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Nilpotent Pieces in Lie Algebras of Exceptional Type in Bad Characteristic

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Abstract

In group theory, a big and important family of infinite groups is given by the algebraic groups. These groups and their structures are already well-understood. In representation theory, the study of the unipotent variety in algebraic groups — and by extension the study of the nilpotent variety in the associated Lie algebra — is of particular interest.

Let G be a connected reductive algebraic group over an algebraically closed field $\mathbf{k}$, and let $\mathrm{Lie}(G)$ be its associated Lie algebra. By now, the orbits in the nilpotent and unipotent variety under the action of G are completely known and can be found for example in a book of Liebeck and Seitz. There exists, however, no uniform description of these orbits that holds in both good and bad characteristic. With this in mind, Lusztig defined a partition of the unipotent variety of G in 2011. Equivalently, one can consider certain subsets of the nilpotent variety of $\mathrm{Lie}(G)$ called the *nilpotent pieces*. This approach appears in the same paper by Lusztig in which he explicitly determines the nilpotent pieces for simple algebraic groups of classical type. The nilpotent pieces for the exceptional groups of type G_2, F_4, E_6, E_7, and E_8 in bad characteristic have not yet been determined. This thesis gives an introduction to the definition of the nilpotent pieces and presents a solution to this problem for groups of type G_2, F_4, E_6, and partly for E_7. The solution relies heavily on computational work which we elaborate on in later chapters.

Zusammenfassung

In der Gruppentheorie bilden sogenannte algebraische Gruppen eine große und wichtige Familie von unendlichen Gruppen. Algebraische Gruppen und ihre Strukturen sind in der Vergangenheit bereits sehr ausführlich untersucht worden. Insbesondere die Struktur der unipotenten Varietät — und damit der nilpotenten Varietät in der asoziierten Lie-Algebra — ist von großem Interesse in der Darstellungstheorie.

Sei G eine zusammenhängende reduktive algebraische Gruppe über einem algebraisch abgeschlossenem Körper $\mathbf{k}$. Sei weiterhin $\mathrm{Lie}(G)$ die zu G asoziierte Lie-Algebra. Zum jetzigen Zeitpunkt sind alle unipotenten und nilpotenten Bahnen unter der Operation einer algebraischen Gruppe bekannt. Diese sind beispielsweise ausführlich in dem Werk von Liebeck und Seitz beschrieben. Allerdings gibt es keine uniforme Beschreibung der Bahnen, die sowohl in guter als auch in schlechter Charakteristik gilt. In Betracht dieser Tatsache definierte Lusztig in 2011 eine Partition der unipotenten Varietät von G. Es ist möglich, stattdessen auch bestimmte Teilmengen der nilpotenten Varietät von $\mathrm{Lie}(G)$ zu betrachten, welche die „nilpotenten pieces" genannt werden. Auch dieser Ansatz wird von Lusztig beschrieben. In demselben Artikel bestimmt Lusztig außerdem die nilpotenten pieces für die klassischen algebraischen Gruppen. In den exzeptionellen Gruppen vom Typ G_2, F_4, E_6, E_7 und E_8 müssen die nilpotenten pieces noch bestimmt werden. Diese Dissertation gibt eine Einführung in die Definition der nilpotenten pieces und stellt eine Lösung für Gruppen vom Typ G_2, F_4, E_6 und teilweise E_7 vor. Die Lösung hängt großteils von programmiertechnischen Verfahren ab, welche in späteren Kapiteln beleuchtet werden.

Preface

In this thesis, we take a closer look at certain subsets of Lie algebras arising from algebraic groups, the so-called *nilpotent pieces*.

Algebraic groups form an important subset of the family of infinite groups and have been extensively studied in the past. Algebraic groups can, for example, be defined as closed subgroups of the general linear group $\mathrm{GL}_n(\mathbf{k})$ for an algebraically closed field $\mathbf{k}$ of arbitrary characteristic. The nilpotent pieces arise from the idea to define a partition of the nilpotent variety in the Lie algebra which is similar to the orbits in the nilpotent variety under the adjoint action of the corresponding algebraic group. The aim is to complete further steps in the proof that the nilpotent pieces form a partition of the nilpotent variety. More precisely, we believe that the following conjecture holds and work towards proving it.

Conjecture A. *Let G be a connected reductive algebraic group over an algebraically closed field $\mathbf{k}$ and let $\mathfrak{g}$ be its Lie algebra. Then the set of nilpotent pieces for $\mathfrak{g}$ is in bijection with the set of nilpotent orbits of a group $\tilde{G}$ defined over $\mathbb{C}$ with the same root datum as G. In particular, any nilpotent piece is given by a union of nilpotent orbits having the same* T-labelling *(see [20, Chapter 10]) as the corresponding orbit in good characteristic.*

The nilpotent pieces have been defined by George Lusztig in a series of papers, [21], [22], [23], and [24]. We will work with the definition of the nilpotent pieces given in [23]. Lusztig originally defined the *unipotent pieces* in [21] and elaborated on the concept in [22], [23], and [24]. However, having computed the nilpotent pieces, it should be within reach to give a description of the unipotent pieces as both definitions rely crucially on the same sets.

Algebraic groups form an important family of groups that gives rise to the finite groups of Lie type which are a large collection of groups appearing in the classification of finite simple groups. As such, we are also interested in the representation theory of the finite groups of Lie type and by extension the structures of algebraic groups, such as the nilpotent orbits. As mentioned above, the idea behind the definition of the nilpotent pieces comes from certain orbits in the Lie algebras of algebraic groups, called the *nilpotent orbits*. These orbits arise from the action of an algebraic group on its Lie algebra given by the *adjoint representation*. The study of the nilpotent orbits is closely related to the unipotent conjugacy classes in algebraic groups and both objects find applications in representation theory. These orbits have been extensively studied and are well-understood. A thorough description can be found in the book by Liebeck and Seitz [20], for instance. We notice that there is no uniform description of the nilpotent orbits which holds in every characteristic. Especially in small characteristic we may get a different number of nilpotent orbits. By defining the nilpotent pieces, we hope to give a more uniform description closely related to the nilpotent orbits "coming from characteristic 0". This problem has already been solved for the *classical* algebraic groups in [23]. It is closely related to the representation theory of algebraic groups and should help to work out specifics for generalised Gelfand–Graev representations in small characteristics, see [12]. In this paper, Geck describes a way to define generalised Gelfand–Graev representations in small characteristic. This relies heavily on a linear map λ being in "sufficiently general position" which can be checked with the knowledge of the nilpotent pieces.

Interestingly, there exist different definitions of partitions of the nilpotent variety. In [13], Hesselink defines a stratification of the nilpotent variety. Clarke and Premet define their own nilpotent pieces in [9] and show that this leads to the same stratification as proposed by Hesselink. In [33], Xue computes nilpotent pieces in $\mathfrak{g}^*$ for groups of type F_4 and G_2 using the definition for the nilpotent pieces in $\mathfrak{g}^*$ proposed by Clarke–Premet in [9]. One would hope that these definitions lead to the same object, and this is indeed the case for algebraic groups of classical type. We expect the equality to hold in the exceptional cases as well.

Conjecture B. *The nilpotent pieces as defined by Clarke–Premet in [9] result in the same nilpotent pieces as defined by Lusztig in [23].*

The main result of this thesis is as stated below.

Theorem A. *Conjecture A and Conjecture B hold for simple algebraic groups of type G_2, F_4, and E_6.*

This thesis is organised as follows. In the first chapter, we give an introduction to algebraic groups and their Lie algebras, presenting results that are already very well understood. Following this, we describe the construction of the nilpotent orbits and give orbit representatives for simple Lie algebras of type $G_2, F_4, E_6,$ and E_7 in the second chapter. Furthermore, we introduce the *weighted Dynkin diagrams* which enable us to describe the nilpotent orbits in characteristic 0 in a concise way. In chapter 3 we define the nilpotent pieces relying heavily on the weighted Dynkin diagrams and examine their properties, as well as give case-free descriptions of certain nilpotent pieces. For instance, it is always possible to explicitly determine the *regular nilpotent piece*, arising from the regular weighted Dynkin diagram without explicit computations.
In Chapter 4 we propose a way to compute the nilpotent pieces with a computer programme written in Magma [2]. In order to write a functioning algorithm, we need to develop a way to compute the action of elements in a connected reductive algebraic group G on the associated Lie algebra. This can be done by writing the group elements in the Bruhat decomposition for a fixed maximal torus T, a fixed Borel subgroup B containing T, a root system Φ with simple roots Π with respect to T and B, and a fixed total ordering on Φ^+. Using the paper by Geck [11], we are able to implement formulas for the action of G on its Lie algebra via the adjoint map on a basis of $\mathrm{Lie}(G)$. This means also that we rely heavily on a given root system and the linear combination of elements in $\mathrm{Lie}(G)$ with respect to the Chevalley basis for this root system.

In the course of Chapter 4, we also work out results on the structure of the nilpotent pieces in order to simplify the computations. At the end of this chapter we describe an algorithm to compute the nilpotent pieces, based on the previous theoretical discussions. This algorithm uses further algebraic structures in order to solve non-linear equation systems, such as Gröbner bases and various constructions in polynomial rings.

We will elaborate on these programming aspects in the following chapters. In Chapter 5 we propose further possible approaches for the computations of the nilpotent pieces, such as an inductive method. In Chapter 6 we describe the implementation in more detail, focusing especially on ways in which to efficiently solve non-linear equation systems.

The results for simple algebraic groups of type $G_2, F_4,$ and $E_6,$ and partly E_7 for characteristic 2 are stated in the final chapter, leading to the main theorem of this thesis.

Contents

1 | Preliminaries

As this work will mainly focus on algebraic groups and their Lie algebras, we start by giving an overview of this extensive area in group theory. It is, unfortunately, not possible to give all proofs and background information on algebraic groups, but the most important results that are necessary to understand this work are stated in this chapter. Further background information can be found in the books of Malle and Testerman [25], Geck [10], and Humphreys [15] for example. The structure of the first chapter will, in most parts, follow the book by Malle and Testerman.
Throughout this text, let $\mathbf{k}$ be an algebraically closed field of arbitrary characteristic unless stated otherwise.

1.1 Algebraic varieties

First, we need to understand the definition of algebraic groups. Algebraic groups arise from a background in algebraic geometry. We keep this section rather short and only define strictly necessary objects.

Definition 1.1 (Affine algebraic variety, [10, Definition 2.1.6])**.** Let $X \neq \emptyset$ be a non-empty set and let $\mathbf{k}$ be a field. Then the set $\mathrm{Maps}(X, \mathbf{k})$ of maps from X to $\mathbf{k}$ is a $\mathbf{k}$-algebra with pointwise defined operations. For each map $f : X \to \mathbf{k}$, that is $f \in \mathrm{Maps}(X, \mathbf{k})$, define

$$\begin{aligned} \varepsilon_x : \mathrm{Maps}(X, \mathbf{k}) &\longrightarrow \mathbf{k} \\ f &\longmapsto f(x). \end{aligned}$$

Let $A[X] \subseteq \mathrm{Maps}(X, \mathbf{k})$ be such that

1. $A[X]$ is a finitely generated $\mathbf{k}$-algebra such that $1 \in A[X]$,

2. for two elements $x, y \in X$ with $x \neq y$ there exists $f \in A[X]$ such that $f(x) \neq f(y)$, and

3. if $\lambda : A[X] \to \mathbf{k}$ is a $\mathbf{k}$-algebra morphism there exists $x \in X$ such that $\lambda = \varepsilon_x$.

The tuple $(X, A[X])$ is called an **affine algebraic variety**, or **affine variety**.

If the choice of $A[X]$ is clear from the context, we will also just refer to the set X as an affine variety. Sometimes we will just write A instead of $A[X]$.
Affine varieties can be turned into topological spaces via the **Zariski topology**.

Definition 1.2 (Zariski Topology, [10, 2.1.7])**.** Let (X, A) be an affine variety. For all subsets $S \subseteq A$ the sets

$$V_X(S) := \{x \in X \mid f(x) = 0 \text{ for all } f \in S\}$$

form the closed sets of the **Zariski topology**. The open sets are given by the complements of the closed sets.

In the terminology of classical algebraic geometry, affine algebraic varieties are also defined as *algebraic sets* together with the induced Zariski topology. A set X is called an **algebraic set** if

$$X = \{(x_1, \dots, x_n) \in \mathbf{k}^n \mid f(x_1, \dots, x_n) = 0 \text{ for all } f \in I\},$$

where $I \lhd \mathbf{k}[X_1, \dots, X_n]$ is an ideal, see for instance [25, 1.1].
Note also that it is possible to define algebraic varieties which are not affine varieties. For instance, it is also possible to define **projective varieties**, see [25, Section 5.1].

Example 1.3.

1. Let $X = \mathbf{k}^n$ and let $A[X] = \mathbf{k}[Y_i \mid 1 \leqslant i \leqslant n]$ for indeterminates Y_i, $i \in \{1, \dots n\}$. Then $(X, A[X])$ is an affine variety, see [10, 2.1.8].

2. Let X be as in (1), let $S \subseteq A[X]$ and let $V = \{x \in \mathbf{k}^n \mid f(x) = 0 \text{ for all } f \in S\}$ be a closed subset of X with respect to the Zariski topology. Let

$$A[V] := {}^{A[X]}\big/_{\{f \in A[X] \mid f(x) = 0 \text{ for all } x \in V\}}.$$

   Then $(V, A[V])$ is also an affine variety, see [10, 2.1.12].

Next, let us define morphisms of affine varieties following the definition in [10].

Definition 1.4 (Morphism of affine varieties, [10, 2.1.6])**.** Let (X, A) and (Y, B) be affine varieties. The map

$$\varphi : X \longrightarrow Y$$

is called **a morphism of affine varieties** if $g \circ \varphi \in A$ for all $g \in B$. In this case we define the **comorphism** of φ as

$$\varphi^* : B \longrightarrow A, \quad g \longmapsto g \circ \varphi.$$

Definition 1.5 (Dominant, [10, Proposition 1.4.15])**.** Let (X, A) and (Y, B) be affine varieties. A morphism

$$\varphi : X \longrightarrow Y$$

is called **dominant** if the image $\varphi(X)$ is dense in Y.

1.2 Algebraic groups

Having defined affine algebraic varieties and a corresponding topology, we can move on to the study of algebraic groups. It turns out that it is possible to endow groups with the structure of algebraic varieties. This indeed gives rise to a mathematically interesting object, an **algebraic group**. We can find out more about their properties by also taking into account their structure as affine varieties, rather than purely looking at them as abstract groups. In the following section, we are going to define these structures and find a way to link a Lie algebra to each algebraic group.

Definition 1.6 (Algebraic groups, [25, Definition 1.1])**.** Let (G, A) be an affine variety and let $(G, \star)$ be a group. There exist maps

$$\mu : G \times G \longrightarrow G, \qquad\qquad \iota : G \longrightarrow G,$$
$$(g, h) \longmapsto g \star h, \qquad\qquad g \longmapsto g^{-1}.$$

If both μ and ι are morphisms of affine varieties, then G is called an **algebraic group**.

Remark 1.7. For two affine varieties (X, A) and (Y, B) over a field k we can take the cartesian product $X \times Y$ endowed with the Zariski variety. Then $(X \times Y, A \otimes_k B)$ is again an affine variety, see [10, 2.1.13]. This shows that taking the product $G \times G$ in the above definition makes sense.

We will give a couple of first examples in order to get a better understanding of algebraic groups. In fact, quite a few groups that are already well-understood are also algebraic groups.

Example 1.8.

1. Let $\mathbb{G}_a := (\mathbf{k}, +)$ be the additive group of the field $\mathbf{k}$. By Example 1.3, $\mathbb{G}_a$ is a variety. It is possible to prove that both μ and ι are morphisms of varieties, therefore $\mathbb{G}_a$ is an algebraic group, see [25, Example 1.2].

2. Let $\mathbb{G}_m := (\mathbf{k}^\times, \cdot)$ be the multiplicative group of the field $\mathbf{k}$. Similar to above, $\mathbb{G}_m$ is also an algebraic group, see [25, Example 1.2].

3. The group of invertible $n \times n$-matrices

$$\mathrm{GL}_n(\mathbf{k}) = \{C \in \mathbf{k}^{n \times n} \mid \det(C) \neq 0\}$$

is an algebraic group. To see this, note that $\mathrm{GL}_n(\mathbf{k})$ is isomorphic to the set $\{(C, y) \in \mathbf{k}^{n \times n} \times \mathbf{k} \mid \det(C)y = 1\}$. Then

$$A[\mathrm{GL}_n(\mathbf{k})] = \mathbf{k}[X_{ij}, Y \mid 1 \leqslant i, j \leqslant n] / \Big(\det\big((X_{ij})_{i,j=1}^n\big)Y - 1\Big),$$

see [25, section 1.1] and Example 1.3 (2).

4. The group $\mathrm{SL}_n(\mathbf{k}) = \{C \in \mathrm{GL}_n(\mathbf{k}) \mid \det(C) = 1\} \subseteq \mathrm{GL}_n(\mathbf{k})$ is an algebraic group and a closed subgroup of $\mathrm{GL}_n(\mathbf{k})$, see [25, section 1.2].

5. In fact, we have the following result: Let $G \leqslant \mathrm{GL}_n(\mathbf{k})$ be a closed subgroup (with respect to the Zariski topology). Then G is an algebraic group. We call these groups **linear algebraic groups**. Each algebraic group as given in Definition 1.6 is isomorphic to a linear algebraic group, see [10, Corollary 2.4.4].

From now on we want to focus on the irreducible components of algebraic groups, which leads us to the concept of *connected groups*.

Definition and Proposition 1.9 (Connected group, [10, Proposition 1.3.13])**.** Recall that a variety is a topological space and that a topological space X is called **irreducible** if X cannot be decomposed as $X = X_1 \cup X_2$ where $X_1, X_2 \neq X$ and $X_1, X_2 \neq \emptyset$ and both X_1 and X_2 are closed.

If G is an irreducible algebraic group, we call G **connected**. Otherwise, G can be written as a decomposition of **irreducible components**, that is, a union of maximal irreducible topological subspaces.
Let G° denote the irreducible component of G containing $1 \in G$. By [10, Proposition 1.3.13 (a)] G° is uniquely determined, G° is an algebraic group, and ${}^{G}/_{G^\circ}$ is a finite group.

The so-called unipotent and semisimple elements play a central part in understanding the structure of algebraic groups.

Definition 1.10 (Unipotent and semisimple elements, [31, Definition 2.1])**.** Let (G, A) be an algebraic group. For every element $g \in G$ we define the map

$$\begin{aligned} \rho_g^* : A &\longrightarrow A, \\ f &\longmapsto (h \mapsto f(hg)). \end{aligned}$$

We call g **unipotent** if ρ_g^* is unipotent, i.e. the map $\rho_g^* - \mathrm{id}_A$ is nilpotent. Recall that an element φ is called nilpotent if there exists $n \in \mathbb{N}$ such that $\varphi^n = 0$.
We call g **semisimple** if ρ_g^* is diagonalisable.

In fact, every element of an algebraic group can be written as a commuting product of a semisimple and a unipotent element.

Proposition 1.11 (Jordan decomposition, [31, Proposition 2.4.1])**.** *For each element $g \in G$ there exists a unipotent element $u_g \in G$ and a semisimple element $s_g \in G$ such that $g = u_g s_g = s_g u_g$. This is called the **Jordan decomposition** of g.*

1.2.1 Lie algebras of algebraic groups

As previously mentioned, it is possible to define a Lie algebra linked to an algebraic group. In order to do so, we will first recall a few facts about Lie algebras, tangent spaces, and derivations.

Definition 1.12 (Lie algebra)**.** A **Lie algebra** is a vector space L over a field $\mathbf{k}$ such that there exists a bilinear product

$$[\ ,\ ] : L \times L \longrightarrow L$$

which fulfils the **Jacobi identity**:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall x, y, z \in L,$$

and is alternating, that is

$$[x, x] = 0, \quad \forall x \in L.$$

Definition 1.13 (Tangent Space, Derivation, [15, 5.1])**.**

1. Let (X, A) be an irreducible variety and $x \in X$. Note that in this case, A is an integral domain, see [15, 1.5]. Define the **localised** ring

$$\mathfrak{O}_x = \left\{ \frac{g}{f} \;\middle|\; f, g \in A, f(x) \neq 0 \right\} \subseteq \mathrm{Frac}(A).$$

Then we have a unique maximal ideal of $\mathfrak{O}_x$ given by

$$\mathfrak{m}_x = \left\{ \frac{g}{f} \,\middle|\, g(x) = 0,\ f(x) \neq 0 \right\} \subseteq \mathfrak{O}_x.$$

We call the dual space of the $\mathbf{k}$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$, denoted by

$$\mathscr{T}(X)_x := \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^*,$$

the **tangent space** of X at x.

2. We call a $\mathbf{k}$-linear map $\delta : A \to A$ a **derivation** if all elements $f, g \in A$ fulfil the **product rule**:

$$\delta(fg) = f\delta(g) + g\delta(f).$$

We denote by $\mathrm{Der}_{\mathbf{k}}(A, A)$ the space of all derivations of A into A.

Remark 1.14. The set of derivations $\mathrm{Der}_{\mathbf{k}}(A, A)$ is a Lie algebra, see [10, Example 1.4.2 (a)].

Let us now consider the connection between Lie algebras and algebraic groups.

Definition 1.15 (Lie algebra of an algebraic group, [15, 9.1])**.** Let (G, A) be an algebraic group. For $x \in G$ let $\lambda_x : A \to A,\ f \mapsto (g \mapsto f(xg))$ be the left translation. We define the **Lie algebra** of G by

$$\mathrm{Lie}(G) := \{\delta \in \mathrm{Der}_{\mathbf{k}}(A, A) \mid \delta\lambda_x = \lambda_x\delta \text{ for all } x \in G\}.$$

Remark 1.16. $\mathrm{Lie}(G)$ is isomorphic to $\mathscr{T}(G)_1$, where we identify $\mathscr{T}(G)_1$ with $\mathscr{T}(G^\circ)_1$, see [15, Theorem 9.1].

Depending on the problem at hand, considering $\mathrm{Lie}(G)$ either as a tangent space or as a set of derivations might yield better results.
As an example, we will give the Lie algebras of two well-understood algebraic groups, $\mathrm{GL}_n(\mathbf{k})$ and $\mathrm{SL}_n(\mathbf{k})$.

Example 1.17 ([25, Example 7.5] and [15, 9.4])**.**

1. Let $G = \mathrm{GL}_n(\mathbf{k})$ and let $\mathfrak{gl}_n(\mathbf{k}) = \mathbf{k}^{n\times n}$ be the Lie algebra with Lie product

$$[A, B] = AB - BA, \text{ for } A, B \in \mathbf{k}^{n\times n}$$

using the usual matrix multiplication. Then $\mathrm{Lie}(G) \simeq \mathfrak{gl}_n(\mathbf{k})$ via the map

$$\mathfrak{gl}_n(\mathbf{k}) \to \mathrm{Lie}(G), \quad X \mapsto (D_X : (T_{ij})_{ij} \mapsto (\sum_{l=1}^{n} T_{il}X_{lj})_{ij}).$$

2. We have $\mathrm{Lie}(\mathrm{SL}_n(\mathbf{k})) \simeq \mathfrak{sl}_n(\mathbf{k}) = \{A \in \mathfrak{gl}_n(k) \mid \mathrm{tr}(A) = 0\}$.

It is possible to define Lie algebra homomorphisms from morphisms of algebraic groups. This may simplify some problems, as passing to the Lie algebra "linearises" some actions.

Definition 1.18 (Differentials and the adjoint representation, [15, 5.4] and [25, Section 7.2]). Let (X, A) and (Y, B) be two varieties and $\varphi : X \to Y$ be a morphism of varieties. Let $x \in X$ and $y := \varphi(x) \in Y$. The composition of φ^* with a map $h \in \mathscr{T}(X)_x$ induces the $\mathbf{k}$-linear map

$$d\varphi_x : \mathscr{T}(X)_x \longrightarrow \mathscr{T}(Y)_y, \quad h \longmapsto h \circ \varphi^*.$$

The map $d\varphi_x$ is called the **differential** of φ at x.
Let G be an algebraic group. For each $x \in G$ we get the morphism of affine varieties

$$\mathrm{Int}_x : G \longrightarrow G, \quad g \longmapsto xgx^{-1},$$

given by conjugation. We write $\mathrm{Ad}(x)$ for the differential $d\mathrm{Int}_x$. Then the map

$$\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathrm{Lie}(G))$$

is called the **adjoint representation** of the group G on its Lie algebra $\mathrm{Lie}(G)$.

Remark 1.19. Following [7, Section 1.3], we see that the adjoint map can also be differentiated. We write $\mathrm{ad} := d\mathrm{Ad}$. Then $\mathrm{ad}(x) \in \mathrm{End}(\mathrm{Lie}(G))$ and in fact

$$\begin{aligned} \mathrm{ad}(x) : \mathrm{Lie}(G) &\longrightarrow \mathrm{Lie}(G), \\ y &\longmapsto [x, y]. \end{aligned}$$

The map ad is also called the **adjoint representation** of the Lie algebra $\mathrm{Lie}(G)$.

Proposition 1.20 (Differential criterion for dominance, [10, Proposition 1.4.15]). *Let $\mathbf{k}$ be an infinite perfect field (e.g. $\mathbf{k}$ algebraically closed) and $V \subseteq \mathbf{k}^n$, $W \subseteq \mathbf{k}^m$ be irreducible algebraic sets. Furthermore, let $\varphi : V \to W$ be a regular map, so there exist $f_1, \ldots, f_m \in \mathbf{k}[X_1, \ldots, X_n]$ such that $\varphi(x) = (f_1(x), \ldots, f_m(x))$ for all $x \in V$.*
Assume that $d\varphi_x : \mathscr{T}(V)_x \to \mathscr{T}(W)_y$ is surjective, where $x \in V$ and $y \in W$ are such that $\dim(V) = \dim(\mathscr{T}(V)_x)$ and $\dim(W) = \dim(\mathscr{T}(W)_y)$. Then φ is dominant.

We will now consider some examples of derivations of well-known morphisms as well as for the adjoint representation of $\mathrm{GL}_n(\mathbf{k})$ and $\mathrm{SL}_n(\mathbf{k})$.

Example 1.21 ([15, Proposition 10.1] and [25, Example 7.13]).

1. Let G be an algebraic group. Consider the maps

$$\begin{aligned} \mu : G \times G &\longrightarrow G, & \iota : G &\longrightarrow G, \\ (g, h) &\longmapsto gh, & g &\longmapsto g^{-1}. \end{aligned}$$

   By Definition 1.6 these maps are morphisms of affine varieties. Their differentials are given by

$$\begin{aligned} d\mu_{(1,1)} : \mathscr{T}(G \times G)_{(1,1)} &\longrightarrow \mathscr{T}(G)_1, & d\iota_1 : \mathscr{T}(G)_1 &\longrightarrow \mathscr{T}(G)_1, \\ (g, h) &\longmapsto g + h, & g &\longmapsto -g. \end{aligned}$$

2. Let $G = \mathrm{GL}_n(\mathbf{k})$. Then one can explicitly compute the adjoint map to be given by $\mathrm{Ad}(g)(x) = gxg^{-1}$ for $g \in G$, $x \in \mathfrak{gl}_n(\mathbf{k})$.
   The adjoint map for $\mathrm{SL}_n(\mathbf{k})$ is simply given by restricting the adjoint map for $\mathrm{GL}_n(\mathbf{k})$.

1.2.2 Structures in algebraic groups

In this section we study algebraic groups by taking a closer look at their structures. This will lead to the Structure Theorem 1.34, which is of central importance when working with algebraic groups. This section follows the book by Malle and Testerman, [25, Chapters 8 and 9], closely.

Definition 1.22 (Torus, character group of a torus, [25, Definitions 3.3 and 3.4]). Let $D_n(\mathbf{k}) \leqslant \mathrm{GL}_n(\mathbf{k})$ denote the subgroup of diagonal matrices, that is

$$D_n(\mathbf{k}) = \{\mathrm{diag}(t_1, \ldots, t_n) \mid t_i \in \mathbf{k}^\times\}.$$

By Example 1.8 (3), $D_n(\mathbf{k})$ is an algebraic group. A group $T \simeq D_n(\mathbf{k})$ is called a **torus**. Note that a torus is an abelian group.
The group

$$X(T) := \{\chi : T \to \mathbb{G}_m \mid \chi \text{ is a homomorphism of algebraic groups}\}$$

is called the **character group** of T.

Example 1.23 ([25, Example 3.5]). The maps $\chi_i : D_n(\mathbf{k}) \to \mathbb{G}_m$ with

$$\chi_i(\mathrm{diag}(t_1, \ldots, t_n)) = t_i \text{ for all } 1 \leqslant i \leqslant n,$$

are characters of T. In fact, every character $\chi \in X(D_n(\mathbf{k}))$ can be written as $\chi = a_1\chi_1 + \ldots + a_n\chi_n$, $a_i \in \mathbb{Z}$. This shows that $X(D_n(\mathbf{k})) \simeq \mathbb{Z}^n$.

If we want to consider structures in G that depend on subgroups such as the maximal tori, it would be helpful if they would not depend on the choice of a maximal torus. Indeed, the following proposition gives the desired result, see [25, Corollary 6.5].

Proposition 1.24. *Let G be an algebraic group. Then all maximal tori in G are conjugates of each other.*

Additional to the maximal tori, we can find subgroups in G that contain a maximal torus. These so-called **Borel subgroups** are defined as follows:

Definition 1.25 (Borel subgroup, [25, Definition 6.3]). Let G be an algebraic group and $B \leqslant G$ such that:

(i) B is closed,

(ii) B is connected,

(iii) B is solvable,

(iv) B is maximal with respect to (i)-(iii).

Then B is called a **Borel subgroup** of G.

As mentioned above, each maximal torus is contained in a Borel subgroup. There is a similar result to Proposition 1.24 for Borel subgroups, see [25, Theorem 6.4 (a)].

Proposition 1.26. *Let G be an algebraic group. Then all Borel subgroups in G are conjugates of each other. Each maximal torus lies in a Borel subgroup of G.*

Example 1.27 ([25, Example 6.7 and 11.4]).

1. Let $G = \mathrm{GL}_n(\mathbf{k})$. Then $D_n(\mathbf{k})$ is a maximal torus and a Borel subgroup B is given by $B_n(\mathbf{k})$, the subgroup of all upper triangular matrices. All unipotent elements in B are given by the upper triangular matrices with 1 on the diagonal, denoted by $U_n(\mathbf{k})$. Then $D_n(\mathbf{k})U_n(\mathbf{k}) = B$.

2. Let $G = \mathrm{SL}_n(\mathbf{k})$. It is possible to prove that $B = G \cap B_n(\mathbf{k})$ is a Borel subgroup, $U = G \cap U_n(\mathbf{k})$ is the subgroup of unipotent elements in B, and $T = G \cap D_n(\mathbf{k})$ is a maximal torus.

Definition 1.28 (Radical, reductive, semisimple, simple, [25, Definition 6.13 and 6.14]). Let G be a connected algebraic group. Define

$$R(G) := \left\langle S \leqslant G \,\middle|\, \begin{array}{c} S \text{ normal, solvable,} \\ \text{closed, connected} \end{array} \right\rangle \leqslant G.$$

The subgroup $R(G)$ is called the **radical** of G. Furthermore, the **unipotent radical** is given by $R_u(G) := \{g \in R(G) \mid g \text{ is unipotent}\}$.
If $R_u(G) = 1$, the group G is called **reductive**. If $R(G) = 1$, G is a **semisimple** algebraic group.
Furthermore, a non-trivial semisimple algebraic group that has no non-trivial proper closed connected normal subgroups is called a **simple group**.

Example 1.29 ([25, Example 6.17]). The group $\mathrm{GL}_n(\mathbf{k})$ is reductive and $\mathrm{SL}_n(\mathbf{k})$ is semisimple.

From now on we consider only reductive connected algebraic groups. We note that for any connected algebraic group G, the factor group ${}^{G}/_{R_u(G)}$ is reductive, see [1, 11.21].

Definition 1.30 (Roots, Weyl Group, [25, Definition 8.1]). Let G be an algebraic group. Let $T \leqslant G$ be a maximal torus and let $\mathfrak{g} := \mathrm{Lie}(G)$ be the Lie algebra of G. We define the **weight spaces** under the action of G on $\mathfrak{g}$ for all $\chi \in X(T)$ as

$$\mathfrak{g}_\chi := \{x \in \mathfrak{g} \mid \mathrm{Ad}(t)(x) = \chi(t)x \text{ for all } t \in T\}.$$

Then the **roots** of G with respect to T are given by the set

$$\Phi(G) := \{\chi \in X(T) \mid \chi \neq 0, \mathfrak{g}_\chi \neq 0\}.$$

Furthermore, we define the **Weyl group** of G with respect to T as

$$W := {}^{N_G(T)}/_{C_G(T)}.$$

If G is connected reductive, $C_G(T) = T$, so in this case the Weyl group is given by ${}^{N_G(T)}/_{T}$, see [25, Corollary 8.13].

Remark 1.31. We can write the Lie algebra of G as a direct sum of the root weight spaces and $\mathfrak{g}_0 = \mathrm{Lie}(T)$. This results in

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi(G)} \mathfrak{g}_\alpha.$$

Example 1.32. Let $G = \mathrm{GL}_n(\mathbf{k})$ and $T = D_n(\mathbf{k})$ as before. For $1 \leqslant i, j \leqslant n$ define maps

$$\chi_{ij} : T \longrightarrow \mathbf{k}^\times, \quad \mathrm{diag}(t_1, \dots, t_n) \mapsto t_i t_j^{-1}.$$

Then $\chi_{ij} \in X(T)$ and

$$\mathfrak{g}_{\chi_{ij}} = \{cE_{ij} \mid c \in \mathbf{k}\}, \quad \text{and } i \neq j,$$

where the entries of $E_{ij} \in \mathbf{k}^{n\times n}$ are given by $(E_{ij})_{k,l} = \delta_{(i,j),(k,l)}$, that is the $E_{ij} \in \mathbf{k}^{n\times n}$ are matrices with entry 1 at the position (i, j) and 0 elsewhere.
In fact, we have $\Phi(\mathrm{GL}_n(\mathbf{k})) = \{\chi_{ij} \mid 1 \leqslant i, j \leqslant n,\ i \neq j\}$ and it is clear that

$$\mathfrak{gl}_n(\mathbf{k}) = \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \mathfrak{g}_{\chi_{ij}} \oplus \mathrm{Lie}(T),$$

where $\mathrm{Lie}(T) = \{\mathrm{diag}(c_1, \dots, c_n) \mid c_1, \dots, c_n \in \mathbf{k}\}$.
One can prove that the corresponding Weyl group is isomorphic to S_n, the symmetric group on n elements, see [25, Example 8.2].

Remark 1.33. The Weyl group W acts on the roots as follows: Let

$$w = n_w C_G(T) \in W$$

where $n_w \in N_G(T)$ and $\alpha \in \Phi(G)$. Then

$$w.\alpha(t) = \alpha(n_w^{-1} t n_w), \quad \text{for all } t \in T,$$

and $\mathrm{Ad}(n_w)(\mathfrak{g}_\alpha) = \mathfrak{g}_{w.\alpha}$, see [25, Chapter 8.1, Proposition 8.4, and Theorem 8.17].

Remark 1.31 suggests the question of whether it is possible to find a similar result for algebraic groups. Indeed, we have already seen that roots and maximal tori seem to play a central role in understanding the structure of algebraic groups. The following theorem, taken from [25, Theorem 8.17], summarises the results on the structure of connected reductive groups.

Theorem 1.34 (Structure theorem for reductive groups)**.** *Let G be a connected reductive algebraic group, $T \leqslant G$ a maximal torus, and $\Phi(G) = \Phi$. Then we have the following properties:*

(i) $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ *with* $\dim(\mathfrak{g}_\alpha) = 1$ *for all* $\alpha \in \Phi$ *and* $\mathfrak{g}_0 = \mathrm{Lie}(T)$.

(ii) For each $\alpha \in \Phi$ there exists a unique morphism of algebraic groups (up to right composition with the multiplication by $c \in \mathbf{k}^\times$) given by

$$u_\alpha : \mathbb{G}_a \longrightarrow G$$

such that $t u_\alpha(c) t^{-1} = u_\alpha(\alpha(t)c)$ for all $t \in T,\ c \in \mathbb{G}_a$.
Furthermore, let $U_\alpha := \mathrm{im}(u_\alpha)$. Then the restriction $u_\alpha : \mathbb{G}_a \to U_\alpha$ is an isomorphism and $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$.

(iii) Let $w \in W = {}^{N_G(T)}/_{C_G(T)}$ and $n_w \in N_G(T)$ such that $n_w C_G(T) = w$. Then $n_w U_\alpha n_w^{-1} = U_{w.\alpha}$.

(iv) $G = \langle T, U_\alpha \mid \alpha \in \Phi \rangle$.

(v) $Z(G) = \bigcap_{\alpha \in \Phi} \ker(\alpha)$.

Definition 1.35 (Root Subgroups)**.** The subgroups U_α, $\alpha \in \Phi$, from Theorem 1.34 are called the **root subgroups** of G with respect to T.

Let V be a real, finite-dimensional vector space. An element $s \in \mathrm{GL}(V)$ is called a **reflection** along $v \in V$ if v is an eigenvector of s with eigenvalue -1 and s fixes a hyperplane of V pointwise. The Weyl group of an algebraic group is generated by reflections, as stated in the next proposition, see also [25, Lemma 8.19 and Proposition 8.20].

Proposition 1.36. *Let $C_\alpha := C_G(\ker(\alpha))$ for each $\alpha \in \Phi$. Define $s_\alpha = n_\alpha C_G(T)$ for $n_\alpha \in N_{C_\alpha}(T) \setminus C_G(T)$ (we have $C_G(T) \leqslant N_{C_\alpha}(T)$ and $[N_{C_\alpha}(T) : C_G(T)] = 2$ so s_α is well-defined). The elements s_α are reflections along α in the vector space $V = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The Weyl group W is generated by the s_α, so $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$.*

If G is a semisimple group, there are stronger results on its structure than stated in Theorem 1.34, see [25, Theorem 8.21].

Proposition 1.37. *Let G be a semisimple algebraic group and let the notation be as in Theorem 1.34. Then*

(i) $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$.

(ii) $G = [G, G]$.

(iii) $G = G_1 \cdots G_r$ *where the G_i are simple algebraic groups that are normal subgroups of G.*

For connected reductive groups we get the following connection with semisimple groups as stated in [25, Corollary 8.22].

Proposition 1.38. *Let G be a connected reductive group. Then*

$$G = [G, G]R(G) = [G, G]Z(G)^\circ$$

and $[G, G]$ is semisimple.

At the end of this section we want to compute the structures in the symplectic group of dimension 4 over an algebraically closed field $\mathbf{k}$.

Example 1.39. The symplectic group in $\mathbf{k}^{4\times 4}$ is defined as

$$\mathrm{Sp}_4(\mathbf{k}) := \{A \in \mathrm{GL}_4(\mathbf{k}) \mid A^{tr} J_4 A = J_4\}, \text{ where } J_4 := \begin{pmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & -1 & . & . \\ -1 & . & . & . \end{pmatrix}.$$

First note that the Lie algebra of $\mathrm{Sp}_4(\mathbf{k})$ is given by

$$\mathrm{Lie}(\mathrm{Sp}_4(\mathbf{k})) = \{A \in M_4(\mathbf{k}) \mid J_4 A + A^{tr} J_4 = 0\},$$

see [20, Lemma 2.7].
We define the subgroup T of $\mathrm{Sp}_4(\mathbf{k})$ as

$$T := \left\{ \begin{pmatrix} t_1 & . & . & . \\ . & t_2 & . & . \\ . & . & t_2^{-1} & . \\ . & . & . & t_1^{-1} \end{pmatrix} \middle| \, t_1, t_2 \in \mathbf{k}^\times \right\}.$$

Clearly, T is abelian and $T \simeq D_2(\mathbf{k})$, so T is a torus. In fact, T is a maximal torus of $\mathrm{Sp}_4(\mathbf{k})$, as for each $A \in \mathrm{Sp}_4(\mathbf{k})$ we have $A\,\mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) = \mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1})A$ if and only if $A \in T$. This shows that there is no abelian subgroup T' containing T as a proper subset.
Next, we want to determine the roots of $\mathrm{Sp}_4(\mathbf{k})$. In order to do so, we compute the adjoint representation of elements in T on $\mathrm{Lie}(\mathrm{Sp}_4(\mathbf{k}))$. As a subgroup of $\mathrm{GL}_4(\mathbf{k})$, the adjoint representation of $\mathrm{Sp}_4(\mathbf{k})$ on its Lie algebra is also given by conjugation, see [25, Example 7.13 and remarks before]. For $A \in \mathrm{Lie}(\mathrm{Sp}_4(\mathbf{k}))$ we therefore have

$$\mathrm{Ad}(\mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}))(A) = (t_i t_j^{-1} a_{i,j})_{i,j=1}^4.$$

By the strucuture of T we have $t_i^{-1} = t_{4-i+1}$ for $i \in \{1, 2, 3, 4\}$. In particular

$$t_1 t_2^{-1} = t_3 t_4^{-1}, \qquad t_1 t_3^{-1} = t_2 t_4^{-1}$$
$$t_2 t_1^{-1} = t_4 t_3^{-1}, \qquad t_3 t_1^{-1} = t_4 t_2^{-1}.$$

This means, we are interested in matrices of the form

$$A_1(a) := \begin{pmatrix} \cdot & a & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -a \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad A_2(b) := \begin{pmatrix} \cdot & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & b \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$
$$A_3(c) := \begin{pmatrix} \cdot & \cdot & \cdot & c \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad A_4(d) := \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & d & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

for $a, b, c, d \in \mathbf{k}$, as

$$\mathrm{Ad}(\mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}))(A_1(a)) = t_1 t_2^{-1} A_1(a),$$
$$\mathrm{Ad}(\mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}))(A_2(b)) = t_1 t_2 A_2(b),$$
$$\mathrm{Ad}(\mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}))(A_3(c)) = 2t_1^2 A_3(c),$$
$$\mathrm{Ad}(\mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}))(A_4(d)) = 2t_2^2 A_4(d).$$

Indeed, we have the characters

$$\chi_1 : T \longrightarrow \mathbf{k}^\times, \quad \begin{pmatrix} t_1 & \cdot & \cdot & \cdot \\ \cdot & t_2 & \cdot & \cdot \\ \cdot & \cdot & t_2^{-1} & \cdot \\ \cdot & \cdot & \cdot & t_1^{-1} \end{pmatrix} \longmapsto t_1,$$
$$\chi_2 : T \longrightarrow \mathbf{k}^\times, \quad \begin{pmatrix} t_1 & \cdot & \cdot & \cdot \\ \cdot & t_2 & \cdot & \cdot \\ \cdot & \cdot & t_2^{-1} & \cdot \\ \cdot & \cdot & \cdot & t_1^{-1} \end{pmatrix} \longmapsto t_2,$$

and therefore the root spaces

$$\mathfrak{g}_{\chi_1 - \chi_2} = \left\{ \begin{pmatrix} \cdot & a & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -a \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| a \in \mathbf{k} \right\}, \qquad \mathfrak{g}_{2\chi_2} = \left\{ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| a \in \mathbf{k} \right\},$$

$$\mathfrak{g}_{\chi_1 + \chi_2} = \left\{ \begin{pmatrix} \cdot & \cdot & a & \cdot \\ \cdot & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| a \in \mathbf{k} \right\}, \qquad \mathfrak{g}_{2\chi_1} = \left\{ \begin{pmatrix} \cdot & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| a \in \mathbf{k} \right\}.$$

Defining the characters $\alpha := \chi_1 - \chi_2$ and $\beta := 2\chi_2$ gives rise to the root system $\Phi = \Phi^+ \sqcup \Phi^-$ where $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$.
Finally, we can compute the root subgroups. Recall that root subgroups for a root

$\chi \in \Phi$ are defined by $U_\chi = \{u_\chi(c) \mid c \in \mathbf{k}\}$ where $u_\chi : \mathbf{k} \to G$ is a homomorphism of algebraic groups and $tu_\chi(c)t^{-1} = u_\chi(\chi(t)c)$ for all $t \in T$ and $c \in \mathbf{k}$. Clearly, the maps of the form $c \mapsto A_i(c) + I_4$ for $i \in \{1, \dots, 4\}$ are of this form, resulting in the root subgroups

$$U_\alpha = \left\{ \left(\begin{smallmatrix} 1 & a & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & -a \\ \cdot & \cdot & \cdot & 1 \end{smallmatrix} \right) \middle| \, a \in \mathbf{k} \right\}, \qquad U_\beta = \left\{ \left(\begin{smallmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & a & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{smallmatrix} \right) \middle| \, a \in \mathbf{k} \right\},$$

$$U_{\alpha+\beta} = \left\{ \left(\begin{smallmatrix} 1 & \cdot & a & \cdot \\ \cdot & 1 & \cdot & a \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{smallmatrix} \right) \middle| \, a \in \mathbf{k} \right\}, \qquad U_{2\alpha+\beta} = \left\{ \left(\begin{smallmatrix} 1 & \cdot & \cdot & a \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{smallmatrix} \right) \middle| \, a \in \mathbf{k} \right\}.$$

1.2.3 Root systems

Root systems can also be considered and studied as abstract objects. As seen in the Structure Theorem 1.34, understanding root systems may lead to a deeper understanding of algebraic groups. In fact, we can prove that there are only nine different types of irreducible root systems that occur for simple algebraic groups. This is an important tool in the classification of these groups.

Definition 1.40 (Abstract root system, Weyl group, [25, Definintion 9.1])**.** Let E be a finite-dimensional real vector space and $\Phi \subseteq E$. Then Φ is an **abstract root system** in E if the following properties are fulfilled:

(R1) Φ is finite, $0 \notin \Phi$, and $\langle \Phi \rangle_{\mathbb{R}} = E$.

(R2) If $c \in \mathbb{R}$ is such that $\alpha, c\alpha \in \Phi$, then $c = \pm 1$.

(R3) For each $\alpha \in \Phi$ there is a reflection $s_\alpha \in \mathrm{GL}(E)$ along α. For each $\beta \in \Phi$ we have $s_\alpha(\beta) \in \Phi$.

(R4) For $\alpha, \beta \in \Phi$, the expression $s_\alpha(\beta) - \beta$ is an integer multiple of α.

The group $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$ is called the **Weyl group** of Φ.

Note that E can be equipped with a positive definite symmetric bilinear form $(\ ,\ )$, left invariant by W, and thus becomes an Euclidean vector space, see [30, 7.1.7] and [25, 9.1].
Since Φ spans the vector space E, we can find a basis for E in Φ, that is every root can be written as a linear combination of elements of a basis in Φ. It turns out that we have an even stronger result, see [25, Definition 9.3 and Proposition A.7].

Definition and Proposition 1.41 (Simple Roots)**.** Let Φ be an abstract root system in E. We call a subset $\Pi \subseteq \Phi$ a set of **simple roots** if Π is a basis of the vector space E and each root $\beta \in \Phi$ can be written as an integer linear combination $\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha$ such that either $c_\alpha \geqslant 0$ for all $\alpha \in \Pi$ or $c_\alpha \leqslant 0$ for all $\alpha \in \Pi$.
Let

$$\Phi^+ := \{\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha \in \Phi \mid c_\alpha \geqslant 0 \ \forall \alpha \in \Pi\}$$

and $\Phi^- := -(\Phi^+)$. The elements of Φ^+ are called the **positive roots** and the elements of Φ^- are the **negative roots**. We have $\Phi = \Phi^+ \sqcup \Phi^-$.
For each root system Φ there exists a set of simple roots.

One way to store information about a root system is the **Cartan matrix**.

Definition 1.42 (Cartan matrix, [8, 6.1]). Let Φ be a root system with simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$. We fix an ordering $(\alpha_1, \dots, \alpha_r)$ and define a matrix by

$$C := (c_{i,j})_{i,j=1}^{r}, \quad c_{i,j} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)},$$

called the **Cartan matrix**.

We will define a few important terms in the context of root systems.

Definition 1.43 (Irreducible and reducible root systems, [25, Section 9]). Let Φ be a root system with simple roots Π. The root system Φ is called **reducible** or **decomposable** if we can write $\Pi = \Pi_1 \sqcup \Pi_2$ for two orthogonal sets Π_1, Π_2 such that $\Phi = (\mathbb{Z}\Pi_1 \cap \Phi) \sqcup (\mathbb{Z}\Pi_2 \cap \Phi)$. Otherwise Φ is called **irreducible** or **indecomposable**.

Definition 1.44 (Height of a root, [25, Definition A.10]). Let Φ be a root system and $\Pi \subseteq \Phi$ be the simple roots. For any root $\beta \in \Phi$ with

$$\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha$$

we define the **height** of β as $\mathrm{ht}(\beta) := \sum_{\alpha \in \Pi} c_\alpha$.

Definition 1.45 (Highest root, [25, Definition B.6]). Let Φ be an indecomposable root system and $\Pi \subseteq \Phi$ be the simple roots. Let $\alpha_0 \in \Phi^+$ with $\alpha_0 = \sum_{\alpha \in \Pi} c_\alpha \alpha$ such that for every other root $\beta \in \Phi^+$ with $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha$ we have $n_\alpha \leqslant c_\alpha$ for all $\alpha \in \Pi$. Then α_0 is called the **highest root**.
It is clear that $\mathrm{ht}(\alpha_0)$ is maximal if α_0 is the highest root.

Remark 1.46. The highest root exists for every indecomposable root system, see [25, Proposition B.5].

Definition 1.47 (Coxeter number, [8, Definition 12.3]). Let Φ be a root system with highest root α_0. The number $c := 1 + \mathrm{ht}(\alpha_0)$ is called the **Coxeter number** of Φ.

Recall that the Weyl group acts on the root system and by extension also on $\langle \Phi \rangle_{\mathbb{R}} = E$.

Definition 1.48 (Fundamental domain, [16, Section 1.12]). A set D of representatives for the orbits of E under the action of the Weyl group W is called a **fundamental domain** for the action of W on E.

It is possible to explicitly write down a fundamental domain for the action of W on E.

Definition and Proposition 1.49 (Fundamental chamber, [25, Theorem A.27]). Let Φ be a root system with Weyl group W, simple roots Π, and $\langle \Phi \rangle_{\mathbb{R}} = E$. Then

$$C := \{v \in E \mid (v, \alpha) > 0 \text{ for all } \alpha \in \Pi\}$$

is called the **fundamental chamber** of W with respect to Π. The closure of C,

$$\bar{C} = \{v \in E \mid (v, \alpha) \geqslant 0 \text{ for all } \alpha \in \Pi\},$$

is a fundamental domain for the action of W on E.

As mentioned before, one can show that the root systems arising from algebraic groups are in fact also abstract root systems, see [25, Proposition 9.2].

Proposition 1.50. *Let G be a connected reductive algebraic group. Then the root system $\Phi \subseteq X(T)$ with respect to a maximal torus $T \leqslant G$ is an abstract root system in the vector space $\langle \Phi \rangle_{\mathbb{R}} \subseteq E := X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. If G is semisimple, we have $\langle \Phi \rangle_{\mathbb{R}} = E$.*

Knowing this, we can combine the structure theorem and the results on abstract root systems in order to get a better picture of the elements in algebraic groups. The result is called the *Bruhat decomposition* and is stated in the following theorem, see [15, Theorem 28.4].

