined by ν and $\mathbf{s}$.

We want to state the definition of the ℓ-quotient map on the level of sequences of β-numbers. Take an ℓ-tuple of sequences of β-numbers $(\beta^{(0)}, ..., \beta^{(\ell-1)})$ with $\beta^{(k)}$ stabilizing with respect to $r_k \in \mathbb{Z}$ for $0 \le k \le \ell - 1$ such that

$$\sum_{k=0}^{\ell-1} r_k = 0 \ . \tag{5.34}$$

We can now construct a new "unified" sequence of β-numbers by taking the set

$$\bigcup_{k=0}^{\ell-1} \left\{ \ell \cdot \left(\beta_i^{(k)} - 1\right) + k + 1 \ \middle| \ i \ge 1 \right\} \tag{5.35}$$

and ordering it into a strictly decreasing sequence of integers forming a new sequence of β-numbers $\boldsymbol{\beta}$. It follows from the arguments preceding Lemma 5.2.5 that $\boldsymbol{\beta}$ stabilizes with respect to $\sum_{k=0}^{\ell-1} r_k = 0$. In essence, the construction (5.35) turns the entries of $\beta^{(k)}$ into those entries of $\boldsymbol{\beta}$ which are congruent to $k+1 \mod \ell$. Therefore, the 1-abacus of $\beta^{(k)}$ forms the column of index k in the ℓ-abacus of $\boldsymbol{\beta}$ by Lemma 5.1.4. An example of this process is displayed in Figure 5.10.

$$\beta^{(0)} = (2, 0, -1, ...)\ ,\ \beta^{(1)} = (2, -1, -2, ...)\ ,\ \beta^{(2)} = (1, -1, -2, ...)$$
$$\boldsymbol{\beta} = (5, 4, 3, -1, -3, -4...)$$

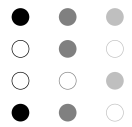


FIGURE 5.10. An example of unifying three sequences of β-numbers.

5.3. J-core operation

The last operation we need to define on abaci in order to parametrize fixed points and symplectic leaves of Calogero–Moser spaces is that of the *J-core*. We call a box u in a partition λ *removable* when $\lambda \setminus \{u\}$ still describes the Young diagram of a partition. The box u is called j-removable when $j \in \mathbb{Z}/\ell\mathbb{Z}$ is the ℓ-residue of $u \in \lambda$.

DEFINITION 5.3.1 ([Gor08a]). For $\lambda \in \mathscr{P}$, $\ell \in \mathbb{Z}_{\geq 0}$, and $J \subseteq \{0, ..., \ell - 1\}$, the *$J$-core* of λ is obtained from λ by successively removing as many j-removable boxes of λ as possible for $j \in J$. We denote this new partition by $\mathrm{Core}_J(\lambda)$ and say λ is a J-core whenever $\mathrm{Core}_J(\lambda) = \lambda$.

On the level of abaci, we can take the ℓ-abacus of a partition λ stabilizing with respect to 0 and identify the set $J \subseteq \{0, ..., \ell - 1\}$ with the corresponding subset of columns of the ℓ-abacus. By Lemma 5.1.4 the J-core of λ is then obtained by iteratively moving active beads in a column of index $j \in J$ one column to the left, looping around whenever $0 \in J$.

EXAMPLE 5.3.2. We have given the $\{0, 2\}$-core of the 3-abacus of $(4, 2, 2, 1)$ in Figure 5.11.

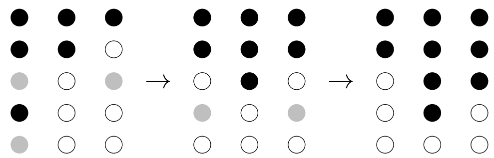


FIGURE 5.11. A visualization of $\mathrm{Core}_{\{0,2\}}((4, 2, 2, 1))$ with movable beads highlighted.

Whenever there is a fixed ℓ-core ν, we will identify the set of ℓ-multipartitions $\mathscr{P}(\ell, n)$ with the ℓ-abaci of partitions in $\mathscr{P}_\nu(|\nu| + n \cdot \ell)$ using the ℓ-quotient map (5.33). We will therefore also talk about the J-core of an element $\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)$.

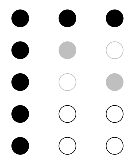
DEFINITION 5.3.3 ([Gor08a]). Fix an ℓ-core ν and $J \subseteq \{0, ..., \ell - 1\}$. We use the bijection $\mathscr{P}(\ell, n) \leftrightarrow \mathscr{P}_\nu(|\nu| + n \cdot \ell)$ to define an equivalence relation $\sim_J$ on $\mathscr{P}(\ell, n)$ by

$$\boldsymbol{\lambda} \sim_J \boldsymbol{\mu} :\Longleftrightarrow \mathrm{Core}_J(\boldsymbol{\lambda}) = \mathrm{Core}_J(\boldsymbol{\mu})\ . \tag{5.36}$$

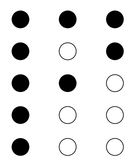
We call the $\sim_J$-equivalences classes *J-classes*.

We will see later in Section 5.5 that it is possible to construct a special parameter of the Calogero–Moser family using an ℓ-core and a set $J \subseteq \{0, ..., \ell - 1\}$ such that the J-classes correspond to Calogero–Moser families. For this reason, we want to describe J-classes combinatorially using *rectangle swaps*. We illustrate this first at an example.

The 3-abacus of $(4, 2, 2, 1)$ of the sequence of β-numbers stabilizing with respect to 0 is given by

(5.37)


with 3-quotient $(\emptyset, \emptyset, (1)) \in \mathscr{P}(3, 1)$. Furthermore, let $J = \{0, 2\}$. We have already highlighted four positions in the abacus that form a "rectangle". They have the property that the two pairs of opposing positions form a marked and an unmarked pair. We can form a new abacus by exchanging the markings to obtain

(5.38)


admitting the 3-quotient $(\emptyset, (1), \emptyset) \in \mathscr{P}(3, 1)$. It is easy to see that

$$\mathrm{Core}_{\{0,2\}}((\emptyset, \emptyset, (1))) = \mathrm{Core}_{\{0,2\}}((\emptyset, (1), \emptyset)) , \tag{5.39}$$

i.e. the two abaci admit the same J-core. Also note that the ℓ-multipartitions both partition the number 1 which means they are in the same J-class. We want to make this phenomenon more precise now.

Definition 5.3.4. Fix some $J \subseteq \{0, ..., \ell - 1\}$. Let $a < b < c < d$ be integers that give positions in an ℓ-abacus. We call the four positions a *J-rectangle* (or simply *rectangle*) if the following conditions are met:

- (i) $a \equiv c \mod \ell$ and $b \equiv d \mod \ell$, i.e. the pairs a, c and b, d are in the same column of the ℓ-abacus,
- (ii) $a - b = c - d$, i.e. together with (i) the position of a relative to b is the same as the position of c relative to d inside the abacus,
- (iii) $\{a + 1, a + 2, ..., b - 1, b\} \mod \ell \subseteq J$, i.e. the columns between a and b belong to J,
- (iv) either the pair a, d is marked or the pair b, c is marked with the respective other pair being unmarked.

We call the operation of switching the marked and unmarked pair in Definition 5.3.4(iv) a *J-rectangle swap* (or simply *rectangle swap*). We have given an example admitting two less well-behaved rectangles in Figure 5.12. As we can see, rectangles can loop around whenever $0 \in J$ and they can span multiple rows and columns.

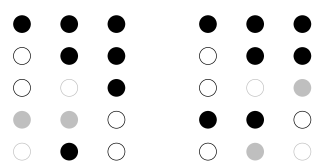


Figure 5.12. Two rectangles for the 3-abacus of $(5, 3, 3, 3, 1, 1)$ stabilizing with respect to 0 and $J = \{0, 2\}$.

By performing the rectangle swaps in Figure 5.12 and reading off the ℓ-quotients, one can check that the resulting ℓ-quotients all partition the same number which in this case is 4. The fact that rectangle swaps do not change the partitioned number holds true in general.

LEMMA 5.3.5. *Rectangle swaps preserve J-classes.*

PROOF. We first need to see that a J-rectangle swap does not change the J-core of the abacus. By Definition 5.3.4(*iii*), we can move beads freely between the two columns affected by the swap without changing the J-core. Also, by Definition 5.3.4(*iv*) the rectangle swap does not change the amount of beads in rows between the positions of the rectangle. So, we get that J-rectangle swaps preserve J-cores.

For the second part we need to see that the number being partitioned by the ℓ-quotient of the ℓ-abacus is preserved by the swap. For this, we can view the swap operation not within rows preserving a J-core but rather as a "jump" of two beads within their respective column. Now, to simulate a bead "jumping" i rows, we need to move beads in that column exactly i times. This number i is *the same* for each column. The movements are the same as in the proof of Proposition 5.2.2 where we described this situation for an ℓ-abacus instead of a 1-abacus. By Definition 5.3.4(*ii*), the number of rows being passed is the same for both columns. Therefore, we get that the number of added and deleted boxes in the ℓ-quotients cancel out and we see that both ℓ-quotients partition the same number. This, together with the preceding discussion gives us the desired result.

■

Since rectangle swaps are reversible, we can form equivalence classes of ℓ-multipartitions that can be obtained from one another by a sequence of rectangle swaps. By the previous lemma, this equivalence relations refines $\sim_J$ given in Definition 5.3.3. We will see later in Section 5.5 that the two equivalence relations are indeed equal. It is therefore sufficient to use only rectangle swaps to fully describe the J-class of an abacus. We call this fact *rectangle swap sufficiency*.

We will detail the connection between the geometry of Calogero–Moser space and the combinatorics of the associated cyclic quiver next as given in [Gor08a; GM09; Mar10].

5.4. Affine Weyl group

Let Q be the cyclic quiver on ℓ vertices given in Section 4.6. One can associate to Q a root datum and a reflection group as in [CB01, Sec. 2]. This construction is independent of the orientation of the arrows of Q and can thus be identified with the affine Dynkin diagram

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, dash] \arrow[drr, dash] & \overset{2}{\bullet} \arrow[r, dash] & \cdots \arrow[r, dash] & \overset{\ell-2}{\bullet} \arrow[r, dash] & \overset{\ell-1}{\bullet} \arrow[dll, dash] \\ & & \underset{0}{\bullet} & & \end{tikzcd} \tag{5.40}$$

of type $\tilde{A}_\ell$. The corresponding Coxeter group $\tilde{\mathfrak{S}}_\ell$ is given by the generators and relations

$$\left\langle \sigma_0, ..., \sigma_{\ell-1} \mid \sigma_i^2 = (\sigma_i\sigma_{i+1})^3 = 1,\ 0 \le i \le \ell - 1 \right\rangle \tag{5.41}$$

where the indices are taken modulo ℓ as will be the assumption throughout. As first described in [Lus00], the group $\tilde{\mathfrak{S}}_\ell$ affords an action on dimension vectors $\mathbf{d} \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$ and parameters $\boldsymbol{\theta} \in \mathbb{C}^{\mathbb{Z}/\ell\mathbb{Z}}$ associated to the quiver variety $\mathcal{X}_{\boldsymbol{\theta}}(\mathbf{d})$ of Q. We follow [Gor08a] and [Prz20] for a more combinatorial treatment.

The action on a dimension vector $\mathbf{d} \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$ is given by

$$\sigma_i * \mathbf{d} = (d_0, ..., d_{i-1}, \quad -d_i + d_{i-1} + d_{i+1} + \delta_{i0}, \quad d_{i+1}, ..., d_{\ell-1}) \tag{5.42}$$

for $0 \leq i \leq \ell - 1$ and δ_{ij} being the Kronecker delta. This action can be translated to ℓ-cores by first mapping

$$\mathscr{C}_\ell \to \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}, \ \nu \mapsto \mathrm{Res}_\ell(\nu) \tag{5.43}$$

where $\mathscr{C}_\ell$ denotes the set of all ℓ-cores. We see here that any ℓ-core affords a dimension vector of Q via the ℓ-residue. The map (5.43) is obviously not surjective but it is injective by [JK81, Thm. 2.7.41], i.e. an ℓ-core is determined uniquely by its ℓ-residue. The action of $\tilde{\mathfrak{S}}_\ell$ on ℓ-cores can be given via addable and removable boxes.

Recall that for a partition $\lambda \in \mathscr{P}$, we said a box $u \in \lambda$ is k-removable (with respect to λ) if u has ℓ-residue k and $\lambda \setminus u$ is still a valid Young diagram. In the same vein, we say a box u is k-*addable* (with respect to λ) if $\lambda \cup \{u\}$ is a valid Young diagram and u is k-removable with respect to $\lambda \cup \{u\}$.

Definition 5.4.1 ([Lee99]). For $0 \leq k \leq \ell - 1$, define the operator

$$\mathbf{T}_k : \mathscr{P} \to \mathscr{P} \tag{5.44}$$

by taking a partition $\lambda \in \mathscr{P}$ and simultaneously adding all k-addable boxes to λ and removing all k-removable boxes from λ.

On the level of abaci, we can view the operation induced by $\mathbf{T}_k$ as exchanging the two columns $k-1$ and k for $k \neq 0$. For $k = 0$, there is a down-shift of column k and up-shift of column $k-1$ as displayed in Figure 5.13. From this perspective, it is also easy to see that the operator $\mathbf{T}_i$ maps ℓ-cores onto ℓ-cores.

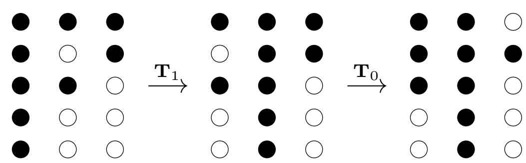


Figure 5.13. Example of the operators $\mathbf{T}_k$ on the 3-abacus of $(4, 2, 2, 1)$ stabilizing with respect to 0.

We let $\tilde{\mathfrak{S}}_\ell$ now act on $\mathscr{P}$ via

$$\sigma_k * \lambda = \mathbf{T}_k(\lambda) \tag{5.45}$$

for $0 \leq k \leq \ell - 1$. This action is well-defined as was first proven in [Lee99, Sec. 4]. It is not difficult to see that the set of all ℓ-cores is given by the orbit $\tilde{\mathfrak{S}}_\ell \cdot \emptyset$ where $\emptyset \in \mathscr{P}$ denotes the empty partition (which is an ℓ-core). From the definition of $\mathbf{T}_k$ we get

$$\sigma_k * (r\delta + \mathbf{d}) = r\delta + \sigma_k * \mathbf{d} \tag{5.46}$$

for $r \in \mathbb{Z}$ where $\delta = (1, ..., 1)$. By [Prz20, Prop. 4.11], the map (5.43) is equivariant with respect to the actions of $\tilde{\mathfrak{S}}_\ell$ in (5.42) and (5.45). Furthermore, by [BM21, Lem. 2.6], every $\mathbf{d} \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$ can be written uniquely as

$$\mathbf{d} = r\delta + \mathrm{Res}_\ell(\nu) \tag{5.47}$$

for some ℓ-core ν. Thus by (5.46), there exists a unique element of the form $r\delta$ in the $\tilde{\mathfrak{S}}_\ell$-orbit of any $\mathbf{d} \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$ and all the elements of that orbit are of the form (5.47) for different ℓ-cores ν.

There is an analogous action of $\tilde{\mathfrak{S}}_n$ on the parameter space $\mathbb{C}^{\mathbb{Z}/\ell\mathbb{Z}}$ given by

$$\sigma_i \cdot \boldsymbol{\theta} = (\theta_0, ..., \theta_{i-2},\ \ \theta_{i-1} + \theta_i,\ -\theta_i,\ \theta_i + \theta_{i+1},\ \ \theta_{i+2} ..., \theta_{\ell-1}) \ , \tag{5.48}$$

i.e. we add θ_i to θ_{i-1} and θ_{i+1}, and replace θ_i by $-\theta_i$. With respect to this action and the scalar product

$$\boldsymbol{\theta} \cdot \mathbf{d} = \sum_{i=0}^{\ell-1} \theta_i \cdot d_i \ , \tag{5.49}$$

we can calculate

$$(\sigma_i \cdot \boldsymbol{\theta}) \cdot (\sigma_i * \mathbf{d}) = \boldsymbol{\theta} \cdot \mathbf{d} - \delta_{0i} \cdot \theta_0 \ . \tag{5.50}$$

We then get this incredibly useful result.

Theorem 5.4.2 ([Lus00, Cor. 3.6]). *Let* $\mathbf{d} \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}, \boldsymbol{\theta} \in \mathbb{C}^{\mathbb{Z}/\ell\mathbb{Z}}$ *with* $\theta_i \neq 0$. *We have an isomorphism of algebraic varieties*

$$\mathcal{X}_{\boldsymbol{\theta}}(\mathbf{d}) \cong \mathcal{X}_{\sigma_i \cdot \boldsymbol{\theta}}(\sigma_i * \mathbf{d}) \ . \tag{5.51}$$

Motivated by this, one can define an equivalence relation $\sim$ on

$$\mathbb{C}^{\mathbb{Z}/\ell\mathbb{Z}} \times \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \tag{5.52}$$

by the transitive closure of

$$(\boldsymbol{\theta}, \mathbf{d}) \sim (\sigma_i \cdot \boldsymbol{\theta}, \sigma_i * \mathbf{d}) \ , \ \theta_i \neq 0 \ . \tag{5.53}$$

Gordon [Gor08a] used a similar result to Theorem 5.4.2 which also preserves the $\mathbb{C}^*$-fixed point structure of the varieties to study the fixed points of $\mathcal{X}_{\boldsymbol{\theta}}(n\delta)$. He accomplished this by transforming a parameter $\boldsymbol{\theta}$ into a parameter $\boldsymbol{\theta}'$ for which the description of fixed points becomes simpler. We will give a short summary of his construction in the next section.

5.5. Parametrization of fixed points

We follow [Gor08a] for $\boldsymbol{\theta} \in \mathbb{Q}^{\mathbb{Z}/\ell\mathbb{Z}}$, but there is an alternative way due to Maksimau [Mak22], who gave a more direct construction of the quiver representations parametrizing the fixed points by adapting the results of [Prz20].

The group $\tilde{\mathfrak{S}}_\ell$ admits a different presentation than the one in (5.41), namely as a semidirect product of the group $\mathfrak{S}_\ell$ and the non-affine coroot lattice $\mathbb{Z}R^\vee$ of type $A_{\ell-1}$. The latter can be identified with $\mathbb{Z}_0^\ell$. For a Calogero–Moser parameter $\mathbf{H}$ in the Gordon basis (cf. (4.24)), we can define

$$\boldsymbol{\theta} = (-h - \sum_{i=1}^{\ell-1} H_i,\ H_1, ..., H_{\ell-1}) \ . \tag{5.54}$$

When we scale $\mathbf{H}$ by setting $h = -1$, we reduce to the parameter set

$$\Theta_1 = \left\{ \boldsymbol{\theta} \in \mathbb{Q}^{\mathbb{Z}/\ell\mathbb{Z}} \ \middle|\ \theta_0 + \cdots + \theta_{\ell-1} = 1 \right\} \ . \tag{5.55}$$

We now choose the point $(1, 0, ..., 0)$ and identify Θ_1 with the weight lattice of $A_{\ell-1}$ via $\boldsymbol{\theta} \mapsto \sum_{i=1}^{\ell-1} \theta_i \varpi_i$ with fundamental weights ϖ_i for $1 \leq i \leq \ell - 1$. The action of $\tilde{\mathfrak{S}}_\ell$ on the coroot lattice now becomes the action of $\tilde{\mathfrak{S}}_\ell$ on parameters from (5.48). For an element

$$\beta^\vee = \sum_{i=1}^{\ell-1} a_i \alpha_i^\vee \in \mathbb{Z}R^\vee \tag{5.56}$$

where the $\alpha_i^\vee$ for $1 \leq i \leq \ell - 1$ are the simple coroots, we get a corresponding translation of Θ_1 denoted by $\tau_{\beta^\vee} \in \tilde{\mathfrak{S}}_\ell$. The dimension vectors of the quiver can be identified with the root lattice of type $\tilde{A}_\ell$ denoted $\mathbb{Z}\Phi$ and then the action of $\tilde{\mathfrak{S}}_\ell$ on $\mathbb{Z}\Phi$ becomes the action on dimension vectors given in (5.42).

Concentrating on the action of the coroot lattice $\mathbb{Z}R^\vee$ first, we start with the pair $(\boldsymbol{\theta}, n\delta)$ and write

$$\boldsymbol{\theta} = \tau_{-\beta^\vee} \cdot \mathbf{1} \tag{5.57}$$

where $\mathbf{1} = \frac{1}{\ell}(1, ..., 1) \in \Theta_1 \subseteq \mathbb{Q}^{\mathbb{Z}/\ell\mathbb{Z}}$. We then get

$$(\boldsymbol{\theta}, n\delta) \sim (\mathbf{1}, \mathbf{d}) \ \text{ where } \mathbf{d} = \tau_{\beta^\vee} * n\delta \ . \tag{5.58}$$

Using the description of $\mathbf{d} \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$ as the ℓ-residue of an ℓ-core (5.47), we have

$$(\boldsymbol{\theta}, n\delta) \sim (\mathbf{1}, \mathrm{Res}_\ell(\nu) + r\delta) \tag{5.59}$$

for some $r \in \mathbb{Z}$ and ℓ-core ν. To ν we can associate the charge $\mathbf{s} \in \mathbb{Z}_0^\ell$ and by [Gor08a, Lem. 7.5(iii)] we have that $\boldsymbol{\theta}$ and $\mathbf{s}$ are related by

$$\boldsymbol{\theta} = \mathbf{1} + (-s_{\ell-1} + s_0, -s_0 + s_1, ..., -s_{\ell-2} + s_{\ell-1}) \ . \tag{5.60}$$

Let now

$$\tau_{\mathbf{s}} : \mathscr{P}(\ell, n) \to \mathscr{P}_\nu(|\nu| + n \cdot \ell) \tag{5.61}$$

denote the ℓ-quotient map (5.33) afforded by ν. Gordon used this map in the proof [Gor08a, Prop. 7.10] to label the fixed points of the Calogero–Moser space which are indexed by the set $\mathscr{P}(\ell, n)$ for a parameter $\boldsymbol{\theta}$ associated to a generic Calogero–Moser parameter $\mathbf{H}$.

When introducing the additional action of $w \in \mathfrak{S}_\ell \leq \tilde{\mathfrak{S}}_\ell$ on Θ_1, the ℓ-quotient map is twisted to

$$\tau_{\mathbf{s}}^w : \mathscr{P}(\ell, n) \to \mathscr{P}_\nu(|\nu| + n \cdot \ell), \ \boldsymbol{\lambda} \mapsto \tau_{\mathbf{s}}(\boldsymbol{\lambda}^w) \tag{5.62}$$

where

$$\boldsymbol{\lambda}^w = (\lambda^{(w \cdot 0)}, ..., \lambda^{(w \cdot (\ell-1))}) \ , \tag{5.63}$$

i.e. the $\lambda^{(j)}$ are permuted by $w \in \mathfrak{S}_ell$. The construction above gives us a parametrization of the alcoves of Θ_1 with respect to the affine reflection action of $\tilde{\mathfrak{S}}_\ell$ using certain 3-tuples

$$(\mathbf{s}, w, \pm) \in \mathbb{Z}_0^\ell \times \mathfrak{S}_\ell \times \{+, -\} \tag{5.64}$$

where the third entry corresponds to a choice of $h = \mp 1$ in the Gordon parameter.

As the final step, Gordon used this twisted ℓ-quotient map to go from a generic parameter to a special parameter on a root hyperplane of $\tilde{\mathfrak{S}}_n$, which are given by

$$\{\boldsymbol{\theta} \cdot \beta = m \mid m \in \mathbb{Z} \ , \ \beta = e_i + \cdots + e_j \ , \ 1 \leq i, j \leq \ell - 1\} \ . \tag{5.65}$$

A point $\boldsymbol{\theta} \in \Theta_1$ in the closure of the alcove of $\mathbf{1} = \frac{1}{\ell}(1, ..., 1) \in \Theta_1$ has a stabilizer subgroup of $\tilde{\mathfrak{S}}_n$ which is a parabolic subgroup generated by the set $\{\sigma_j \mid j \in J\}$ for some $J \subseteq \{0, ..., \ell - 1\}$. The set J is called the *type* of $\boldsymbol{\theta}$ and extend this definition to all points of Θ_1 by using the $\tilde{\mathfrak{S}}_\ell$-action. Gordon then proved this result.

Proposition 5.5.1 ([Gor08a, Prop. 8.3]). *Let $\mathbf{H} = (-1, H_1, ..., H_{\ell-1})$ be a Calogero–Moser parameter in the Gordon basis and let*

$$\boldsymbol{\theta} = (1 - H_1 - \cdots - H_{\ell-1}, H_1, ..., H_{\ell-1}) \in \Theta_1 \tag{5.66}$$

be a point of type $J \subseteq \{0, ..., \ell - 1\}$ in an alcove labeled by $(\mathbf{s}, w, +)$. We then have that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}(\ell, n)$ are in the same Calogero–Moser $\mathbf{H}$-family if and only if

$$\mathrm{Core}_J(\tau_{\mathbf{s}}^w(\boldsymbol{\lambda})) = \mathrm{Core}_J(\tau_{\mathbf{s}}^w(\boldsymbol{\mu})) \ . \tag{5.67}$$

We immediately get a useful corollary (cf. Sec. 5.3).

COROLLARY 5.5.2. *Rectangle swaps are sufficient for generating the Calogero–Moser partition.*

PROOF. By Theorem 4.1.4, the family relations are transitive closures of families attached to generic points on hyperplanes.

For a single hyperplane, we only have two columns between which we may perform rectangle swaps, i.e. we can restrict to type B. Let $J = \{1\}$. Two 2-abaci then have the same J-core, charge, and bipartition size if and only if they contain the same rows up to permutation. Using rectangle swaps, we can permute the rows of the 2-abaci and translate them into one another, which gives us rectangle swap sufficiency for type B and $J = \{1\}$. The argument holds true for $J = \{0\}$ as well using a shift in charge.

Taking the transitive closure over all hyperplanes of a given Calogero–Moser parameter gives the same equivalence relation in both cases.

■

As detailed in [Gor08a], the nondegenerate Calogero–Moser hyperplanes are given by a specific subset of the root hyperplanes of the action of $\tilde{\mathfrak{S}}_\ell$ on Θ_1, namely

$$\{\boldsymbol{\theta} \cdot \beta = m \mid -n+1 \leq m \leq n-1,\ \beta = e_i + \cdots + e_j,\ 1 \leq i, j \leq \ell - 1\}\,. \tag{5.68}$$

5.5.1. Families via charged residues. Martino used Proposition 5.5.1 and results in [GM09] in [Mar10] to give a different combinatorial model to determine the Calogero–Moser families. The constructions in [Mar10] are based on [BK02, Sec. 3.A] and we summarize them here. Define the ∞-*residue* of a box in a Young diagram with matrix coordinates (i, j) as $j - i \in \mathbb{Z}$. We extend this to the whole partition and define the ∞-*residue* (or simply *residue*) of $\lambda \in \mathscr{P}$ by

$$\mathrm{Res}_\infty : \mathscr{P} \to \mathbb{Z}_{\geq 0}^{\mathbb{Z}} \tag{5.69}$$

(also denoted by Res) as the vector that counts the different ∞-residues of boxes of λ. To be able to write down the ∞-residue more easily, one identifies $\mathbb{Z}_{\geq 0}^{\mathbb{Z}}$ with $\mathbb{Z}_{\geq 0}[t, t^{-1}]$ by defining

$$\mathrm{Res}(\lambda, t) = \sum_{(i,j) \in \lambda} t^{j-i} \tag{5.70}$$

An example is illustrated in Figure 5.14. Note that the residue of a partition λ detemines λ uniquely.

0	1	2	3
−1	0		
−2	−1		
−3			

$$\mathrm{Res}((4,2,2,1), t) = t^{-3} + t^{-2} + 2 \cdot t^{-1} + 2 + t + t^2 + t^3$$

FIGURE 5.14. The residue of $\lambda = (4, 2, 2, 1)$.

For $m \in \mathbb{Z}, \lambda \in \mathscr{P}$, the *$m$-shifted residue* of λ is given by

$$\mathrm{Res}^m(\lambda, t) = t^m \cdot \mathrm{Res}(\lambda, t) \tag{5.71}$$

Using shifted residues, this construction can be extended to ℓ-multipartitions by defining for

$$\mathbf{m} = (m_0, ..., m_{\ell-1}) \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \text{ and } \boldsymbol{\lambda} \in \mathscr{P}(\ell, n) \tag{5.72}$$

the $\mathbf{m}$-*shifted residue* of $\boldsymbol{\lambda}$ by

$$\operatorname{Res}^{\mathbf{m}}(\boldsymbol{\lambda},t)=\sum_{k=0}^{\ell-1}\operatorname{Res}^{m_k}(\lambda^{(k)},t)=\sum_{k=0}^{\ell-1}t^{m_k}\cdot\operatorname{Res}(\lambda^{(k)},t)\ . \tag{5.73}$$

Define next for a rational Calogero–Moser parameter[1]

$$\mathbf{H}=(1,H_1,H_2,...,H_{\ell-1})\in\mathbb{Q}^{\mathbb{Z}/\ell\mathbb{Z}} \tag{5.74}$$

the vector

$$\mathbf{m_H}=(0,d\cdot H_1,d\cdot(H_1+H_2),...,d\cdot(H_1+\cdots+H_{\ell-1})) \tag{5.75}$$

where d is some fixed integer such that $d\cdot\mathbf{H}\in\mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$. Martino then proved the following analogue of [BK02, Thm. 3.13].

Theorem 5.5.3 ([Mar10, Thm. 3.13]). *Let* $\mathbf{H}=(1,H_1,H_2,...,H_{\ell-1})$ *be a rational Calogero–Moser parameter and let* $\mathbf{m_H}$ *and* d *be defined as in* (5.75). *We then have that* $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}(\ell,n)$ *are in the same Calogero–Moser* $\mathbf{H}$*-family if and only if*

$$\operatorname{Res}^{\mathbf{m_H}}(\boldsymbol{\lambda},t^d)=\operatorname{Res}^{\mathbf{m_H}}(\boldsymbol{\lambda},t^d)\ . \tag{5.76}$$

It is sufficient to work with rational parameters here, since every hyperplane has a point with rational Gordon coordinates in generic position. This is because the hyperplane equations with respect to the Gordon basis (cf. Thm. 4.2.1) are a system of linear equations with rational coefficients, therefore admitting a rational solution whenever one exists. We illustrate the utility of Theorem 5.5.3 with an example.

