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ON THE MRÓZ MODEL

Martin Brokate, Klaus Dressler, Pavel Krejci

UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Arbeitsgruppe Technomathematik
Postfach 3049
W-6750 Kaiserslautern

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Martin Brokate
Fachbereich Mathematik
Universität Kaiserslautern
6750 Kaiserslautern
Germany

Klaus Dressler
Tecmath GmbH
6750 Kaiserslautern 25
Germany

Pavel Krejčí*
Institute of Mathematics
Academy of Sciences
Žitná 25
11567 Prague
Czechoslovakia

Abstract

We treat the mathematical properties of the one parameter version of the Mróz model for plastic flow. We present continuity results and an energy inequality for the hardening rule and discuss different versions of the flow rule regarding their relation to the second law of thermodynamics.

1 Introduction

In a material body, which undergoes plastic deformation, the current values of the stress and the strain tensor at a given point no longer uniquely determine each other, and one must take into account some aspects of the time history of either the stress or the strain. This is commonly achieved through the classical and well established concept of a yield surface. But since a single fixed yield surface does not describe correctly many experimentally observed phenomena, a lot of modifications and generalizations have been developed. There are various hardening rules which specify translations and changes in size or shape of the yield surface, and there are multi-surface theories to allow for a more complex memory of the past history.

In this paper, we study a particular model whose origin is usually attributed to Mróz [11]. It employs a one parameter family of yield surfaces embedded into each other and subjected to a kinematic hardening rule due to Prager and Ziegler. Chu [3], [4] has already observed that, in the case of spherical yield surfaces (in the space of stress deviators, as usual), the hardening rule of the Mróz model leads to a surprisingly

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simple structure of memory, although the movement of the individual surfaces may be somewhat involved. Actually, in the uniaxial case one obtains the memory structure of the scalar hysteresis model due to Prandtl [12], Preisach [13], and Ishlinskii [6], which has been studied recently by mathematicians, see e.g. [7], [2] and [8]. This constitutes the starting point of our mathematical analysis of the Mróz hardening rule in section 2. We present there a rigorous definition as well as some continuity and regularity theorems. Such theorems are important for both theoretical and computational investigations, since they show in which sense the Mróz model is well posed. As the proofs of the theorems are somewhat involved, we delegate them to the sections 4 and 5. In section 3, we discuss the flow rule of the Mróz model. It turns out that the standard von Mises normality rule, if applied indifferently, may lead to a violation of the second law of thermodynamics. In analogy to the vector Ishlinskii model of [9], we present a different flow rule and study its energy dissipation properties.

Our original motivation to study the Mróz model came from low cycle fatigue analysis, where one wants to estimate the life span of a certain workpiece subject to loads of varying size and direction. In order to do this, one has to find out which features of a (very long) sequence of different loads are relevant for the accumulation of damage. While this question seems to be largely open for multiaxial or multipoint loading, there are various established procedures in the case of scalar loads. Among the more successful ones is the so-called rainflow counting method due to Endo (see [10] for a reprint of his original papers), which basically identifies and counts the nested hysteresis loops in the stress-strain diagram and then computes an estimate for the total damage from this count. The rainflow method is intimately related to the memory structure of the uniaxial hysteresis model cited above, and hence also of the uniaxial Mróz model. Therefore, a detailed analysis of the Mróz model also helps in the development and analysis of a vector version of the rainflow method. This aspect, however, will not be pursued in this paper.

2 The hardening rule

A mathematical formulation of the constitutive stress-strain relation of plastic flow in terms of yield surfaces usually has three ingredients:

- A *yield condition* to specify the form of the yield surface(s).
- A *hardening rule* to describe their time evolution.
- A *flow rule* to characterize the plastic strain.