Theorem 1.51 (Bruhat decomposition)**.** *Let G be a connected reductive algebraic group over $\mathbf{k}$ and $W = {}^{N_G(T)}/_{C_G(T)}$ the Weyl group of G with respect to a maximal torus $T \subseteq G$. Furthermore, let Φ be the root system of G with respect to T.*
The Bruhat decomposition of an element $g \in G$ is given by $g = u'tn_w u$ where

1. *$n_w \in N_G(T)$ is a representative of $w \in W$,*
2. *$u \in \prod_{\alpha \in \Phi^+} U_\alpha =: U$,*
3. *$t \in T$, and*
4. *$u' \in \prod_{\substack{\alpha \in \Phi^+ \\ w^{-1}.\alpha \in \Phi^-}} U_\alpha =: U_w^-$.*

This decomposition is uniquely defined for any fixed choice of the n_w, $w \in W$. In particular, we have

$$G = \bigsqcup_{w \in W} B n_w B,$$

where B is a Borel subgroup such that $T \subseteq B$. In fact, $TU = UT$ is such a Borel subgroup.

Knowing this, we are interested in classifying the *irreducible components* of a root system, which in turn will lead us to the classification of the root systems of simple algebraic groups.

Graphically, the irreducible root systems can be described by their **Dynkin diagrams**:

Definition 1.52 (Dynkin Diagrams, [25, Section 9.1])**.** Let Φ be an irreducible root system with simple roots Π. Define a diagram with $|\Pi|$ nodes where each node corresponds to a simple root.
Two nodes corresponding to the simple roots $\alpha, \beta \in \Pi$ are connected by $m-2$ edges if $\operatorname{ord}(s_\alpha s_\beta) = m$, for $m \in \{2,3,4\}$, and 3 edges if $\operatorname{ord}(s_\alpha s_\beta) = 6$, where s_α and s_β are the reflections along α and β respectively.
If two nodes corresponding to $\alpha, \beta \in \Pi$ are connected and α, β have different length, we draw an arrow on the edge connecting the nodes. The arrow points in the direction of the shorter root.

Theorem 1.53 (Classification of Irreducible Root Systems)**.** *Let Φ be an irreducible root system in a Euclidean vector space E. Then Φ is described by one of the Dynkin diagrams in Figure 1.1.*

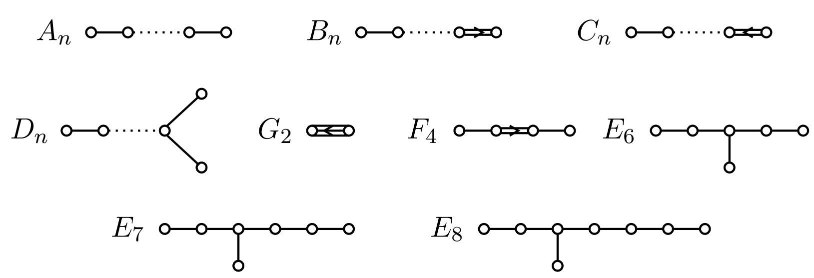
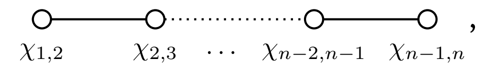


Figure 1.1: The Dynkin diagrams of irreducible root systems

Example 1.54. Let $G = \mathrm{SL}_n(\mathbf{k})$. We already know by Example 1.27 that a maximal torus in $G = \mathrm{SL}_n(\mathbf{k})$ is given by $T = D_n(\mathbf{k}) \cap \mathrm{SL}_n(\mathbf{k})$. We have the root system $\Phi = \{\chi_{ij} \mid 1 \leqslant i,j \leqslant n,\ i \neq j\}$ with the same notation as in Example 1.32. Then $\Pi := \{\chi_{i,i+1} \mid 1 \leqslant i \leqslant n-1\}$ is a set of simple roots for Φ. We have $\mathrm{ord}(s_\alpha s_\beta) = |(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi^+|$ for two simple roots $\alpha, \beta \in \Pi$ by [25, Example 9.5]. Note that

$$|(\mathbb{Z}\chi_{i,i+1} + \mathbb{Z}\chi_{j,j+1}) \cap \Phi^+| = 3 \text{ if } j = i+1,$$
$$|(\mathbb{Z}\chi_{i,i+1} + \mathbb{Z}\chi_{j,j+1}) \cap \Phi^+| = 2 \text{ if } j > i+1.$$

This gives us the following diagram:



so the roots of G form an irreducible root system of type A_{n-1}.

An important tool in understanding the multiplication of group elements is the commutator formula which describes the form of the commutator of two elements in the root subgroups. This is stated, for example, in [15, Lemma 32.5.] and [25, Theorem 11.8].

Proposition 1.55 (Commutator formula)**.** *Let Φ be a root system of a connected reductive group G over an algebraically closed field $\mathbf{k}$ and fix a total ordering in Φ compatible with addition. Let $\alpha, \beta \in \Phi$ be two roots such that $\alpha \neq \pm\beta$. Then there exist $c^{m,n}_{\alpha,\beta} \in \mathbb{Z}$ and isomorphisms u_γ for every $\gamma \in \Phi$ as in Theorem 1.34 (ii) such that*

$$[u_\alpha(c_1), u_\beta(c_2)] = \prod_{\substack{m,n>0 \\ m\alpha+n\beta\in\Phi}} u_{m\alpha+n\beta}(c^{m,n}_{\alpha,\beta} c_1^m c_2^n) \quad \textit{for all } c_1,\ c_2 \in \mathbf{k}.$$

In Lie algebras we would like to understand how the Lie product $[e_\alpha, e_\beta]$ behaves for $\alpha, \beta \in \Phi$ and $\langle e_\alpha \rangle = \mathfrak{g}_\alpha$ generating the root weight space of the roots $\alpha \in \Phi$. Indeed, there is a similar formula, see [5, Ch. VIII, §2, n°4].

Proposition 1.56. *Let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ be the decomposition of a Lie algebra with root system Φ. We have a basis $\{h_i \mid i \in \{1, \ldots, |\Pi|\}\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ where Π is a system of simple roots and the e_α generate the one-dimensional spaces $\mathfrak{g}_\alpha$. For the Lie product we have*

$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha,\beta} e_{\alpha+\beta}, & \textit{if } \alpha+\beta \in \Phi \\ 0, & \textit{if } \alpha+\beta \notin \Phi \cup \{0\} \end{cases}$$

and $[e_\alpha, e_{-\alpha}] \in \mathfrak{t}$. The $N_{\alpha,\beta}$ are constants in $\mathbf{k}^\times$ depending on the root system.

1.2.4 Parabolic subgroups

Sometimes it is a good idea to consider smaller subgroups with their own root system in an algebraic group G. This can be done by defining the *parabolic subgroups*. This section mainly follows the book by Malle and Testerman, [25, Chapter 12].
For this section, let G be a connected reductive group, $T \leqslant G$ a maximal torus, $B \leqslant G$ a Borel subgroup such that $T \leqslant B$, and Φ a root system with respect to T with simple roots Π defined by B. Furthermore, let $W = {}^{N_G(T)}/_{T}$ denote the Weyl subgroup with respect to the maximal torus T.
Parabolic subgroups can be defined via a corresponding root system, see [25, Section 12.1].

Definition 1.57 (Parabolic Weyl subgroups, parabolic root subsystems)**.**
Let $S := \{s_\alpha \mid \alpha \in \Pi\}$ be the set of simple reflections defined by the simple roots Π from above. Fix a subset $I \subseteq S$. Then the subgroup $W_I := \langle s \in I \rangle \leqslant W$ is called a **standard parabolic subgroup** of W. Any subgroup of W conjugate to W_I is called a **parabolic subgroup** of W.
Let $\Pi_I := \{\alpha \in \Pi \mid s_\alpha \in I\}$ and $\Phi_I := \Phi \cap \sum_{\alpha\in\Pi_I} \mathbb{Z}\alpha$. Then Φ_I is the corresponding **parabolic subsystem** of the roots.

It is indeed the case that the parabolic subsystems form their own root systems and Weyl groups, see [25, Proposition 12.1].

Proposition 1.58. *Let the notation be as in Definition 1.57. Then Φ_I is a root system in $\mathbb{R}\Phi_I$ with simple roots Π_I and Weyl group W_I.*

Using these root subsystems, it is possible to define parabolic subgroups in algebraic groups.

Definition 1.59 (Parabolic subgroup, [25, Example 12.4])**.** Let the notation be as in Definition 1.57. Then $P_I := BW_IB = \bigsqcup_{w\in W_I} Bn_wB$ is a subgroup of G, called a **standard parabolic subgroup**. A **parabolic subgroup** of G is a subgroup which is conjugate to P_I for a subset $I \subseteq S$.

We summarise a few results on parabolic subgroups, as stated in [25, Proposition 12.2].

Proposition 1.60. *Let the notation be as in Definition 1.57. We have the following properties of the parabolic subgroups.*

1. *P_I is a closed, connected, self-normalising subgroup of G which contains B.*
2. *The P_I are not conjugate to each other. In particular, if $P_I = P_J$ for two subsets $I, J \subseteq S$, then $I = J$.*
3. *We have $P_I = \langle T, U_\alpha \mid \alpha \in \Phi^+ \cup \Phi_I \rangle$.*
4. *Every parabolic subgroup contains a Borel subgroup and every subgroup containing a Borel subgroup is a parabolic subgroup.*

We see that a parabolic subgroup in itself does not just contain root subgroups for $\alpha \in \Phi_I$ but also for all $\alpha \in \Phi^+$. In order to argue inductively by using groups with smaller root systems, we would like to know the role of the group $\langle T, U_\alpha \mid \alpha \in \Phi_I \rangle$ in P_I. Indeed, this gives rise to a rather interesting decomposition of the subgroup P_I.

Proposition 1.61 (Levi decomposition, see [25, Proposition 12.6]). *Let P_I be a parabolic subgroup of G for some subset $I \subseteq S$. Define two subgroups of P_I as follows.*

$$U_I := \prod_{\alpha \in \Phi^+ \setminus \Phi_I} U_\alpha = \langle U_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_I \rangle,$$
$$L_I := \langle T, U_\alpha \mid \alpha \in \Phi_I \rangle.$$

Then the unipotent radical satisfies $R_u(P_I) = U_I$, and L_I is a complement to U_I, that is $P_I = U_I \rtimes L_I$. Furthermore, L_I is a connected reductive group with root system Φ_I.

Definition 1.62 (Levi decomposition, [25, Definition 12.7]).
The decomposition $P_I = U_I \rtimes L_I$ is called the **Levi decomposition** of P_I and L_I is the **standard Levi complement** of P_I. All subgroups conjugate to L_I are called **Levi subgroups** of G.

1.2.5 Formulas for the adjoint representation

Following [11], we can give explicit formulas of a semisimple algebraic group G acting on its Lie algebra via the adjoint representation. Let Φ be the root system of G with respect to a maximal torus $T \subseteq G$ and $\Pi \subseteq \Phi$ a set of simple roots.
For $\alpha, \beta \in \Phi$ define the integers $p_{\beta,\alpha}, q_{\beta,\alpha} \in \mathbb{N}$ in the following way:

$$\begin{aligned} &\beta + p_{\beta,\alpha}\alpha \in \Phi \quad \text{and} \quad \beta + (p_{\beta,\alpha}+1)\alpha \notin \Phi, \\ &\beta - q_{\beta,\alpha}\alpha \in \Phi \quad \text{and} \quad \beta - (q_{\beta,\alpha}+1)\alpha \notin \Phi. \end{aligned} \tag{1.1}$$

Then $s_\alpha(\beta) = \beta - (q_{\beta,\alpha} - p_{\beta,\alpha})\alpha$, see [4, VI, §1, no. 1.3, Proposition 9].
It is possible to fix a Chevalley basis $C = \{h_i \mid i \in \{1, \ldots, |\Pi|\}\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ where $\langle e_\alpha \rangle = \mathfrak{g}_\alpha$ for each $\alpha \in \Phi$, such that we can explicitly describe the action of G on this basis. In order to do this, we define the map

$$\exp : \mathrm{Lie}(G) \to \mathrm{GL}(\mathrm{Lie}(G)),\ x \mapsto \sum_{i \geqslant 0} \frac{1}{i!}\mathrm{ad}^i(x).$$

Following [11], we will list the action of each factor in the Bruhat decomposition, using the same notation as in Definition 1.51. Additionally, fix an ordering $(\alpha_1, \ldots, \alpha_n)$ of the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$.

1. Action of an element $u_\alpha(c_\alpha) \in U_\alpha$ on $\mathfrak{g}$ where $\alpha \in \Phi^+$ and $c_\alpha \in \mathbf{k}$. Let $x \in \mathfrak{g}$. Then

$$\begin{aligned} \mathrm{Ad}(u_\alpha(c_\alpha))(x) &= \exp(c_\alpha e_\alpha)(x) \\ &= \sum_{i \geqslant 0} \frac{c_\alpha^i}{i!} e_\alpha^i(x), \end{aligned}$$

   where we set $e_\alpha(x) := [e_\alpha, x]$ and $e_\alpha^i(x) := [e_\alpha^{i-1}, x]$ for $i > 1$, by [11, 4.10. and Section 5].
   Note that we are dividing by $i!$ in this formula. In particular, we need to be careful if the characteristic of $\mathbf{k}$ is positive: In this case we will first compute $\mathrm{Ad}(u_\alpha(c_\alpha))(x)$ in characteristic 0 and then reduce modulo the characteristic of

$\mathbf{k}$. This is possible since the coefficients $\frac{c_\alpha^i}{i!}$ are integers, see [11, 4.10. and Corollary 5.6.].
Alternatively, we can also describe the action of the elements $u_\alpha(c_\alpha)$ for roots α such that $\alpha \in \Pi$ or $-\alpha \in \Pi$ on the Lie algebra elements e_β for $\beta \in \Phi$ as follows by [11, 4.10]:

$$\mathrm{Ad}(u_\alpha(c_\alpha))(e_\beta) = \begin{cases} \sum_{\substack{k\geqslant 0 \\ \beta+k\alpha\in\Phi}} \binom{k+q_{\beta,\alpha}}{k} c_\alpha^k e_{\beta+k\alpha} & \text{if } \alpha \in \Phi^+, \alpha \neq \pm\beta, \\ e_\alpha & \text{if } \alpha \in \Phi^+, \alpha = \beta, \\ e_\beta + c_\alpha h_i + c_\alpha^2 e_\alpha & \text{if } \alpha \in \Phi^+, \alpha = -\beta, \\ & \text{and } \alpha = \alpha_i \in \Pi, \\ \sum_{\substack{k\geqslant 0 \\ \beta-k\alpha\in\Phi}} \binom{k+p_{\beta,\alpha}}{k} c_\alpha^k e_{\beta-k\alpha} & \text{if } \alpha \in \Phi^-, \alpha \neq \pm\beta, \\ e_\alpha & \text{if } \alpha \in \Phi^-, \alpha = \beta, \\ e_\beta + c_\alpha h_i + c_\alpha^2 e_\alpha & \text{if } \alpha \in \Phi^-, \alpha = -\beta, \\ & \text{and } -\alpha = \alpha_i \in \Pi. \end{cases}$$

2. Action of a representative $n_w \in N_G(T) \subseteq G$ of the element $w \in W$. First, define the map $n_\alpha(c) : \mathfrak{g} \to \mathfrak{g}$ for $c \in \mathbf{k}^\times$ and $\alpha \in \Pi$ by

$$n_\alpha(c) := \exp(ce_\alpha)\exp(-c^{-1}e_{-\alpha})\exp(ce_\alpha).$$

There exist elements $h_i \in \mathfrak{t}$ for $i \in \{1, \ldots, |\Pi|\}$ that form a basis of $\mathfrak{t}$, that is $\mathfrak{t} = \langle h_i \mid i \in \{1, \ldots, |\Pi|\}\rangle$, and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ as in [11, Definition 5.2.]. Then for $i \in \{1, \ldots, |\Pi|\}$

$$n_{\alpha_i}(c)(h_j) = h_j - \left| 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \right| h_i,$$

$$n_{\alpha_i}(c)(e_\alpha) = \begin{cases} c^{-2} e_{-\alpha_i} & \alpha = \alpha_i, \\ c^2 e_{\alpha_i} & \alpha = -\alpha_i, \\ -(-1)^{q_{\alpha,\alpha_i}+1} c^{-2\frac{(\alpha_i,\alpha)}{(\alpha_i,\alpha_i)}} e_{s_{\alpha_i}(\alpha)} & \text{else.} \end{cases}$$

Here $2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ is the (i,j)-th entry of the Cartan matrix of G and q_{α,α_i} is as in (1.1). In this case we have $\mathrm{Ad}(n_{s_\alpha})(x) = n_\alpha(1)(x)$ for all $x \in \mathfrak{g}$. Alternatively, the element n_{s_α} can also be described by $n_{s_\alpha} = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$, see [7, Section 1.9, p. 19].

3. Action of an element of the torus T on $\mathfrak{g}$. For every element in T we can find $c \in \mathbf{k}^\times$ and $\alpha \in \Pi$ such that the action of this element can be represented by the map $h_\alpha(c) := n_\alpha(c)n_\alpha(-1)$, see [6, Theorem 12.1.1].

Note that these actions are all defined up to sign, depending on the concrete choice of the Chevalley basis, see [11, Lemma 5.1 and Definition 5.2].

1.3 The classification of semisimple algebraic groups

As mentioned before, an important tool in the classification of semisimple algebraic groups is given by their root systems. However, it is possible to have two non-isomorphic semisimple groups with the same root system, as shown by the following example, see [25, Example 9.9].

Example 1.63 (Root systems of $\mathrm{SL}_2(\mathbf{k})$ and $\mathrm{PGL}_2(\mathbf{k})$).

(i) Let $G = \mathrm{SL}_2(\mathbf{k})$ and $T = D_2(\mathbf{k}) \cap G$. Then the character group of T is generated by the element χ, where
$$\chi\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = t.$$
Furthermore, let $\alpha \in X(T)$ such that $\alpha\left(\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)\right) = t^2$. Then $\Phi(G) = \{\pm\alpha\}$ as seen in Example 1.54 and $\mathbb{Z}\Phi = \langle 2\chi \rangle$, $X(T) = \mathbb{Z}\chi$.

(ii) Let $G = \mathrm{PGL}_2(\mathbf{k}) = {}^{\mathrm{GL}_2(\mathbf{k})}/_{Z(\mathrm{GL}_2(\mathbf{k}))}$. We have the homomorphism
$$\begin{aligned} \overline{} : \mathrm{GL}_2(\mathbf{k}) &\longrightarrow \mathrm{PGL}_2(\mathbf{k}), \\ A &\longmapsto \overline{A} = AZ(\mathrm{GL}_2(\mathbf{k})). \end{aligned}$$
Then a maximal torus of $\mathrm{PGL}_2(\mathbf{k})$ is given by $T = \overline{D_2(\mathbf{k})}$, the image of $D_2(\mathbf{k})$ under $\overline{}$. The root system is given by $\Phi = \{\pm\beta\}$ where
$$\beta\left(\overline{\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}}\right) = t, \quad \text{for } t \in \mathbf{k}^\times$$
and $X(T) = \mathbb{Z}\Phi$.

We see that $\Phi(\mathrm{SL}_2(\mathbf{k}))$ is isomorphic to $\Phi(\mathrm{PGL}_2(\mathbf{k}))$. However, the indices $[X(T) : \mathbb{Z}\Phi]$ are not the same.

We would therefore like to define some structures that differ if the groups are not isomorphic.

Definition and Proposition 1.64 (Cocharacters and Coroots, [25, Definition 3.4 and Lemma 8.19])**.** Let G be a connected reductive algebraic group and $T \subseteq G$ a maximal torus. We define the **cocharacters** of T by
$$Y(T) := \{\gamma : \mathbb{G}_m \to G \mid \gamma \text{ is a homomorphism of algebraic groups}\}.$$
The Weyl group $W = {}^{N_G(T)}/_T$ of G with respect to T acts on $Y(T)$ as follows: Let $w \in W$ and $n_w \in N_G(T)$ be a fixed corresponding representative in G. Let $\gamma \in Y(T)$ and $c \in \mathbf{k}$. Then
$$w.\gamma(c) = n_w \gamma(c) n_w^{-1}.$$
We have a map
$$\langle\ ,\ \rangle : X(T) \times Y(T) \longrightarrow \mathbb{Z}$$
where $\langle \chi, \gamma \rangle$ is defined by $\chi(\gamma(t)) = t^{\langle \chi, \gamma \rangle}$ for all $t \in \mathbb{G}_m$.
It is possible to prove that this map induces isomorphisms $\mathrm{Hom}(X(T), \mathbb{Z}) \simeq Y(T)$ and $\mathrm{Hom}(Y(T), \mathbb{Z}) \simeq X(T)$, see [7, 1.9]. Furthermore, let Φ be the root system of G with respect to T and let $\alpha \in \Phi$. Then there is a unique element $\alpha^\vee \in Y(T)$ such that
$$\begin{aligned} s_\alpha.\chi &= \chi - \langle \chi, \alpha^\vee \rangle \alpha, \quad &&\text{for all } \chi \in X(T) \text{ and} \\ s_\alpha.\gamma &= \gamma - \langle \alpha, \gamma \rangle \alpha^\vee, \quad &&\text{for all } \gamma \in Y(T). \end{aligned}$$
In particular, $\langle \alpha, \alpha^\vee \rangle = 2$.
The element $\alpha^\vee$ is called the **coroot** of α and $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ is the set of all **coroots** of Φ.

Example 1.65 (Coroots of $\mathrm{SL}_2(\mathbf{k})$ and $\mathrm{PGL}_2(\mathbf{k})$, [25, Example 9.9])**.** We continue Example 1.63, using the same notation as before.

(i) Let $G = \mathrm{SL}_2(\mathbf{k})$. Recall that $\Phi(G) = \{\pm\alpha\}$.
The coroot of α is given by

$$\alpha^\vee(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \text{ for } t \in \mathbf{k}^\vee.$$

Then $\mathbb{Z}\Phi = \langle 2\chi \rangle$, $X(T) = \mathbb{Z}\chi$ and $\mathbb{Z}\Phi^\vee = Y(T)$.

(ii) Let $G = \mathrm{PGL}_2(\mathbf{k})$ and $\Phi = \{\pm\beta\}$. The coroot of β is given by

$$\beta^\vee(t) = \overline{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}} = \overline{\begin{pmatrix} t^2 & 0 \\ 0 & 1 \end{pmatrix}}.$$

Therefore $\mathbb{Z}\Phi^\vee = 2Y(T)$.

In particular, we see that both $\mathrm{SL}_2(\mathbf{k})$ and $\mathrm{PGL}_2(\mathbf{k})$ have the same root system, but there are still differences in the relations of (co-)roots and (co-)characters. This motivates the following definition:

Definition 1.66 (Root Datum, [25, Definition 9.10])**.**
A quadruple $(X, \Phi, Y, \Phi^\vee)$ is called a **root datum** if:

(i) $X \simeq \mathbb{Z}^n \simeq Y$ and there is a map $\langle\, , \rangle : X \times Y \to \mathbb{Z}$ such that each homomorphism $X \to \mathbb{Z}$ is of the form $\chi \mapsto \langle \chi, \gamma \rangle$ for a unique $\gamma \in Y$ and each homomorphism $Y \to \mathbb{Z}$ is of the form $\gamma \mapsto \langle \chi, \gamma \rangle$ for a unique $\chi \in X$.

(ii) $\Phi \subseteq X,\ \Phi^\vee \subseteq Y$ are abstract root systems in $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$, respectively $\mathbb{Z}\Phi^\vee \otimes_{\mathbb{Z}} \mathbb{R}$.

(iii) There exists a bijection $\Phi \to \Phi^\vee,\ \alpha \mapsto \alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$.

(iv) The reflections s_α of the root system Φ and $s_{\alpha^\vee}$ of the root system $\Phi^\vee$ are given by

$$\begin{aligned} s_\alpha.\chi &= \chi - \langle \chi, \alpha^\vee \rangle \alpha, \quad \text{for all } \chi \in X \text{ and} \\ s_{\alpha^\vee}.\gamma &= \gamma - \langle \alpha, \gamma \rangle \alpha^\vee, \quad \text{for all } \gamma \in Y. \end{aligned}$$

It remains to see that the corresponding structures in algebraic groups form a root datum, which is indeed the case, see [25, Proposition 9.11].

Proposition 1.67. *Let Φ be a root system of a connected reductive algebraic group G with respect to a maximal torus $T \leqslant G$ with Weyl group W. Let $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ be the set of coroots of Φ. Then the quadruple $(X(T), \Phi, Y(T), \Phi^\vee)$ is a root datum.*

Finally, we need to prove that different root data do indeed define non-isomorphic simple algebraic groups and vice versa. This result is stated in Chevalley's classification theorem, see [25, Theorem 9.13]:

Theorem 1.68 (Chevalley's Classification Theorem)**.** *Two semisimple algebraic groups are isomorphic if and only if they have isomorphic root data. For each root datum there exists a semisimple algebraic group and this group is simple if and only if its root system is irreducible.*

1.3.1 Isogenies and isogeny types

We have already seen that each connected reductive algebraic group has a root system and that groups with the same root system are not necessarily isomorphic. The question that poses itself in this context, is whether there is a way to compare groups with the same root system. In order to solve this problem, we are going to define special homomorphisms of algebraic groups, so-called *isogenies*. For this section let G be a semisimple algebraic group.
First, we are going to look at the different types of groups that exist for each root system. Recall that $\Phi \subseteq X := X(T)$ and as $\langle \Phi \rangle_{\mathbb{R}} = E \supseteq X$ the group $\mathbb{Z}\Phi$ has finite index in E, see Proposition 1.50.
Let $\Omega := \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$. We can consider $\mathbb{Z}\Phi \subseteq X \subseteq \Omega$ via the homomorphism

$$X \simeq \mathrm{Hom}(Y(T), \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z}) = \Omega.$$

We recall that $\mathbb{Z}\Phi^\vee$ has finite index in $Y(T)$, so the homomorphism is injective.

Definition 1.69 (Fundamental group, simply connected, adjoint, [25, Definition 9.14])**.** Let $\Lambda := \Lambda(\Phi) := \Omega/\mathbb{Z}\Phi$. This group does not depend on X and therefore not on the torus T or the group G. The set $\Lambda(\Phi)$ is called the **fundamental group** of Φ and $\Lambda(G) := \Omega/X$ is called the **fundamental group** of the semisimple group G. If $X = \Omega$ (i.e. $\Lambda(G) = 1$), then G is called **simply connected**. If $X = \mathbb{Z}\Phi$, then G is of **adjoint type**. We write G_{ad} for an adjoint algebraic group and G_{sc} for a simply connected algebraic group with root system Φ.

Note also that all root data with a fixed root system Φ are classified by the subgroups $X/\mathbb{Z}\Phi \leqslant \Omega/\mathbb{Z}\Phi$ of the fundamental group of Φ.
We are interested in finding homomorphisms between the different types of semisimple algebraic groups G for a given root system Φ. These maps are called *isogenies* and the types of algebraic groups for a given root system are called *isogeny types*.

Definition 1.70 (Isogeny)**.** Let $\varphi : G \longrightarrow H$ be a surjective homomorphism of algebraic groups. If the kernel of φ is finite, φ is called an **isogeny**. If G is connected, the kernel lies in the centre of G. If G is also reductive, then $\ker \varphi$ lies in all maximal tori of G, see also [30, 9.6.1].

We have the following properties for isogenies of algebraic groups, see [30, 9.6.1,9.6.3] and [1, Corollary 22.5]:

Proposition 1.71. *Let G and H be two connected reductive algebraic groups with maximal tori T_G and T_H and corresponding root data $\Psi_G = (X_G, \Phi_G, Y_G, \Phi_G^\vee)$ of G and $\Psi_H = (X_H, \Phi_H, Y_H, \Phi_H^\vee)$ of H. Let $\varphi : G \longrightarrow H$ be an isogeny. We have the following properties.*

1. *If $\varphi(T_G) = T_H$, the isogeny φ defines two homomorphisms $f : X_H \to X_G$ and $f^\vee : Y_G \to Y_H$ of the character and cocharacter groups respectively.*
   Then $\langle \chi, f^\vee(\lambda) \rangle = \langle f(\chi), \lambda \rangle$ for all $\chi \in X_H$ and $\lambda \in Y_G$.

2. *There is a bijection $\rho : \Phi_G \to \Phi_H$ with $\varphi(U_\alpha) = U_{\rho(\alpha)}$ where the U_α denote the root subgroups in G or H.*

3. *For ρ as above we have $f(\rho(\alpha)) = q(\alpha)\alpha$ and $f^\vee(\alpha^\vee) = q(\alpha)(\rho(\alpha))^\vee$ where $q(\alpha)$ is a power of the characteristic p of $\mathbf{k}$ (or 1 if $\operatorname{char}(\mathbf{k}) = 0$).*
   *The isogeny φ is called **central** if $q(\alpha) = 1$ for all $\alpha \in \Phi$.*

4. *If φ is central, then $\operatorname{im}(d\varphi)$ contains all nilpotent elements of $\operatorname{Lie}(H) = \mathfrak{h}$. We have $\mathfrak{h} = d\varphi(\mathfrak{g}) + \operatorname{Lie}(T_H)$.*

It is in fact true that we can find an isogeny between groups of different isogeny types with the same root system. This result is stated in the isogeny theorem, see for example [30, Theorem 9.6.5].

Theorem 1.72 (Isogeny theorem)**.** *Let (f, ρ, q) be as in Proposition 1.71. Then there exists an isogeny $\varphi : G \to H$ with $\varphi(T_G) = T_H$. If φ' is another isogeny fulfilling these properties, then there exists an element $t \in T_G$ such that $\varphi' = \varphi(tgt^{-1})$ for all $g \in G$.*
In particular, there are isogenies π_1 and π_2 with

$$G_{\mathrm{sc}} \xrightarrow{\pi_1} G \xrightarrow{\pi_2} G_{\mathrm{ad}}.$$

1.4 Good and bad primes

Recall that the commutator formula given in Proposition 1.55 depends (amongst other things) on constants $c^{n,m}_{\alpha,\beta} \in \mathbb{Z}$. In small prime characteristic it can happen that $c^{n,m}_{\alpha,\beta} = 0$ for fixed $\alpha, \beta \in \Phi$ and $n, m \in \mathbb{N}$ but $c^{n,m}_{\alpha,\beta} \neq 0$ for $\operatorname{char}(\mathbf{k}) = 0$. Certainly, this depends on the root system. It follows therefore that the calculations in these cases are different than for characteristic 0.

Definition 1.73 (Bad primes, [25, Definition 14.14])**.** Let G be a simple algebraic group. Depending on the root system of an algebraic group G, the following primes are called **bad primes**:

type of G	bad primes
A_n	–
B_n $(n \geqslant 3)$, D_n $(n \geqslant 4)$, C_n $(n \geqslant 2)$	2
G_2, F_4, E_6, E_7	2, 3
E_8	2, 3, 5

Table 1.2: Bad primes of simple algebraic groups

If p is a bad prime and $\operatorname{char}(\mathbf{k}) = p$, then the characteristic of $\mathbf{k}$ is called **bad** for G. Otherwise, the characteristic is called **good** for G. If G is not simple, the bad primes for G are the bad primes of the irreducible components of the root system of G.

Example 1.74 (Bad primes in $\operatorname{Sp}_4(\mathbf{k})$)**.** The prime $p = 2$ is a bad prime for $G = \operatorname{Sp}_4(\mathbf{k})$, as the root system has type C_2, see for instance [25, Table 9.2].
We compute the commutator of two elements in $\operatorname{Sp}_4(\mathbf{k})$, where $a,\, b \in \mathbf{k}^\times$:

$$\left[\begin{pmatrix} 1 & a & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & -a \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \begin{pmatrix} 1 & \cdot & b & \cdot \\ \cdot & 1 & \cdot & b \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & a & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & -a \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & b & \cdot \\ \cdot & 1 & \cdot & b \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & -a & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & a \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & -b & \cdot \\ \cdot & 1 & \cdot & -b \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cdot & \cdot & 2ab \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

It is easy to see that the matrices $\begin{pmatrix} 1 & a & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & -a \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \cdot & b & \cdot \\ \cdot & 1 & \cdot & b \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$ only commute if $\mathrm{char}(\mathbf{k}) = 2$.

2 | Nilpotent orbits

In this chapter we give an overview of the theory of nilpotent orbits. Let G be a connected reductive algebraic group and $\mathfrak{g} := \mathrm{Lie}(G)$ be its Lie algebra. As noted in Definition 1.18, G acts on $\mathfrak{g}$ via the adjoint representation Ad. Let $\mathscr{N}_\mathfrak{g} \subseteq \mathfrak{g}$ be the **nilpotent variety** consisting of all nilpotent elements in $\mathfrak{g}$. Then Ad fixes $\mathscr{N}_\mathfrak{g}$ and we can consider the action of G on the nilpotent variety. The orbits under this action are called the **nilpotent orbits** and have been extensively studied. By now, all nilpotent orbits are known for each type of root system regardless of the characteristic of $\mathbf{k}$. For the description see for example [20]. In good characteristic the nilpotent orbits can be parametrised by objects called the *weighted Dynkin diagrams*. Note that we have to consider the cases where $\mathrm{char}(\mathbf{k})$ is bad for G separate from the cases in good characteristic for the description of the nilpotent orbits. This means that we do not always get a uniform description of the orbits in each type.
We additionally give a list of explicit representatives of the orbits in the exceptional groups, as this will prove useful in later chapters.

2.1 Weighted Dynkin diagrams

In the following section let G be defined over an algebraically closed field $\mathbf{k}$, such that the characteristic of $\mathbf{k}$ is good for G. In this case the nilpotent orbits of $\mathfrak{g}$ can be described by certain diagrams, called the *weighted Dynkin diagrams*.
Let $0 \neq e \in \mathfrak{g} = \mathrm{Lie}(G)$ be a nilpotent element. The construction of the weighted Dynkin diagrams relies on embedding e in a 3-dimensional subalgebra $\langle e, h, f\rangle$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbf{k})$. This will in turn define a linear map from the roots of G, $\Phi(G)$, to $\mathbb{Z}$ which gives rise to the weighted diagram of the orbit of e.
In order to do so, we first need the fact that any nilpotent element can indeed be embedded in a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbf{k})$. This is a result of the Jacobson–Morozov theorem.

Theorem 2.1 (Jacobson–Morozov, [7, Theorem 5.3.2])**.** *Let G be a simple algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic 0 or a good prime p for G. Let $\mathfrak{g} := \mathrm{Lie}(G)$ and $e \in \mathfrak{g}$, $e \neq 0$ be a nilpotent element such that $\mathrm{ad}(e)^m = 0$ for some $m \in \mathbb{N}$. In the case of $\mathrm{char}(\mathbf{k}) = p$, we additionally assume that $m \leqslant p - 2$. Then there exist elements h, $f \in \mathfrak{g}$ with $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, that is, e can be embedded in a subalgebra $\langle e, h, f\rangle \subseteq \mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbf{k})$.*

It follows therefore, that each nilpotent element $e \in \mathfrak{g}, e \neq 0$ can be embedded in a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbf{k})$ if $\mathrm{char}(\mathbf{k}) = 0$. This poses the question what happens if there are two such subalgebras. We cannot expect them to be the same. However, following [7, Proposition 5.5.10] we see that they are in the same G-orbit.

Proposition 2.2. *Let* $\mathrm{char}(\mathbf{k}) = 0$ *or* $\mathrm{char}(\mathbf{k}) = p > 3(c-1)$ *where* c *is the Coxeter number of G. Let* $\langle e, h, f\rangle$ *and* $\langle e, h_1, f_1\rangle$ *be two subalgebras of* $\mathfrak{g}$ *as in Theorem 2.1. Then there exists an element* $g \in C_G(e)^\circ$ *such that* $\mathrm{Ad}(g)(h) = h_1$ *and* $\mathrm{Ad}(g)(f) = f_1$*, that is* $\langle e, h, f\rangle$ *and* $\langle e, h_1, f_1\rangle$ *are in the same* G*-orbit.*

In fact, it follows that the Lie algebra $\mathfrak{g}$ is a direct sum of irreducible $\mathfrak{sl}_2(\mathbf{k})$-modules as stated in [7, Theorem 5.4.8].

Theorem 2.3. *Let* V *be a finite-dimensional* $\mathfrak{sl}_2(\mathbf{k})$*-module affording the representation* ρ*. We use the notation from Theorem 2.1. Suppose there is a positive integer* $m \geqslant 2$ *such that* $\rho(e^{m-1}) = 0$, $\rho(f^{m-1}) = 0$*. Suppose that* $p \neq 2$ *and* $m \leqslant p$ *if* $\mathrm{char}(\mathbf{k}) = p$*. If* $\mathrm{char}(\mathbf{k}) = 0$*, we do not need any restrictions on* m*. Then* V *is a direct sum of irreducible submodules each of which affords one of the representations* ρ_j *for some* $j \leqslant m-1$*. Here the representations* ρ_j *are defined as follows: Let* $x_1, \ldots, x_j$ *be a basis of an irreducible* $\mathfrak{sl}_2(\mathbf{k})$*-module. Then*

$$\begin{aligned} \rho_j(e).x_i &= x_{i+1}, & i &= 1,2,\ldots,j-1, & \rho_j(e).x_j &= 0, \\ \rho_j(h).x_i &= (2i-j-1)x_i, & i &= 1,2,\ldots,j, & & \\ \rho_j(f).x_{i+1} &= i(j-i)x_i, & i &= 1,2,\ldots,j-1, & \rho_j(f).x_1 &= 0, \end{aligned} \tag{2.1}$$

as defined in [7, Section 5.4].

Setting $V := \mathfrak{g}$ in the above theorem, it follows that $\mathfrak{g}$ is a direct sum of irreducible $\mathfrak{sl}_2(\mathbf{k})$-modules with a basis $x_1, \ldots, x_j$ and representations ρ_j of $\mathfrak{sl}_2(\mathbf{k})$ as above.
Let $c \in \mathbf{k}$ and $x_1, \ldots, x_j$ be a basis of an irreducible $\mathfrak{sl}_2(\mathbf{k})$-module in $\mathfrak{g}$ as above. There is a homomorphism of algebraic groups, $\gamma : \mathbf{k}^\times \to G$, such that the elements in its image act on this basis by

$$\gamma(c).x_i = c^{2i-j-1}x_i, \quad c \in \mathbf{k}^\times \tag{2.2}$$

and $\gamma(c)$ describes an action of $\left(\begin{smallmatrix} c & 0 \\ 0 & c^{-1}\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{k})$ on $\mathfrak{g}$, see [7, Proposition 5.5.6].
Therefore, if $\mathrm{char}(\mathbf{k}) = 0$ or $m \leqslant \mathrm{char}(\mathbf{k}) - 2$, we can define such a map γ for each nilpotent element $e \in \mathfrak{g}$. The following result holds by [7, Theorem 5.5.11].

Proposition 2.4. *Let* $\mathrm{char}(\mathbf{k}) = 0$ *or* $\mathrm{char}(\mathbf{k}) = p > 3(c-1)$ *where* c *is the Coxeter number of G. Then there is a bijection between the non-zero nilpotent orbits of* $\mathfrak{g}$ *and the* G*-orbits of the subalgebras of* $\mathfrak{g}$ *isomorphic to* $\mathfrak{sl}_2(\mathbf{k})$*.*
By Pommerening [27] and [28], as well as Premet [26], this result also holds if $\mathrm{char}(\mathbf{k})$ *is good for* G*.*

Proof. We follow the proof in [7].
Define a map

$$\begin{aligned} \phi : {}_G\backslash(\mathscr{N}_\mathfrak{g} \setminus \{0\}) &\longrightarrow {}_G\backslash\{\text{subalgebras isomorphic to } \mathfrak{sl}_2(\mathbf{k})\}, \\ \mathscr{O}_e &\longmapsto G.\langle e, h, f\rangle. \end{aligned}$$

First note that each non-zero nilpotent element $e \in \mathfrak{g}$ is contained in a subalgebra $\langle e, h, f\rangle \subseteq \mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbf{k})$, which follows from the Jacobson-Morozov Theorem 2.1. By Proposition 2.2, this subalgebra is in the same G-orbit as any other subalgebra $\langle e, f_1, h_1\rangle$ isomorphic to $\langle e, h, f\rangle$. This shows that ϕ is well-defined. In order to see that ϕ is surjective, note that each subalgebra in $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbf{k})$ is of the form

$\langle e,f,h\rangle$. Now this subalgebra $\langle e,f,h\rangle$ is the Lie algebra of a subgroup $H \subseteq G$ where H is isomorphic to $\mathrm{SL}_2(\mathbf{k})$ or $\mathrm{PGL}_2(\mathbf{k})$ (see for example [7, 5.5.5]). In particular, e is a non-zero nilpotent element in $\mathrm{Lie}(H)$ and therefore also in $\mathrm{Lie}(G)$. Thus, for each subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbf{k})$, we can find $e \in \mathfrak{g}$, e nilpotent, such that $\phi(\mathscr{O}_e) = G.\mathfrak{h}$.
Now suppose $\phi(\mathscr{O}_e) = \phi(\mathscr{O}_{e'})$ for two non-zero nilpotent elements $e, e' \in \mathfrak{g}$. In $\mathfrak{gl}_2(\mathbf{k})$ each non-zero nilpotent element is conjugate to $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ under the action of $\mathrm{GL}_2(\mathbf{k})$, i.e. e, e', and $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ are in the same orbit under this action. Since

$$C_{\mathrm{GL}_2(\mathbf{k})}\left(\begin{pmatrix}0&0\\1&0\end{pmatrix}\right) = \left\{\begin{pmatrix}a&0\\c&a\end{pmatrix}\middle|\, c\in\mathbf{k}, a\in\mathbf{k}^\times\right\},$$

we have $\mathrm{GL}_2(\mathbf{k}) = \mathrm{SL}_2(\mathbf{k}).C_{\mathrm{GL}_2(\mathbf{k})}\left(\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\right)$. Thus, any two non-zero nilpotent elements in a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbf{k})$ are conjugate under $\mathrm{SL}_2(\mathbf{k})$, and therefore under G which contains a subgroup related to E and either isomorphic to $\mathrm{SL}_2(\mathbf{k})$ or $\mathrm{PGL}_2(\mathbf{k})$ by [7, 5.5.6]. This means that $\mathscr{O}_e = \mathscr{O}_{e'}$, so ϕ is injective. □

Let $\gamma : \mathbf{k}^\times \to G$ be a homomorphism of algebraic groups. If we choose a maximal torus $T \subseteq G$ such that $\mathrm{im}(\gamma) \subseteq T$, the map γ gives rise to a linear map

$$\eta_\gamma : \Phi \longrightarrow \mathbb{Z}, \quad \alpha \longmapsto \langle\alpha,\gamma\rangle. \tag{2.3}$$

Here $\gamma\circ\alpha : \mathbf{k}^\times \to \mathbf{k}^\times, c \mapsto c^n$ for some $n\in\mathbb{Z}$. We set $\langle\alpha,\gamma\rangle := n$. If γ is the homomorphism from (2.2), there exists a set of simple roots $\Pi \subseteq \Phi$ such that $\eta_\gamma(\Pi) \subseteq \{0,1,2\}$ by [7, Proposition 5.6.6]. We will briefly repeat the proof of this fact here.

Lemma 2.5. *Let $\eta : \Phi \to \mathbb{Z}$ be a linear map, that is $\eta(-\alpha) = -\eta(\alpha)$ and $\eta(\alpha+\beta) = \eta(\alpha)+\eta(\beta)$ for all $\alpha,\beta\in\Phi$ such that $\alpha+\beta\in\Phi$. Then there exists a system of simple roots $\Pi \subset \Phi$ such that $\eta(\alpha) \geqslant 0$ for all $\alpha\in\Pi$. In fact we have $\eta(\alpha)\in\{0,1,2\}$ if η corresponds to the map $\gamma \in Y(T)$ and γ is as in* (2.2)*.*

Proof. Note that η is determined by its values on the simple roots and can be uniquely extended to an element in $\mathrm{Hom}(X,\mathbb{Z})$ where $X = X(T) = \mathbb{Z}\Phi$. As $\mathrm{Hom}(X,\mathbb{Z}) \simeq Y = Y(T)$, we also have a unique cocharacter $\gamma\in Y$ to which η maps under this bijection such that $\eta(\alpha) = \langle\alpha,\gamma\rangle$ for all $\alpha\in\Phi$. Now choose a system of simple roots Π such that $\langle\alpha,\gamma\rangle \geqslant 0$ for all $\alpha\in\Pi$. This is possible, as this means that γ is in the closure of the fundamental chamber, which is a fundamental domain for the action of the Weyl group W of Φ on Y, see Proposition 1.49.
Suppose γ is as in (2.2). For each $i\in\mathbb{Z}$ define the sets

$$\mathfrak{g}_i := \{x\in\mathfrak{g} \mid \gamma(c).x = c^i x \text{ for all } c\in\mathbf{k}^\times\}. \tag{2.4}$$

These sets form a grading of the Lie algebra $\mathfrak{g}$, i.e. $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_i$. Let $j\in\mathbb{Z}$ and $\alpha\in\Phi$ with $\langle\alpha,\gamma\rangle = j$. Then $e_\alpha\in\mathfrak{g}_j$, where $\langle e_\alpha\rangle = \mathfrak{g}_\alpha$, as

$$\gamma(c).e_\alpha = \mathrm{Ad}(\gamma(c))(e_\alpha) = \alpha(\gamma(c))e_\alpha = c^{\langle\alpha,\gamma\rangle}e_\alpha$$

for all $c\in\mathbf{k}^\times$. Recall, that this is because $\gamma\in Y$ is a cocharacter and therefore $\gamma(c)\in T$ for all $c\in\mathbf{k}^\times$. Note that $[h,e] = 2e$, $[h,f] = -2f$, and $[e,f] = h$. By restricting the

$\langle e, f, h\rangle$-module $\mathfrak{g}$ to $\langle e, f, h\rangle$ and taking the basis $x_1 := f, x_2 := h, x_3 := -2e$ we get the action

$$\begin{aligned} e.x_1 &= [e,f] = h = x_2, & f.x_1 &= [f,f] = 0,\\ e.x_2 &= [e,h] = -2e = x_3, & f.x_2 &= [f,h] = 2f = 1\cdot(3-1)x_1,\\ e.x_3 &= [e,-2e] = 0, & f.x_3 &= [f,-2e] = 2h, \end{aligned}$$

$$\begin{aligned} h.x_1 &= [h,f] = -2f = (2\cdot 1 - 3 - 1)x_1,\\ h.x_2 &= [h,h] = 0 = (2\cdot 2 - 3 - 1)x_2,\\ h.x_3 &= [h,-2e] = -4e = (2\cdot 3 - 3 - 1)x_3. \end{aligned}$$

This means that we get a $\langle e, f, h\rangle$-module with representation ρ_3. Then

$$\begin{aligned} \gamma(c).e &= \gamma(c).(-\frac{1}{2}x_3) = -\frac{1}{2}c^{2\cdot 3-3-1}x_3 = c^2 e,\\ \gamma(c).f &= \gamma(c).x_1 = c^{2\cdot 1-3-1}x_1 = c^{-2}f,\\ \gamma(c).h &= \gamma(c).x_2 = c^{2\cdot 2-3-1}x_2 = c^0 h. \end{aligned}$$

So $e \in \mathfrak{g}_2$, $f \in \mathfrak{g}_{-2}$, and $h \in \mathfrak{g}_0$. Therefore,

$$f = \sum_{\substack{\beta\in\Phi\\ \gamma(\beta)=-2}} \lambda_\beta e_\beta \quad \text{for } \lambda_\beta \in \mathbf{k}.$$

In particular, all roots β occurring in the above linear combination are negative roots, as $\gamma(\alpha) \geqslant 0$ for all simple roots α, and therefore for all positive roots. Then

$$[f, e_\alpha] = \sum_{\substack{\beta\in\Phi\\ \gamma(\beta)=-2}} \lambda_\beta [e_\beta, e_\alpha] \in \mathfrak{t} \oplus \bigoplus_{\substack{\alpha+\beta\in\Phi\\ \gamma(\beta)=-2}} \mathfrak{g}_{\alpha+\beta}$$

by Proposition 1.56. As $\alpha \in \Pi$ is a simple root, this is a linear combination of elements in $\mathfrak{t} = \mathrm{Lie}(T)$ and of $e_{\beta'}$, where β' is a negative root. Therefore, $[f, e_\alpha] \in \bigoplus_{i\leqslant 0}\mathfrak{g}_i$.
We first suppose that $[f, e_\alpha] \neq 0$. The above calculation and Proposition 1.56 show that $[f, e_\alpha] \in \mathfrak{g}_{j-2}$ for $e_\alpha \in \mathfrak{g}_j$ and so $j - 2 \leqslant 0$, that is $j \leqslant 2$. Then the only possibilities are $j \in \{0, 1, 2\}$.
If conversely $[f, e_\alpha] = 0$, we have $e_\alpha \in C_{\mathfrak{g}}(f) \subseteq \bigoplus_{i\leqslant 0}\mathfrak{g}_i$. This follows because we can write $\mathfrak{g} = \bigoplus V_r$, where the V_r are the irreducible $\langle e, f, h\rangle$-submodules. If we take the standard basis $\{x_{r,1}, \ldots, x_{r,j_r}\}$ as in Theorem 2.3 for each V_r, there exists exactly one vector $x_{r,1}$ in the centraliser of f for each V_r. To see this, recall by (2.1) that

$$\rho_{j_r}(f).x_{r,i+1} = i(j_r - i)x_{r,i}$$

for $1 \leqslant i \leqslant j_r - 1$, and so $x_{r,i} \in C_{\mathfrak{g}}(f)$ if $[f, x_{r,i}] = 0$, i.e. $i = 1$. We have $\gamma(c).x_{r,1} = c^{1-j_r}x_{r,1}$, so $x_{r,1} \in \mathfrak{g}_{1-j_r}$ and surely $j_r \geqslant 1$.
Then $e_\alpha \in \mathfrak{g}_j \cap \bigoplus_{i\leqslant 0}\mathfrak{g}_i$ with $j \geqslant 0$. This means that $j = 0$ and the claim follows. □

We fix such a system of simple roots Π in order to define the *weighted Dynkin diagrams*, see [7, Text after Proposition 5.6.6.].

Definition 2.6 (Weighted Dynkin diagrams)**.** Let γ be as above. Then we can define the linear map $\gamma : \Phi \to \mathbb{Z}$, where $\gamma(\alpha) = \langle \gamma, \alpha \rangle$, as before.
As γ is a linear map, γ is determined by its values on the set of simple roots. This means that, instead of giving γ, we can take the Dynkin diagram corresponding to Φ and assign to each node for a simple root $\alpha \in \Pi$ the value $\gamma(\alpha)$. By Lemma 2.5, we can find a system of simple roots such that the nodes are labelled by $0, 1$ or 2. The resulting diagram is called the **weighted Dynkin diagram** of γ.

Note that this construction still requires some restrictions if $\mathrm{char}(\mathbf{k}) = p$, even if p is good for G, as it relies heavily on the Jacobson–Morozov Theorem 2.1. However, this classification also works in any good characteristic due to the work of Pommerening [27] and [28] and Premet [26].