Example 5.5.4. The Calogero–Moser partition for B_2 at parameter $\mathbf{H}=(h,H_1)=(1,-1)$ is given by

$$\begin{array}{c|cccc|c}
\boxed{0\,|\,1}\,,\emptyset & \begin{array}{|c|}\hline 0\\\hline -1\\\hline\end{array}\,,\emptyset & \boxed{0}\,,\boxed{-1} & \emptyset,\boxed{-1\,|\,0} & & \emptyset,\begin{array}{|c|}\hline -1\\\hline -2\\\hline\end{array} \\[2ex]
((2),\emptyset) & ((1,1),\emptyset) & ((1),(1)) & (\emptyset,(2)) & & (\emptyset,(1,1))
\end{array}\ . \tag{5.77}$$

We can extend the action of $\tilde{\mathfrak{S}}_\ell$ on $\boldsymbol{\theta}$ to $\mathbf{H}$ and then to the derived charge

$$\mathbf{m}(\boldsymbol{\theta}):=(0,\theta_1,\theta_1+\theta_2,...,\sum_{i=1}^{\ell-1}\theta_i) \tag{5.78}$$

where one can compute

$$\mathbf{m}(\sigma_i\cdot\boldsymbol{\theta})=\mathbf{m}^{s_i}(\boldsymbol{\theta}) \tag{5.79}$$

for $1\le i\le\ell-1$ where $\mathbf{m}^{s_i}$ is the permutation of the vector $\mathbf{m}$ with the transposition $s_i\in\mathfrak{S}_\ell$. Also, the charge version of the hyperplanes with $h=1$ are given by

$$m_i-m_j\in\{-(n-1),...,n-1\}\ . \tag{5.80}$$

We close this section by giving a full picture of the type B case in the language of Proposition 5.5.1. For $\mathbf{H}=(-1,H_1)$, we define $\boldsymbol{\theta}=(1-H_1,H_1)\in\Theta_1$. The action of $\tilde{\mathfrak{S}}_2\cong\mathfrak{S}_2\ltimes\mathbb{Z}_0^2$ on Θ_1 is then given by

$$\sigma_1\mapsto\pi_1\cdot(-1,1)\,,\quad\sigma_0\mapsto\pi_1\cdot(1,-1)\ . \tag{5.81}$$

where $\mathfrak{S}_2=\langle\pi_1\rangle$. This presentation agrees with (5.60). For the Calogero–Moser hyperplanes (5.68), we only have the choice $\beta=e_1$ and we index the hyperplanes by $m\in\mathbb{Z}$. We can start with the basepoint $\mathbf{1}=(\frac{1}{2},\frac{1}{2})$ and apply the reflections

[1]There is a sign error in [Mar10] which we correct by setting $h=1$.

to obtain the labeling of our chambers. The type of the hyperplane is given by the conjugate reflection hyperplane of the chamber containing **1**. We obtain the picture in Figure 5.15. Here, we obtain the Calogero–Moser hyperplanes (resp. chambers) with respect to Θ_1 by intersecting the horizontal lines (resp. the space in between the horizontal lines) with Θ_1.

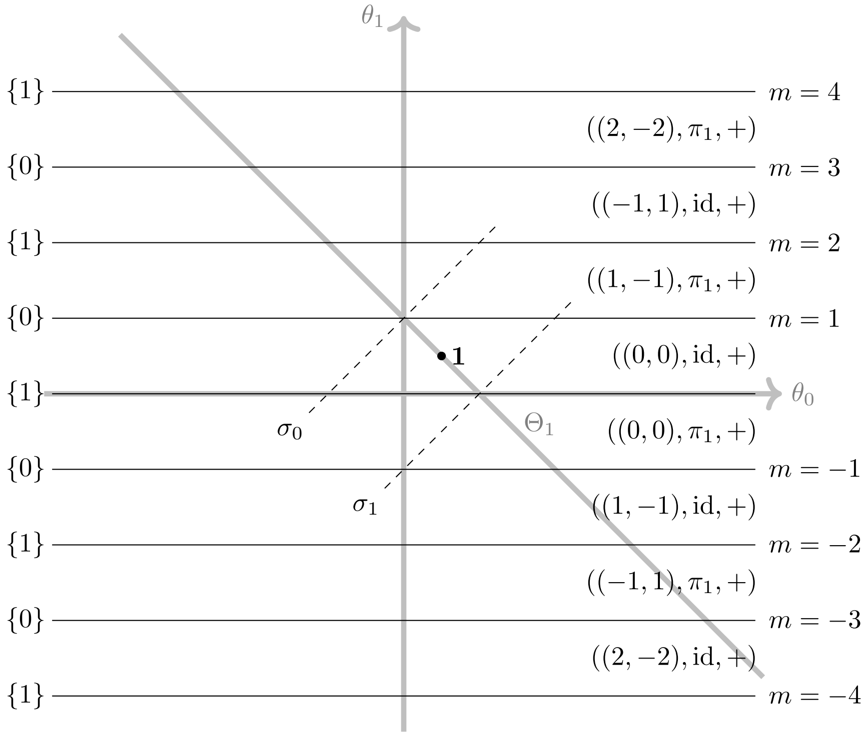


FIGURE 5.15. The transformed Calogero–Moser chambers in type B.

Now pick the point $\boldsymbol{\theta} = (-1, 2) \in \Theta_1$ lying on the hyperplane of $m = 2$ of type $J = \{1\}$. We want to investigate one B_3 family with respect to $\boldsymbol{\theta}$ coming from either Calogero–Moser chamber adjacent to the hyperplane of $\boldsymbol{\theta}$. We first give the 2-abacus of the family J-core with respect to the chamber $((-1,1),\mathrm{id},+)$, i.e. we start with the 2-core abacus

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \circ & \bullet \\ \circ & \bullet \end{array} \tag{5.82}$$

with corresponding 2-core partition $\nu = (2, 1)$. By Theorem 5.5.3 we can calculate that there is only one family of B_3 for $J = \{1\}$ and it is given by the abaci

$$\begin{array}{cc} \bullet & \bullet \\ \circ & \bullet \\ \circ & \bullet \\ \circ & \bullet \\ \bullet & \circ \\ \\ ((3),\emptyset) \end{array} \quad \begin{array}{cc} \bullet & \bullet \\ \circ & \bullet \\ \circ & \bullet \\ \bullet & \circ \\ \circ & \bullet \\ \\ ((2),(1)) \end{array} \quad \begin{array}{cc} \bullet & \bullet \\ \circ & \bullet \\ \bullet & \circ \\ \circ & \bullet \\ \circ & \bullet \\ \\ ((1),(1^2)) \end{array} \quad \begin{array}{cc} \bullet & \bullet \\ \bullet & \circ \\ \circ & \bullet \\ \circ & \bullet \\ \circ & \bullet \\ \\ (\emptyset,(1^3)) \end{array} \tag{5.83}$$

which all share the $\{1\}$-core

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \circ \\ \bullet & \circ \\ \bullet & \circ \\ \bullet & \circ \end{array} \tag{5.84}$$

with corresponding 2-core partition $\nu' = (3, 2, 1)$. Alternatively, we can view the same point $\boldsymbol{\theta} = (-1, 2)$ with respect to the chamber $((1, -1), \pi_1, +)$. That chamber has the 2-core abacus

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \circ \\ \bullet & \circ \end{array} \tag{5.85}$$

with corresponding 2-core partition $\nu = (1)$. This 2-abacus is the same as for the previous chamber $((-1, 1), \mathrm{id}, +)$ with the columns swapped. Since the 2-quotient map of $((1, -1), \pi_1, +)$ swaps the columns in the same way, we obtain the same $\{1\}$-core calculations.

5.6. Parametrization of symplectic leaves

The symplectic leaves of the Calogero–Moser space $\mathsf{X}_{\mathbf{c}}$ of a complex reflection group W of type $G(\ell, 1, n)$ have recently been parametrized by Maksimau in [Mak22]. We give a short summary of his results and methods in this section.

We say a parameter $\boldsymbol{\theta} \in \mathbb{C}^{\mathbb{Z}/\ell\mathbb{Z}}$ is *J-standard* for some $J \subseteq \{0, ..., \ell - 1\}$ if $\boldsymbol{\theta}$ is contained in the closure of the alcove of $\mathbf{1} = \frac{1}{\ell}(1, ..., 1)$ and $\boldsymbol{\theta}$ is of type J as described in Section 4.4. Since the closure of this alcove contains a fundamental domain for the action of $\tilde{\mathfrak{S}}_\ell$, we have that each pair $(\boldsymbol{\theta}, \mathbf{d}) \in \mathbb{C}^{\mathbb{Z}/\ell\mathbb{Z}} \times \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$ is equivalent to a pair $(\boldsymbol{\theta}', \mathbf{d}')$ such that $\boldsymbol{\theta}'$ is J-standard for some J. We then also have that J is the set of entries of $\boldsymbol{\theta}'$ which are 0.

Now, start again with the pair $(\boldsymbol{\theta}, n\delta)$ where $\delta = (1, ..., 1) \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$, and transform it into a pair $(\boldsymbol{\theta}', \mathbf{d})$ such that $\boldsymbol{\theta}'$ is J-standard for some J. By (5.46), we can write $\mathbf{d}$ uniquely as

$$\mathbf{d} = \mathrm{Res}_\ell(\nu) + n\delta \tag{5.86}$$

for some ℓ-core ν. Maksimau constructed in [Mak22, Sec. 3.G] a representation A_μ of the preprojective algebra $\Pi_{\boldsymbol{\theta}}$ (cf. Sec. 4.6) for each element $\mu \in \mathscr{P}_\nu(|\nu| + n \cdot \ell)$, i.e. for each element of $\mathscr{P}(\ell, n)$ under the corresponding ℓ-quotient map. The semisimplification of A_μ denoted by A'_μ decomposes as

$$A'_\mu = A_\lambda \oplus \bigoplus_{u \in \mu/\lambda} L(\alpha_{\mathrm{Res}_\ell(u)}) \tag{5.87}$$

where $\lambda = \mathrm{Core}_J(\mu)$ is the J-core of μ and $L(\alpha_i)$ is the unique simple representation of Q that has dimension 1 at node $i \in \mathbb{Z}/\ell\mathbb{Z}$ and dimension 0 elsewhere.

By [Mak22, Lem. 3.25], the representation $A_{\mathrm{Core}_J(\mu)}$ is fixed by the $\mathbb{C}^*$-action (4.82) and by Corollary 3.27 of *loc. cit.* two representations A_{μ_1}, A_{μ_2} are isomorphic if and only if μ_1, μ_2 have the same J-core. Lastly, by (5.87) and [Mak22, Exm. 3.26], we have

$$\dim^{\mathrm{reg}} A_\mu = \mathrm{Res}_\ell(\mathrm{Core}_J(\mu)) \tag{5.88}$$

where we used the definition of $\dim^{\mathrm{reg}}$ from (4.91). By Lemma 4.6.12, we now have that two points

$$[A_{\mu_1}], [A_{\mu_2}] \in \mathcal{X}_{\boldsymbol{\theta}}(n\delta) \tag{5.89}$$

given by quiver representations A_{μ_1}, A_{μ_2} lie on the same symplectic leaf of $\mathcal{X}_{\boldsymbol{\theta}}(n\delta)$ if and only if

$$\mathrm{Core}_\ell(\mathrm{Core}_J(\boldsymbol{\mu}_1)) = \mathrm{Core}_\ell(\mathrm{Core}_J(\boldsymbol{\mu}_2)) \tag{5.90}$$

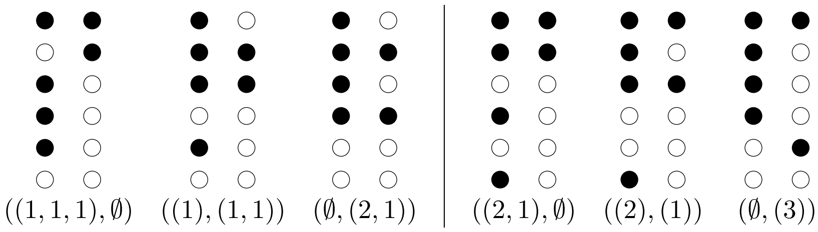
where $\boldsymbol{\mu}_i$ is the ℓ-quotient of μ_i for $i = 1, 2$. By Lemma 3.31 of *loc. cit.*, we can write (5.88) as

$$\mathrm{Res}_\ell(\mathrm{Core}_J(\mu)) = \mathrm{Res}_\ell(\nu') + n'\delta = \mathbf{d}' \tag{5.91}$$

where ν' is an ℓ-core that is also a J-core, and n' is a nonnegative integer. This means, by Remark 3.31 of *loc. cit.*, that the dimension of the symplectic leaf $\mathcal{L}(A_\mu)$ is $2n'$, i.e. equal to the rank of the ℓ-quotient of $\mathrm{Core}_J(\mu)$. To see this procedure in action, we give an example.

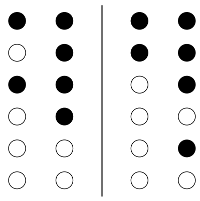
EXAMPLE 5.6.1. For $W = B_3$ and $\boldsymbol{\theta} = (2, -1)$ we are in the $J = \{0\}$ wall of the chamber $((1, -1), \mathrm{id}, +)$ by Figure 5.15. We obtain the two families

(5.92)



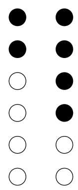
with family J-cores

(5.93)



which both have ℓ-core

(5.94)



and therefore lie on the same symplectic leaf. This symplectic leaf has dimension 2 since the ℓ-quotients of the J-cores in (5.93) are $((1), \emptyset)$ and $(\emptyset, (1))$ which are both of rank 1.

Our aim in the upcoming sections is to combine this model of ℓ-cores attached to symplectic leaves with the cuspidal module induction described in Section 4.5. The main idea is that we can interpret ℓ-cores of family J-cores as the J-core of a cuspidal family attached to a zero-dimensional symplectic leaf.

To close this section, we want discuss the relationship between the constructions we have seen so far. To relate the phenomenon described in this section to the theory of Bellamy described in Section 4.5, we need to show that the bijection between $\mathscr{P}(\ell, n)$ and $\mathscr{P}_\nu(|\nu| + n \cdot \ell)$ commutes with the labelling of $\mathbb{C}^*$-fixed points of $\mathcal{X}_{\boldsymbol{\theta}_{\mathbf{H}}}(n\delta)$ and $\mathsf{X}_{\mathbf{H}}$ with respect the isomorphism of these varieties mentioned in Theorem 4.6.10. We capture this in the following proposition.

Proposition 5.6.2. *Let* $\mathbf{H}$ *be a Calogero–Moser parameter in Gordon basis with associated* ℓ*-quotient map* τ*. Then the diagram*

$$\begin{tikzcd} \{[A_\mu] \mid \mu \in \mathscr{P}_\nu(|\nu| + n\cdot\ell)\} \arrow[r, leftrightarrow, "\tau"] \arrow[d, hook] & \{\mathrm{Ann}_{\mathsf{Z}_{\mathbf{c}}}(L_{\mathbf{c}}(\boldsymbol{\lambda})) \mid \boldsymbol{\lambda} \in \mathscr{P}(\ell, n)\} \arrow[d, hook] \\ \mathcal{X}_{\boldsymbol{\theta}_{\mathbf{H}}}(n\delta) \arrow[r, leftrightarrow, "\sim"] & \mathsf{X}_{\mathbf{H}} \end{tikzcd} \tag{5.95}$$

commutes.

Proof. For smooth parameter, this has been shown in [Prz20]. Note that in *loc. cit.*, there is a twist of parametrization by reversing the order of the columns of the abaci and replacing the ℓ-core ν by ${}^t\nu$ before taking the ℓ-quotient, as well as transposing the partition $\mu \in \mathscr{P}_{{}^t\nu}(|{}^t\nu| + n\cdot\ell)$. We get back to our conventions by using Lemma 5.2.6.

In the special parameter case, some quiver representations become isomorphic and some maximal ideals of $\mathsf{Z}_{\mathbf{c}}$ coincide. The fact that both sides agree on which, follows by comparing the degenerations (5.67) and (5.87).

■

All in all though, Proposition 5.6.2 only gives *evidence* for a connection between Maksimau's and Bellamy's theories, as there is too little known about the nature of cuspidal module induction to make more precise statements. However, we can circumvent this obstacle by looking at groups with relatively simple Calogero–Moser spaces, e.g. the smallest nontrivial wreath product family B_n.

5.7. Cuspidal family induction for type B

We want to describe a combinatorial counterpart to cuspidal module induction (cf. Sec. 4.5) by using the combinatorics admitted by Maksimau's labelling of symplectic leaves (cf. Sec. 5.6), which turn out to model this induction phenomenon quite nicely.

We will be focussing first on the complex reflection group of type B_n. As has been shown already in [Mar06, Sec. 5.1], the Calogero–Moser space of B_n contains at most one zero-dimensional symplectic leaf, therefore this symplectic leaf is unique whenever it exists. Recall that we can normalize the Etingof–Ginzburg coordinate κ to be 1 (cf. (4.22)). By the results in Section 4.2, special Calogero–Moser parameters in the Etingof–Ginzburg basis are then given by

$$\mathbf{c} = (\kappa, c_1) = (1, m) \tag{5.96}$$

for an integer m with $|m| < n$. The point $(1, m)$ corresponds to the point of Θ_1 lying on the hyperplane indexed by m in Figure 5.15. Since these points lie on exactly one hyperplane each, they afford a finest nontrivial Calogero–Moser partition by the discussion in Section 4.1. The cuspidal families in type B have been completely classified by Bellamy and Thiel.

Theorem 5.7.1 ([BT16, Thm. 6.21 + Thm. 6.26]). *Let* $\mathbf{c} = (\kappa, c_1) = (1, m)$ *be a special Calogero–Moser parameter of* B_n*. There exists a cuspidal* $\mathbf{c}$*-family of* B_n *if and only if there exists a positive integer* k *such that*

$$k \cdot (m + k) = n\ . \tag{5.97}$$

The family members are then given by partitions $(\lambda, \lambda^\dagger)$*, where* $\lambda \subseteq (k^{k+m})$ *and* $\lambda^\dagger \subseteq ((k+m)^k)$ *are such that they all have the same* $\mathbf{m}$*-charged residue as* $((k^{k+m}), \emptyset)$*.*

We want to illustrate Theorem 5.7.1 using abaci but before we do that, we introduce a slight shift to our conventions. We have seen in Figure 5.15 that for two neighboring chambers

$$(\mathbf{s},\sigma,+),(\mathbf{s}',\sigma',+)\in\mathbb{Z}_0^2\times\mathfrak{S}_2\times\{+\} \tag{5.98}$$

the maximal entry of $\mathbf{s},\mathbf{s}'$ is changed, and the elements $\sigma,\sigma'\in\mathfrak{S}_2$ differ as well. This is because a 2-core ν is of the form

$$\nu=(n,n-1,...,1) \tag{5.99}$$

with a partition that looks like a staircase. For these staircase partitions, all rows either end in the 2-residue 0 or they all end in 2-residue 1 depending on the parity of n. This means, the active beads need to change rows whenever we increase the size of our ℓ-core, i.e. whenever we apply the operator $\mathbf{T}_i$ (cf. Defn. 5.4.1) for $i=0,1$.

Because it can be difficult to keep track of the change in permutations and charges that happen when crossing a wall to a neighboring chamber, we will use a different convention for B_n: whenever we have a chamber with permutation $\pi_1\in\mathfrak{S}_2$, we will shift the sequence of β-numbers of the 2-core of that chamber by 1, making the sequence of β-numbers now stabilize to $1\in\mathbb{Z}/2\mathbb{Z}$. This way, our 2-cores can no longer be labeled by a charge $\mathbf{s}\in\mathbb{Z}_0^2$, so we also change that convention and choose $\mathbf{s}=(s_0,s_1)$ such that $s_0=0$. To preserve 2-quotients, we need to replace π_1 by id. The change in labels from Figure 5.15 is given in Figure 5.16.

old	new
$\vdots$	$\vdots$
$((2,-2),\pi_1,+)$	$((0,3),\mathrm{id},+)$
$((-1,1),\mathrm{id},+)$	$((0,2),\mathrm{id},+)$
$((1,-1),\pi_1,+)$	$((0,1),\mathrm{id},+)$
$((0,0),\mathrm{id},+)$	$((0,0),\mathrm{id},+)$
$((0,0),\pi_1,+)$	$((0,-1),\mathrm{id},+)$
$((1,-1),\mathrm{id},+)$	$((0,-2),\mathrm{id},+)$
$((-1,1),\pi_1,+)$	$(0,-3),\mathrm{id},+)$
$((2,-2),\mathrm{id},+)$	$(0,-4),\mathrm{id},+)$
$\vdots$	$\vdots$

FIGURE 5.16. New labelings for chambers of type B.

Note that a shift in sequences of β-numbers also changes how the J-core operation visually looks in the abacus, i.e. a $\{0\}$-core in the abacus of a shifted sequence of β-numbers looks like a $\{1\}$-core. We call an operation on a shifted sequence of

β-numbers that uses the column index set J of a non-shifted sequence of β-numbers a *visual J-core operation.* This means, we would need to perform two different visual J-core operations when approaching the wall from two neighboring chambers to obtain the same family. In summary, we get the following statement.

LEMMA 5.7.2. *The Calogero–Moser families of B_n associated to the parameter $\mathbf{c} = (1, m)$ are given by visual $\{1\}$-core operations on the abaci of $\mathscr{P}(2, n)$ with charge $(0, m)$.*

This new parametrization of chambers and J-cores is more in line with the classic theory of Lusztig families (cf. Sec. 5.11).

Fix now some special Calogero–Moser parameter $\mathbf{c} = (1, m)$, $m \in \mathbb{Z}$. To better illustrate the abacus, we demand $m \geq 0$. For the $m \leq 0$ construction, simply swap the columns in the end. Let now $\mathcal{F}^{\mathrm{cusp}}$ be a cuspidal Calogero–Moser family of the group B_n. We start by constructing the family member

$$((k^{k+m}), \emptyset) \in \mathcal{F}^{\mathrm{cusp}} \tag{5.100}$$

where k is the unique positive integer that solves the equation $k \cdot (m + k) = n$. The number k exists by Theorem 5.7.1. When we start with the ℓ-core abacus of charge $(0, m)$, we obtain

$$\begin{array}{l} \bullet \;\; \bullet \\ \left.\begin{array}{cc} \circ & \bullet \\ \vdots & \vdots \\ \circ & \bullet \end{array}\right\} m \end{array} . \tag{5.101}$$

When we construct $\boldsymbol{\lambda} = ((k^{k+m}), \emptyset)$ and the respective cuspidal family J-core out of (5.101) we get (starting with the ℓ-core)

$$\begin{array}{r} \bullet \;\; \bullet \\ k\left\{\begin{array}{cc} \bullet & \bullet \\ \vdots & \vdots \\ \bullet & \bullet \end{array}\right. \\ \left.\begin{array}{cc} \circ & \bullet \\ \vdots & \vdots \\ \circ & \bullet \end{array}\right\} m \end{array} \overset{\boldsymbol{\lambda}}{\rightsquigarrow} \begin{array}{r} \bullet \;\; \bullet \\ k\left\{\begin{array}{cc} \circ & \bullet \\ \vdots & \vdots \\ \circ & \bullet \end{array}\right. \\ \left.\begin{array}{cc} \circ & \bullet \\ \vdots & \vdots \\ \circ & \bullet \end{array}\right\} m \\ k\left\{\begin{array}{cc} \bullet & \circ \\ \vdots & \vdots \\ \bullet & \circ \end{array}\right. \end{array} \overset{J\text{-core}}{\rightsquigarrow} \begin{array}{r} \bullet \;\; \bullet \\ k\left\{\begin{array}{cc} \bullet & \circ \\ \vdots & \vdots \\ \bullet & \circ \end{array}\right. \\ \left.\begin{array}{cc} \bullet & \circ \\ \vdots & \vdots \\ \bullet & \circ \end{array}\right\} m \\ k\left\{\begin{array}{cc} \bullet & \circ \\ \vdots & \vdots \\ \bullet & \circ \end{array}\right. \end{array} . \tag{5.102}$$

From the middle abacus in (5.102), we can see that there are $\binom{m+2k}{k}$ different possibilities of dividing the active beads between the two columns to construct the other family members. As described in [BT16, Sec. 6] the members of $\mathcal{F}^{\mathrm{cusp}}$ are given by dividing (k^{m+k}) into a sub-partition $\lambda \subseteq (k^{m+k})$ and skew shape $\lambda' = (k^{m+k})/\lambda$, and then mirroring λ' at a point to obtain $\lambda^{\dagger} \subseteq ((k+m)^k)$. This is best illustrated in an example.

EXAMPLE 5.7.3. For B_6 and special parameter $\mathbf{c} = (1, 1)$ there is a unique cuspidal family for $k = 2$ since $6 = 2 \cdot (1 + 2)$. There are $\binom{1+2\cdot 2}{2} = 10$ different family members, which we have displayed in Figure 5.17. For the sake of clarity, we have highlighted the skew shape in this example only.

To make the leap from cuspidal family J-cores to non-cuspidal families, we first observe that a (visual) J-core contains only *full rows* ($\bullet\bullet$), *empty rows* ($\circ\circ$), and $\bullet\circ$-rows. Performing the 2-core operation on an abacus that contains only these

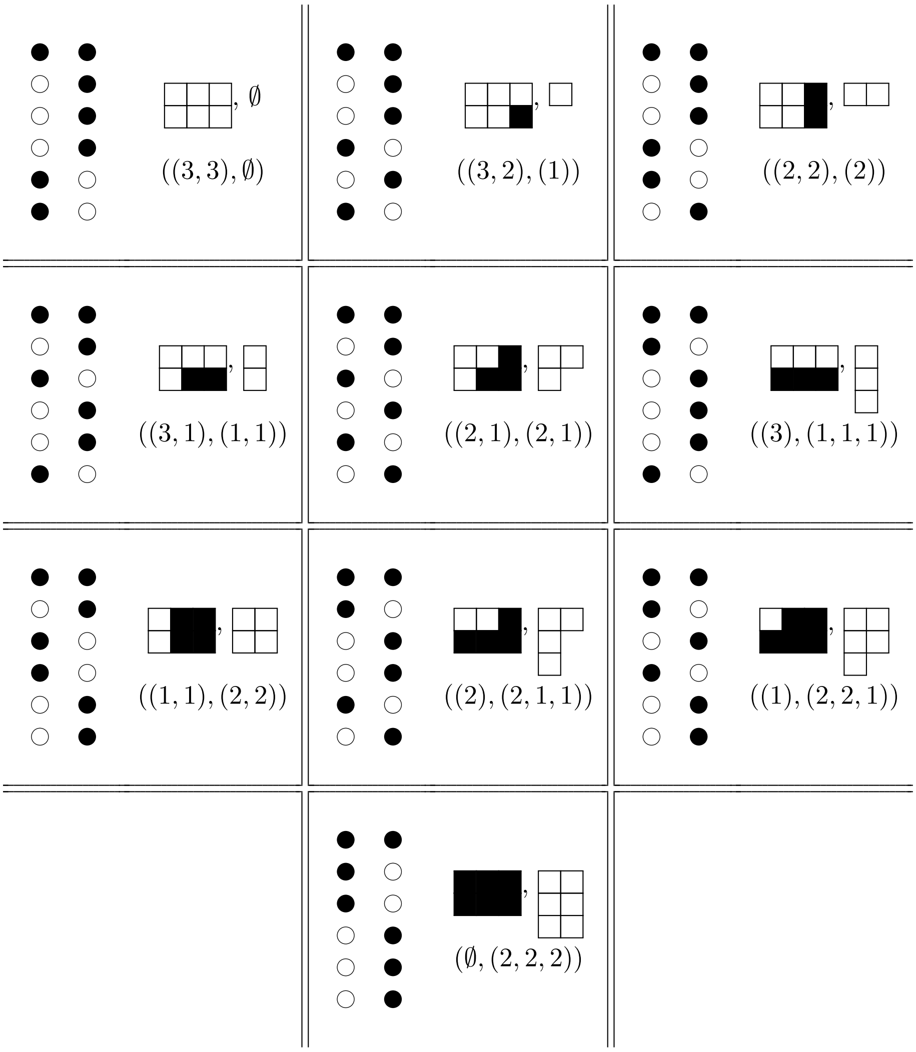


FIGURE 5.17. Cuspidal type B family for $n = 6, m = 1$, and $k = 2$ with highlighted skew shapes.

three types of rows can now be viewed as a row permutation of the abacus. This is because in this case, 2-cores essentially sort the rows by the amounts of beads they contain. Because this new abacus still contains no rows of the form (○●), it is again a J-core. These arguments extend to ℓ-abaci for $\ell > 2$ and we get the following result (see [Mak22, Lem. 2.11] for a different proof).

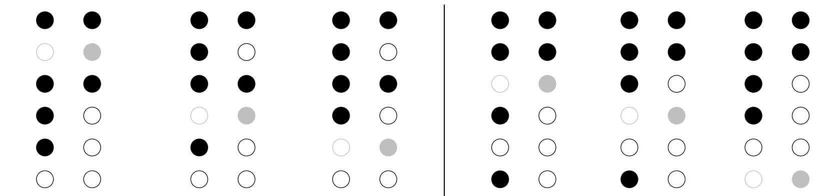
LEMMA 5.7.4. *The ℓ-core of a J-core is still a J-core.*

This brings us to the heart of our construction: given a family $\mathcal{F}$ of B_n, we are now able to view the 2-core of the corresponding family J-core again as the J-core of a new family $\mathcal{F}'$. We will fill in all the details now.

Start with a Calogero–Moser family $\mathcal{F}$ of B_n. Take a family member $\boldsymbol{\lambda} \in \mathcal{F}$ and produce from the 2-abacus of $\boldsymbol{\lambda}$ the family J-core. As described in the previous paragraph, this J-core contains only ●●-rows, ○○-rows, and ●○-rows. Some of the ●○-rows in the abacus of the J-core have already been in the abacus of $\boldsymbol{\lambda}$

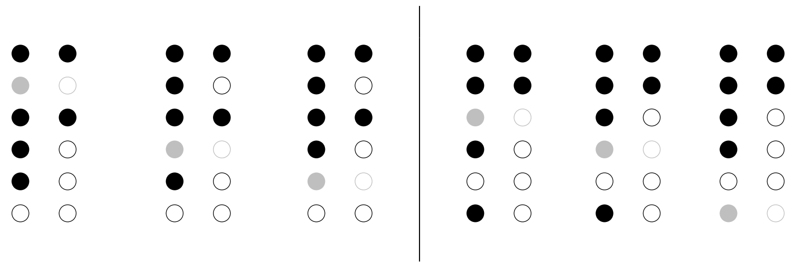
while the others have been constructed by performing J-core operations on ○●-rows belonging to the abacus of $\boldsymbol{\lambda}$. We can now mark these new ●○-rows, perform the 2-core operation, and reverse the J-core operation for these marked rows only. We adapt Example 5.6.1 to visual J-cores.