Usually, hardening rules determine the yield surface evolution from the time history of the stress. Let us therefore consider a stress function $\sigma : [0,T] \to \mathbf{T}$, where we denote by $\mathbf{T}$ the space of symmetric 3×3 tensors endowed with the scalar product

$$< \xi, \eta > \;=\; \xi_{ij}\, \eta_{ij}$$

and with the norm

$$|\xi| \;=\; < \xi, \xi >^{\frac{1}{2}} . \tag{1}$$

In plastic constitutive laws, only the deviatoric part of the stress tensor

$$\sigma_{ij}^{(d)} = \sigma_{ij} - \bar{\sigma}\,\delta_{ij},$$

with the pressure $\bar{\sigma} = \frac{1}{3}\sigma_{ii}$, plays any role. Consequently, $\mathbf{T}$ is decomposed into the orthogonal direct sum

$$\mathbf{T} = \mathbf{T_{dia}} \oplus \mathbf{T_{dev}}$$

of spaces of diagonal tensors

$$\mathbf{T_{dia}} = \{\,\xi \in \mathbf{T} : \xi_{ij} = \lambda\delta_{ij} \text{ for some } \lambda \in \mathbf{R}\}$$

and of deviatoric tensors

$$\mathbf{T_{dev}} = \{\,\xi \in \mathbf{T} : \xi_{ii} = 0\,\}.$$

In conformity with the von Mises yield criterion, our yield surfaces are spheres in $\mathbf{T_{dev}}$. Specifically, we consider the one parameter time dependent family $S_r(t)$ of spheres in $\mathbf{T_{dev}}$ with radius r and center $\phi(t,r)$, namely

$$S_r(t) = \{\,\xi \in \mathbf{T_{dev}} : |\,\xi - \phi(t,r)\,| = r\,\}.$$

We further denote by

$$E_r(t) = \{\,\xi \in \mathbf{T_{dev}} : |\,\xi - \phi(t,r)\,| < r\,\}$$

the region of elasticity of the yield surface $S_r(t)$. To describe the time evolution of the yield surfaces, we have to define the yield center function $\phi = \phi(t,r)$ for any given stress deviator $\sigma^{(d)} = \sigma^{(d)}(t)$. First, we require that the stress deviator always remains within every yield surface. This means that

$$|\,\sigma^{(d)}(t) - \phi(t,r)\,| \le r \qquad \text{for any } t \in [0,T],\ r > 0. \tag{2}$$

Next, a yield surface should not move when the stress deviator lies in its interior, so

$$\frac{\partial}{\partial t}\phi(t,r) = 0 \qquad \text{if } |\,\sigma^{(d)}(t) - \phi(t,r)\,| < r. \tag{3}$$

While the conditions (2) and (3) are common to most yield surface models, the *nonintersection condition*

$$E_{r_1}(t) \subset E_{r_2}(t) \qquad \text{for any } 0 < r_1 < r_2,\ t \in [0,T], \tag{4}$$

constitutes the distinctive feature of the model of Mróz. It can be equivalently rewritten as

$$|\,\phi(t,r_1) - \phi(t,r_2)\,| \le r_2 - r_1 \qquad \text{for any } 0 < r_1 < r_2,\ t \in [0,T]. \tag{5}$$

Finally, let us assume that initially the yield surfaces are concentric around 0, so

$$\phi(0-,r) = 0 \qquad \text{for any } r > 0. \tag{6}$$

We will see that the conditions (2) – (6) uniquely define a function ϕ and therefore completely specify the movement of the yield surfaces for a given stress deviator $\sigma^{(d)} =$

$\sigma^{(d)}(t)$. We call the map $\sigma^{(d)} \to \phi$ the *(continuous) Mróz hardening rule* . It is a kinematic hardening rule, since the yield surfaces move but do not change shape; nevertheless it is able to model anisotropic material behaviour through its memory stored in the function $\phi(t,\cdot)$ at time t.

For the mathematical treatment of the Mróz hardening rule, it is completely immaterial that we are working with deviatoric stresses as inputs, except for the scalar product structure of $\mathbf{T_{dev}}$. To emphasize this fact, we replace in the following the space $\mathbf{T_{dev}}$ with an arbitrary separable Hilbert space U and denote the input function by u instead of $\sigma^{(d)}$. But actually, nothing is lost if the reader always interprets U as the deviatoric plane and u as the stress deviator.