Example 2.7 (Weighted Dynkin diagrams of type G_2)**.** For type G_2 we have the following weighted Dynkin diagrams, taken from [7, 13.1, p. 401].

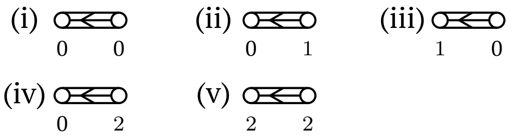


Figure 2.1: The weighted Dynkin diagrams of type G_2

Remark 2.8. Even though the weighted Dynkin diagrams arise from the above construction in good characteristic, we can define corresponding maps $\delta : \Phi \longrightarrow \mathbb{Z}$ such that $\delta(\alpha)$ corresponds to the weight of the node belonging to the simple root $\alpha \in \Pi$ in every characteristic. In the next section we will see how to define so-called T*-labellings* which are just the weighted Dynkin diagrams in good characteristic.

It remains to see that each weighted Dynkin diagram is uniquely determined by a nilpotent orbit. This fact is for example proved in the book of Carter, see [7, Propositions 5.6.7 and 5.6.8].

Proposition 2.9. *The weighted Dynkin diagram defined by a nilpotent element $e \in \mathfrak{g}$ is uniquely determined by the nilpotent orbit of e in $\mathfrak{g}$. If $e, e_1 \in \mathfrak{g}$ are two nilpotent elements then the weighted Dynkin diagrams of e and e_1 are the same if and only if e and e_1 are in the same nilpotent orbit.*

For later use we will define a parabolic subgroup determined by a nilpotent element e and prove that the centraliser of e in G is contained in this subgroup. We suppose that $\mathrm{char}(\mathbf{k})$ is good for G such that the above construction is possible. Recall that a nilpotent element $e \in \mathfrak{g}$ gives rise to a map $\gamma : \mathbf{k}^\times \to G$. We fix a root system with simple roots $\Pi \subseteq \Phi$ such that $\gamma(\alpha) \in \{0, 1, 2\}$ for each $\alpha \in \Pi$. This gives rise to a parabolic subgroup

$$G^{\gamma}_{\geqslant 0} := \langle T, U_\alpha \mid \alpha \in \Phi \text{ with } \langle \alpha, \gamma \rangle \geqslant 0 \rangle. \tag{2.5}$$

We will see this group again in the next section. With respect to this parabolic subgroup the following proposition holds, see [7, Proposition 5.7.1].

Proposition 2.10. *Let the notation be as before and* $\mathrm{char}(\mathbf{k}) = 0$ *or* $\mathrm{char}(\mathbf{k}) = p > 3(c-1)$ *for the Coxeter number* c. *Then*

1. $G^{\gamma}_{\geqslant 0}$ *is uniquely determined by* e.
2. $C_G(e) \subseteq G^{\gamma}_{\geqslant 0}$.

2.2 *T*-labellings

In [20, Chapter 10] there is an alternative definition for the parametrisation of the nilpotent orbits, called a T**-labelling**. In fact, the T-labellings have the same labels as the weighted Dynkin diagrams in good characteristic, see the introduction of [20]. This approach allows us to define weighted diagrams for nilpotent orbits in bad characteristic as well.

Definition 2.11. Let G be a simple algebraic group over $\mathbf{k}$ and $T \subseteq G$ be a maximal torus. Let Φ be the root system of G with respect to T and for each $\alpha \in \Phi$ we have the weight vector $e_\alpha \in \mathfrak{g}$ generating the root space $\mathfrak{g}_\alpha$. Let $\gamma : \mathbf{k}^\times \to T$ be a cocharacter. If γ is not trivial, the image $S := \gamma(\mathbf{k}^\times) \subseteq T$ is a one-dimensional torus in G. Then $\mathrm{Ad}(\gamma(c))(e_\alpha) = c^{n_\alpha} e_\alpha$ for all $c \in \mathbf{k}^\times$ and some $n_\alpha \in \mathbb{Z}$. As stated in Lemma 2.5, it is possible to choose a set of simple roots $\Pi \subseteq \Phi$ such that $\gamma(\alpha) \in \{0, 1, 2\}$ for all $\alpha \in \Pi$ in good characteristic. In bad characteristic this is a result in [20] as stated below. The corresponding Dynkin diagram of G with each node α labelled by n_α is called the **labelled diagram** for S.

Let $e \in \mathfrak{g}$. Then there exists a one-dimensional torus $S = \{\gamma(c) \mid c \in \mathbf{k}^\times\}$ such that $\mathrm{Ad}(\gamma(c))(e) = c^2 e$ for all $c \in \mathbf{k}^\times$. This follows from [20, Section 5.1 and 16.1]. This torus will determine a T-labelling corresponding to the nilpotent orbit of the element e. As mentioned above, these labellings are exactly the weighted Dynkin diagrams in good characteristic. Each labelling determines a parabolic subgroup P as follows: Let

$$P = \langle T, U_\alpha \mid \alpha \in \Phi,\, n_\alpha \geqslant 0 \rangle,$$

using the notation in Definition 2.11. Then P is a parabolic subgroup of G with Levi subgroup $L = \langle T, U_\alpha \mid n_\alpha = 0 \rangle$ and unipotent radical $Q = \prod_{n_\alpha > 0} U_\alpha$. This is just the subgroup $G^{\gamma}_{\geqslant 0}$ from (2.5). For representatives of nilpotent orbits in $\mathfrak{g}$ we get the following result, see [20, Theorem 1, Lemma 2.29, Theorem 9.1, Lemma 15.3, Lemma 15.4, and 16.1.1]. Note that in good characteristic this is just Proposition 2.10.

Proposition 2.12. *Let* $e \in \mathfrak{g} \setminus \{0\}$ *be a nilpotent orbit representative. Then there exists a one-dimensional torus* S *giving rise to a labelling and a parabolic subgroup* $P = QL$ *as above. Furthermore,*

$$e \in \mathrm{Lie}(Q)_{\geqslant 2} = \bigoplus_{\substack{\alpha \in \Phi \\ n_\alpha \geqslant 2}} \mathfrak{g}_\alpha$$

and $C_G(e) \subseteq P$.
If e *is a representative of an orbit occurring both in good and bad characteristic, then we even have*

$$e \in \mathrm{Lie}(Q)_2 = \bigoplus_{\substack{\alpha \in \Phi \\ n_\alpha = 2}} \mathfrak{g}_\alpha.$$

2.3 Orbit Types

In this section we explore the different kinds of nilpotent orbits and how they are denoted. We start with the classical groups and move on to the exceptional groups. We will give representatives of the orbits occurring only in the exceptional groups, as well as a complete list of representatives in exceptional groups.
Let G be an exceptional simple algebraic group over an algebraically closed field $\mathbf{k}$. Let $e \in \mathfrak{g}$ be a representative of a nilpotent orbit. Then $e = \sum_{k=1}^{r} e_k$ where each e_k lies in a subgroup L_k of G such that $[L, L] = L' := L_1 \cdots L_r$ is a commuting product of simple factors L_k and L is a Levi subgroup of G. The elements e_k are distinguished in the L_k, see [20, chapter 9]. Here, we refer to $e \in \mathfrak{g}$ as **distinguished** if the connected group $C_G(e)^\circ$ is unipotent. The element e is denoted by the sequence of the $\mathrm{Lie}(L_k)$. If e is the representative of an **exceptional orbit**, that is, an orbit only occurring in bad characteristic, we denote the orbit type by $(L)_p$.

2.3.1 In classical groups

For classical groups we fix a notation for the classes as follows. J_n denotes a nilpotent Jordan block of dimension n, so

$$J_n := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & \mathbf{0} & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \in \mathbf{k}^{n \times n}.$$

The nilpotent orbits with a representative in Jordan form $J_{n+1}, J_{2n+1}, J_{2n}$, or (J_1, J_{2n-1}) will be denoted by A_n, B_n, C_n, or D_n respectively.
For $G = \mathrm{SO}_{2n}(\mathbf{k})$ with root system type D_n the distinguished nilpotent class with Jordan form $(J_{2i+1}, J_{2n-2i-1})$, where $1 \leqslant i < \frac{n-1}{2}$, will be denoted by $D_n(a_i)$.
Furthermore, we define three kinds of indecomposable modules, see [20, 5.1]. In the following list, let V be a finite-dimensional vector space over $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) = 2$, and $G = \mathrm{Sp}(V)$ or $O(V)$. G preserves a non-degenerate symmetric bilinear form $(\ ,\ )$ on V and if $G = O(V)$, additionally a quadratic form Q. Let $e \in \mathfrak{g} = \mathrm{Lie}(G)$ be a nilpotent element. Then $V \downarrow e$ is a direct sum of modules isomorphic to the modules introduced below. Furthermore, let $S = \{\gamma(c) \mid c \in \mathbf{k}^\times\}$ be a one-dimensional torus corresponding to e as described below Definition 2.11.

1. For the group $G = \mathrm{Sp}(V)$ define the module $V(m)$ as a non-degenerate space of dimension m, where m is even, and a basis $\{v_{-m+1}, v_{-m+3}, \ldots, v_{m-3}, v_{m-1}\}$, where

$$(v_i, v_j) = \begin{cases} 0 & \text{if } j \neq -i, \\ 1 & \text{if } j = -i. \end{cases}$$

   For a one-dimensional torus $S = \{\gamma(c) \mid c \in \mathbf{k}^\times\}$ as in Section 2.2 and $e \in \mathfrak{g}$ we have the action

$$\begin{aligned} \gamma(c).v_i &= c^i v_i, \quad \text{for all } i \in \{-m+1, -m+3, \ldots, m-3, m-1\}, \\ e.v_i &= v_{i+2}, \quad \text{for } i < m-1 \text{ and } e.v_{m-1} = 0. \end{aligned}$$

2. In $G = \mathrm{Sp}(V)$, the module $W(m)$ is a non-degenerate space of dimension $2m$ with a basis

$$B := \{r_{-m+1}, r_{-m+3}, \ldots, r_{m-3}, r_{m-1}\} \cup \{s_{-m+1}, s_{-m+3}, \ldots, s_{m-3}, s_{m-1}\},$$

where for $v,\ w \in B$

$$(v, w) = \begin{cases} 1 & \text{if } v = r_i,\ w = s_{-i} \text{ or } v = s_i,\ w = r_{-i}, \\ 0 & \text{otherwise.} \end{cases}$$

The operation of the one-dimensional torus S and the nilpotent element e is given by

$$\begin{aligned} \gamma(c).r_i &= c^i r_i, \quad \text{for all } i \in \{-m+1, -m+3, \ldots, m-3, m-1\}, \\ \gamma(c).s_i &= c^i s_i, \quad \text{for all } i \in \{-m+1, -m+3, \ldots, m-3, m-1\}, \\ e.r_i &= r_{i+2}, \quad \text{for } i < m-1 \text{ and } e.r_{m-1} = 0, \\ e.s_i &= s_{i+2}, \quad \text{for } i < m-1 \text{ and } e.s_{m-1} = 0. \end{aligned}$$

For $G = O(V)$ the space $W(m)$ is defined in the same way with $Q(s_i) = Q(r_i) = 0$. Here $Q : V \to \mathbf{k}$ is a fixed non-degenerate quadratic form with the associated symmetric bilinear form $(\ ,\ ) : V \times V \to \mathbf{k}$.

3. $W_l(m)$ is defined for $0 < l < \frac{1}{2}m$ and is a non-degenerate space of dimension $2m$.
For $G = \mathrm{Sp}(V)$ we define $W_l(m)$ as the vector space with a basis

$$B := \{v_{-2l+1}, \ldots, v_{2l-1}, v_{2l+1}, \ldots, v_{2m-2l-1}\} \cup \{w_{-2m+2l+1}, \ldots, w_{2l-3}, w_{2l-1}\}.$$

The subspace spanned by all w_j is totally singular, while the subspace spanned by all v_i has a radical. This radical is given by the subspace spanned by the v_i for $i = 2l+1, \ldots, 2m-2l-1$. The quotient is the non-degenerate space spanned by images of v_i for $i = -(2l-1), \ldots, 2l-1$.
For two elements $v, w \in B$ we have

$$(v, w) = \begin{cases} 1 & \text{if } v = v_i,\ w = v_{-i} \text{ for } -(2l-1) \leqslant i \leqslant 2l-1, \\ 0 & \text{if } v = v_i,\ w = v_j \text{ for } i,\ j \text{ not as above}, \\ 1 & \text{if } v = v_i,\ w = w_{-i} \text{ for all } i, \\ 0 & \text{if } v = v_i,\ w = w_j \text{ and } i \neq j. \end{cases}$$

The operation of the one-dimensional torus S and the nilpotent element e is given by

$$\begin{aligned} \gamma(c).v_i &= c^i v_i, \quad \text{for all } i, \\ \gamma(c).w_j &= c^j w_j, \quad \text{for all } j, \\ e.v_i &= v_{i+2}, \quad \text{for } i < 2m-2l-1 \text{ and } e.v_{2m-2l-1} = 0, \\ e.w_j &= w_{j+2}, \quad \text{for } j < 2l-1 \text{ and } e.w_{2l-1} = 0. \end{aligned}$$

In the case of $G = O(V)$ let a basis for $W_l(m)$ be given by

$$B := \{v_{-2l+2}, \ldots, v_{-2}, v_0, v_2, \ldots, v_{2m-2l}\} \cup \{w_{-2m+2l}, \ldots, w_0, w_2, \ldots w_{2l-2}\}.$$

Here, the subspaces spanned by all v_i and by all w_j, respectively, are singular under the bilinear form Q. Furthermore, $Q(v_0) = 1$, $Q(v_i) = Q(w_j) = 0$ for all i and j. For $(\, , \,)$ we have $(v_i, w_{-i}) = 1$ and $(v_i, w_j) = 0$ if $i \neq -j$. The operation of the one-dimensional torus S and the nilpotent element e is given by

$$\begin{aligned} \gamma(c).v_i &= c^i v_i, \quad \text{for all } i, \\ \gamma(c).w_j &= c^j w_j, \quad \text{for all } j, \\ e.v_i &= v_{i+2}, \quad i < 2m - 2l \text{ and } e.v_{2m-2l} = 0, \\ e.w_j &= w_{j+2}, \quad j < 2l - 2 \text{ and } e.w_{2l-2} = 0. \end{aligned}$$

Let $G = \mathrm{SO}(V)$ with $\dim V = 2n$ and $p = 2$. Then we denote the G-class of a nilpotent element $e \in \mathfrak{g}$ where $V \downarrow e = W_l(n)$ by $D_n(a_{n-l})$. For $n = l$ we get the class D_n.
In type C_3 there is a distinguished class corresponding to $V(4) + V(4)$, see [20, Lemma 3.12 and Proposition 5.3]. This class is labelled by $C_3(a_1)$.

2.3.2 In exceptional groups

In exceptional groups the labels are given as in Table 2.3, which also states the representatives in the Lie algebra, see [20, Table 13.3, Table 14.1, and Table 16.2].

Notation

Before we state the orbit representatives, we give a few remarks about the notation we use. Let $\beta \in \Phi$ be a root and $e_\beta \in \mathfrak{g}$ be a nilpotent element such that $\langle e_\beta \rangle = \mathfrak{g}_\beta$ and the e_β form (together with a basis $\{h_i \mid i \in \{1, \ldots, |\Pi|\}\}$ of $\mathrm{Lie}(T)$) a Chevalley basis of $\mathfrak{g}$. We fix a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ and an ordering $(\alpha_1, \ldots, \alpha_r)$. We can write each $\beta \in \Phi$ as $\beta = \sum_{i=1}^{r} \lambda_i \alpha_i$, that is, as a linear combination of the α_i with coefficients $\lambda_i \in \mathbb{Z}$. Instead of e_β we then write $e_{1^{\lambda_1}, 2^{\lambda_2}, \ldots, r^{\lambda_r}}$. If $\lambda_j = 0$ for some $j \in \{1, \ldots, r\}$, we will omit the term j^{λ_j} in this list. For $\lambda_j = 1$ for some $j \in \{1, \ldots, r\}$ we will simply write j instead of j^1.
The order of the $\alpha_i \in \Pi$ is chosen as stated in the following diagrams.

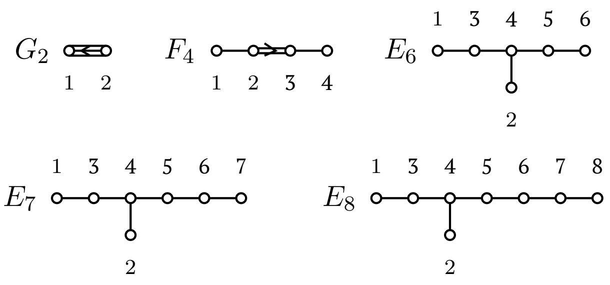


Figure 2.2: Numbering of the nodes in the exceptional Dynkin diagrams

We give a list of the exceptional nilpotent orbit representatives in the exceptional groups, taken from [20, Table 13.3, Table 14.1, and Table 16.2]. Also note that the representatives $\bar{e}$ in the exceptional classes $(L)_p$ are given by $\bar{e} = e + e_\alpha$ where $e \in L$ and $\alpha \in \Phi$, [20, Theorem 14.1 and text after]. In the following table the element e_α is given in brackets.

G	label of nilpotent orbit	orbit representative	T-labelling
G_2	$(\tilde{A}_1)_3$ $(p=3)$	$e_{1^2,2}+(e_{1^3,2^2})$	1 0
	G_2	e_1+e_2	2 2
	$G_2(a_1)$	$e_2+e_{1^3,2}$	0 2
F_4	$(\tilde{A}_1)_2$	$e_{1,2^2,3^3,4^2}+(e_{1^2,2^3,3^4,4^2})$	0 0 0 1
	$(\tilde{A}_2)_2$	$e_{2,3^2,4}+e_{1,2,3,4}+(e_{1^2,2^3,3^4,4^2})$	0 0 0 2
	$(B_2)_2$	$e_{2,3^2,4^2}+e_{1,2,3}+(e_{1,2^2,3^2})$	2 0 0 1
	$(C_3)_2$	$e_{1,2,3}+e_{2,3^2}+e_4+(e_{1,2^2,3^2,4^2})$ $(p=2)$	1 0 1 2
	$(C_3(a_1))_2$	$e_{1,2,3}+e_{2,3^2,4^2}+e_{2,3^2}+(e_{1,2^2,3^2,4^2})$ $(p=2)$	1 0 1 0
	$(\tilde{A}_2A_1)_2$	$e_{2,3,4}+e_{1,2,3^2,4}+e_{1,2^2,3^2}+(e_{2,3^2,4^2})$ $(p=2)$	0 1 0 1
	F_4	$e_1+e_2+e_3+e_4$	2 2 2 2
	$F_4(a_1)$	$e_1+e_2+e_{2,3}+e_{3,4}$	2 2 0 2
	$F_4(a_2)$	$e_{1,2}+e_{2,3^2}+e_4+e_{3,4}$	0 2 0 2
	$F_4(a_3)$	$e_2+e_{1,2}+e_{2,3^2}+e_{1,2,3^2,4^2}$	0 2 0 0
E_6	E_6	$e_1+e_2+e_3+e_4+e_5+e_6$	2 2 2 2 2 2
	$E_6(a_1)$	$e_1+e_3+e_{2,4}+e_{3,4}+e_5+e_6$	2 2 0 2 2 2
	$E_6(a_3)$	$e_1+e_{3,4}+e_{2,4}+e_{2,4,5}+e_{2,3,4,5}+e_{5,6}$	2 0 2 0 2 0
E_7	$(A_6)_2$	$e_{5,6}+e_{6,7}+e_{1,3,4}+e_{2,3,4}+e_{3,4,5}+e_{2,4,5}+(e_{1,2,3^2,4^2,5})$ $(p=2)$	0 0 2 0 2 0 0
	E_7	$e_1+e_2+e_3+e_4+e_5+e_6+e_7$	2 2 2 2 2 2 2
	$E_7(a_1)$	$e_1+e_2+e_{2,4}+e_{3,4}+e_5+e_6+e_7$	2 2 0 2 2 2 2

	$E_7(a_2)$	$e_1 + e_2 + e_3 + e_{2,4} + e_{4,5} + e_{5,6} + e_{6,7}$	2 2 0 2 0 2 2
	$E_7(a_3)$	$e_1 + e_{2,4} + e_{3,4} + e_{2,4,5} + e_{2,3,4,5} + e_{5,6} + e_7$	2 0 2 0 2 2 0
	$E_7(a_4)$	$e_1 + e_{2,3,4} + e_{3,4,5} + e_{2,4,5} + e_{4,5,6} + e_{3,4,5,6} + e_{6,7}$	2 0 2 0 0 2 0
	$E_7(a_5)$	$e_{1,3,4} + e_{2,3,4} + e_{1,3,4,5,6} + e_{2,4,5} + e_{4,5,6} + e_{5,6,7} + e_{3,4,5,6}$	0 0 2 0 0 2 0
E_8	$(D_7)_2$	$e_1 + e_{2,3,4} + e_{3,4,5} + e_{2,4,5} + e_{4,5,6} + e_{5,6,7} + e_{6,7,8} + (e_{1,2,3,4,5,6,7,8})$ $(p = 2)$	2 1 0 1 1 0 1 1
	$(D_7(a_1))_2$	$e_5 + e_{4,5} + e_{2,3,4^2,5,6,7} + e_{1,3} + e_{2,4,5,6} + e_{3,4,5,6} + e_{7,8} + (e_8)$ $(p = 2)$	2 0 0 2 0 0 2 0
	$(D_5A_2)_2$	$e_{1,2,3,4,5} + e_{2,3,4^2,5} + e_{1,3,4,5,6} + e_{2,3,4,5,6} + e_{3,4,5,6,7} + e_{2,4,5,6,7} + e_{7,8} + (e_{6,7,8})$ $(p = 2)$	0 0 0 2 0 0 2 0
	$(A_7)_3$	$e_{5,6,7} + e_{1,2,3,4} + e_{1,3,4,5} + e_{3,4,5,6} + e_{2,4,5,6} + e_{2,3,4^2,5} + e_{6,7,8} + (e_{4,5,6,7,8})$ $(p = 3)$	1 0 1 0 1 1 0 0
	E_8	$e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8$	2 2 2 2 2 2 2 2
	$E_8(a_1)$	$e_1 + e_2 + e_{2,4} + e_{3,4} + e_5 + e_6 + e_7 + e_8$	2 2 0 2 2 2 2 2
	$E_8(a_2)$	$e_1 + e_2 + e_3 + e_{2,4} + e_{4,5} + e_{5,6} + e_{6,7} + e_8$	2 2 0 2 0 2 2 2
	$E_8(a_3)$	$e_{1,3} + e_{2,4} + e_{3,4} + e_{4,5} + e_{3,4,5} + e_{5,6} + e_7 + e_8$	2 0 2 0 2 2 2 0
	$E_8(a_4)$	$e_{1,3} + e_{2,4} + e_{3,4} + e_{4,5} + e_{3,4,5} + e_{5,6} + e_{6,7} + e_{7,8}$	2 0 2 0 2 0 2 0

	$E_8(a_5)$	$e_{1,3}+e_{2,3,4}+e_{3,4,5}+e_{2,4,5}+e_{4,5,6}+e_{2,4,5,6}+e_{6,7}+e_{7,8}$	2 0 2 0 0 2 0 0
	$E_8(a_6)$	$e_{1,3,4}+e_{1,2,3,4}+e_{1,3,4,5}+e_{2,3,4,5}+e_{1,2,3,4,5,6}+e_{4,5,6}+e_{6,7}+e_{7,8}$	0 0 2 0 0 2 0 0
	$E_8(a_7)$	$e_{1,2,3,4,5}+e_{2,3,4,5,6}+e_{2,3,4^2,5,6}+e_{1,2,3,4^2,5,6}+e_{3,4,5,6,7}+e_{1,2,3,4^2,5,6,7}+e_{1,2,3^2,4^2,5,6,7}+e_{4,5,6,7,8}$	0 0 0 2 0 0 0 0
	$E_8(b_4)$	$e_{1,3}+e_{2,3,4}+e_{3,4,5}+e_{2,4,5}+e_{5,6,7}+e_{4,5,6}+e_7+e_8$	2 0 2 0 0 2 2 0
	$E_8(b_5)$	$e_{1,3,4}+e_{2,3,4}+e_{1,3,4,5}+e_{2,4,5}+e_{4,5,6}+e_{2,3,4,5,6}+e_{5,6,7}+e_8$	0 0 2 0 0 2 2 0
	$E_8(b_6)$	$e_{1,2,3,4}+e_{1,3,4,5}+e_{2,3,4,5}+e_{2,4,5,6}+e_{4,5,6,7}+e_{3,4,5,6}+e_{7,8}+e_{5,6,7,8}$ $(p \neq 2)$	0 0 2 0 0 0 2 0
		$e_{1,2,3,4}+e_{1,3,4,5}+e_{2,3,4,5}+e_{2,4,5,6}+e_{4,5,6,7}+e_{3,4,5,6}+e_{7,8}+e_{6,7,8}$ $(p=2)$	0 0 2 0 0 0 2 0

Table 2.3: Some nilpotent orbit representatives in exceptional groups

A table of every nilpotent orbit representatives for each exceptional group can be found in the Appendix A.1. These are also the representatives used in the programme described in Chapter 4.

3 | The nilpotent pieces

In the following chapter we give a first definition of the nilpotent pieces as stated in Lusztig's paper [23]. In order to do so, we define a set of maps $\mathscr{D}_G$ and see that the maps in $\mathscr{D}_G$ are in fact the weighted Dynkin diagrams in good characteristic. These maps give rise to a grading of $\mathfrak{g}$ and define certain subsets $\mathfrak{g}_2^{\delta!}$ for each map $\delta \in \mathscr{D}_G$. Provided these sets are known, we can compute the nilpotent pieces. In certain cases, it is possible to find the nilpotent piece linked to a weighted Dynkin diagram in a case-free way. These pieces are described at the end of this chapter.
Let $\mathbf{k}$ be an algebraically closed field with $\mathrm{char}(\mathbf{k}) = p$ prime or $\mathrm{char}(\mathbf{k}) = 0$ and let G be a connected reductive algebraic group over $\mathbf{k}$ with Lie algebra $\mathfrak{g}$. We fix a maximal torus $T \subseteq G$ and denote by $\Phi \subseteq X(T)$ the corresponding root system consisting of characters of T.

3.1 The set $\mathscr{D}_G$

Consider a homomorphism of algebraic groups $\delta : \mathbf{k}^\times \longrightarrow G$ mapping an element $c \in \mathbf{k}^\times$ to an element $t_c \in T \subseteq G$. Then $\delta(\mathbf{k}^\times) \subseteq T$ and we can apply the roots in Φ to elements in $\mathrm{im}(\delta)$.
Let $\alpha \in \Phi$, and consider $\alpha \circ \delta : \mathbf{k}^\times \longrightarrow \mathbf{k}^\times$. This is a homomorphism of algebraic groups and therefore there exists $n \in \mathbb{Z}$ such that $(\alpha \circ \delta)(c) = c^n$ for all $c \in \mathbf{k}^\times$, as stated in Chapter 2, (2.3). Let $\langle \alpha, \delta \rangle := n$. This defines a bilinear map on $\Phi \times \mathrm{Hom}(\mathbf{k}^\times, G)$. Recall from the previous chapter that we can define a linear map

$$\eta_\delta : \Phi \longrightarrow \mathbb{Z}, \quad \alpha \longmapsto \langle \alpha, \delta \rangle. \tag{3.1}$$

In the following section we are interested in a particular subset of these maps η_δ as above. We follow the construction of this subset as given in [23, 1.1].
Let G' be a connected reductive algebraic group defined over $\mathbb{C}$ of the same type as G, that is, G' has the same root datum but is defined over $\mathbb{C}$. There exists a bijection between the sets of orbits ${}_G\backslash \mathrm{Hom}(\mathbf{k}^\times, G)$ and ${}_{G'}\backslash \mathrm{Hom}(\mathbb{C}^\times, G')$.

Proposition 3.1. *Let G be a reductive connected algebraic group over the algebraically closed field $\mathbf{k}$ and let G' be a reductive connected group of the same type as G over $\mathbb{C}$. Both G and G' act on $\mathrm{Hom}(\mathbf{k}^\times, G)$, respectively $\mathrm{Hom}(\mathbb{C}^\times, G')$, via conjugation. There is a bijection between the set of orbits ${}_G\backslash \mathrm{Hom}(\mathbf{k}^\times, G)$ and ${}_{G'}\backslash \mathrm{Hom}(\mathbb{C}^\times, G')$.*

Proof. First note that for a fixed maximal torus $T \subseteq G$ the set $Y_G(T) := \mathrm{Hom}(\mathbf{k}^\times, T)$ is the cocharacter group. Similarly, let $T' \subseteq G'$ be a maximal torus and $Y_{G'}(T') := \mathrm{Hom}(\mathbb{C}^\times, T')$. Instead of $Y_G(T)$ and $Y_{G'}(T')$ we will also write Y_G and $Y_{G'}$. Both $N_G(T)$ and $N_{G'}(T')$ act on Y_G and $Y_{G'}$, respectively, by conjugation.

Let $W = \langle s_\alpha \mid \alpha \in \Phi\rangle$ be the Weyl group of G. As G' is of the same type as G, the group W is (up to isomorphism) also the Weyl group of G'. In this proof we will also write W' for the Weyl group of G'.
Now $W = {}^{N_G(T)}/_T$ acts on T by conjugation and therefore W also acts on Y_G, and by the same argument on $Y_{G'}$, as follows: Let $\gamma : \mathbf{k}^\times \to T$ be an element in Y_G. For any $w \in W$ we define the action of w on γ by $(w.\gamma)(c) := n_w\gamma(c)n_w^{-1}$ for $c \in \mathbf{k}^\times$ and $n_w \in N_G(T)$ a representative of w, see Remark 1.33. Note that the orbits of W on Y_G and of W' on $Y_{G'}$ are in natural bijection. We therefore construct a bijection between ${}_G\backslash^{\mathrm{Hom}(\mathbf{k}^\times, G)}$ and ${}_W\backslash^{Y_G}$ and by the same argument between ${}_{G'}\backslash^{\mathrm{Hom}(\mathbb{C}^\times, G')}$ and ${}_{W'}\backslash^{Y_{G'}}$. The bijection between ${}_G\backslash^{\mathrm{Hom}(\mathbf{k}^\times, G)}$ and ${}_{G'}\backslash^{\mathrm{Hom}(\mathbb{C}^\times, G')}$ follows from the existence of the above bijections.
The bijection between ${}_G\backslash^{\mathrm{Hom}(\mathbf{k}^\times, G)}$ and ${}_W\backslash^{Y_G}$ is given by

$$\begin{aligned} {}_G\backslash^{\mathrm{Hom}(\mathbf{k}^\times, G)} &\longleftrightarrow {}_W\backslash^{Y_G} \\ \mathscr{O}_\delta^G &\longmapsto \mathscr{O}_{\delta'}^W, \end{aligned} \tag{3.2}$$

where $\mathscr{O}_\delta^G$ denotes the orbit with a representative δ in ${}_G\backslash^{\mathrm{Hom}(\mathbf{k}^\times, G)}$, and $\mathscr{O}_{\delta'}^W$ is the orbit in ${}_W\backslash^{Y_G}$ with a representative δ' such that $\delta' \in \mathscr{O}_\delta^G$.
The image of $\mathbf{k}^\times$ under δ is a subgroup of some maximal torus $\tilde{T}$ in G. Thus, there exists $g \in G$ such that $\delta' := g\delta g^{-1}$ and the image of $\mathbf{k}^\times$ under δ' is in T, so we can in fact define this map.
As each element $\delta' \in Y_G$ is also contained in $\mathrm{Hom}(\mathbf{k}^\times, G)$, this map is clearly surjective. In order to see that the map (3.2) is well-defined, we follow the proof of [7, Proposition 3.7.1]. Let δ and δ' be in the same orbit in ${}_G\backslash^{\mathrm{Hom}(\mathbf{k}^\times, G)}$, that is, there exists $g \in G$ such that $g\delta(c)g^{-1} = \delta'(c)$ for all $c \in \mathbf{k}^\times$. Assume further that both $\mathrm{im}(\delta)$ and $\mathrm{im}(\delta')$ are subsets of the maximal torus T. We need to show that $\mathscr{O}_\delta^W = \mathscr{O}_{\delta'}^W$. Using the Bruhat decomposition and notation from Theorem 1.51, we write $g = u'tn_wu$. Then $g\delta(c) = \delta'(c)g$, so $u'tn_wu\delta(c) = \delta'(c)u'tn_wu$ and

$$\begin{aligned} u'tn_wu\delta(c) &= u'tn_w\delta(c)n_w^{-1}n_w\delta(c)^{-1}u\delta(c) \text{ and} \\ \delta'(c)u'tn_wu &= \delta'(c)u'\delta'(c)^{-1}\delta'(c)tn_wu \text{ so} \\ u'tn_w\delta(c)n_w^{-1}n_w\delta(c)^{-1}u\delta(c) &= \delta'(c)u'\delta'(c)^{-1}\delta'(c)tn_wu. \end{aligned}$$

Set $t_1 := tn_w\delta(c)n_w^{-1} \in T$, $u_1 := \delta(c)^{-1}u\delta(c) \in U$, $u_2 := \delta'(c)u'\delta'(c)^{-1} \in U$, and $t_2 := \delta'(c)t \in T$. Since T is abelian, it follows that $t_2 = \delta'(c)t = t\delta'(c)$. The above equation can therefore be written as

$$u't_1n_wu_1 = u_2t_2n_wu.$$

Since the Bruhat decomposition is unique, it follows that $u' = u_2$, $u = u_1$, and in particular $t_1 = t_2$, that is, $tn_w\delta(c)n_w^{-1} = t\delta'(c)$ and therefore $n_w\delta(c)n_w^{-1} = \delta'(c)$. This shows that δ and δ' are in the same orbit under the action of W.
Finally, we need to prove that this map is injective. Suppose there are two orbits $\mathscr{O}_\delta^G$ and $\mathscr{O}_{\tilde{\delta}}^G$ in $\mathrm{Hom}(\mathbf{k}^\times, G)$ such that their images agree, that is, $\mathscr{O}_{\delta'}^W = \mathscr{O}_{\tilde{\delta}'}^W$ for two representatives $\delta' \in \mathscr{O}_\delta^G$ and $\tilde{\delta}' \in \mathscr{O}_{\tilde{\delta}}^G$ such that $\delta'(\mathbf{k}^\times) \subseteq T$ and $\tilde{\delta}'(\mathbf{k}^\times) \subseteq T$. Then δ' and $\tilde{\delta}'$ are in the same W-orbit and therefore in the same G-orbit. As shown above, δ and δ', as well as $\tilde{\delta}$ and $\tilde{\delta}'$, are also in the same W-orbit. By transitivity it follows that $\mathscr{O}_\delta^G = \mathscr{O}_{\tilde{\delta}}^G$. □

Let

$$\mathscr{D}_{G'} := \left\{ f \in \operatorname{Hom}(\mathbb{C}^\times, G') \;\middle|\; \begin{array}{l} \text{there exists } h \in \operatorname{Hom}(\mathrm{SL}_2(\mathbb{C}), G') \text{ s.t.} \\ h\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) = f(a) \text{ for all } a \in \mathbb{C}^\times \end{array} \right\}.$$

We define a similar set $\mathscr{D}_G \subseteq Y_G$ for the group G in the following way: An element $\delta \in \operatorname{Hom}(\mathbf{k}^\times, G)$ is contained in $\mathscr{D}_G$ if we find a representative $\delta' \in \mathscr{D}_{G'}$ of the orbit in ${}_{G'}\backslash \operatorname{Hom}(\mathbb{C}^\times, G')$ corresponding to the orbit of δ in ${}_{G}\backslash \operatorname{Hom}(\mathbf{k}^\times, G)$.
We note that $\eta_\delta(\alpha) = \eta_{\delta'}(\alpha)$ for all roots $\alpha \in \Phi$ in this case for the right representatives δ, δ' of an orbit in ${}_{W}\backslash Y_G \simeq {}_{W'}\backslash Y_{G'}$.

Remark 3.2. Recall the construction of the weighted Dynkin diagrams in Section 2.1. We take a map γ as in (2.2). Choose a maximal torus $T \subseteq G$ such that $\operatorname{im}(\gamma) \subseteq T$ and $\Pi \subseteq \Phi$ such that $\eta_\gamma(\Pi) \subseteq \{0, 1, 2\}$, see Lemma 2.5.

1. From the definition of the weighted Dynkin diagrams and the map $\gamma : \mathbf{k}^\times \to G$ it is clear that each such map γ is contained in $\mathscr{D}_G$. Conversely, let $\delta \in \mathscr{D}_G$. Then δ defines a representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathfrak{g}' := \operatorname{Lie}(G')$ by

$$\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right).x = \operatorname{Ad}(\delta'(a))(x)$$

   for $a \in \mathbf{k}^\times$, $x \in \mathfrak{g}'$, and $\delta' \in \mathscr{D}_{G'}$ being the corresponding element to $\delta \in \mathscr{D}_G$ as defined above. Recall that $\varphi\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) = \delta'(a)$ and

$$\mathrm{SL}_2(\mathbb{C}) \xrightarrow{\varphi} G' \xrightarrow{\operatorname{Ad}} \mathrm{GL}(\mathfrak{g}').$$

   Note that this representation acts like one of the maps ρ_j in (2.1) on irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules in $\mathfrak{g}'$ by Theorem 2.3. In particular, δ acts on $\mathfrak{g}$ like a map arising from a nilpotent element $e \in \mathfrak{g}'$ (recall that the nilpotent elements can be embedded in a subspace isomorphic to $\mathfrak{sl}_2(\mathbb{C})$). This shows that δ gives rise to a weighted Dynkin diagram.

2. Let $\delta \in \mathscr{D}_G$ and let $\delta' \in \mathscr{D}_{G'}$ be the corresponding map. We can find a system of simple roots Π such that $\eta_{\delta'}(\alpha) \in \{0, 1, 2\}$ for all $\alpha \in \Pi$, see (1) and Lemma 2.5. We can therefore — up to conjugation — focus on the maps δ' such that $\eta_{\delta'}(\alpha) \in \{0, 1, 2\}$ for a fixed system of simple roots $\Pi \subseteq \Phi$. To see this, let $n_w \in N_{G'}(T')$ be a representative of $w \in W' = {}^{N_{G'}(T')}/_{T'}$. Then w acts on a root $\alpha \in \Phi$ by $w.\alpha(t) = \alpha(n_w^{-1} t n_w)$ for all $t \in T$. As W' acts transitively on the systems of simple roots, see [25, Theorem A.22], the claim follows.
   Now η_δ is conjugate to a map that acts on the root system Φ as $\eta_{\delta'}$ acts on the root system Φ. This means that each orbit in $\mathscr{D}_G$ contains a map $\delta \in \mathscr{D}_G$ such that $\eta_\delta(\alpha) \in \{0, 1, 2\}$ for a fixed system of simple roots $\Pi \subseteq \Phi$.

3. For the nilpotent element $e = 0$ we choose the map $\delta \in \mathscr{D}_G$ such that $\eta_\delta(\alpha) = 0$ for all $\alpha \in \Phi$.

Definition 3.3. Let γ be as in (2.2), the map corresponding to δ under the bijection in Section 3.1. We will call δ the map **arising** from the weighted Dynkin diagram of γ if $\eta_\delta(\alpha) = \eta_\gamma(\alpha)$ for all $\alpha \in \Pi$ (and hence all $\alpha \in \Phi$).
By abuse of notation we will also denote η_δ by δ for all $\delta \in \mathscr{D}_G$.

3.2 The sets $\mathfrak{g}_i^\delta,\ \mathfrak{g}_{\geqslant i}^\delta$, and $\mathfrak{g}_2^{\delta!}$

Following [23, Section 1], we will define certain subsets of the Lie algebra $\mathfrak{g}$ which will eventually lead us to the definition of the nilpotent pieces.
From now on, let the characteristic of $\mathbf{k}$ be arbitrary. Let $\delta \in \mathscr{D}_G$ with $\delta(\mathbf{k}^\times) \subseteq T$, and $i \in \mathbb{Z}$. We can define subspaces of the Lie algebra $\mathfrak{g}$ depending on the weighted Dynkin diagram corresponding to δ. These subspaces are crucial in the definition of the *nilpotent pieces*, whose union will prove to be the nilpotent variety.
As G is a connected reductive group, we have $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ where $\mathfrak{t} = \mathrm{Lie}(T)$ and the $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{Ad}(t)(x) = \alpha(t)x \text{ for all } t \in T\} = \mathrm{Lie}(U_\alpha)$ are the one-dimensional weight spaces of the roots, see Theorem 1.34.
We define for $i \in \mathbb{Z}$

$$\mathfrak{g}_i^\delta := \{x \in \mathfrak{g} \mid \mathrm{Ad}(\delta(a))(x) = a^i x \text{ for all } a \in \mathbf{k}^\times\}.$$

Note that these are just the sets defined in (2.4) in good characteristic. Clearly, we have $\mathfrak{g}_i^\delta = \bigoplus_{\substack{\alpha\in\Phi\\ \delta(\alpha)=i}} \mathfrak{g}_\alpha$ for all $i \in \mathbb{Z}\setminus\{0\}$. For two sets $\mathfrak{g}_i^\delta, \mathfrak{g}_j^\delta$ let $x \in \mathfrak{g}_i^\delta$ and $y \in \mathfrak{g}_j^\delta$. Then the Lie product of x and y is in $\mathfrak{g}_{i+j}^\delta$:

$$\mathrm{Ad}(\delta(a))([x,y]) = [\mathrm{Ad}(\delta(a))(x), \mathrm{Ad}(\delta(a))(y)] = [a^i x, a^j y] = a^{i+j}[x,y].$$

If $i = 0$, we have $\mathfrak{g}_0^\delta = \mathfrak{t} \oplus \bigoplus_{\substack{\alpha\in\Phi\\ \delta(\alpha)=0}} \mathfrak{g}_\alpha$. Note that for $i \neq 0$ the set $\mathfrak{g}_i^\delta$ is not a Lie algebra: We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for two roots $\alpha, \beta \in \Phi$ such that $\alpha+\beta \in \Phi$. In particular, $\delta(\alpha+\beta) = 2i$ if $\mathfrak{g}_\alpha, \mathfrak{g}_\beta \subseteq \mathfrak{g}_i^\delta$, as shown above.
Similarly, for $i \in \mathbb{Z}$ define the sets

$$\mathfrak{g}_{\geqslant i}^\delta := \bigoplus_{j\geqslant i} \mathfrak{g}_j^\delta.$$

By the same argument as above, we can see that for $i \geqslant 0$ the $\mathfrak{g}_{\geqslant i}^\delta$ are in fact Lie subalgebras of $\mathfrak{g}$.
We can define the corresponding subgroups $G_{\geqslant i}^\delta$ of G for $i \geqslant 0$, such that $\mathrm{Lie}(G_{\geqslant i}^\delta) = \mathfrak{g}_{\geqslant i}^\delta$ by

$$G_{\geqslant i}^\delta := \langle U_\alpha \mid \ \alpha \in \Phi, \langle\alpha,\delta\rangle \geqslant i\rangle \quad \text{if } i > 0,$$

and

$$G_{\geqslant 0}^\delta := \langle T, U_\alpha \mid \ \alpha \in \Phi, \langle\alpha,\delta\rangle \geqslant 0\rangle.$$

Again, note that we have defined this set already in (2.5). In particular, $G_{\geqslant 0}^\delta$ is a parabolic subgroup of G with the Levi subgroup $G_0^\delta = \langle T, U_\alpha \mid \ \alpha \in \Phi, \langle\alpha,\delta\rangle = 0\rangle$. This follows easily from Definition 1.59. If we fix the set $I := \{s_\alpha \mid \alpha \in \Pi, \langle\alpha,\delta\rangle \geqslant 0\}$, then $G_{\geqslant 0}^\delta = P_I$.
We have an equivalence relation on the set $\mathscr{D}_G$ where $\delta \simeq \delta'$ if $\mathfrak{g}_{\geqslant i}^\delta = \mathfrak{g}_{\geqslant i}^{\delta'}$ for all $i \in \mathbb{Z}$. Write

$$\triangle_\delta := \{\delta' \in \mathscr{D}_G \mid \mathfrak{g}_{\geqslant i}^\delta = \mathfrak{g}_{\geqslant i}^{\delta'} \text{ for all } i \in \mathbb{Z}\}$$

for the equivalence class of a map $\delta \in \mathscr{D}_G$.

Definition 3.4 ($\mathfrak{g}_2^{\delta !}$)**.** Let $x \in \mathfrak{g}$ be a nilpotent element. The subgroup $C_G(x) = \{g \in G \mid \mathrm{Ad}(g)(x) = x\}$ of G defines the stabiliser of x in G. For each $\delta \in \mathscr{D}_G$ we define the sets

$$\mathfrak{g}_2^{\delta !} := \{x \in \mathfrak{g}_2^{\delta} \mid C_G(x) \subseteq G_{\geqslant 0}^{\delta}\}.$$

Note that in general, $0 \notin \mathfrak{g}_2^{\delta !}$, so $\mathfrak{g}_2^{\delta !}$ is not a subspace of $\mathfrak{g}$.

From now on we may also write $\mathfrak{g}_i^{\triangle}$, $\mathfrak{g}_{\geqslant i}^{\triangle}$ instead of $\mathfrak{g}_i^{\delta}$, $\mathfrak{g}_{\geqslant i}^{\delta}$ where $\triangle = \triangle_\delta$.
We have the obvious isomorphism of vector spaces

$$\mathfrak{g}_2^{\triangle} \xrightarrow{\sim} \mathfrak{g}_{\geqslant 2}^{\triangle}/\mathfrak{g}_{\geqslant 3}^{\triangle}.$$

Let $\Sigma^{\triangle}$ be the image of $\mathfrak{g}_2^{\delta !}$ under this isomorphism. Furthermore, with the natural map

$$\pi : \mathfrak{g}_{\geqslant 2}^{\triangle} \longrightarrow \mathfrak{g}_{\geqslant 2}^{\triangle}/\mathfrak{g}_{\geqslant 3}^{\triangle}$$

we can define the set $\sigma^{\triangle} := \pi^{-1}(\Sigma^{\triangle})$. We let $\blacktriangle_\delta$ be the G-orbit of $\triangle_\delta$ via the conjugation action of G, that is, $\blacktriangle_\delta = \{\triangle_{\delta'} \mid g\delta'(c)g^{-1} = \delta(c) \text{ for all } c \in \mathbf{k}^{\times}\}$. The following definition is due to Lusztig, [23, A.6.], defining the central objects of this thesis, the *nilpotent pieces*.

Definition 3.5 (Nilpotent Pieces)**.** The sets

$$\mathscr{N}_{\mathfrak{g}}^{\blacktriangle_\delta} = \bigcup_{\triangle \in \blacktriangle_\delta} \sigma^{\triangle}$$

are the **nilpotent pieces** in $\mathfrak{g} = \mathrm{Lie}(G)$.

Remark 3.6. Note that each orbit $\blacktriangle$ contains all the maps $\delta \in \mathscr{D}_G$ parametrising exactly one nilpotent orbit.
Therefore, if $\delta \in \mathscr{D}_G$ arises from a weighted Dynkin diagram, we will also write $\mathscr{N}_{\mathfrak{g}}^{\delta}$.

In good characteristic it is relatively easy to see that each nilpotent piece $\mathscr{N}_{\mathfrak{g}}^{\blacktriangle_\delta}$ is just the nilpotent orbit corresponding to the weighted Dynkin diagram δ. We will prove this in Proposition 3.11. Furthermore, the nilpotent pieces are explicitly known for the classical groups in all characteristics and we will state them later in Section 3.4. However, we do not know the nilpotent pieces in bad characteristic for simple groups of exceptional type yet, i.e. for G of type G_2, F_4, E_6, or E_7 for $p = 2, 3$, and G of type E_8 for $p = 2, 3$ or 5.

For further computations we note that it is enough to assume that G is a semisimple adjoint group.

Proposition 3.7. *Let G be a connected reductive algebraic group. Then the nilpotent pieces of G are the same as those of the semisimple adjoint group of the same type.*

Proof. Let G_{ad} be an adjoint group of the same type as G. We first show that the nilpotent pieces in $\mathfrak{g}_{\mathrm{ad}} := \mathrm{Lie}(G_{\mathrm{ad}})$ are the same as in $\mathfrak{g}$.
There exists a central isogeny $\pi : G \to G_{\mathrm{ad}}$ by Theorem 1.72 with differential $d\pi : \mathfrak{g} \to \mathfrak{g}_{\mathrm{ad}}$. By Proposition 1.71 $d\pi$ is injective on each $\mathfrak{g}_\alpha := \mathrm{Lie}(U_\alpha)$ and by the same proposition $\mathfrak{g}_{\mathrm{ad}} = d\pi(\mathfrak{g}) + \mathfrak{t}'$ where $\mathfrak{t}' = \mathrm{Lie}(T')$, and $T' \subseteq G_{\mathrm{ad}}$ is a maximal torus in

G_{ad}.
Let $x \in \mathfrak{g}$ such that $x \in \mathfrak{g}_2^{\delta!}$ for some $\delta \in \mathscr{D}_G$. For each $g \in G$ we have

$$\begin{aligned}\mathrm{Ad}(\pi(g))(d\pi(x)) &= (d(\mathrm{Int}_{\pi(g)}) \circ d\pi)(x) = d(\mathrm{Int}_{\pi(g)} \circ \pi)(x)\\ &= d(\pi \circ \mathrm{Int}_g)(x) = d\pi(\mathrm{Ad}(g)(x)).\end{aligned}$$

Since $x \in \bigoplus_{\substack{\alpha\in\Phi \\ \delta(\alpha)=2}} \mathfrak{g}_\alpha$ is nilpotent, we have $\mathrm{Ad}(g)(x) = x$ if and only if $d\pi(\mathrm{Ad}(g)(x)) = d\pi(x)$, i.e. $\mathrm{Ad}(\pi(g))(d\pi(x)) = d\pi(x)$, and every nilpotent element in $\mathfrak{g}_{\mathrm{ad}}$ can be written as $d\pi(y)$ for some nilpotent element $y \in \mathfrak{g}$ by Proposition 1.71 (4). For $x \in \mathfrak{g}_2^{\delta!}$ we have $\mathrm{Ad}(g)(x) = x$ only for $g \in C_G(x) \subseteq G_{\geqslant 0}^\delta$. As π is an isogeny, π induces a bijection ρ between the root systems of G and G_{ad} where $\pi(U_\alpha) = U_{\rho(\alpha)}$ by Proposition 1.71. In particular, $\pi(g) \in (G_{\mathrm{ad}})_{\geqslant 0}^{\tilde{\delta}}$ with $\tilde{\delta}(\alpha) = \delta(\rho^{-1}(\alpha))$ and so $d\pi(x) \in \mathfrak{g}_2^{\tilde{\delta}!}$. As π restricted to U_α is an isomorphism onto its image, the claim is also true in the other direction.
We now show the reduction to semisimple groups. If G is not semisimple, we can write $G = [G,G]Z(G)^\circ$ and $[G,G]$ is semisimple, see Proposition 1.38. Thus, $G/Z(G)^\circ = [G,G]$. By [25, Theorem 7.9] it follows that $\mathrm{Lie}(G/Z(G)^\circ) \simeq \mathrm{Lie}(G)/\mathrm{Lie}(Z(G)^\circ)$. This means that for every $x \in \mathrm{Lie}(G)$ we have $x = x_1 + x_2$ for $x_1 \in \mathrm{Lie}([G,G])$ and $x_2 \in \mathrm{Lie}(Z(G)^\circ)$, so x_2 is semisimple. By [30, Section 4.4.19 and Theorem 4.4.20] every element in the Lie algebra can uniquely be written as the sum of a nilpotent and a semisimple element. Therefore, if $x \in \mathfrak{g}$ is nilpotent, we have $x \in \mathrm{Lie}([G,G]) =: \mathfrak{g}'$. We have $C_G(x) = C_{[G,G]}(x)Z(G)^\circ \subseteq G_{\geqslant 0}^\delta$ if and only if $C_{[G,G]}(x) \subseteq [G,G]_{\geqslant 0}^\delta$ for a weighted Dynkin diagram δ and therefore for all $\delta \in \mathscr{D}_G$. □

It is helpful to prove that each nilpotent piece consists of a union of nilpotent orbits. To verify this, let $x \in \mathscr{N}_\mathfrak{g}^\blacktriangle$. By Definition 3.5 we have $x \in \sigma^\triangle$ for a $\triangle \in \blacktriangle$. Furthermore, let $y = \mathrm{Ad}(g)(x)$ for some $g \in G$. Then $y \in \mathrm{Ad}(g)(\sigma^\triangle)$. We want to see that $\mathrm{Ad}(g)(\sigma^\triangle) \subseteq \mathscr{N}_\mathfrak{g}^\blacktriangle$. In order to do so, we write $\sigma^\triangle = \mathfrak{g}_2^{\delta!} \oplus \mathfrak{g}_{\geqslant 3}^\triangle$ for some $\delta \in \triangle$. This means that we have to consider $\mathrm{Ad}(g)(\mathfrak{g}_2^{\delta!})$ and $\mathrm{Ad}(g)(\mathfrak{g}_{\geqslant 3}^\triangle)$.