EXAMPLE 5.7.5. For $W = B_3$ and $\mathbf{c} = (1, -1)$, we use the visual J-core method to obtain the two families where the positions affected by the J-core operation are highlighted

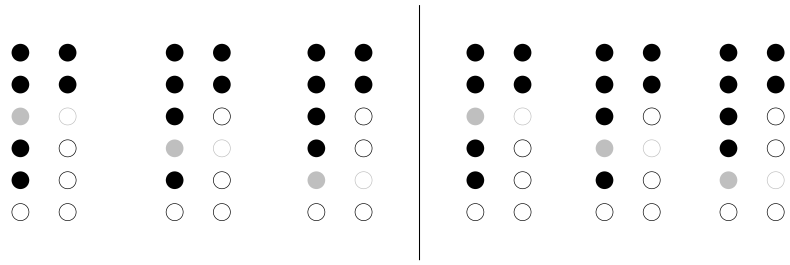


$((1,1,1),\emptyset)$ $((1),(1,1))$ $(\emptyset,(2,1))$ | $((2,1),\emptyset)$ $((2),(1))$ $(\emptyset,(3))$

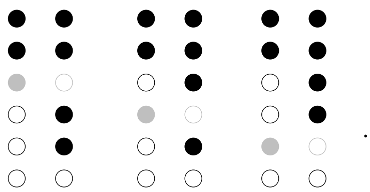
with three marked versions each of the two family J-cores



which have marked ℓ-cores

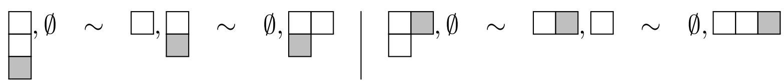


which are pairwise equal as marked abaci. Reversing the J-core operation for the respective first, second, and third family member of either family gives the cuspidal family



$((1,1),\emptyset)$ $((1),(1))$ $(\emptyset,(2))$

Translating this process to Young diagrams, we get



where the boxes that were added to the cuspidal family are highlighted.

By the construction of cuspidal families (5.102), we have that cuspidal family J-cores are already ℓ-cores. Here, the reversal of J-core operations on an ℓ-core of a J-core gives us a bijection between the members of the original family and the members of the corresponding cuspidal family.

Let $\mathcal{F}$ be family of B_n on a symplectic leaf $\mathcal{L} \subseteq \mathsf{X}_{\mathbf{c}}(B_n)$ of dimension $2r$. By the parametrization of Section 5.6, the rank of the ℓ-quotient of the J-core of $\mathcal{F}$ is r, and furthermore, the difference in the rank of $\mathcal{F}$ and the associated cuspidal family $\mathcal{F}^{\mathrm{cusp}}$ is also r. The latter fact can be read off the corresponding abaci. We therefore associate to a family $\mathcal{F}$ of B_n lying on $\mathcal{L}$ a cuspidal family of B_{n-r}.

On the other hand, by the discussions in Section 4.5, we have that cuspidal module induction associates to a point on $\mathcal{L}$ also a cuspidal family of B_{n-r} where $2r = \dim \mathcal{L}$. Therefore, the two theories agree on the parameter $\mathbf{c}$ and group W that give us the Calogero–Moser space $\mathsf{X}_{\mathbf{c}}(W)$ in which we need to find a cuspidal point. Because a Calogero–Moser space of type B contains at most one cuspidal family, we have that the two theories associate to a non-cuspidal type B family the same cuspidal family.

Definition 5.7.6. We call the process of associating a cuspidal family to a non-cuspidal family as described above *cuspidal family induction* of type B.

The preceding discussion gives us a first connection to cuspidal module induction.

Lemma 5.7.7. *In type B, cuspidal module induction and cuspidal family induction associate to a given family $\mathcal{F}$ the same cuspidal family $\mathcal{F}^{\mathrm{cusp}}$.*

On the level of abaci, cuspidal family induction amounts to adding full and empty rows to abaci of cuspidal family members. This becomes apparent in Example 5.7.5. Furthermore, this method affords an explicit bijection

$$\varphi : \mathcal{F}^{\mathrm{cusp}} \to \mathcal{F} \tag{5.103}$$

for a cuspidal family $\mathcal{F}^{\mathrm{cusp}}$ and non-cuspidal family $\mathcal{F}$. The map φ satisfies

$$\boldsymbol{\lambda} \subseteq \varphi(\boldsymbol{\lambda}) \tag{5.104}$$

for $\boldsymbol{\lambda} \in \mathcal{F}^{\mathrm{cusp}}$. Another bijection is implicitly given by cuspidal module induction (cf. Sec. 4.5). For type B, we have now seen that both bijections are between the same pairs of families but we do not know if they are equal on a family member level. To prove that this holds at least for certain family members, we will use the explicit structure of the characters of a cuspidal family.

Let W be a complex reflection group of type $G(\ell, 1, n)$ and let $\mathbf{c}$ be a Calogero–Moser parameter. From the general construction of simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$ given in Section 1.2, we know that each simple module $L_{\mathbf{c}}(\boldsymbol{\lambda})$ is nonnegatively graded with degree zero piece isomorphic to $\boldsymbol{\lambda}$ as a W-module. Because cuspidal module induction uses ungraded modules, we can really only use that $\boldsymbol{\lambda}$ is a constituent of $L_{\mathbf{c}}(\boldsymbol{\lambda})$. This, however, is enough in certain cases.

Definition 5.7.8. We call a simple $\overline{\mathsf{H}}_{\mathbf{c}}$-module $L_{\mathbf{c}}(\boldsymbol{\lambda})$ *rigid* if $L_{\mathbf{c}}(\boldsymbol{\lambda})$ is already irreducible as a W-module, i.e. we have

$$L_{\mathbf{c}}(\boldsymbol{\lambda}) \cong \boldsymbol{\lambda}\ . \tag{5.105}$$

Rigid modules have been used by Bellamy and Thiel in [BT16] to obtain a classification of cuspidal families in type B, which we have already seen in Theorem 5.7.1. They showed that irreducible W-representations can not be induced from representations of parabolic subgroups of W. Therefore, rigid modules can only ever appear in cuspidal families. In type B, it is even the case that *all* cuspidal families contain rigid modules.

THEOREM 5.7.9 ([BT16, Thm. 6.24]). *Let* $\mathbf{c} = (1, m)$ *be a special Calogero–Moser parameter of* B_n *such that* $n = k \cdot (m + k)$ *for some* $k \geq 0$*. Then there exist exactly two rigid* $\overline{\mathsf{H}}_{\mathbf{c}}$*-modules* $L_{\mathbf{c}}(\boldsymbol{\lambda})$*, namely for*

$$\boldsymbol{\lambda} = ((k^{k+m}), \emptyset)\ , \quad \boldsymbol{\lambda} = (\emptyset, (k + m)^k)\ . \tag{5.106}$$

Our plan now is to "tag" rigid $\overline{\mathsf{H}}_{\mathbf{c}}$-modules by their irreducible W-character to be able to uniquely identify their induced family members in the non-cuspidal families.

5.8. Durfee induction

Denote by $W_{\ell,n}$ the complex reflection group of type $G(\ell, 1, n)$. Recall from the end of Chapter 2 that the induction of a $W_{\ell,n}$-representation indexed by an ℓ-multipartition $\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)$ is given by

$$\operatorname{Ind}_{W_{\ell,n}}^{W_{\ell,n+1}} \boldsymbol{\lambda} = \bigoplus_{\substack{\boldsymbol{\mu} \in \mathscr{P}(\ell,n+1) \\ |\boldsymbol{\mu} \,/\, \boldsymbol{\lambda}| = 1}} \boldsymbol{\mu}\ . \tag{5.107}$$

where $\boldsymbol{\mu} \,/\, \boldsymbol{\lambda}$ denotes the *multi-skew shape* given by

$$\boldsymbol{\mu} \,/\, \boldsymbol{\lambda} = (\mu^{(0)}/\lambda^{(0)}, ..., \mu^{(\ell-1)}/\lambda^{(\ell-1)}) \tag{5.108}$$

for two ℓ-multipartitions $\boldsymbol{\lambda} = (\lambda^{(0)}, ..., \lambda^{(\ell-1)}), \boldsymbol{\mu} = (\mu^{(0)}, ..., \mu^{(\ell-1)})$. This result is known as the *Ariki–Koike branching rule* and it is proven in [AK94, Cor. 3.12].

Now, let $\mathcal{F}$, $\mathcal{F}^{\text{cusp}}$ be two families in type B such that $\mathcal{F}^{\text{cusp}}$ is a cuspidal family inducing the non-cuspidal family $\mathcal{F}$. We can then ask ourselves which element $\boldsymbol{\lambda} \in \mathcal{F}$ is induced from one of the rigid family members $\boldsymbol{\lambda}' \in \mathcal{F}^{\text{cusp}}$. By (5.107) and Theorem 4.5.3, we know that all constituents of $L(\boldsymbol{\lambda})$ must have $\boldsymbol{\lambda}'$ as a subpartition. We will show that this condition already determines $\boldsymbol{\lambda}$ uniquely in certain cases.

DEFINITION 5.8.1 ([And71]). Let $\lambda \in \mathscr{P}$ be a partition, $m \in \mathbb{Z}$. The *m-Durfee rectangle* of the Young diagram of λ is the maximal subdiagram $\mu \subseteq \lambda$ of the form

$$\mu = (k^{k+m}) = (k, ..., k) \tag{5.109}$$

for some $k \geq 0$. We then say λ has *m-Durfee number k*. We extend the definition of the m-Durfee rectangle to ℓ-multipartitions that contain exactly one nonempty partition.

Note that the m in Definition 5.8.1 can be negative. For positive m, the m-Durfee rectangle has m more rows than columns. For negative m, the m-Durfee rectangle has $|m|$ more columns than rows.

For special Calogero–Moser parameter $\mathbf{c} = (1, m)$, every cuspidal $\mathbf{c}$-family in type B has a unique family member $(\lambda, \emptyset)$ such that λ is already an m-Durfee rectangle.

THEOREM 5.8.2. *For* $m \in \mathbb{Z}$ *let* $\mathbf{c} = (1, m)$ *be a Calogero–Moser parameter for* B_n *and* $\boldsymbol{\lambda} = (\lambda, \emptyset) \in \mathscr{P}(2, n)$*. Then* $L_{\mathbf{c}}(\boldsymbol{\lambda})$ *is induced from the unique cuspidal family member* $\boldsymbol{\mu} = (\mu, \emptyset)$ *that has the same m-Durfee number as* $\boldsymbol{\lambda}$*.*

PROOF. Let k be the m-Durfee number of λ and $n' := k \cdot (m + k) = |\boldsymbol{\mu}|$. First, we need to prove that the family of $\boldsymbol{\lambda}$ is *family* induced from the family of $\boldsymbol{\mu}$. To see that, we take a look at the abacus of $\boldsymbol{\mu}$ (displayed in Figure 5.18).

Now, we only have to see that we can construct all possible λ with m-Durfee number k by moving black beads in the first column downwards without changing the ℓ-core of the J-core. Because of the structure of the abacus, this is equivalent to a row permutation. The m-Durfee number k stays constant because to increase

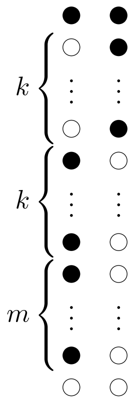


FIGURE 5.18. Abacus of an m-Durfee rectangle in type B.

k we would have to construct a new gap above the $k+m$ active beads of μ. This though would create an active bead in the 1-abacus of the second column and the resulting 2-quotient would not be of the form $(\nu, \emptyset)$. This means that $\boldsymbol{\lambda}$ is family induced from $\boldsymbol{\mu}$.

Because the families of $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ family induce each other, we know that they module induce each other by Lemma 5.7.7. To see that $L_{\mathbf{c}}(\boldsymbol{\mu})$ must then module induce $L_{\mathbf{c}}(\boldsymbol{\lambda})$, we take a look at the box $u \in \lambda$ with matrix coordinates $(k+m, k)$. For the cuspidal family of $\boldsymbol{\mu}$, this box only appears in $\boldsymbol{\mu}$ and no other family member. Now, that also holds for the induced family of $\boldsymbol{\lambda}$ since this box disappears as soon as we perform one rectangle swap: in the abacus this box corresponds to the $k+m^{\text{th}}$ active bead (counted from the bottom) having k gaps above it. When we do a rectangle swap, we lose an active bead below the $k+m^{\text{th}}$ bead and we lose a gap above it. This turns the lost gap that is now a bead into the new $k+m^{\text{th}}$ bead. This new $k+m^{\text{th}}$ bead has now less than k gaps above it. Therefore, $\boldsymbol{\lambda}$ is the only family member that contains the box $(k+m, k)$ in the first partition, i.e. $\boldsymbol{\lambda}$ is the only family member that has $\boldsymbol{\mu}$ as a sub-partition.

Since $L_{\mathbf{c}}(\boldsymbol{\mu}) \cong \boldsymbol{\mu}$ by Theorem 5.7.1, we have by the Ariki–Koike branching rule that all irreducible constituents of

$$\operatorname{Ind}_{B_n}^{B_{n'}} L_{\mathbf{c}}(\boldsymbol{\mu}) = \operatorname{Ind}_{B_{n'}}^{B_n} \boldsymbol{\mu} \tag{5.110}$$

must have the box $(k+m, k)$ in their first partition. But every simple module $L_{\mathbf{c}}(\boldsymbol{\nu})$ of a family member $\boldsymbol{\nu}$ of $\boldsymbol{\lambda}$ has at least one constituent that does not have the box $(k+m, k)$, e.g. $\boldsymbol{\nu}$ itself. Therefore $\boldsymbol{\mu}$ can only module induce $\boldsymbol{\lambda}$.

■

DEFINITION 5.8.3. We call the inductive process of Theorem 5.8.2 *Durfee induction.*

We have formulated Durfee induction for bipartitions of the form $(\lambda, \emptyset)$, but it holds for $(\emptyset, \lambda)$ as well, one simply needs to replace m-Durfee rectangles by $-m$-Durfee rectangles. The same replacement works for the following corollary.

COROLLARY 5.8.4. *For $m \in \mathbb{Z}$ let $\mathbf{c} = (1, m)$ be a Calogero–Moser parameter for B_n and $\boldsymbol{\lambda} = (\lambda, \emptyset) \in \mathscr{P}(2, n)$ with m-Durfee number k. Furthermore, let $n' = k \cdot (k+m)$ and $\boldsymbol{\mu} = ((k^{k+m}), \emptyset)$. We then have*

$$L_{\mathbf{c}}(\boldsymbol{\lambda}) \cong \operatorname{Ind}_{B_{n'}}^{B_n} \boldsymbol{\mu} \tag{5.111}$$

as ungraded W-modules with dimension

$$\dim L_{\mathbf{c}}(\boldsymbol{\lambda}) = n! \prod_{i=0}^{m+k-1} \frac{i!}{(k+i)!} \cdot \frac{2^n n!}{2^{n'} n'!} \tag{5.112}$$

which is a product of $\dim \boldsymbol{\mu}$ *as given by the hook-length formula (cf. Prop 2.1.18) and the index* $[B_n : B_{n'}]$.

Note that when $\boldsymbol{\lambda}$ has m-Durfee number 0, then $\boldsymbol{\mu}$ is the empty bipartition corresponding to the trivial module of the trivial group B_0, and we get the regular representation for $L_{\mathbf{c}}(\boldsymbol{\lambda})$.

The modules in Corollary 5.8.4 include the trivial representation of B_n which is given by the bipartition

$$\mathbf{triv} = ((n), \emptyset) \in \mathscr{P}(2, n) \ . \tag{5.113}$$

We record this fact separately.

Corollary 5.8.5. *For $m \in \mathbb{Z}$ let $\mathbf{c} = (1, m)$ be a Calogero–Moser parameter for B_n and denote by $\mathbf{triv}_n$ the trivial representation of B_n. We have*

$$L_{\mathbf{c}}(\mathbf{triv}_n) \cong \operatorname{Ind}_{B_{n'}}^{B_n} \mathbf{triv}_{n'} \tag{5.114}$$

as ungraded W-modules with dimension

$$\dim L_{\mathbf{c}}(\mathbf{triv}_n) = \frac{2^n n!}{2^{n'} n'!} \ . \tag{5.115}$$

We can extend the result in Corollary 5.8.4 to the families induced by the 3-member family $((1,1), \emptyset) \sim ((1),(1)) \sim (\emptyset, (2))$ in equal Calogero–Moser parameter to work out the W-module structure of the non-rigid family member. In general though, we cannot extend the method used above to non-rigid family members. This can be seen in the following example.

Example 5.8.6. For parameter $\mathbf{c} = (1, -2)$ and the group B_3, we have a cuspidal family of the form

$$\begin{array}{ccccccc} \square\square\square, \emptyset & \sim & \square\square, \square & \sim & \square, \begin{array}{l}\square\\ \square\end{array} & \sim & \emptyset, \begin{array}{l}\square\\ \square\\ \square\end{array} \\ ((3), \emptyset) & & ((2),(1)) & & ((1),(1,1)) & & (\emptyset,(1,1,1)) \end{array} \ . \tag{5.116}$$

The first and last bipartition correspond to the rigid modules $L_{\mathbf{c}}(((3), \emptyset))$ and $L_{\mathbf{c}}((\emptyset, (1,1,1)))$. One can compute using CHAMP [Thi15] that the simple modules $L_{\mathbf{c}}(((2),(1)))$ and $L_{\mathbf{c}}(((1),(1,1)))$ of the two middle bipartitions both have

$$\begin{array}{c} \square, \square\square \\ ((1),(2)) \end{array} \tag{5.117}$$

as a constituent. When we now use cuspidal family induction to construct the B_4 family

$$\begin{array}{ccccccc} \square\square\square\blacksquare, \emptyset & \sim & \square\square, \square\blacksquare & \sim & \square, \begin{array}{l}\square\blacksquare\\ \square\end{array} & \sim & \emptyset, \begin{array}{l}\square\blacksquare\\ \square\\ \square\end{array} \\ ((4), \emptyset) & & ((2),(2)) & & ((1),(2,1)) & & (\emptyset,(2,1,1)) \end{array} \ , \tag{5.118}$$

we have that the middle family members are both constituents of $\operatorname{Ind}_{B_3}^{B_4}(\square, \square\square)$. We therefore cannot "tag" the $\overline{\mathsf{H}}_{\mathbf{c}}$-modules $L_{\mathbf{c}}(((2),(2)))$, $L_{\mathbf{c}}(((1),(2,1)))$ by their namesake constituents.

5.9. Cuspidal family induction for type $G(\ell, 1, n)$

We want to generalize the type B theory of Section 5.7 to the general $G(\ell, 1, n)$ case. By Lemma 5.7.4, we already know that ℓ-cores of J-cores are again J-cores. What we do not know, is how we can interpret this new J-core as the family J-core of a suitable cuspidal family.

In type B, we have interpreted the 2-core of a J-core as a row permutation. This method can be extended to $\ell > 2$ as well: when we exchange two neighboring rows of an ℓ-abacus containing k_1 and k_2 beads respectively, the new ℓ-quotient differs from the old one by

$$k := |k_1 - k_2| \tag{5.119}$$

boxes. Furthermore, the J-core of the new ℓ-abacus differs from the J-core of the previous ℓ-abacus by the same exchange of rows whenever $0 \notin J$. Because k is already determined by the J-core, we can perform this operation on all members of a family simultaneously to obtain the members of a new family. Since this process is reversible, the new family has the same number of family members. We can now permute rows and sort them by the number of beads that appear in them to obtain this fact.

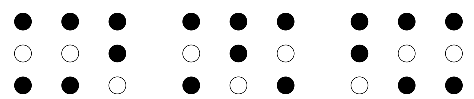
LEMMA 5.9.1. *To a non-cuspidal* $\mathbf{c}$*-family* $\mathcal{F}$*, we can associate a cuspidal* $\mathbf{c}$*-family* $\mathcal{F}^{\mathrm{cusp}}$ *whose family* J*-core is equal to the* ℓ*-core of the family* J*-core of* $\mathcal{F}$*. Furthermore, we have* $|\mathcal{F}| = |\mathcal{F}^{\mathrm{cusp}}|$*.*

The exchange of neighboring rows affords a bijection between a family $\mathcal{F}$ and its associated cuspidal family $\mathcal{F}^{\mathrm{cusp}}$. However, it is easy to see that this bijection is not well-behaved in the sense that $\boldsymbol{\lambda} \in \mathcal{F}$ may be mapped to a partition $\boldsymbol{\lambda}' \in \mathcal{F}^{\mathrm{cusp}}$ such that $\boldsymbol{\lambda}' \not\subseteq \boldsymbol{\lambda}$. Since we expect cuspidal module induction (cf. Sec. 4.5) to afford such a well-behaved bijection, we want this condition to hold for our combinatorial counterpart as well.

In type B, it was possible to construct a well-behaved bijection because we can ignore full and empty rows in the 2-abacus of a J-core to obtain the 2-abacus of a cuspidal family J-core. In this new J-core, we can then reverse the J-core operation. However, this is not possible for $\ell > 2$. We want to illustrate this briefly.

EXAMPLE 5.9.2. Take the empty 3-core given by the charge $(0, 0, 0)$ and let $J = \{1, 2\}$, i.e. we can move beads from the second and third column of the abacus. We have the $G(3, 1, 2)$-family

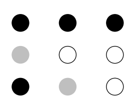
(5.120)



$((1), (1), \emptyset)$ $((1), \emptyset, (1))$ $(\emptyset, (1), (1))$

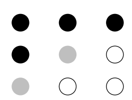
with family J-core

(5.121)



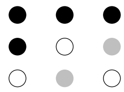
where we have marked the beads that moved for the family member $((1), \emptyset, (1))$. Taking the 3-core by exchanging rows gives us

(5.122)



and after re-doing the J-core operation we have

(5.123)



$$(\emptyset, (1), \emptyset)$$

which is not a sub-partition of $((1), \emptyset, (1))$.

We expect that it is possible to choose a bijection between family members that sends partitions to sub-partitions. What we want is for this bijection to be natural in some combinatorial way. To achieve this, we will use $\mathfrak{sl}_\infty$-crystal operators. We will give the relevant definitions here, for a more detailed discussion we refer the reader to [HK02] and [LM04].

First, we will reduce the problem to a smaller class of ℓ-cores and charges.

Definition 5.9.3. The abacus of an ℓ-core ν is called *staircase abacus* (or simply *staircase*) if the corresponding charge $\mathbf{s} \in \mathbb{Z}_0^\ell$ of ν is a weakly decreasing sequence.

Example 5.9.4. We have given two examples of staircase abaci in Figure 5.19.

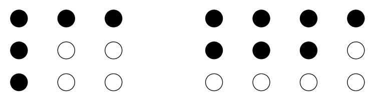


Figure 5.19. Two staircase abaci.

The abacus of every ℓ-core is equivalent to a staircase abacus up to permutation of the columns which induces a permutation of the $\lambda^{(j)}$.

To make the J-core operations easier to visualize, we set

$$J = \{1, ..., \ell - 1\} . \tag{5.124}$$

This way, the cuspidal family J-core corresponding to a staircase charge is again a staircase charge.

The restriction (5.124) is not a real constraint since by Martino's characterization of families by residue (cf. Thm. 5.5.3) we can split the J-core operations for the parameter

$$\mathbf{H} = (1, H_1, H_2, ..., H_{\ell-1}) \in \mathbb{Q}^{\mathbb{Z}/\ell\mathbb{Z}} \tag{5.125}$$

by grouping the entries of

$$\mathbf{m_H} = (0, d \cdot H_1, d \cdot (H_1 + H_2), ..., d \cdot (H_1 + \cdots + H_{\ell-1})) \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}} \tag{5.126}$$

in terms of their equivalence class modulo $d \in \mathbb{Z}$. This way, we obtain d different groups of columns forming d different (possibly empty) abaci. Lastly, we can shift the sequence of β-numbers corresponding to the ℓ-core as we did in Section 5.7 for type B such that $0 \notin J$.

Fix now a charge $\mathbf{s}$ of a staircase abacus and let $J = \{1, ..., \ell - 1\}$. We want to describe two operators f_i, e_i on charged ℓ-multipartitions, i.e. ℓ-abaci, that respectively add and remove a specific box from the ℓ-multipartition. The rule by which this specific box is determined is called the *signature rule* and we refer the readers to [HK02, Ch. 4.4] and [JMMO91] for a more detailed discussion and the relationship to the crystal structure of Fock spaces.

Consider $\Sigma = \mathbb{Z}$ as an an alphabet with associative word monoid Σ^*, i.e. Σ^* is the set of all finite sequences of elements of $\mathbb{Z}$ with concatenation as a product and the empty word $\emptyset \in \Sigma^*$ as the neutral element. We can now define operators

$$f_i\,,\ e_i \quad \text{for } i \in \mathbb{Z} \tag{5.127}$$

on Σ^* as follows. For $w \in \Sigma^*$ and $i \in \mathbb{Z}$, denote by w_i the subword of w consisting only of the entries of w equal to i or $i+1$. Then, replace all instances of i in w_i with a closed bracket ")" and all instances of $i+1$ with an open bracket "(". Now, we iteratively cancel "()" pairs of brackets and obtain an expression of the form

$$)\cdots)(\cdots(\,. \tag{5.128}$$

We now define $f_i.w$ as the word w' obtained from w by replacing the letter $i \in w$ that corresponds to the rightmost ")" in (5.128) by an $i+1$. We have $f_i.w = \emptyset$ whenever there is no ")" left after cancellation. Conversely, the operator e_i replaces the $i+1$ of the leftmost ")" by an i and $e_i.w = \emptyset$ when there is no "(" in (5.128). It is easy to see that

$$f_i.(e_i.w) = e_i.(f_i.w) = w \tag{5.129}$$

whenever $e_i.w \neq \emptyset \neq f_i.w$.

This operation is usually visualized in terms of abaci, see for example [Ger18]. Take two neighboring rows of an abacus that have row index i and $i{+}1$ and associate to each column a bracket expression by the rule

$$\genfrac{}{}{0pt}{}{\bullet}{\circ} \to \text{“)”}\,,\quad \genfrac{}{}{0pt}{}{\circ}{\bullet} \to \text{“(”}\,,\quad \genfrac{}{}{0pt}{}{\bullet}{\bullet} \to \text{“()”}\,,\quad \genfrac{}{}{0pt}{}{\circ}{\circ} \to \text{“}\varepsilon\text{”} \tag{5.130}$$

where ε denotes the empty bracket expression. Next, we concatenate the ℓ expressions corresponding to the ℓ columns and perform the bracket cancellation as described previously. We then have f_i replace the rightmost $\genfrac{}{}{0pt}{}{\bullet}{\circ}$ by $\genfrac{}{}{0pt}{}{\circ}{\bullet}$, and have e_i be the inverse operation. An example of this is given in Figure 5.20.

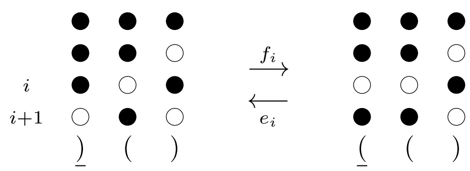


FIGURE 5.20. Application of crystallographic operators f_i, e_i.

Take now a Calogero–Moser family $\mathcal{F} \subseteq \mathscr{P}(\ell, n)$ whose family J-core has k_i beads in the row of index i, and has k_{i+1} beads in the row of index $i+1$. Furthermore, we demand

$$k_i - k_{i+1} =: k > 0\,, \tag{5.131}$$

i.e. the i^{th} row contains more beads than the $(i{+}1)^{\text{st}}$ row. We identify each element $\boldsymbol{\lambda} \in \mathcal{F}$ with the charged ℓ-abacus of $\boldsymbol{\lambda}$ and define the new partition

$$\boldsymbol{\lambda}' := f_i^k.\boldsymbol{\lambda} \in \mathscr{P}(\ell, n+k)\,. \tag{5.132}$$

Since we have $f_i^k.\boldsymbol{\lambda} \neq \emptyset$, the J-core of $\boldsymbol{\lambda}'$ is obtained from the J-core of $\boldsymbol{\lambda}$ by exchanging row i and $i+1$. By the discussion preceding Lemma 5.9.1, we now know that each $\boldsymbol{\lambda}'$ is a member of the same family $\mathcal{F}'$ with $|\mathcal{F}'| = |\mathcal{F}|$. Because the map

$$\mathcal{F} \to \mathcal{F}',\ \boldsymbol{\lambda} \mapsto f_i^k.\boldsymbol{\lambda} \tag{5.133}$$

has a left inverse given by the operator e_i^k, it is injective and thus already bijective. Furthermore, this bijection fulfills

$$\boldsymbol{\lambda} \subseteq f_i^k.\boldsymbol{\lambda} \tag{5.134}$$

by the definition of the operation of f_i. We can extend this relation transitively and associate to any family $\mathcal{F}$ a cuspidal family $\mathcal{F}'$ whose family J-core is obtained from $\mathcal{F}$ by "sorting" the rows of the J-core of $\mathcal{F}$ by the number of beads they contain.

Definition 5.9.5. We call the process of associating a cuspidal family to a non-cuspidal family as described above *cuspidal family induction* of type $G(\ell, 1, n)$.

It is easy to see that this definition reduces to Definition 5.7.6 for type B when we set $\ell = 2$.

As in type B, we can again argue that cuspidal module induction from Section 4.5 and this cuspidal family induction based on the theory in Section 5.6 associate to the same Calogero–Moser space associated to the group $G(\ell, 1, n-r)$ where $2r$ is the dimension of the symplectic leaf associated to the fixed point of $\mathcal{F}$. The problem now is the fact that for a given Calogero–Moser space $\mathsf{X}_{\mathbf{c}}$ of type $G(\ell, 1, n)$ where $\ell > 2$, it is possible for $\mathsf{X}_{\mathbf{c}}$ to contain multiple zero-dimensional symplectic leaves and therefore multiple cuspidal families (cf. Sec. 5.10). Furthermore, if we knew the associated cuspidal family for module induction, we still would not know which cuspidal family member induces which non-cuspidal family member. Combinatorially, we have a unique associated cuspidal family member by cuspidal family induction but there is too much left unknown about cuspidal module induction to connect the two. This leads us to the following conjectures.

Conjecture 5.9.6. *Cuspidal module induction and cuspidal family induction agree, i.e. they associate to a family $\mathcal{F}$ the same cuspidal family $\mathcal{F}^{\mathrm{cusp}}$.*

Conjecture 5.9.7. *Assume Conjecture 5.9.6 to be true. Let $\mathcal{F}$ be a family with associated cuspidal family $\mathcal{F}^{\mathrm{cusp}}$. Then both cuspidal module induction and cuspidal family induction afford the same bijective map $\mathcal{F} \to \mathcal{F}'$*

There are certain corner cases where it is possible to extend the type B results in Section 5.8 to $G(\ell, 1, n)$, e.g. whenever there is a unique cuspidal family. However, in this case one still needs to use some property of the characters of cuspidal family members, e.g. rigidity.