To start with the formal theory, we define an appropriate function space Ψ for the memory in order to have $\phi(t,\cdot) \in \Psi$. Due to (3), the yield surface with radius r will move away from zero only if the norm of the stress deviator exceeds the value r.

Therefore, we adopt the following definition.

Definition 2.1 *Let U be a Hilbert space with the scalar product $<\cdot,\cdot>$ and the norm* (1) . *We call U the input space and its elements input values. We define the space Ψ of admissible memory states by*

$$\begin{aligned} \Psi \;=\; & \{ \psi \mid \psi : [0,\infty) \to U \,,\, |\psi(r) - \psi(s)| \le |r - s| \text{ for any } r,s \ge 0, \\ & \text{and there exists } R > 0 \text{ with } \psi(r) = 0 \text{ for any } r \ge R \} . \end{aligned} \tag{7}$$

For $\psi \in \Psi$ and $r > 0$, we interpret $\psi(r)$ as the center of the yield surface with radius r, and $\psi(0)$ as the current input value (compare (2)). Moreover, we call $\psi(r)$ a corner of the memory state ψ, if ψ is not differentiable at r.

Actually, Ψ is a metric space if we consider it as a subset of the space of bounded continuous functions on the nonnegative real numbers with values in U, endowed with the norm

$$\|\psi\|_\infty \;=\; \sup_{r\ge 0} |\psi(r)| . \tag{8}$$

We will now describe the memory update, i.e. the movement of the yield surfaces, for a given memory state ψ, if the input changes from its current value $\psi(0)$ to a new value v along a straight line in the input space U. We first note formally that there is a smallest radius $\alpha(v,\psi)$ such that the new input value v does not lie outside any yield surface with radius $r \ge \alpha(v,\psi)$.

Lemma 2.2 *Let $\psi \in \Psi$ and $v \in U$ be given. Then*

$$\alpha(v,\psi) \;=\; \min \{ r \ge 0 : |\psi(r) - v| = r \} \tag{9}$$

is well defined, and

$$r \;<\; |\psi(r) - v| \qquad \text{if and only if } 0 \le r < \alpha(v,\psi) . \tag{10}$$

Proof: Since

$$\Big| |\psi(r) - v| - |\psi(s) - v| \Big| \;\le\; |r - s|$$

for any $r,s \ge 0$, the function

$$f(r) \;=\; r - |\psi(r) - v|$$

is nondecreasing, continuous, and satisfies $f(0) \leq 0$ as well as $\lim_{r\to\infty} f(r) = \infty$, so all assertions follow. □

Due to (3), no yield surface with radius $r \geq \alpha(v, \psi)$ should move if the input value changes from $\psi(0)$ to v along a straight line. On the other hand, the yield surfaces with smaller radius should move so as to form a new memory state in Ψ as well as to include the value v. Because of (9), their centers have to arrange themselves along the straight line connecting v and the center of the surface with radius $\alpha(v, \psi)$ with a common normal at the common boundary point v, see figure 1. Therefore, the following

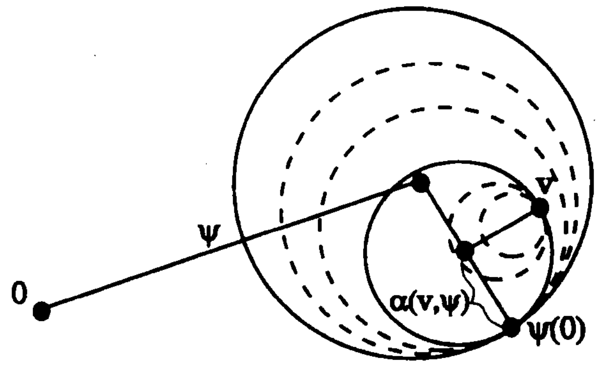

Figure 1: Arrangement of the yield surfaces.

definition specifies the unique hardening rule compatible with (2) – (6).