Lemma 3.8. *Let $g \in G$ and $i \in \mathbb{Z}$. Let $\delta \in \mathscr{D}_G$ and $\triangle := \triangle_\delta$. Then*

(i) $\mathrm{Ad}(g)(\mathfrak{g}_{\geqslant i}^\triangle) = \mathfrak{g}_{\geqslant i}^{g.\triangle}$ *and*

(ii) $\mathrm{Ad}(g)(\mathfrak{g}_2^{\delta!}) = \mathfrak{g}_2^{(g.\delta)!}$.

Proof. (i) As $\mathfrak{g}_{\geqslant i}^\triangle = \bigoplus_{j\geqslant i} \mathfrak{g}_j^\triangle$ for all $g \in G$, it is enough to show that $\mathrm{Ad}(g)(\mathfrak{g}_j^\triangle) = \mathfrak{g}_j^{g.\triangle}$. We have

$$\begin{aligned}\mathrm{Ad}(g)(\mathfrak{g}_j^\triangle) &= \mathrm{Ad}(g)(\mathfrak{g}_j^\delta)\\ &= \{\mathrm{Ad}(g)(x) \in \mathfrak{g} \mid \mathrm{Ad}(\delta(a))(x) = a^j x \text{ for all } a \in \mathbf{k}^\times\}\\ &= \{x \in \mathfrak{g} \mid \mathrm{Ad}(\delta(a)g^{-1})(x) = a^j \mathrm{Ad}(g^{-1})(x) \text{ for all } a \in \mathbf{k}^\times\}\\ &= \{x \in \mathfrak{g} \mid \mathrm{Ad}(g\delta(a)g^{-1})(x) = a^j x \text{ for all } a \in \mathbf{k}^\times\}\\ &= \mathfrak{g}_j^{g.\triangle},\end{aligned}$$

and therefore $\mathrm{Ad}(\mathfrak{g}^{\triangle}_{\geqslant i}) = \bigoplus_{j\geqslant i} \mathrm{Ad}(\mathfrak{g}^{\triangle}_{j}) = \bigoplus_{j\geqslant i} \mathfrak{g}^{g.\triangle}_{j} = \mathfrak{g}^{g.\triangle}_{\geqslant i}$.

(ii) Firstly, we have

$$\begin{aligned} C_G(\mathrm{Ad}(g)(x)) &= \{h \in G \mid \mathrm{Ad}(hg)(x) = \mathrm{Ad}(g)(x)\} \\ &= \{h \in G \mid \mathrm{Ad}(g^{-1}hg)(x) = x\} \\ &= \{ghg^{-1} \in G \mid \mathrm{Ad}(h)(x) = x\} \\ &= gC_G(x)g^{-1}. \end{aligned}$$

Recall from Section 3.2 that $G^{\triangle}_{\geqslant 0}$ is the well-defined parabolic subgroup of G such that $\mathrm{Lie}(G^{\triangle}_{\geqslant 0}) = \mathfrak{g}^{\triangle}_{\geqslant 0}$. Then $\mathrm{Lie}(G^{g.\triangle}_{\geqslant 0}) = \mathfrak{g}^{g.\triangle}_{\geqslant 0}$ and by (i) $\mathfrak{g}^{g.\triangle}_{\geqslant 0} = \mathrm{Ad}(g)(\mathfrak{g}^{\triangle}_{\geqslant 0})$.
It follows, that $\mathrm{Lie}(G^{g.\triangle}_{\geqslant 0}) = \mathrm{Ad}(g)(\mathrm{Lie}(G^{\triangle}_{\geqslant 0})) = \mathrm{Lie}(gG^{\triangle}_{\geqslant 0}g^{-1})$, so $G^{g.\triangle}_{\geqslant 0} = gG^{\triangle}_{\geqslant 0}g^{-1}$.
Finally, this shows that

$$\begin{aligned} \mathrm{Ad}(g)(\mathfrak{g}^{\delta !}_{2}) &= \{\mathrm{Ad}(g)(x) \mid x \in \mathfrak{g}^{\delta}_{2},\ C_G(x) \subseteq G^{\delta}_{\geqslant 0}\} \\ &= \{\mathrm{Ad}(g)(x) \mid x \in \mathfrak{g}^{\triangle}_{2},\ C_G(x) \subseteq G^{\triangle}_{\geqslant 0}\} \\ &= \{x \mid x \in \mathrm{Ad}(g)(\mathfrak{g}^{\triangle}_{2}),\ C_G(\mathrm{Ad}(g^{-1})(x)) \subseteq G^{\triangle}_{\geqslant 0}\} \\ &= \{x \mid x \in \mathfrak{g}^{g.\triangle}_{2},\ g^{-1}C_G(x)g \subseteq G^{\triangle}_{\geqslant 0}\} \\ &= \{x \in \mathfrak{g}^{g.\triangle}_{2} \mid C_G(x) \subseteq gG^{\triangle}_{\geqslant 0}g^{-1}\} \\ &= \{x \in \mathfrak{g}^{g.\triangle}_{2} \mid C_G(x) \subseteq G^{g.\triangle}_{\geqslant 0}\} \\ &= \mathfrak{g}^{(g.\delta)!}_{2}. \end{aligned}$$

□

Corollary 3.9. *If $x \in \mathfrak{g}$ is nilpotent and $\mathscr{O}_x$ is the G-orbit of x, then $x \in \mathscr{N}^{\blacktriangle}_{\mathfrak{g}}$ if and only if $\mathscr{O}_x \subseteq \mathscr{N}^{\blacktriangle}_{\mathfrak{g}}$.*
In order to compute the nilpotent pieces, it is therefore enough to check for each nilpotent orbit in $\mathfrak{g}$ whether a chosen representative of this orbit lies in a given piece.

Proof. Suppose $x \in \mathscr{N}^{\blacktriangle}_{\mathfrak{g}}$. Then $x \in \mathrm{Ad}(g)(\sigma^{\triangle})$ for some $g \in G$ and $\triangle \in \blacktriangle$. Let $y \in \mathscr{O}_x$, $y = \mathrm{Ad}(g')(x)$ for some $g' \in G$.
By (i) and (ii) in Lemma 3.8 we have $y \in \mathrm{Ad}(g'^{-1}g)(\sigma^{\triangle}) = \sigma^{g'^{-1}g.\triangle}$ and so $y \in \mathscr{N}^{\blacktriangle}_{\mathfrak{g}}$.
The converse statement is clearly true. □

Example 3.10 (The nilpotent pieces in $\mathrm{Sp}_4(\mathbf{k})$)**.** We want to compute the nilpotent pieces in $\mathrm{Sp}_4(\mathbf{k})$ for both $\mathrm{char}(\mathbf{k}) \neq 2$, that is good characteristic, and $\mathrm{char}(\mathbf{k}) = 2$.
The group $G = \mathrm{Sp}_4(\mathbf{k})$ is defined as

$$\mathrm{Sp}_4(\mathbf{k}) := \{A \in \mathrm{GL}_4(\mathbf{k}) \mid A^{tr} J_4 A = J_4\}, \text{ where } J_4 := \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & -1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{pmatrix}.$$

We recall the structures from Example 1.39. The maximal torus is given by

$$T := \left\{ \begin{pmatrix} t_1 & \cdot & \cdot & \cdot \\ \cdot & t_2 & \cdot & \cdot \\ \cdot & \cdot & t_2^{-1} & \cdot \\ \cdot & \cdot & \cdot & t_1^{-1} \end{pmatrix} \middle| \, t_1, t_2 \in \mathbf{k}^{\times} \right\}.$$

We choose a Borel subgroup containing T by taking $B := \mathrm{Sp}_4(\mathbf{k}) \cap B_4(\mathbf{k})$, where

$$B_4(\mathbf{k}) := \left\{ \begin{pmatrix} * & * & * & * \\ \cdot & * & * & * \\ \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_4(\mathbf{k}).$$

This is a Borel subgroup by [25, Example 6.7.(4)]. Furthermore, we have two homomorphisms of algebraic groups given by

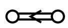
$$\alpha : T \longrightarrow \mathbf{k}^\times, \quad \begin{pmatrix} t_1 & \cdot & \cdot & \cdot \\ \cdot & t_2 & \cdot & \cdot \\ \cdot & \cdot & t_2^{-1} & \cdot \\ \cdot & \cdot & \cdot & t_1^{-1} \end{pmatrix} \longmapsto t_1 t_2^{-1},$$

$$\beta : T \longrightarrow \mathbf{k}^\times, \quad \begin{pmatrix} t_1 & \cdot & \cdot & \cdot \\ \cdot & t_2 & \cdot & \cdot \\ \cdot & \cdot & t_2^{-1} & \cdot \\ \cdot & \cdot & \cdot & t_1^{-1} \end{pmatrix} \longmapsto t_2^2.$$

One can easily see that α and β are indeed roots, see also Example 1.39, as there exist weight spaces which are non-zero and given by

$$\mathfrak{g}_\alpha = \left\{ \begin{pmatrix} \cdot & a & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -a \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| \, a \in \mathbf{k} \right\}, \qquad \mathfrak{g}_\beta = \left\{ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| \, a \in \mathbf{k} \right\}.$$

This results in the root system $\Phi := \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$ with simple roots $\Pi := \{\alpha, \beta\}$ and Dynkin diagram


α β .

This gives us the following root spaces

$$\mathfrak{g}_\alpha = \left\{ \begin{pmatrix} \cdot & a & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -a \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| \, a \in \mathbf{k} \right\}, \qquad \mathfrak{g}_\beta = \left\{ \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| \, a \in \mathbf{k} \right\},$$

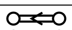
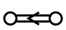
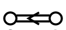
$$\mathfrak{g}_{\alpha+\beta} = \left\{ \begin{pmatrix} \cdot & \cdot & a & \cdot \\ \cdot & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| \, a \in \mathbf{k} \right\}, \qquad \mathfrak{g}_{2\alpha+\beta} = \left\{ \begin{pmatrix} \cdot & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \middle| \, a \in \mathbf{k} \right\},$$

and the root subgroups

$$U_\alpha = \left\{ \begin{pmatrix} 1 & a & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & -a \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \middle| \, a \in \mathbf{k} \right\}, \qquad U_\beta = \left\{ \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & a & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \middle| \, a \in \mathbf{k} \right\},$$

$$U_{\alpha+\beta} = \left\{ \begin{pmatrix} 1 & \cdot & a & \cdot \\ \cdot & 1 & \cdot & a \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \middle| \, a \in \mathbf{k} \right\}, \qquad U_{2\alpha+\beta} = \left\{ \begin{pmatrix} 1 & \cdot & \cdot & a \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \middle| \, a \in \mathbf{k} \right\}.$$

The group $G = \mathrm{Sp}_4(\mathbf{k})$ has four nilpotent orbits in good characteristic described by the following weighted Dynkin diagrams, see [7, Section 13.1]. We also give the sets $\mathfrak{g}_2^\delta$ and $\mathfrak{g}_2^{\delta!}$ which are easy to compute in this case. We let $\langle e_\gamma \rangle = \mathfrak{g}_\gamma$ for all roots $\gamma \in \Phi$.

weighted Dynkin diagram δ	nilpotent orbit representative	$\mathfrak{g}_2^\delta$	$\mathfrak{g}_2^{\delta!}$
 0 0	$x_1 := 0$	$\{0\}$	$\{0\}$
 1 0	$x_2 := \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\mathfrak{g}_{2\alpha+\beta}$	$\mathfrak{g}_{2\alpha+\beta} \setminus \{0\}$
 0 2	$x_3 := \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta}$	$\mathfrak{g}_2^\delta \setminus (\mathfrak{g}_{2\alpha+\beta} \cup \{a_1 e_\beta + a_1 a_2 e_{\alpha+\beta} + a_1 a_2^2 e_{2\alpha+\beta} \mid a_1, a_2 \in \mathbf{k}\})$
 2 2	$x_4 := \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\mathfrak{g}_\alpha + \mathfrak{g}_\beta$	$(\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta) \setminus (\mathfrak{g}_\alpha \cup \mathfrak{g}_\beta)$

In characteristic 2 we have the following orbit representatives, taken from [18, Table 7]:

$$\left.\begin{aligned} x_1&:=0\\ x_2&:=\begin{pmatrix} \cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot \end{pmatrix}\\ x_3&:=\begin{pmatrix} \cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot \end{pmatrix}\\ x_4&:=\begin{pmatrix} \cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&-1\\ \cdot&\cdot&\cdot&\cdot \end{pmatrix} \end{aligned}\right\}\text{also orbit representatives in good characteristic}$$

$$x_5 := \begin{pmatrix} \cdot&\cdot&\cdot&1\\ \cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

Note that for $A := \begin{pmatrix} -z&\frac{z}{2}&\cdot&\cdot\\ -1&-\frac{1}{2}&\cdot&\cdot\\ \cdot&\cdot&-1&-\frac{1}{2}\\ \cdot&\cdot&\frac{1}{z}&-\frac{1}{2z} \end{pmatrix} \in \mathrm{Sp}_4(\mathbf{k})$, $z^2 = -1$, and $\mathrm{char}(\mathbf{k}) \neq 2$ we have $Ax_3A^{-1} = x_5$, that is, x_3 and x_5 are in the same orbit if $\mathrm{char}(\mathbf{k}) \neq 2$.

The aim now is to figure out which orbit has representatives in the sets $\mathfrak{g}_2^{\delta!} \oplus \mathfrak{g}_{\geqslant 3}^{\delta}$ for all weighted Dynkin diagrams δ. Note that we have to check the sets $(\mathfrak{g}_2^{\delta!} \oplus \mathfrak{g}_{\geqslant 3}^{\delta}) \cap \mathscr{O}_x$ for each representative x, even if we have already found another orbit $\mathscr{O}_y$ such that $(\mathfrak{g}_2^{\delta!} \oplus \mathfrak{g}_{\geqslant 3}^{\delta}) \cap \mathscr{O}_y \neq \emptyset$. In most cases we can just check whether $\mathfrak{g}_2^{\delta!} \cap \mathscr{O}_x$ is non-empty, as $\mathfrak{g}_{\geqslant 3}^{\delta}$ is the empty set. In fact, $\mathfrak{g}_{\geqslant 3}^{\delta}$ is only non-empty for the weighted Dynkin diagram $\underset{2}{\circ}\Leftarrow\underset{2}{\circ}$.

It is easy to see that in both $\mathrm{char}(\mathbf{k}) = 2$ and $\mathrm{char}(\mathbf{k}) \neq 2$

- $x_1 \in \mathfrak{g}_2^{\delta!}$ for $\underset{0}{\circ}\Leftarrow\underset{0}{\circ}$
- $x_2 \in \mathfrak{g}_\beta$ and $\begin{pmatrix} \cdot&1&\cdot&\cdot\\ -1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&-1\\ \cdot&\cdot&1&\cdot \end{pmatrix} x_2 \begin{pmatrix} \cdot&-1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1\\ \cdot&\cdot&-1&\cdot \end{pmatrix} = \begin{pmatrix} \cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot \end{pmatrix} \in \mathfrak{g}_2^{\delta!}$ for $\underset{1}{\circ}\Leftarrow\underset{0}{\circ}$,
- $x_3 \in \mathfrak{g}_2^{\delta!}$ for $\underset{0}{\circ}\Leftarrow\underset{2}{\circ}$,
- $x_4 \in \mathfrak{g}_2^{\delta!}$ for $\underset{2}{\circ}\Leftarrow\underset{2}{\circ}$,
- $x_5 \in \mathfrak{g}_2^{\delta!}$ for $\underset{0}{\circ}\Leftarrow\underset{2}{\circ}$.

We now take a closer look at the sets $\mathfrak{g}_2^{\delta!}$ in arbitrary characteristic.

- For $\mathfrak{g}_{2\alpha+\beta} \setminus \{0\}$ every element is of the form $\begin{pmatrix} \cdot&\cdot&\cdot&a\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot \end{pmatrix}$ where $a \neq 0$. It is clear that in this case $\mathfrak{g}_2^{\delta!}$ only contains elements from one orbit.
- The set

  $$(\mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta}) \setminus (\mathfrak{g}_{2\alpha+\beta} \cup \{a_1e_\beta + a_1a_2e_{\alpha+\beta} + a_1a_2^2e_{2\alpha+\beta} \mid a_1, a_2 \in \mathbf{k}\})$$

  contains elements of the form $\begin{pmatrix} \cdot&\cdot&b&c\\ \cdot&\cdot&a&b\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot \end{pmatrix}$, where $a, b, c \in \mathbf{k}$ and $(a, b, c) \neq (d, de, de^2)$, $(a, b) \neq (0, 0)$ for elements $d, e \in \mathbf{k}$. In particular, $ca - b^2 \neq 0$.

Suppose first that $a = 0$. Then $b \neq 0$ and

$$C := \begin{pmatrix} b^{-1} & -cb^{-2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \in \mathrm{Sp}_4(\mathbf{k})$$

is such that $C\begin{pmatrix} \cdot & \cdot & b & c \\ \cdot & \cdot & \cdot & b \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}C^{-1} = x_3$.

Next let $b = 0$. Then $a \neq 0$ and $c \neq 0$. Let $t_1, t_2 \in \mathbf{k}$ be such that $t_1^2 = c^{-1}$ and $t_2^2 = a^{-1}$ (this is possible since $\mathbf{k}$ is algebraically closed). Then

$$C := \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \in \mathrm{Sp}_4(\mathbf{k})$$

is such that $C\begin{pmatrix} \cdot & \cdot & \cdot & c \\ \cdot & \cdot & a & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}C^{-1} = x_3$.

Now let $a, b \neq 0$ and $c = 0$. Let $z \in \mathbf{k}$ be such that $z^2 = -1$ and $x \in \mathbf{k}$ such that $x^2 = -a$. We choose

$$C := \begin{pmatrix} xb^{-1} & x^{-1} & 0 & 0 \\ 0 & zx^{-1} & 0 & 0 \\ 0 & 0 & -zx & zbx^{-1} \\ 0 & 0 & 0 & bx^{-1} \end{pmatrix}.$$

In this case we have $C\begin{pmatrix} \cdot & \cdot & b & \cdot \\ \cdot & \cdot & a & b \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}C^{-1} = x_5$.

Finally suppose that $a, b, c \neq 0$. Set $d := ca - b^2$ and choose $z \in \mathbf{k}$ such that $z^2 = c$ as well as $x \in \mathbf{k}$ such that $x^2 = (ac - b^2)c^{-1}$. Then

$$C := \begin{pmatrix} z^{-1} & 0 & 0 & 0 \\ -bd^{-1}x & xcd^{-1} & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & bz^{-1} & z \end{pmatrix} \in \mathrm{Sp}_4(\mathbf{k})$$

is such that $C\begin{pmatrix} \cdot & \cdot & b & c \\ \cdot & \cdot & a & b \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}C^{-1} = x_5$.

In this case we see that the only orbits included in $\mathfrak{g}_2^{\delta!}$ are those of x_3 and x_5. In the case of $\mathrm{char}(\mathbf{k}) \neq 2$ this is just one orbit.

- Let $\mathfrak{g}_2^{\delta!} = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta) \setminus (\mathfrak{g}_\alpha \cup \mathfrak{g}_\beta)$. Elements in $\mathfrak{g}_2^{\delta!}$ are of the form $\begin{pmatrix} \cdot & a & \cdot & \cdot \\ \cdot & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & -a \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ for $a, b \in \mathbf{k} \setminus \{0\}$. Let $t_2 \in \mathbf{k}$ be such that $t_2^2 = b^{-1}$ and define

$$C := \begin{pmatrix} t_2a^{-1} & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_2^{-1}a \end{pmatrix} \in \mathrm{Sp}_4(\mathbf{k}).$$

Then $C\begin{pmatrix} \cdot & a & \cdot & \cdot \\ \cdot & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & -a \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}C^{-1} = x_4$. However, note that in this case, unlike in the other cases, we need to actually check the orbits of the elements in $\mathfrak{g}_2^{\delta!} \oplus \mathfrak{g}_{\geqslant 3}^\delta$, as $\mathfrak{g}_{\geqslant 3}^\delta \neq \emptyset$. This means that we also have to check elements of the form $\begin{pmatrix} \cdot & a & c & d \\ \cdot & \cdot & b & c \\ \cdot & \cdot & \cdot & -a \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ for $a, b \in \mathbf{k} \setminus \{0\}$ and $c, d \in \mathbf{k}$.

Indeed, let $z \in \mathbf{k}$ be such that $z^2 = a^2 b$ and let $y \in \mathbf{k}$ such that $y^2 = -b(db - c^2)$. Then we find a matrix

$$C := \begin{pmatrix} z^{-1} & z^{-1}a^{-1}b^{-2}(-cab+yz) & 0 & 0 \\ 0 & az^{-1} & ya^{-1}b^{-1} & (dab^2-abc^2+zyc)(z^{-1}a^{-1}b^{-2}) \\ 0 & 0 & z^{-1}ab & (cab-yz)z^{-1}b^{-1} \\ 0 & 0 & 0 & z \end{pmatrix} \in \mathrm{Sp}_4(\mathbf{k})$$

such that $C\begin{pmatrix} . & a & c & d \\ . & . & b & c \\ . & . & . & -a \\ . & . & . & . \end{pmatrix}C^{-1} = x_4$.

We have successfully computed the nilpotent pieces in $\mathrm{Sp}_4(\mathbf{k})$ and give the result in the following table.

δ	$\mathcal{N}_{\mathfrak{g}}^{\blacktriangle\delta}$
0 0	$\{0\}$
1 0	$\mathcal{O}_{x_2}$
0 2	$\mathcal{O}_{x_3} \sqcup \mathcal{O}_{x_5}, p = 2$ $\mathcal{O}_{x_3} = \mathcal{O}_{x_5}, p \neq 2$
2 2	$\mathcal{O}_{x_4}$

In good characteristic the nilpotent orbits in $\mathfrak{g}$ correspond bijectively to the weighted Dynkin diagrams, see Proposition 2.9. It would therefore be reasonable to expect each nilpotent piece $\mathcal{N}_{\mathfrak{g}}^{\blacktriangle\delta}$ to be just the nilpotent orbit corresponding to the weighted Dynkin diagram δ. Indeed, this is true and it is possible to give a case-free proof of the fact. The results follows from [23, Section 1.2(a)].

Proposition 3.11 (The nilpotent pieces in good characteristic). *Let* $\mathrm{char}(\mathbf{k})$ *be good for the simple algebraic group* G*. Let* δ *be a weighted Dynkin diagram and* $\mathcal{O}_\delta$ *the corresponding nilpotent orbit in* $\mathfrak{g}$*. Then the nilpotent piece* $\mathcal{N}_{\mathfrak{g}}^{\blacktriangle\delta}$*, parametrised by* δ*, is given by* $\mathcal{O}_\delta$*.*

Proof. Let $\mathrm{char}(\mathbf{k})$ be good for G and $e \in \mathfrak{g}$ be an element in the nilpotent orbit corresponding to the weighted Dynkin diagram arising from a map $\delta \in \triangle_\delta \in \blacktriangle$. By the proof of Lemma 2.5 or Proposition 2.12 we have $\mathrm{Ad}(g)(e) \in \mathfrak{g}_2^\delta$ for a suitable element $g \in G$. As $\mathcal{N}_{\mathfrak{g}}^{\delta}$ is a union of nilpotent orbits, it is enough to assume that $e \in \mathfrak{g}_2^\delta$. Note that then the orbit of e under the action of $C_G(\delta(\mathbf{k}^\times)) = G_0^\delta$ is a subset of $\mathfrak{g}_2^\delta$. If $\mathcal{O}_e^\circ$ is this orbit, we directly get that $\mathcal{O}_e^\circ \subseteq \mathfrak{g}_2^{\delta!}$ by Proposition 2.10. For the other inclusion we follow [23, Section 1.2].

We want to show that any element $x \in \mathfrak{g}_2^{\delta!}$ lies in $\mathcal{O}_e^\circ$. It is enough to show that $\mathrm{Ad}(G_0^\delta)(x)$ is a dense subset of $\mathfrak{g}_2^\delta$. Then $\mathrm{Ad}(G_0^\delta)(x) = \mathcal{O}_e^\circ$ as there exists only one open dense orbit of G_0^δ in $\mathfrak{g}_2^\delta$ and $\mathcal{O}_e^\circ$ is also open and dense in $\mathfrak{g}_2^\delta$ by [7, Proposition 5.6.2]. Now $\mathrm{Ad}(G_0^\delta)(x)$ is a dense subset of $\mathfrak{g}_2^\delta$ if $\mathrm{ad} : \mathfrak{g}_0^\delta \to \mathfrak{g}_2^\delta, y \mapsto [x, y]$ is surjective. Then as $\mathrm{ad} = d\mathrm{Ad}$, the map defined by Ad is dominant. This is because the sets $\mathfrak{g}_i^\delta$ are algebraic sets as the sets of zeros of polynomial equations, and therefore we can apply Proposition 1.20.

In order to see this, let $\kappa(\, , \,)$ be the Killing form. In good characteristic the Killing form is non-degenerate. For $y \in \mathfrak{g}_{-2}^\delta$ note that $[x, y] \in \mathfrak{g}_0^\delta$. Then $0 = \kappa(y, [x, z])$ for all $z \in \mathfrak{g}_0^\delta$ means that $\kappa(z, [x, y]) = 0$ for all $z \in \mathfrak{g}_0^\delta$. Then we have by the non-degeneracy of κ that $[x, y] = 0$, so $y \in C_{\mathfrak{g}}(x) \subseteq \bigoplus_{i \geqslant 0} \mathfrak{g}_i^\delta$. This is because in good characteristic we

have $C_{\mathfrak{g}}(x) = \mathrm{Lie}(C_G(x))$ for all $x \in \mathfrak{g}$ by [17, Theorem 3.10]. As $y \in \mathfrak{g}^{\delta}_{-2} \cap \bigoplus_{i \geqslant 0} \mathfrak{g}^{\delta}_i$, it follows that $y = 0$. Thus, we have shown that $\kappa : \mathfrak{g}^{\delta}_{-2} \times [x, \mathfrak{g}^{\delta}_0] \to \mathbf{k}$ is non-degenerate. Now $[x, \mathfrak{g}^{\delta}_0]$ is a subspace of $\mathfrak{g}^{\delta}_2$ and $\kappa : \mathfrak{g}^{\delta}_{-2} \times \mathfrak{g}^{\delta}_2 \to \mathbf{k}$ is non-degenerate. Then it must follow that $[x, \mathfrak{g}^{\delta}_0] = \mathfrak{g}^{\delta}_2$. Consequently, $\mathrm{ad} : \mathfrak{g}^{\delta}_0 \to \mathfrak{g}^{\delta}_2$, $y \mapsto [x, y]$ is surjective and we are done.
This means that in good characteristic each piece $\mathscr{N}_{\mathfrak{g}}^{\blacktriangle}$ is given by the nilpotent orbit in G of an element corresponding to the weighted Dynkin diagram arising from $\delta \in \triangle_\delta \in \blacktriangle$. □

3.3 Alternative definition of nilpotent pieces

As remarked upon in the introduction, there exists an alternative definition of the nilpotent pieces, given in [9] and originally defined in [21] for the unipotent variety. We use the same notation as before. Additionally, let $\tilde{H}^{\blacktriangle}(\mathfrak{g}) = \bigcup_{\triangle \in \blacktriangle} \mathfrak{g}^{\triangle}_{\geqslant 2}$.

Definition 3.12 (Nilpotent CP-Pieces, [9, Section 7.1]). Let

$$H^{\blacktriangle}(\mathfrak{g}) := \tilde{H}^{\blacktriangle}(\mathfrak{g}) \setminus \bigcup_{\blacktriangle'} \tilde{H}^{\blacktriangle'}(\mathfrak{g})$$

where we take the union over all G-orbits $\blacktriangle'$ of the sets $\triangle_\delta := \{\delta' \in \mathscr{D}_G \mid \mathfrak{g}^{\delta}_{\geqslant i} = \mathfrak{g}^{\delta'}_{\geqslant i}$ for all $i \in \mathbb{Z}\}$ such that $\tilde{H}^{\blacktriangle'}(\mathfrak{g}) \subsetneq \tilde{H}^{\blacktriangle}(\mathfrak{g})$. Then the sets $H^{\blacktriangle}(\mathfrak{g})$ are the **nilpotent CP-pieces** in $\mathfrak{g}$.

One can show that CP-pieces are disjoint and form a partition of the nilpotent variety in $\mathfrak{g}$, see [9, Theorem 7]. It is in fact true that the nilpotent pieces defined by Clarke–Premet agree with the nilpotent pieces defined by Lusztig if G is of classical type. Note that the CP-pieces come from the stratification of the nullcone, defined by Hesselink in [13], see [9, Theorem 5]. In [33], Xue computes the nilpotent pieces in $\mathfrak{g}^*$, using the definition of Clarke–Premet. As it is not clear whether the nilpotent pieces as introduced by Lusztig agree with the CP-pieces, the nilpotent pieces still have to be computed in $\mathfrak{g}$. In [34] Xue describes the Springer correspondence for the types G_2 and F_4 and uses it to compute nilpotent orbit representatives in $\mathfrak{g}$. We cannot find the nilpotent pieces from $\mathfrak{g}^*$ under this correspondence, as they are computed by using the CP-definition.

Theorem 3.13 ([9, 7.3, Remark 1]). *If G is simple of classical type A, B, C, or D in any characteristic, we have $H^{\blacktriangle}(\mathfrak{g}) = \mathscr{N}_{\mathfrak{g}}^{\blacktriangle}$ for all orbits $\blacktriangle$.*

Again, this problem has not been solved for G of exceptional type in bad characteristic yet, but we hope for the nilpotent pieces and the CP-pieces to agree in all cases.

Remark 3.14. If the nilpotent variety is a disjoint union of the nilpotent pieces, they agree with the nilpotent CP-pieces, as shown in [9, Remark 7.2.1].

Finally, there is an equivalent way to define the nilpotent pieces as defined by Lusztig. This definition will prove helpful when computing the nilpotent pieces in exceptional groups.
Let $x \in \mathfrak{g}$ be a nilpotent element and $\delta \in \mathscr{D}_G$ correspond to a weighted Dynkin diagram. As we are interested in the set $\mathfrak{g}^{\delta}_2$ we will define the "parts" of an element $x \in \mathfrak{g}$ that lie in it.

Definition 3.15 ($\mathfrak{g}_i^\delta$-part)**.** Let $x \in \mathfrak{g}$ be a nilpotent element such that we can write $x = \sum_{\alpha\in\Phi} \lambda_\alpha e_\alpha$ where each e_α generates the subalgebra $\mathfrak{g}_\alpha$ and $\lambda_\alpha \in \mathbf{k}$ for $\alpha \in \Phi$. Then we define for $i \in \mathbb{Z}$ the element

$$[x]_{\mathfrak{g}_i^\delta} := \sum_{\delta(\alpha)=i} \lambda_\alpha e_\alpha$$

as the $\boldsymbol{\mathfrak{g}_i^\delta}$**-part** of x. In particular, we have $x = [x]_{\mathfrak{g}_i^\delta} + \sum_{\substack{\alpha\in\Phi \\ \delta(\alpha)\neq i}} \lambda_\alpha e_\alpha$ for all $i \in \mathbb{Z}$.

We can use this in order to give the alternative definition of the nilpotent pieces defined by Lusztig.

Definition 3.16. Let G be a reductive connected algebraic group. Furthermore, let

(i) $\delta : \Phi \longrightarrow \mathbb{Z}$ be a linear map describing a nilpotent orbit in good characteristic, that is, δ is a weighted Dynkin diagram,

(ii) R be a set of representatives of the nilpotent orbits in $\mathfrak{g}$, and

(iii) $R_\delta \subseteq R$ such that $x \in R_\delta$ if there exists an element $g \in G$ that satisfies

  (i) $\mathrm{Ad}(g)(x) \in \mathfrak{g}^\delta_{\geqslant 2}$ and

  (ii) $C_G([x]_{\mathfrak{g}_2^\delta}) \subseteq \langle T, U_\alpha \mid \delta(\alpha) \geqslant 0\rangle$.

Then $\mathscr{N}_\mathfrak{g}^\delta = \bigsqcup_{x\in R_\delta} \mathscr{O}_x$.

We prove that this definition does indeed agree with Definition 3.5.

Lemma 3.17. *The above definition and Definition 3.5 are equivalent.*

Proof. In order to see that Definition 3.16 is equivalent to Definition 3.5 of the nilpotent pieces, we will proceed as follows. First assume $x \in R_\delta$ for a fixed weighted Dynkin diagram δ. Then there exists $g \in G$ such that $\mathrm{Ad}(g)(x) = [\mathrm{Ad}(g)(x)]_{\mathfrak{g}_2^\delta} + y$ for some $y \in \mathfrak{g}^\delta_{\geqslant 3}$ and $[\mathrm{Ad}(g)(x)]_{\mathfrak{g}_2^\delta} \in \mathfrak{g}_2^{\delta !}$. Let

$$\pi : \mathfrak{g}^\triangle_{\geqslant 2} \longrightarrow \mathfrak{g}^\triangle_{\geqslant 2}/\mathfrak{g}^\triangle_{\geqslant 3}$$

be the natural epimorphism. Then by definition

$$\mathrm{Ad}(g)(x) \in \pi^{-1}([\mathrm{Ad}(g)(x)]_{\mathfrak{g}_2^\delta} + \mathfrak{g}^\delta_{\geqslant 3}) \subseteq \sigma^\delta$$

and so $x \in \mathscr{N}_\mathfrak{g}^\delta$ by Lusztig's definition.
Conversely, assume that $x \in \mathscr{N}_\mathfrak{g}^\delta$. Then $x \in \mathrm{Ad}(g)(\sigma^\delta)$ for some $g \in G$, that is $\mathrm{Ad}(g)(x) = [\mathrm{Ad}(g)(x)]_{\mathfrak{g}_2^\delta} + y$ for some $y \in \mathfrak{g}^\delta_{\geqslant 3}$ and $[\mathrm{Ad}(g)(x)]_{\mathfrak{g}_2^\delta} \in \mathfrak{g}_2^{\delta !}$. But then $x \in R_\delta$ if we choose x to be a nilpotent orbit representative of its orbit. This shows that both definitions are equivalent. $\square$

3.4 Nilpotent pieces in classical groups for characteristic 2

As in good characteristic, the nilpotent pieces for classical groups in bad characteristic, that is $\mathrm{char}(\mathbf{k}) = 2$, have already been computed by Lusztig in [23, 1.6-1.8]. It is worth noting that the nilpotent pieces are not given explicitly as a union of nilpotent orbits but in the form of the sets $\mathfrak{g}_2^{\delta !}$.
As there is no bad characteristic for simple groups of type A_n, we will focus on groups of type B_n, C_n, and D_n. Hence, it is enough to give the results for the groups $\mathrm{Sp}_n(\mathbf{k})$ and $\mathrm{SO}_n(\mathbf{k})$. Let V be an n-dimensional $\mathbf{k}$-vector space with $\mathrm{char}(\mathbf{k}) = 2$.
We follow Lusztig, [23, 1.4 and 1.5], in stating these results.

3.4.1 Sp$_n$(k)

Let n be even. Define the group $\mathrm{Sp}(V)$ by taking a non-degenerate symplectic form $(\,,\,) : V \times V \to \mathbf{k}$ and defining $\mathrm{Sp}(V) = \{A \in \mathrm{GL}(V) \mid A \text{ preserves } (\,,\,)\}$. The Lie algebra of $\mathrm{Sp}(V)$ is given by

$$\mathfrak{s}(V) := \{A \in \mathrm{End}(V) \mid (Av, v') + (v, Av') = 0 \text{ for all } v, v' \in V\}.$$

We say a $\mathbb{Z}$-grading of $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is ***s*-good** if $\dim V_i = \dim V_{-i} \geqslant \dim V_{-i-2}$ for all $i \geqslant 0$ and $\dim V_i$ is even if i is even. Furthermore, we want $(V_i, V_j) = 0$ for $i + j = 0$. Note that in this case we can define $\delta \in \mathscr{D}_G$ by the $\mathbb{Z}$-grading such that $\delta(a)|_{V_r} = a^r \mathrm{id}$ for all $a \in \mathbf{k}^\times$ and $r \in \mathbb{Z}$.
With respect to this grading we define

$$\mathrm{End}(V)_2 := \{A \in \mathfrak{g} \mid A(V_r) \subseteq V_{r+2}\},$$
$$\mathrm{End}(V)_2^\circ := \{A \in \mathfrak{g} \mid A(V_r) \subseteq V_{r+2},\ A^n : V_{-n} \to V_n \text{ is an isomorphism}\}.$$

Finally, set $\mathfrak{s}(V)_2 = \mathfrak{s}(V) \cap \mathrm{End}(V)_2$. Then $\mathfrak{g}_2^\delta = \mathfrak{s}(V)_2$ and $\mathfrak{g}_2^{\delta !} = \mathfrak{s}(V)_2 \cap \mathrm{End}(V)_2^\circ$ by [23, 1.4].

3.4.2 SO$_n$(k)

First of all, consider the group $O(V)$. For $O(V)$ we take a non-degenerate quadratic form $Q : V \to \mathbf{k}$ such that $(v, v') = Q(v + v') - Q(v) - Q(v')$ for all $v, v' \in V$ where $(\,,\,) : V \times V \to \mathbf{k}$ is the associated symmetric bilinear form. Let R be the radical of $(\,,\,)$. Then $O(V) = \{A \in \mathrm{GL}(V) \mid Q(A(v)) = Q(v) \text{ for all } v \in V\}$.
Now let $G = \mathrm{SO}(V)$ be the identity component of $O(V)$. The Lie algebra of G is given by $\mathfrak{g} = \mathfrak{o}(V) = \{A \in \mathrm{End}(V) \mid (Av, v) = 0 \text{ for all } v \in V,\ A|_R = 0\}$.
A $\mathbb{Z}$-grading $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is called ***o*-good** if:

- $\dim V_i = \dim V_{-i} \geqslant \dim V_{-i-2}$ for all $i \geqslant 0$,
- $\dim V_i$ is even for any odd i,
- $(V_i, V_j) = 0$ when $i + j \neq 0$, and
- $Q|_{V_i} = 0$ for $i \neq 0$.

As before, define sets

$$\begin{aligned} \mathrm{End}(V)_2 &:= \{A \in \mathfrak{g} \mid A(V_r) \subseteq V_{r+2}\}, \\ \mathfrak{o}(V)_2 &:= \mathfrak{o}(V) \cap \mathrm{End}(V)_2, \\ \mathfrak{o}(V)_2^\circ &:= \left\{ A \in \mathfrak{o}(V)_2 \;\middle|\; \begin{array}{l} A^{\frac{n}{2}} : V_{-n} \to V_0 \text{ is injective and} \\ Q|_{A^{\frac{n}{2}}(V_{-n})} \text{ is non-degenerate if } n \text{ is even,} \\ A^n : V_{-n} \to V_n \text{ is an isomorphism if } n \text{ is odd} \end{array} \right\}. \end{aligned}$$

Then $\mathfrak{g}_2^{\delta !} = \mathfrak{o}(V)_2^\circ$ by [23, 1.5].
The previous results lead to the following theorem on the nilpotent pieces in classical groups.

Theorem 3.18 ([23, A.6]). *The nilpotent pieces $\mathcal{N}_\mathfrak{g}^\blacktriangle$ form a partition of the nilpotent variety $\mathcal{N}_\mathfrak{g}$ of $\mathfrak{g}$ if G is simple of classical type $A, B, C,$ or D in any characteristic.*

3.5 Some special pieces

We are now in the position to describe a few nilpotent pieces. Every nilpotent piece not described in this section will have to be computed case-by-case.

3.5.1 The diagonal cases

Let $e \in \mathfrak{g}$ be a nilpotent orbit representative, where $\mathfrak{g} := \mathrm{Lie}(G)$ for a group G of exceptional type. Then e is either in an **exceptional orbit**, that is, an orbit which only occurs in bad characteristic, or e is in a non-exceptional orbit, see [20, Theorem 9.1 and Tables 22.1.1-22.1.5]. Now each orbit gives rise to a so-called T-labelling of the Dynkin diagram which in good characteristic is the weighted Dynkin diagram δ_e, see Definition 2.11. We will refer to the cases in which we check whether $e \in \mathcal{N}_\mathfrak{g}^{\blacktriangle \delta_e}$ as the **diagonal cases**.

Lemma 3.19. *Let $e \in \mathfrak{g}$ be a representative of a nilpotent orbit, where $\mathfrak{g}$ is the Lie algebra of a group G of exceptional type. Let δ_e be the T-labelling for the orbit of e, as noted above. Then $e \in \mathcal{N}_\mathfrak{g}^{\blacktriangle \delta_e}$.*

Proof. First assume e is in a non-exceptional orbit. By Proposition 2.12 we have $e \in \mathfrak{g}_2^{\delta_e}$. Furthermore, the same Proposition 2.12 shows that $C_G(e) \subseteq G_{\geqslant 0}^{\delta_e}$. But then it follows automatically that $e \in \mathfrak{g}_2^{\delta !}$ by definition, so $e \in \mathcal{N}_\mathfrak{g}^{\blacktriangle \delta_e}$ as claimed.
Now suppose e is the representative of an exceptional orbit. Then we can write $e = \bar{e} + e_\alpha$ for some root $\alpha \in \Phi^+$, see [20, Chapter 14, text after Theorem 14.1]. Here $\bar{e}$ is the nilpotent representative of the non-exceptional orbit with the same labelling δ_e. Considering the representatives in Table 2.3, we see that there are two options. If the representative e is in $\mathfrak{g}_2^{\delta_e}$, then $C_G(e) \subseteq G_{\geqslant 0}^{\delta_e}$ and therefore $e \in \mathfrak{g}_2^{\delta_e !}$ by [20, Theorem 14.1]. Otherwise, $e = \bar{e} + e_\alpha$ where $[e]_{\mathfrak{g}_2^{\delta_e}} = \bar{e}$ as we can see by checking all cases in Table 2.3. As $\bar{e}$ is a standard distinguished element with $\bar{e} \in \mathfrak{g}_2^{\delta_e}$, it follows that $C_G(\bar{e}) \subseteq G_{\geqslant 0}^{\delta_e}$ by [20, Lemma 2.26]. This means that $[e]_{\mathfrak{g}_2^{\delta_e}} = \bar{e} \in \mathfrak{g}_2^{\delta_e !}$. In both cases we have $e \in \mathcal{N}_\mathfrak{g}^{\blacktriangle \delta_e}$. □

Remark 3.20. This shows in particular that

$$\mathscr{N}_{\mathfrak{g}} = \bigcup_{\delta} \mathscr{N}_{\mathfrak{g}}^{\delta}$$

where δ runs over all weighted Dynkin diagrams. Note that it does not follow that the nilpotent pieces are disjoint.

3.5.2 The regular piece

Let δ be the map corresponding to the weighted Dynkin diagram with weight 2 for every simple root. This is known to always parametrise a nilpotent orbit, see [7, Chapter 13.1]. We will call this the **regular diagram** and the corresponding piece the **regular piece**. In this case, we have $\mathfrak{g}^{\triangle}_{\geqslant 2} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ and $\mathfrak{g}^{\triangle}_2 = \bigoplus_{\alpha\in\Pi} \mathfrak{g}_\alpha$.

Proposition 3.21 (The regular piece). *Let $\mathscr{N}_{\mathfrak{g}}^{\blacktriangle}$ be the regular piece. Then $\mathscr{N}_{\mathfrak{g}}^{\blacktriangle} = \mathscr{O}_x$, the nilpotent orbit of $x = \sum_{\alpha\in\Pi} e_\alpha$ with $0 \neq e_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in \Pi$.*

Proof. Let $\delta \in\triangle\in \blacktriangle$ correspond to the regular diagram and suppose that $y \in \mathfrak{g}$ is a nilpotent element such that $y = \sum_{\beta\in\Pi} \lambda_\beta e_\beta \in \mathfrak{g}^{\delta}_2$ with $\lambda_\alpha = 0$ for some $\alpha \in \Pi$. We fix this α for the rest of the proof. For $\gamma, \beta \in \Phi$ let $p_{\beta,\gamma}, q_{\beta,\gamma} \in \mathbb{N}$ be so that

$$\begin{aligned}
&\beta + p_{\beta,\gamma}\gamma \in \Phi && \text{and} && \beta + (p_{\beta,\gamma} + 1)\gamma \notin \Phi,\\
&\beta - q_{\beta,\gamma}\gamma \in \Phi && \text{and} && \beta - (q_{\beta,\gamma} + 1)\gamma \notin \Phi,
\end{aligned}$$

see [4, VI, §1, no. 1.3, Proposition 9] and (1.1). For $\gamma, \beta \in \Pi$ and $\beta \neq \gamma$ we have $q_{\beta,\gamma} = 0$.
Let $t \in T$ and $\alpha \in \Pi$ fixed as above, such that $\beta(t) = (-1)^{p_{\beta,\alpha}}$ for all $\beta \in \Pi \setminus \{\alpha\}$ and $\alpha(t) = -1$. Then set $g := u_\alpha(1)tn_{s_\alpha}u_\alpha(-1)$ by the Bruhat decomposition 1.51. This choice is possible by Dedekind's theorem ([19, Chapter VIII, §4]) and from [15, Lemma 16.2 C]. As $n_{s_\alpha} = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$ and $u_{-\alpha}(-1) \notin G^{\delta}_{\geqslant 0}$ we have $n_{s_\alpha} \notin G^{\delta}_{\geqslant 0}$. However, $u_\alpha(1)$, t, and $u_\alpha(-1)$ are elements in $G^{\delta}_{\geqslant 0}$, so $g \notin G^{\delta}_{\geqslant 0}$. We want to show that $\mathrm{Ad}(g)(y) = y$.
To compute the action of Ad on $\mathfrak{g}$ we use the formulas from Section 1.2.5.
$\mathrm{Ad}(u_\alpha(-1))(y)$: Recall that $q_{\beta,\alpha} = 0$, so $\binom{k+q_{\beta,\alpha}}{k} = \binom{k}{k} = 1$. We have

$$\begin{aligned}
\mathrm{Ad}\left(u_\alpha(-1)\right)(y) &= \mathrm{Ad}(u_\alpha(-1))(\sum_{\beta\in\Pi} \lambda_\beta e_\beta)\\
&= \sum_{\beta\in\Pi} \lambda_\beta \mathrm{Ad}(u_\alpha(-1))(e_\beta)\\
&= \sum_{\beta\in\Pi} \lambda_\beta \sum_{\substack{k\geqslant 0\\ \beta+k\alpha\in\Phi}} \binom{k+q_{\beta,\alpha}}{k} (-1)^k e_{\beta+k\alpha}\\
&= \sum_{\beta\in\Pi} \lambda_\beta \sum_{p_{\beta,\alpha}\geqslant k\geqslant 0} (-1)^k e_{\beta+k\alpha} =: y'
\end{aligned}$$

by Section 1.2.5

$\mathrm{Ad}(n_{s_\alpha})(y')$: We have $s_\alpha(\beta) = \beta - (q_{\beta,\alpha} - p_{\beta,\alpha})\alpha = \beta + p_{\beta,\alpha}\alpha$ for $\alpha, \beta \in \Pi$, see Section 1.2.5. Then

$$\begin{aligned}\mathrm{Ad}(n_{s_\alpha})(y') &= \mathrm{Ad}(n_{s_\alpha})\big(\sum_{\substack{\beta\in\Pi,\\ p_{\beta,\alpha}\geqslant k\geqslant 0}} \lambda_\beta(-1)^k e_{\beta+k\alpha}\big)\\ &= \sum_{\substack{\beta\in\Pi,\\ p_{\beta,\alpha}\geqslant k\geqslant 0}} \lambda_\beta(-1)^k \mathrm{Ad}(n_{s_\alpha})(e_{\beta+k\alpha})\\ &= \sum_{\substack{\beta\in\Pi,\\ p_{\beta,\alpha}\geqslant k\geqslant 0}} \lambda_\beta(-1)^k(-1)^{p_{\beta,\alpha}-k} e_{\beta+(p_{\beta,\alpha}-k)\alpha}\\ &= \sum_{\substack{\beta\in\Pi,\\ p_{\beta,\alpha}\geqslant k\geqslant 0}} \lambda_\beta(-1)^{p_{\beta,\alpha}} e_{\beta+(p_{\beta,\alpha}-k)\alpha} =: y''\end{aligned}$$

by Section 1.2.5 (2).
$\mathrm{Ad}(t)(y'')$: An element $t' \in T$ acts on the elements $e_\alpha \in \mathfrak{g}$ by $\mathrm{Ad}(t')(e_\alpha) = \alpha(t')e_\alpha$. It follows that with the above choice for t we have

$$\begin{aligned}\mathrm{Ad}(t)(y'') &= \mathrm{Ad}(t)\big(\sum_{\substack{\beta\in\Pi,\\ p_{\beta,\alpha}\geqslant k\geqslant 0}} \lambda_\beta(-1)^{p_{\beta,\alpha}} e_{\beta+(p_{\beta,\alpha}-k)\alpha}\big)\\ &= \sum_{\substack{\beta\in\Pi,\\ p_{\beta,\alpha}\geqslant k\geqslant 0}} \lambda_\beta(-1)^{p_{\beta,\alpha}} \mathrm{Ad}(t)(e_{\beta+(p_{\beta,\alpha}-k)\alpha})\\ &= \sum_{\substack{\beta\in\Pi,\\ p_{\beta,\alpha}\geqslant k\geqslant 0}} \lambda_\beta(-1)^{p_{\beta,\alpha}} \beta(t)\alpha(t)^{p_{\beta,\alpha}-k} e_{\beta+(p_{\beta,\alpha}-k)\alpha}\\ &= \sum_{\substack{\beta\in\Pi,\\ p_{\beta,\alpha}\geqslant k\geqslant 0}} \lambda_\beta(-1)^{p_{\beta,\alpha}-k} e_{\beta+(p_{\beta,\alpha}-k)\alpha} =: y'''.\end{aligned}$$

Now

$$\sum_{p_{\beta,\alpha}\geqslant k\geqslant 0} (-1)^k e_{\beta+k\alpha} = \sum_{p_{\beta,\alpha}\geqslant k\geqslant 0} (-1)^{p_{\beta,\alpha}-k} e_{\beta+(p_{\beta,\alpha}-k)\alpha},$$

therefore $y''' = y'$ by our choice for t.
$\mathrm{Ad}(u_\alpha(1))(y''')$: Finally,

$$\mathrm{Ad}(u_\alpha(1))(y''') = \mathrm{Ad}(u_\alpha(1))(y') = \mathrm{Ad}(u_\alpha(1))(\mathrm{Ad}(u_\alpha(-1))(y)) = y.$$

This shows that $g \in C_G(y)$, so $y \notin \mathfrak{g}_2^{\delta!}$.