In the next section, we want to discuss the possibility of classifying cuspidal families and rigid modules in type $G(\ell, 1, n)$.

5.10. Cuspidal families and rigid modules for type $G(\ell, 1, n)$

In type B, we have seen the classification of cuspidal families by Bellamy and Thiel in Theorem 5.7.1, which relies on classifying rigid modules (cf. Thm. 5.7.9). In type $G(\ell, 1, n)$ however, the cuspidal families are in general much less structured. Additionally, we we will see later in this section that not all cuspidal families contain a rigid family member and it is therefore impossible to detect all cuspidal families this way (cf. Exm. 5.10.7). Our goal will be to give some more structure to the zoo of cuspidal families and rigid modules in type $G(\ell, 1, n)$ by describing special cases in more detail.

Let ν_ℓ be a staircase ℓ-core with associated charge $\mathbf{s}_\ell$ and let ν_J be a cuspidal J-core of a family of elements in $\mathscr{P}_{\nu_\ell}(|\nu_\ell| + n \cdot \ell)$ with associated staircase $\mathbf{s}_J$. The vector

$$\mathbf{s}_J - \mathbf{s}_\ell =: \mathbf{s}_\Delta = (s_0, \ldots, s_{\ell-1}) \tag{5.135}$$

describes the number of beads that got removed or added to the respective column of the abacus. We will use $\mathbf{s}_\Delta$ to describe some cuspidal families in more detail.

5.10.1. Disjoint type B. Let $\mathcal{P} \subseteq \{0, ..., \ell-1\}^2$ be the set of all pairs (i, j) for which the difference in the entries s_i, s_j of $\mathbf{s}_\ell$ is "small enough", i.e.

$$s_i - s_j =: s_{ij} \in \{-n+1, ..., n-1\} \ . \tag{5.136}$$

Those are the column pairs where J-core operations are possible by (5.80). Lastly, we demand that all pairs $(i, j) \in \mathcal{P}$ are disjoint. Together with the fact that ν_ℓ is a staircase ℓ-core, this means that $(i, j) = (i, i+1)$ for all pairs in $\mathcal{P}$. Each pair now describes an isolated part of the ℓ-abacus that must behave like a type B cuspidal family for ν_J to be an ℓ-core as well. This means there exist k_{ij} such that

$$n = \sum_{(i,j)\in\mathcal{P}} k_{ij} \cdot (s_{ij} + k_{ij}) \ . \tag{5.137}$$

The family members are given by the cartesian product of the type B families involved.

EXAMPLE 5.10.1. Let $W = G(4, 1, 4)$ and $\mathbf{s}_\ell = (3, 2, -2, -3)$. We then have the cuspidal J-core $\mathbf{s}_J = (4, 1, -1, -4)$ with cuspidal family

$$\begin{gathered} ((1,1), \emptyset, (1,1), \emptyset) \ , \ ((1,1), \emptyset, (1), (1)) \ , \ ((1,1), \emptyset, \emptyset, (2)) \ , \\ ((1), (1), (1,1), \emptyset) \ , \ ((1), (1), (1), (1)) \ , \ ((1), (1), \emptyset, (2)) \ , \\ (\emptyset, (2), (1,1), \emptyset) \ , \ (\emptyset, (2), (1), (1)) \ , \ (\emptyset, (2), \emptyset, (2)) \ . \end{gathered} \tag{5.138}$$

The associated abaci for $((1,1), \emptyset, \emptyset, (2))$ can be seen in Figure 5.21.

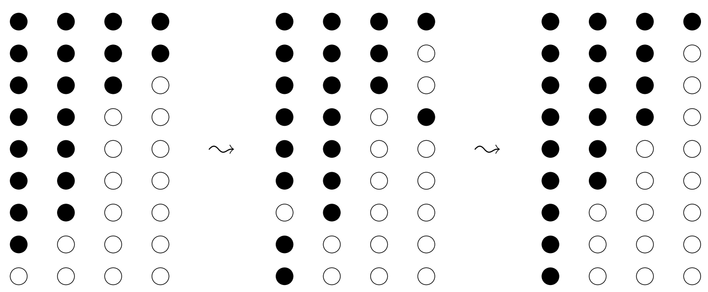


FIGURE 5.21. Abaci for the family member $((1,1), \emptyset, \emptyset, (2))$ in Example 5.10.1.

5.10.2. Blocks of ℓ-abaci. It is easy to see from the nature of J-core operations that for $\mathbf{s}_\Delta = (s_0, ..., s_{\ell-1}) \in \mathbb{Z}_0^\ell$, the partial sums of $\mathbf{s}_\Delta$ are nonnegative, i.e.

$$\sum_{i=0}^{k} s_i \geq 0 \quad \text{for} \quad 0 \leq k \leq \ell - 1 \ .$$

Using this fact, we can define a block structure on the charge $\mathbf{s}$. Let $I = \{k_1, ..., k_r\}$ be the set of indices for which we have $\sum_{i=0}^{k} s_i = 0$. Since we have $k_r = \ell - 1$, we can divide the set $\{0, ..., \ell-1\}$ into r subsets or *blocks* of the form

$$\{0, ..., k_1\}, \{k_1 + 1, ..., k_2\}, \ \ldots \ , \{k_{r-1} + 1, ..., \ell - 1\} \ . \tag{5.139}$$

Note that J-core operations only work within a block because the number of beads per block stays constant when performing J-core operations. Therefore, we can concentrate on constructing cuspidal families that admit one block.

There are two obvious special cases of blocks. The first one being given by the condition

$$s_i \geq 0 \quad \text{for all } 0 \leq i \leq \ell - 2, \text{ and } s_{\ell-1} = -\sum_{i=0}^{\ell-2} s_i < 0 \,. \tag{5.140}$$

We can then construct a distinguished family member of the form

$$(\emptyset, ..., \emptyset, \lambda) \tag{5.141}$$

by first defining

$$\overline{m}_i = s_i - s_{\ell-1} \,. \tag{5.142}$$

Then $\mathbf{s}$ gives us a cuspidal family for all $\mathscr{P}(\ell, n)$ with $n \in \mathbb{N}$ of the form

$$n = \sum_{i=0}^{\ell-2} s_i \cdot \left(\overline{m}_i + \sum_{j=0}^{i} s_j \right) \tag{5.143}$$

with distinguished family member

$$\left(\emptyset, ..., \emptyset, \left((\overline{m}_0 + s_0)^{s_0}, (\overline{m}_1 + s_0 + s_1)^{s_1}, ..., (\overline{m}_{\ell-2} + \sum_{i=0}^{\ell-2} s_i)^{s_{\ell-2}} \right) \right) \,. \tag{5.144}$$

EXAMPLE 5.10.2. Let $W = G(4, 1, 13)$ and $\mathbf{s}_\ell = (2, 1, -1, -2)$. We then have the cuspidal J-core $\mathbf{s}_J = (3, 1, 1, -5)$ with distinguished family member

$$\boldsymbol{\lambda} = (\emptyset, \emptyset, \emptyset, (5, 4, 4)) \,. \tag{5.145}$$

The associated abaci for $\boldsymbol{\lambda}$ can be seen in Figure 5.22.

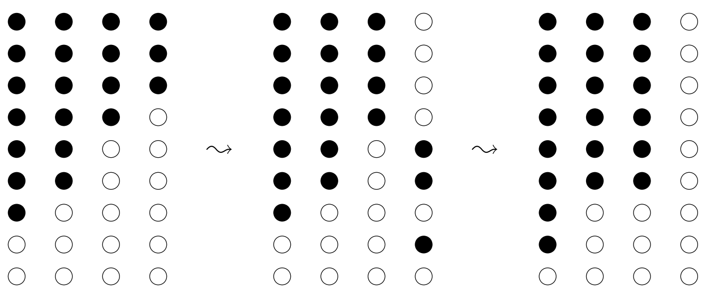


FIGURE 5.22. Abaci for the family member $(\emptyset, \emptyset, (5, 4, 4))$ Example 5.10.2.

An analogous construction can be done when we replace the inequalities in equation (5.140) by

$$s_i \leq 0 \quad \text{for all } 0 \leq i \leq \ell - 2, \text{ and } s_{\ell-1} = -\sum_{i=0}^{\ell-2} s_i > 0 \tag{5.146}$$

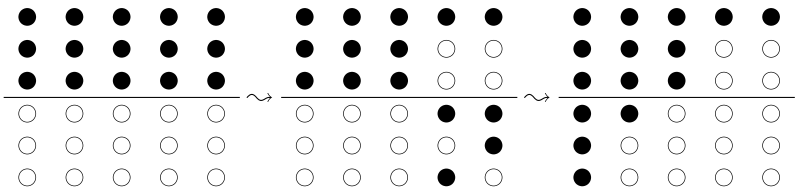
with a distinguished family member of the form

$$(\emptyset, \lambda^{(1)}, ..., \lambda^{(\ell-1)}) \,. \tag{5.147}$$

5.10.3. Zero-charge. We want to take a look at the staircase

$$\mathbf{s}_\ell = (0, ..., 0) \tag{5.148}$$

which we call *zero-charge*. Because $\mathbf{s}_J$ always corresponds to a staircase abacus for cuspidal J-cores, we can parametrize the cuspidal families by pairs of partitions (λ, μ) which are described respectively by the beads in positive rows and gaps in nonpositive rows. Let $\mathbf{s}_J^+$ (resp. $\mathbf{s}_J^-$) be the subvector of $\mathbf{s}_J$ of positive (resp. negative) entries. Because $\mathbf{s}_J$ is a staircase, these subvectors are partitions, so we can define $\lambda^+ = \mathbf{s}_J^+$ and $\lambda = \mathrm{rev}(-\mathbf{s}_J^-)$ where $\mathrm{rev}(x)$ denotes the reverse of a vector x. See Figure 5.23 for an example.



$\lambda = (3, 1), \quad \mu = (2, 2)$

FIGURE 5.23. Partitions from J-core charge $(3, 1, 0, -2, -2)$ with zero-charge ℓ-core.

The partitions λ and μ of a cuspidal zero-charge family are related in the way that $\lambda \trianglerighteq \mu$ in the dominance order of partitions (cf. Defn. 2.1.9). This is because we can move boxes in row i of μ up to row $i' < i$ by sliding the bead in the corresponding column down until that bead is moved past the beads in the columns next to it. Another restriction on (λ, μ) is $l(\lambda) + l(\lambda) \leq \ell$ where $l(\cdot)$ is the length function on partitions. We summarize this here.

LEMMA 5.10.3. *For zero-charge, cuspidal families of $G(\ell, 1, n)$ are indexed by pairs of partitions (λ, μ) that satisfy the following conditions:*

(*i*) $|\lambda| = |\mu|$,
(*ii*) $l(\lambda) + l(\mu) \leq \ell$,
(*iii*) $\lambda \trianglerighteq \mu$,

where $\trianglelefteq$ is the dominance order on partitions.

By using the pair $({}^t\lambda, {}^t\mu)$ instead of (λ, μ) we are able to transform the conditions in Lemma 5.10.3 into linear equations describing a polytope.

PROPOSITION 5.10.4. *For a fixed n, we view partitions as elements of $\mathbb{R}^n$ and $({}^t\lambda, {}^t\mu)$ in $\mathbb{R}^{2n}$. The linear equations*

(*i*) ${}^t\lambda_i \geq {}^t\lambda_{i+1}$,
(*ii*) ${}^t\mu_i \geq {}^t\mu_{i+1}$,
(*iii*) $|{}^t\lambda| = |{}^t\mu|$,
(*iv*) ${}^t\lambda_1 + {}^t\mu_1 \leq \ell$,
(*v*) ${}^t\lambda \trianglelefteq {}^t\mu$,
(*vi*) $\sum_{i=1}^n (2i-1) \cdot {}^t\mu_i + \sum_{j=1}^i \left({}^t\mu_j - {}^t\lambda_j\right) = n$

define a polytope in $\mathbb{R}^{2n}$, and the number of its integer lattice points is the number of cuspidal families of $G(\ell, 1, n)$ in zero-charge.

Proof. The conditions (i)-(v) have been explained in the preceding paragraphs. To see that condition (vi) is the correct one, we divide the sum into two parts. The first part

$$\sum_{i=1}^{n}(2i-1)\cdot {}^t\mu_i \tag{5.149}$$

describes the process of moving the beads from ${}^t\mu$ to the positive rows of the ℓ-abacus to create the pair $({}^t\mu, {}^t\mu)$. Assuming we have already moved the first $i-1$ rows of ${}^t\mu$ to their correct position, we need to move the i^{th} row of ${}^t\mu$ first to row 0 of the abacus by performing $(i-1)\cdot {}^t\mu_i$ movements. Afterwards, we need to move the ${}^t\mu_i$ beads of the first i rows upwards by 1 which gives us (5.149).

The second summand is

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\left({}^t\mu_j - {}^t\lambda_j\right) \tag{5.150}$$

and it described the number of bead movements needed to transform ${}^t\mu$ into ${}^t\lambda$. The inner sum represents the difference in the partial sums of the $\trianglelefteq$ order which is exactly the amount of beads that need to be moved from ${}^t\mu_i$ to ${}^t\mu_{i+1}$. The whole equation (vi) could also be written as

$$\sum_{i=1}^{n}(n+i)\cdot {}^t\mu_i - (n-i+1)\cdot {}^t\lambda_i = n\ . \tag{5.151}$$

Finally, we get all possible cuspidal families since an element of $\mathbb{R}^n$ that fulfills condition (vi) has at most $\sqrt{n}$ nonzero entries.

■

The two-step process from the proof above is visualized for the previous example in Figure 5.24.

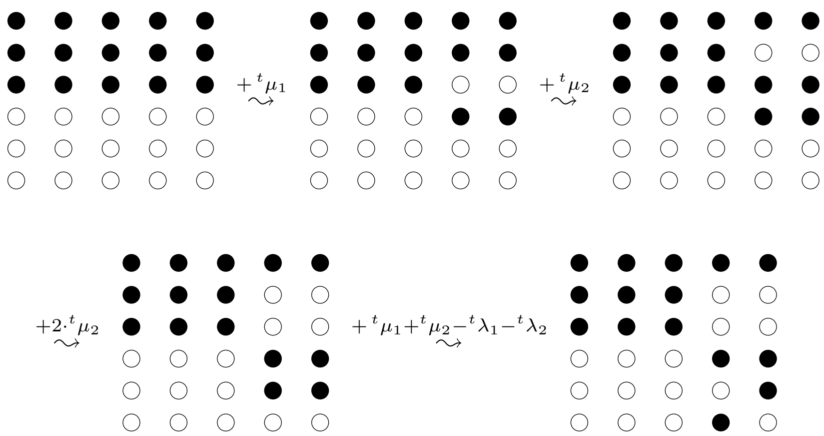


Figure 5.24. Counting the movements in the proof of Proposition 5.10.4 for $\lambda, \mu = ((3,1),(2,2))$.

5.10.4. Rigid modules for $G(\ell, 1, n)$. Recall that a simple module of a restricted rational Cherednik algebra $\overline{\mathsf{H}}_{\mathbf{c}}(W)$ is called *rigid* if it is already irreducible as a graded W-module. In [BT16], the authors have shown that an irreducible W-module cannot be induced from a parabolic subgroup of W and, using the description of symplectic leaves given in [Mar06], they have furthermore shown that each cuspidal type B family contains a rigid module and classified the cuspidal families by classifying the rigid modules.

In type $G(\ell,1,n)$ however, not every cuspidal family contains a rigid module. This can, for example, be computed using CHAMP [Thi15], or by using theoretical methods (cf. Exa. 5.10.7). It is therefore impossible to classify all cuspidal families using the same strategy as in type B. What we can still do, is to adapt the methods in [BT16] to $G(\ell,1,n)$ to study a smaller class of rigid modules. This would be a development of the toolset needed to give generalizations of the results in Section 5.8, e.g. the ungraded W-character of $L_{\mathbf{c}}(\mathbf{triv})$.

By [BT16, Lem. 4.10] we know that an $\overline{\mathsf{H}}_{\mathbf{c}}$-module $L_{\mathbf{c}}(\boldsymbol{\lambda})$ is rigid if and only if all commutators $[x,y] \in \overline{\mathsf{H}}_{\mathbf{c}}$ act by 0 for $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$. Using the commutator relations (4.18) and (4.19) from Section 4.2, we obtain for $L_{\mathbf{c}}(\boldsymbol{\lambda}) \cong \boldsymbol{\lambda} \in \operatorname{Irr} W$ the conditions

$$[x_i,y_j]\cdot v = -\sum_{k=0}^{\ell-1} \kappa\frac{1}{2}\zeta^k s_{ij}\gamma_i^{-k}\gamma_j^k \cdot v = 0 \ , \tag{5.152}$$

$$[x_i,y_i]\cdot v = \sum_{k=1}^{\ell-1} c_k\gamma_i^k \cdot v + \sum_{k=1}^{\ell-1}\sum_{i\neq j} \kappa\frac{1}{2}s_{ij}\gamma_i^{-k}\gamma_j^k \cdot v = 0 \tag{5.153}$$

for all $v \in \boldsymbol{\lambda}$ where we have used the Etingof–Ginzburg coordinates κ, c_k introduced in Section 4.2. Following the notation in Definition 2.3.4, we can view $\boldsymbol{\lambda} = (\lambda^{(0)},...,\lambda^{(\ell-1)})$ as being induced from

$$\boldsymbol{\lambda}^{\otimes} := \bigotimes_{j=0}^{\ell-1} U_j \wr \lambda^{(j)} \in \operatorname{Irr} \prod_{j=0}^{\ell-1} G(\ell,1,|\lambda^{(j)}|) \ . \tag{5.154}$$

Assume now there exist indices $p, p+1 \in \{0,...,\ell-1\}$ taken modulo ℓ such that

$$\lambda^{(p)} \neq \emptyset \neq \lambda^{(p+1)} \ . \tag{5.155}$$

Following the arguments in [BT16, Thm 6.24] we can now find integers $i,j \in \{0,...,\ell-1\}$ with $i<j$ such that

$$\gamma_i \cdot v = \zeta^p v \ , \quad \gamma_j \cdot v = \zeta^{p+1} v \tag{5.156}$$

for all $v \in \boldsymbol{\lambda}^{\otimes}$. Here, we have that γ_i, γ_j are in the respective group factors i,j of (5.154). Looking at the commutator $[x_j,y_i]$ in place of (5.152), we obtain

$$\begin{aligned}[x_j,y_i]\cdot v &= -\sum_{k=0}^{\ell-1} \kappa\frac{1}{2}\zeta^k s_{ij}\gamma_j^{-k}\gamma_i^k \cdot v \\ &= -\sum_{k=0}^{\ell-1} \kappa\frac{1}{2}\zeta^k s_{ij}\zeta^{-(p+1)\cdot k}\zeta^{p\cdot k} \cdot v \\ &= -\sum_{k=0}^{\ell-1} \kappa\frac{1}{2}s_{ij} \cdot v \\ &= -\ell\kappa\frac{1}{2}s_{ij}\cdot v \ ,\end{aligned} \tag{5.157}$$

which only vanishes for $v=0$. Therefore, the condition in (5.155) cannot be fulfilled for rigid $\boldsymbol{\lambda}$. We record this fact in a lemma.

LEMMA 5.10.5. *Let $\boldsymbol{\lambda} = (\lambda^{(0)},...,\lambda^{(\ell-1)})$ parametrize a rigid $\overline{\mathsf{H}}_{\mathbf{c}}$-module. We then have*

$$\lambda^{(i)} \neq \emptyset \implies \lambda^{(i-1)} = \lambda^{(i+1)} = \emptyset \tag{5.158}$$

for all $i \in \mathbb{Z}/\ell\mathbb{Z}$, i.e. nonempty partitions are always surrounded by empty ones.

The result above describes the familiar look of rigid modules in the type B case where we obtain rigid modules of the form $(\lambda, \emptyset)$ and $(\emptyset, {}^t\lambda)$ for some rectangle λ. We can see rectangles appearing in the $G(\ell, 1, n)$ case as well.

PROPOSITION 5.10.6. *For a Calogero–Moser parameter* $\mathbf{H} = (1, H_1, ..., H_{\ell-1})$, *all rigid modules of the form* $\boldsymbol{\lambda} = (\emptyset, ..., \emptyset, \lambda^{(p)}, \emptyset, ..., \emptyset)$ *must be equal to*

$$(\emptyset, ..., \emptyset, (H_p + k)^k, \emptyset, ..., \emptyset) \tag{5.159}$$

for some $k \geq 0$.

PROOF. Take a rigid module of the form

$$\boldsymbol{\lambda} = (\emptyset, ..., \emptyset, \lambda^{(p)}, \emptyset, ..., \emptyset) \ . \tag{5.160}$$

With the same arguments as before we have for (5.152)

$$[x_j, y_i] \cdot v = -\sum_{k=0}^{\ell-1} \kappa \frac{1}{2} \zeta^k s_{ij} \gamma_j^{-k} \gamma_i^k \cdot v = \left(\sum_{k=0}^{\ell-1} \zeta^k \right) \kappa \frac{1}{2} s_{ij} \cdot v = 0 \tag{5.161}$$

which is what we desired. For the second commutator (5.153), we obtain

$$\begin{aligned} [x_i, y_i] &= \sum_{k=1}^{\ell-1} c_k \gamma_i^k \cdot v + \sum_{k=0}^{\ell-1} \sum_{i \neq j} \frac{1}{2} \kappa s_{ij} \gamma_i^{-k} \gamma_j^k \cdot v \\ &= \sum_{k=1}^{\ell-1} c_k \gamma_i^k \cdot v + \sum_{k=0}^{\ell-1} \sum_{i \neq j} \frac{1}{2} \kappa s_{ij} \cdot v \\ &= \sum_{k=1}^{\ell-1} c_k \zeta_\ell^{p \cdot k} \cdot v + \sum_{k=0}^{\ell-1} \sum_{j \neq i} \frac{1}{2} \kappa s_{ij} \cdot v \\ &= \ell H_p \cdot v + \ell \sum_{i \neq j} h s_{ij} \cdot v \end{aligned}$$

where we used the Gordon basis (4.24). All in all, the expression above is equal to 0 if and only if

$$h \sum_{i \neq j} s_{ij} \cdot v = -H_p v \ . \tag{5.162}$$

By following the arguments in the proof of [BT16] and setting $h = 1$, we have that $\boldsymbol{\lambda}$ must be of the form

$$\boldsymbol{\lambda} = (\emptyset, ..., \emptyset, (H_p + k)^k, \emptyset, ..., \emptyset) \tag{5.163}$$

for some $k \geq 0$. The module $\boldsymbol{\lambda}$ is now rigid for parameter $\mathbf{c}$ and the group $G(\ell, 1, n')$ where $n' = k \cdot (H_p + k)$.

∎

Using Lemma 5.10.5 and Proposition 5.10.6, we can give examples of cuspidal families containing no rigid modules.

EXAMPLE 5.10.7. For $W = G(3, 1, 6)$ and 3-core charge $\mathbf{s} = (2, -1, -1)$, we have a cuspidal family $\mathcal{F}$ containing the family member $(\emptyset, \emptyset, (4, 2))$. By Lemma 5.10.5 for $\ell = 3$, all rigid modules of $\mathcal{F}$ have exactly one nonempty partition $\lambda^{(p)}$. Then, by Proposition 5.10.6, the partition $\lambda^{(p)}$ must be a rectangle. When we compute the elements of $\mathcal{F}$, we see that there is no family member with exactly one nonempty partition that is also a rectangle. Therefore, $\mathcal{F}$ is a cuspidal family which does not contain a rigid module.

One possible way to extend the classification of cuspidal families is to find a family member $L_{\mathbf{c}}(\lambda)$ in each cuspidal family that cannot be induced as a W-module from a parabolic subgroup of W. This, however, seems to be very difficult to prove, although one could start by gathering evidence for the existence of such a family member computationally.

We have now seen how we can use cuspidal family induction to compute characters of simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$ in type B and possibly type $G(\ell,1,n)$. Next, we want to take a step back and work out connections between cuspidal family induction and other established theories to see if our inductive phenomenon appears naturally elsewhere.

5.11. Comparison to j-induction

The complex reflection group W of type B_n affords a real reflection representation making W into a finite Coxeter group. For such a finite Coxeter group W, Lusztig defined *constructible characters* in [Lus03, Ch. 22] in order to study characters afforded by Kazhdan–Lusztig left cells. We want to review the relevant combinatorial framework associated to constructible characters here and relate these notions to the combinatorics of abaci developed in the previous sections. We follow the discussion in [BT16].

Similar to before, the theory for type B depends on the choice of two real numbers a, b. We want to study the nondegenerate case so we demand $a \neq 0$. By [BT16, Lem. 2.5] we can rescale such that $a = 1$. Because we are interested in nontrivial family structures, we will additionally demand that b is an integer. The case where b is negative can be derived from the positive case by twisting the families by the character $((1^n), \emptyset) \in \mathscr{P}(2,n)$ which when tensored with $(\lambda,\mu) \in \operatorname{Irr} B_n$ gives the irreducible character $({}^t\lambda, {}^t\mu)$. So, let $a = 1$ and $b \in \mathbb{Z}_{\geq 0}$.

For a bipartition $\boldsymbol{\lambda} = (\lambda,\mu) \in \mathscr{P}(2,n)$ we define the integer sequences

$$\begin{aligned} \beta_i &= \lambda_{N+b-i+1} + i - 1 \quad \text{for } i \in [1, N+b]\ , \\ \gamma_j &= \mu_{N-j+1} + j - 1 \qquad \text{for } j \in [1, N]\ , \end{aligned} \tag{5.164}$$

for some large enough integer N, e.g $N = n$. Note that we extend the partitions with trailing 0's as per usual and we define $[x,y]$ as the discrete interval $\{x, x+1, ..., y-1, y\}$ for $x, y \in \mathbb{Z}$. We can display the sequences (5.164) as a *symbol*

$$\mathcal{S}^N_{1,b}(\boldsymbol{\lambda}) = \begin{pmatrix} \beta_1 & \cdots & \beta_N & \cdots & \beta_{N+b} \\ \gamma_1 & \cdots & \gamma_N & & \end{pmatrix}\ . \tag{5.165}$$

The *content* of a symbol $\mathcal{S}$ is given by the polynomial

$$\operatorname{cont}(\mathcal{S}) = \sum_{k \in \mathcal{S}} x^k \tag{5.166}$$

where x is an indeterminate over $\mathbb{Z}$. Lusztig then defined the $\mathbf{a}$-*function*

$$\mathbf{a} : \mathscr{P}(2,n) \to \mathbb{Z}\ , \tag{5.167}$$

which associates to a bipartition $\boldsymbol{\lambda}$ the $\mathbf{a}$-*value* $\mathbf{a}_{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$. A combinatorial description of the $\mathbf{a}$-function can be given by using the entries of the symbol attached to $\boldsymbol{\lambda}$. We give the $\mathbf{a}$-value in the form found in [GJ11, Exm. 1.3.9]. For a symbol $\mathcal{S}^N_{1,b}(\boldsymbol{\lambda})$, we denote by

$$Z^N_{1,b}(\boldsymbol{\lambda}) = (z_1(\boldsymbol{\lambda}), ..., z_{2N+b}(\boldsymbol{\lambda})) \tag{5.168}$$

the weakly decreasing list of entries of $\mathcal{S}^N_{a,b}(\boldsymbol{\lambda})$. We then define

$$\mathbf{a}_{\boldsymbol{\lambda}} = \sum_{i=1}^{2N+b} (i-1) \cdot z_i(\boldsymbol{\lambda}) - \sum_{i=1}^{2N+b} (i-1) \cdot z_i(\boldsymbol{\emptyset})\ . \tag{5.169}$$

The number $\mathbf{a}_{\boldsymbol{\lambda}}$ is independent of N by [Lus03, 22.14(b)] and we have

$$\mathbf{a}_{\mathbf{triv}} = 0 \tag{5.170}$$

where $\mathbf{triv} = ((n), \emptyset)$ is the partition corresponding to the trivial B_n-representation.

Lusztig now used the **a**-values of bipartitions to relate representations of B_n with representations of parabolic subgroups of B_n. By first identifying $\mathscr{P}(2,n)$ with $\operatorname{Irr} B_n$, we obtain the **a**-value of irreducible characters of B_n. We then extend this definition additively to arbitrary B_n-characters. Next, we want to extend the **a**-value to characters of parabolic subgroups of B_n.

Let P be a parabolic subgroup of B_n, i.e. P stabilizes a linear subspace in the real reflection representation of B_n. By Theorem 4.5.6 we have (up to B_n-conjugacy)

$$P = B_{n-k} \times \mathfrak{S}_\alpha \tag{5.171}$$

where $0 \leq k \leq n$, and

$$\mathfrak{S}_\alpha = \prod_{i \geq 1} \mathfrak{S}_{\alpha_i} \tag{5.172}$$

is the Young subgroup of $\mathfrak{S}_l$ corresponding to $\alpha \in \mathscr{P}(l)$ for $l \leq k$. Lusztig then extended the definition of **a**-values to characters of P by defining the **a**-value for symmetric group characters and then using (5.171): for $\nu \in \operatorname{Irr} \mathfrak{S}_n$ let as in [GJ11, Exm. 1.3.8]

$$\mathbf{a}_\nu = \sum_{i \geq 1} (i-1) \cdot \nu_i \tag{5.173}$$

which is the b-invariant of λ from Definition 2.1.13. Since P is of the form $B_{n-k} \times \mathfrak{S}_\alpha$, an irreducible character η of P is parametrized by

$$\boldsymbol{\lambda} \times \prod_i \nu^{(i)} \tag{5.174}$$

for some $\boldsymbol{\lambda} \in \operatorname{Irr} B_n$ and $\nu^{(i)} \in \operatorname{Irr} \mathfrak{S}_{\alpha_i}$ for all i. We define

$$\mathbf{a}_{\boldsymbol{\lambda} \times \prod_i \nu^{(i)}} := \mathbf{a}_{\boldsymbol{\lambda}} + \sum_i \mathbf{a}_{\nu^{(i)}} \tag{5.175}$$

and extend this definition additively to all characters of P. Note that groups of the form (5.171) are closed under taking parabolic subgroups. We can now give Lusztig's definition of **j**-induction.