Definition 2.3 *We define an operator $G : U \times \Psi \to \Psi$ by*

$$G(v, \psi)(r) = \begin{cases} \psi(r), & \text{if } r \geq \alpha(v, \psi) \\ v + \dfrac{r}{\alpha(v, \psi)} \left(\psi(\alpha(v, \psi)) - v \right), & \text{otherwise} \end{cases} \tag{11}$$

for any $r \geq 0$ and any $v \in U$, $\psi \in \Psi$, where $\alpha(v, \psi)$ is defined in lemma 2.2. We call G the Mróz hardening rule.

To update a piecewise linear memory state ψ with the rule (11), we insert a (possibly degenerate) corner P at $\psi(\alpha(v, \psi))$ and connect it to the point v, thereby discarding the piecewise linear segment from $\psi(0)$ to P. We present in figure 2 the resulting possible corner structures for various input values. There we imagine the input value v traveling along a straight line, and we may think of P moving along the old state ψ while eating up corners in the process. We also see, and easily check formally, that no yield surface can stay fixed while a larger one is moving. In terms of the operator G, this means that

$$G(v, \psi)(r) \neq \psi(r) \quad \Rightarrow \quad G(v, \psi)(s) \neq \psi(s) \quad \text{for any } s \leq r . \tag{12}$$

Next, let us consider a piecewise linear input function $u : [0, T] \to U$ represented by a sequence $\{u_k\}$ of input values. Naturally, we apply definition 2.3 successively to obtain the corresponding movement of the yield surfaces.

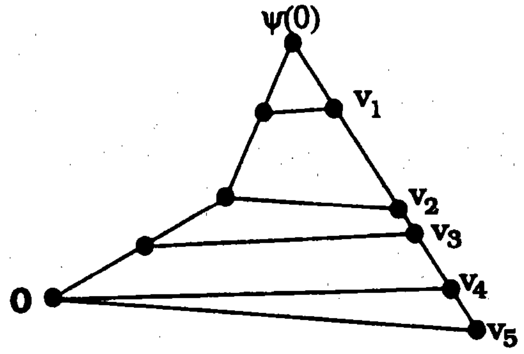

Figure 2: Update of the memory state.

Definition 2.4 *(i) For any sequence $\{u_k\}$, $k = 0, 1, 2, \ldots$ of input values in U we define the corresponding sequence of memory states $\{\phi_k\}$ in Ψ by*

$$\phi_k \;=\; G(u_k, \phi_{k-1}), \qquad \phi_{-1} \;=\; 0. \tag{13}$$

(ii) Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a partition of $[0, T]$ and $u : [0, T] \to U$ be the piecewise linear interpolate for the values $u(t_k) = u_k$ with $u_k \in U$. Then we define the memory state function $\phi : [0, T] \times [0, \infty) \to U$ by

$$\begin{aligned} \phi(t_k, r) &= \phi_k(r), && 0 \le k \le n,\ r \ge 0, \\ \phi(t, r) &= G(u(t), \phi_k))(r), && t \in (t_k, t_{k+1}),\ r \ge 0. \end{aligned} \tag{14}$$

We write (14) in operator notation as

$$\phi \;=\; F(u). \tag{15}$$

This is justified since for any piecewise linear $u : [0, T] \to U$, the function ϕ in (14) does not depend on the choice of the partition as long as u is linear within each interval.

From the definitions (2.3) and (2.4) it is obvious that the memory states $\{\phi_k\}$ generated from (13) are piecewise linear curves with finitely many corners in the space U. They have the length

$$L(\phi_k) \;:=\; \int_0^\infty |\phi_k'(r)|\, dr \;=\; \max_{0 \le j \le k} |u_j| \tag{16}$$

and satisfy

$$\phi_k(r) \;=\; 0 \qquad \text{if } r > L(\phi_k), \tag{17}$$

as well as

$$|\phi_k'(r)| \;=\; 1 \qquad \text{if } r < L(\phi_k), \tag{18}$$

except in corners, of course. We also note two other obvious consequences of the definitions above related to the storage and deletion of corners.