Conversely, let $x = \sum_{\gamma\in\Pi} e_\gamma$ as above and $g \in G \setminus G^\delta_{\geqslant 0}$. We can write $g = u'tn_w u$ by the Bruhat decomposition 1.51.
Let $w \neq 1$. Then, as $x \in \mathfrak{g}_2^\delta$, it follows that

$$\begin{aligned}x' &:= \mathrm{Ad}(u)(x) \in \mathfrak{g}^\delta_{\geqslant 2} \quad \text{and}\\ x'' &:= \mathrm{Ad}(n_w)(x') \in \mathfrak{g}^\delta_{\geqslant 2} \oplus \bigoplus_{\beta\in\Phi^-} \mathfrak{g}_\beta\end{aligned}$$

as there is at least one $\gamma \in \Phi^+$ with $\lambda_\gamma \neq 0$ such that $w.\gamma \in \Phi^-$ for $x' = \sum_{\beta \in \Phi^+} \lambda_\beta e_\beta$, $\langle e_\beta \rangle = \mathfrak{g}_\beta$, and $\lambda_\beta \in \mathbf{k}$ (otherwise $w.\Phi^+ \subseteq \Phi^+$ which is only possible for $w = \mathrm{id}$, see for example [25, Theorem A.22]). In particular, $\mathrm{Ad}(n_w)(x') \notin \mathfrak{g}^\delta_{\geqslant 2}$. But then $\mathrm{Ad}(u't)(x'') \notin \mathfrak{g}^\delta_2$ and so there is no element $g \in G \setminus G^\delta_{\geqslant 0}$ that centralises x.
As we can find $t \in T$ such that $\lambda_\beta = \beta(t)$ for all $\beta \in \Pi$ and $\lambda_\beta \in \mathbf{k}^\times$ (again by [19, Chapter VIII, §4] and [15, Lemma 16.2 C]), it follows that

$$\mathfrak{g}^{\delta !}_2 = \{\sum_{\beta \in \Pi} \lambda_\beta e_\beta \mid \lambda_\beta \neq 0 \text{ for all } \beta \in \Pi\}.$$

This proves the claim. □

3.5.3 The highest root piece

Lemma 3.22. *Let $\gamma \in \Phi$ be the highest root. By checking the weighted Dynkin diagrams for G of exceptional type, one can see that there exists a diagram δ_γ such that $\mathfrak{g}^{\delta_\gamma}_2 = \mathfrak{g}_\gamma$. We will call this the **highest root diagram** and the corresponding piece the **highest root piece**. In this case, $\mathscr{N}^{\delta_\gamma}_{\mathfrak{g}} = \mathscr{O}_\gamma$.*

Proof. The highest root diagram corresponds to a non-exceptional orbit and therefore we have by Lemma 3.19 that $e_\gamma \in \mathscr{N}^{\delta_\gamma}_{\mathfrak{g}}$. In fact, since $\mathfrak{g}^{\delta_\gamma}_2 = \mathfrak{g}_\gamma$ this proves that $\mathfrak{g}^{\delta_\gamma !}_2 = \mathfrak{g}_\gamma \setminus \{0\}$ and therefore $\mathscr{N}^{\delta_\gamma}_{\mathfrak{g}} = \mathscr{O}_\gamma$, the orbit with representative e_γ.

□

4 | A computational approach

This chapter focuses on finding the nilpotent pieces by using computational methods. In order to compute the nilpotent pieces as defined in Definition 3.5, we will first present a few results on the action of G on its Lie algebra and in particular on the sets $\mathfrak{g}_2^{\delta !}$. Having computed the nilpotent pieces in the Lie algebras of exceptional type, it should be within reach to prove that Lusztig's nilpotent pieces agree with the CP–pieces as noted in Remark 3.14.
As before, G will denote a connected reductive algebraic group over the algebraically closed field k and $\mathfrak{g}$ is the Lie algebra of G. We will write δ for a weighted Dynkin diagram where $\delta \in \triangle_\delta \in \blacktriangle_\delta$ with notation as in Section 3.

4.1 Computing the action of Ad

As mentioned in section 1.2.5 we can compute the action of G via Ad on $\mathfrak{g}$ up to sign by following [11]. We will consider the actions of a unipotent element in B, an element of the torus and a Weyl group representative respectively. As the action of Ad on $\mathfrak{g}$ is linear, it is enough to examine this action on the elements e_α where $\langle e_\alpha \rangle = \mathfrak{g}_\alpha$ for $\alpha \in \Phi$ since every nilpotent element has a conjugate which is a linear combination of the elements e_α.

4.2 Practical aspects

Our aim is to develop an algorithm to compute the nilpotent pieces. In order to do so, we would like to decide whether a nilpotent orbit representative $x \in \mathfrak{g}$ is contained in a nilpotent piece $\mathscr{N}_{\mathfrak{g}}^{\blacktriangle}$ for a given orbit $\blacktriangle$. This can be done by checking two things, as we have already seen in Definition 3.16.

1. Check whether there exists some element $g \in G$ such that $\mathrm{Ad}(g)(x) \in \mathfrak{g}_{\geqslant 2}^{\delta}$ for $\delta \in \triangle \in \blacktriangle$. If not, then x cannot lie in $\mathscr{N}_{\mathfrak{g}}^{\blacktriangle}$.

2. If the condition in (1) is fulfilled, we need to take a closer look at the $\mathfrak{g}_2^{\delta}$-part of $\mathrm{Ad}(g)(x)$. By Definition 3.15 we have

$$\mathrm{Ad}(g)(x) = [\mathrm{Ad}(g)(x)]_{\mathfrak{g}_2^{\delta}} + \sum_{i \geqslant 3} [\mathrm{Ad}(g)(x)]_{\mathfrak{g}_i^{\delta}},$$

   and so $\mathrm{Ad}(g)(x) \in \sigma^{\delta} \subseteq \mathscr{N}_{\mathfrak{g}}^{\blacktriangle}$ if $[\mathrm{Ad}(g)(x)]_{\mathfrak{g}_2^{\delta}} \in \mathfrak{g}_2^{\delta !}$. Conversely, suppose that there is no $g \in G$ such that $[\mathrm{Ad}(g)(x)]_{\mathfrak{g}_2^{\delta}} \in \mathfrak{g}_2^{\delta !}$. Then $x \notin \mathscr{N}_{\mathfrak{g}}^{\blacktriangle}$ as otherwise we would have $x \in \sigma^{h.\delta}$ for some $h \in G$, $h.\delta \in h.\triangle \in \blacktriangle$ and by Lemma 3.8 we would have

$\mathrm{Ad}(h^{-1})(x) \in \sigma^\delta$. In particular, $[\mathrm{Ad}(h^{-1})(x)]_{\mathfrak{g}_2^\delta} \in \mathfrak{g}_2^{\delta!}$ which is a contradiction. We can therefore concentrate on deciding whether $[\mathrm{Ad}(g)(x)]_{\mathfrak{g}_2^\delta} \in \mathfrak{g}_2^{\delta!}$.

As G is a connected reductive group, every element $g \in G$ can be written uniquely as $g = u'tn_w u$ by the Bruhat decomposition, see Definition 1.51. Furthermore, every element $u \in \prod_{\alpha\in\Phi^+} U_\alpha$ can be written as $u = \prod_{\alpha\in\Phi^+} u_\alpha(c_\alpha)$ for fixed isomorphisms $u_\alpha : \mathbf{k}^\times \to U_\alpha, c_\alpha \in \mathbf{k}^\times$, and a fixed and total ordering of Φ^+ as in Theorem 1.34. Thus, after choosing a fixed set of preimages $n_w, w \in W$, the elements in G can be parametrised by $w \in W$, $c_\alpha, c'_\alpha \in \mathbf{k}^\times$ such that $u = \prod_{\alpha\in\Phi^+} u_\alpha(c_\alpha)$, $u' = \prod_{\substack{\alpha\in\Phi^+\\ w.\alpha\in\Phi^-}} u_\alpha(c'_\alpha)$, and $t \in T$.
Let $g_w := u'tn_w u \in G$ as above. Since W is finite, it can be possible to decide whether $\mathrm{Ad}(g_w)(x) \in \mathfrak{g}^\delta_{\geqslant 2}$ for each $w \in W$. In practice, we might encounter restraints such as memory space or time, that might make it difficult to compute the whole Weyl group W or decide whether $\mathrm{Ad}(g_w)(x) \in \mathfrak{g}^\delta_{\geqslant 2}$.
Following Section 1.2.5 we can compute

$$\mathrm{Ad}(g_w)(x) = \mathrm{Ad}(u'tn_w u)(x) = \sum_{\beta\in\Phi} \lambda_{w,\beta} e_\beta + \sum_{i=1}^{|\Pi|} \mu_{w,i} h_i, \tag{4.1}$$

where the h_i form a basis of $\mathrm{Lie}(T) = \mathfrak{t}$ and $\lambda_{w,\beta}, \mu_{w,i} \in \mathbf{k}$ depend on g_w and can be determined by the rules in Section 1.2.5.
To check whether we can choose g_w such that $\mathrm{Ad}(g_w)(x) \in \mathfrak{g}^\delta_{\geqslant 2}$ we need to solve the system of non-linear equations given by (4.1):

$$\begin{aligned} \lambda_{w,\beta}(c_\alpha, d_\gamma, c'_{\alpha'})_{\substack{\alpha,\alpha'\in\Phi^+\\ \gamma\in\Pi}} &= 0 \quad \text{for all } \beta \in \Phi \text{ with } \delta(\beta) \leqslant 1, \\ \mu_{w,i}(c_\alpha, d_\gamma, c'_{\alpha'})_{\substack{\alpha,\alpha'\in\Phi^+\\ \gamma\in\Pi}} &= 0 \quad \text{for all } i \in \{1, \dots, |\Pi|\}, \end{aligned} \tag{4.2}$$

where $\lambda_{w,\beta}, \mu_{w,i}$ are polynomials in the variables $c_\alpha, c'_{\alpha'}$, and d_γ determined by

$$u = \prod_{\alpha\in\Phi^+} u_\alpha(c_\alpha), \quad u' = \prod_{\substack{\alpha'\in\Phi^+\\ w^{-1}.\alpha'\in\Phi^-}} u_{\alpha'}(c'_{\alpha'}), \text{ and } \quad t = \prod_{\gamma\in\Pi} h_\gamma(d_\gamma)$$

as given in Section 1.2.5 (3). Using Gröbner bases, we can decide whether the system can be solved and determine a solution. In Section 6.3 we will take a closer look at how to solve these systems of non-linear equations.
If $\mathrm{Ad}(g_w)(x) \in \mathfrak{g}^\delta_{\geqslant 2}$, we continue with step (2) from the above list, otherwise we check whether $\mathrm{Ad}(g_w)(x) \in \mathfrak{g}^\delta_{\geqslant 2}$ for the next $w \in W$.
It would be desirable to simplify the elements g_w without losing any information. First of all, we would like to compute $\mathrm{Ad}(g_w)(x)$ only for certain elements $w \in W$. To this end we define Weyl group elements $w \in W$ of *weight* 0.

Definition 4.1. Let $\delta \in \mathscr{D}_G$ be a map arising from a weighted Dynkin diagram and let $w = s_{\alpha_1} \cdots s_{\alpha_r} \in W$ be a reduced expression of w and $\alpha_1, \dots, \alpha_r \in \Pi$. Then we say that w has **weight 0** if $\delta(\alpha_1) = \cdots = \delta(\alpha_r) = 0$. These elements form a subgroup of W which we will denote by $W_0^\delta := \langle s_\alpha \mid \alpha \in \Pi,\ \delta(\alpha) = 0\rangle$.

We can show that n_w fixes the set $\mathfrak{g}^\delta_{\geqslant 2}$ if w has weight 0. It turns out that an even stronger result is true, as stated in the following lemma.

Lemma 4.2. *Let $\delta \in \mathscr{D}_G$ be a map arising from a weighted Dynkin diagram. Let $g_w = u'tn_wu \in G$ as in Definition 1.51 and $y \in \mathfrak{g}^\delta_{\geqslant 2}$. Then:*

(i) The element g_w is contained in $G^\delta_{\geqslant 0}$ if and only if $w \in W^\delta_0$.

(ii) If $g_w \in G^\delta_{\geqslant 0}$, then $Ad(g_w)(y) \in \mathfrak{g}^\delta_{\geqslant 2}$.

In particular, $\mathrm{Ad}(g_w)(y) \in \mathfrak{g}^\delta_{\geqslant 2}$ *if and only if* $\mathrm{Ad}(n_wu)(y) \in \mathfrak{g}^\delta_{\geqslant 2}$.

Proof.

(i) Write $g_w = u'tn_wu$ using the Bruhat decomposition with the notation from Definition 1.51. Clearly, any elements $u \in U$ and $t \in T$ are contained in $G^\delta_{\geqslant 0}$, so $u, u', t \in G^\delta_{\geqslant 0}$ and therefore $g_w \in G^\delta_{\geqslant 0}$ if and only if $n_w \in G^\delta_{\geqslant 0}$. Furthermore, let $s_\alpha \in W$ be the reflection for the root $\alpha \in \Phi$. Then we can choose $n_{s_\alpha} = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$, see Section 1.2.5. Note also that $s_\alpha = s_{-\alpha}$ in W, therefore $n_{s_\alpha} = n_{s_{-\alpha}}$. It follows that if $\delta(\alpha) \geqslant 0$ we have $u_\alpha(1) \in G^\delta_{\geqslant 0}$ and therefore $n_{s_\alpha} = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1) \in G^\delta_{\geqslant 0}$ if and only if $\delta(\alpha) = 0$. If, on the other hand, $\delta(\alpha) \leqslant 0$, we have $u_{-\alpha}(1) \in G^\delta_{\geqslant 0}$ as well as $t \in G^\delta_{\geqslant 0}$ and therefore $n_{s_\alpha} = n_{s_{-\alpha}}t = u_{-\alpha}(1)u_\alpha(-1)u_{-\alpha}(1)t \in G^\delta_{\geqslant 0}$ if and only if $\delta(\alpha) = 0$. This means that for $w = s_{\alpha_1} \cdots s_{\alpha_r}$ we have $n_w \in G^\delta_{\geqslant 0}$ if $\delta(\alpha_1) = \cdots = \delta(\alpha_r) = 0$, i.e. if w has weight 0.
Conversely, let $w = s_{\alpha_1} \cdots s_{\alpha_r}$ be a reduced form of $w \in W$. Since δ arises from a weighted Dynkin diagram, we have $\delta(\alpha) \geqslant 0$ for all $\alpha \in \Phi^+$. Then $G^\delta_{\geqslant 0} = P_I = BW_IB$ is a standard parabolic subgroup, where $\Phi_I = \{\alpha \in \Phi \mid \delta(\alpha) = 0\}$ and $W_I = \langle s_\alpha \mid \alpha \in \Phi_I \rangle$. Clearly, $W^\delta_0 = W_I$. By the uniqueness of the Bruhat decomposition in Theorem 1.51, $n_w \in G^\delta_{\geqslant 0}$ if and only if $w \in W_I$. Therefore, $\delta(\alpha_i) = 0$ for all $i \in \{1, \ldots, r\}$ if $n_w \in G^\delta_{\geqslant 0}$. This proves the claim.

(ii) To see that $\mathrm{Ad}(g)(y) \in \mathfrak{g}^\delta_{\geqslant 2}$, it is enough to show that for each $e_\beta \in \mathfrak{g}_\beta$ with $\beta \in \Phi$ such that $\delta(\beta) \geqslant 2$ the following claims are true:

(1) $\mathrm{Ad}(u_\alpha(c_\alpha))(e_\beta) \in \mathfrak{g}^\delta_{\geqslant 2}$ for all $c_\alpha \in \mathbf{k}$,

(2) $\mathrm{Ad}(t)(e_\beta) \in \mathfrak{g}^\delta_{\geqslant 2}$ for all $t \in T$, and

(3) $\mathrm{Ad}(n_w)(e_\beta) \in \mathfrak{g}^\delta_{\geqslant 2}$ for all $n_w \in G^\delta_{\geqslant 0}$.

To (1): We have

$$\mathrm{Ad}(u_\alpha(t_\alpha))(e_\beta) = \sum_{\substack{\beta+k\alpha\in\Phi \\ k\geqslant 0}} t^k_\alpha c_{k,\alpha,\beta} e_{\beta+k\alpha} \in \mathfrak{g}^\delta_{\geqslant 2},$$

for some $t_\alpha, c_{k,\alpha,\beta} \in \mathbf{k}$ and $\delta(\beta + k\alpha) = \delta(\beta) + k\delta(\alpha) \geqslant 2$ as both $k \geqslant 0$ and $\delta(\alpha) \geqslant 0$ for $\alpha \in \Phi$.

To (2): For $t \in T$ we have $\mathrm{Ad}(t)(e_\beta) = \beta(t)e_\beta \in \mathfrak{g}_\beta$, so the action of T stabilises $\mathfrak{g}^\delta_{\geqslant 2}$.

To (3): By Section 1.2.5 (2) we have $\mathrm{Ad}(n_w)(\mathfrak{g}_\beta) = \mathfrak{g}_{w.\beta}$. As before, write $w = s_{\alpha_1} \cdots s_{\alpha_r}$ where $\delta(\alpha_1) = \cdots = \delta(\alpha_r) = 0$ and $\alpha_i \in \Pi$ for all i (since $\delta(\beta) \geqslant 2$, the reflection s_β is not in this product). For two roots $\alpha, \beta \in \Phi$ we have $s_\alpha(\beta) = \beta + k_{\alpha,\beta}\alpha$ for some $k_{\alpha,\beta} \in \mathbb{Z}$. If $\delta(\alpha) = 0$, then $\delta(s_\alpha(\beta)) = \delta(\beta)$. Iteratively, we get $\delta(w(\beta)) = \delta(\beta)$ and so $\mathrm{Ad}(n_w)(\mathfrak{g}_\beta) = \mathfrak{g}_{w(\beta)} \in \mathfrak{g}^\delta_{\geqslant 2}$.

Finally, let $w \in W$ be an arbitrary element of the Weyl group and suppose that $\mathrm{Ad}(g_w)(y) = x \in \mathfrak{g}^{\delta}_{\geqslant 2}$.
As $u't \in G^{\delta}_{\geqslant 0}$, so is $(u't)^{-1} \in G^{\delta}_{\geqslant 0}$ and therefore by the above calculations $\mathrm{Ad}(n_w u)(y) = \mathrm{Ad}((u't)^{-1})(x) \in \mathfrak{g}^{\delta}_{\geqslant 2}$.
Conversely, suppose that $\mathrm{Ad}(n_w u)(y) = x \in \mathfrak{g}^{\delta}_{\geqslant 2}$. Since, as before, $u't \in G^{\delta}_{\geqslant 0}$, it follows that $\mathrm{Ad}(u'tn_w u)(y) = \mathrm{Ad}(u't)(x) \in \mathfrak{g}^{\delta}_{\geqslant 2}$. This proves the claim. □

In particular, this shows that for all elements $g \in G^{\delta}_{\geqslant 0}$ we do not have to check whether $\mathrm{Ad}(g)(y)$ is in $\mathfrak{g}^{\delta}_{\geqslant 2}$, as the above lemma states that this is the case whenever $y \in \mathfrak{g}^{\delta}_{\geqslant 2}$.

Corollary 4.3. *Let* $u = u'u''$ *for* $u \in U$, $u' \in \prod_{\substack{\alpha\in\Phi^+ \\ w.\alpha\in\Phi^+}} U_\alpha$ *and* $u'' \in U^-_w$. *Then it is enough to check whether* $\mathrm{Ad}(n_w u'')(y) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ *in order to see that* $\mathrm{Ad}(n_w u)(y) \in \mathfrak{g}^{\delta}_{\geqslant 2}$.

Proof. By Theorem 1.34 we have $n_w u = n_w u' n_w^{-1} n_w u''$ and $n_w u' n_w^{-1} \in U \subseteq G^{\delta}_{\geqslant 0}$. So $n_w u = (n_w u' n_w^{-1}) n_w u''$ and by the above lemma the claim follows. □

Another direct consequence is the following: Suppose we already know that

$$\mathrm{Ad}(n_w u)(y) \in \mathfrak{g}^{\delta}_{\geqslant 2} \quad \text{or} \quad \mathrm{Ad}(n_w u)(y) \notin \mathfrak{g}^{\delta}_{\geqslant 2}$$

for some $w \in W$. Then for all $w' \in W$ with weight 0 we have $\mathrm{Ad}(n_{w'} n_w u)(y) = \mathrm{Ad}(t n_{w'w} u)(y) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ (resp. $\mathrm{Ad}(t n_{w'w} u)(y) \notin \mathfrak{g}^{\delta}_{\geqslant 2}$) for some $t \in T$ and therefore $Ad(n_{w'w} u)(y) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ (resp. $\mathrm{Ad}(n_{w'w} u)(y) \notin \mathfrak{g}^{\delta}_{\geqslant 2}$).
There is a similar result when we consider the $\mathfrak{g}^{\delta}_2$-part and the set $\mathfrak{g}^{\delta!}_2$. From now on we will only consider $\delta \in \mathscr{D}_G$ where δ arises from a weighted Dynkin diagram.

Lemma 4.4. *Let* $x \in \mathfrak{g}$ *be a nilpotent element and* $w \in W, u \in U^-_w$ *with* $\mathrm{Ad}(n_w u)(x) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ *and* $[\mathrm{Ad}(n_w u)(x)]_{\mathfrak{g}^{\delta}_2} \notin \mathfrak{g}^{\delta!}_2$.
Then we have $[\mathrm{Ad}(n_{w'w} u)(x)]_{\mathfrak{g}^{\delta}_2} \notin \mathfrak{g}^{\delta!}_2$ *for any* $w' \in W^{\delta}_0$.

Proof. As $\mathrm{Ad}(n_w u)(x) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ it follows that $\mathrm{Ad}(n_{w'w} u)(x) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ for any $w' \in W^{\delta}_0$ by Lemma 4.2.
For $g \in G^{\delta}_0$ we have (by a similar calculation as in the proof of Lemma 4.2) $\mathrm{Ad}(g)(\mathfrak{g}^{\delta}_i) = \mathfrak{g}^{\delta}_i$ for each $i \in \mathbb{Z}$. Note that for the representatives $n_w, n_{w'}, n_{w'w} \in N_G(T)$ we have $n_{w'w} = t n_{w'} n_w$ for some $t \in T$. Then

$$[\mathrm{Ad}(n_{w'w} u)(x)]_{\mathfrak{g}^{\delta}_2} = \mathrm{Ad}(t n_{w'})([\mathrm{Ad}(n_w u)(x)]_{\mathfrak{g}^{\delta}_2})$$

since $n_{w'} \in G^{\delta}_0$.
To simplify notation let $y := [\mathrm{Ad}(n_w u)(x)]_{\mathfrak{g}^{\delta}_2}$ and $y' := [\mathrm{Ad}(n_{w'w} u)(x)]_{\mathfrak{g}^{\delta}_2}$. Suppose that there exists $g \in G \setminus G^{\delta}_{\geqslant 0}$ with $\mathrm{Ad}(g)(y) = y$, so $g \in C_G(y)$. Then $\mathrm{Ad}(t n_{w'} g n_{w'}^{-1} t^{-1})(y') = y'$. As $t n_{w'} \in G^{\delta}_{\geqslant 0}$ and $g \notin G^{\delta}_{\geqslant 0}$ we have $t n_{w'} g n_{w'}^{-1} t^{-1} \notin G^{\delta}_{\geqslant 0}$ and so $y' \notin \mathfrak{g}^{\delta!}_2$. □

This lemma simplifies the calculations further: It is enough to check whether $\mathrm{Ad}(n_w u) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ for only one element $w \in W$ of every right coset of W^{δ}_0. However, we do need some further results to justify focusing on elements of the form $\mathrm{Ad}(n_w u)(x)$.

Lemma 4.5. *Let* $x \in \mathfrak{g}^{\delta}_{\geqslant 2}$, $u = \prod_{\beta\in\Phi^+} u_\beta(c_\beta) \in U$, $c_\beta \in \mathbf{k}$ *and* $\delta(\beta) \geqslant 0$ *for all* $\alpha \in \Phi^+$.
Let $\tilde{u} := \prod_{\substack{\beta\in\Phi^+ \\ \delta(\beta)=0}} u_\beta(c_\beta) \in U$.
Then $[\mathrm{Ad}(u)(x)]_{\mathfrak{g}^{\delta}_2} = \mathrm{Ad}(\tilde{u})([x]_{\mathfrak{g}^{\delta}_2})$.

Proof. We have $x = \sum_{\substack{\alpha\in\Phi^+\\ \delta(\alpha)\geqslant 2}} \lambda_\alpha e_\alpha$ for suitable $\lambda_\alpha \in \mathbf{k}$. Furthermore, let $\beta \in \Phi^+$ and consider $\mathrm{Ad}(u_\beta(c))(x)$ for some $c \in \mathbf{k}$. We have

$$\mathrm{Ad}(u_\beta(c))(x) = \sum_{\substack{\alpha\in\Phi^+\\ \delta(\alpha)\geqslant 2}} \lambda_\alpha \sum_{\substack{k\geqslant 0,\\ \alpha+k\beta\in\Phi}} c_{\alpha,k,\beta} e_{\alpha+k\beta}$$

and

$$\mathrm{Ad}(u_\beta(c))([x]_{g_2^\delta}) = \sum_{\substack{\alpha\in\Phi^+\\ \delta(\alpha)=2}} \lambda_\alpha \sum_{\substack{k\geqslant 0,\\ \alpha+k\beta\in\Phi}} c_{\alpha,k,\beta} e_{\alpha+k\beta},$$

for some $c_{\alpha,k,\beta} \in \mathbf{k}$. As $\beta \in \Phi^+$ we have either $\delta(\beta) = 0$ or $\delta(\beta) > 0$.
In the first case, $\delta(\alpha + k\beta) = \delta(\alpha)$ for all $k \in \mathbb{Z}_{\geqslant 0}$ and in the second case we have $\delta(\alpha + k\beta) = \delta(\alpha) + k\delta(\beta) > \delta(\alpha)$ for all $k \in \mathbb{Z}_{>0}$.
So $[\mathrm{Ad}(u_\beta(c))(x)]_{g_2^\delta} = \mathrm{Ad}(u_\beta(c))([x]_{g_2^\delta}) \in \mathfrak{g}_2^\delta$ if $\delta(\beta) = 0$.
Suppose $\delta(\beta) \neq 0$. Then

$$\begin{aligned}\mathrm{Ad}(u_\beta(c))(x) = &\sum_{\substack{\alpha\in\Phi^+\\ \delta(\alpha)=2}} \big(\lambda_\alpha e_\alpha + \lambda_\alpha \sum_{\substack{k\geqslant 0,\\ \alpha+k\beta\in\Phi}} c_{\alpha,k,\beta} e_{\alpha+k\beta}\big)\\ &+ \sum_{\substack{\alpha\in\Phi^+\\ \delta(\alpha)>2}} \lambda_\alpha \sum_{\substack{k\geqslant 0,\\ \alpha+k\beta\in\Phi}} c_{\alpha,k,\beta} e_{\alpha+k\beta},\end{aligned}$$

and so $[\mathrm{Ad}(u_\beta(c))(x)]_{g_2^\delta} = \sum_{\substack{\alpha\in\Phi^+\\ \delta(\alpha)=2}} \lambda_\alpha e_\alpha = [x]_{g_2^\delta}$.
The claim follows inductively. $\square$

Proposition 4.6. *We use the same notation as in Lemma 4.5. Suppose that*

$$[\mathrm{Ad}(n_w u)(x)]_{\mathfrak{g}_2^\delta} \notin \mathfrak{g}_2^{\delta !}.$$

(i) Then for all $u' \in U$ and $t \in T$ it follows that $[\mathrm{Ad}(u' t n_w u)(x)]_{\mathfrak{g}_2^\delta} \notin \mathfrak{g}_2^{\delta !}$ as well.

(ii) For $w_0 \in W_0^\delta$ we have $[\mathrm{Ad}(n_{w_0 w} u)(x)]_{\mathfrak{g}_2^\delta} \notin \mathfrak{g}_2^{\delta !}$.

It is therefore enough to check whether $[\mathrm{Ad}(n_w u)(x)]_{\mathfrak{g}_2^\delta}$ is in $\mathfrak{g}_2^{\delta !}$ for w contained in a right transversal of W_0 in W in order to decide if $[\mathrm{Ad}(u' t n_{w_0 w} u)(x)]_{\mathfrak{g}_2^\delta}$ is in $\mathfrak{g}_2^{\delta !}$ for all $u' \in U$, $t \in T$ and $w_0 \in W_0^\delta$.

Proof. Let $y := [\mathrm{Ad}(n_w u)(x)]_{\mathfrak{g}_2^\delta}$.

(i) As $t \in T$ we have $[\mathrm{Ad}(t n_w u)(x)]_{\mathfrak{g}_2^\delta} = \mathrm{Ad}(t)([\mathrm{Ad}(n_w u)(x)]_{\mathfrak{g}_2^\delta}) = \mathrm{Ad}(t)(y)$. By the Lemma 4.5 we can write $[\mathrm{Ad}(u' t n_w u)(x)]_{\mathfrak{g}_2^\delta} = \mathrm{Ad}(\tilde{u}t)(y)$ for $\tilde{u}$ as in Lemma 4.5. As $y \notin \mathfrak{g}_2^{\delta !}$ there exists $g \in G \backslash G_{\geqslant 0}^\delta$ such that $\mathrm{Ad}(g)(y) = y$. Then $h := \tilde{u} t g (\tilde{u} t)^{-1} \in G \setminus G_{\geqslant 0}^\delta$ as $\tilde{u} t \in G_{\geqslant 0}^\delta$ and $\mathrm{Ad}(h)(\mathrm{Ad}(\tilde{u}t)(y)) = \mathrm{Ad}(\tilde{u}t)(y)$ which proves (i).

(ii) As we have seen before, $n_{w'w} = t n_{w'} n_w$ for some $t \in T$ which stabilises the sets $\mathfrak{g}_i^\delta$. Since $w_0 \in W_0^\delta$ we have $n_{w_0} \in G_0^\delta$ and therefore $[\mathrm{Ad}(n_{w_0 w} u)(x)]_{\mathfrak{g}_2^\delta} =$

$\mathrm{Ad}(tn_{w_0})(y)$ as seen in the proof of Lemma 4.4. By the same argument as in (i) there exists $g \in G \setminus G^{\delta}_{\geqslant 0}$ such that $\mathrm{Ad}(g)(y) = y$ and therefore

$$\mathrm{Ad}(tn_{w_0}gn_{w_0}^{-1}t^{-1})([\mathrm{Ad}(n_{w_0w}u)(x)]_{\mathfrak{g}_2^\delta}) = [\mathrm{Ad}(n_{w_0w}u)(x)]_{\mathfrak{g}_2^\delta} = \mathrm{Ad}(tn_{w_0})(y).$$

As $tn_{w_0}gn_{w_0}^{-1}t^{-1} \notin G^{\delta}_{\geqslant 0}$ it follows that the element $[\mathrm{Ad}(n_{w_0w}u)(x)]_{\mathfrak{g}_2^\delta}$ is not contained in $\mathfrak{g}_2^{\delta !}$.

□

In the next section, we will use the results we just proved to formulate an algorithm that will decide which nilpotent orbits are contained in a given nilpotent piece.

4.3 Description of the algorithm to compute the nilpotent pieces

To determine which nilpotent orbits make up a piece $\mathscr{N}_{\mathfrak{g}}^{\blacktriangle}$ we fix a map $\delta \in \triangle \in \blacktriangle$ arising from a weighted Dynkin diagram, and for each nilpotent orbit $\mathscr{O}$ in $\mathfrak{g}$ check whether $\mathscr{O} \subseteq \mathscr{N}_{\mathfrak{g}}^{\blacktriangle}$. We will use the results from the previous sections to simplify the computations and proceed in several steps, checking if Definition 3.16 is fulfilled.
We fix the following notation: Let $x \in \mathfrak{g}$ denote a nilpotent orbit representative, where x is chosen such that $x = \sum_{\alpha\in\Phi^+} \lambda_\alpha e_\alpha$ for suitable $\lambda_\alpha \in \mathbf{k}$. Define tuples

$$\begin{aligned} c &:= (c_\alpha)_{\alpha\in\Phi^+}, \\ c' &:= c'_w := (c'_\alpha)_{\substack{\alpha\in\Phi^+ \\ w^{-1}.\alpha\in\Phi^-}} \quad \text{for } w \in W, \text{ and} \\ d &:= (d_\alpha)_{\alpha\in\Pi}, \end{aligned}$$

where the entries are elements in a function field over $\mathbf{k}$, which we will denote by F. Let $g_w(c, d, c') := u'(c')t(d)n_w u(c)$ be the Bruhat decomposition of an element in G over this function field, where

$$\begin{aligned} u'(c') &:= \prod_{\substack{\alpha\in\Phi^+ \\ w^{-1}.\alpha\in\Phi^-}} u_\alpha(c'_\alpha), \\ t(d) &:= \prod_{\alpha\in\Pi} h_\alpha(d_\alpha), \text{ and} \\ u(c) &:= \prod_{\alpha\in\Phi^+} u_\alpha(c_\alpha) \end{aligned}$$

with the h_α defined as in Section 1.2.5 and for a fixed ordering on Φ^+. We write

$$x_w(c, d, c') := \mathrm{Ad}(u'(c')t(d)n_w u(c))(x)$$

for the action of $g_w(c, d, c')$ on x. Finally, let $\lambda_{w,\beta}(c, d, c') \in F$ for $\beta \in \Phi$ and $\mu_{w,i}(c, d, c') \in F$ for $i \in \{1, \dots, |\Pi|\}$ be defined as in (4.1):

$$\mathrm{Ad}(g_w(c, d, c'))(x) = \sum_{\beta\in\Phi} \lambda_{w,\beta}(c, d, c')e_\beta + \sum_{i=1}^{|\Pi|} \mu_{w,i}(c, d, c')h_i.$$

Step 1: Let δ arise from a fixed weighted Dynkin diagram and $x = \sum_{\alpha \in \Phi^+} \lambda_\alpha e_\alpha \in \mathfrak{g}$ for suitable $\lambda_\alpha \in \mathbf{k}$ be a nilpotent element and a representative of a nilpotent orbit, denoted by $\mathscr{O}_x$.
Fix $w \in W$. We want to check if for an arbitrary element $g_w(c, d, c') \in Bn_wB$ we have $\mathrm{Ad}(g_w(c, d, c'))(x) = x_w(c, d, c') \in \mathfrak{g}^\delta_{\geqslant 2}$. In this manner, it is possible to compute the Ad-action of every element in G on x.
By Corollary 4.3, it is enough to check if $\mathrm{Ad}(n_w\tilde{u})(x) \in \mathfrak{g}^\delta_{\geqslant 2}$ where

$$\tilde{u} = u_{\alpha_1}(\tilde{c}_{\alpha_1}) \cdots u_{\alpha_r}(\tilde{c}_{\alpha_r})$$

denotes an element in U, parametrised by the elements $\tilde{c}_{\alpha_1}, \ldots, \tilde{c}_{\alpha_r} \in \mathbf{k}$ for some $r \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_r \in \Phi^+$ such that $w.\alpha_i \notin \Phi^+$ for all $i = 1, \ldots, r$. From now on we will use the notation c_{α_i} instead of $\tilde{c}_{\alpha_i}$ for easier reading and let $x_w(c) := x_w(c_{\alpha_1}, \ldots, c_{\alpha_r}) := \mathrm{Ad}(n_w\tilde{u})(x)$ be the element that depends on the c_{α_i}.
In order to check if $\mathrm{Ad}(n_w\tilde{u})(x) \in \mathfrak{g}^\delta_{\geqslant 2}$, we first compute the action of $n_w\tilde{u}$ on x as in Section 1.2.5. Following this, the resulting system of non-linear equations for the c_{α_i} (see (4.2)) can be solved by computing a Gröbner basis of this system. For this we use the standard algorithm in Magma, see [3, Section 112.4.3] with a reverse lexicographical ordering. In order to speed this process up, we will first check for variables that occur in linear equations and solve for those variables. We will apply this every time we compute a Gröbner basis. This approach is described in Chapter 6 in more detail. Note that in general the solution (if there exists one) will still depend on some of the c_{α_i}. This means that we will have to check whether $[\mathrm{Ad}(n_w\tilde{u})(x)]_{\mathfrak{g}^\delta_2} \in \mathfrak{g}^{\delta !}_2$ for each solution.

Step 2: Suppose there exists a solution such that $\mathrm{Ad}(n_w\tilde{u})(x) \in \mathfrak{g}^\delta_{\geqslant 2}$. As mentioned in Step 1 this solution will still depend on some of the c_{α_i}. Let $y_w := x_w(\varepsilon_1, \ldots, \varepsilon_r)$, $\varepsilon_i \in \{0, 1, \ldots, \mathrm{char}(\mathbf{k}) - 1\}$ if $\mathrm{char}(\mathbf{k}) \neq 0$, and $\varepsilon_i \in \{0, 1, 2 \ldots\}$ otherwise, be the element arising from a particular solution where we replaced the c_{α_i} in $\tilde{u} = u_{\alpha_1}(c_{\alpha_1}) \cdots u_{\alpha_r}(c_{\alpha_r})$ by the solution in which we set the remaining c_{α_i} to zero if possible or otherwise another fixed element in the prime field of $\mathbf{k}$. This depends on which solution is possible, so that we do not divide by zero when replacing the c_{α_i}.

Step 3: Let $x'_w(c) := [x_w(c)]_{\mathfrak{g}^\delta_2}$ and $y'_w := [y_w]_{\mathfrak{g}^\delta_2}$ by setting the coefficients of the basis elements not contained in $\mathfrak{g}^\delta_2$ to zero.

Step 4: Check if there is an element $g_{y'_w} \in G^\delta_{\geqslant 0}$ such that we have $x'_w(c) = \mathrm{Ad}(g_{y'_w})(y'_w)$. As in Step 1 we will use Section 1.2.5 to compute $\mathrm{Ad}(g_{y'_w})(y'_w)$ and determine a Gröbner basis to solve the resulting system. Note that the solution $g_{y'_w}$ will depend on the variables c_{α_i} that $x'_w(c)$ depends on.

Case 1: If such an element $g_{y'_w} \in G^\delta_{\geqslant 0}$ with $x'_w(c) = \mathrm{Ad}(g_{y'_w})(y'_w)$ exists, we can focus our computations on $x'_w(c)$:

Case 1a: If there is no $g' \in G \setminus G^\delta_{\geqslant 0}$ such that $\mathrm{Ad}(g')(y'_w) = y'_w$, then $y'_w \in \mathfrak{g}^{\delta !}_2$ and so both $y_w \in \mathscr{N}^{\blacktriangle}_{\mathfrak{g}}$ and $x \in \mathscr{N}^{\blacktriangle}_{\mathfrak{g}}$.

Case 1b: There is $g' \in G \setminus G^\delta_{\geqslant 0}$ such that $\mathrm{Ad}(g')(y'_w) = y'_w$. Then $y'_w \notin$

$\mathfrak{g}_2^{\delta !}$. As $x'_w(c) = \mathrm{Ad}(g_{y'_w})(y'_w)$ we have

$$\mathrm{Ad}(g' g_{y'_w}^{-1})(x'_w(c)) = \mathrm{Ad}(g')(y'_w) = y'_w$$

and it follows that $\mathrm{Ad}(g_{y'_w} g' g_{y'_w}^{-1})(x'_w(c)) = x'_w(c)$.
As before $g_{y'_w} g' g_{y'_w}^{-1} \notin G_{\geqslant 0}^{\delta}$, so $x'_w(c) \notin \mathfrak{g}_2^{\delta !}$.

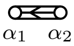
Case 2: If there is no $g_{y'_w} \in G_{\geqslant 0}^{\delta}$ such that $x'_w(c) = \mathrm{Ad}(g_{y'_w})(y'_w)$, we first check whether Case 1b holds for y'_w. If it does, we check the same thing for $x'_w(c)$, i.e. if there is an element $g' \in G \setminus G_{\geqslant 0}^{\delta}$ such that $\mathrm{Ad}(g')(x'_w(c)) = x'_w(c)$. Note that g' may depend on the variables in $x'_w(c)$, so we need to make sure that this solution holds for all possible values of the c_{α_i}.
Sometimes we might find such a $g_{y'_w} \in G_{\geqslant 0}^{\delta}$ only for certain values of the c_{α_i}. If this is the case, we will have to check everything in Step 4 for the values of the $\bar{c}_{\alpha_i} := c_{\alpha_i}$ for which no $g_{y'_w} \in G_{\geqslant 0}^{\delta}$ as above exists. For these cases we need to check separately whether $x''_w(c) \in \mathfrak{g}_2^{\delta !}$ for the arising elements $x''_w(c) := [x(\bar{c}_{\alpha_1}, \ldots, \bar{c}_{\alpha_r})]_{\mathfrak{g}_2^{\delta}}$ depending on the $\bar{c}_{\alpha_i}$.

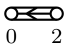
Step 5: For each $w'' \in W$ first check whether w'' has weight 0, i.e. $w'' \in W_0^{\delta}$. If this is not the case, let $g_{w''} \in B n_{w''} B$ and note that $g_{w''} \notin G_{\geqslant 0}^{\delta}$ by Lemma 4.2. We want to check whether $g_{w''}$ centralises y'_w, so as before we compute $\mathrm{Ad}(g_{w''})(y'_w)$ and solve the system $y'_w = \mathrm{Ad}(g_{w''})(y'_w)$ by using Gröbner bases. If this system has no solution, there is a possibility that $y'_w \in \mathfrak{g}_2^{\delta !}$ and we repeat Step 5 for the next element $w''' \in W$.
If there is a solution, we take the elements $x''_w(c)$ from Step 4 for which we found no $g \in G_{\geqslant 0}^{\delta}$ such that $\mathrm{Ad}(g)(y'_w) = x''_w(c)$. We will check whether there is a solution for $x''_w(c) = \mathrm{Ad}(g_{w''})(x''_w(c))$ in each case. If we find a solution, it immediately follows that $x''_w(c) \notin \mathfrak{g}_2^{\delta !}$ and we move on to the next $w''' \in W$ in Step 1. If we have checked each $w''' \in W$, it follows that this particular orbit is not contained in the piece.
If there exists $x''_w(c)$ as in Step 4, Case 2, that is, there is no element $g \in G_{\geqslant 0}^{\delta}$ with $x''_w(c) = \mathrm{Ad}(g)(y'_w)$, we need to check if there is a solution to $x''_w(c) = \mathrm{Ad}(g_{w'''})(x''_w(c))$ for each $w''' \in W$ for which we have already seen that $y'_w \neq \mathrm{Ad}(g_{w''})(y'_w)$. If so, then $x''_w(c) \notin \mathfrak{g}_2^{\delta !}$ and we move to the next $w''' \in W$ in Step 1. Otherwise we continue Step 5 by replacing y'_w with $x''_w(c)$.
If we do not find a solution for any element in W in Step 5, we have successfully proved that the centraliser of $x'_w(c)$ is contained in $G_{\geqslant 0}^{\delta}$ and therefore the orbit $\mathscr{O}_x$ is contained in the piece. We can move on to check the next orbit. As we already know that certain orbits are contained in certain pieces, see Lemma 3.19, we expect that in most cases $\mathscr{O}_x$ will not be contained in $\mathscr{N}_{\mathfrak{g}}^{\delta}$, see also Conjecture A. This means we can run (and try to optimise) this algorithm keeping in mind that the most likely outcome is that there is an element $g \notin G_{\geqslant 0}^{\delta}$ centralising $[\mathrm{Ad}(n_w u')(x)]_{\mathfrak{g}_2^{\delta}}$.

There are a few ways in which to simplify the computations further: If $n_{\tilde{w}} \in G_0^{\delta}$ for a fixed $\tilde{w} \in W$, we check if $\mathrm{Ad}(n_{\tilde{w}})(y'_w) = y'_w$ where y'_w is as in Step 3 above. If this is the case and we know that there exists a $g \in G \setminus G_{\geqslant 0}^{\delta}$ such that $\mathrm{Ad}(g)(y'_w) \neq y'_w$, then for

$$h \in \{g n_{\tilde{w}}^{-1}, g n_{\tilde{w}}, n_{\tilde{w}} g, n_{\tilde{w}}^{-1} g, n_{\tilde{w}} g n_{\tilde{w}}^{-1}, n_{\tilde{w}} g n_{\tilde{w}}, n_{\tilde{w}}^{-1} g n_{\tilde{w}}^{-1}, n_{\tilde{w}}^{-1} g n_{\tilde{w}}\}$$

it follows that $\mathrm{Ad}(h)(y'_w) \neq y'_w$ by Lemma 4.4. This means that we do not have to check whether the above elements stabilise y'_w. Especially in groups with smaller root systems, it usually takes longer to compute the Bruhat decomposition of the elements in the above set than to compute the nilpotent pieces without it. We will therefore not always use this approach.
Similarly, if $\tilde{u} \in U := \prod_{\alpha\in\Phi^+} U_\alpha$ with $\mathrm{Ad}(\tilde{u})(y'_w) = y'_w$, it is enough to check whether $\mathrm{Ad}(\tilde{u}g)(y'_w) = y'_w$ or $\mathrm{Ad}(g\tilde{u})(y'_w) = y'_w$. This means we may choose $\tilde{u}$ in the centralizer of y'_w in such a way that g depends on fewer variables c_α: If $g = u'tn_wu$ as in Definition 1.51 where $u' = u_{\alpha_1}(c'_{\alpha_1})\cdots u_{\alpha_r}(c'_{\alpha_r})$ and $u = u_{\alpha_1}(c_{\alpha_1})\cdots u_{\alpha_r}(c_{\alpha_r})$ for $c_{\alpha_i}, c'_{\alpha_i} \in \mathbf{k}$, then choose $\tilde{u}_1, \tilde{u}_2 \in U$ such that $v' := \tilde{u}_1u'$ and $v := \tilde{u}_2u$ depend on fewer variables $c_{\alpha_i}, c'_{\alpha_i} \in F$. It will now be easier to compute $\mathrm{Ad}(\tilde{u}_1g_w\tilde{u}_2)(y'_w) = y'_w$.

Example 4.7. Consider a simple algebraic group of type G_2 over a field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) = 3$, simple roots $\Pi = \{\alpha_1, \alpha_2\}$, and Dynkin diagram 

.
We have already seen that 

is a weighted Dynkin diagram. We want to see whether the orbit described by $x := e_{\alpha_1}$ is contained in the piece for this weighted Dynkin diagram. The Weyl group has only 12 elements and we only have to consider a transversal of the subgroup generated by $W_0^\delta = \langle s_{\alpha_1}\rangle$. We choose the elements

$$\{\mathrm{id}, s_{\alpha_2}, s_{\alpha_2}s_{\alpha_1}, s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}, (s_{\alpha_2}s_{\alpha_1})^2, (s_{\alpha_2}s_{\alpha_1})^2 s_{\alpha_2}\}.$$

We will apply the algorithm for the action of $w = s_{\alpha_2}$. We have the set of positive roots $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$.

Step 1: Check if $\mathrm{Ad}(n_wu_{\alpha_2}(c_{\alpha_2}))(x) \in \mathfrak{g}^\delta_{\geqslant 2}$. This is enough, as $w.\alpha \in \Phi^+$ for all $\alpha \in \Phi^+$ with $\alpha \neq \alpha_2$. Then $\mathrm{Ad}(n_wu_{\alpha_2}(c_{\alpha_2}))(x) = c_{\alpha_2}e_{\alpha_1} + e_{\alpha_1+\alpha_2}$. This element is in $\mathfrak{g}^\delta_{\geqslant 2}$ if $c_{\alpha_2} = 0$.
This means $x_w(c) := \mathrm{Ad}(n_w)(x) = e_{\alpha_1+\alpha_2}$. In this case, $x_w(c)$ does not depend on any variables, so we can skip replacing them and move on to Step 3.

Step 3: In fact, $\delta(\alpha_1+\alpha_2) = 2$, so $x'_w(c) = x$.

Step 4: As $y'_w = x'_w(c)$ we can skip this step.