Definition 5.11.1. Let P, P' be parabolic subgroups of B_n such that $P' \leq P$ and let η be a character of P'. The **j**-*induction* (or *truncated induction*) of η to P is the character of P given by

$$\mathbf{j}_{P'}^P \eta = \sum_{\substack{\chi \in \operatorname{Irr}(P) \\ \mathbf{a}_\chi = \mathbf{a}_\eta}} \langle \operatorname{Ind}_{P'}^P \eta, \chi \rangle \cdot \chi \ , \tag{5.176}$$

i.e. we induce η to P and collect all irreducible constituents with the same **a**-value as η where $\langle \cdot, \cdot \rangle$ denotes the inner product of characters.

Note that **j**-induction is transitive and we can restrict to the case when P' is a maximal parabolic subgroup of P, e.g. for $P = B_n$ we have P' the form $B_{n-k} \times \mathfrak{S}_k$ for some k. Next, Lusztig defined a set of characters of B_n by starting with the trivial character of the trivial group and **j**-inducing to parabolic subgroups of B_n.

Definition 5.11.2. Let P be a parabolic subgroup of B_n. The set of *constructible characters* of P is defined as

$$\operatorname{Con}(P) = \left\{ \mathbf{j}_{P'}^P \eta \ , \ \operatorname{sgn} \otimes \mathbf{j}_{P'}^P \eta \mid \eta \in \operatorname{Con}(P'), \ P' \leq P \text{ parabolic} \right\} \tag{5.177}$$

where $\operatorname{sgn} \operatorname{Irr} P$ is the sign character of P which sends each Coxeter generator of P to -1. For the trivial parabolic subgroup $\{1\}$, we define $\operatorname{Con}(\{1\}) = \{\mathbf{triv}\}$ where $\mathbf{triv}$ is the trivial character.

The set $\operatorname{Con}(B_n)$ can be used to define families of $\operatorname{Irr} B_n$. We write $\boldsymbol{\lambda} \sim \boldsymbol{\mu}$ if there exists a constructible character $\eta \in \operatorname{Con}(B_n)$ such that $\langle \eta, \boldsymbol{\lambda} \rangle, \langle \eta, \boldsymbol{\mu} \rangle \neq 0$. Denote by $\sim_L$ the transitive closure of $\sim$ and we call an equivalence class of $\sim_L$ a *Lusztig family* of B_n (with respect to the parameters a, b). The following theorem can be derived from results in [Lus03, Ch. 22] as has been done in [BT16, Thm. 6.11], or alternatively one may specialize the more general result [BK02, Thm. 3.13] together with [Chl07, Thm. 3.11].

THEOREM 5.11.3 (Lusztig). *For $b \geq 0$, and large enough integer N, two bipartitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ belong to the same Lusztig family if and only if their respective symbols $\mathcal{S}^N_{1,b}(\boldsymbol{\lambda})$, $\mathcal{S}^N_{1,b}(\boldsymbol{\mu})$ have the same content, i.e.* $\operatorname{cont}(\mathcal{S}^N_{1,b}(\boldsymbol{\lambda})) = \operatorname{cont}(\mathcal{S}^N_{1,b}(\boldsymbol{\mu}))$.

Because of the result above, we may also write $\operatorname{Con}(\mathcal{F})$ when referring to the content of the symbols attached to a Lusztig family $\mathcal{F}$.

The definition of constructible characters and thus Lusztig families works for the parabolic subgroup $\mathfrak{S}_n \leq B_n$ as well, for which Lusztig showed in [Lus03, Lem. 22.4] that

$$\operatorname{Con}(\mathfrak{S}_n) = \operatorname{Irr} \mathfrak{S}_n \ , \tag{5.178}$$

i.e. every constructible character is irreducible and thus all Lusztig families of $\mathfrak{S}_n$ are singletons. This means the Lusztig families of $B_{n-k} \times \mathfrak{S}_k$ are given by

$$\{\mathcal{F} \mid \mathcal{F} \text{ is a Lusztig family of } B_{n-k}\} \times \mathscr{P}(k) \tag{5.179}$$

where $\mathscr{P}(k)$ parametrizes $\operatorname{Con}(\mathfrak{S}_n) = \operatorname{Irr}(\mathfrak{S}_k)$.

Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be a family of B_n (resp. $B_{n'}$) for $n \geq n'$. We say $\mathcal{F}$ is $\mathbf{j}$-*induced* from $\mathcal{F}'$ if there exists a character $\nu \in \operatorname{Irr} \mathfrak{S}_{n-n'}$ such that $\mathbf{j}$-induction restricts to a bijection

$$\mathcal{F}' \times \{\nu\} \to \mathcal{F} \ , \ \boldsymbol{\lambda} \times \nu \mapsto \mathbf{j}^{B_n}_{B_{n'} \times \mathfrak{S}_{n-n'}} \boldsymbol{\lambda} \times \nu \ . \tag{5.180}$$

This bijection also carries over to the family

$$\operatorname{sgn} \otimes \mathcal{F} := \{\operatorname{sgn} \otimes \boldsymbol{\lambda} \mid \boldsymbol{\lambda} \in \mathcal{F}\} \tag{5.181}$$

and we say $\operatorname{sgn} \otimes \mathcal{F}$ is $\mathbf{j}$-induced from $\mathcal{F}'$ as well. Note that the sgn character is given by $(\emptyset, (1^n)) \in \mathscr{P}(2, n)$ and

$$\operatorname{sgn} \otimes (\lambda, \mu) = ({}^t\mu, {}^t\lambda) \tag{5.182}$$

for $(\lambda, \mu) \in \operatorname{Irr} B_n$. The induction (5.180) has been given in [Lus03, Sec. 22.15] and in more detail in the proof of [BT16, Thm. 6.21]. We will visualize the induction at the end of this section using abaci.

The Lusztig families that are not $\mathbf{j}$-induced from other Lusztig families are called *cuspidal*. A classification of cuspidal families in type B have again been derived in [BT16, Sec. 6.12] summarizing results in [Lus03, Ch. 22].

THEOREM 5.11.4. *Let $\mathcal{F}$ be a Lusztig family and* $\operatorname{Con}(\mathcal{F}) = \sum_{k \geq 0} i_k \cdot x^k$ *the content of $\mathcal{F}$. We then have that $\mathcal{F}$ is cuspidal if and only if $i_k \geq i_{k+1}$ for all $k \in \mathbb{N}$, i.e. the exponents in* $\operatorname{Con}(\mathcal{F})$ *form a weakly decreasing sequence of nonnegative integers.*

We will now relate the theory of $\mathbf{j}$-induction of Lusztig families to cuspidal family induction of Calogero–Moser families. Let us begin by translating the combinatorics of symbols to β-numbers and abaci. We can transform the indices $1 \leq i \leq N+b$, $1 \leq j \leq N$ in (5.164) by

$$i' = N + b - i + 1 \ , \quad j' = N - j + 1 \tag{5.183}$$

which simply reverses the order in which the sequences are listed, and we obtain

$$\begin{aligned} \beta_{i'} &= \lambda_{i'} + N + b - i' \quad \text{for } i' \in [1, N+b] \ , \\ \gamma_{j'} &= \mu_{j'} + N - j' \qquad \text{for } j' \in [1, N] \ . \end{aligned} \tag{5.184}$$

Now, these new sequences are just the positive segments of the sequences of β-numbers of λ and μ stabilizing with respect to $N+b$ and N, respectively. From the proof of [Lus03, 22.14(b)] we know that the **a**-value does not change if we replace N by $N+1$. Note that the **a**-value also stays constant if we subtract 1 from each entry in the sequence since these changes would cancel out in the difference (5.169). Now, both subtracting the symbol entries by 1 as well as replacing N with $N+1$ amounts to extending the index sets to

$$i' \in [1, N+b+1] \ , \ j' \in [1, N+1] \ . \tag{5.185}$$

This way, we can take the “direct limit” of symbols and extend the definition of the **a**-function to sequences of β-numbers and 2-abaci. The only condition is that the stabilizing numbers of the sequences of β-numbers have difference b. The corresponding charge of the 2-core would be given by $(b, 0)$. To a sequence of β-numbers, we could associate a formal power series analogous to the content. It is easy to see that after this transformation and extension, Theorem 5.11.3 is still valid.

Over on the Calogero–Moser side, we can now form the 2-abacus associated to the Calogero–Moser parameter $\mathbf{c} = (\kappa, c_1) = (1, b)$ and charge $\mathbf{s} = (0, b)$. We obtain the same 2-abaci as from the symbols of the transformed sequences (5.184) with the conventions introduced in Section 5.7. Now, the visual $\{1\}$-core of a 2-abacus is simply counting the number of beads in each row of the abacus the same way the content of a symbol does. We thus obtain [BT16, Cor. 6.13].

PROPOSITION 5.11.5. *For $b \in \mathbb{Z}_{\geq 0}$ and type B, the Lusztig families with respect to the parameters $(a = 1, b)$ and the Calogero–Moser families associated to the Calogero–Moser parameter $\mathbf{c} = (1, b)$ are equal.*

When the content of a family is weakly decreasing, we have that the corresponding 2-abacus of that family J-core more beads in row i than row $i+1$ for all $i \in \mathbb{Z}$. This is precisely the condition that the family J-core is already a 2-core, i.e. cuspidal. We retrieve [BT16, Thm. 6.26].

PROPOSITION 5.11.6. *For $b \geq 0$ and type B, the cuspidal Lusztig families with respect to the parameters $(a = 1, b)$ and the cuspidal Calogero–Moser families associated to the Calogero–Moser parameter $\mathbf{c} = (1, b)$ are equal.*

We now want to use abaci to describe the bijection [Lus03, Sec. 22.15] which is afforded by **j**-induction. Take a family $\mathcal{F}$ of the group B_n for some n. Take the sequence

$$Z^N_{1,b}(\boldsymbol{\lambda}) = (z_1(\boldsymbol{\lambda}), ..., z_{2N+b}(\boldsymbol{\lambda})) \tag{5.186}$$

from (5.168) and increase the k entries $z_1(\boldsymbol{\lambda}), ..., z_k(\boldsymbol{\lambda})$ with the p largest values each by 1. If the p^{th} largest entry appears twice, we have to increase both entries of $Z^N_{1,b}(\boldsymbol{\lambda})$. We obtain a new symbol with entries

$$z_1(\boldsymbol{\lambda}) + 1, ..., z_k(\boldsymbol{\lambda}) + 1, z_{k+1}(\boldsymbol{\lambda}), ..., z_{2N+b}(\boldsymbol{\lambda}) \tag{5.187}$$

corresponding to a bipartition $\boldsymbol{\lambda}' \in \mathscr{P}(2, n+k)$. We can see that the $\mathbf{a}$-value (5.169) becomes

$$\mathbf{a}_{\boldsymbol{\lambda}'} = \sum_{i=1}^{k} (i-1)\cdot(z_i(\boldsymbol{\lambda})+1) + \sum_{i=k+1}^{2N+b} (i-1)\cdot z_i(\boldsymbol{\lambda}) - \sum_{i=1}^{2N+b} (i-1)\cdot z_i(\boldsymbol{\emptyset}) \tag{5.188}$$

$$= \mathbf{a}_{\boldsymbol{\lambda}} + \sum_{i=1}^{k} (i-1) \tag{5.189}$$

$$= \mathbf{a}_{\boldsymbol{\lambda}} + \mathbf{a}_{(1^k)} \tag{5.190}$$

where (1^k) denotes the partition $(1, ..., 1) \in \mathscr{P}(k)$ in multiplicative notation. This inductive process works for all family members of $\mathcal{F}$ and it is clear that the new bipartitions all have the same content, i.e. they belong to the same family $\mathcal{F}'$. Furthermore, $\mathbf{j}$-induction induces a bijection

$$\mathcal{F} \times \{(1^k)\} \to \mathcal{F}' \ , \ \boldsymbol{\lambda} \times (1^k) \mapsto \mathbf{j}_{B_n \times \mathfrak{S}_k}^{B_{n+k}} \boldsymbol{\lambda} \times (1^k) = \boldsymbol{\lambda}' \tag{5.191}$$

and also one between the families $\mathcal{F} \to \operatorname{sgn} \otimes \mathcal{F}'$.

On the level of abaci, we choose some $p \in \mathbb{Z}_+$ and take all beads in the p bottommost occupied rows and shift each down by 1 position. This is equivalent to inserting an empty row above these p rows. With this method, we can now arbitrarily insert ○○-rows in the abacus to induce new families. As mentioned previously, the family members of $\operatorname{sgn} \otimes \mathcal{F}'$ are obtained from the ones of $\mathcal{F}$ by

$$\mathcal{F} \to \operatorname{sgn} \otimes \mathcal{F} \ , (\lambda, \mu) \mapsto \operatorname{sgn} \otimes (\lambda, \mu) = ({}^t\mu, {}^t\lambda) \tag{5.192}$$

Take now a family $\mathcal{F}$ obtained from a cuspidal family $\mathcal{F}^{\text{cusp}}$ by inserting an empty row somewhere in the abacus of $\mathcal{F}^{\text{cusp}}$. For $\boldsymbol{\lambda} \in \mathcal{F}$, we obtain the abacus of $\operatorname{sgn} \otimes \boldsymbol{\lambda}$ by using Lemma 5.2.6, i.e. we swap the two columns, replacing each marked position by an unmarked position (and vice versa), and flip the abacus at the horizontal axis to get the black beads on top (see Figure 5.25). Note that the 2-core does not change from this operation.

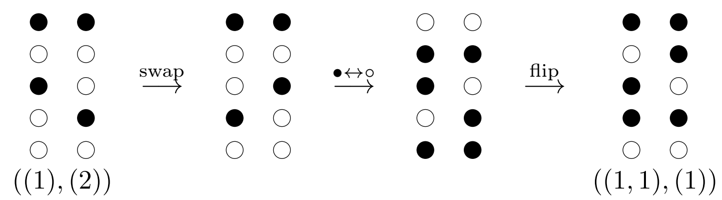


FIGURE 5.25. Abacus operations for $\operatorname{sgn} \otimes ((1), (2)) = ((1, 1), (1))$.

Applying this process to the abacus of a cuspidal family member gives us again a member of the same cuspidal family. For non-cuspidal families, i.e. an abacus with inserted full and empty rows in between active beads, this process turns ●●-rows into ○○-rows and vice versa. This means the abaci of $\operatorname{sgn} \otimes \mathcal{F}$ have a full row inserted where $\mathcal{F}$ has had an empty row inserted. Iterating this, we can insert full and empty rows into abaci of all family members. This means that $\mathbf{j}$-induction and cuspidal family induction associate the same families to one another. The bijections agree on a family member level whenever we insert empty rows. For full rows, the bijection afforded by $\mathbf{j}$-induction is twisted by the sgn character which we then have to reverse. We obtain the following result.

PROPOSITION 5.11.7. *In type B, Lusztig's* $\mathbf{j}$*-induction and cuspidal family induction agree.*

We have now seen that the Calogero–Moser combinatorics of abaci and cuspidal family induction specialize to Lusztig's symbols and **j**-induction. From the discussion above, we can tell that **j**-induction induces a bijection between constituents of constructible characters as well. How to best generalize constructible characters to higher wreath products is still unknown but one could expect cuspidal family induction to play a similar role as **j**-induction in the type B case. We want to review possible generalizations of Lusztig's work next and see how these theories interface with the induction phenomena we have seen thus far.

5.12. Constructible and cellular characters

There are several algebras associated to $W = C_\ell \wr \mathfrak{S}_n$ that afford similar W-characters as Lusztig's constructible characters, which come from the theory of Hecke algebras associated to Coxeter groups (see [GJ11] for more details). In that vein, there is also the notion of *Kazhdan–Lusztig cellular characters* of B_n discussed for example in [Lus03] which are proven to be equal to Lusztig's constructible characters of B_n for $a = b$.

Another notion of cellular characters has been developed in [BR17] and comes from rational Cherednik algebras at $t = 0$. They are called *Calogero–Moser cellular characters* and they are conjectured to coincide with Kazhdan–Lusztig cellular characters for $W = B_n$, therefore generalizing these characters to complex reflection groups.

The third theory which associates to $C_\ell \wr \mathfrak{S}_n$ a notion of constructible characters comes from the the quantum group $\mathcal{U}_q(\mathfrak{sl}_\infty)$. These characters are called *Leclerc–Miyachi constructible characters* and they can be computed quite easily using the algorithm in [LT00].

There have been some connections made between the characters of these three worlds: in type B, we have by [LM04] that constructible characters associated to Hecke algebras coincide with the ones associated to quantum groups. We are going to review parts of this construction later. For W the dihedral group, which is a non-Weyl Coxeter group and a complex reflection group of type $G(\ell, \ell, 2)$, the Calogero–Moser cellular characters agree with Kazhdan–Lusztig cellular characters by [Bon18]. Lacabanne proved in [Lac20, Thm. 4.3] that Leclerc–Miyachi constructible characters and Calogero–Moser cellular characters conicide for $W = G(\ell, 1, 2)$ and any parameter $\mathbf{c}$. Furthermore, in asymptotic charge $\mathbf{s}$, i.e.

$$s_0 >> s_1 >> \dots >> s_{\ell-1}\ , \tag{5.193}$$

Lacabanne proved in [Lac20, Sec. 1+2] for $W = G(\ell, 1, n)$ that the Leclerc–Miyachi constructible characters and the Calogero–Moser cellular characters are both irreducible and thus coincide. Lacabanne conjectured that this is true for all charges.

We are going to review some explicit constructions of these characters to see how traces of cuspidal family induction and the rest of our combinatorial theory can be observed.

5.12.1. Leclerc–Miyachi constructible characters.

We start by defining $\mathcal{U}_q(\mathfrak{sl}_\infty)$, i.e. the quantized universal enveloping algebra of the affine Lie algebra $\mathfrak{sl}_\infty$ associated to the Dynkin diagram of type A_∞. Throughout this section, we adapt the definition and discussion found in [LT00] to the A_∞ case using the arguments in [LM04, Sec. 2.7]. The book [HK02] is a general reference for the theories of this section as well.

Let q be a complex indeterminate. Let $\mathcal{U}_q(\mathfrak{sl}_\infty)$ be the $\mathbb{C}(q)$-algebra generated by the symbols

$$e_i, f_i, q^{\varepsilon_i}, q^{-\varepsilon_i} \tag{5.194}$$

for $i \in \mathbb{Z}$ subject to the relations

$$q^{\varepsilon_i} q^{-\varepsilon_i} = q^{-\varepsilon_i} q^{\varepsilon_i} = 1 \ , \quad [q^{\varepsilon_i}, q^{\varepsilon_j}] = 0 \ , \tag{5.195}$$

$$q^{\varepsilon_i} e_j q^{-\varepsilon_i} = \begin{cases} q e_j & \text{if } i = j \ , \\ q^{-1} e_j & \text{if } i = j+1 \ , \\ e_j & \text{else} \ , \end{cases} \tag{5.196}$$

$$q^{\varepsilon_i} f_j q^{-\varepsilon_i} = \begin{cases} q^{-1} f_j & \text{if } i = j \ , \\ q f_j & \text{if } i = j+1 \ , \\ f_j & \text{else} \ , \end{cases} \tag{5.197}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{\varepsilon_i} q^{-\varepsilon_{i+1}} - q^{-\varepsilon_i} q^{\varepsilon_{i+1}}}{q - q^{-1}} \ , \tag{5.198}$$

$$[e_i, e_j] = 0 = [f_i, f_j] \qquad \text{if } |i-j| > 1 \ , \tag{5.199}$$

$$e_j e_i^2 - (q+q^{-1}) e_i e_j e_i + e_i^2 e_j = 0 \qquad \text{if } |i-j| = 1 \ , \tag{5.200}$$

$$f_j f_i^2 - (q+q^{-1}) f_i f_j f_i + f_i^2 f_j = 0 \qquad \text{if } |i-j| = 1 \ . \tag{5.201}$$

By (5.195) we will identify $\langle \varepsilon_i, -\varepsilon_i \mid i \in \mathbb{Z} \rangle$ with $\mathbb{Z}^{\mathbb{Z}}$.

The representation theory of $\mathcal{U}_q(\mathfrak{sl}_\infty)$ resembles that of $\mathcal{U}(\mathfrak{sl}_\infty)$: let M be a $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module and let $\mu \in \mathbb{Z}^{\mathbb{Z}}$ be an integral vector. We can form the *weight space* M_μ of M by

$$M_\mu = \{ v \in M \mid q^{\varepsilon_i}.v = q^{\mu_i} v \ , \ i \in \mathbb{Z} \} \tag{5.202}$$

and we say μ is a *weight* of M if M_μ is nonzero. By (5.196) and (5.197) we have

$$e_i.M_\mu \subset M_{\mu+\alpha_i} \ , \quad f_i.M_\mu \subset M_{\mu-\alpha_i} \tag{5.203}$$

where $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \mathbb{Z}^{\mathbb{Z}}$ for all $i \in \mathbb{Z}$. A vector v is called *highest weight vector* if

$$e_i.v = 0 \tag{5.204}$$

for all $i \in \mathbb{Z}$. The module M is called *highest weight module* if M contains a unique highest weight vector v such that

$$M = \mathcal{U}_q(\mathfrak{sl}_\infty).v \ . \tag{5.205}$$

The weights λ for which we can associate a unique highest weight module $V(\lambda)$ are called *dominant weights*. A class of examples of dominant weights are given by the *fundamental weights*

$$\Lambda_i = \sum_{j \leq i} \varepsilon_i \in \mathbb{Z}^{\mathbb{Z}} \ . \tag{5.206}$$

Let $V(\Lambda_s)$ be the highest weight module associated to Λ_s for some $s \in \mathbb{Z}$. The index set $\{j \mid j \leq s\}$ of the summands $(\varepsilon_j)_{j \leq s}$ of Λ_s form the entries of the sequence of β-numbers of $\emptyset \in \mathscr{P}$ stabilizing with respect to s. For highest weight modules of fundamental weights, all occurring weight spaces can be associated with sequences of β-numbers stabilizing with respect to s in the same way. We can in turn identify this set of all β-numbers with the set of partitions $\mathscr{P}$. Furthermore, it is well-known that the module $V(\Lambda_s)$ is the direct sum of its weight spaces which are all 1-dimensional. We therefore have a basis of $V(\Lambda_s)$ of the form

$$\left\{ v(\mu) \mid V(\Lambda_s)_\mu \neq 0 \ , \ \mu \in \mathbb{Z}^{\mathbb{Z}} \right\} \ . \tag{5.207}$$

where $v(\mu)$ is some nonzero vector inside $V(\Lambda_s)_\mu$.

Let μ be a weight with corresponding sequence of β-numbers $\beta(\mu)$. Adding (resp. subtracting) α_i to μ as done in (5.203) gives us a weight corresponding to a sequence of β-numbers only if

$$i \notin \beta(\mu), \ i+1 \in \beta(\mu) \ (\text{resp. } i \in \beta(\mu), \ i+1 \notin \beta(\mu)) \ . \tag{5.208}$$

Therefore, the operator f_i (resp. e_i) annihilates the vector $v(\mu)$ when the condition is not met since the space $V(\mu+\alpha)$ (resp. $V(\mu-\alpha)$) is empty then. This gives us the full action of $\mathcal{U}_q(\mathfrak{sl}_\infty)$ on the weight space basis (5.207) of $V(\Lambda_s)$.

When associating the respective 1-abaci to the sequences of β-numbers of weights of $V(\Lambda_s)$, we see by (5.203) that f_i moves a bead in row i down 1 position and e_i reverses that movement.

REMARK 5.12.1. Note that we lose the information about the stabilizing number by working with 1-abaci but by the automorphism of the affine Dynkin diagram of type A_∞ which sends a node i to $i+1$, we get back the same module structure.

Let now $\mathbf{s} = (s_0, ..., s_{\ell-1}) \in \mathbb{Z}^{\mathbb{Z}/\ell\mathbb{Z}}$ be the charge of a staircase abacus where we impose no restriction on the sum of the entries of $\mathbf{s}$. The weight

$$\Lambda_{\mathbf{s}} = \Lambda_{s_0} + \cdots + \Lambda_{s_{\ell-1}} \tag{5.209}$$

is a dominant weight as well and we can associate to $\Lambda_{\mathbf{s}}$ a highest weight module $V(\Lambda_{\mathbf{s}})$. We want to describe the structure of $V(\Lambda_{\mathbf{s}})$ by an embedding into a different module, namely

$$F(\Lambda_{\mathbf{s}}) = V(\Lambda_{s_0}) \otimes \cdots \otimes V(\Lambda_{s_{\ell-1}}) \tag{5.210}$$

which is called a *Fock space of level* ℓ. For two $\mathcal{U}_q(\mathfrak{sl}_\infty)$-modules U, V, the action of $\mathcal{U}_q(\mathfrak{sl}_\infty)$ on the tensor product $U \otimes V$ is given by

$$\begin{aligned} q^{\varepsilon_i}.(u \otimes v) &= q^{\varepsilon_i}.u \otimes q^{\varepsilon_i}.v \\ e_i.(u \otimes v) &= e_i.u \otimes v + q^{-\varepsilon_i} q^{\varepsilon_{i+1}}.u \otimes e_i.v \\ f_i.(u \otimes v) &= f_i.u \otimes q^{\varepsilon_i} q^{-\varepsilon_{i+1}} v + u \otimes f_i.v \end{aligned} \tag{5.211}$$

for $u \in U, v \in V$. To embed $V(\Lambda_{\mathbf{s}})$ into $F(\Lambda_{\mathbf{s}})$ let

$$\mathbf{v} = v_0 \otimes \cdots \otimes v_{\ell-1} \in F(\Lambda_{\mathbf{s}}) \tag{5.212}$$

be the tensor product of the highest weight vectors $v_i \in V(\Lambda_{s_i})$ for $i \in \mathbb{Z}/\ell\mathbb{Z}$. The vector $\mathbf{v}$ is a highest weight vector of weight $\Lambda_{\mathbf{s}}$ and therefore mapping the highest weight vector $v(\Lambda_{\mathbf{s}}) \in V(\Lambda_{\mathbf{s}})$ onto $\mathbf{v} \in F(\Lambda_{\mathbf{s}})$ gives us an embedding of $\mathcal{U}_q(\mathfrak{sl}_\infty)$-modules

$$V(\Lambda_{\mathbf{s}}) \hookrightarrow F(\Lambda_{\mathbf{s}}) \, . \tag{5.213}$$

So, let us describe $F(\Lambda_{\mathbf{s}})$ by using abaci again.

As a vector space, we know that $F(\Lambda_{\mathbf{s}})$ has a basis indexed by the cartesian product of bases of the $V(\Lambda_i)$ for $i \in \mathbb{Z}/\ell\mathbb{Z}$. Thus we get a basis of $F(\Lambda_{\mathbf{s}})$ parametrized by

$$\mathscr{B}_{\mathbf{s}} := \mathscr{B}_{s_1} \times \cdots \times \mathscr{B}_{s_{\ell-1}} \tag{5.214}$$

where $\mathscr{B}_s$ denotes the set of all sequences of β-numbers stabilizing with respect to $s \in \mathbb{Z}$. We can now associate to an element of $\mathscr{B}_{\mathbf{s}}$ an ℓ-abacus and represent $F(\Lambda_{\mathbf{s}})$ as $\mathbb{C}(q)$-linear combinations of ℓ-abaci of charge $\mathbf{s}$. When the charge is clear from the context, we identify the ℓ-abaci with their respective ℓ-quotients and $F(\Lambda_{\mathbf{s}})$ with the $\mathbb{C}(q)$-vector space generated by the set of all ℓ-multipartitions

$$\bigcup_{k \geq 0} \mathscr{P}(\ell, k) = \mathscr{P} \times \cdots \times \mathscr{P} = \mathscr{P}^\ell \, . \tag{5.215}$$

We want to find a "nice" basis of $V(\Lambda_{\mathbf{s}})$ inside $F(\Lambda_{\mathbf{s}})$. In the classical case of $\mathcal{U}(\mathfrak{sl}_n)$, we have fundamental weights of the form

$$\Lambda_i = (\underbrace{1, ..., 1}_{i}, \underbrace{0, ..., 0}_{n-i}) \tag{5.216}$$

for $i \in \{1, ..., n-1\}$. We can identify a dominant weight

$$\lambda = a_1\Lambda_1 + \cdots + a_{n-1}\Lambda_{n-1} \in \mathbb{N}^n \ , \tag{5.217}$$

where $a_1, ..., a_{n-1}$ are nonnegative integers, with a partition of length smaller than n via the partial sums

$$(a_1 + \cdots + a_{n-1},\ a_2 + \cdots + a_{n-1},\ ...,\ a_{n-1}) \in \mathscr{P} \tag{5.218}$$

and we denote this partition by λ as well. We can now give a basis of the irreducible $\mathfrak{sl}_n$-module $V(\lambda)$ which is parametrized by the semistandard Young tableaux of shape λ filled with integers from the set $\{1, ..., n\}$ (see [Ful97, Sec. 8.2+8.3]). To this end, we want to review the construction of the *global crystal basis* in [LT00] which is a basis

$$(G(T))_T \subseteq V(\Lambda_{\mathbf{s}}) \hookrightarrow F(\Lambda_{\mathbf{s}}) \tag{5.219}$$

indexed by semistandard Young tableaux of shape $\Lambda_{\mathbf{s}}$ with integers $\{1, ..., n\}$.

This construction extends to the $\mathcal{U}_q(\mathfrak{sl}_\infty)$ case as well by [LM04, Sec. 2.7], for which we give the abacus interpretation here. First, we can view the beads of a staircase ℓ-abacus as an "infinite Young diagram" that is made up of an infinite sequence of rows with ℓ boxes, a finite sequence of rows with less than ℓ boxes, and trailing rows with 0 boxes (see Figure 5.26 for an example).

(5.220)

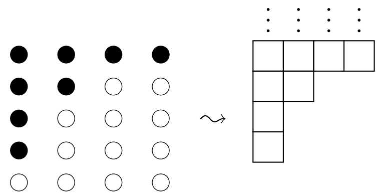


FIGURE 5.26. Infinite Young diagram corresponding to the staircase with charge $\mathbf{s} = (2, 0, -1, -1) \in \mathbb{Z}_0^\ell$.

Next, we define a filling on that Young diagram by filling each of the boxes with the row label of the bead corresponding to that box. An example is displayed in Figure 5.27.

(5.221)

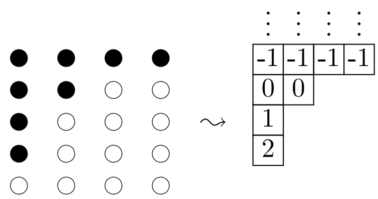


FIGURE 5.27. Infinite filling corresponding to the staircase with charge $\mathbf{s} = (2, 0, -1, -1) \in \mathbb{Z}_0^\ell$.