Lemma 2.5 *Let the memory states $\{\phi_k\}$ be generated by the input values $\{u_k\}$.*
(i) If $\phi_k(r)$ is a corner, then $\phi_{k-1}(s) = \phi_k(s)$ for any $s \geq r$.
(ii) If $\alpha(\phi_k, u_{k+1}) \geq \alpha(\phi_{k-1}, u_k)$, then

$$G(u_{k+1}, \phi_k) = \phi_{k+1} = G(u_{k+1}, \phi_{k-1}), \tag{19}$$

so the memory due to the input value u_k is deleted. □

Our first main result shows that the Mróz hardening rule is well posed. More precisely, the operator F is $\frac{1}{2}$-Hölder continuous with respect to the sup norm.

Theorem 2.6 *The map F defined by (15) can be extended to an operator*

$$F : C(0,T;U) \to C(0,T;\Psi), \tag{20}$$

and we have

$$\max_{\substack{0 \leq t \leq T \\ r \geq 0}} |\phi(t,r) - \psi(t,r)| \leq \left(2R \max_{0 \leq t \leq T} |u(t) - v(t)| \right)^{\frac{1}{2}} \tag{21}$$

for any $u, v \in C(0,T;U)$, where $\phi = F(u)$, $\psi = F(v)$, and

$$R = \max\{ |u(t)|, |v(t)| : 0 \leq t \leq T \}.$$

Proof: This will be given in section 4. □

The exponent $\frac{1}{2}$ in equation (21) cannot be improved in the vector case, i.e. if dim $U \geq 2$. In figure 3 we see an example where the linear interpolates u, v of (u_0, u_1) and (v_0, v_1) satisfy (we assume $R \geq 3\delta$)

$$\| u - v \|_\infty = \delta, \quad \| F(u) - F(v) \|_\infty \geq (R\delta)^{\frac{1}{2}}.$$

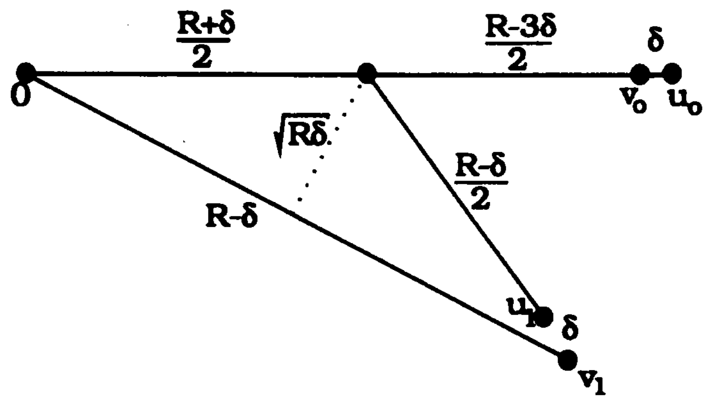

Figure 3: Optimality of the Hölder exponent.

In the scalar case dim $U = 1$, the operator F is Lipschitz continuous, see [2],[7]. Moreover, if the input function u is Lipschitz continuous (with respect to time), then so is the memory state function $\phi = F(u)$. Again, this is no longer true in the vector case. The counterexample 2.9 below shows that $\partial_t\phi(t,u)$, i.e. the partial time derivative of the motion of the centers of the yield surfaces, in general does not lie in L^p if $p > 1$. For $p = 1$, the question is open. Actually, we do not even know whether $\partial_t\,\phi(t,u)$ exists almost everywhere if the input function u is not piecewise linear. We do however have some positive results. Let us denote by $C^\alpha(0,T;U)$ the space of α–Hölder continuous functions, where $0 < \alpha \leq 1$. Then the following theorem holds.