Step 5: We simplify this computation by finding elements $u \in U$ and $t \in T$ which centralise $x_w(c)$. In order to do so, we compute the elements $u \in U$ and $t \in T$ for which $\mathrm{Ad}(u)(x_w(c)) = x_w(c)$, $\mathrm{Ad}(t)(x_w(c)) = x_w(c)$. This is true for $u = u_{\alpha_2}(c'_{\alpha_2})u_{\alpha_1+\alpha_2}(c'_{\alpha_1+\alpha_2})$ and $t = h_{\alpha_1}(d_{\alpha_1})$ with $c'_{\alpha_2}, c'_{\alpha_1+\alpha_2} \in \mathbf{k}$ and $d_{\alpha_1} \in \mathbf{k}^\times$. That means we only have to consider torus elements without the factor $h_{\alpha_1}(c) \in T$ and unipotent elements without the factors

$$u_{\alpha_2}(c'_{\alpha_2})u_{\alpha_1+\alpha_2}(c'_{\alpha_1+\alpha_2}) \in U.$$

Consider the element $g_w(c, d, c') := u'h_{\alpha_2}(d_{\alpha_2})n_{s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}}u$, for

$$u' = u_{\alpha_1}(\tilde{c}'_{\alpha_1})u_{2\alpha_1+\alpha_2}(\tilde{c}'_{2\alpha_1+\alpha_2})u_{3\alpha_1+\alpha_2}(\tilde{c}'_{3\alpha_1+\alpha_2}),\ \text{and}$$
$$u = u_{\alpha_1}(c'_{\alpha_1})u_{2\alpha_1+\alpha_2}(c'_{2\alpha_1+\alpha_2})u_{3\alpha_1+\alpha_2}(c'_{3\alpha_1+\alpha_2})u_{3\alpha_1+2\alpha_2}(c'_{3\alpha_1+2\alpha_2}),$$

with $\tilde{c}'_{\alpha_1}, \tilde{c}'_{2\alpha_1+\alpha_2}, \tilde{c}'_{3\alpha_1+\alpha_2}, c'_{\alpha_1}, c'_{2\alpha_1+\alpha_2}, c'_{3\alpha_1+\alpha_2}, c'_{3\alpha_1+2\alpha_2} \in \mathbf{k}$. Then $\mathrm{Ad}(g_w)(x_w(c)) = \lambda_{\alpha_1} e_{\alpha_1} + \lambda_{\alpha_1+\alpha_2} e_{\alpha_1+\alpha_2} + \lambda_{2\alpha_1+\alpha_2} e_{2\alpha_1+\alpha_2} + \lambda_{-\alpha_1} e_{-\alpha_1} + \mu_1 h_1$ with

$$\begin{aligned}
\lambda_{\alpha_1} &= \frac{2}{d_{\alpha_2}} c'_{\alpha_1} \tilde{c}'^2_{\alpha_1}, \\
\lambda_{\alpha_1+\alpha_2} &= \frac{2}{d_{\alpha_2}} c'_{\alpha_1} \tilde{c}'^2_{2\alpha_1+\alpha_2} + 2d^3_{\alpha_2}, \\
\lambda_{2\alpha_1+\alpha_2} &= \frac{2}{d_{\alpha_2}} c'_{\alpha_1} \tilde{c}'_{\alpha_1} \tilde{c}'_{2\alpha_1+\alpha_2} + \frac{2}{d_{\alpha_2}} c'_{\alpha_1} \tilde{c}'_{\alpha_1} \tilde{c}'_{3\alpha_1+\alpha_2} + d^3_{\alpha_2} \tilde{c}'_{\alpha_1}, \\
\lambda_{-\alpha_1} &= \frac{1}{d_{\alpha_2}} c'_{\alpha_1}, \\
\mu_1 &= \frac{1}{d_{\alpha_2}} c'_{\alpha_1} \tilde{c}'_{\alpha_1}.
\end{aligned}$$

Thus, in order to see if there are values of the variables such that $\mathrm{Ad}(g_w)(y) = y$, we need to solve the system

$$\begin{aligned}
\lambda_{\alpha_1} &= 0, \\
\lambda_{\alpha_1+\alpha_2} &= 1, \\
\lambda_{2\alpha_1+\alpha_2} &= 0, \\
\lambda_{-\alpha_1} &= 0, \\
\mu_1 &= 0.
\end{aligned}$$

We find a solution given by

$$d_{\alpha_2} = 2, \tilde{c}'_{\alpha_1} = c'_{\alpha_1} = 0, \text{ and } \tilde{c}'_{2\alpha_1+\alpha_2}, \tilde{c}'_{3\alpha_1+\alpha_2}, c'_{2\alpha_1+\alpha_2}, c'_{3\alpha_1+\alpha_2}, c'_{3\alpha_1+2\alpha_2} \in \mathbf{k}.$$

That means $x_w(c) \notin \mathfrak{g}_2^{\delta !}$.
Note that all steps have to be repeated for every element w in the transversal in order to be able to conclude that $x \notin \mathscr{N}_{\mathfrak{g}}^{\delta}$.

A pseudocode to compute the nilpotent pieces

We summarise this section with the following pseudocode and graphic. The code is rather brief and only sketches the most important steps in the algorithm.

Algorithm 1 Computation of the nilpotent pieces

1: **procedure** Pieces(x, S, W, δ)
INPUT:
x - nilpotent orbit representative
S - type of root system
W - Weyl group
δ - weighted Dynkin diagram
OUTPUT: result whether x is in $\mathscr{N}_{\mathfrak{g}}^{\delta}$
2: find all elements $u \in U = \prod_{\alpha \in \Phi^+} U_\alpha$ s.t. $\mathrm{Ad}(u)(x) = x$ *// we get U from the root system*

save the indices i of $u_{\alpha_i}(c_i)$ in a list Lu where $u_{\alpha_i}(c_i)$ is a factor in u such that c_i can be chosen arbitrarily and $\mathrm{Ad}(u_{\alpha_i}(c_i))(x) = x$
save all $w \in W$ with weight 0 in a list L
compute a right transversal T of all the elements of weight 0 in W
$L2 := [\,]$
for all $w \in T, w \notin L2$ **do**
$y := \mathrm{Ad}(n_w u)(x)$, u not containing any factors u_{α_i}, $i \in Lu$
if $y \in \mathfrak{g}^{\delta}_{\geqslant 2}$ **then**
$z \leftarrow$ replace all variables c_i in y by 0 or 1 // *in practice these two values are usually enough. If not, we have to choose another value in the prime field.*
$y'_w := [y]_{\mathfrak{g}^{\delta}_2}$
$z' := [z]_{\mathfrak{g}^{\delta}_2}$
check if there exists $g \in G^{\delta}_{\geqslant 0}$ such that $\mathrm{Ad}(g)(z') = y'_w$
if there exist values of the c_i such that there exists no such g **then**
$z_1 \leftarrow$ values of the c_i such that there exists no such g
end if
for all $w' \in W$ **do**
if $w' \notin L$ **then**
if $\mathrm{Ad}(g_{w'})(z') \neq z'$ **then**
go to the next w'
else
check the same for all z_1
if $\mathrm{Ad}(g_{w'})(z_1) \neq z_1$ **then**
go to the next w'
else
go to the next w and save all $g.w$ for $g \in G_{\geqslant 0}$ in $L2$
end if
end if
end if
end for
else
go to the next w and save all $g.w$ for $g \in G_{\geqslant 0}$ in $L2$
end if
end for
if for all $w \in T$ there exists $w' \in W \setminus L$ with $\mathrm{Ad}(g_{w'})(z') = z'$ **then**
return "x not in piece"
else
return "x in piece"
end if
end procedure

We also give a schematic representation of the programme for a better overview. Note that this is a rather simplified version of the programme, as it does not display every step in great detail.

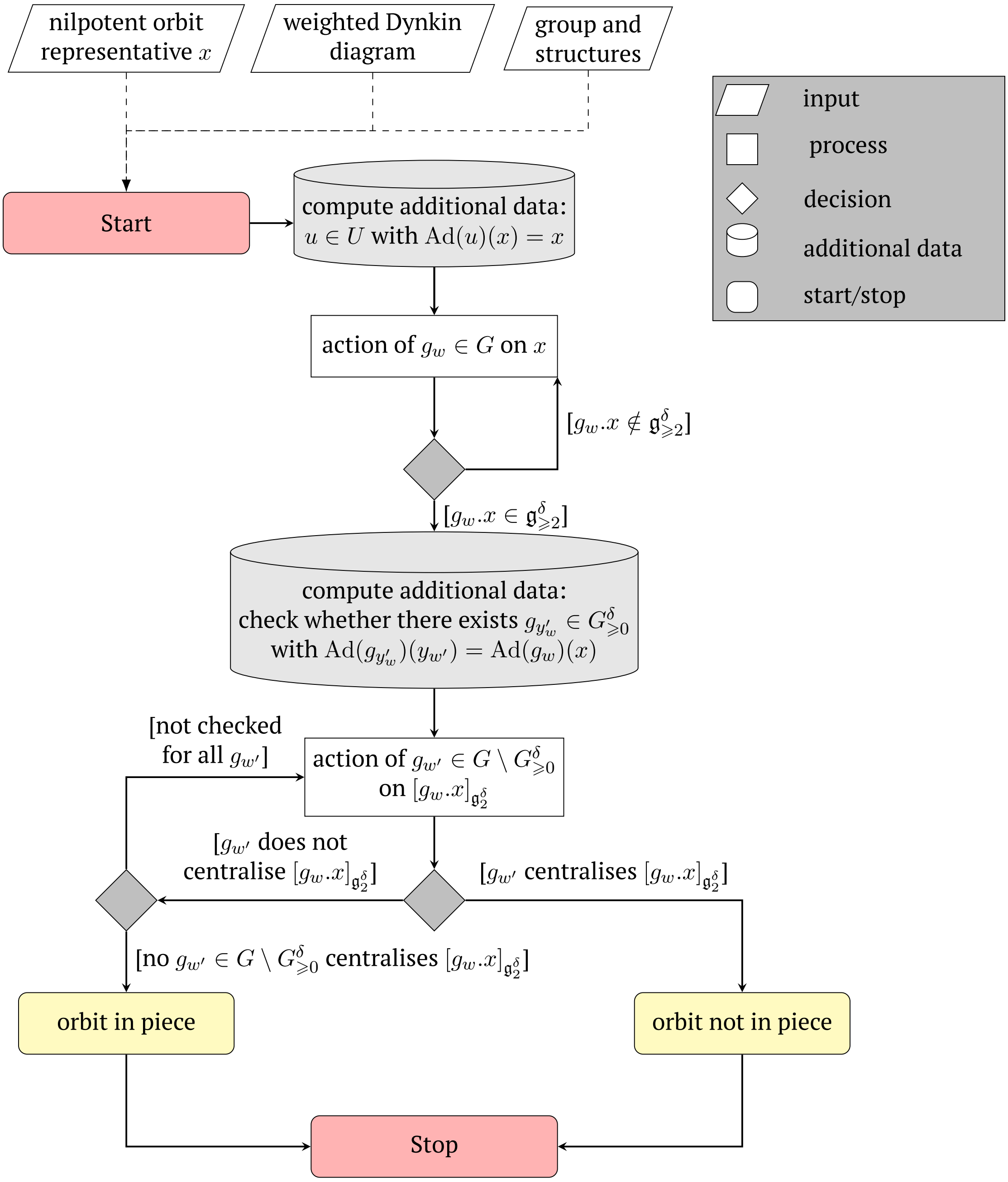


Figure 4.1: Schematic representation of the algorithm to compute the nilpotent pieces

5 | Alternative approaches

In this chapter we present some alternative approaches to the problem of computing the nilpotent pieces. The first approach will not lead to viable solutions or give any great advantages compared to the algorithm introduced in the previous chapter.

5.1 Computing the centralizers of the orbit representatives

One alternative approach is to focus on computing the sets $\mathfrak{g}_2^{\delta !}$. For this we would have to compute the centralizer of each nilpotent orbit representative which is not exceptional. We recall the fact that $C_G(x) \subseteq G_{\geqslant 0}^{\delta}$ where δ is the weighted Dynkin diagram for the orbit of x in good characteristic, see Proposition 2.12. However, this does not necessarily mean we can easily solve the problem.
Once $C_G(x)$ is known, we can compute the centraliser of any element $y := \mathrm{Ad}(g)(x)$ in the orbit of x for $g \in G$. The centraliser is given by $gC_G(x)g^{-1}$. In order to decide whether y is in $\mathfrak{g}_2^{\delta !}$, it is enough to find one element not contained in $G_{\geqslant 0}^{\delta}$ (provided $y \in \mathfrak{g}_2^{\delta}$). Note, however, that g does depend on indeterminates defined over $\mathbf{k}$ and so do the elements in $C_G(x)$ that we conjugate with g. In particular, we have to check for values of the variables where denominators in the solution turn to zero. This step turns out to take too much computation time in general, so that even in type F_4 it is not always possible to arrive at a solution. We will look at a small example in $\mathrm{Sp}_4(\mathbf{k})$.

Example 5.1. Consider the orbit representative $x_3 := \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$. We can compute (by using the Bruhat decomposition) that for $t_1, t_2 \in \mathbf{k}^{\times}$ and $a_1, a_2, a_3, a_4, b_1 \in \mathbf{k}$ the matrices

$$C_1 := \begin{pmatrix} t_1 & t_1 a_1 & t_1(a_1a_2+a_3) & t_1(a_1a_3+a_4) \\ 0 & t_1^{-1} & a_2 t_1^{-1} & a_3 t_1^{-1} \\ 0 & 0 & t_1 & -t_1 a_1 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix}, \text{ where } a_1 = 0 \text{ if } \mathrm{char}(\mathbf{k}) \neq 2, \text{ and}$$

$$C_2 := \begin{pmatrix} -b_1 t_1 & (b_1 t_2^2 a_1 - 1) t_2^{-1} & (b_1 t_2^2 a_1 (a_1 a_2 - a_3) - a_2) t_2^{-1} & (b_1 (a_1 a_3 - a_4) t_2^2 a_1 - a_3) t_2^{-1} \\ -t_2 & -t_2 a_1 & -t_2(a_1 a_2 + a_3) & -t_2(a_1 a_3 + a_4) \\ 0 & 0 & -b_1 t_2 (2 b_1 t_2^2 a_1 - 1)^{-1} & -(b_1 t_2^2 a_1 + 1)(2 a_1 b_1 t_2^3 - t_2)^{-1} \\ 0 & 0 & t_2 (2 b_1 t_2^2 a_1 - 1)^{-1} & -t_2 a_1 (2 b_1 t_2^2 a_1 - 1)^{-1} \end{pmatrix},$$

where $a_1 = b_1 = 0$ if $\mathrm{char}(\mathbf{k}) \neq 2$,

centralise x_3. Each element centralising x_3 is of the form of one of these matrices. Clearly, these matrices are already dependent on a number of variables which will complicate further computations. We now need to figure out if the orbit of x_3 intersects with one of the sets $\mathfrak{g}_2^{\delta !}$ for all weighted Dynkin diagrams δ. In order to do so, we compute $\mathrm{Ad}(g_w)(x_3)$ where

$$g_w = u_\alpha(c'_\alpha)u_\beta(c'_\beta)u_{\alpha+\beta}(c'_{\alpha+\beta})u_{2\alpha+\beta}(c'_{2\alpha+\beta})tn_w u_\alpha(c_\alpha)u_\beta(c_\beta)u_{\alpha+\beta}(c_{\alpha+\beta})u_{2\alpha+\beta}(c_{2\alpha+\beta})$$

and $c'_\gamma = 0$ if $w.\gamma \in \Phi^+$. Furthermore, $w \in W$, where W is the the Weyl group of $\mathrm{Sp}_4(\mathbf{k})$, $t \in T$, and $c_\gamma, c'_\gamma \in \mathbf{k}$ for all $\gamma \in \Phi$. The result is a linear combination of elements of the Chevalley basis. We then have to solve a system of non-linear equations arising as in (4.2), where we set the basis coefficients in the linear combination of the basis to zero if the basis element is not in $\bigoplus_{\substack{\alpha\in\Phi^+ \\ \delta(\alpha)=2}} \mathfrak{g}_\alpha$.

Here, this means that we want to find a solution such that $\mathrm{Ad}(g_w)(x_3) \in \mathfrak{g}_2^\delta$. If there is a solution, we will evaluate g_w at the elements in the solution, leading to an element $g'_w \in G$. Then the centraliser of $\mathrm{Ad}(g'_w)(x_3)$ can be easily computed as

$$C_{\mathrm{Sp}_4(\mathbf{k})}\left(\mathrm{Ad}(g'_w)(x_3)\right) = \left\{ g'_w C_1 g'^{-1}_w \;\middle|\; \begin{array}{l} t_1, t_2 \in \mathbf{k}^\times, \\ a_1, a_2, a_3, a_4 \in \mathbf{k}, \\ a_1 = 0 \text{ if } \mathrm{char}(\mathbf{k}) \neq 2 \end{array} \right\} \bigcup \left\{ g'_w C_2 g'^{-1}_w \;\middle|\; \begin{array}{l} t_1, t_2 \in \mathbf{k}^\times, \\ a_1, a_2, a_3, a_4, b_1 \in \mathbf{k}, \\ a_1 = b_1 = 0 \text{ if } \mathrm{char}(\mathbf{k}) \neq 2 \end{array} \right\}.$$

Thus, it is easy to decide whether $\mathrm{Ad}(g'_w)(x_3)$ is contained in $\mathfrak{g}_2^{\delta !}$ or not. For instance, consider the element g_w for $w = 1$. Then $g_w = tu_\alpha(c_\alpha)u_\beta(c_\beta)u_{\alpha+\beta}(c_{\alpha+\beta})u_{2\alpha+\beta}(c_{2\alpha+\beta})$, or as a matrix

$$g_w = \begin{pmatrix} t_1 & t_1c_\alpha & t_1(c_\alpha c_\beta + c_{\alpha+\beta}) & t_1(c_\alpha c_{\alpha+\beta} + c_{2\alpha+\beta}) \\ 0 & t_1 & t_2 c_\beta & t_2 c_{\alpha+\beta} \\ 0 & 0 & t_2^{-1} & -c_\beta t_2^{-1} \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix},$$

where $t_1, t_2 \in \mathbf{k}^\times$ and $c_\alpha, c_\beta, c_{\alpha+\beta}, c_{2\alpha+\beta} \in \mathbf{k}$. Thus,

$$\mathrm{Ad}(g_w)(x_3) = \begin{pmatrix} 0 & 0 & t_1t_2 & 2t_1^2a_1 \\ 0 & 0 & 0 & t_1t_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By Example 3.10, the result is contained in $\mathfrak{g}_2^\delta$ only for the weighted Dynkin diagram δ given by $\underset{0}{\circ}\Leftarrow\underset{2}{\circ}$, as $t_1t_2 \neq 0$. Then the centraliser of $\mathrm{Ad}(g_w)(x_3)$ is given by matrices of the form $g_wC_1g_w^{-1}$ and $g_wC_2g_w^{-1}$ with $t_1, t_2 \in \mathbf{k}^\times$, $a_1, a_2, a_3, a_4, b_1 \in \mathbf{k}$, and $a_1 = 0$ if $\mathrm{char}(\mathbf{k}) \neq 2$, respectively $a_1 = b_1 = 0$ if $\mathrm{char}(\mathbf{k}) \neq 2$. All matrices of these forms are contained in $G_{\geqslant 0}^\delta$. This is because g_w itself is contained in $G_{\geqslant 0}^\delta$, as well as C_1 and C_2. Thus the orbit of x_3 has a non-trivial intersection with $\mathfrak{g}_2^{\delta !}$.

Finally, we have to decide whether there are elements $x \in \mathfrak{g}_2^{\delta !} \cap \mathscr{O}_{x_3}$ such that elements of the form $x + y$ for $y \in \mathfrak{g}_{\geqslant 3}^\delta$ are not in the orbit of x_3. However, note that in this case $\mathfrak{g}_{\geqslant 3}^\delta$ is empty. Otherwise, we would have to check whether general elements of the form $x_3 + y$, $y \in \mathfrak{g}_{\geqslant 3}^\delta$ are in the same orbit as x_3.

5.2 The nilpotent pieces in Levi subgroups

As it is easier (and faster) to compute the nilpotent pieces for groups of smaller rank, the question arises whether we can use these results in order to simplify the calculations. We note for instance that simple groups of type E_7 contain parabolic subgroups

P with their Levi subgroups having root systems of type A_6, E_6, and D_6, which can easily be seen by the Dynkin diagram:

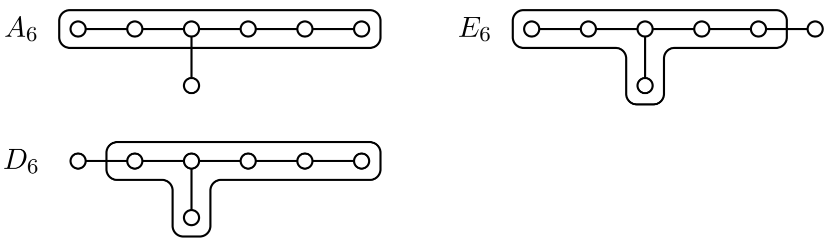


Figure 5.1: Dynkin diagrams of E_7 with subsystems A_6, E_6, and D_6

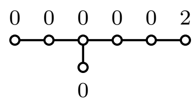
There are several difficulties to consider when comparing the nilpotent pieces in Levi subgroups G_0 to the nilpotent pieces in the bigger group G.

1. Note that the weighted Dynkin diagrams of G_0 and G are in general not compatible with respect to their weights.

2. We need to know which nilpotent G_0-orbits are included in which nilpotent G-orbits. As there are usually more G-orbits than G_0-orbits, there will be G-orbits that do not contain any G_0-orbits. For instance, in the case of E_7, simple groups of type E_6, A_6, and D_6 possess fewer nilpotent orbits than a simple group of type E_7. For the orbits not containing an orbit of a smaller rank subgroup, we may not be able to use any results from G_0 in order to help with the calculations in G.

3. Suppose we find a nilpotent orbit representative $x \in \mathrm{Lie}(G_0) \subseteq \mathfrak{g}$ where $G_0 \leqslant G$ is a Levi subgroup of which we already know the nilpotent pieces. Suppose further that we can find a map $\tilde{\delta}$ conjugate to δ for a weighted Dynkin diagram $\delta \in \mathscr{D}_G$ of G, such that $\tilde{\delta}$ agrees with a weighted Dynkin diagram on the simple roots of G_0. In this case we know whether for $g \in G_0$ we have $[\mathrm{Ad}(g)(x)]_{\mathfrak{g}_2^{\tilde{\delta}}} \in \mathfrak{g}_2^{\delta!}$. However, there seems to be no easy way to extend this to $g \in G \setminus G_0$. It does not even seem possible to only consider elements in a (right or left) transversal of G_0 in G. The problem is the following. Suppose $g = g_1 g_2$ for an element $g \in G$ with $g_1 \in L$ and $g_2 \in G_0$ where L is a right transversal of ${}^{G}/{}_{G_0}$. Then we would have to decide whether $\mathrm{Ad}(g_1 g_2)(x) \in \mathfrak{g}^{\delta}_{\geqslant 2}$. Note that even if $\mathrm{Ad}(g_2)(x) \notin \mathfrak{g}^{\delta}_{\geqslant 2}$, we might have $\mathrm{Ad}(g_1 g_2)(x) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ (or vice versa). It is therefore not enough to focus on a transversal.

Example 5.2. To illustrate this problem, consider a simple algebraic group G of type E_7 with simple roots $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_7\}$ corresponding to the nodes in the weighted Dynkin diagram as in Figure 2.2, and a Levi subgroup G_0 of type D_6. Let $\{\alpha_1, \dots, \alpha_6\}$ be the simple roots of the root system of G_0. Let δ arise from the weighted Dynkin diagram



.

Let $x_2 = e_{1^2,2^2,3^2,4^4,5^3,6^2,7} \in \mathfrak{g}$ be a nilpotent orbit representative, using the same notation as described in Section 2.3.2. Then we can compute the action of elements n_w and $n_{w'}$ on x_2, where $w = s_{\alpha_7} \in W$ and $w' = s_{\alpha_7} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1} \in W$. Note that in a transversal of ${}^{G}/_{G_0}$ both elements are in the same coset. However, $\mathrm{Ad}(n_w)(x_2) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ and $\mathrm{Ad}(n_{w'})(x_2) \notin \mathfrak{g}^{\delta}_{\geqslant 2}$.

The first problem can be solved as follows. Recall, that the weighted Dynkin diagrams are a way to describe one representative $\delta \in \mathscr{D}_G$ under the action of G. It is not strictly necessary to use the map δ induced by the weighted Dynkin diagram in our calculations. Any map $g.\delta$ in the same G-orbit on $\mathscr{D}_G$ can be used in the same way, unless specified otherwise in Chapter 4. The easiest way to find other maps in the orbit of δ is to compute them under the action of W.

Lemma 5.3. *Let $\delta : \Phi \longrightarrow \mathbb{Z}$ be a map induced from $\delta \in \mathscr{D}_G$. Let $w \in W$ and $n_w \in G$ be a representative. Then $n_w.\delta(\alpha) = \delta(w^{-1}.\alpha)$ for all $\alpha \in \Phi$.*

Proof. We recall the definition of δ: we have $\delta(\alpha) = \langle \alpha, \delta \rangle$ where $\alpha(\delta(c)) = c^{\langle \alpha, \delta \rangle}$ for all $c \in \mathbf{k}$. Then $n_w.\delta(\alpha) = \langle \alpha, n_w.\delta \rangle$, so compute

$$\alpha(n_w.\delta(c)) = \alpha(n_w \delta(c) n_w^{-1}) = n_w^{-1}.\alpha(\delta(c)) = (w^{-1}.\alpha)(\delta(c)).$$

This shows that $\langle \alpha, n_w.\delta \rangle = \langle w^{-1}.\alpha, \delta \rangle$, and hence the claim follows. □

Lemma 5.4. *Let δ be a weighted Dynkin diagram for G and $\delta_0 \in \mathscr{D}_G$ such that $\delta_0 \sim \delta$ with $n_w.\delta = \delta_0$ for some $w \in W$. Furthermore, let $y \in \mathfrak{g}^{\delta}_{\geqslant 2}$ be a nilpotent element. Then*

$$[y]_{\mathfrak{g}_2^\delta} = \mathrm{Ad}(n_w^{-1}) \left([\mathrm{Ad}(n_w)(y)]_{\mathfrak{g}_2^{\delta_0}} \right).$$

Proof. Write $y = \sum_{\substack{\alpha \in \Phi \\ \delta(\alpha) \geqslant 2}} \lambda_\alpha e_\alpha$ where $\lambda_\alpha \in \mathbf{k}$ and $\langle e_\alpha \rangle = \mathfrak{g}_\alpha$. Then $[y]_{\mathfrak{g}_2^\delta} = \sum_{\substack{\alpha \in \Phi \\ \delta(\alpha) = 2}} \lambda_\alpha e_\alpha$. Note that as $y \in \mathfrak{g}^{\delta}_{\geqslant 2}$ we have $\mathrm{Ad}(n_w)(y) \in \mathrm{Ad}(n_w)(\mathfrak{g}^{\delta}_{\geqslant 2}) = \mathfrak{g}^{\delta_0}_{\geqslant 2}$ by Lemma 3.8 (i). It is therefore possible to compute $[\mathrm{Ad}(n_w)(y)]_{\mathfrak{g}_2^{\delta_0}}$.
On the other hand, we have

$$\begin{aligned} \mathrm{Ad}(n_w)(y) &= \sum_{\substack{\alpha \in \Phi \\ \delta(\alpha) \geqslant 2}} \lambda_\alpha (-1)^{k_{\alpha,w}} e_{w.\alpha} \\ &= \sum_{\substack{\alpha \in \Phi \\ \delta_0(w.\alpha) \geqslant 2}} \lambda_\alpha (-1)^{k_{\alpha,w}} e_{w.\alpha}, \end{aligned}$$

where $k_{\alpha,w} \in \mathbb{Z}$ arises from the action of n_w as described in Section 1.2.5. As the exact value of $k_{\alpha,w}$ is not important for this calculation, we do not need to specify it.
So $[\mathrm{Ad}(n_w)(y)]_{\mathfrak{g}_2^{\delta_0}} = \sum_{\substack{\alpha \in \Phi \\ \delta_0(w.\alpha) = 2}} \lambda_\alpha (-1)^{k_{\alpha,w}} e_{w.\alpha}$. Finally, we have

$$\begin{aligned} \mathrm{Ad}(n_w^{-1}) \left([\mathrm{Ad}(n_w)(y)]_{\mathfrak{g}_2^{\delta_0}} \right) &= \sum_{\substack{\alpha \in \Phi \\ \delta_0(w.\alpha) = 2}} \lambda_\alpha (-1)^{k_{\alpha,w}} (-1)^{k_{\alpha,w}} e_{w^{-1}.(w.\alpha)} \\ &= \sum_{\substack{\alpha \in \Phi \\ \delta_0(w.\alpha) = 2}} \lambda_\alpha e_\alpha \\ &= \sum_{\substack{\alpha \in \Phi \\ \delta(\alpha) = 2}} \lambda_\alpha e_\alpha = [y]_{\mathfrak{g}_2^\delta}. \end{aligned}$$

This completes the proof. □

The two previous lemmas can be used to simplify the computations if the nilpotent pieces in smaller root systems are already known.

Proposition 5.5. *Let G be a connected reductive algebraic group with root system $\Phi := \Phi(G)$. Let $G_0 \subseteq G$ be a closed connected reductive algebraic group of maximal rank in G with root system $\Phi_0 := \Phi(G_0) \subseteq \Phi$.*
Suppose that the nilpotent pieces in $\mathfrak{g}_0 := \mathrm{Lie}(G_0)$ are already known for each weighted Dynkin diagram δ_0, and that for each nilpotent orbit representative $x_0 \in \mathfrak{g}_0$ we know the nilpotent orbit representative $x \in \mathfrak{g}$ such that $x_0 \in \mathscr{O}_x$.
Suppose further that there exists $w \in W$ such that $\delta_0' := n_w\delta$ with $\delta_0'(\alpha) = \delta_0(\alpha)$ for all $\alpha \in \Phi_0$, where δ is a weighted Dynkin diagram of G and δ_0 is a weighted Dynkin diagram of G_0.
Fix a nilpotent orbit representative $x \in \mathfrak{g}$. If for each nilpotent orbit representative $x_0 \in \mathscr{O}_x \cap \mathfrak{g}_0$ it is known that $\mathscr{O}_{x_0} \cap \mathfrak{g}_0 \not\subseteq \mathscr{N}_{\mathfrak{g}_0}^{\delta_0}$, then $[y]_{\mathfrak{g}_2^\delta} \notin \mathfrak{g}_2^{\delta!}$ for all $y \in \mathscr{O}_x \cap \mathfrak{g}_{\geqslant 2}^\delta$ where $\mathrm{Ad}(n_w)(y) \in \mathfrak{g}_0$.

Proof. Let $y \in \mathscr{O}_x \cap \mathfrak{g}_{\geqslant 2}^\delta$ with $\mathrm{Ad}(n_w)(y) \in \mathfrak{g}_0$. By assumption, $\mathrm{Ad}(n_w)(y) \notin \mathscr{N}_{\mathfrak{g}_0}^{\delta_0}$. As in the proof of Lemma 5.4, it follows that $y_w' := \mathrm{Ad}(n_w)(y) \in \mathfrak{g}_{\geqslant 2}^{\delta_0'} \cap \mathfrak{g}_0$. We already know that $C_{G_0}\left([y_w']_{(\mathfrak{g}_0)_2^{\delta_0}}\right) \not\subseteq (G_0)_{\geqslant 0}^{\delta_0}$ as we assume that $\mathscr{O}_{x_0} \cap \mathfrak{g}_0 \not\subseteq \mathscr{N}_{\mathfrak{g}_0}^{\delta_0}$. This means that there exists an element $g_{w'} = u't n_{w'} u \in G_0$ with $w' = s_{\alpha_1} \cdots s_{\alpha_r}$, $\alpha_i \in \Phi_0$ and $\mathrm{Ad}(g_{w'})(y_w') = (y_w')$. Now w' does not have weight 0 with respect to δ_0' by the definition of δ_0'. It follows that $g_{w'} \notin G_{\geqslant 0}^{\delta_0'}$ as an element of G by Lemma 4.2. Therefore, $C_G\left([y_w']_{\mathfrak{g}_2^{\delta_0'}}\right) \not\subseteq (G)_{\geqslant 0}^{\delta_0'}$. Note that, since $y_w' \in \mathfrak{g}_0$, we have $[y_w']_{(\mathfrak{g}_0)_2^{\delta_0}} = [y_w']_{(\mathfrak{g})_2^{\delta_0'}}$.
The fact that $C_G\left([y_w']_{\mathfrak{g}_2^{\delta_0'}}\right)$ is not a subset of $G_{\geqslant 0}^{\delta_0'}$ is equivalent to

$$n_w^{-1} C_G\left([y_w']_{\mathfrak{g}_2^{\delta_0'}}\right) n_w \not\subseteq n_w^{-1} G_{\geqslant 0}^{\delta_0'} n_w = G_{\geqslant 0}^{n_w^{-1}.\delta_0'} = G_{\geqslant 0}^{\delta}.$$

Finally, note that

$$\begin{aligned} n_w^{-1} C_G\left([y_w']_{\mathfrak{g}_2^{\delta_0'}}\right) n_w &= C_G\left(\mathrm{Ad}(n_w^{-1})\left([\mathrm{Ad}(n_w)(y)]_{\mathfrak{g}_2^{\delta_0'}}\right)\right) \\ &= C_G\left([y]_{\mathfrak{g}_2^\delta}\right) \end{aligned}$$

by Lemma 5.4. So it follows that $C_G\left([y]_{\mathfrak{g}_2^\delta}\right) \not\subseteq G_{\geqslant 0}^\delta$, that is, $[y]_{\mathfrak{g}_2^\delta} \notin \mathfrak{g}_2^{\delta!}$ as claimed. □

In order to use the result of this proposition, we need to know two things:

1. Orbit inclusion of the smaller Lie algebras in the orbits of the Lie algebra.
2. The weighted Dynkin diagrams δ_0 of G such that δ_0 is in the same G-orbit as a weighted Dynkin diagram $\delta \in \mathscr{D}_G$.

We can compute these by using the help of the programme from the previous section. The results are stated in the Appendix in Tables A.5 – A.8. This makes the computation easier once the results for (1) and (2) are known. Note however, that both (1) and (2) have to be computed for every type of root system.

6 | Implementation

In this chapter we take a closer look at implementing the ideas developed in the preceding chapters. For the implementation Magma V2.26-6 ([2]) was used on Gentoo Linux, version 5.15.32. Examining the implementation in more detail, we notice that there are two main difficulties to consider, namely

1. the implementation of the action of the simple algebraic group G on its Lie algebra $\mathfrak{g}$ and
2. solving non-linear equation systems with many indeterminates (e.g. in E_7 we may have up to 78 indeterminates to consider).

Additionally, we cannot work over a finite field or a field of characteristic 0, as the solution of the non-linear equation systems is sometimes in the algebraic closure of a field of positive characteristic. Keeping this in mind, the choice of the underlying programming language is rather limited (unless we would like to programme a rather large set of structures and algorithms ourself). We have therefore decided to use Magma, [2], as the underlying programming language.
We will use the following notation, similar to Section 4.3. Let $x \in \mathfrak{g}$ denote a nilpotent orbit representative. Define tuples

$$\begin{aligned} c &:= (c_\alpha)_{\alpha\in\Phi^+}, \\ c' &:= c'_w := (c'_\alpha)_{\substack{\alpha\in\Phi^+ \\ w^{-1}.\alpha\in\Phi^-}}, \quad \text{for } w \in W, \text{ and} \\ d &:= (d_\alpha)_{\alpha\in\Pi}, \end{aligned} \tag{6.1}$$

where the entries are indeterminates in a function field over $\mathbf{k}$. Let $g_w := u'(c')t(d)n_w u(c)$ be the Bruhat decomposition of an element in a group of the same type as G but defined over a function field, where

$$\begin{aligned} u'(c') &:= \prod_{\substack{\alpha\in\Phi^+ \\ w^{-1}.\alpha\in\Phi^-}} u_\alpha(c'_\alpha), \\ t(d) &:= \prod_{\alpha\in\Pi} h_\alpha(d_\alpha), \text{ and} \\ u(c) &:= \prod_{\alpha\in\Phi^+} u_\alpha(c_\alpha) \end{aligned}$$

with the h_α defined as in Section 1.2.5 and for a fixed total ordering on Φ^+. We write

$$x_w(c, d, c') := \mathrm{Ad}(u'(c')t(d)n_w u(c))(x)$$

for the action of g_w on x, where we consider x as an element in the Lie algebra over the function field.

Furthermore, let $x'_w(c,d,c') := [x_w(c,d,c')]_{\mathfrak{g}_2^\delta}$ and write $y_w := x_w(c_\varepsilon, d_\varepsilon, c'_\varepsilon)$ with

$$\begin{aligned} c_\varepsilon &= (c_{\varepsilon_\alpha})_{\alpha\in\Phi^+}, \\ d_\varepsilon &= (d_{\varepsilon_\gamma})_{\gamma\in\Pi}, \text{ and} \\ c_\varepsilon &= (c'_{\varepsilon_\beta})_{\substack{\beta\in\Phi^+ \\ w^{-1}.\alpha\in\Phi^-}} \end{aligned}$$

where $c_{\varepsilon_\alpha}, c'_{\varepsilon_\beta}, d_{\varepsilon_\gamma} \in \{0,1,\dots,\mathrm{char}(\mathbf{k})-1\}$ are fixed elements, such that we do not divide by zero when computing y_w for $\alpha\in\Phi^+$, $\beta\in\Phi^+$ with $w^{-1}.\beta\in\Phi^-$, and $\gamma\in\Pi$. If $\mathrm{char}(\mathbf{k}) = 0$ we will check whether this is possible for $c_{\varepsilon_\alpha}, c'_{\varepsilon_\beta}, d_{\varepsilon_\gamma} \in \{0,1\}$. If not, we will not replace the corresponding entries in c, d, or c'. For the $\mathfrak{g}_2^\delta$-part of y write $y'_w := [y_w]_{\mathfrak{g}_2^\delta}$.

Finally, let $\lambda_{w,\beta}(c,d,c') \in \mathbf{k}$ for $\beta\in\Phi$ and $\mu_{w,i}(c,d,c') \in \mathbf{k}$ for $i\in\{1,\dots,|\Pi|\}$ be defined as in (4.1):

$$\mathrm{Ad}(g_w)(x_w(c,d,c')) = \sum_{\beta\in\Phi} \lambda_{w,\beta}(c,d,c')e_\beta + \sum_{i=1}^{|\Pi|} \mu_{w,i}(c,d,c')h_i.$$

This chapter will be organised as follows: We first look at the Ad-action of G on $\mathfrak{g}$. Considering the fact that we need free variables in the computations, we next describe the construction of the polynomial rings containing these elements. In the last step, we describe how to best solve the arising non-linear equation systems.

6.1 Implementation of the Ad-action of G on $\mathfrak{g}$

While there exists an implementation of the Ad-action of G on $\mathfrak{g}$ in Magma, this does not work if G (and therefore also $\mathfrak{g}$) is defined over certain polynomial rings. Unfortunately, as noted above, we need to be able to use indeterminates and are therefore forced to implement the action of G on $\mathfrak{g}$ ourselves. This is done with the help of the results described in Section 1.2.5. In order to minimise computational work, we compute the action of each $n_{\alpha_i}, h_{\alpha_i}$, and $u := \prod_{\alpha\in\Phi^+} u_\alpha(c_\alpha)$ on a Chevalley basis of the Lie algebra. Here, $\alpha_i \in \Pi = \{\alpha_1,\dots,\alpha_r\}$ and the c_α are indeterminates defined over the field $\mathbf{k}$ for all $\alpha\in\Phi$. In order to compute the action of G on $\mathfrak{g}$, we can use the results of this action on the Chevalley basis as determined earlier. The action on an arbitrary element in $\mathfrak{g}$ is computed by linear extension without having to explicitly calculate the Ad-action again. Especially in groups with bigger root systems this saves a lot of time, as the calculation on the basis elements has to be done only once. In comparison, computing the Ad-action by using linear combinations of the results on the basis elements is a lot easier and faster than computing the action anew each time.

6.2 Polynomial rings, function fields and algebraic closures

In order to compute the action of G on the Lie algebra and to solve certain equations, it is necessary to define polynomial rings and function fields containing the indeterminates of the actions from (6.1). The construction is done as follows:

1. As a first step, we need to compute the action of an element $g_w \in G$ on a nilpotent orbit representative $e \in \mathfrak{g}$. The naive approach is to construct a polynomial ring over the finite field $\mathbb{F}_p$ and add indeterminates c_α, c'_α and d_β with $\alpha \in \Phi^+$, $\beta \in \Pi$, using the same notation as above. We note, however, that the indeterminates d_β, parametrising torus elements, need to be invertible. Therefore, we need to first define a function field $f := \mathbb{F}_p(d_\beta \mid \beta \in \Pi)$, containing the indeterminates d_β. Note that mathematically we would like to use the ring of Laurent polynomials,
$$\mathbb{F}_p[c_\alpha, c'_\beta, d_\gamma, d_\gamma^{-1} \mid \alpha, \beta \in \Phi^+, \gamma \in \Pi].$$
However, at the time of programming this, it was not possible to use this construction in Magma and we use the function field construction instead.
In the second step, we can define the polynomial ring $P := f[c_\alpha, c'_\alpha \mid \alpha \in \Phi^+]$. This will allow us to compute $\mathrm{Ad}(g_w)(x) = x_w(c, d, c')$.

2. We need to decide whether $x_w(c, d, c')$ is in $\mathfrak{g}^\delta_{\geqslant 2}$. This results in a system of equations, setting each coefficient $\lambda_\alpha(c, d, c')$ of $e_\alpha \in \mathfrak{g}_\alpha$ to zero if $\delta(\alpha) > 2$. However, the solutions may be in an algebraic closure of an underlying finite field. Additionally, we want the indeterminates in d to be considered for the solution. This means that (once we have the equation system) we redefine everything over the polynomial ring
$$P := \mathbb{F}_p[c_\alpha, c'_\beta, d_\gamma \mid \alpha, \beta \in \Phi^+, \gamma \in \Pi].$$
Note that each equation needs to be multiplied by the common denominator of its coefficients in order to do so. The solution will in most cases still depend on indeterminates which we also call (by an abuse of notation) c, d, c'. We need to take the algebraic closure of $\mathbb{F}_p$ in order to compute the solutions that are not in $\mathbb{F}_p$.

3. Suppose, we know that $x_w(c, d, c') \in \mathfrak{g}^\delta_{\geqslant 2}$. Next we want to compute the centraliser of $x'_w(c, d, c') := [x_w(c, d, c')]_{\mathfrak{g}^\delta_2}$ for all possible values for $c, d,$ and c'. However, if we try to compute a Gröbner basis over a polynomial ring containing these indeterminates, the solution might depend on explicit values of c, d, and c'. In order to prevent this from happening, we need to redefine $x_w(c, d, c')$ over a function field containing the indeterminates c, d, and c'. Additionally, we would like to have the algebraic closure containing the solution from (2). As this proves to be difficult when solving further non-linear equation systems, we instead add the minimal polynomials (given in the corresponding indeterminates) of elements in the algebraic closure (and not in $\mathbf{k}$) to the equation systems. Then we define the corresponding indeterminates over a polynomial ring, not a function field. We now have redefined $x_w(c, d, c')$ over $P := f[X_i]$ where $f = \mathbb{F}_p(c, d, c')$ is a function field containing the free variables from the solution in (2), and the X_i are the indeterminates with solution in an algebraic closure.

4. Finally, we want to have a similar construction as in (1), so that we can compute the action of a $g_{w'}$ on $x'_w(c, d, c') := [x_w(c, d, c')]_{\mathfrak{g}^\delta_2}$ and solve the system arising from the equation $\mathrm{Ad}(g_{w'})(x'_w(c, d, c')) = x'_w(c, d, c')$. We therefore define a polynomial ring over a function field over f in the same number of indeterminates as in (1), adding the X_i from the previous step to the polynomial ring.

Note that this brings also another set of difficulties with it: as the construction of the polynomial rings and function fields is rather complicated, it is not easy to change the ring or field over which the indeterminates are defined. Mostly, the polynomial ring will be defined as a variation of something as follows:

$$\begin{array}{c} P := F[c_\alpha, c'_\alpha \mid \alpha \in \Phi^+] \\ | \\ F := A(d_\alpha \mid \alpha \in \Pi) \\ | \\ A \text{ algebraic closure of } f \\ | \\ f := K(\tilde{c}_\alpha, \tilde{c}'_\beta, \tilde{d}_\gamma \mid \alpha, \beta \in \Phi^+,\ \gamma \in \Pi) \\ | \\ K \text{ either finite field } \mathbb{F}_p \text{ or } \mathbb{Q} \end{array}$$

We need these constructions in different parts of the programme, depending on which problem needs to be solved. Note for instance, that Gröbner bases will only provide a solution for the indeterminates in a polynomial ring, not the indeterminates in any function fields contained in the polynomial ring.

6.3 Solving non-linear equation systems

Having described a way in which to define the elements over polynomial rings, we are now in the position to attempt to find a solution for the non-linear equation systems described in Section 4.3. This part proves to be the most difficult one. Even for smaller root systems, the naive computation of Gröbner bases may require too much memory space. It is therefore necessary to find several ways in which to simplify the system of non-linear equations that need to be solved. We have already seen a few ways in which to do this in the previous section and in Chapter 4.
Note also that we have different problems for each of which we need to solve a system of non-linear equations. We will look at each problem separately and remark upon the ways in which to simplify the arising Gröbner bases.

6.3.1 General remarks

For the computation of Gröbner bases, we use the command `GroebnerBasis(I)` in Magma, where `I` is an ideal in a polynomial ring. We use the construction as detailed in [3, Section 112.4.3] without any further parameters. The Gröbner basis returned by Magma is unique and sorted with respect to a fixed monomial ordering, see [3, Section 112.4.2]. In the case of this programme we have fixed the graded reverse lexicographical order “`grevlex`” see [3, Section 112.2.3].

In general, solving non-linear systems of equations can be simplified as follows: Checking each equation, it may turn out that in some equations certain indeterminates only occur as a linear factor. In this case we can easily solve this equation for one of the indeterminates occurring linearly. It is then possible to replace this variable in

the system by the solution which will be saved in a separate list in the programme. Not only does this reduce the number of equations contained in the system, but also the number of indeterminates. In fact, if we reduce the number of indeterminates in the polynomial ring over which the system is defined to only the indeterminates contained in this system, it is easier to find a Gröbner basis.
Furthermore, we note that some equations can be factorised into several polynomials. In this case we get a new equation system for each factor. This in turn means we only have to check the systems until one of them yields a solution. However, we would need to check every equation system to be sure that no equation yields a solution. If we expect this to be the case, it is better not to factorise the equation.

Note furthermore, (as already remarked upon in the previous section) that the non-linear equation systems arise from the action of an element $g_w = u'tn_wu$ in the Bruhat-decomposition (see (4.2)). It is usually the case for an element $x \in \mathfrak{g}$ that there exists a subset $\Phi_x^+ \subseteq \Phi^+$ such that $\prod_{\alpha\in\Phi_x^+} u_\alpha(c_\alpha) \in C_G(x)$ for any choice of $c_\alpha \in \mathbf{k}$. We will compute this subset Φ_x^+ for each x that gives rise to a non-linear equation system. This in turn will enable us to solve a system arising from the action of $g_w = \tilde{u}'tn_w\tilde{u}$ on x instead. In this case we have

$$\tilde{u} = \prod_{\alpha\in\Phi^+\setminus\Phi_x^+} u_\alpha(c_\alpha) \text{ and } \tilde{u}' = \prod_{\substack{\alpha\in\Phi^+\setminus\Phi_x^+\\ w^{-1}.\alpha\in\Phi^-}} u_\alpha(c'_\alpha).$$

It is clear that this system contains less indeterminates and should consequently be easier to solve.
The same approach can be used when considering the torus elements $t \in T$. Recall that we can parametrise a torus element by a product $t = \prod_{\alpha\in\Pi} h_\alpha(d_\alpha)$, $d_\alpha \in \mathbf{k}^\times$. As before, we can check for which simple roots $\alpha \in \Pi$ an orbit representative $x \in \mathfrak{g}$ is centralised by $h_\alpha(d_\alpha)$ (regardless of the choice of $d_\alpha \in \mathbf{k}^\times$). Let $\Pi_x \subseteq \Pi$ denote the set of simple roots fulfilling this property. In this case, we may instead consider $t' = \prod_{\alpha\in\Pi\setminus\Pi_x} h_\alpha(d_\alpha)$, $d_\alpha \in \mathbf{k}^\times$, see Section 4.3.

A closer look at the torus variables

It is also worth noting that — even if a solution of an equation system exists — it does not necessarily mean that the equation system is solvable for every choice of variables. For instance, the variables describing the action of the torus on $\mathfrak{g}$ cannot be zero. Thus, we have to check that this does not occur. Additionally, note that these variables must be invertible when computing the action of g_w on x. It is therefore necessary that we first compute the action of g_w on x (including variables) by defining a polynomial ring over a function field over $\mathbb{F}_p$. The variables describing the action of the torus will be contained in the function field while all other variables are in the polynomial ring, see Section 6.2. In a second step, once we have computed the system of equations, we will multiply by all denominators containing indeterminates over the function field such that we can now consider the whole system over a polynomial ring. This is necessary so we can apply the Gröbner basis algorithm without disregarding the torus variables. Note that in this case we also need to ensure that none of the denominators we multiplied with will be zero in the solution.

Computing the solutions from the Gröbner basis

Having computed the Gröbner basis, we are still faced with the task of finding the ac-

tual solution of the system. One problem is that, in general, we will have to solve the system while considering that some variables can be chosen freely.
We proceed as follows. We start with the last element with respect to the reverse degree lexicographic order in the list of the Gröbner basis and continue up until the first element. We check if this is a univariate polynomial, in which case the solution will be given by its roots. Note that the roots have to be computed in the algebraic closure of $\mathbb{F}_p$, that is, in practice we define $k^{(1)} := \mathbb{F}_p(a_{1,1}, \dots, a_{1,n_1})$ and $k^{(i)} := k^{(i-1)}(a_{i,1}, \dots, a_{i,n_i})$ for $i > 1$ for each equation we solve, where the $a_{i,j}$ are the roots of the ith polynomial. Consequently, we have to redefine all elements of the Gröbner basis over $k^{(i)}$ in each step. For each solution a_i we will get a new basis where we evaluate the remaining Gröbner basis elements at a_i. We then proceed as before for each new arising basis.
Should we come across a polynomial that is not univariate, we will have to choose one indeterminate with respect to which we compute the roots of the polynomial. In order to do that, we will redefine the elements of the Gröbner basis over an algebraic closure of a function field containing all other indeterminates in the polynomial and proceed as before. This will however lead to the problem that at some point there might be a polynomial in the Gröbner basis only containing indeterminates defined over the function field. In this case we have to chose one of the indeterminates, redefine this indeterminate over the polynomial ring, and find the roots with respect to this indeterminate. We need to take care to replace this indeterminate by its roots in the solutions already computed.

6.3.2 Free variables and preventing division by zero

Suppose we are at a point where we have to check whether there exists an element $g_w \in G$ such that $\mathrm{Ad}(g_w)(x'_w(c,d,c')) = x'_w(c,d,c')$ (respectively for y'_w or $x''_w(c)$ by the notation in Section 4.3). This means that we are in Step 5 in Section 4.3. Note that we want this equality to hold regardless of the choices for $c, d,$ and c'. However, the solution g_w may depend on the choice of our variables. In general, this will be no problem unless we divide by a term containing one of the indeterminates. To ensure that we do find a solution for any possible choice of $c, d,$ and c', we proceed as follows:

1. Find the solution from the Gröbner basis arising from the system
$$\mathrm{Ad}(g_w)(x'_w(c,d,c')) = x'_w(c,d,c')$$
as described before.

2. Check each term in the solution for division by a term with an indeterminate contained in $c, d,$ or c'. Note that these terms may also be *hidden* in an element defined over an algebraic closure. For these elements we have to check the minimal polynomial instead.

3. Save the values for the indeterminates for which division by zero occurs in a list L_0.

4. For each value $z \in L_0$ we compute $x'_w(z) := [x_w(z)]_{\mathfrak{g}_2^\delta}$ and recursively check everything again for $x'_w(z)$ starting from (1).
If there is a solution such that L_0 is empty, we can go up one step in the recursion

and continue with the programme.
Should we not find a solution for $x'_w(z)$, set $y' := x'_w(z)$ and continue checking for all the Weyl group elements $w \in W$ if there exists $g_w \in G \setminus G^{\delta}_{\geqslant 0}$ that centralises y'. If we find such a g_w, proceed as before. Otherwise we have $y \in \mathscr{N}_{\mathfrak{g}}^{\blacktriangle}$.