More generally, we can associate to any ℓ-abacus with staircase ℓ-core ν a filling of a Young diagram with infinite rows associated to ν. We do this by fixing the association between boxes of the Young diagram and beads of the staircase, then constructing an ℓ-multipartition out of the staircase abacus, and filling the boxes of the Young diagram with the row index of the new position of the bead that is associated to that box. We again give an example of this in Figure 5.28. Note that these fillings always strictly increase along columns but not necessarily weakly along rows.

$$\begin{array}{cccc} \bullet & \bullet & \circ & \bullet \\ \bullet & \circ & \circ & \circ \\ \circ & \bullet & \bullet & \circ \\ \bullet & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ \end{array} \rightsquigarrow \begin{array}{|c|c|c|c|} \vdots & \vdots & \vdots & \vdots \\ \hline -1 & -1 & 1 & -1 \\ \hline 0 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \tag{5.222}$$

FIGURE 5.28. Infinite filling corresponding to the staircase with charge $\mathbf{s} = (2, 0, -1, -1) \in \mathbb{Z}_0^\ell$.

Compared to the $\mathfrak{sl}_n$-case, we now allow all integers in our fillings. There is, however, a finiteness condition: only a finite amount of boxes can have a different filling compared to the filling of the ℓ-abacus of $\boldsymbol{\emptyset} \in \mathscr{P}^\ell$. We can now describe the basis of the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module $V(\Lambda_{\mathbf{s}})$.

DEFINITION 5.12.2. We say an ℓ-abacus with staircase ℓ-core is *semistandard* if the associated tableaux T is a semistandard Young tableaux.

Here, the definition of infinite semistandard Young tableaux is the same as in the classic case. We get the following result.

LEMMA 5.12.3 ([LM04, Sec. 2.7]). *There is a basis of the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module $V(\Lambda_{\mathbf{s}})$ parametrized by semistandard ℓ-abaci.*

In [Kas91; Kas92] Kashiwara studied the *global crystal basis*

$$\mathcal{B} = (G(T))_T \subseteq V(\Lambda_{\mathbf{s}}) \hookrightarrow F(\Lambda_{\mathbf{s}}) \tag{5.223}$$

indexed by semistandard Young tableaux. In [LT00], Leclerc and Toffin have then given an algorithm to compute $\mathcal{B}$ by giving an intermediate basis $(A(T))_T$ and a triangular base change to $(G(T))_T$. We will review parts of their algorithm now.

To compute the intermediate basis $(A(T))_T$, we first transform a semistandard Young tableau T into the highest weight tableau of shape $\operatorname{sh} T$, i.e. the tableau containing only the number i in row i. Let i_1 be the smallest integer such that there exists a box filled with $i_1 + 1$ in row $s \leq i_1$ of T. Let s_1 be the smallest such row and r_1 be the number of times $i + 1$ appears in row s_1. Replace now the r_1 entries $i + 1$ by i and repeat this process until T is highest weight. We obtain the two finite sequences $(i_j)_j$ and $(r_j)_j$ and define

$$A(T) = f_{i_1}^{(r_1)} \cdots f_{i_k}^{(r_k)}.v_{\mathbf{s}} \tag{5.224}$$

where $v_{\mathbf{s}}$ is the highest weight vector of $V(\Lambda_{\mathbf{s}})$ and

$$f^{(d)} = \frac{f^d}{[d]!} \tag{5.225}$$

for an integer d and the q-factorial

$$[d]! = \prod_{i=1}^{d} [i] = \prod_{i=1}^{d} \frac{q^i - q^{-i}}{q - q^{-1}} \ . \tag{5.226}$$

We have computed an example of this intermediate basis in Figure 5.29.

The $(G(T))_T$ basis elements are now a linear combination of the $(A(T))_T$ with triangular base change matrix (see [LT00, Sec. 4.2] for all the details). For a tableau T, we can identify the abacus summands of $G(T)$ with their respective ℓ-quotients and by the definition of the characters $(A(T))_T$ we have

$$G(T) = \sum_{\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)} a_{\boldsymbol{\lambda}}(q)\, \boldsymbol{\lambda} \ , \quad a_{T,\boldsymbol{\lambda}}(q) \in \mathbb{C}(q) \text{ for all } \boldsymbol{\lambda} \in \mathscr{P}(\ell, n) \ , \tag{5.227}$$

i.e. the ℓ-multipartitions appearing in the sum all partition the same positive integer n. It is true that $a_{T,\boldsymbol{\lambda}}(1) \in \mathbb{N}$ for all tableaux T and $\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)$, so by setting $q = 1$ we can interpret $G(T)$ as a character of $C_\ell \wr \mathfrak{S}_n$. These characters are called *Leclerc–Miyachi constructible characters.*

We now give the connection to Lusztig's constructible characters in type B as discussed in [LM04]. Let $\ell = 2$, $n \in \mathbb{N}$, and

$$\Lambda_{\mathbf{s}} = \Lambda_{s+m} + \Lambda_s \tag{5.228}$$

for some $s \in \mathbb{Z}, m \in \mathbb{N}$. Let $\mathcal{B}$ be the global crystal basis of $V(\Lambda_{\mathbf{s}})$ and let $\mathcal{B}|_n \subseteq \mathcal{B}$ be the set of basis elements corresponding to semistandard ℓ-abaci with ℓ-quotients inside $\mathscr{P}(\ell, n)$.

Theorem 5.12.4 ([LM04, Thm. 10]). *By setting $q = 1$, the elements of $\mathcal{B}|_n$ coincide with Lusztig's constructible characters of type B associated to the parameter $(a, b) = (1, m)$.*

For type B, there is an explicit construction of the elements of $\mathcal{B}|_n$ given in [LM04] which we reiterate now. For a given 2-abacus, we first define an injective map between the beads in the right column to the beads in the left column.

Definition 5.12.5. For a standard 2-abacus with sequences of β-numbers $\beta^{(0)}$ and $\beta^{(1)}$ we define an injective map

$$\psi : \beta^{(1)} \to \beta^{(0)} \tag{5.229}$$

as follows: starting with $i = 0$, we map each element $x \in \beta^{(1)}$ to $x - i$ if possible. We then continue by successively increasing i until all elements of $\beta^{(1)}$ are mapped.

Note that the map in Definition 5.12.5 is indeed well-defined whenever the abacus is standard. Furthermore, since two sequences of β-numbers only differ by at a finite amount of entries, the infinite abacus is reduced to a finite symbol after mapping $x \mapsto x$ in the step $i = 0$.

Let ψ be associated to a standard 2-abacus as in Definition 5.12.5 and denote by $\boldsymbol{\lambda} \in \mathscr{P}(2, n)$ the corresponding 2-quotient. Whenever $\psi(j) \neq j$, we say that $(\psi(j), j)$ is a *pair* of ψ. By the construction of ψ we know that the beads of a pair together with their neighboring positions form a rectangle in the 2-abacus. Let k be the number of pairs of ψ. Denote by

$$\mathcal{C}(\boldsymbol{\lambda}) \subseteq \mathscr{P}(2, n) \tag{5.230}$$

the set of 2^k bipartitions obtained by performing rectangle swaps of pairs of ψ.

Theorem 5.12.6 ([LM04, Thm. 10]). *Let $(G(T))_T$ be a global crystal basis of a highest weight module $V(\Lambda_{\mathbf{s}})$ inside a level 2 Fock space $F(\Lambda_{\mathbf{s}})$. We then have*

$$G(T) = \sum_{\boldsymbol{\mu} \in \mathcal{C}(\boldsymbol{\lambda})} q^{n_{\boldsymbol{\lambda}}(\boldsymbol{\mu})} \, \boldsymbol{\mu} \tag{5.231}$$

where $\boldsymbol{\lambda}$ is the 2-quotient associated to T and $n_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ is the number of rectangle swaps needed to obtain $\boldsymbol{\mu}$ from $\boldsymbol{\lambda}$.

Now that we have seen the preliminary constructions of [LT00], we can briefly compare them with our theory of cuspidal family induction.

First of all, because the bead pairs that we constructed are invariant under insertion of empty and full rows into the 2-abacus, we can see here again that cuspidal family induction of type B provides a natural bijection between *sets* of constructible characters.

This result extends to type $G(\ell, 1, n)$ for $\ell > 2$. By the discussions in [HK02, Sec. 4.4], the cuspidal family induction as defined in Section 5.9 maps standard symbols to standard symbols. Therefore, for two families $\mathcal{F}, \mathcal{F}'$ that are induced from one another, we get a bijection between the set of family members in each of the families whose symbols are standard. This gives us a bijection between the sets of Leclerc–Miyachi constructible characters associated to $\mathcal{F}, \mathcal{F}'$. We will record this result.

LEMMA 5.12.7. *Let $\mathcal{F}$ be a non-cuspidal Calogero–Moser family of rank n which is induced from the cuspidal Calogero–Moser family $\mathcal{F}'$ of rank n'. When we label each Leclerc–Miyachi constructible character of $G(\ell,1,n)$ and $G(\ell,1,n')$ by their respective standard symbol, we have that cuspidal family induction affords a bijection between the Leclerc–Miyachi constructible characters associated to standard symbols of family members of each of $\mathcal{F}, \mathcal{F}'$.*

The bijection in Lemma 5.12.7 does, however, not restrict to a family member level as seen in the following example.

EXAMPLE 5.12.8. Fix the staircase charge $(2,1,1)$ for the groups $G(3,1,2)$ and $G(3,1,3)$. The group $G(3,1,2)$ admits a cuspidal family $\mathcal{F}^{\mathrm{cusp}}$ in that charge which induces a family $\mathcal{F}$ of $G(3,1,3)$ via cuspidal family induction.

In Figure 5.29, we computed the $q=1$-specialized intermediate basis $(A(T))_T$ for the two families by the rules outlined in this section. The characters $(A(T))_T$ in column [1] are already equal to the characters $(G(T))_T$ by the total order in [LT00, Sec. 2.2]. But since these characters $(A(T))_T$ have a different number of irreducible summands they do not afford a bijection on the level of family members.

There should be some combinatorial ℓ-analogue of the exchange method in type B and we expect cuspidal family induction to have some connection to it.

5.12.2. Calogero–Moser cellular characters. We follow [Lac20, Sec. 1] for this discussion. The original work on the Gaudin algebra Calogero–Moser cellular characters can be found in [BR17]. Let W be a complex reflection group of type $G(\ell,1,n)$ with reflection representation $\mathfrak{h}$ and set of complex reflections $\mathcal{S} \subseteq W$. Denote by $\mathfrak{h}^{\mathrm{reg}}$ the Zariski-open subset of $\mathfrak{h}$ obtained by removing the reflection hyperplanes of $\mathfrak{h}$. Let $\mathbf{c} : \mathcal{S} \to \mathbb{C}$ be a Calogero–Moser parameter for W. We define

$$\mathscr{D}_y = \sum_{s\in\mathcal{S}} \mathbf{c}(s)\det(s)\frac{\langle y,\alpha_s\rangle}{\alpha_s} s \in \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]W \tag{5.232}$$

for $y \in \mathfrak{h}$ where $\alpha_s \in \mathrm{Im}(s - \mathrm{id}_{\mathfrak{h}^*}) \subseteq \mathfrak{h}^*$ is the root of the reflection $s \in \mathcal{S}$. The *Gaudin algebra* is given by

$$\mathrm{Gau}_{\mathbf{c}}(W) = \langle \mathscr{D}_y \mid y \in \mathfrak{h}\rangle_{\mathbb{C}[V]} \tag{5.233}$$

and by [BR17, Sec. 13.4.B], the $\mathbb{C}[V]$-algebra $\mathrm{Gau}_{\mathbf{c}}(W)$ is commutative. To an irreducible character $L \in \mathrm{Irr}(\mathbb{C}(V)\,\mathrm{Gau}_{\mathbf{c}}(W))$, Bonnafé and Rouquier have associated a W-character by

$$\gamma_L^{\mathrm{Gau}_{\mathbf{c}}(W)} = \sum_{\boldsymbol{\lambda}\in\mathrm{Irr}\,W}\left[\mathrm{Res}^{\mathbb{C}(V)W}_{\mathbb{C}(V)\,\mathrm{Gau}_{\mathbf{c}}(W)}(\mathbb{C}(V)\,\boldsymbol{\lambda}) : L\right]\boldsymbol{\lambda}\,, \tag{5.234}$$

i.e. we restrict each irreducible character from W to $\mathrm{Gau}_{\mathbf{c}}(W)$ and count the multiplicities of a fixed irreducible character associated to the Gaudin algebra.

DEFINITION 5.12.9 ([BR17]). For a Calogero–Moser parameter $\mathbf{c}$ we call the set of W-characters

$$\left\{\gamma_L^{\mathrm{Gau}_{\mathbf{c}}(W)} \;\middle|\; L \in \mathrm{Irr}(\mathbb{C}(V)\,\mathrm{Gau}_{\mathbf{c}}(W))\right\} \tag{5.235}$$

the *Calogero–Moser $\mathbf{c}$-cellular characters.*

	[1]	[2]	[3]	[4]	[5]
Partition	$(\emptyset,(1),(1))$	$(\emptyset,\emptyset,(2))$	$(\emptyset,(1,1),\emptyset)$	$((1),\emptyset,(1))$	$((1,1),\emptyset,\emptyset)$
Abacus	● ● ○ ● ○ ● ○ ● ○	● ● ○ ● ● ○ ○ ○ ●	● ○ ● ● ● ○ ○ ● ○	● ● ○ ○ ● ● ● ○ ○	○ ● ● ● ● ○ ● ○ ○
Tableau	$\begin{smallmatrix}1&1&2\\2&3\end{smallmatrix}$	$\begin{smallmatrix}1&1&3\\2&2\end{smallmatrix}$	$\begin{smallmatrix}1&2&1\\2&3\end{smallmatrix}$	$\begin{smallmatrix}1&1&2\\3&2\end{smallmatrix}$	$\begin{smallmatrix}2&1&1\\3&2\end{smallmatrix}$
Standard	✓	✓	×	×	×
Operators	$f_1^{(1)} f_2^{(1)}$	$f_2^{(1)} f_1^{(1)}$	×	×	×
$A(T)$	[1]⊕[3]⊕[4]⊕[5]	[1]⊕[2]⊕[4]	×	×	×
Partition	$((1),(1),(1))$	$(\emptyset,(1),(2))$	$((1),(1,1),\emptyset)$	$((1),\emptyset,(2))$	$((1,1),(1),\emptyset)$
Abacus	● ● ○ ○ ○ ● ● ● ○	● ● ○ ● ○ ○ ○ ● ●	● ○ ● ○ ● ○ ● ● ○	● ● ○ ○ ● ○ ● ○ ●	○ ● ● ● ○ ○ ● ● ○
Tableau	$\begin{smallmatrix}1&1&2\\3&3\end{smallmatrix}$	$\begin{smallmatrix}1&1&3\\1&2\end{smallmatrix}$	$\begin{smallmatrix}1&2&1\\3&3\end{smallmatrix}$	$\begin{smallmatrix}1&2&1\\3&3\end{smallmatrix}$	$\begin{smallmatrix}2&1&1\\3&3\end{smallmatrix}$
Standard	✓	✓	×	×	×
Operators	$f_1^{(1)} f_2^{(2)}$	$f_2^{(2)} f_1^{(1)}$	×	×	×
$A(T)$	[1]⊕[3]⊕[5]	[1]⊕[2]⊕[4]	×	×	×

FIGURE 5.29. Two families for charge $(1,1,0)$ the characters $(A(T))_T$ for the standard tableaux of that shape.

It is conjectured in [BR17, Sec. 15.2.A Con. L] that the Calogero–Moser **c**-cellular characters coincide with Kazhdan–Lusztig cellular characters for type B.

In [Lac20], the Calogero–Moser **c**-cellular characters have been studied using a different set of characters which we will reiterate now. We call

$$\mathbf{eu}_{\mathbf{c},n} = \sum_{s\in\mathcal{S}} \mathbf{c}(s)\det(s)\cdot s \in \mathbb{C}W \tag{5.236}$$

the *Euler element* of $W = G(\ell,1,n)$. The element $\mathbf{eu}_{\mathbf{c},n}$ is central in $\mathbb{C}W$. We define the *Jucys-Murphy element* of $G(\ell,1,n)$ associated to $\mathbf{c}$ by

$$J_k = \mathbf{eu}_{\mathbf{c},k} - \mathbf{eu}_{\mathbf{c},k-1} = \sum_{\substack{s\in\operatorname{Ref}(G(\ell,1,k))\\ s\notin\operatorname{Ref}(G(\ell,1,k-1))}} \mathbf{c}(s)\det(s)\cdot s\ . \tag{5.237}$$

where $\operatorname{Ref}(G)$ denotes the set of complex reflections of a complex reflection group G.

As with the Gaudin algebra, the *Jucys–Murphy algebra*

$$\mathrm{JM}_{\mathbf{c}}(W) = \langle J_i \mid 1 \leq i \leq n\rangle_{\mathbb{C}} \tag{5.238}$$

is commutative by [Lac20, Lem. 1.6]. Since $\mathrm{JM}_{\mathbf{c}}(W)$ is a subalgebra of $\mathbb{C}W$, we have a $\mathrm{JM}_{\mathbf{c}}(W)$-action on W-modules. This action is simultaneously diagonalizable and the eigenvalues have been computed by Lacabanne. Recall that there exists for $\boldsymbol{\lambda} \in \operatorname{Irr} W$ a basis $(v_T)_T$ indexed by standard multitableaux of shape $\boldsymbol{\lambda}$ filled with the numbers $1,...,n$ (cf. Sec. 2.3).

PROPOSITION 5.12.10 ([Lac20, Prop. 1.7]). *Let $\boldsymbol{\lambda} \in \operatorname{Irr} W$ with tableau basis $(v_T)_T$ and let $\mathbf{c}$ be a Calogero–Moser parameter associated to the charge $(m_0, m_1, ..., m_{\ell-1})$ (cf. Sec. 5.5). We then have*

$$J_p \cdot v_T = \ell \cdot (m_k + (j-i)) \cdot v_T \quad \text{for } 1 \leq p \leq n \tag{5.239}$$

where the number p appears in the tableau T in box (i,j) of $\lambda^{(k)}$.

In the same way as with the Gaudin algebra, Lacabanne has defined for an irreducible character $L \in \operatorname{Irr}(\mathrm{JM}_{\mathbf{c}}(W))$ the W-character

$$\gamma_L^{\mathrm{JM}_{\mathbf{c}}(W)} = \sum_{\boldsymbol{\lambda} \in \operatorname{Irr} W} \left[\operatorname{Res}^W_{\mathrm{JM}_{\mathbf{c}}(W)}(\boldsymbol{\lambda}) : L\right] \boldsymbol{\lambda} \ . \tag{5.240}$$

DEFINITION 5.12.11 ([Lac20]). For a Calogero–Moser parameter $\mathbf{c}$ we call the set of W-characters

$$\left\{\gamma_L^{\mathrm{JM}_{\mathbf{c}}(W)} \;\middle|\; L \in \operatorname{Irr}(\mathrm{JM}_{\mathbf{c}}(W))\right\} \tag{5.241}$$

the *Jucys–Murphy* $\mathbf{c}$*-cellular characters.*

To a set of characters we can associate a family structure on $\mathscr{P}(\ell, n)$ in the same way we did with Lusztig's constructible characters in Section 5.11. For each set of cellular or constructible characters, we write $\boldsymbol{\lambda} \sim_{\mathrm{Gau}} \boldsymbol{\mu}$ (resp. $\boldsymbol{\lambda} \sim_{\mathrm{JM}} \boldsymbol{\mu}$) for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}(\ell, n)$ if there exists a Calogero–Moser cellular character (resp. Jucys–Murphy cellular character) γ such that $[\boldsymbol{\lambda} : \gamma] \neq 0 \neq [\boldsymbol{\mu} : \gamma]$. Now take the transitive closure of $\sim$ to obtain an equivalence relation. We call the respective equivalence relations *Gaudin partition* $\mathfrak{F}_{\mathrm{Gau}}$ and *Jucys–Murphy partition* $\mathfrak{F}_{\mathrm{JM}}$ of $\operatorname{Irr} W$. The equivalence classes are called *Gaudin families* and *Jucys–Murphy families*, respectively.

By [Lac20, Thm. 1.10] we have that Jucys–Murphy $\mathbf{c}$-cellular character are sums of Calogero–Moser $\mathbf{c}$-celullar characters and thus the Gaudin partition refines the Jucys–Murphy partition.

The Jucys–Murphy characters can be constructed using Proposition 5.12.10. We can parametrize $L \in \operatorname{Irr} \mathrm{JM}_{\mathbf{c}}(W)$ by a list of integers

$$(a_1, ..., a_n) \in \mathbb{Z}^n \tag{5.242}$$

such that a_i is equal to the charged residue of the box with filled with $i \in \{1, ..., n\}$. Thus, two ℓ-multipartitions are constituents of the same Jucys–Murphy cellular character if and only if there exist two multitableaux T_1, T_2 with respective shapes $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ such that the box with filling i has the same charged residue in T_1 and T_2 for all $i \in \{1, ..., n\}$. The existence of such a filling implies that the partitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ must already have the same charged residue polynomial (cf. Sec. 5.5) and therefore belong to the same Calogero–Moser family. This means, the Jucys–Murphys partition refines the Calogero–Moser partition. All in all, we denote these refinements by

$$\mathfrak{F}_{\mathrm{Gau}} \leq \mathfrak{F}_{\mathrm{JM}} \leq \mathfrak{F}_{\mathrm{CM}} \tag{5.243}$$

where $\mathfrak{F}_{\mathrm{CM}}$ denotes the Calogero–Moser partition of $\operatorname{Irr} W$ and $\leq$ denotes the refinement relation in the set partition poset of $\operatorname{Irr} W$. For one of the relations in (5.243) relations, we can already show equality.

PROPOSITION 5.12.12. *We have $\mathfrak{F}_{\mathrm{JM}} = \mathfrak{F}_{\mathrm{CM}}$.*

PROOF. To see that CM-families also refine JM-families we use rectangle swaps. We say a rectangle in an abacus is *minimal* if there is no other rectangle inside of the rectangle that uses the same two columns. For example, if our rectangle has a ●○-row and a ○●-row, all rows in between the two are ○○- or ●●-rows. Now, when we perform a rectangle swap using a minimal rectangle using columns $i, j \in \mathbb{Z}/\ell\mathbb{Z}$ of the ℓ-abacus, we remove a skew shape from the partition $\lambda^{(j)}$ and add the same skew shape to the partition $\lambda^{(i)}$ corresponding to the other column. This is true since the two columns in the rectangle are equal because of the minimal property. That means we can fill the relocated skew diagram with the biggest entries of our tableau and move the skew diagram in between the two family members. The rest of the two family members' tableaux is filled in the same way. Thus we have that for a minimal rectangle swap we can construct a **c**-cellular Jucys–Murphy character that connects them.

Now, it is not difficult to see that we can construct all rectangle swaps using minimal rectangle swaps. We then use that rectangle swaps are sufficient for generating all family members by Corollary 5.5.2 and we are done. ■

We have illustrated the fact that minimal rectangle swaps are sufficient to construct any rectangle swap at an example in Figure 5.30

$$\begin{matrix}\bullet&\bullet\\\bullet&\circ\\\bullet&\circ\\\circ&\bullet\\\circ&\bullet\\\circ&\bullet\end{matrix}\;\to\;\begin{matrix}\bullet&\bullet\\\bullet&\circ\\\circ&\bullet\\\bullet&\circ\\\circ&\bullet\\\circ&\bullet\end{matrix}\;\to\;\begin{matrix}\bullet&\bullet\\\bullet&\circ\\\circ&\bullet\\\circ&\bullet\\\bullet&\circ\\\circ&\bullet\end{matrix}\;\to\;\begin{matrix}\bullet&\bullet\\\bullet&\circ\\\circ&\bullet\\\circ&\bullet\\\circ&\bullet\\\bullet&\circ\end{matrix}\;\to\;\begin{matrix}\bullet&\bullet\\\circ&\bullet\\\bullet&\circ\\\circ&\bullet\\\circ&\bullet\\\bullet&\circ\end{matrix}$$

FIGURE 5.30. Exchanging the first and last rows using only minimal rectangle swaps.

As was the case with Lusztig's constructible characters, we can induce type B Jucys–Murphy characters as well. Let $W = B_n$, let $\mathcal{F}$ a cuspidal family of W, and let $\mathcal{F}'$ be a family that is cuspidal family induced from $\mathcal{F}$. Furthermore let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$) be the set of Jucys–Murphy cellular characters γ_L associated to $\mathcal{F}$ (resp. $\mathcal{F}'$).

PROPOSITION 5.12.13. *For each $\gamma \in \Gamma_{\mathcal{F}}$ we can construct $\gamma' \in \Gamma_{\mathcal{F}'}$ such that their respective unique constituents are in bijection via type B cuspidal family induction.*

PROOF. We prove this for the cuspidal induction exchanging two rows of the J-core abacus of $\mathcal{F}$ by looking at the 3 cases separately.

Case 1 $\begin{smallmatrix}\circ&\bullet\\\circ&\circ\end{smallmatrix} \to \begin{smallmatrix}\circ&\circ\\\circ&\bullet\end{smallmatrix}$: In this induction we add one box to the multipartition. For constructing the tableaux that afford γ', we can simply add the new entry to that box. Now we just have to show, that no new multipartitions have tableaux that afford the same character. But this is clear since the biggest entry of the tableaux has to be in the rightmost box of some row and there is only one box of the correct residue by Lemma 5.1.4. Since the position of the largest entry is fixed and the box must be removable, removing the box gives a tableaux that affords γ which means there are no new constituents.

Case 2 $\begin{smallmatrix}\bullet&\bullet\\\bullet&\circ\end{smallmatrix} \to \begin{smallmatrix}\bullet&\circ\\\bullet&\bullet\end{smallmatrix}$: Again, the biggest entry has to be in the rightmost box of some row with the correct residue of which there are two. But since the biggest entry cannot be dominated, it must be in a removable box of which there is only one. We are again in Case 1.

Case 3 $\begin{smallmatrix}\bullet&\bullet\\\circ&\circ\end{smallmatrix} \to \begin{smallmatrix}\circ&\circ\\\bullet&\bullet\end{smallmatrix}$: Here, we add one box to each of the parts of the multipartition and construct γ' the same way as in Case 1. Note that we get a factor 2 for the multiplicity are two possibilities of placing the two new entries. To see that there are no new constituents, we argue that the two new entries have to occupy the newly added boxes. The two entries cannot be in the same part of the bitableau since they would have to be placed in the same diagonal of boxes because of their equal residue. This is not possible in a standard Young tableaux for two entries that differ by 1. Now, since they are in different parts of the bitableau, we can argue as in Case 1.

■

For $\ell > 2$ not all Jucys–Murphy characters are induced from cuspidal ones. For the charge $\mathbf{s} = (0,0,0)$, the cuspidal family of $G(3,1,4)$ containing $(\emptyset, \emptyset, (2,2))$ has no cellular characters with 15 unique irreducible constituents. But there exists an induced family of $G(3,1,5)$ that has a cellular character that has 15 unique irreducible constituents. Therefore, this cellular character of $G(3,1,5)$ is not induced from the cuspidal ones via cuspidal family induction.

5.12.3. Special characters and minimal b-invariants. Recall that in Section 3.3 we defined for $\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)$ the b-invariant of $\boldsymbol{\lambda}$ by

$$b(\boldsymbol{\lambda}) = \sum_{i=0}^{\ell-1} (i-1) \cdot |\lambda^{(i)}| + \ell \cdot \sum_{j=0}^{\ell-1} \sum_{i \geq 1} (i-1) \cdot \lambda_i^{(j)} \ . \tag{5.244}$$

In [Bon15] Bonnafé augmented the work of Lusztig [Lus03] to obtain the following result which we formulate for type B.

Theorem 5.12.14 ([Bon15]). *Let $\mathcal{F}$ be a Calogero-Moser family, γ a Calogero-Moser cellular character for B_n.*

(*i*) *There exists a unique $\boldsymbol{\lambda}_{\mathcal{F}} \in \mathcal{F}$ with minimal b-invariant.*
(*ii*) *There is a unique constituent $\boldsymbol{\lambda}_\gamma$ with minimal b-invariant among the constituents of γ. We have $[\boldsymbol{\lambda}_\gamma : \gamma] = 1$.*

We call the character $\lambda_{\mathcal{F}}$ the *special character* of $\mathcal{F}$.

Proposition 5.12.15. *For a cuspidal family $\mathcal{F}_{\mathrm{cusp}}$ of type B at charge $(0,m)$ with $m \geq 0$, the special character is*

$$\boldsymbol{\lambda}_{\mathcal{F}} = (\lambda_{\mathcal{F}}^{(0)}, \lambda_{\mathcal{F}}^{(1)}) = ((m+k, m+k-1, ..., m+1), (k-1, k-2, ..., 1)) \ . \tag{5.245}$$

The special characters of non-cuspidal families are cuspidal family induced from the special characters of their respective cuspidal family.

Proof. The form of $\boldsymbol{\lambda}_{\mathcal{F}}$ is given by *special symbols* in [Bon15, Defn. 3]. The special symbol is constructed by first removing full and empty rows from the symbol, which is in accordance with cuspidal family induction of type B.

■

In the case that $m < 0$ we need to replace $\lambda_{\mathcal{F}}^{(i)}$ in (5.245) by its transpose for $i = 0, 1$. Proposition 5.12.15 does not hold in general for $\ell > 2$. A counterexample is given by the groups $G(3,1,2)$ and $G(3,1,3)$ in charge $(1,0,0)$ where $((1),(1),\emptyset)$ does not induce $((1),(1),(1))$.

We have now seen that cuspidal family induction does not seem to extend directly to the case $\ell > 2$. To use cuspidal family induction in order to describe cellular characters and other structures related to Calogero–Moser families, we would probably need additional combinatorial models such as a generalized pairing method for $G(\ell, 1, n)$.

So far, the combinatorial representation theory of $\overline{\mathsf{H}}_{\mathbf{c}}$ in the generic parameter and the special parameter seem very disparate. In the next section, we will bring these theories closer together by revisiting wreath Macdonald symmetric functions.

5.13. Wreath Macdonald polynomials

We want to take a step back now compare our models for the generic parameter case and the special parameter case. In generic parameter, we have used Haiman's wreath Macdonald polynomials to describe the characters of simple $\overline{\mathsf{H}}_{\mathbf{c}}$-modules (cf. Sec. 3.5). These polynomials, however, are only defined for generic $\mathbf{c}$. The $q = t$-specialized wreath Macdonald polynomials are equal for all Calogero–Moser chambers but their bigraded versions can differ from chamber to chamber.

The question we want to study in this section is what happens to the wreath Macdonald polynomials when crossing a special parameter hyperplane $\mathcal{H}$ and if there is a connection to the Calogero–Moser families associated to $\mathcal{H}$. We will start by reviewing the discussion on various orders on $\mathscr{P}(\ell, n)$ found in [Gor03; Prz20], then review once more the definition of the wreath Macdonald polynomial [Hai03] before stating our Conjecture 5.13.2. Afterwards, we will prove the conjecture for certain hyperplane crossings.