Theorem 2.7 *Let $u \in C(0,T;U)$ be given, set $\phi = F(u)$. Then we have*

$$|\phi(t,r) - \phi(s,r)| \;\leq\; (2\|\,u\,\|_\infty \max_{\tau\in[s,t]} |u(\tau) - u(s)|)^{\frac{1}{2}} \tag{22}$$

for any $s,t \in [0,T]$ and any $r > 0$. In particular, if $u \in C^\alpha(0,T;U)$, then $\phi(\cdot,r) \in C^{\alpha/2}(0,T;U)$.

Proof: Fix $t, s \in [0,T]$, $s \leq t$, and define $v \in C(0,T;U)$ by $v = u$ on $[0,s]$ and $v = u(s)$ on $[s,t]$. Then Theorem 2.6 implies that, setting $R = \|\,u\,\|_\infty$,

$$\begin{aligned} |\phi(t,r) - \phi(s,r)| &= |(Fu)(t,r) - (Fv)(t,r)| \;\leq\; \sqrt{2R \max_{0\leq\tau\leq t} |u(\tau) - v(\tau)|} \\ &\leq \sqrt{2R \max_{s\leq\tau\leq t} |u(\tau) - u(s)|} \;\leq\; \sqrt{2R}\,C|t-s|^{\alpha/2}, \end{aligned}$$

so (22) follows. If now $u \in C^\alpha(0,T;U)$, then $|u(t) - u(s)| \leq C|t-s|^\alpha$ for some C independent from t and s, and (22) implies that

$$|\phi(t,r) - \phi(s,r)| \;\leq\; \sqrt{2R}\,C|t-s|^{\alpha/2}\,.\square$$

We also have an estimate for $\partial_t\,\phi(t,r)$, if the input function u is piecewise linear. We will use this result later to derive continuity properties of the stress–strain law. As usual, we denote by $W^{1,1}(0,T;U)$ the Sobolev space of functions with values in U whose first derivative is Bochner integrable.

Theorem 2.8 *For any piecewise linear $u \in W^{1,1}(0,T;U)$, the function $t \mapsto \phi(t,r) = (Fu)(t,r)$ is an element of $W^{1,1}(0,T;U)$ and satisfies*

$$\int_0^T |\,\partial_t\,\phi(t,r)\,|\,dt \quad\leq\quad 3\int_0^T |\,u'(t)\,|\,dt\,. \tag{23}$$

Proof: This will be given in section 5. $\square$

We now present the example of a Lipschitz continuous input function u whose corresponding state function ϕ does not have a time derivative in any L^p, $p > 1$.

Example 2.9 For $r = 1$ and a specific $T > 0$ to be defined below, we construct a function $u : [0,T] \to \mathbf{R}^2$ such that u is Lipschitz continuous, $u'(t) = 1$ a.e., but $\partial_t\phi(t,r) \notin L^p(0,T;U)$ for all $p > 1$. (By rescaling and embedding, the example is easily extended to arbitrary values of r, T and arbitrary input spaces U with $\dim U \geq 2$.) The idea is to let the input value $u(t)$ run through a sequence of loops ($A \to B_n \to C_n \to A$) of decreasing size but bounded total length T. A single loop is shown in figure 4. We

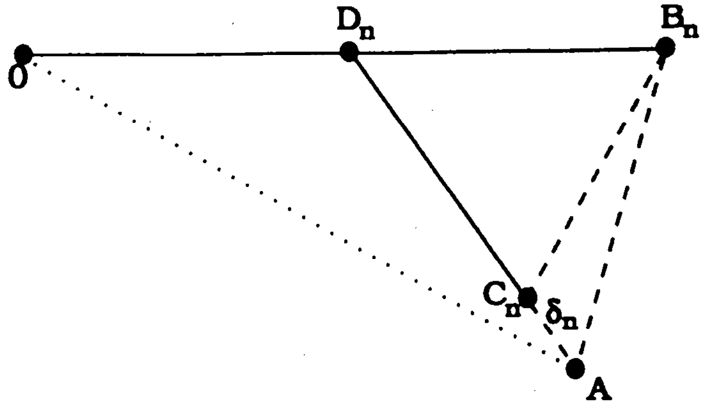

Figure 4: Construction of the counterexample.