6.3.3 Deciding whether an element is in $\mathfrak{g}^{\delta}_{\geqslant 2}$

As seen in Section 4.3, we first want to decide whether $\mathrm{Ad}(g_w)(x) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ for a given representative of a nilpotent orbit $x \in \mathfrak{g}$ and an element of the Weyl group $w \in W$. This system is still relatively easy to solve: We have seen that it is enough to check whether $\mathrm{Ad}(n_w\tilde{u}')(x) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ where $\tilde{u}'$ is as in Section 6.3.1 (see also Corollary 4.3). In this case the polynomial ring we work over is not too complex, as it is only the polynomial ring over $\mathbf{k}$ (in fact we will first find the Gröbner basis over $\mathbb{F}_p$ where $p = \mathrm{char}(\mathbf{k})$ and in a later step solve for the indeterminates over the algebraic closure of $\mathbb{F}_p$). We proceed by the steps given in "General remarks", 6.3.1. Should there exist a solution, we need to decide whether $[\mathrm{Ad}(n_w\tilde{u}')(x)]_{\mathfrak{g}^{\delta}_2} \in \mathfrak{g}^{\delta !}_2$.

6.3.4 Deciding whether an element is in $\mathfrak{g}^{\delta !}_2$

Firstly, we want to decide whether the solutions are in $\mathfrak{g}^{\delta !}_2$ for all possible solutions $x_w(c, d, c')$ as detailed in Section 6.3.3 in the above step. We will therefore compute $\mathrm{Ad}(n_w\tilde{u}')(x)$, where we have replaced the indeterminates in $\tilde{u}'$ by their solutions such that $\mathrm{Ad}(n_w\tilde{u}')(x) \in \mathfrak{g}^{\delta}_{\geqslant 2}$ and $\tilde{u}'$ contains only these factors $u_\alpha(c_\alpha)$ with $w.\alpha \in \Phi^-$ as in Section 4.3. If there are free variables in the solution, we will define a function field containing these indeterminates and use them instead as described in Section 6.2 (3). Note that the following calculations will be more complex if a function field is involved. We will therefore also define one specific fixed element y_w, in which each free variable is replaced by an element in $\mathbf{k}$, depending on whether one of the options leads to division by zero. In practice we will only check this for the elements $\{0, 1, \dots, \mathrm{char}(\mathbf{k}) - 1\}$ or $\{0, 1\}$ if $\mathrm{char}(\mathbf{k}) = 0$. If every option results in division by zero, this indeterminate cannot be replaced. The idea is to check for each element $w \in W$ whether $g_w = u'tn_wu$ centralises (for arbitrary u', t, u) this fixed representative. Should this be the case, we need to check the same thing for the representative x depending on the indeterminates c, c', and d. We can skip this step if it turns out that these elements are in the same orbit under the action of some element $g_w \in G^{\delta}_{\geqslant 0}$, as described in Section 4.3. We check whether $\mathrm{Ad}(g_w)(y_w) = x_w(c, d, c')$ recursively: Should we find a solution for the resulting Gröbner basis, we need to ensure that this solution for g_w holds for all choices of variables in $x_w(c, d, c')$. In particular, we need to make sure that division by zero does not occur. In order to check this, we look at the resulting solutions and consider all denominators, proceeding as in Section 6.3.3. If there are variables for which the denominators turn zero, we save the values for these variables. For each value, we evaluate the new element (it now depends on less indeterminates) and repeat the process. If at some point the process returns that there is no solution for the choice of variables, we save them in a list. In this case, we need to check whether an element $g_{w'} \in G$ also centralizes $x_w(c, d, c')$ evaluated at these values if $g_{w'}$ centralised y_w in the first place.
Note that this way of solving the problem might not always be the best; sometimes it is better to just decide whether $\mathrm{Ad}(g_{w'})(x_w(c, d, c')) = x_w(c, d, c')$ (that is whether

$x_w(c, d, c')$ is centralised by an element $g_{w'}$) for all values of the free variables. This is in particular the case when some variables are contained in the algebraic closure but not in the prime field.
It may be the case that an element $x_w(c, d, c') \in \mathfrak{g}$ is already centralised by some $g_{w'} = u'tn_wu \in G$ where $t = 1$. Let

$$\Pi_x := \{\alpha \in \Pi \mid h_\alpha(d_\alpha) \in C_G(x_w(c, d, c')) \text{ for all } d_\alpha \in \mathbf{k}^\times\},$$

see Section 6.3.1. We will compute Π_x and try solving the equations first for $t = 1$. We will successively add one factor $h_\alpha(c_\alpha)$ to the previously used t, where $\alpha \notin \Pi_x$, until we find a solution or we have added all such factors (this may not always be the fastest way, however note that proceeding in this manner may provide us with a solution where it was difficult for the computer to solve the system before).

7 | Results

In this last chapter we present the results of the programme described in the previous chapters. We fix the following notation for this chapter: Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots of a simple algebraic group G for some $r \in \mathbb{N}$. Denote the elements e_α spanning $\mathfrak{g}_\alpha$, $\alpha \in \Phi$, $\mathfrak{g} = \mathrm{Lie}(G)$ by $e_{1^{i_1},2^{i_2},\dots,r^{i_r}} := e_{i_1\alpha_1+i_2\alpha_2+\cdots+i_r\alpha_r}$, $i_j \in \mathbb{Z}$ and $i_1\alpha_1 + i_2\alpha_2 + \cdots + i_r\alpha_r \in \Phi$. If $i_j = 0$, we will simply leave j^{i_j} out, see also the notation in Section 2.3.2. If we can prove that the nilpotent pieces in a Lie algebra $\mathfrak{g}$ are disjoint, the nilpotent pieces agree with the CP-pieces. For the results stated in this chapter, we repeat Remark 3.14 about the CP-pieces.

Remark 7.1. Note that by [9, Theorem 7.3] the CP-pieces agree with the nilpotent pieces if the nilpotent pieces form a partition of the nilpotent variety $\mathscr{N}_\mathfrak{g}$.

7.1 G_2

The weighted Dynkin diagrams of type G_2 are as in Figure 7.1, see [7, 13.1, p. 401].

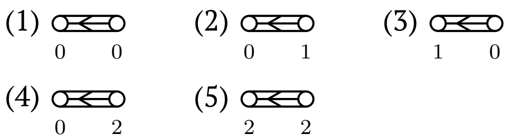
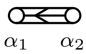


Figure 7.1: The weighted Dynkin diagrams of type G_2

Furthermore, the root system is given by

$$\Phi := \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2), \pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\},$$

where the simple roots $\{\alpha_1, \alpha_2\}$ are as in the following Dynkin diagram: 

.

We begin with the case where $\mathrm{char}(\mathbf{k}) = 2$. By [32], we have the following orbit representatives, where we let $\mathfrak{g}_\gamma = \langle e_\gamma \rangle$ for $\gamma \in \Phi$. Recall that we have already given the nilpotent orbit representatives in table A.1.

1. $x_1 := 0$,
2. $x_2 := e_{\alpha_1} + e_{\alpha_2}$,
3. $x_3 := e_{\alpha_1} + e_{2\alpha_1+\alpha_2}$,
4. $x_4 := e_{\alpha_1}$,
5. $x_5 := e_{\alpha_2}$.

Let $\mathscr{O}_i$ be the orbit corresponding to the representative x_i, $i \in \{1, \dots, 5\}$, and $\mathscr{N}_i$ be the nilpotent piece corresponding to the weighted Dynkin diagram (i), $i \in \{1, \dots, 5\}$, as in Figure 7.1. This results in the following nilpotent pieces:

1. $\mathscr{N}_1 := \{0\} = \mathscr{O}_1$,
2. $\mathscr{N}_2 := \mathscr{O}_5$,
3. $\mathscr{N}_3 := \mathscr{O}_4$,
4. $\mathscr{N}_4 := \mathscr{O}_3$,
5. $\mathscr{N}_5 := \mathscr{O}_2$.

If $\mathrm{char}(\mathbf{k}) = 3$, the following nilpotent orbit representatives are again given by [32] using the same notation as above:

1. $x_1 := 0$,
2. $x_2 := e_{\alpha_1} + e_{\alpha_2}$,
3. $x_3 := e_{\alpha_2} + e_{\alpha_1+\alpha_2}$,
4. $x_4 := e_{\alpha_2} + e_{2\alpha_1+\alpha_2}$,
5. $x_5 := e_{\alpha_1}$,
6. $x_6 := e_{\alpha_2}$.

Then we get the following pieces:

1. $\mathscr{N}_1 := \{0\} = \mathscr{O}_1$,
2. $\mathscr{N}_2 := \mathscr{O}_6$,
3. $\mathscr{N}_3 := \mathscr{O}_3 \cup \mathscr{O}_5$,
4. $\mathscr{N}_4 := \mathscr{O}_4$,
5. $\mathscr{N}_5 := \mathscr{O}_2$.

Note that in good characteristic, the representatives x_3 and x_5 are in the same nilpotent orbit, as $\mathrm{Ad}(u_{2\alpha_1+\alpha_2}\left(-\frac{1}{3}\right) n_{(s_{\alpha_2}s_{\alpha_1})^2})(x_3) = x_5$.
In order to give a better overview, we give the above results in the form of a table. If not otherwise stated, the nilpotent orbit representatives are true for all characteristics. In case (iii) we get a union of two orbits in characteristic 3, and the additional representative is denoted by $x_{3,2}$. We remark that the elements $e_{\alpha_1} + e_{2\alpha_1+\alpha_2}$ and $e_{\alpha_2} + e_{2\alpha_1+\alpha_2}$ are in the same nilpotent orbit. In the list above, we have chosen the nilpotent orbit representatives as in [32], which is why they differ.

	Weighted Dynkin Diagram	Nilpotent orbit representative
(i)	0 0	$x_1 := 0$
(ii)	0 1	$x_2 := e_{\alpha_2}$
(iii)	1 0	$x_3 := e_{\alpha_1}$ $x_{3,2} := e_{\alpha_2} + e_{\alpha_1+\alpha_2}$, $p = 3$
(iv)	0 2	$x_4 := e_{\alpha_1} + e_{2\alpha_1+\alpha_2}$
(v)	2 2	$x_5 := e_{\alpha_1} + e_{\alpha_2}$

Table 7.2: Nilpotent pieces in G_2

We summarise the above results as follows:

Theorem 7.2 (Nilpotent pieces for G_2 in characteristic 2 and 3)**.** *We use the same notation as above and let*

1. $x_1 := 0$,
2. $x_2 := e_{\alpha_1} + e_{\alpha_2}$,
3. $x_3 := e_{\alpha_2} + e_{\alpha_1+\alpha_2}$,
4. $x_4 := e_{\alpha_2} + e_{2\alpha_1+\alpha_2}$,
5. $x_5 := e_{\alpha_1}$,
6. $x_6 := e_{\alpha_2}$.

Then the nilpotent pieces for G_2 with respect to the weighted Dynkin diagrams are given by

1. $\mathscr{N}_1 := \{0\} = \mathscr{O}_{x_1}$,
2. $\mathscr{N}_2 := \mathscr{O}_{x_6}$,
3. $\mathscr{N}_3 := \mathscr{O}_{x_5}$,
4. $\mathscr{N}_4 := \mathscr{O}_{x_4}$,
5. $\mathscr{N}_5 := \mathscr{O}_{x_2}$

if $\mathrm{char}(\mathbf{k}) = 2$ *and by*

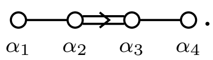
1. $\mathscr{N}_1 := \{0\} = \mathscr{O}_{x_1}$,
2. $\mathscr{N}_2 := \mathscr{O}_{x_6}$,
3. $\mathscr{N}_3 := \mathscr{O}_{x_3} \cup \mathscr{O}_{x_5}$,
4. $\mathscr{N}_4 := \mathscr{O}_{x_4}$,
5. $\mathscr{N}_5 := \mathscr{O}_{x_2}$

if $\mathrm{char}(\mathbf{k}) = 3$. *In particular, the nilpotent pieces form a partition of the nilpotent variety and therefore agree with the CP-pieces by Remark 7.1.*

Remark 7.3. In his paper [13], Hesselink describes a stratification of the nullcone of the Lie algebra of G_2. The results are exactly the nilpotent pieces as computed here, coming from an entirely different definition. But this stratification is just the CP-pieces, see [9, Theorem, Section 5].

7.2 F_4

We proceed as above by listing the weighted Dynkin diagrams for F_4 as well as the orbit representatives in good and bad characteristic. Note that the orbit representatives in good characteristic are the same as in characteristic 3. For the orbit representatives see [20, Table 22.1.4] and [29], and Table A.2. The simple roots are as given in the Dynkin diagram:



As the nilpotent orbit representatives are the same in good characteristic and for characteristic 3, these cases are not distinguished. In characteristic 2 we get additional orbits, which are denoted by $x_{i,2}$ if they are in the same orbit as x_i in good characteristic. All orbit representatives in good characteristic (or characteristic 3) are also orbit representatives in characteristic 2. Applying the programme described in Section 4 results in the following pieces in characteristic 2 (with the same notation as for G_2):

1. $\mathscr{N}_1 = \mathscr{O}_1$,
2. $\mathscr{N}_2 = \mathscr{O}_2$,
3. $\mathscr{N}_3 = \mathscr{O}_3 \cup \mathscr{O}_{3,2}$,
4. $\mathscr{N}_4 = \mathscr{O}_4$,
5. $\mathscr{N}_5 = \mathscr{O}_5$,
6. $\mathscr{N}_6 = \mathscr{O}_6 \cup \mathscr{O}_{6,2}$,
7. $\mathscr{N}_7 = \mathscr{O}_7$,
8. $\mathscr{N}_8 = \mathscr{O}_8 \cup \mathscr{O}_{8,2}$,
9. $\mathscr{N}_9 = \mathscr{O}_9 \cup \mathscr{O}_{9,2}$,
10. $\mathscr{N}_{10} = \mathscr{O}_{10} \cup \mathscr{O}_{10,2}$,
11. $\mathscr{N}_{11} = \mathscr{O}_{11}$,
12. $\mathscr{N}_{12} = \mathscr{O}_{12}$,
13. $\mathscr{N}_{13} = \mathscr{O}_{13} \cup \mathscr{O}_{13,2}$,
14. $\mathscr{N}_{14} = \mathscr{O}_{14}$,
15. $\mathscr{N}_{15} = \mathscr{O}_{15}$,
16. $\mathscr{N}_{16} = \mathscr{O}_{16}$,

where $\mathscr{N}_{16}$ follows by Proposition 3.21.
Note that each of the pairs of nilpotent orbit representatives x_3 and $x_{3,2}$, as well as x_6 and $x_{6,2}$, x_8 and $x_{8,2}$, x_9 and $x_{9,2}$, x_{10} and $x_{10,2}$, and x_{13} and $x_{13,2}$ are in the same nilpotent orbits in good characteristic. In particular, the nilpotent pieces contain the same nilpotent orbits (or the nilpotent orbits they split into) as in good characteristic. More precisely we can give explicit group elements sending one element to the other. We will use the same notation for the root subgroup elements $u_\alpha(c_\alpha)$, $c_\alpha \in \mathbf{k}, \alpha \in \Phi$, as for the e_α, see Section 2.3.2.

$$\begin{aligned}
\mathrm{Ad}\left(h_4(1)u_{1,2,3}\left(-\frac{1}{2}\right)\right)(x_3) &= x_{3,2}\\
\mathrm{Ad}\left(h_1(-1)h_2(-1)u_{1,2^2,3^3,4}\left(-\frac{1}{2}\right)\right)(x_6) &= x_{6,2}\\
\mathrm{Ad}\left(u_{\alpha_3}(1)h_1(-1)u_3\left(\frac{1}{2}\right)\right)(x_8) &= x_{8,2}\\
\mathrm{Ad}\left(h_1(-1)h_2(-1)n_{s_{\alpha_2}}u_3\left(-\frac{1}{2}\right)u_{1,2,3^2,4}\left(\frac{1}{8}\right)u_{2,3^2,4^2}\left(-\frac{1}{2}\right)\right)(x_9) &= x_{9,2}\\
\mathrm{Ad}\left(u_{3,4}(z)h_1(1)h_3(-1)h_4(z)u_{2,3}\left(\frac{1}{2}\right)\right)(x_{10}) &= x_{10,2}\\
\mathrm{Ad}\left(h_1(-1)h_3(1)h_4(-1)n_{s_{\alpha_2}}u_{1,2^2,3^2,4}\left(-\frac{1}{2}\right)\right)(x_{13}) &= x_{13,2}.
\end{aligned}$$

Here we choose $z \in \mathbf{k}$ such that $z^2 = -1$. We expect this to be a pattern that should hold for the other exceptional groups as well. In characteristic 3 we use the same orbit representatives (i.e. mapping the coefficients of the linear combinations of basis elements into $\mathbf{k}$). Applying the programme results in the same pieces as in characteristic 0, i.e. we have $\mathscr{N}_i = \mathscr{O}_i$ for all $i = 1, \ldots, 16$ with the same notation as above. As before, we also state the results in the form of a table.

	Weighted Dynkin diagram	Nilpotent orbit representative
(i)	0 0 0 0	$x_1 := 0$
(ii)	1 0 0 0	$x_2 := e_1$
(iii)	0 0 0 1	$x_3 := e_{1,2^2,3^3,4^2}$ $x_{3,2} := e_{1,2^2,3^3,4^2} + e_{1^2,2^3,3^4,4^2}$, $p = 2$
(iv)	0 1 0 0	$x_4 := e_{1,2^2,3^3,4} + e_{1,2^2,3^2,4^2}$
(v)	2 0 0 0	$x_5 := e_{1,2^2,3^2} + e_{1,2,3^2,4^2}$
(vi)	0 0 0 2	$x_6 := e_{1,2,3,4} + e_{2,3^2,4}$ $x_{6,2} := e_{1,2,3,4} + e_{2,3^2,4} + e_{1^2,2^3,3^4,4^2}$, $p = 2$
(vii)	0 0 1 0	$x_7 := e_{1,2^2,3^2} + e_{1,2,3^2,4} + e_{2,3^2,4^2}$
(viii)	2 0 0 1	$x_8 := e_{1,2,3} + e_{2,3^2,4^2}$ $x_{8,2} := e_{1,2} + e_{1,2,3^2} + e_{2,3^2,4^2}$, $p = 2$
(ix)	0 1 0 1	$x_9 := e_{1,2,3} + e_{2,3^2,4} + e_{1,2^2,3^2,4^2}$ $x_{9,2} := e_{1,2,3} + e_{2,3^2,4} + e_{1,2,3^2,4^2} + e_{1,2^2,3^2}$, $p = 2$
(x)	1 0 1 0	$x_{10} := e_{1,2,3} + e_{2,3^2,4} + e_{2,3^2,4^2}$ $x_{10,2} := e_{1,2} + e_{2,3^2,4} + e_{1,2,3^2,4^2} + e_{1,2^2,3^4,4^2}$, $p = 2$
(xi)	0 2 0 0	$x_{11} := e_{1,2,3} + e_{1,2,3^2} + e_{2,3,4} + e_{2^2,3^2,4^2}$
(xii)	2 2 0 0	$x_{12} := e_1 + e_{2,3} + e_{2,3,4} + e_{2,3^2,4^2}$
(xiii)	1 0 1 2	$x_{13} := e_4 + e_{1,2,3} + e_{2,3^2}$ $x_{13,2} := e_4 + e_{1,2,3} + e_{2,3^2} + e_{1,2^2,3^2,4^2}$, $p = 2$
(xiv)	0 2 0 2	$x_{14} := e_{1,2} + e_{2,3} + e_{3,4} + e_{1,2,3^2}$

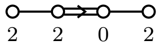
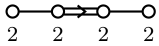
(xv)		$x_{15} := e_1 + e_4 + e_{2,3} + e_{2,3^2}$
(xvi)		$x_{16} := e_1 + e_2 + e_3 + e_4$

Table 7.3: Nilpotent pieces in F_4

Theorem 7.4 (Nilpotent pieces for F_4 in characteristic 2 and 3). *We use the same notation as above. Then the nilpotent pieces for F_4 in characteristic 2 are given by $\mathcal{N}_i = \mathcal{O}_i \cup \mathcal{O}_{i,2}$ for $i \in \{3, 6, 8, 9, 10, 13\}$ and $\mathcal{N}_i = \mathcal{O}_i$ otherwise.*
In characteristic 3 we get $\mathcal{N}_i = \mathcal{O}_i$ for all $1 \leqslant i \leqslant 16$. In particular, the nilpotent pieces form a partition of the nilpotent variety and therefore agree with the CP-pieces by Remark 7.1.

7.3 E_6

If the root system is of type E_6, we get the same number of orbits in good and bad characteristic. In fact we can choose the "same" orbit representatives for each characteristic, where the coefficients of the e_α are either 0 or 1. They can be found in Table A.3. Here, the roots $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ are the simple roots of Φ as denoted in the Dynkin diagram:

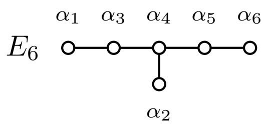
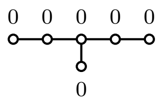
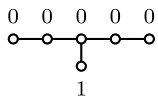
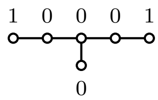
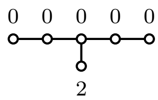


Figure 7.4: The Dynkin diagram of type E_6

The table below lists the nilpotent orbit representatives contained in the nilpotent pieces that are described by the weighted Dynkin diagrams in the left column.

	Weighted Dynkin Diagram	Nilpotent orbit representative
(i)		$x_1 := 0$
(ii)		$x_2 := e_{1,2^2,3^2,4^3,5^2,6}$
(iii)		$x_3 := e_{1,2,3^2,4^2,5,6} + e_{1,2,3,4^2,5^2,6}$
(iv)		$x_4 := e_{2,3,4,5,6} + e_{1,2,3,4^2,5}$

(v)	0 0 1 0 0 / 0	$x_5 := e_{1,2,3^2,4^2,5} + e_{1,2,3,4^2,5,6} + e_{2,3,4^2,5^2,6}$
(vi)	1 0 0 0 1 / 1	$x_6 := e_{1,3,4,5,6} + e_{2,3,4,5,6} + e_{1,2,3,4^2,5}$
(vii)	1 0 0 0 1 / 2	$x_7 := e_{2,3,4} + e_{2,4,5} + e_{1,3,4,5,6}$
(viii)	0 1 0 1 0 / 0	$x_8 := e_{1,2,3,4,5} + e_{1,3,4,5,6} + e_{2,3,4^2,5} + e_{2,3,4,5,6}$
(ix)	2 0 0 0 2 / 0	$x_9 := e_{1,2,3,4} + e_{1,3,4,5} + e_{2,4,5,6} + e_{3,4,5,6}$
(x)	0 1 0 1 0 / 1	$x_{10} := e_{3,4,5} + e_{1,2,3,4} + e_{2,4,5,6} + e_{1,3,4,5,6}$
(xi)	2 0 0 0 2 / 2	$x_{11} := e_{5,6} + e_{1,3,4} + e_{2,3,4} + e_{2,4,5}$
(xii)	0 0 2 0 0 / 2	$x_{12} := e_2 + e_{1,3,4} + e_{3,4,5} + e_{4,5,6}$
(xiii)	0 0 2 0 0 / 0	$x_{13} := e_{3,4,5} + e_{4,5,6} + e_{1,2,3,4} + e_{2,4,5,6}$
(xiv)	1 0 1 0 1 / 0	$x_{14} := e_{1,2,3,4} + e_{1,3,4,5} + e_{2,4,5,6} + e_{3,4,5,6} + e_{2,3,4^2,5}$
(xv)	1 1 0 1 1 / 1	$x_{15} := e_{5,6} + e_{1,3,4} + e_{2,3,4} + e_{2,4,5} + e_{3,4,5}$
(xvi)	2 1 0 1 2 / 1	$x_{16} := e_1 + e_6 + e_{2,3,4} + e_{2,4,5} + e_{3,4,5}$
(xvii)	2 0 2 0 2 / 2	$x_{17} := e_2 + e_6 + e_{1,3} + e_{3,4} + e_{4,5}$
(xviii)	1 1 0 1 1 / 2	$x_{18} := e_{1,3} + e_{2,4} + e_{5,6} + e_{3,4,5} + e_{4,5,6}$
(ixx)	2 2 0 2 2 / 2	$x_{19} := e_1 + e_3 + e_{2,4} + e_{4,5} + e_5 + e_6$

(xx)	2 0 2 0 2 0	$x_{20} := e_1 + e_{3,4} + e_{2,4} + e_{2,4,5} + e_{2,3,4,5} + e_{5,6}$
(xxi)	2 2 2 2 2 2	$x_{21} := e_1 + e_2 + e_3 + e_4 + e_5 + e_6$

Table 7.5: Nilpotent pieces in E_6

Applying the programme results in the same pieces as in characteristic 0, i.e. we have $\mathscr{N}_i = \mathscr{O}_i$ for all $i = 1, \dots, 21$ with the same notation as above.

Theorem 7.5 (Nilpotent pieces for E_6 in characteristic 2 and 3)**.** *We use the same notation as above. Then the nilpotent pieces for E_6 in both characteristic 2 and 3 are given by $\mathscr{N}_i = \mathscr{O}_i$ for all $1 \leqslant i \leqslant 21$. In particular, the nilpotent pieces form a partition of the nilpotent variety and therefore agree with the CP-pieces by Remark 7.1.*

7.4 E_7

Unfortunately, the computations for type E_7 in this version of the programme are still too complex to yield results for every piece. However, it was possible to compute some of the nilpotent pieces in characteristic 2, the solution of which will be given below. We use the same notation as in E_6 and the nilpotent orbit representatives are the ones given in Magma for non-exceptional type and are taken from the book of Liebeck and Seitz [20] otherwise.
The following table displays the weighted Dynkin diagrams and in the right column the nilpotent orbit representatives in the nilpotent pieces corresponding to the respective weighted Dynkin diagrams. The rows that have not been computed yet have been left out.

	Weighted Dynkin Diagram E_7	Nilpotent orbit representative
(i)	0 0 0 0 0 0 0	$x_1 := 0$
(ii)	1 0 0 0 0 0 0	$x_2 := e_{1^2,2^2,3^3,4^4,5^3,6^2,7}$
(iii)	0 0 0 0 1 0 0	$x_3 := e_{1,2^2,3^2,4^3,5^2,6^2,7} + e_{1,2,3^2,4^3,5^3,6^2,7}$
(iv)	0 0 0 0 0 2 0	$x_4 := e_{1,2,3^2,4^2,5,6,7} + e_{1,2,3,4^2,5^2,6,7} + e_{2,3,4^2,5^2,6^2,7}$

(v)	0 1 0 0 0 0 0	$x_5 := e_{1,2^2,3^2,4^3,5^2,6} + e_{1,2,3^2,4^3,5^2,6,7} + e_{1,2,3^2,4^2,5^2,6^2,7}$
(vi)	2 0 0 0 0 0 0	$x_6 := e_{1,2,3,4^2,5,6,7} + e_{1,2,3^2,4^2,5^2,6}$
(vii)	0 0 0 0 0 1 1	$x_7 := e_{1,2,3^2,4^2,5,6,7} + e_{1,2,3,4^2,5^2,6,7} + e_{2,3,4^2,5^2,6^2,7} + e_{1,2^2,3^2,4^3,5^2,6}$
(viii)	1 0 0 0 1 0 0	$x_8 := e_{1,2,3,4^2,5,6,7} + e_{1,2,3^2,4^2,5^2,6} + e_{2,3,4^2,5^2,6^2,7}$
(ix)	0 0 1 0 0 0 0	$x_9 := e_{1,2,3^2,4^2,5,6} + e_{1,2,3,4^2,5^2,6} + e_{1,2,3,4^2,5,6,7} + e_{2,3,4^2,5^2,6,7}$
(xi)	0 0 0 0 2 0 0	$x_{11} := e_{1,3,4,5,6,7} + e_{2,3,4,5,6,7} + e_{1,2,3,4^2,5,6} + e_{2,3,4^2,5^2,6}$
(xii)	0 0 0 0 0 0 2	$x_{12} := e_{1,2,3^2,4^2,5,6} + e_{1,2,3,4^2,5^2,6} + e_{1,2,3,4^2,5,6,7} + e_{2,3,4^2,5^2,6,7}$
(xiii)	2 0 0 0 0 2 0	$x_{13} := e_{2,4,5,6,7} + e_{3,4,5,6,7} + e_{1,2,3,4^2,5} + e_{1,2,3,4,5,6}$
(xvi)	0 2 0 0 0 0 0	$x_{16} := e_{2,3,4,5,6} + e_{3,4,5,6,7} + e_{1,2,3,4^2,5} + e_{1,3,4,5,6,7}$

Table 7.6: Nilpotent pieces in E_7 for $\mathrm{char}(\mathbf{k}) = 2$

A | Appendix

A.1 Nilpotent orbit representatives

We write down the representatives of the nilpotent orbits in the case of bad characteristic for the exceptional groups of Lie type for later use. They are taken from [20, Tables 22.1.1 – 22.1.5] as well as from the nilpotent orbit representatives given in Magma [2]. The orbit representatives in F_4 are taken from [29, Table 1]. Note that in bad characteristic one has to check that each of the representatives given by Magma defines a different orbit. For instance, the representatives given in Magma for the orbits of type A_5 and $E_6(a_3)$ in the simple group of type E_6 are in the same orbit in characteristic 2 and we have to choose a different representative for the orbit of type $E_6(a_3)$.

A.1.1 G_2

Class	Weighted Dynkin Diagram	Nilpotent orbit representative
1	0 0	$x_1 := 0$
A_1	0 1	$x_2 := e_2$
$\tilde{A}_1$	1 0	$x_3 := e_1$, $p \neq 3$
$(\tilde{A}_1)_3$		$x_{3,2} := e_2 + e_{1,2}$, $p = 3$
$G_2(a_1)$	0 2	$x_4 := e_2 + e_{1^3,2}$
G_2	2 2	$x_5 := e_1 + e_2$

Table A.1: Nilpotent orbits in G_2

A.1.2 F_4

Class	Weighted Dynkin diagram	Nilpotent orbit representative
1	0 0 0 0	$x_1 := 0$
A_1	1 0 0 0	$x_2 := e_1$

$\tilde{A}_1$ $(\tilde{A}_1)_2$	0 0 0 1	$x_3 := e_{1,2^2,3^3,4^2}$ $x_{3,2} := e_{1,2^2,3^3,4^2} + e_{1^2,2^3,3^4,4^2}$, $p = 2$
$A_1\tilde{A}_1$	0 1 0 0	$x_4 := e_{1,2^2,3^3,4} + e_{1,2^2,3^2,4^2}$
A_2	2 0 0 0	$x_5 := e_{1,2^2,3^2} + e_{1,2,3^2,4^2}$
$\tilde{A}_2$ $(\tilde{A}_2)_2$	0 0 0 2	$x_6 := e_{1,2,3,4} + e_{2,3^2,4}$ $x_{6,2} := e_{1,2,3,4} + e_{2,3^2,4} + e_{1^2,2^3,3^4,4^2}$, $p = 2$
$A_2\tilde{A}_1$	0 0 1 0	$x_7 := e_{1,2^2,3^2} + e_{1,2,3^2,4} + e_{2,3^2,4^2}$
B_2 $(B_2)_2$	2 0 0 1 2 0 0 1	$x_8 := e_{1,2,3} + e_{2,3^2,4^2}$ $x_{8,2} := e_{1,2} + e_{1,2,3^2} + e_{2,3^2,4^2}$, $p = 2$
$\tilde{A}_2A_1$ $(\tilde{A}_2A_1)_2$	0 1 0 1	$x_9 := e_{1,2,3} + e_{2,3^2,4} + e_{1,2^2,3^2,4^2}$ $x_{9,2} := e_{1,2,3} + e_{2,3^2,4} + e_{1,2,3^2,4^2} + e_{1,2^2,3^2}$, $p = 2$
$C_3(a_1)$ $(C_3(a_1))_2$	1 0 1 0	$x_{10} := e_{1,2,3} + e_{2,3^2,4} + e_{2,3^2,4^2}$ $x_{10,2} := e_{1,2} + e_{2,3^2,4} + e_{1,2,3^2,4^2} + e_{1,2^2,3^4,4^2}$, $p = 2$
$F_4(a_3)$	0 2 0 0	$x_{11} := e_{1,2,3} + e_{1,2,3^2} + e_{2,3,4} + e_{2^2,3^2,4^2}$
B_3	2 2 0 0	$x_{12} := e_1 + e_{2,3} + e_{2,3,4} + e_{2,3^2,4^2}$
C_3 $(C_3)_2$	1 0 1 2	$x_{13} := e_4 + e_{1,2,3} + e_{2,3^2}$ $x_{13,2} := e_4 + e_{1,2,3} + e_{2,3^2} + e_{1,2^2,3^2,4^2}$, $p = 2$
$F_4(a_2)$	0 2 0 2	$x_{14} := e_{1,2} + e_{2,3} + e_{3,4} + e_{1,2,3^2}$
$F_4(a_1)$	2 2 0 2	$x_{15} := e_1 + e_4 + e_{2,3} + e_{2,3^2}$
F_4	2 2 2 2	$x_{16} := e_1 + e_2 + e_3 + e_4$

Table A.2: Nilpotent orbits in F_4

A.1.3 E_6

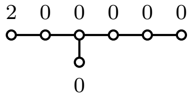
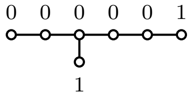
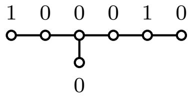
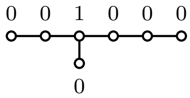
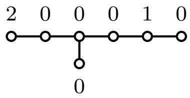
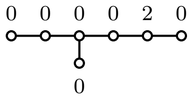
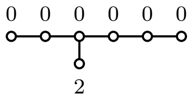
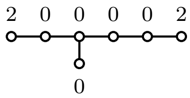
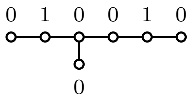
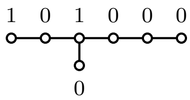
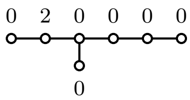
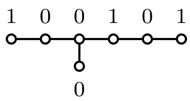
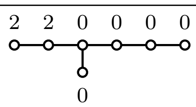
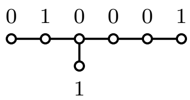
	Weighted Dynkin diagram	Nilpotent orbit representative
1	0 0 0 0 0 0	$x_1 := 0$
A_1	0 0 0 0 0 1	$x_2 := e_{1,2^2,3^2,4^3,5^2,6}$
A_1^2	1 0 0 0 1 0	$x_3 := e_{1,2,3^2,4^2,5,6} + e_{1,2,3,4^2,5^2,6}$
A_2	0 0 0 0 0 2	$x_4 := e_{2,3,4,5,6} + e_{1,2,3,4^2,5}$
A_1^3	0 0 1 0 0 0	$x_5 := e_{1,2,3^2,4^2,5} + e_{1,2,3,4^2,5,6} + e_{2,3,4^2,5^2,6}$
A_2A_1	1 0 0 0 1 1	$x_6 := e_{1,3,4,5,6} + e_{2,3,4,5,6} + e_{1,2,3,4^2,5}$
A_3	1 0 0 0 1 2	$x_7 := e_{2,3,4} + e_{2,4,5} + e_{1,3,4,5,6}$
$A_2A_1^2$	0 1 0 1 0 0	$x_8 := e_{1,2,3,4,5} + e_{1,3,4,5,6} + e_{2,3,4^2,5} + e_{2,3,4,5,6}$
A_2^2	2 0 0 0 2 0	$x_9 := e_{1,2,3,4} + e_{1,3,4,5} + e_{2,4,5,6} + e_{3,4,5,6}$
A_3A_1	0 1 0 1 0 1	$x_{10} := e_{3,4,5} + e_{1,2,3,4} + e_{2,4,5,6} + e_{1,3,4,5,6}$
A_4	2 0 0 0 2 2	$x_{11} := e_{5,6} + e_{1,3,4} + e_{2,3,4} + e_{2,4,5}$
D_4	0 0 2 0 0 2	$x_{12} := e_2 + e_{1,3,4} + e_{3,4,5} + e_{4,5,6}$
$D_4(a_1)$	0 0 2 0 0 0	$x_{13} := e_{3,4,5} + e_{4,5,6} + e_{1,2,3,4} + e_{2,4,5,6}$
$A_2^2A_1$	1 0 1 0 1 0	$x_{14} := e_{1,2,3,4} + e_{1,3,4,5} + e_{2,4,5,6} + e_{3,4,5,6} + e_{2,3,4^2,5}$

A_4A_1	1 1 0 1 1; 1	$x_{15} := e_{5,6} + e_{1,3,4} + e_{2,3,4} + e_{2,4,5} + e_{3,4,5}$
A_5	2 1 0 1 2; 1	$x_{16} := e_1 + e_6 + e_{2,3,4} + e_{2,4,5} + e_{3,4,5}$
D_5	2 0 2 0 2; 2	$x_{17} := e_2 + e_6 + e_{1,3} + e_{3,4} + e_{4,5}$
$D_5(a_1)$	1 1 0 1 1; 2	$x_{18} := e_{1,3} + e_{2,4} + e_{5,6} + e_{3,4,5} + e_{4,5,6}$
$E_6(a_1)$	2 2 0 2 2; 2	$x_{19} := e_1 + e_3 + e_{2,4} + e_{4,5} + e_5 + e_6$
$E_6(a_3)$	2 0 2 0 2; 0	$x_{20} := e_1 + e_{3,4} + e_{2,4} + e_{2,4,5} + e_{2,3,4,5} + e_{5,6}$
E_6	2 2 2 2 2; 2	$x_{21} := e_1 + e_2 + e_3 + e_4 + e_5 + e_6$

Table A.3: Nilpotent orbits in E_6

A.1.4 E_7

Class	Weighted Dynkin Diagram E_7	Nilpotent orbit representative
1	0 0 0 0 0 0; 0	$x_1 := 0$
A_1	1 0 0 0 0 0; 0	$x_2 := e_{1^2,2^2,3^3,4^4,5^3,6^2,7}$
A_1^2	0 0 0 0 1 0; 0	$x_3 := e_{1,2^2,3^2,4^3,5^2,6^2,7} + e_{1,2,3^2,4^3,5^3,6^2,7}$
$(A_1^3)^{(1)}$	0 0 0 0 0 2; 0	$x_4 := e_{1,2,3^2,4^2,5,6,7} + e_{1,2,3,4^2,5^2,6,7} + e_{2,3,4^2,5^2,6^2,7}$
$(A_1^3)^{(2)}$	0 1 0 0 0 0; 0	$x_5 := e_{1,2^2,3^2,4^3,5^2,6} + e_{1,2,3^2,4^3,5^2,6,7} + e_{1,2,3^2,4^2,5^2,6^2,7}$

A_2		$x_6 := e_{1,2,3,4^2,5,6,7} + e_{1,2,3^2,4^2,5^2,6}$
A_1^4		$x_7 := e_{1,2,3^2,4^2,5,6,7} + e_{1,2,3,4^2,5^2,6,7} + e_{2,3,4^2,5^2,6^2,7} + e_{1,2^2,3^2,4^3,5^2,6}$
A_2A_1		$x_8 := e_{1,2,3,4^2,5,6,7} + e_{1,2,3^2,4^2,5^2,6} + e_{2,3,4^2,5^2,6^2,7}$
$A_2A_1^2$		$x_9 := e_{1,2,3^2,4^2,5,6} + e_{1,2,3,4^2,5^2,6} + e_{1,2,3,4^2,5,6,7} + e_{2,3,4^2,5^2,6,7}$
A_3		$x_{10} := e_{1,2,3,4} + e_{1,3,4,5} + e_{2,3,4^2,5^2,6^2,7}$
A_2^2		$x_{11} := e_{1,3,4,5,6,7} + e_{2,3,4,5,6,7} + e_{1,2,3,4^2,5,6} + e_{2,3,4^2,5^2,6}$
$A_2A_1^3$		$x_{12} := e_{1,2,3^2,4^2,5,6} + e_{1,2,3,4^2,5^2,6} + e_{1,2,3,4^2,5,6,7} + e_{2,3,4^2,5^2,6,7}$
$(A_3A_1)^{(1)}$		$x_{13} := e_{2,4,5,6,7} + e_{3,4,5,6,7} + e_{1,2,3,4^2,5} + e_{1,2,3,4,5,6}$
$A_2^2A_1$		$x_{14} := e_{1,3,4,5,6,7} + e_{2,3,4,5,6,7} + e_{1,2,3^2,4^2,5} + e_{1,2,3,4^2,5,6} + e_{2,3,4^2,5^2,6}$
$(A_3A_1)^{(2)}$		$x_{15} := e_{2,3,4^2,5} + e_{1,2,3,4,5,6} + e_{1,3,4,5,6,7} + e_{2,3,4^2,5^2,6^2,7}$
$D_4(a_1)$		$x_{16} := e_{2,3,4,5,6} + e_{3,4,5,6,7} + e_{1,2,3,4^2,5} + e_{1,3,4,5,6,7}$
$A_3A_1^2$		$x_{17} := e_{2,4,5,6,7} + e_{3,4,5,6,7} + e_{1,2,3,4^2,5} + e_{1,2,3,4,5,6} + e_{2,3,4^2,5^2,6}$
D_4		$x_{18} := e_1 + e_{2,3,4^2,5} + e_{2,3,4,5,6} + e_{3,4,5,6,7}$
$D_4(a_1)A_1$		$x_{19} := e_{2,3,4,5,6} + e_{2,4,5,6,7} + e_{3,4,5,6,7} + e_{1,2,3,4^2,5} + e_{1,3,4,5,6,7}$

A_3A_2 $(A_3A_2)_2$	0 0 1 0 1 0 0	$x_{20} := e_{1,3,4,5,6} + e_{2,3,4,5,6} + e_{2,4,5,6,7} + e_{3,4,5,6,7} + e_{1,2,3,4^2,5}$ $x_{20,2} := e_7 + e_{6,5,4} + e_{6,5,4,3,2} + e_{5,4,3} + e_{4,2} + e_{7,6}$
A_4	2 0 0 0 2 0 0	$x_{21} := e_{1,2,3,4} + e_{1,3,4,5} + e_{4,5,6,7} + e_{2,3,4,5,6}$
$A_3A_2A_1$	0 0 0 2 0 0 0	$x_{22} := e_{1,2,3,4,5} + e_{1,3,4,5,6} + e_{2,3,4^2,5} + e_{2,3,4,5,6} + e_{2,4,5,6,7} + e_{3,4,5,6,7}$
$(A_5)^{(1)}$	2 0 0 0 2 2 0	$x_{23} := e_7 + e_{1,2,3,4} + e_{1,3,4,5} + e_{2,4,5,6} + e_{3,4,5,6}$
D_4A_1	2 1 0 0 0 1 1	$x_{24} := e_1 + e_{2,3,4^2,5} + e_{2,3,4,5,6} + e_{2,4,5,6,7} + e_{3,4,5,6,7}$
A_4A_1	1 0 1 0 1 0 0	$x_{25} := e_{1,2,3,4} + e_{1,3,4,5} + e_{4,5,6,7} + e_{2,3,4^2,5} + e_{2,3,4,5,6}$
$D_5(a_1)$	2 0 1 0 1 0 0	$x_{26} := e_{1,3} + e_{2,4,5,6} + e_{4,5,6,7} + e_{2,3,4^2,5} + e_{3,4,5,6,7}$
A_4A_2	0 0 2 0 0 0 0	$x_{27} := e_{1,2,3,4} + e_{1,3,4,5} + e_{2,3,4,5} + e_{2,4,5,6} + e_{3,4,5,6} + e_{4,5,6,7}$
$(A_5)^{(2)}$	1 0 1 0 2 0 0	$x_{28} := e_{5,6} + e_{6,7} + e_{1,2,3,4} + e_{1,3,4,5} + e_{2,3,4^2,5}$
A_5A_1	1 0 1 0 1 2 0	$x_{29} := e_7 + e_{1,2,3,4} + e_{1,3,4,5} + e_{2,4,5,6} + e_{3,4,5,6} + e_{2,3,4^2,5}$
$D_5(a_1)A_1$	2 0 0 2 0 0 0	$x_{30} := e_{3,4,5} + e_{4,5,6} + e_{5,6,7} + e_{1,2,3,4} + e_{2,3,4,5,6} + e_{2,4,5,6,7}$
$D_6(a_2)$	0 1 0 1 0 2 1	$x_{31} := e_{6,7} + e_{2,4,5} + e_{3,4,5} + e_{1,2,3,4} + e_{3,4,5,6} + e_{1,3,4,5,6}$
$E_6(a_3)$	0 2 0 0 2 0 0	$x_{32} := e_{1,3,4} + e_{2,3,4} + e_{4,5,6} + e_{5,6,7} + e_{2,3,4,5} + e_{1,2,3,4,5}$

D_5	2 2 0 0 2 0 0	x_{33}:= $e_1+e_{2,3,4}+e_{3,4,5}+e_{4,5,6}+e_{5,6,7}$
$E_7(a_5)$	0 0 2 0 0 2 0	x_{34}:= $e_{1,3,4}+e_{2,3,4}+e_{2,4,5}+e_{4,5,6}+e_{5,6,7}+e_{3,4,5,6}+e_{1,3,4,5,6}$
A_6 $(A_6)_2$	0 0 2 0 2 0 0	x_{35}:= $e_{5,6}+e_{6,7}+e_{1,3,4}+e_{2,3,4}+e_{2,4,5}+e_{3,4,5}$ $x_{35,2}$:= $e_{5,6}+e_{6,7}+e_{1,3,4}+e_{2,3,4}+e_{3,4,5}+e_{2,4,5}+e_{1,2,3^2,4^2,5}$
D_5A_1	2 1 0 1 1 0 1	x_{36}:= $e_1+e_{2,3,4}+e_{2,4,5}+e_{3,4,5}+e_{4,5,6}+e_{5,6,7}$
$D_6(a_1)$	2 1 0 1 0 2 1	x_{37}:= $e_1+e_{6,7}+e_{2,3,4}+e_{2,4,5}+e_{3,4,5}+e_{3,4,5,6}$
$E_7(a_4)$	2 0 2 0 0 2 0	x_{38}:= $e_{1,3}+e_{4,5}+e_{6,7}+e_{2,3,4}+e_{2,3,4,5}+e_{2,4,5,6}+e_{3,4,5,6}$
D_6	2 1 0 1 2 2 1	x_{39}:= $e_1+e_6+e_7+e_{2,3,4}+e_{2,4,5}+e_{3,4,5}$
$E_6(a_1)$	2 0 2 0 2 0 0	x_{40}:= $e_{1,3}+e_{2,4}+e_{4,5}+e_{5,6}+e_{6,7}+e_{3,4,5}$
E_6	2 2 2 0 2 0 0	x_{41}:= $e_1+e_3+e_{2,4}+e_{4,5}+e_{5,6}+e_{6,7}$
$E_7(a_3)$	2 0 2 0 2 2 0	x_{42}:= $e_1+e_{2,4}+e_{3,4}+e_{2,4,5}+e_{2,3,4,5}+e_{5,6}+e_7$
$E_7(a_2)$	2 2 0 2 0 2 2	x_{43}:= $e_1+e_2+e_3+e_{2,4}+e_{4,5}+e_{5,6}+e_{6,7}$
$E_7(a_1)$	2 2 0 2 2 2 2	x_{44}:= $e_1+e_3+e_{2,4}+e_{3,4}+e_5+e_6+e_7$
E_7	2 2 2 2 2 2 2	x_{45}:= $e_1+e_2+e_3+e_4+e_5+e_6+e_7$

Table A.4: Nilpotent orbits in E_7

A.2 Orbit inclusions

We give the orbit inclusions needed for the calculations in Section 5.2. These inclusions have been computed using parts of the programme. We check whether the representatives are in the same orbit in E_7. This can be made easier by taking the orbit representatives in E_7 and checking whether we can find an element in the group, such that they are sent to an element in the Lie algebra of the parabolic subgroup. Then we can check whether these elements are in the same orbit in the smaller Lie algebra.