Given a partially ordered set $(P, \preceq)$, one defines the *Hasse diagram* $H = (V, A)$ of $(P, \preceq)$ as the directed graph with vertex set $V = P$ and arcs $a \to b$ whenever $a \succeq b$ and there exists no $m \in P \setminus \{a, b\}$ such that $a \succeq m \succeq b$. We will use Hasse diagrams to visualize partial orders on $\mathscr{P}(\ell, n)$ for some positive integers ℓ, n. Note that the definition of our orders are twisted slightly compared to [Gor08a; Prz20]. This is done to have the orders refine one another.

The *c-order* has been studied in the context of rational Cherednik algebras $\mathsf{H}_{t,\mathbf{c}}$ "at $t = 1$" to describe the highest weight structure of the category $\mathcal{O}$ corresponding to $\mathsf{H}_{1,\mathbf{c}}$ (cf. [GGOR03, Thm. 2.19]). We mostly follow the discussions in [Gor08a] and [Prz20]. Let

$$\mathbf{H} = (h, H_1, ..., H_{\ell-1}) \in \mathbb{Q}^{\mathbb{Z}/\ell\mathbb{Z}} \tag{5.246}$$

be a rational Calogero–Moser parameter in the Gordon basis (cf. Sec. 4.2). Define a function $\mathscr{P}(\ell, n) \to \mathbb{Q}$ that sends $\boldsymbol{\lambda} = (\lambda^{(0)}, ..., \lambda^{(\ell-1)}) \in \mathscr{P}(\ell, n)$ to

$$\begin{aligned} c_{\mathbf{H}}(\boldsymbol{\lambda}) = \ell \cdot & \sum_{r=1}^{\ell-1} |\lambda^{(r)}| \cdot (H_1 + \cdots + H_r) \\ & - \ell \cdot \left(\frac{n(n-1)}{2} + \sum_{r=0}^{\ell-1} b(\lambda^{(r)}) + b({}^t\lambda^{(r)}) \right) \cdot h \end{aligned} \tag{5.247}$$

where $b(\lambda)$ is the b-invariant of $\lambda \in \mathscr{P}$ (cf. Sec. 2.1). This function is called the *c-function* and it is used to define the c-order $<_{\mathbf{H}}$ with respect to $\mathbf{H}$ by

$$\boldsymbol{\lambda} <_{\mathbf{H}} \boldsymbol{\mu} \Longleftrightarrow c_{\mathbf{H}}(\boldsymbol{\lambda}) < c_{\mathbf{H}}(\boldsymbol{\mu}) \ \text{ for } \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}(\ell, n)\ . \tag{5.248}$$

For an ℓ-multipartition $\boldsymbol{\lambda}$ the sum $c_{\mathbf{H}}(\boldsymbol{\lambda})$ can be split in two parts. The $\mathbf{H}$-dependent part

$$\ell \cdot \sum_{r=1}^{\ell-1} |\lambda^{(r)}| \cdot (H_1 + \cdots + H_r) \tag{5.249}$$

which describes an $\mathbf{H}$-weighted count of the boxes in $\boldsymbol{\lambda}$, and a part that only depends on h and $\boldsymbol{\lambda}$ equal to

$$\ell \cdot \left(\frac{n(n-1)}{2} + \sum_{r=0}^{\ell-1} b(\lambda^{(r)}) - b({}^t\lambda^{(r)}) \right) \cdot h \tag{5.250}$$

which is a difference of b-invariants. This sum is 0 when $\boldsymbol{\lambda}$ consists of *symmetric* partitions, i.e. $\lambda^{(i)} = {}^t\lambda^{(i)}$ for all i. Therefore, one can view this second part as a measure for how nonsymmetric the partitions in $\boldsymbol{\lambda}$ are. All in all, two multipartitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ that consist of symmetric partitions such that

$$|\lambda^{(r)}| = |\mu^{(r)}| \tag{5.251}$$

for all $r \in \{0, ..., \ell - 1\}$ have equal $c_{\mathbf{H}}$-values, i.e.

$$c_{\mathbf{H}}(\boldsymbol{\lambda}) = c_{\mathbf{H}}(\boldsymbol{\mu}) \ , \tag{5.252}$$

for all $\mathbf{H} \in \mathbb{Q}^{\mathbb{Z}/\ell\mathbb{Z}}$. Therefore $<_{\mathbf{H}}$ is not a total order as has already been discussed in [Prz20]. Note the set $\mathscr{P}(k)$ does not contain two different symmetric partitions for $k \leq 7$.

Recall that $\mathscr{C}$ denotes the set of complex valued, W-equivariant functions on the set $\mathcal{S} \subseteq W$ of complex reflections of W. For $W = C_\ell \wr \mathfrak{S}_n$, we can identify $\mathscr{C}$ with $\mathbb{C}^\ell$. For a fixed partition $\boldsymbol{\nu} \in \mathscr{P}(\ell, n)$, we can define the function

$$c(\boldsymbol{\nu}) : \mathscr{C} \to \mathbb{C} \ , \mathbf{H} \to c_{\mathbf{H}}(\boldsymbol{\nu}) \ . \tag{5.253}$$

Because two such functions $c(\boldsymbol{\lambda})$, $c(\boldsymbol{\mu})$ are polynomial, they are equal for all $\mathbf{H} \in \mathscr{C}$ if and only if they are equal on a Zariski-open subset of $\mathscr{C}$. Furthermore, since the function $c(\boldsymbol{\lambda}) - c(\boldsymbol{\mu})$ is a linear map we have that the c-order "degenerates" on a finite union of hyperplanes in $\mathscr{C}$, i.e. generically comparable multipartitions become incomparable on those hyperplanes. We call these hyperplanes *c-hyperplanes* and the corresponding chambers of $\mathscr{C}$ the *c-chambers*. It is easy to see that the order induced by the c-function is constant inside c-chambers. The following result is due to Gordon.

THEOREM 5.13.1 ([Gor08a, Thm. 4.5]). *The Calogero–Moser hyperplanes of $\mathscr{C}$ are a subset of the c-hyperplanes, or, equivalently, the Calogero–Moser chambers are a union of c-chambers.*

In the type B case the chamber decompositions for $h \neq 0$ agree and we obtain the four Hasse diagrams displayed in Figure 5.31 (see Figure 5.15 for the parametrization of type B chambers).

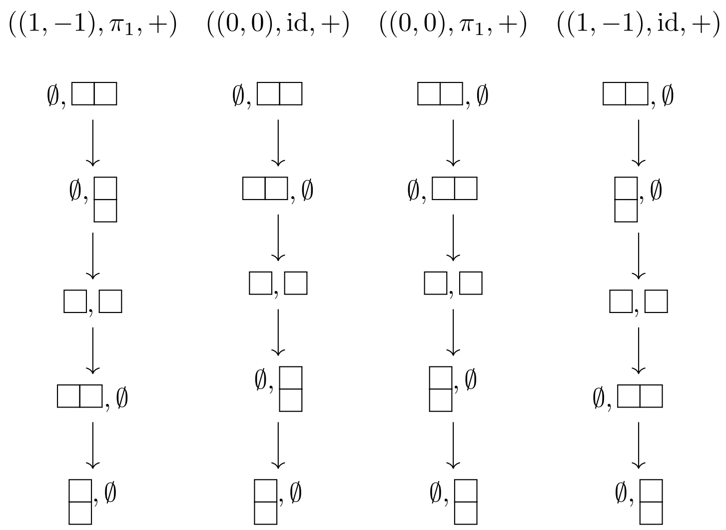


FIGURE 5.31. The four c-orders for B_2 and $h = -1$.

The second order is given by the ℓ-quotient map which Gordon studied in [Gor08a] building on discussions in [Hai03]. Recall that for a Calogero–Moser parameter $\mathbf{H}$ we have associated in Section 5.5 a Calogero–Moser chamber $(\mathbf{s}, w, +)$

and a bijective map

$$\tau_{\mathbf{s}}^{w} : \mathscr{P}(\ell,n) \to \mathscr{P}_{\nu}(|\nu| + n\cdot\ell)\ ,\boldsymbol{\lambda} \mapsto \tau_{\mathbf{s}}(\boldsymbol{\lambda}^{w}) \tag{5.254}$$

where ν is the ℓ-core corresponding to $\mathbf{s} \in \mathbb{Z}_0^{\ell}$. We can now define the *combinatorial order*

$$\boldsymbol{\lambda} \preceq_{\mathbf{H}}^{\mathsf{com}} \boldsymbol{\mu} \iff \tau_{\mathbf{s}}(\boldsymbol{\lambda}^{w}) \trianglelefteq \tau_{\mathbf{s}}(\boldsymbol{\mu}^{w}) \tag{5.255}$$

where $\boldsymbol{\lambda}, \boldsymbol{\mu}$ are ℓ-multipartitions and $\trianglelefteq$ is the dominance order on partitions (cf. Sec. 2.1). For B_2, this order is total and equal to the c-order given in Figure 5.31. In fact, it is true in general that the c-order refines the combinatorial order, i.e. we have for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}(\ell,n)$ that

$$\boldsymbol{\lambda} \preceq_{\mathbf{H}}^{\mathsf{com}} \boldsymbol{\mu} \implies \boldsymbol{\lambda} <_{\mathbf{H}} \boldsymbol{\mu}\ . \tag{5.256}$$

This result can essentially be found in the proof of [Gor08a, Prop. 8.3(*ii*)]. The combinatorial order respects the highest weight structure of $\mathsf{H}_{1,\mathbf{c}}$-mod as well by [DG10].

The third order $\preceq_{\mathbf{H}}^{\mathsf{geo}}$ is defined geometrically via attraction sets on varieties associated to Calogero–Moser space (see [Gor08a, Sec. 5.4] for all the details). By [Prz20, Cor. 10.6] the geometric order refines the combinatorial one such that we have

$$\boldsymbol{\lambda} \preceq_{\mathbf{H}}^{\mathsf{geo}} \boldsymbol{\mu} \implies \boldsymbol{\lambda} \preceq_{\mathbf{H}}^{\mathsf{com}} \boldsymbol{\mu} \tag{5.257}$$

for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}(\ell,n)$. Note as stated in [Prz20] the proof of (5.257) in [Gor08a, Prop. 7.10] is false and we only get one implication.

Fix now a generic Calogero–Moser parameter $\mathbf{H}$. Recall from Section 3.4 the definition of Haiman's wreath Macdonald symmetric function $H_{\boldsymbol{\lambda}}^{\preceq}(x;q,t)$ associated to a partial order $\preceq$ and a multipartition $\boldsymbol{\lambda}$. The symmetric functions $H_{\boldsymbol{\lambda}}^{\preceq}(x;q,t)$ are precisely given by the combinatorial order $\preceq_{\mathbf{H}}^{\mathsf{com}}$ which we will from now on denote simply by $\preceq$.

For $\boldsymbol{\lambda} \in \mathscr{P}(\ell,n)$, we define $\mathscr{P}_{\succeq\boldsymbol{\lambda}}, \mathscr{P}_{\preceq\boldsymbol{\lambda}} \subseteq \mathscr{P}(\ell,n)$ by

$$\mathscr{P}_{\succeq\boldsymbol{\lambda}} = \{\boldsymbol{\mu} \in \mathscr{P}(\ell,n) \mid \boldsymbol{\mu} \succeq \boldsymbol{\lambda}\}\ , \tag{5.258}$$

$$\mathscr{P}_{\preceq\boldsymbol{\lambda}} = \{\boldsymbol{\mu} \in \mathscr{P}(\ell,n) \mid \boldsymbol{\mu} \preceq \boldsymbol{\lambda}\}\ . \tag{5.259}$$

We can now rewrite the conditions in Lemma 3.4.4 as

$$H_{\boldsymbol{\lambda}}^{\preceq} \in \left\langle s_{\boldsymbol{\mu}}^{q} \middle| \boldsymbol{\mu} \in \mathscr{P}_{\succeq\boldsymbol{\lambda}} \right\rangle_{\mathbb{Q}(q,t)} \cap \left\langle s_{\boldsymbol{\mu}}^{t^{-1}} \middle| \boldsymbol{\mu} \in \mathscr{P}_{\preceq\boldsymbol{\lambda}} \right\rangle_{\mathbb{Q}(q,t)} \tag{5.260}$$

where

$$s_{\boldsymbol{\nu}}^{\alpha} := \prod_{j=0}^{\ell-1} s_{\nu^{(j)}}\Big[Z_{\alpha}^{(j)}/(1-\alpha^{\ell})\Big] \tag{5.261}$$

for $\boldsymbol{\nu} = (\nu^{(0)}, ..., \nu^{(\ell-1)})$ and $\alpha \in \mathbb{C}(q,t)$.

By the uniqueness of wreath Macdonald polynomials, the intersection in (5.260) describes a 1-dimensional $\mathbb{Q}(q,t)$-vector space. This means if we have two combinatorial orders $\preceq, \preceq'$, i.e. orders of the form $\preceq_{\mathbf{H}}^{\mathsf{com}}$ for some $\mathbf{H}$, and $\boldsymbol{\lambda} \in \mathscr{P}(\ell,n)$ such that

$$\mathscr{P}_{\succeq\boldsymbol{\lambda}} \subseteq \mathscr{P}_{\succeq'\boldsymbol{\lambda}} \text{ and } \mathscr{P}_{\preceq\boldsymbol{\lambda}} \subseteq \mathscr{P}_{\preceq'\boldsymbol{\lambda}} \tag{5.262}$$

it is easy to see that the 1-dimensional spaces must coincide and the respective wreath Macdonald polynomials must then be equal. This will be our strategy in detecting changes and equalities of wreath Macdonald polynomials when crossing a wall. We conjecture the following.

CONJECTURE 5.13.2. *Let $\preceq, \preceq'$ be two combinatorial orders whose chambers share a wall given by a hyperplane $\mathcal{H}$. Then the wreath Macdonald polynomials $H^{\preceq}_{\boldsymbol{\lambda}}$ and $H^{\preceq'}_{\boldsymbol{\lambda}}$ are equal whenever $\boldsymbol{\lambda} \in \mathscr{P}(\ell, n)$ belongs to a singleton family of the Calogero–Moser partition associated to a point of $\mathcal{H}$ in general position.*

REMARK 5.13.3. What we really want to describe is not the dominating sets (5.262), but rather the *support* of $H_{\boldsymbol{\lambda}}$, i.e. the partitions contributing to the linear combinations in (5.260). The support, however, is much more difficult to attain. It is possible that the support could be given by the geometrical order $\preceq^{\mathsf{geo}}_{\mathbf{H}}$.

Now that we have reviewed the preliminaries from [Gor08a; Prz20; Hai03], we are ready to start working on Conjecture 5.13.2. We focus on complex reflection groups of type B in this chapter, but some of the results can be adapted to certain charges for higher wreath products.

The conjecture was verified by me computationally for all hyperplanes in the cases $W = B_2, B_3$. Let us now give a theoretical approach to Conjecture 5.13.2. Fix a type B Calogero–Moser parameter $\mathbf{H} = (-1, H_1)$. The c-order then reduces to

$$c_{\mathbf{H}}(\boldsymbol{\lambda}) = 2 \cdot |\lambda^{(1)}| \cdot H_1 + 2\left(\binom{n}{2} + b(\lambda^{(0)}) + b(\lambda^{(1)}) - b({}^t\lambda^{(0)}) - b({}^t\lambda^{(1)})\right) \tag{5.263}$$

which we abbreviate to

$$c_{\mathbf{H}}(\boldsymbol{\lambda}) = 2 \cdot |\lambda^{(1)}| \cdot H_1 + 2 \cdot f(\boldsymbol{\lambda}) \tag{5.264}$$

where $f(\boldsymbol{\lambda}) \in \mathbb{Z}$ does not depend on $\mathbf{H}$. The formula (5.264) shows that for a family $\mathcal{F}$ corresponding to a special parameter $H_1 \in \mathbb{Z}$, the c-order within $\mathcal{F}$ reverses under the wall-crossing between $H_1 \pm \varepsilon$ for some $\varepsilon > 0$.

This property of family members being "close" to one another in Hasse diagrams holds in the combinatorial order $\preceq^{\mathsf{com}}_{\mathbf{H}}$ as well. Recall that a rectangle in an abacus is said to be *minimal* if the two columns of the rectangle only contain ●●- and ○○-rows. In a cuspidal family, minimal rectangles span exactly 2 rows. Such a minimal rectangle swap moves a single box in the Young diagram to an adjacent row. This movement gives us an edge in the Hasse diagram. Because minimal rectangle swaps are sufficient to construct all family members, we obtain the following lemma.

LEMMA 5.13.4. *Let $\preceq$ be a combinatorial order in type B and let $\mathcal{F}^{\mathrm{cusp}} \subseteq \mathscr{P}(2, n)$ be a cuspidal family associated to a wall of the chamber of $\preceq$. Then the Hasse diagram of $(\mathcal{F}^{\mathrm{cusp}}, \preceq)$ embeds into the Hasse diagram of $(\mathscr{P}(\ell, n), \preceq)$.*

When we insert full and empty rows into the 2-abacus to construct a non-cuspidal family $\mathcal{F}$, we can still construct the Hasse diagram of $(\mathcal{F}^{\mathrm{cusp}}, \preceq)$ using minimal rectangle swaps although these diagrams do not necessarily embed into the Hasse diagram of $(\mathscr{P}(\ell, n), \preceq)$.

We can describe the effect of some wall-crossings on the Hasse diagram of $(\mathcal{F}, \preceq)$ already.

DEFINITION 5.13.5. For a cuspidal family $\mathcal{F}$ of type B and $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}) \in \mathcal{F}$ we define ${}^T\boldsymbol{\lambda} = ({}^t\lambda^{(1)}, {}^t\lambda^{(0)})$ where ${}^t\lambda$ is the usual transpose. We extend this definition to all families via cuspidal family induction.

LEMMA 5.13.6. *Let $\mathcal{F}$ be a type B family with cuspidal family of size $k \cdot (k+m)$. Crossing the $(-1, m)$-wall induces an isomorphism of the respective Hasse diagrams of $\mathcal{F}$ given by ${}^T(\cdot)$.*

PROOF. This behavior is best seen in a $\{1\}$-wall where the change in the order is given by swapping the two columns of the 2-abaci. There, a rectangle swap that *increases* the partition with respect to the dominance order turns into a rectangle swap that *decreases* the partition after swapping the column. We can reduce the case of a $\{0\}$-wall to a $\{1\}$-wall by shifting the sequences of β-numbers.

■

We can use Lemma 5.13.6 to describe the wall-crossing of the outmost wall $m = -n+1$. This wall has a unique family given by

$$\mathcal{F}_{-n+1} = \{((n), \emptyset), ..., ((n-k), (1^k)), ..., (\emptyset, (1^n))\} . \tag{5.265}$$

This family is also cuspidal. For $m < -n+1$, we are in the *asymptotic* chamber of B_n. The asymptotic chamber for $G(\ell, 1, n)$ is given by a charge

$$s_0 >> s_1 >> ... >> s_{\ell-1} \tag{5.266}$$

and it affords a combinatorial order which is equal to the dominance order on ℓ-multipartitions given by

$$\begin{aligned} \boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu} \iff & \sum_{k=0}^{j} |\lambda^{(k)}| \leq \sum_{k=0}^{j} |\mu^{(k)}| \text{ and} \\ & \sum_{i=1}^{s} \lambda_i^{(j+1)} \leq \mu_i^{(j+1)} \text{ for all } 0 \leq j < \ell - 1 \text{ , } s \in \mathbb{N} \end{aligned} \tag{5.267}$$

for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}(\ell, n)$.

PROPOSITION 5.13.7. *Crossing the $m = -n+1$ wall with cuspidal family $\mathcal{F}$ changes the combinatorial order from $\preceq$ to $\preceq'$ as follows:*

(i) *For $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \in \mathcal{F}$ the order reverses,*
(ii) *for $\boldsymbol{\lambda}, \boldsymbol{\mu} \notin \mathcal{F}$ the order stays the same,*
(iii) *for $\boldsymbol{\lambda} \notin \mathcal{F}, \boldsymbol{\rho}_0 \in \mathcal{F}$ with $\boldsymbol{\lambda} \preceq \boldsymbol{\rho}_0$ we get $\boldsymbol{\lambda} \preceq' \boldsymbol{\rho}$ for all $\boldsymbol{\rho} \in \mathcal{F}$. The same goes for $\succeq$.*

PROOF. The case (i) has been dealt with in Lemma 5.13.6. The case (ii) is clear when looking at the bead diagrams since for any partitions not in the family, the order stays equal to the dominance order of the asymptotic chamber.

Let us look at case (iii) and show that for $\boldsymbol{\lambda} \notin \mathcal{F}$ and $\boldsymbol{\rho}_0 \in \mathcal{F}$ with $\boldsymbol{\lambda} \succeq \boldsymbol{\rho}_0$ that domination is preserved after the wall-crossing. This, together with (i) and the transitivity of $\preceq$ gives us the domination of all family members. We prove this by examining the bead diagrams for the chambers

$$((0, m), \mathrm{id}, +) \to ((m, 0), \pi_1, +) \tag{5.268}$$

under crossing a $\{1\}$-wall. This case generalizes sufficiently to all others after suitable shifts of sequences of β-numbers.

Let $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}) \notin \mathcal{F}, \boldsymbol{\rho}_0 \in \mathcal{F}$ and $\boldsymbol{\lambda} \succeq \boldsymbol{\rho}_0$. First note that $\lambda_1^{(1)} > 1$ because for $\lambda^{(1)} \leq 1$ the $n+1$ biggest partitions are all the family members. Now, because $\boldsymbol{\lambda}$ is in a singleton family, the bead diagram of $\boldsymbol{\lambda}$ contains only ○○-, ○●- and ●●-rows. Furthermore, there are no ○○-rows directly above ●●-rows and thus no J-core operations for the left column. The process of wall-crossing can thus be seen as the J-core operation removing all m removable boxes of the diagram corresponding to the beads in the right column of the 2-abacus.

For the bead diagram of $\boldsymbol{\rho}_0$ the exchange of columns has the effect of removing $m+1$ boxes and adding back 1. Because $\lambda_1^{(1)}$ is at least 2, we have

$$\tau(\boldsymbol{\lambda})_1 \geq \tau(\boldsymbol{\rho}_0)_1 + 2 \tag{5.269}$$

where τ is the ℓ-quotient map associated to the chamber $((0, m), \mathrm{id}, +)$.

When crossing a wall, we remove all $\{1\}$-removable boxes from $\tau(\boldsymbol{\lambda})$ and $\tau(\boldsymbol{\rho}_0)$ and add a $\{1\}$-addable box to $\tau(\boldsymbol{\rho}_0)$ in some row. The diagram of $\tau(\boldsymbol{\lambda})$ only has removable boxes of residue 1. The diagram of $\tau(\boldsymbol{\rho}_0)$ has at most $n+1$ rows. Since the column swap affects $n+1$ rows of the diagram we have for each row a removal of a box or an addition (of which there is only one) in each row. Now, even if the diagram of $\tau(\boldsymbol{\lambda})$ also has the first n rows affected by a box removal, domination is still preserved.

The result for $\boldsymbol{\lambda} \preceq \boldsymbol{\rho}_0$ now follows since the sets $\mathcal{F}$ and $\mathscr{P}(n,2)\backslash\mathcal{F}$ are stabilized by ${}^T(\cdot)$ which reverses the order $\preceq$.

■

In summary, the elements of $\mathcal{F}_{-n+1}$ form a connected subgraph of the asymptotic Hasse diagram of $(\mathscr{P}(\ell,n),\trianglelefteq)$. After crossing the $m=-n+1$ wall, the elements of $\mathcal{F}_{-n+1}$ still form a connected subgraph of the Hasse diagram but now only the biggest and smallest elements have neighbors that are not family members. We get a proof of Conjecture 5.13.2 for $W=B_n$ and the hyperplane $m=-n+1$ by using Proposition 5.13.7 and following the discussion after (5.260).

Corollary 5.13.8. *For $\boldsymbol{\lambda} \notin \mathcal{F}_{-n+1}$, the wreath Macdonald polynomial $H_{\boldsymbol{\lambda}}$ does not change under crossing the wall given by $m=-n+1$.*

We close this section by discussing Proposition 5.13.7 in the context of general non-cuspidal families. For the wall-crossing $m=1$ in $n=3$, we have the non-cuspidal family

$$((3),\emptyset)\sim((2),(1))\sim(\emptyset,(2,1)) \tag{5.270}$$

induced by the cuspidal family

$$((2),\emptyset)\sim((1),(1))\sim(\emptyset,(1,1)) \tag{5.271}$$

and we have that the non-family member partition $((2),(1))$ is related to some but not all non-cuspidal family members. This means that Proposition 5.13.7 does not extend to this case.

The proposition also does not hold for other cuspidal families in general. For the wall $m=1$, there is the family given by

$$((2,2,2),\emptyset)\sim ... \sim(\emptyset,(3,3)) \tag{5.272}$$

which has partial relations to other multipartitions. Another example is given by the $m=0$ wall and the family

$$((2,2),\emptyset)\sim ... \sim(\emptyset,(2,2)) \tag{5.273}$$

which has some but not all of its members related to $((3,1),\emptyset)$ (resp. $(\emptyset,(3,1))$) on either side as the wall-crossing is described by the Hasse diagram isomorphism afforded by π_1.

CHAPTER 6

Summary and Outlook

Let us review our journey through the combinatorics of wreath product groups $C_\ell \wr \mathfrak{S}_n$ in the context of restricted rational Cherednik algebras.

We began Chapter 1 with a description of $W = C_\ell \wr \mathfrak{S}_n$ as a complex reflection group of type $G(\ell, 1, n)$. Following [EG02] and [Gor03], a finite dimensional algebra $\overline{\mathsf{H}}_{\mathbf{c}}$ was introduced called the restricted rational Cherednik algebra. We have summarized the parametrization of the simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$ by the set $\operatorname{Irr} W$ (cf. [Gor03]). Because any $\overline{\mathsf{H}}_{\mathbf{c}}$-module admits a graded W-module structure, a natural question emerges which goes back to the inception of $\overline{\mathsf{H}}_{\mathbf{c}}$: what is the $\mathbf{c}$-dependent graded W-character of the simple $\overline{\mathsf{H}}_{\mathbf{c}}$-module $L_{\mathbf{c}}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \operatorname{Irr} W$?

From the results in [Gor03; EG02; Thi16; Thi18], it became apparent very quickly that the description of the W-characters of simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$ splits into two very different cases: the generic parameter and the special parameter.

In order to solve the generic parameter case, we looked at multipartition combinatorics and multisymmetric function theory as they relate to the representation theory of W. We give a slightly technical, but straightforward generalization of the methods in [Gor03] for $\mathfrak{S}_n$ to obtain the desired description of the simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$ as a basis of symmetric functions (cf. Thm. 3.5.2). Using a result by Wen [Wen19], we were able to relate our characters to the wreath Macdonald polynomials by Haiman [Hai03] (cf. Cor. 3.5.3).

Now, because the complex reflection groups of type $G(\ell, 1, n)$ and G_4 are the only ones admitting a generically smooth Calogero–Moser space (cf. Thm. 4.3.7), our approach is unlikely to work out for the generic parameter case of other groups such as $G(\ell, d, n)$ for $d > 1$. There is, however, a result by Griffeth [Gri14] wherein a connection between type A Macdonald polynomials and bigraded characters of modules of rational Cherednik algebras is made. For $\ell > 2$, this construction could be generalized as described in Section 2.20 of *loc. cit.* to strengthen the connection between Macdonald polynomials and characters of modules of rational Cherednik algebras.

After having described the generic parameter case, we started in Chapter 4 with the special parameter. We first defined Calogero–Moser families and reviewed some of the combinatorial constructions associated to them, namely hyperplane equations and semi-continuity (cf. Sec. 4.2). We then defined the Calogero–Moser space $\mathsf{X}_{\mathbf{c}}$ using the center of the rational Cherednik algebra following [EG02], and dove into the geometric properties of $\mathsf{X}_{\mathbf{c}}$ which become central to the discussion of the special parameter case. Associated to such a Calogero–Moser space, there are geometric data such as $\mathbb{C}^*$-fixed points and symplectic leaves, which we reviewed following the discussions in [EG02] and [Bel11].

We then moved on to the most important aspect of the geometric representation theory of $\overline{\mathsf{H}}_{\mathbf{c}}$: cuspidal module induction. This inductive process was first detailed in [Bel11] and it combined together everything we knew thus far about the geometry of $\mathsf{X}_{\mathbf{c}}$. Cuspidal module induction became our main plan of attack to construct the graded W-characters of simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$.

To actually be able to work with these complicated geometric features of $\mathsf{X}_{\mathbf{c}}$, we reviewed the long-established connection between Calogero–Moser spaces and quiver varieties (cf. Sec. 4.6).

We started off Chapter 5 by reviewing the definition of abaci, which served as the lense through which we analyzed all combinatorial structures appearing in the special parameter case. We used abaci together with work by [Mak22], [Mar10], and [GM09] to propose a combinatorial model for cuspidal module induction called cuspidal family induction (cf. Sec. 5.7, Sec. 5.9). In type B, we were able to use cuspidal family induction to describe the ungraded W-characters of bipartitions of the form $(\lambda, \emptyset)$ (cf. Cor. 5.8.4) which includes the trivial representation $((n), \emptyset)$ (cf. Cor 5.8.5). Afterwards, we proposed a generalization of cuspidal family induction to $G(\ell, 1, n)$ for $\ell > 2$ (cf. Sec. 5.9). This part of the thesis affords a natural question which we formulate as a conjecture here (cf. Sec. 5.9).

CONJECTURE. *For W of type $G(\ell, 1, n)$, cuspidal module induction and cuspidal family induction agree.*

In type B, we used rigid modules to "tag" family members and follow them through the inductive process. This, however, is not always possible since there are no rigid modules in cuspidal families of $G(\ell, 1, n)$ in general (cf. Exm. 5.10.7). One way to circumvent this fact could be to prove that family members are already nonisomorphic as ungraded W-modules. Afterwards, we would have to find some alternative structure to rigidity that makes modules unable to be induced from modules of parabolic subgroups. This does become very difficult though since cuspidal families in type $G(\ell, 1, n)$ are hard to classify or endow with a unifying structure (cf. Sec. 5.10).

Next, we have reviewed the constructions of various sets of constructible and cellular characters found in [Lus03; LM04; BR17; Lac20] and studied their connection to cuspidal family induction. In type B, there are very strong connections to the aforementioned characters and $\mathbf{j}$-induction, though we have found few evidence for a "naive" connection between the characters and the inductive phenomena in type $G(\ell, 1, n)$ for $\ell > 2$. We nonetheless still believe that there is a way to interpret cuspidal family induction in the context of these characters.