fix a point A in the plane with $|A| = 2$ and choose points B_n, C_n, D_n such that for some given $\delta_n > 0$

$$|D_n| = 1, \quad |A - D_n| = 1 + \delta_n, \quad B_n = 2D_n,$$

$$C_n = \frac{\delta_n}{1+\delta_n} D_n + \frac{1}{1+\delta_n} A_n .$$

We easily compute that $2 < A, D_n >= 4 - 2\delta_n - \delta_n^2$ and therefore

$$|B_n - A|^2 = 2\delta_n(2+\delta_n), \quad |C_n - B_n|^2 \leq 4\delta_n, \quad |A - C_n|^2 = \delta_n^2 . \tag{24}$$

Let us define δ_n for $n \geq 3$ by

$$\delta_n = n^{-\frac{2}{2-p_n}}, \quad p_n = 1 + \frac{\epsilon_n}{1+\epsilon_n}, \quad \epsilon_n = \frac{2\ln(\ln n)}{\ln n} . \tag{25}$$

Then the sum T of the loop lengths for $n = 3, 4, \ldots$ can be estimated as

$$\begin{aligned} T &= \sum_{n=3}^{\infty}(|B_n - A| + |C_n - B_n| + |A - C_n|) \leq 4\sum_{n=3}^{\infty}\sqrt{\delta_n} + (1+\sqrt{2})\sum_{n=3}^{\infty}\delta_n \\ &= 4\sum_{n=3}^{\infty}\frac{1}{n\ln^2 n} + (1+\sqrt{2})\sum_{n=3}^{\infty}\frac{1}{n^2\ln^4 n} < +\infty . \end{aligned}$$

We define $u : [0,T] \to \mathbf{R}^2$ as the linear interpolate of the sequence $A, B_3, C_3, A, B_4, \ldots$ satisfying $|u'| = 1$ a.e.. If I_n denotes the time interval during which $u(t)$ moves from C_n to A, we obtain, using (24) and (25),

$$\delta_n^{p-1}\int_{I_n} |\partial_t\phi(t,1)|^p\,dt \geq \left(\int_{I_n} |\partial_t\phi(t,1)|\,dt\right)^p \geq \left(\frac{1}{2}|B_n - A|\right)^p \geq \delta_n^{p/2},$$

so

$$\int_0^T |\partial_t\phi(t,1)|^p\,dt \geq \sum_{n=3}^{\infty}\delta_n^{1-p+p/2} = \sum_{n=3}^{\infty} n^{-\frac{2-p}{2-p_n}} = +\infty .$$

The example is complete. □

3 The flow rule

In plastic flow theory, the flow rule serves to determine the plastic strain ϵ^p from the current value of stress and the current position of the yield surfaces. The total strain ϵ is given by

$$\epsilon = \epsilon^p + \epsilon^e, \quad \epsilon^e = A\sigma, \tag{26}$$

where the elastic strain ϵ^e is obtained from the stress via Hooke's law expressed with the symmetric positive definite matrix A. Chu [3], [4] presents a flow rule which has the following two properties:

- The plastic strain rate tensor points in the direction of the outward normal $n(t)$ common to the active yield surfaces.
- For uniaxial stress, the standard stabilized stress–strain behaviour characterized by Masing's law and the memory properties of the uniaxial version of the Mróz hardening rule (which is in fact identical with Prandtl's model in [12]) is obtained.