Orbit Type	Orbit representative E_7	Orbit representative E_6	Orbit representative A_6	Orbit representative D_6
1	$x_1 = 0$	$y_1 = 0$	$z_1 = 0$	$w_1 = 0$
A_1	$x_2 = e_{1^2,2^2,3^3,4^4,5^3,6^2,7}$	$y_2 = e_{1,2^2,3^2,4^3,5^2,6}$	$z_2 = e_{1,3,4,5,6,7}$	$w_2 = e_{2,3,4^2,5^2,6^2,7}$
A_1^2	$x_3 = e_{1,2^2,3^2,4^3,5^2,6^2,7}$ $+ e_{1,2,3^2,4^3,5^3,6^2,7}$	$y_3 = e_{1,2,3^2,4^2,5,6}$ $+ e_{1,2,3,4^2,5^2,6}$	$z_3 = e_{1,3,4,5,6}$ $+ e_{3,4,5,6,7}$	$w_{3,1} = e_{3,4,5,6,7}$ $+ e_{2,4,5,6,7}$ $w_{3,2} = e_{2,3,4^2,5,6,7}$ $+ e_{2,3,4^2,5^2,6}$
$(A_1^3)^{(1)}$	$x_4 = e_{1,2,3^2,4^2,5,6,7}$ $+ e_{1,2,3,4^2,5^2,6,7}$ $+ e_{2,3,4^2,5^2,6^2,7}$	–	–	$w_4 = e_{2,4,5,6,7}$ $+ e_{2,3,4,5,6}$ $+ e_{2,3,4^2,5}$
$(A_1^3)^{(2)}$	$x_5 = e_{1,2^2,3^2,4^3,5^2,6}$ $+ e_{1,2,3^2,4^3,5^2,6,7}$ $+ e_{1,2,3^2,4^2,5^2,6^2,7}$	$y_5 = e_{1,2,3^2,4^2,5}$ $+ e_{1,2,3,4^2,5,6}$ $+ e_{2,3,4^2,5^2,6}$	$z_5 = e_{1,3,4,5}$ $+ e_{3,4,5,6,7}$ $+ e_{4,5,6,7}$	$w_{5,1} = e_{3,4,5,6,7}$ $+ e_{2,3,4,5,6}$ $+ e_{2,3,4^2,5}$ $w_{5,2} = e_{3,4,5,6,7}$ $+ e_{2,4,5,6,7}$ $+ e_{2,3,4^2,5^2,6}$
A_2	$x_6 = e_{1,2,3,4^2,5,6,7}$ $+ e_{1,2,3^2,4^2,5^2,6}$	$y_6 = e_{2,3,4,5,6}$ $+ e_{1,2,3,4^2,5}$	$z_6 = e_{1,3,4}$ $+ e_{5,6,7}$	$w_6 = e_{2,4,5,6}$ $+ e_{3,4,5,6,7}$
A_1^4	$x_7 = e_{1,2,3^2,4^2,5,6,7}$ $+ e_{1,2,3,4^2,5^2,6,7}$ $+ e_{2,3,4^2,5^2,6^2,7}$ $+ e_{1,2^2,3^2,4^3,5^2,6}$	–	–	$w_7 = e_{3,4,5,6,7}$ $+ e_{2,4,5,6,7}$ $+ e_{2,3,4,5,6}$ $+ e_{2,3,4^2,5}$
A_2A_1	$x_8 = e_{1,2,3,4^2,5,6,7}$ $+ e_{1,2,3^2,4^2,5^2,6}$ $+ e_{2,3,4^2,5^2,6^2,7}$	$y_8 = e_{1,3,4,5,6}$ $+ e_{2,3,4,5,6}$ $+ e_{1,2,3,4^2,5}$	$z_8 = e_{1,3,4}$ $+ e_{5,6,7}$ $+ e_{3,4,5,6}$	$w_8 = e_{2,4,5,6}$ $+ e_{3,4,5,6,7}$ $+ e_{2,3,4^2,5}$
$A_2A_1^2$	$x_9 = e_{1,2,3^2,4^2,5,6}$ $+ e_{1,2,3,4^2,5^2,6}$ $+ e_{1,2,3,4^2,5,6,7}$ $+ e_{2,3,4^2,5^2,6,7}$	$y_9 = e_{1,2,3,4,5}$ $+ e_{1,3,4,5,6}$ $+ e_{2,3,4^2,5}$ $+ e_{2,3,4,5,6}$	$z_9 = e_{1,3,4}$ $+ e_{3,4,5}$ $+ e_{4,5,6}$ $+ e_{5,6,7}$	$w_9 = e_{4,5,6,7}$ $+ e_{3,4,5,6}$ $+ e_{2,4,5,6}$ $+ e_{2,3,4,5}$

A_3	$x_{10} = e_{1,2,3,4}$ $+ e_{1,3,4,5}$ $+ e_{2,3,4^2,5^2,6^2,7}$	$y_{10} = e_{2,3,4}$ $+ e_{2,4,5}$ $+ e_{1,3,4,5,6}$	$z_{10} = e_1 + e_7$ $+ e_{3,4,5,6}$	$w_{10,1} = e_{6,7}$ $+ e_{5,6}$ $+ e_{2,3,4^2,5}$ $w_{10,2} = e_7$ $+ e_{3,4,5,6}$ $+ e_{2,4,5,6}$
A_2^2	$x_{11} = e_{1,3,4,5,6,7}$ $+ e_{2,3,4,5,6,7}$ $+ e_{1,2,3,4^2,5,6}$ $+ e_{2,3,4^2,5^2,6}$	$y_{11} = e_{1,2,3,4}$ $+ e_{1,3,4,5}$ $+ e_{2,4,5,6}$ $+ e_{3,4,5,6}$	$z_{11} = e_{5,6} + e_{6,7}$ $+ e_{1,3,4}$ $+ e_{3,4,5}$	$w_{11} = e_{2,4,5}$ $+ e_{2,3,4}$ $+ e_{4,5,6,7}$ $+ e_{3,4,5,6}$
$A_2A_1^3$	$x_{12} = e_{1,2,3^2,4^2,5,6}$ $+ e_{1,2,3,4^2,5^2,6}$ $+ e_{1,2,3,4^2,5,6,7}$ $+ e_{2,3,4^2,5^2,6,7}$	–	–	–
$(A_3A_1)^{(1)}$	$x_{13} = e_{2,4,5,6,7}$ $+ e_{3,4,5,6,7}$ $+ e_{1,2,3,4^2,5}$ $+ e_{1,2,3,4,5,6}$	–	–	$w_{13} = e_{5,4,7}$ $+ e_{4,5,6}$ $+ e_{2,4,5}$ $+ e_{2,3,4}$
$A_2^2A_1$	$x_{14} = e_{1,3,4,5,6,7}$ $+ e_{2,3,4,5,6,7}$ $+ e_{1,2,3^2,4^2,5}$ $+ e_{1,2,3,4^2,5,6}$ $+ e_{2,3,4^2,5^2,6}$	$y_{14} = e_{1,2,3,4}$ $+ e_{1,3,4,5}$ $+ e_{2,4,5,6}$ $+ e_{3,4,5,6}$ $+ e_{2,3,4^2,5}$	–	–
$(A_3A_1)^{(2)}$	$x_{15} = e_{2,3,4^2,5}$ $+ e_{1,2,3,4,5,6}$ $+ e_{1,3,4,5,6,7}$ $+ e_{2,3,4^2,5^2,6^2,7}$	$y_{15} = e_{3,4,5}$ $+ e_{1,2,3,4}$ $+ e_{2,4,5,6}$ $+ e_{1,3,4,5,6}$	$z_{15} = e_{1,3} + e_{4,5}$ $+ e_{6,7}$ $+ e_{3,4,5,6}$	$w_{15,1} = e_{5,6,7}$ $+ e_{4,5,6}$ $+ e_{3,4,5}$ $+ e_{2,3,4}$ $w_{15,2} = e_7$ $+ e_{3,4,5,6}$ $+ e_{2,4,5,6}$ $+ e_{2,3,4^2,5}$
$D_4(a_1)$	$x_{16} = e_{2,3,4,5,6}$ $+ e_{3,4,5,6,7}$ $+ e_{1,2,3,4^2,5}$ $+ e_{1,3,4,5,6,7}$	$y_{16} = e_{3,4,5}$ $+ e_{4,5,6}$ $+ e_{1,2,3,4}$ $+ e_{2,4,5,6}$	–	$w_{16} = e_{6,7}$ $+ e_{3,4,5}$ $+ e_{2,4,5}$ $+ e_{2,4,5,6}$
$A_3A_1^2$	$x_{17} = e_{2,4,5,6,7}$ $+ e_{3,4,5,6,7}$ $+ e_{1,2,3,4^2,5}$ $+ e_{1,2,3,4,5,6}$ $+ e_{2,3,4^2,5^2,6}$	–	–	$w_{17} = e_{5,6,7}$ $+ e_{4,5,6}$ $+ e_{3,4,5}$ $+ e_{2,4,5}$ $+ e_{2,3,4}$

D_4	$x_{18} = e_1$ $+ e_{2,3,4^2,5}$ $+ e_{2,3,4,5,6}$ $+ e_{3,4,5,6,7}$	$y_{18} = e_2$ $+ e_{1,3,4}$ $+ e_{3,4,5}$ $+ e_{4,5,6}$	–	$w_{18} = e_7 + e_6$ $+ e_{3,4,5}$ $+ e_{2,4,5}$
$D_4(a_1)A_1$	$x_{19} = e_{2,3,4,5,6}$ $+ e_{2,4,5,6,7}$ $+ e_{3,4,5,6,7}$ $+ e_{1,2,3,4^2,5}$ $+ e_{1,3,4,5,6,7}$	–	–	$w_{19} = e_{6,7}$ $+ e_{3,4,5}$ $+ e_{2,4,5}$ $+ e_{2,3,4}$ $+ e_{2,4,5,6}$
A_3A_2	$x_{20} = e_{1,3,4,5,6}$ $+ e_{2,3,4,5,6}$ $+ e_{2,4,5,6,7}$ $+ e_{3,4,5,6,7}$ $+ e_{1,2,3,4^2,5}$	–	$z_{20} = e_{1,3}$ $+ e_{3,4}$ $+ e_{4,5}$ $+ e_{5,6}$ $+ e_{6,7}$	$w_{20} = e_{3,4}$ $+ e_{2,4}$ $+ e_{5,6,7}$ $+ e_{4,5,6}$ $+ e_{2,3,4,5}$
$(A_3A_2)_2$	$x_{21} = e_7 + e_{2,4}$ $+ e_{6,7}$ $+ e_{4,5,6}$ $+ e_{2,3,4,5}$ $+ e_{2,3,4,5,6}$	–	–	$w_{21} = e_7$ $+ e_{4,5,6}$ $+ e_{2,3,4,5,6}$ $+ e_{3,4,5}$ $+ e_{2,4} + e_{6,7}$
A_4	$x_{22} = e_{1,2,3,4}$ $+ e_{1,3,4,5}$ $+ e_{4,5,6,7}$ $+ e_{2,3,4,5,6}$	$y_{22} = e_{5,6}$ $+ e_{1,3,4}$ $+ e_{2,3,4}$ $+ e_{2,4,5}$	$z_{22} = e_1 + e_7$ $+ e_{3,4} + e_{5,6}$	$w_{22} = e_{6,7} + e_{5,6}$ $+ e_{2,4} + e_{3,4,5}$
$A_3A_2A_1$	$x_{23} = e_{1,2,3,4,5}$ $+ e_{1,3,4,5,6}$ $+ e_{2,3,4^2,5}$ $+ e_{2,3,4,5,6}$ $+ e_{2,4,5,6,7}$ $+ e_{3,4,5,6,7}$	–	–	–
$(A_5)^{(1)}$	$x_{24} = e_7$ $+ e_{1,2,3,4}$ $+ e_{1,3,4,5}$ $+ e_{2,4,5,6}$ $+ e_{3,4,5,6}$	–	–	$w_{24} = e_2 + e_{6,7}$ $+ e_{5,6} + e_{4,5}$ $+ e_{3,4}$
D_4A_1	$x_{25} = e_1$ $+ e_{2,3,4^2,5}$ $+ e_{2,3,4,5,6}$ $+ e_{2,4,5,6,7}$ $+ e_{3,4,5,6,7}$	–	–	$w_{25} = e_7 + e_6$ $+ e_{3,4,5}$ $+ e_{2,4,5}$ $+ e_{2,3,4}$
A_4A_1	$x_{26} = e_{1,2,3,4}$ $+ e_{1,3,4,5}$ $+ e_{4,5,6,7}$ $+ e_{2,3,4^2,5}$ $+ e_{2,3,4,5,6}$	$y_{26} = e_{5,6}$ $+ e_{1,3,4}$ $+ e_{2,3,4}$ $+ e_{2,4,5}$ $+ e_{3,4,5}$	$z_{26} = e_1 + e_7$ $+ e_{3,4} + e_{4,5}$ $+ e_{5,6}$	–

$D_5(a_1)$	$x_{27} = e_{1,3} + e_{2,4,5,6} + e_{4,5,6,7} + e_{2,3,4^2,5} + e_{3,4,5,6,7}$	$y_{27} = e_{1,3} + e_{2,4} + e_{5,6} + e_{3,4,5} + e_{4,5,6}$	–	$w_{27} = e_7 + e_{5,6} + e_{3,4} + e_{2,4} + e_{2,4,5}$
A_4A_2	$x_{28} = e_{1,2,3,4} + e_{1,3,4,5} + e_{2,3,4,5} + e_{2,4,5,6} + e_{3,4,5,6} + e_{4,5,6,7}$	–	–	–
$(A_5)^{(2)}$	$x_{29} = e_{5,6} + e_{6,7} + e_{1,2,3,4} + e_{1,3,4,5} + e_{2,3,4^2,5}$	$y_{29} = e_1 + e_6 + e_{2,3,4} + e_{2,4,5} + e_{3,4,5}$	$z_{29} = e_1 + e_3 + e_4 + e_6 + e_{4,5}$	$w_{29} = e_3 + e_{6,7} + e_{5,6} + e_{4,5} + e_{2,4}$
A_5A_1	$x_{30} = e_7 + e_{1,2,3,4} + e_{1,3,4,5} + e_{2,4,5,6} + e_{3,4,5,6} + e_{2,3,4^2,5}$	–	–	–
$D_5(a_1)A_1$	$x_{31} = e_{3,4,5} + e_{4,5,6} + e_{5,6,7} + e_{1,2,3,4} + e_{2,3,4,5,6} + e_{2,4,5,6,7}$	–	–	–
$D_6(a_2)$	$x_{32} = e_{6,7} + e_{2,4,5} + e_{3,4,5} + e_{1,2,3,4} + e_{3,4,5,6} + e_{1,3,4,5,6}$	–	–	$w_{32} = e_3 + e_2 + e_{6,7} + e_{5,6} + e_{4,5} + e_{2,4}$
$E_6(a_3)$	$x_{33} = e_{1,3,4} + e_{2,3,4} + e_{4,5,6} + e_{5,6,7} + e_{2,3,4,5} + e_{1,2,3,4,5}$	$y_{33} = e_1 + e_{3,4} + e_{2,4} + e_{2,4,5} + e_{2,3,4,5} + e_{5,6}$	–	–
D_5	$x_{34} = e_1 + e_{2,3,4} + e_{3,4,5} + e_{4,5,6} + e_{5,6,7}$	$y_{34} = e_2 + e_6 + e_{1,3} + e_{3,4} + e_{4,5}$	–	$w_{34} = e_7 + e_6 + e_5 + e_{3,4} + e_{2,4}$

$E_7(a_5)$	$x_{35} = e_{1,3,4} + e_{2,3,4} + e_{2,4,5} + e_{4,5,6} + e_{5,6,7} + e_{3,4,5,6} + e_{1,2,3,4,5,6}$	–	–	–
A_6	$x_{36} = e_{5,6} + e_{6,7} + e_{1,3,4} + e_{2,3,4} + e_{2,4,5} + e_{3,4,5}$	–	$z_{36} = e_1 + e_3 + e_4 + e_5 + e_6 + e_7$	–
$(A_6)_2$	$x_{37} = e_{5,6} + e_{6,7} + e_{1,3,4} + e_{2,3,4} + e_{e,4,5} + e_{3,4,5} + e_{1,2,3^2,4^2,5}$	–	–	–
D_5A_1	$x_{38} = e_1 + e_{2,3,4} + e_{2,4,5} + e_{3,4,5} + e_{4,5,6} + e_{5,6,7}$	–	–	–
$D_6(a_1)$	$x_{39} = e_1 + e_{6,7} + e_{2,3,4} + e_{2,4,5} + e_{3,4,5} + e_{3,4,5,6}$	–	–	$w_{39} = e_7 + e_6 + e_3 + e_2 + e_{4,5} + e_{2,4}$
$E_7(a_4)$	$x_{40} = e_{1,3} + e_{4,5} + e_{6,7} + e_{2,3,4} + e_{2,3,4,5} + e_{2,4,5,6} + e_{3,4,5,6}$	–	–	–
D_6	$x_{41} = e_1 + e_6 + e_7 + e_{2,3,4} + e_{2,4,5} + e_{3,4,5}$	–	–	$w_{41} = e_7 + e_6 + e_5 + e_4 + e_3 + e_2$
$E_6(a_1)$	$x_{42} = e_{1,3} + e_{2,4} + e_{4,5} + e_{5,6} + e_{6,7} + e_{3,4,5}$	$y_{42} = e_1 + e_3 + e_{2,4} + e_{4,5} + e_5 + e_6$	–	–
E_6	$x_{43} = e_1 + e_3 + e_{2,4} + e_{4,5} + e_{5,6} + e_{6,7}$	$y_{43} = e_1 + e_2 + e_3 + e_4 + e_5 + e_6$	–	–

$E_7(a_3)$	$x_{44} = e_7 + e_{1,3} + e_{2,4} + e_{3,4} + e_{4,5} + e_{5,6} + e_{2,3,4,5}$	–	–	–
$E_7(a_2)$	$x_{45} = e_1 + e_3 + e_7 + e_{2,4} + e_{4,5} + e_{5,6} + e_{4,5,6}$	–	–	–
$E_7(a_1)$	$x_{46} = e_1 + e_3 + e_5 + e_6 + e_7 + e_{2,4} + e_{4,5}$	–	–	–
E_7	$x_{47} = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$	–	–	–

Table A.5: Orbit inclusions for E_7

A.3 Parabolic subgroups in E_7

We first give alternative "weighted Dynkin diagrams" computed as in Lemma 5.3 in E_7 that agree with weighted Dynkin diagrams on the respective nodes of Levi subgroups of type E_6, A_6, and D_6. We will give the elements n_w as a list of integers $[i_1, \dots, i_r]$ if $n_w = n_{s_{\alpha_{i_1}}} \cdots n_{s_{\alpha_{i_r}}}$ where the α_i are the simple roots with the ordering as in Figure 2.2. Note that in some cases, we get several nilpotent orbits in the underlying Lie algebras of type E_6, A_6, and D_6 for one nilpotent orbit in the Lie algebra of type E_7.

A.3.1 For type E_6

	Weighted Dynkin Diagram δ of E_7	Weighted Dynkin Diagram δ_0' for E_6	n_w with $\delta_0' = n_w.\delta$	belongs to orbit in E_6
(i)	0 0 0 0 0 0 0	0 0 0 0 0 0 0	[]	1
(ii)	1 0 0 0 0 0 0	0 0 0 0 0 −1 1	$[7, 6, 5, 4, 3, 1]$	A_1
(iii)	0 0 0 0 1 0 0	1 0 0 0 1 −2 0	$[7, 6, 5, 4, 3, 2, 4, 5, 6]$	A_1^2
(iv)	0 0 0 0 0 2 0	0 0 0 0 0 2 0	[]	1

(v)	0 1 0 0 0 0 / 0	0 0 1 0 0 −2 / 0	$[2, 4, 2]$	A_1^3
(vi)	2 0 0 0 0 0 / 0	0 0 0 0 0 −2 / 2	$[7, 6, 5, 4, 3, 1]$	A_2
(vii)	0 0 0 0 0 1 / 1	0 0 0 0 0 1 / 1	$[]$	A_1
(viii)	1 0 0 0 1 0 / 0	1 0 0 0 1 −3 / 1	$[7, 6, 5, 4, 3, 2, 4, 5, 6, 1]$	A_2A_1
(ix)	0 0 1 0 0 0 / 0	0 1 0 1 0 −3 / 0	$[7, 6, 5, 4, 3, 2, 4, 5, 1, 3, 4]$	$A_2A_1^2$
(x)	2 0 0 0 1 0 / 0	1 0 0 0 1 −4 / 2	$[7, 6, 5, 4, 3, 2, 4, 5, 6, 1]$	A_3
(xi)	0 0 0 0 2 0 / 0	2 0 0 0 2 −4 / 0	$[7, 6, 5, 4, 3, 2, 4, 5, 6]$	A_2^2
(xii)	0 0 0 0 0 0 / 2	0 0 0 0 0 0 / 2	$[]$	A_2
(xiii)	2 0 0 0 0 2 / 0	2 0 0 0 2 −2 / 0	$[6]$	A_2^2
(xiv)	0 1 0 0 1 0 / 0	1 0 1 0 1 −4 / 0	$[7, 6, 5, 4, 3, 2, 4, 5, 6, 1, 3]$	$A_2^2A_1$
(xv)	1 0 1 0 0 0 / 0	0 1 0 1 0 −4 / 1	$[4, 5, 6, 7, 1]$	A_3A_1
(xvi)	0 2 0 0 0 0 / 0	0 0 2 0 0 −4 / 0	$[2, 4, 2]$	$D_4(a_1)$
(xvii)	1 0 0 1 0 1 / 0	1 0 1 0 1 −2 / 0	$[1, 3]$	$A_2^2A_1$
(xviii)	2 2 0 0 0 0 / 0	0 0 2 0 0 −6 / 2	$[3, 1, 4, 2, 3, 4, 5, 6, 7]$	D_4
(ixx)	0 1 0 0 0 1 / 1	1 0 0 0 1 −2 / 2	$[2, 4, 3]$	A_3

(xx)	0 0 1 0 1 0 0	0 1 0 1 0 −2 1	[6, 7, 6]	$(A_3A_1)^{(2)}$
(xxi)	2 0 0 0 2 0 0	2 0 0 0 2 −6 2	[7, 6, 5, 4, 3, 2, 4, 5, 6, 1]	A_4
(xxii)	0 0 0 2 0 0 0	0 0 2 0 0 −2 0	[1]	$D_4(a_1)$
(xxiii)	2 0 0 0 2 2 0	2 0 0 0 2 2 0	[]	$A_2A_1^2$
(xxiv)	2 1 0 0 0 1 1	–	–	–
(xxv)	1 0 1 0 1 0 0	1 1 0 1 1 −6 1	[7, 6, 5, 4, 3, 2, 4, 5, 6, 1, 3, 4]	A_4A_1
(xxvi)	2 0 1 0 1 0 0	1 1 0 1 1 −7 2	[4, 5, 6, 3, 2, 4, 5]	A_4A_1
(xxvii)	0 0 2 0 0 0 0	0 0 2 0 0 0 0	[]	$D_4(a_1)$
(xxviii)	1 0 1 0 2 0 0	2 1 0 1 2 −8 1	[4, 5, 6, 4, 3, 2, 4, 5]	A_5
(xxix)	1 0 1 0 1 2 0	1 0 1 0 1 2 0	[]	$A_2^2A_1$
(xxx)	2 0 0 2 0 0 0	0 0 2 0 0 −4 2	[5, 6, 2]	D_4
(xxxi)	0 1 0 1 0 2 1	0 1 0 1 0 2 1	[]	A_3A_1
		2 1 0 1 2 −6 1	[3, 4, 5, 2, 4, 3]	A_5
(xxxii)	0 2 0 0 2 0 0	2 0 2 0 2 −8 0	[4, 5, 2, 4, 1]	$E_6(a_3)$
(xxxiii)	2 2 0 0 2 0 0	2 0 2 0 2 −10 2	[5, 6, 1, 3, 4, 1]	D_5

(xxxiv)	$0\ 0\ 2\ 0\ 0\ 2$ / 0	$0\ 0\ 2\ 0\ 0\ 2$ / 0	[]	D_4
		$2\ 0\ 2\ 0\ 2\ -6$ / 0	$[5, 6, 2, 4]$	$E_6(a_3)$
(xxxv)	$0\ 0\ 2\ 0\ 2\ 0$ / 0	–	–	–
(xxxvi)	$2\ 1\ 0\ 1\ 1\ 0$ / 1	–	–	–
(xxxvii)	$2\ 1\ 0\ 1\ 0\ 2$ / 1	$2\ 1\ 0\ 1\ 2\ -2$ / 1	$[2]$	A_5
(xxxviii)	$2\ 0\ 2\ 0\ 0\ 2$ / 0	$2\ 0\ 2\ 0\ 2\ -8$ / 2	$[5]$	D_5
		$2\ 0\ 2\ 0\ 2\ -8$ / 2	$[7, 5, 6, 3, 2]$	
(xxxix)	$2\ 1\ 0\ 1\ 2\ 2$ / 1	$2\ 1\ 0\ 1\ 2\ 2$ / 1	[]	A_5
(xl)	$2\ 0\ 2\ 0\ 2\ 0$ / 0	$2\ 2\ 0\ 2\ 2\ -12$ / 2	$[4, 5, 6, 4, 3, 2, 4, 5]$	$E_6(a_1)$
(xli)	$2\ 2\ 2\ 0\ 2\ 0$ / 0	$2\ 2\ 2\ 2\ 2\ -16$ / 2	$[1, 3, 4, 5, 6, 5, 3, 4, 3]$	E_6
(xlii)	$2\ 0\ 2\ 0\ 2\ 2$ / 0	$2\ 0\ 2\ 0\ 2\ 2$ / 0	[]	$E_6(a_3)$
		$2\ 2\ 0\ 2\ 2\ -6$ / 2	$[6, 7, 1]$	$E_6(a_1)$
(xliii)	$2\ 2\ 0\ 2\ 0\ 2$ / 2	$2\ 2\ 0\ 2\ 2\ -2$ / 2	$[2]$	$E_6(a_1)$
		$2\ 2\ 2\ 2\ 2\ -14$ / 2	$[7, 6, 4, 3, 2, 4, 3, 1]$	E_6

(xliv)	2 2 0 2 2 2 2	2 2 0 2 2 2 2	[]	$E_6(a_1)$
		2 2 0 2 2 2 2	$[5, 3, 4]$	
(xlv)	2 2 2 2 2 2 2	2 2 2 2 2 2 2	[]	E_6

Table A.6: Corresponding weighted Dynkin diagrams for E_6 in E_7

A.3.2 For type A_6

	Weighted Dynkin Diagram δ of E_7	Weighted Dynkin Diagram δ_0' for A_6	n_w with $\delta_0' = n_w.\delta$	belongs to orbit in A_6
(i)	0 0 0 0 0 0 0	0 0 0 0 0 0 0	[]	1
(ii)	1 0 0 0 0 0 0	1 0 0 0 0 1 −1	$[7, 1]$	A_1
(iii)	0 0 0 0 1 0 0	0 1 0 0 1 0 −2	$[2, 4, 5, 6]$	A_1^2
(iv)	0 0 0 0 0 2 0	–	–	–
(v)	0 1 0 0 0 0 0	0 0 1 1 0 0 −3	$[7, 5, 6, 5, 3]$	$(A_1^3)^{(2)}$
(vi)	2 0 0 0 0 0 0	2 0 0 0 0 2 −2	$[7, 1]$	A_2
(vii)	0 0 0 0 0 1 1	–	–	–
(viii)	1 0 0 0 1 0 0	1 1 0 0 1 1 −3	$[7, 4, 5, 4, 3, 2]$	A_2A_1
(ix)	0 0 1 0 0 0 0	1 0 1 1 0 1 −4	$[5, 6, 7, 2, 4, 5, 4]$	$A_2A_1^2$

(x)	$\begin{smallmatrix} 2&0&0&0&1&0 \\ &&0&&& \end{smallmatrix}$	$\begin{smallmatrix} 2&1&0&0&1&2 \\ &&-4&&& \end{smallmatrix}$	$[7,4,5,4,3,2]$	A_3
(xi)	$\begin{smallmatrix} 0&0&0&0&2&0 \\ &&0&&& \end{smallmatrix}$	$\begin{smallmatrix} 0&2&0&0&2&0 \\ &&-4&&& \end{smallmatrix}$	$[2,4,5,6]$	A_2^2
(xii)	$\begin{smallmatrix} 0&0&0&0&0&0 \\ &&2&&& \end{smallmatrix}$	$\begin{smallmatrix} 0&0&0&0&0&0 \\ &&2&&& \end{smallmatrix}$	$[]$	1
		$\begin{smallmatrix} 0&0&0&0&0&0 \\ &&-2&&& \end{smallmatrix}$	$[7,6,5,4]$	
(xiii)	$\begin{smallmatrix} 2&0&0&0&0&2 \\ &&0&&& \end{smallmatrix}$	$\begin{smallmatrix} 2&0&0&0&0&2 \\ &&0&&& \end{smallmatrix}$	$[]$	A_2
		$\begin{smallmatrix} 2&0&0&0&0&2 \\ &&-4&&& \end{smallmatrix}$	$[5,2,4,3,2,1]$	
(xiv)	$\begin{smallmatrix} 0&1&0&0&1&0 \\ &&0&&& \end{smallmatrix}$	$\begin{smallmatrix} 0&1&0&0&1&0 \\ &&0&&& \end{smallmatrix}$	$[]$	A_1^2
		$\begin{smallmatrix} 0&1&0&0&1&0 \\ &&-4&&& \end{smallmatrix}$	$[5,4,3,2,4,5,6,3,$ $4,5,2,1]$	$(A_3A_1)^{(2)}$
(xv)	$\begin{smallmatrix} 1&0&1&0&0&0 \\ &&0&&& \end{smallmatrix}$	$\begin{smallmatrix} 2&0&1&1&0&2 \\ &&-5&&& \end{smallmatrix}$	$[6,5,1,3,2,4,2,1]$	$(A_3A_1)^{(2)}$
(xvi)	$\begin{smallmatrix} 0&2&0&0&0&0 \\ &&0&&& \end{smallmatrix}$	–	–	–
(xvii)	$\begin{smallmatrix} 1&0&0&1&0&1 \\ &&0&&& \end{smallmatrix}$	$\begin{smallmatrix} 1&1&0&0&1&1 \\ &&-1&&& \end{smallmatrix}$	$[7,2]$	A_2A_1
		$\begin{smallmatrix} 1&1&0&0&1&1 \\ &&-5&&& \end{smallmatrix}$	$[5,6,4,5,3,4,1,3,$ $2,4,5,4,3,2]$	
(xviii)	$\begin{smallmatrix} 2&2&0&0&0&0 \\ &&0&&& \end{smallmatrix}$	–	–	–
(ixx)	$\begin{smallmatrix} 0&1&0&0&0&1 \\ &&1&&& \end{smallmatrix}$	$\begin{smallmatrix} 1&0&1&1&0&1 \\ &&-2&&& \end{smallmatrix}$	$[5,2]$	A_2A_1
		$\begin{smallmatrix} 1&0&1&1&0&1 \\ &&-6&&& \end{smallmatrix}$	$[4,2,3,1,5,4,2,7,$ $6,5,4,2,3,4]$	

(xx)	0 0 1 0 1 0 0	1 1 1 1 1 1 −6	[1, 3, 4, 5, 6, 7, 5, 6, 1]	A_3A_2
(xxi)	2 0 0 0 2 0 0	2 2 0 0 2 2 −6	[7, 4, 5, 4, 3, 2]	A_4
(xxii)	0 0 0 2 0 0 0	0 2 0 0 2 0 −2	[4]	$(A_2)^2$
		0 2 0 0 2 0 −6	[6, 4, 1, 3, 2, 4, 5, 2]	
(xxiii)	2 0 0 0 2 2 0	–	–	–
(xxiv)	2 1 0 0 0 1 1	–	–	–
(xxv)	1 0 1 0 1 0 0	2 1 1 1 1 2 −7	[3, 4, 5, 6, 7, 6, 1, 3, 2, 4, 5, 2]	A_4A_1
(xxvi)	2 0 1 0 1 0 0	–	–	–
(xxvii)	0 0 2 0 0 0 0	–	–	–
(xxviii)	1 0 1 0 2 0 0	2 2 1 1 2 2 −9	[3, 4, 5, 6, 7, 6, 1, 3, 2, 4, 5, 2]	$A_5^{(2)}$
(xxix)	1 0 1 0 1 2 0	–	–	–
(xxx)	2 0 0 2 0 0 0	–	–	–
(xxxi)	0 1 0 1 0 2 1	–	–	–
(xxxii)	0 2 0 0 2 0 0	0 2 0 0 2 0 0	[]	$(A_2)^2$
		0 2 0 0 2 0 −8	[5, 4, 3, 2, 4, 5, 6, 3, 4, 5, 2, 1]	

(xxxiii)	2 2 0 0 2 0 0	–	–	–
(xxxiv)	0 0 2 0 0 2 0	–	–	–
(xxxv)	0 0 2 0 2 0 0	2 2 2 2 2 2 -12	$[1,3,4,5,6,7,5,6,1]$	A_6
(xxxvi)	2 1 0 1 1 0 1	–	–	–
(xxxvii)	2 1 0 1 0 2 1	–	–	–
(xxxviii)	2 0 2 0 0 2 0	2 2 2 2 2 2 -10	$[6,3,2,4,5,3,1]$	A_6
		2 2 2 2 2 2 -14	$[5,6,7,1,3,4,5,1,3,4,3,2,1]$	
(xxxix)	2 1 0 1 2 2 1	–	–	–
(xl)	2 0 2 0 2 0 0	2 2 2 2 2 2 -8	$[7,5,6,5,4]$	A_6
		2 2 2 2 2 2 -16	$[5,6,7,5,6,1,3,4,5,1,3,2,4,1,3,1]$	
(xli)	2 2 2 0 2 0 0	–	–	–
(xlii)	2 0 2 0 2 2 0	–	–	–
(xliii)	2 2 0 2 0 2 2	–	–	–
(xliv)	2 2 0 2 2 2 2	2 2 2 2 2 2 -2	$[6]$	A_6
		2 2 2 2 2 2 -22	$[6,2,4,5,3,4,1,3,2,4,5,6,7,6,5,4,3,2,4,5,6,1,3,4,5,3,2,4,1,3]$	

(xlv)	2 2 2 2 2 2 2	2 2 2 2 2 2 2	[]	A_6

Table A.7: Corresponding weighted Dynkin diagrams for A_6 in E_7

A.3.3 For type D_6

	Weighted Dynkin Diagram δ of E_7	Weighted Dynkin Diagram δ_0' for D_6	n_w with $\delta_0' = n_w.\delta$	belongs to orbit in D_6
(i)	0 0 0 0 0 0 0	0 0 0 0 0 0 0	[]	1
(ii)	1 0 0 0 0 0 0	−1 0 0 0 1 0 0	$[4, 5]$	A_1
(iii)	0 0 0 0 1 0 0	−2 0 1 0 0 0 0	$[1, 3, 4]$	A_1^2
		−1 0 0 0 0 2 0	$[6, 7, 3, 4]$	A_1^2
(iv)	0 0 0 0 0 2 0	−2 0 0 0 0 0 2	$[6, 7]$	$(A_1^3)^{(1)}$
(v)	0 1 0 0 0 0 0	−3 2 0 0 0 0 0	$[6, 7, 6, 5, 3]$	$(A_1^3)^{(2)}$
		−2 0 0 1 0 1 0	$[7, 1, 3, 2]$	$(A_1^3)^{(2)}$
(vi)	2 0 0 0 0 0 0	−2 0 0 0 2 0 0	$[4, 5]$	A_2
(vii)	0 0 0 0 0 1 1	−3 1 0 0 0 1 1	$[5, 4, 3, 1]$	A_1^4
(viii)	1 0 0 0 1 0 0	−3 0 1 0 1 0 0	$[6, 7, 3, 4, 2]$	A_2A_1
(ix)	0 0 1 0 0 0 0	−3 0 0 2 0 0 0	$[5, 6, 7, 4, 2, 1]$	$A_2A_1^2$

(x)	2 0 0 0 1 0 / 0	−4 0 1 0 2 0 / 0	$[6,7,3,4,2]$	A_3
		−3 0 0 0 2 2 / 0	$[6,7,2,4,5,1]$	
(xi)	0 0 0 0 2 0 / 0	−4 0 2 0 0 0 / 0	$[1,3,4]$	A_2^2
(xii)	0 0 0 0 0 0 / 2	0 0 0 0 0 0 / 2	$[]$	1
		−2 0 0 2 0 0 / 0	$[2]$	$A_2A_1^2$
		−4 0 0 2 0 0 / 0	$[3,4,5,4]$	
		−4 0 0 0 0 0 / 2	$[6,7,5,3,2]$	$(A_1^3)^{(2)}$
(xiii)	2 0 0 0 0 2 / 0	2 0 0 0 0 2 / 0	$[]$	A_1^2
		−2 0 0 0 2 2 / 0	$[5,6,4]$	A_3
		−4 0 0 0 2 0 / 2	$[3,4,1]$	$(A_3A_1)^{(1)}$
		−4 0 0 0 2 2 / 0	$[7,4,5,6,3]$	A_3
(xiv)	0 1 0 0 1 0 / 0	−1 0 1 0 1 0 / 0	$[5]$	A_2A_1
		−3 0 2 0 0 0 / 0	$[5,6,7,5,1]$	A_2^2
		−5 0 2 0 0 0 / 0	$[3,4,1,3,2,4,5,3,4,2]$	
		−5 0 1 0 1 0 / 0	$[5,6,7,4,5,3,2,4,1,3,2]$	A_2A_1

(xv)	1 0 1 0 0 0 / 0	−5 2 0 0 2 0 / 0	$[2,4,5,6,7,6,5,4]$	$(A_3A_1)^{(2)}$
		−4 0 1 0 1 2 / 0	$[5,6,7,3,2,4,2]$	$(A_3A_1)^{(2)}$
(xvi)	0 2 0 0 0 0 / 0	−4 0 0 2 0 2 / 0	$[7,1,3,2]$	$D_4(a_1)$
(xvii)	1 0 0 1 0 1 / 0	−5 1 0 1 1 0 / 1	$[5,6,3,4,5,2,4,2]$	$A_3A_1^2$
(xviii)	2 2 0 0 0 0 / 0	−6 0 0 2 2 2 / 0	$[2,4,5,6,4]$	D_4
(ixx)	0 1 0 0 0 1 / 1	−5 1 0 1 0 2 / 1	$[7,4,5,6,2,4,5,4]$	$D_4(a_1)A_1$, $(A_3A_2)_2$
(xx)	0 0 1 0 1 0 / 0	−5 0 2 0 0 2 / 0	$[4,5,6,5,2,4,1,3]$	A_3A_2
(xxi)	2 0 0 0 2 0 / 0	−6 0 2 0 2 0 / 0	$[6,7,3,4,2]$	A_4
(xxii)	0 0 0 2 0 0 / 0	0 0 0 2 0 0 / 0	$[]$	$A_2A_1^2$
		−2 0 0 0 2 0 / 2	$[5,6]$	$(A_3A_1)^{(1)}$
		−4 0 2 0 0 2 / 0	$[6,7,3]$	A_3A_2
		−6 0 2 0 0 2 / 0	$[2,4,5,6,7,5]$	A_3A_2
		−6 0 0 0 2 0 / 2	$[6,7,6,4,5,1,3]$	$(A_3A_1)^{(1)}$
		−6 0 0 2 0 0 / 0	$[7,1,3,4,5,6,4,2]$	$A_2A_1^2$
(xxiii)	2 0 0 0 2 2 / 0	−8 0 2 0 2 0 / 2	$[7,6,5,2,4,2]$	$A_5^{(1)}$

(xxiv)	$2\ 1\ 0\ 0\ 0\ 1;\ 1$	$-7\ 1\ 0\ 1\ 2\ 2;\ 1$	$[3,4,5,6,7,5,2,4,2]$	D_4A_1
(xxv)	$1\ 0\ 1\ 0\ 1\ 0;\ 0$	$1\ 0\ 1\ 0\ 1\ 0;\ 0$	$[]$	A_2A_1
		$-2\ 1\ 0\ 1\ 1\ 0;\ 1$	$[6,3]$	$A_3A_1^2$
		$-5\ 0\ 2\ 0\ 2\ 0;\ 0$	$[7,6,1,3,4,5,2,4]$	A_4
		$-7\ 0\ 2\ 0\ 2\ 0;\ 0$	$[1,3,2,4,5,6,4,3,2,4,1,3,2]$	
		$-8\ 1\ 0\ 1\ 1\ 0;\ 1$	$[1,3,4,5,6,4,3,2,4,5,4,3,2,4,1,3]$	$A_3A_1^2$
		$-7\ 0\ 1\ 0\ 1\ 0;\ 0$	$[6,2,4,5,3,4,1,3,2,4,5,6,7,6,4,5,3]$	A_2A_1
(xxvi)	$2\ 0\ 1\ 0\ 1\ 0;\ 0$	$-7\ 0\ 2\ 0\ 2\ 2;\ 0$	$[4,3,2,4,5,3,2,4,3,2]$	$D_5(a_1)$
(xxvii)	$0\ 0\ 2\ 0\ 0\ 0;\ 0$	$0\ 0\ 2\ 0\ 0\ 0;\ 0$	$[]$	$A_2A_1^2$
		$0\ 0\ 2\ 0\ 0\ 0;\ 0$	$[7,2,4]$	
		$0\ 0\ 2\ 0\ 0\ 0;\ 0$	$[7,2,4,5,6,5,4,2]$	
(xxviii)	$1\ 0\ 1\ 0\ 2\ 0;\ 0$	$-9\ 2\ 2\ 0\ 2\ 0;\ 0$	$[6,7,6,5,4,1,3,2,4,5]$	$A_5^{(2)}$
(xxix)	$1\ 0\ 1\ 0\ 1\ 2;\ 0$	$1\ 0\ 1\ 0\ 1\ 2;\ 0$	$[]$	$(A_3A_1)^{(2)}$
		$-7\ 0\ 2\ 0\ 2\ 0;\ 2$	$[7,3,4,5,2,4,3,1]$	$A_5^{(1)}$
		$-9\ 0\ 2\ 0\ 2\ 0;\ 2$	$[5,6,7,5,6,3,4,5,4,3,2,4,2,1]$	

(xxix)	1 0 1 0 1 2 0	−9 0 1 0 1 2 0	[7, 6, 4, 5, 3, 4, 1, 3, 2, 4, 5, 6, 4, 5, 3, 2, 4, 2, 1]	$(A_3A_1)^2$
(xxx)	2 0 0 2 0 0 0	2 0 0 2 0 0 0	[]	$(A_2A_1)^2$
		−6 0 2 0 2 2 0	[4, 5, 6, 7, 2]	$D_5(a_1)$
		−8 0 2 0 2 2 0	[5, 1, 3, 2, 4, 3, 1]	
		−8 0 0 2 0 0 0	[7, 2, 4, 5, 6, 4, 3, 2, 4, 5, 2]	$(A_2A_1)^2$
(xxxi)	0 1 0 1 0 2 1	−9 2 0 2 0 2 2	[3, 2, 4, 5, 6, 5, 4, 3 , 2, 4, 3]	$D_6(a_2)$
(xxxii)	0 2 0 0 2 0 0	0 2 0 0 2 0 0	[]	$(A_3A_1)^{(2)}$
		−2 0 2 0 2 0 0	[5]	A_4
		−8 2 2 0 2 0 0	[7, 5, 6, 4, 5, 2]	$A_5^{(2)}$
		−10 2 2 0 2 0 0	[3, 4, 5, 6, 5, 3, 2, 4]	
		−10 0 2 0 2 0 0	[5, 6, 7, 4, 5, 3, 2, 4, 1, 3, 2]	A_4
		−10 2 0 0 2 0 0	[3, 2, 4, 5, 6, 7, 3, 2, 4, 2, 3, 2]	$(A_3A_1)^{(2)}$
(xxxiii)	2 2 0 0 2 0 0	−10 0 2 2 2 2 0	[7, 6, 3, 2, 4, 5, 4, 2]	D_5
(xxxiv)	0 0 2 0 0 2 0	0 0 2 0 0 2 0	[]	A_3A_2
		−4 0 2 0 2 2 0	[7, 4, 3]	$D_5(a_1)$

(xxxiv)	0 0 2 0 0 2 0	−6 0 2 0 2 0 2	$[4, 5, 4, 1]$	$A_5^{(1)}$
		−8 2 0 2 0 2 2	$[5, 6, 7, 6, 3, 2]$	$D_6(a_2)$
		−10 2 0 2 0 2 2	$[2, 4, 5, 6, 7, 6, 4, 1]$	
		−10 0 2 0 2 0 2	$[3, 4, 1, 3, 2, 4, 5, 3, 4, 2]$	$A_5^{(1)}$
		−10 0 2 0 2 2 0	$[3, 2, 4, 5, 6, 7, 6, 3, 4, 1]$	$D_5(a_1)$
		−10 0 2 0 0 2 0	$[6, 7, 3, 4, 5, 6, 3, 4, 5, 2, 4, 1]$	A_3A_2
(xxxv)	0 0 2 0 2 0 0	0 0 2 0 2 0 0	$[]$	A_4
		−12 0 2 0 2 0 0	$[3, 4, 5, 6, 7, 6, 4, 1, 3, 2, 4, 5, 1]$	
(xxxvi)	2 1 0 1 1 0 1	2 1 0 1 1 0 1	$[]$	$A_3A_1^2$
		−9 0 2 2 2 2 0	$[7, 6, 5, 4, 3, 2, 4, 5, 6, 2, 1]$	D_5
		−11 0 2 2 2 2 0	$[4, 5, 6, 7, 4, 3, 2, 4, 5, 6, 4, 3, 2, 4, 5]$	
		−12 1 0 1 1 0 1	$[4, 1, 3, 2, 4, 5, 6, 2, 4, 5, 3, 4, 1, 3, 2, 4, 5, 4, 3, 2, 4, 1, 3]$	$A_3A_1^2$
(xxxvii)	2 1 0 1 0 2 1	−11 2 0 2 2 2 2	$[3, 4, 5, 6, 7, 5, 4, 2, 3, 2, 4, 5, 6]$	$D_6(a_1)$
(xxxviii)	2 0 2 0 0 2 0	2 0 2 0 0 2 0	$[]$	A_3A_2
		−4 2 0 2 0 2 2	$[1, 3]$	$D_6(a_2)$

(xxxviii)	$2\ 0\ 2\ 0\ 0\ 2$ 0	$-10\ 2\ 0\ 2\ 2\ 2$ 2	$[5,6,5,3,4,2,1]$	$D_6(a_1)$
		$-12\ 2\ 0\ 2\ 2\ 2$ 2	$[3,4,5,6,7,6,5,3,4,3]$	
		$-14\ 2\ 0\ 2\ 0\ 2$ 2	$[7,6,5,4,1,3,2,4,5,1,3,2,4,1]$	$D_6(a_2)$
(xxxix)	$2\ 1\ 0\ 1\ 2\ 2$ 1	$-15\ 2\ 2\ 2\ 2\ 2$ 2	$[6,3,2,4,5,3,4,1,3,2,4,3,2,1]$	D_6
(xl)	$2\ 0\ 2\ 0\ 2\ 0$ 0	$2\ 0\ 2\ 0\ 2\ 0$ 0	$[]$	A_4
		$-2\ 2\ 2\ 0\ 2\ 0$ 0	$[7]$	$A_5^{(2)}$
		$-16\ 2\ 2\ 0\ 2\ 0$ 0	$[1,3,4,5,6,7,3,2,4,5,6,4,5,3,4,1]$	
		$-14\ 0\ 2\ 0\ 2\ 0$ 0	$[6,2,4,5,3,4,1,3,2,4,5,6,7,6,4,5,3]$	A_4
(xli)	$2\ 2\ 2\ 0\ 2\ 0$ 0	$2\ 2\ 2\ 0\ 2\ 0$ 0	$[]$	$A_5^{(2)}$
		$-20\ 2\ 2\ 0\ 2\ 0$ 0	$[2,4,5,3,4,1,3,2,4,5,6,7,5,2,4,1,3,2,1]$	
(xlii)	$2\ 0\ 2\ 0\ 2\ 2$ 0	$2\ 0\ 2\ 0\ 2\ 2$ 0	$[]$	$D_5(a_1)$
		$-4\ 2\ 0\ 2\ 2\ 2$ 2	$[4,1]$	$D_6(a_1)$
		$-14\ 2\ 2\ 2\ 2\ 2$ 2	$[3,4,5,6,5,4,1,3,1]$	D_6
		$-16\ 2\ 2\ 2\ 2\ 2$ 2	$[7,6,1,3,4,5,3,2,4,1,3,2]$	
		$-18\ 2\ 0\ 2\ 2\ 2$ 2	$[1,3,2,4,5,6,7,6,3,4,5,4,3,2,4,1,3,2]$	$D_6(a_1)$

(xlii)	2 0 2 0 2 2; 0	−16 0 2 0 2 2; 0	[7, 6, 4, 5, 3, 4, 1, 3, 2, 4, 5, 6, 4, 5, 3, 2, 4, 2, 1]	$D_5(a_1)$
(xliii)	2 2 0 2 0 2; 2	2 2 0 2 0 2; 2	[]	$D_6(a_1)$
		−8 2 2 2 2 2; 2	[4, 5, 6, 1]	D_6
		−22 2 2 2 2 2; 2	[4, 1, 3, 2, 4, 5, 6, 7, 6, 4, 3, 2, 4, 5, 4, 3, 2, 4, 1]	
		−20 2 0 2 0 2; 2	[6, 2, 4, 5, 3, 4, 1, 3, 2, 4, 5, 6, 3, 2, 4, 5, 4, 1, 3, 2, 4, 1, 3]	$D_6(a_1)$
(xliv)	2 2 0 2 2 2; 2	2 2 0 2 2 2; 2	[]	$D_6(a_1)$
		−4 2 2 2 2 2; 2	[5, 4]	D_6
		−26 2 2 2 2 2; 2	[3, 2, 4, 5, 6, 7, 6, 3, 4, 5, 3, 4, 1, 3, 2, 4, 5, 6, 3, 2, 4, 5, 2, 4, 1]	
(xlv)	2 2 2 2 2 2; 2	2 2 2 2 2 2; 2	[]	D_6

Table A.8: Corresponding weighted Dynkin diagrams for D_6 in E_7

Bibliography

[1] A. Borel, *Linear algebraic groups*. (Second enlarged edition), Springer-Verlag, New York, 1991.

[2] W. Bosma, J. J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. **24** (1997), 235–265.

[3] W. Bosma, J. J. Cannon, C. Fieker, A. Steel (eds.), Handbook of Magma functions, Version 2.26-4 (2021), 6347 pages.

[4] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4, 5 et 6*. Hermann, Paris, 1968.

[5] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 7 et 8*. Hermann, Paris, 1975.

[6] R. W. Carter, *Simple groups of Lie type*. Wiley, New York, 1972; reprinted 1989 as Wiley Classics Library Edition.

[7] R. W. Carter, *Finite groups of Lie type: Conjugacy classes and complex characters*. Wiley, New York, 1985.

[8] R. W. Carter, *Lie algebras of finite and affine type*. Cambridge University Press, Cambridge, 2005.

[9] M. C. Clarke and A. Premet, The Hesselink stratification of nullcones and base change. Invent. math. **191** (2013), 631–669.

[10] M. Geck, *An introduction to algebraic geometry and algebraic groups*. Oxford Graduate Texts in Mathematics, 10. Oxford University Press, Oxford, 2003.

[11] M. Geck, On the construction of semisimple Lie algebras and Chevalley groups. Proc. Amer. Math. Soc. **145**:8 (2017), 3233–3247.

[12] M. Geck, Generalised Gelfand-Graev representations in bad characteristic? Transform. Groups **26** (2021), 305–326.

[13] W. Hesselink, The nullcone of the Lie algebra of G_2, Indag. Math. **30**:4 (2019), 623–648.

[14] D. F. Holt and N. Spaltenstein, Nilpotent orbits of exceptional Lie algebras over algebraically closed fields of bad characteristic. J. Austral Math. Soc. (Series A), **38** (1985), 330–350.

[15] J. E. Humphreys, *Linear algebraic groups*. Springer-Verlag, New York, 1975.

[16] J. E. Humphreys, *Reflection groups and Coxeter groups*. Cambridge University Press, Cambridge, 1990.

[17] J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*. Math. Surveys Monogr. **43**, American Mathematical Society, Providence, 2011.

[18] M. Korhonen, D. Stewart, A. Thomas, (2022) Representatives for unipotent classes and nilpotent orbits. Comm. Algebra **50:4** (2022), 1641–1661.

[19] S. Lang, *Algebra*. (2nd edition), Addison-Wesley Publishing Company, 1984.

[20] M. W. Liebeck and G. M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*. American Mathematical Society, 2012.

[21] G. Lusztig, Unipotent elements in small characteristic. Transform. Groups **10** (2005), no. 3–4, 449–487.

[22] G. Lusztig, Unipotent elements in small characteristic II. Transform. Groups **13** (2008), no. 3–4, 773–797.

[23] G. Lusztig, Unipotent elements in small characteristic III. J. Algebra **329** (2011), 163–189.

[24] G. Lusztig, Unipotent elements in small characteristic IV. Transform. Groups **15** (2010), no. 4, 921–936.

[25] G. Malle and D. Testerman, *Linear algebraic groups and finite groups of Lie type*. Cambridge University Press, Cambridge, 2011.

[26] A. Premet, Nilpotent orbits in good characteristic and the Kempf–Rousseau theory (Special issue celebrating the 80th birthday of Robert Steinberg). J. Algebra **260** (2003), 338–366.

[27] K. Pommerening, Über die unipotenten Klassen reduktiver Gruppen. J. Algebra **49** (1977), 525–536.

[28] K. Pommerening, Über die unipotenten Klassen reduktiver Gruppen II. J. Algebra **65** (1980), 373–398.

[29] N. Spaltenstein, Nilpotent classes in Lie algebras of type F_4 over fields of characteristic 2. J. Fac. Sci. Univ. Tokyo. Sect. 1 A, Mathematics **30** (1984), 517–524.

[30] T. A. Springer, *Linear algebraic groups.* Second edition. Progress in Mathematics, 9. Birkhäuser, Boston, 1998.

[31] R. Steinberg, *Conjugacy classes in algebraic groups*. Lecture Notes in Mathematics, vol. 366. Springer-Verlag, Berlin–New York, 1974.

[32] U. Stuhler, Unipotente und nilpotente Klassen in einfachen Gruppen und Liealgebren vom Typ G_2. Indag. Math. (Proceedings) **74** (1971), 365–378.

[33] T. Xue, Nilpotent coadjoint orbits in small characteristic. J. Algebra **397** (2014), 111–140.

[34] T. Xue, Springer correspondence for exceptional Lie algebras and their duals in small characteristic. J. Lie Theory **27** (2017), no.2, 357–375.

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