Lastly, we have made an attempt to connect the combinatorics of generic and special parameters by studying Haiman's wreath Macdonald polynomials in Section 5.13. We have proved that when crossing the outermost hyperplane $\mathcal{H}$ in type B, the wreath Macdonald polynomials associated to singleton families on $\mathcal{H}$ stay constant. We conjectured that this holds true for $\ell > 2$ and any hyperplane (cf. Con. 5.13.2).

Another approach to use with symmetric functions are combinatorial statistics: often times when studying symmetric functions and their respective transition matrices, combinatorial statistics on Young tableaux appear that give rise to exponents of indeterminates in these transition matrix. One prominent example would be the Kostka polynomials $K_{\mu,\lambda}(t)$ given by the *charge* statistic (cf. [Mac95, Thm. 6.5]). Now, it may be possible to define a statistic for the t, t-Kostka–Macdonald polynomials we have studied here, and afterwards degenerate this statistic by only looking at a $\mathbf{c}$-dependent subset of Young tableaux.

Finally, we could focus our attention away from the transition matrix C_L, which describes the decomposition of simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$ into simple modules of W, but rather the matrix D_Δ, which describes the decomposition of standard modules of $\overline{\mathsf{H}}_{\mathbf{c}}$ into simple modules of $\overline{\mathsf{H}}_{\mathbf{c}}$. The latter matrix has the advantage of block-diagonal structure. Furthermore, it would allow for a graded approach to the problem instead of the ungraded version that cuspidal module induction provides.

APPENDIX A

Parameters and Hyperplanes

The rational Cherednik algebra has originally been defined for a larger class of groups: the *symplectic reflection groups*. We will review the associated constructions now. Let V be a symplectic vector space with symplectic form ω and let $\mathrm{Sp}(V)$ be the subspace of endomorphisms $f \in \mathrm{End}(V)$ that keep the form ω invariant, i.e.

$$\omega(f.v, f.w) = \omega(v, w) \ . \tag{A.1}$$

An element $s \in \mathrm{Sp}(V)$ is called *symplectic reflection* if

$$\dim(\mathrm{Im}(s - \mathrm{id}_V)) = 2 \ , \tag{A.2}$$

i.e. s fixes pointwise a space $\mathcal{H}_s \leq V$ of codimension 2, which we call *symplectic hyperplane* of s. The set of symplectic reflections of Γ is again denoted by $\mathcal{S}$. For a group Γ with faithful representation $\Gamma \to \mathrm{Sp}(V)$, we identify the elements of Γ with their respective image in $\mathrm{Sp}(V)$. If Γ is generated by its symplectic reflections, we say Γ is a *symplectic reflection group*.

Let $\mathscr{C}$ denote the vector space of Γ-equivariant maps

$$\mathbf{c} : \mathcal{S} \to \mathbb{C} \ , \tag{A.3}$$

i.e. $\mathbf{c}(s) = \mathbf{c}(t)$ when s and t are conjugate in Γ. Denote by ω_s the restriction of ω to the image of $s \in \mathcal{S}$. The rational Cherednik algebra of Γ with parameter $\mathbf{c}$ is the quotient of the *(complex) skew tensor algebra* $T_{\mathbb{C}}(V) \rtimes \Gamma$ (cf. Defn. 1.2.1) by the commutator relation

$$[v, w] = \sum_{s \in \mathcal{S}} \mathbf{c}(s)\omega_s(v, w)s \in \mathbb{C}\Gamma \tag{A.4}$$

for $v, w \in V$.

We will survey a number of bases of the space $\mathscr{C}$ associated to symplectic reflection groups, complex reflection groups, and finally wreath product complex reflection groups.

A.1. Symplectic reflection groups

The following construction can be found in [Bel16, Sec. 3.2] or [BST18, Sec. 3.2]. Let Γ be a symplectic reflection group with symplectic reflections $\mathcal{S} \subseteq \Gamma$. For $s \in \mathcal{S}$, we denote by $[s] \in \mathcal{S}/\Gamma$ the Γ-conjugacy class of s. For $s \in \mathcal{S}$, let

$$\mathcal{H}_s = \mathrm{Ker}(s - \mathrm{id}_V) \tag{A.5}$$

denote the "reflecting hyperplane" (of codimension 2) of s. Denote by $\mathcal{A}$ the set of all such reflecting hyperplanes of Γ. For $\mathcal{H} \in \mathcal{A}$, denote by $\mathcal{S}_{\mathcal{H}} \subseteq \mathcal{S}$ the set of reflections with reflecting hyperplane $\mathcal{H}$ and let $\Gamma_{\mathcal{H}} \leq \Gamma$ denote the stabilizer subgroup of a reflecting hyperplane $\mathcal{H}$. We have

$$\Gamma_{\mathcal{H}} = \mathcal{S}_{\mathcal{H}} \cup \{\mathrm{id}\} \ . \tag{A.6}$$

The group Γ acts on $\mathcal{A}$ by

$$g.\mathcal{H}_s = \mathcal{H}_{gsg^{-1}} \tag{A.7}$$

and we denote by $[\mathcal{H}] \in \mathcal{A}/\Gamma$ the class $\mathcal{H} \in \mathcal{A}$. Note that

$$\Gamma_{g.\mathcal{H}} = g\Gamma_{\mathcal{H}}g^{-1} \tag{A.8}$$

and we obtain a well-defined map

$$\mathcal{S}/\Gamma \to \mathcal{A}/\Gamma,\ [s] \mapsto [\mathcal{H}_s] \tag{A.9}$$

Define

$$(\mathcal{S}/\Gamma)_{[\mathcal{H}]} := \{[s] \in \mathcal{S}/\Gamma \mid \mathcal{H}_s \in [\mathcal{H}]\} \tag{A.10}$$

to get a decomposition

$$\mathcal{S}/\Gamma = \coprod_{[\mathcal{H}]\in\mathcal{A}/\Gamma} (\mathcal{S}/\Gamma)_{[\mathcal{H}]} \ . \tag{A.11}$$

There is now a well-defined bijection

$$(\mathcal{S}/\Gamma)_{[\mathcal{H}]} \leftrightarrow \mathrm{Cl}(\Gamma_{\mathcal{H}}) \setminus \{\mathrm{id}\} \ . \tag{A.12}$$

This is because two elements $s, t \in \mathcal{S}_{\mathcal{H}}$ are Γ-conjugate if and only if they are already $\Gamma_{\mathcal{H}}$-conjugate.

We denote the set $\mathrm{Cl}(\Gamma) \setminus \{\mathrm{id}\}$ by $\mathrm{Cl}_* \Gamma_{\mathcal{H}}$. There is a (non-canonical) bijection

$$\mathrm{Cl}_* \Gamma_{\mathcal{H}} \leftrightarrow \mathrm{Irr}_* \Gamma_{\mathcal{H}} \tag{A.13}$$

where similarly $\mathrm{Irr}_* \Gamma_{\mathcal{H}}$ denotes the set of nontrivial irreducible characters of $\Gamma_{\mathcal{H}}$. The group Γ acts on

$$\hat{\mathcal{A}} := \mathcal{A} \times \mathrm{Irr}\,\Gamma_{\mathcal{H}} = \{(\mathcal{H}, \eta) \mid \mathcal{H} \in \mathcal{A}, \eta \in \mathrm{Irr}\,\Gamma_{\mathcal{H}}\} \tag{A.14}$$

by defining for $\gamma \in \mathrm{Irr}\,\Gamma_{\mathcal{H}}$, $g \in \Gamma$ the irreducible character ${}^g\eta \in \mathrm{Irr}\,\Gamma_{g.\mathcal{H}}$ by

$${}^g\eta(ghg^{-1}) := \eta(h) \tag{A.15}$$

and identifying $\mathrm{Irr}\,\Gamma_{\mathcal{H}}$ with $\mathrm{Irr}\,\Gamma_{g.\mathcal{H}}$. We denote by $[\mathcal{H}, \eta] \in \hat{\mathcal{A}}/\Gamma$ the equivalence class of $(\mathcal{H}, \eta) \in \hat{\mathcal{A}}$. For $[\mathcal{H}, \eta] \in \hat{\mathcal{A}}/\Gamma$ and $[s] \in \mathcal{S}/\Gamma$ with $[\mathcal{H}_s] = [\mathcal{H}]$, we have that $\eta(s)$ is well-defined. This is again because two elements $s, t \in \mathcal{S}_{\mathcal{H}}$ are Γ-conjugate if and only if they are already $\Gamma_{\mathcal{H}}$-conjugate. Lastly, denote by $\hat{\mathcal{A}}_* \subset \mathcal{A}$ the subset of all $[\mathcal{H}, \eta]$ where η is nontrivial. The action of Γ on $\hat{\mathcal{A}}$ restricts to $\hat{\mathcal{A}}_*$.

A.1.1. Etingof–Ginzburg c. The *Etingof–Ginzburg basis* [EG02] of $\mathscr{C}$ is denoted by

$$(\mathbf{c}_{[s]})_{[s]\in\mathcal{S}/W} \tag{A.16}$$

with coordinates

$$(c_{[s]})_{[s]\in\mathcal{S}/W} \tag{A.17}$$

and it is defined as

$$\begin{aligned} \mathbf{c}_{[s]} &: \mathcal{S} \to \mathbb{C} \\ r &\mapsto \mathbf{c}_{[s]}(r) = \begin{cases} 1, & r \in [s] \\ 0 & \text{else} \end{cases} \end{aligned} \tag{A.18}$$

for $[s] \in \mathcal{S}/W$. A general element $\mathbf{c} \in \mathscr{C}$ is written as

$$\mathbf{c} = \sum_{[s]\in\mathcal{S}/W} c_{[s]}\mathbf{c}_{[s]} \ . \tag{A.19}$$

A.1.2. McKay K. The *McKay basis* [Bel16; BST18, as **k**] of $\mathscr{C}$ is denoted by

$$(\mathbf{K}_{[\mathcal{H},\eta]})_{[\mathcal{H},\eta]\in\hat{\mathcal{A}}_*/\Gamma} \tag{A.20}$$

with coordinates

$$(K_{[\mathcal{H},\eta]})_{[\mathcal{H},\eta]\in\hat{\mathcal{A}}_*/\Gamma} \tag{A.21}$$

and it is defined as

$$\mathbf{K}_{[\mathcal{H},\eta]} := \frac{1}{|\Gamma_{\mathcal{H}}|}\sum_{[s]\in(\mathcal{S}/\Gamma)_{[\mathcal{H}]}} \eta(s^{-1})\cdot\mathbf{c}_{[s]} = \frac{1}{|\Gamma_{\mathcal{H}}|}\sum_{[s]\in\mathrm{Cl}_*\,\Gamma_{\mathcal{H}}} \eta(s^{-1})\cdot\mathbf{c}_{[s]}\ . \tag{A.22}$$

for $[\mathcal{H},\eta]\in\mathcal{A}_*/\Gamma$.

We can divide the McKay basis vectors into "blocks" by $[\mathcal{H}]\in\mathcal{A}/\Gamma$. When we include the vectors $(\mathbf{K}_{[\mathcal{H},\mathbf{triv}]})_{H\in\mathcal{A}}$ defined as in (A.22), the blocks become linearly dependent. This is because for $\mathcal{H}\in\mathcal{A}$ we have

$$\begin{aligned}\sum_{\eta\in\mathrm{Irr}\,\Gamma_{\mathcal{H}}}\eta(1)\cdot\mathbf{K}_{[\mathcal{H},\eta]} &= \sum_{\eta\in\mathrm{Irr}\,\Gamma_{\mathcal{H}}}\eta(1)\cdot\frac{1}{|\Gamma_{\mathcal{H}}|}\sum_{[s]\in\mathrm{Cl}_*\,\Gamma_{\mathcal{H}}}\eta(s^{-1})\cdot\mathbf{c}_{[s]}\\ &= \frac{1}{|\Gamma_{\mathcal{H}}|}\sum_{[s]\in\mathrm{Cl}_*\,\Gamma_{\mathcal{H}}}\left(\sum_{\eta\in\mathrm{Irr}\,\Gamma_{\mathcal{H}}}\eta(1)\eta(s^{-1})\right)\mathbf{c}_{[s]}\\ &= 0\end{aligned} \tag{A.23}$$

by the orthogonality relations of the character table of $\Gamma_{\mathcal{H}}$. We have for a general element of $\mathscr{C}$ and block $[\mathcal{H}]$ the expression

$$\sum_{\eta\in\mathrm{Irr}_*\,\Gamma_{\mathcal{H}}}K_{[\mathcal{H},\eta]}\mathbf{K}_{[\mathcal{H},\eta]} = \sum_{\eta\in\mathrm{Irr}_*\,\Gamma_{\mathcal{H}}}K_{[\mathcal{H},\eta]}\left(\frac{1}{|\Gamma_{\mathcal{H}}|}\sum_{[s]\in(\mathcal{S}/\Gamma)_{[\mathcal{H}]}}\eta(s^{-1})\mathbf{c}_{[s]}\right) \tag{A.24}$$

$$= \sum_{[s]\in\mathrm{Cl}_*\,\Gamma_{\mathcal{H}}}\left(\frac{1}{|\Gamma_{\mathcal{H}}|}\sum_{\eta\in(\mathcal{S}/\Gamma)_{[\mathcal{H}]}}\eta(s^{-1})K_{[\mathcal{H},\eta]}\right)\mathbf{c}_{[s]}\ . \tag{A.25}$$

which gives us the coordinate transformation

$$c_{[s]} = \frac{1}{|\Gamma_{\mathcal{H}}|}\sum_{\eta\in(\mathcal{S}/\Gamma)_{[\mathcal{H}]}}\eta(s^{-1})K_{[\mathcal{H},\eta]}\ . \tag{A.26}$$

A.2. Complex reflection groups

Let Γ be a symplectic reflection group with faithful symplectic reflection representation V. We say Γ is irreducible as a symplectic reflection group if V cannot be decomposed as a direct sum of two nontrivial symplectic representations of Γ. A pairing (Γ, V) which is symplectically irreducible might not be irreducible as a *complex* reflection group. In this case, we can write V as a direct sum

$$V = \mathfrak{h}\oplus\mathfrak{h}^* \tag{A.27}$$

such that $\mathfrak{h}$ is a faithful complex reflection representation of Γ with dual $\mathfrak{h}^*$. The symplectic reflections of Γ on V are the same as the complex reflections of Γ on $\mathfrak{h}$ and

$$\ker(s-\mathrm{id}_V) = \ker(s-\mathrm{id}_{\mathfrak{h}})\oplus\ker(s-\mathrm{id}_{\mathfrak{h}^*})\ , \tag{A.28}$$

i.e. the symplectic reflection "hyperplanes" are given by a direct sum of complex reflection hyperplanes. Furthermore, the stabilizer of a symplectic reflection hyperplane is the stabilizer of any one of its complex reflection hyperplane summands.

We can thus reduce the discussion of the "symplectified" complex reflection groups to the case of ordinary complex reflection groups. The notations will stay the same.

Let $(W, \mathfrak{h})$ be a complex reflection group and let $\mathcal{H}$ be a complex reflection hyperplane. It is well-known that the stabilizer subgroup $W_{\mathcal{H}}$ is cyclic. As seen in Section 2.3, the elements of $W_{\mathcal{H}}$ are now in canonical bijection with $\operatorname{Irr} W$ when choosing a generator $s_{\mathcal{H}} \in W_{\mathcal{H}}$ with

$$\det(s_{\mathcal{H}}) = \zeta_{|W_{\mathcal{H}}|} = \exp\left(\frac{2\pi i}{|W_{\mathcal{H}}|}\right) . \tag{A.29}$$

We have

$$\mathcal{S} = \{s_{\mathcal{H}}^j \mid \mathcal{H} \in \mathcal{A} \, , \; 1 \leq j \leq |W_{\mathcal{H}}| - 1\} . \tag{A.30}$$

For $0 \leq i \leq |W_{\mathcal{H}}| - 1$, we define the irreducible $W_{\mathcal{H}}$-character

$$\eta_i : W_{\mathcal{H}} \to \mathbb{C}, \; s \to \det(s)^i \tag{A.31}$$

which for $s = s_{\mathcal{H}}^j \in W_{\mathcal{H}}$ becomes $\det(s_{\mathcal{H}})^{i \cdot j}$. The irreducible characters of $W_{\mathcal{H}}$ are given by

$$\eta_0, \ldots, \eta_{|W_{\mathcal{H}}|-1} \tag{A.32}$$

where we consider the indices modulo $|W_{\mathcal{H}}|$. Note that $\eta_i(1) = 1$ for all i.

A.2.1. Etingof–Ginzburg c. The *Etingof–Ginzburg basis* [EG02] of $\mathscr{C}$ in the complex reflection setting does not change from Section A.1.1.

A.2.2. McKay K. The *McKay basis* [BST18; Bel16, as $\mathbf{k}$] of $\mathscr{C}$ in the complex reflection setting is denoted by

$$(\mathbf{K}_{[\mathcal{H}],i})_{[\mathcal{H}] \in \mathcal{A}/W, 1 \leq i \leq |W_{\mathcal{H}}|-1} \tag{A.33}$$

with coordinates

$$(K_{[\mathcal{H}],i})_{[\mathcal{H}] \in \mathcal{A}/W, 1 \leq i \leq |W_{\mathcal{H}}|-1} \tag{A.34}$$

and it is defined as

$$\mathbf{K}_{[\mathcal{H}],i} = \mathbf{K}_{[\mathcal{H},\eta_i]} \; , \quad K_{[\mathcal{H}],i} = K_{[\mathcal{H}],\eta_i} \; . \tag{A.35}$$

We have for a general element of $\mathscr{C}$ and block $[\mathcal{H}]$ the expression

$$\sum_{i=1}^{|W_{\mathcal{H}}|-1} K_{[\mathcal{H}],i} \mathbf{K}_{[\mathcal{H}],i} = \sum_{i=1}^{|W_{\mathcal{H}}|-1} K_{[\mathcal{H}],i} \left(\frac{1}{|W_{\mathcal{H}}|} \sum_{[s] \in \mathrm{Cl}_* W_{\mathcal{H}}} \det(s)^{-i} \mathbf{c}_{[s]} \right) \tag{A.36}$$

$$= \sum_{j=1}^{|W_{\mathcal{H}}|-1} \left(\frac{1}{|W_{\mathcal{H}}|} \sum_{i=1}^{|W_{\mathcal{H}}|-1} \zeta_{|W_{\mathcal{H}}|}^{-ij} K_{[\mathcal{H},\eta]} \right) \mathbf{c}_{[s]} \; . \tag{A.37}$$

which gives us for the coordinate transformation

$$c_{[s]} = \frac{1}{|W_{\mathcal{H}}|} \sum_{i=1}^{|W_{\mathcal{H}}|-1} \zeta_{|W_{\mathcal{H}}|}^{-ij} K_{[\mathcal{H},\eta]} \; . \tag{A.38}$$

A.2.3. GGOR k. The *GGOR basis* [GGOR03], [BST18, as $\boldsymbol{\kappa}$] of $\mathscr{C}$ is denoted by

$$(\mathbf{k}_{[\mathcal{H}],i})_{[\mathcal{H}]\in\mathcal{A}/W, 1\le i\le |W_{\mathcal{H}}|-1} \tag{A.39}$$

with coordinates

$$(k_{[\mathcal{H}],i})_{[\mathcal{H}]\in\mathcal{A}/W, 1\le i\le |W_{\mathcal{H}}|-1} \tag{A.40}$$

and it is defined as

$$\mathbf{k}_{[\mathcal{H}],i} := \sum_{[s]\in \mathrm{Cl}_* W_{\mathcal{H}}} \left(\det(s)^{i-1} - \det(s)^i\right) \mathbf{c}_{[s]} \tag{A.41}$$

$$= \sum_{k=1}^{|W_{\mathcal{H}}|-1} \left(\zeta_\ell^{(i-1)k} - \zeta_\ell^{ik}\right) \mathbf{c}_{[s]} \tag{A.42}$$

In [GGOR03], the authors have included the element $\mathbf{k}_{[\mathcal{H}],0} = \mathbf{k}_{[\mathcal{H}],|W_{\mathcal{H}}|}$ which they defined to be the zero vector in $\mathscr{C}$. However, when we define $\mathbf{k}_{[\mathcal{H}],0}$ analogously to (A.41), we get the relation

$$\sum_{i=0}^{|W_{\mathcal{H}}|-1} \mathbf{k}_{[\mathcal{H}],i} = 0 \tag{A.43}$$

with a straightforward computation. When including the vectors $(\mathbf{k}_{[\mathcal{H}],0})_{\mathcal{H}\in\mathcal{A}}$ together with their respective coordinates $(k_{[\mathcal{H}],0})_{\mathcal{H}\in\mathcal{A}}$, we restrict ourselves to the parameter space given by the relation

$$\sum_{i=0}^{|W_{\mathcal{H}}|-1} k_{[\mathcal{H}],i} = 0\ . \tag{A.44}$$

Doing this, we get a nice operation of the Namikawa–Weyl group on the $\mathscr{C}$ (cf. [BST18, Lem. 4.1]). We have for a general element of $\mathscr{C}$ and block $[\mathcal{H}]$ the expression

$$\sum_{i=0}^{\ell-1} k_{[\mathcal{H}],i}\mathbf{k}_{[\mathcal{H}],i} = \sum_{i=0}^{\ell-1} k_{[\mathcal{H}],i} \left(\sum_{[s]\in \mathrm{Cl}_* W_{\mathcal{H}}} \left(\det(s)^{i-1} - \det(s)^i\right) \mathbf{c}_{[s]} \right) \tag{A.45}$$

$$= \sum_{[s]\in \mathrm{Cl}_* W_{\mathcal{H}}} \left(\sum_{i=0}^{\ell-1} k_{[\mathcal{H}],i} \det(s)^{i-1} - \sum_{i=0}^{\ell-1} k_{[\mathcal{H}],i} \det(s)^{i} \right) \mathbf{c}_{[s]} \tag{A.46}$$

$$= \sum_{[s]\in \mathrm{Cl}_* W_{\mathcal{H}}} \left(\sum_{i=0}^{\ell-1} k_{[\mathcal{H}],i+1} \det(s)^{i} - \sum_{i=0}^{\ell-1} k_{[\mathcal{H}],i} \det(s)^{i} \right) \mathbf{c}_{[s]} \tag{A.47}$$

$$= \sum_{k=1}^{\ell-1} \left(\sum_{i=0}^{\ell-1} \det(s)^i (k_{[\mathcal{H}],i+1} - k_{[\mathcal{H}],i}) \right) \mathbf{c}_{[\mathcal{H}],k}\ . \tag{A.48}$$

which gives us for the coordinate transformation

$$c_{[s]} = \sum_{i=0}^{\ell-1} \det(s)^i (k_{[\mathcal{H}],i+1} - k_{[\mathcal{H}],i})\ . \tag{A.49}$$

The relationship between the sum condition GGOR parameter and the McKay parameter is

$$\sum_{i=0}^{|W_{\mathcal{H}}|-1} k_{[\mathcal{H}],i}\mathbf{k}_{[\mathcal{H}],i} = \sum_{i=0}^{|W_{\mathcal{H}}|-1} |W_{\mathcal{H}}|(k_{[\mathcal{H}],i+1} - k_{[\mathcal{H}],i}) \cdot \mathbf{K}_{[\mathcal{H}],-i} \tag{A.50}$$

since both sides are equal to (A.48), and we obtain the coordinate transformation

$$K_{[\mathcal{H}],-i} = |W_{\mathcal{H}}|(k_{[\mathcal{H}],i+1} - k_{[\mathcal{H}],i}) \; . \tag{A.51}$$

A.3. Wreath product groups

The wreath product complex reflection groups of type $G(\ell, 1, n)$ for some ℓ, n have been discussed in Section 4.2. We reiterate the notations here. Let W be the complex reflection group generated by the complex reflections

$$\gamma_j := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \zeta & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} , \quad s_{i,i+1} := \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \tag{A.52}$$

for $1 \leq j \leq n$, $1 \leq i < n$ where $s_{i,i+1}$ transposes rows i and $i+1$ of the identity matrix. The remaining complex reflections of W are given by

$$\begin{gathered} \gamma_j^{-k} s_{ij} \gamma_j^k = s_{ij} \gamma_i^{-k} \gamma_j^k \; , \\ \gamma_j^k , \end{gathered} \tag{A.53}$$

where $1 \leq k \leq \ell - 1$, $1 \leq i < j \leq n$, and

$$s_{ij} = s_{i,i+1} \cdot s_{i+1,i+2} \cdots s_{j-2,j-1} \cdot s_{j-1,j} \cdot s_{j-2,j-1} \cdots s_{i+1,i+2} \cdot s_{i,i+1} \tag{A.54}$$

is the transposition of row i and j of the identity matrix. We have ℓ different conjugacy classes of reflections, namely

$$\begin{aligned} \mathcal{S}_0 &:= \{ s_{ij} \gamma_i^k \gamma_j^{-k} \mid 1 \leq i \neq j \leq n, 0 \leq k \leq \ell - 1 \} \; , \\ \mathcal{S}_k &:= \{ \gamma_j^k \mid 1 \leq j \leq n \} \; , \quad 1 \leq k \leq \ell - 1 \; , \end{aligned} \tag{A.55}$$

with the product of their nonzero matrix entries forming a separating invariant. There are two hyperplanes up to W-conjugation, namely

$$\begin{aligned} \mathcal{H}_0 &:= \mathrm{Ker}(1 - s_{12}) = (1, -1, 0, ..., 0)^{\perp} \; , \\ \mathcal{H}_1 &:= \mathrm{Ker}(1 - \gamma_1) = (1, 0, ..., 0)^{\perp} \; , \end{aligned} \tag{A.56}$$

where $\mathcal{H}_0$ is conjugated to the hyperplanes of elements of $\mathcal{S}_0$, and $\mathcal{H}_1$ is conjugated to the hyperplanes of elements of $\mathcal{S}_1, ..., \mathcal{S}_{\ell-1}$. The stabilizer subgroups are

$$\begin{aligned} W_{\mathcal{H}_0} &:= W_0 := \{1, s_{12}\} \cong C_2 \; , \\ W_{\mathcal{H}_1} &:= W_1 := \{ \gamma_1^k \mid 0 \leq k \leq \ell - 1 \} \cong C_\ell \; . \end{aligned} \tag{A.57}$$

A.3.1. Etingof–Ginzburg c.

The *Etingof–Ginzburg basis* [EG02] of $\mathscr{C}$ in the wreath product setting is denoted by

$$\begin{aligned} \boldsymbol{\kappa} &:= \mathbf{c}_{[s]}, \quad s \in \mathcal{S}_0 \; , \\ \mathbf{c}_k &:= \mathbf{c}_{[s]}, \quad s \in \mathcal{S}_k \; , \quad 1 \leq k \leq \ell - 1 \end{aligned} \tag{A.58}$$

with coordinates

$$\begin{aligned} \kappa &:= c_{[s]}, \quad s \in \mathcal{S}_0 \; , \\ c_k &:= c_{[s]}, \quad s \in \mathcal{S}_k \; , \quad 1 \leq k \leq \ell - 1 \; . \end{aligned} \tag{A.59}$$

A.3.2. McKay K.

The *McKay basis* [Bel16; BST18, as $\mathbf{k}$] of $\mathscr{C}$ in the wreath product setting does not change from Section A.2.2.

A.3.3. GGOR k.

The *GGOR basis* [GGOR03] of $\mathscr{C}$ in the wreath product setting does not change from Section A.2.3.

A.3.4. Gordon H. The *Gordon basis* [Gor08a], [BST18, as $\boldsymbol{\kappa}$] of $\mathscr{C}$ is denoted by

$$\mathbf{h}\ ,\ (\mathbf{H}_i)_{1 \le i \le \ell-1} \tag{A.60}$$

with coordinates

$$h\ ,\ (H_i)_{1 \le i \le \ell-1} \tag{A.61}$$

and it is defined as

$$\begin{aligned} \mathbf{h} &:= 2 \cdot \boldsymbol{\kappa}, \\ \mathbf{H}_i &:= \sum_{k=0}^{\ell-1} \zeta_\ell^{-ik} \mathbf{c}_k\ ,\ 1 \le i \le \ell - 1 \end{aligned} \tag{A.62}$$

We have for a general element of $\mathscr{C}$ the expression

$$h\mathbf{h} + \sum_{i=0}^{\ell-1} H_i \mathbf{H}_i = 2h\boldsymbol{\kappa} + \sum_{k=1}^{\ell-1} \left(\sum_{i=0}^{\ell-1} H_i \zeta_\ell^{-ik} \right) \mathbf{c}_k \tag{A.63}$$

which gives us the coordinate transformation

$$h = \frac{1}{2}\kappa\ ,\ c_k = \sum_{i=0}^{\ell-1} \zeta_\ell^{-ik} H_i\ . \tag{A.64}$$

A.4. Hyperplane equations for $G(\ell, 1, n)$

The set $\mathsf{BlEx}(\overline{\mathsf{H}}(W))$ is a union of hyperplanes (cf. Sec. 1.4). For W of type $G(\ell, 1, n)$, we give the hyperplane equations in the different parameter bases.

A.4.1. Etingof–Ginzburg c.

$$\begin{gathered} \kappa = 0\ ,\ \text{and}\ \ 2\left(c_1 \cdot \sum_{t=i}^{j} \zeta_\ell^{-t} + \cdots + c_{\ell-1} \sum_{t=i}^{j} \zeta_\ell^{-(\ell-1)\cdot t} \right) + m\ell\kappa = 0\ , \\ 1 \le i \le j \le \ell - 1\ ,\quad -n < m < n\ . \end{gathered} \tag{A.65}$$

A.4.2. McKay K.

$$\begin{gathered} K_{[\mathcal{H}_0],1} = 0\ ,\ \text{and}\ \ 4(K_{[\mathcal{H}_1],i} + \cdots + K_{[\mathcal{H}_1],j}) + m\ell K_{[\mathcal{H}_0],1} = 0\ , \\ 1 \le i \le j \le \ell - 1\ ,\quad -n < m < n\ . \end{gathered} \tag{A.66}$$

A.4.3. GGOR k.

$$\begin{gathered} k_{[\mathcal{H}_0],1} = 0\ ,\ \text{and}\ \ 2(k_{[\mathcal{H}_0],i} - k_{[\mathcal{H}_0],j}) + m(k_{[\mathcal{H}_0],0} - k_{[\mathcal{H}_0],1}) = 0\ , \\ 1 \le i < j \le \ell - 1\ ,\quad -n < m < n\ . \end{gathered} \tag{A.67}$$

A.4.4. Gordon H.

$$\begin{gathered} h = 0\ ,\ \text{and}\ \ (H_i + \cdots + H_j) + mh = 0\ , \\ 1 \le i \le j \le \ell - 1\ ,\quad -n < m < n\ . \end{gathered} \tag{A.68}$$

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