These properties result in the formula

$$\dot{\epsilon}^{(p)}(t) = f'(a(t)) < n(t), \dot{\sigma}^{(d)}(t) > n(t), \tag{27}$$

where $a(t)$ is the radius of the largest active yield surface

$$a(t) = \max\{r : r \geq 0, |\phi(t,r) - \sigma^{(d)}(t)| = r\}, \quad n(t) = -\frac{\partial}{\partial r}\phi(t,0), \tag{28}$$

and the function $|\epsilon^p| = f(|\sigma^{(d)}|)$ with $f(0) = f'(0) = 0$, $f'' \geq 0$, denotes the stabilized uniaxial initial stress–strain curve.

It is useful to rewrite the flow rule (27) in a derivative–free form. To this end, let us introduce an auxiliary function $\psi = \psi(t,r)$ via the Stieltjes integral

$$\psi(t,r) = \phi(0,r) + \int_0^t \frac{\sigma^{(d)}(\tau) - \phi(\tau,r)}{r} < \frac{\sigma^{(d)}(\tau) - \phi(\tau,r)}{r}, d_\tau\phi(\tau,r) > . \tag{29}$$

A straightforward computation shows that equation (27) together with the initial condition

$$\epsilon^p(0) = \frac{f(a(0))}{a(0)}\sigma^{(d)}(0), \quad a(0) = |\sigma^{(d)}(0)|, \tag{30}$$

is equivalent to the formula

$$\epsilon^p(t) = \int_0^\infty \psi(t,r)\eta(r)\,dr, \quad \eta(r) := f''(r), \tag{31}$$

if the functions occuring are smooth enough. This follows since we have $\partial_t\phi(t,r) = 0$ for $r > a(t)$ and

$$< \partial_t\phi(t,r), n(t) > = < \dot{\sigma}^{(d)}(t) - r\dot{n}(t), n(t) > = < \dot{\sigma}^{(d)}(t), n(t) > \tag{32}$$

for $r < a(t)$. We obtain a continuity result for this version of the flow rule.

Proposition 3.1 *The hardening rule $\phi = F(\sigma^{(d)})$ from definition 2.4 together with (29) and (31) defines an operator $\epsilon^p = M(\sigma^{(d)})$,*

$$M : C(0,T;\mathbf{T_{dev}}) \cap BV(0,T;\mathbf{T_{dev}}) \rightarrow C(0,T;\mathbf{T_{dev}}) \cap BV(0,T;\mathbf{T_{dev}}).$$

Moreover, for any sequence $(u_n)_{n\in N}$ in $C(0,T;\mathbf{T_{dev}}) \cap BV(0,T;\mathbf{T_{dev}})$ with uniformly bounded variation and $\| u_n - \sigma^{(d)} \|_\infty \rightarrow 0$ we obtain that $M(u_n)$ has uniformly bounded variation and that $\| M(u_n) - M(\sigma^{(d)}) \|_\infty \rightarrow 0$.

Proof: This follows from Theorems 2.6 and 2.8 together with the convergence result of [5], Theorem II.15.3 and its consequences. □

As many other extensions and refinements of the basic yield surface model for plastic flow, the flow rule (27) is obtained from a mixture of various guiding principles and as such is not a priori consistent with the framework of thermodynamics. In particular, one has to impose additional restrictions in order to exclude a violation of the second law. Let

$$W(t) = \int_0^t < \dot\epsilon(\tau), \sigma(\tau) > d\tau \tag{33}$$

denote the total mechanical work. According to (26) and (27), this is decomposed as $W(t) = W^e(t) + W^p(t)$, where

$$W^e(t) = \int_0^t < \dot\epsilon^{(e)}(\tau), \sigma(\tau) > d\tau = \left[\frac{1}{2} < A\sigma(\tau), \sigma(\tau) >\right]_0^t, \tag{34}$$

$$W^p(t) = \int_0^t < \dot\epsilon^{(p)}(\tau), \sigma(\tau) > d\tau = \int_0^t f'(a(\tau)) < \dot\sigma^{(d)}(\tau), n(\tau) >< \sigma^{(d)}(\tau), n(\tau) > d\tau \tag{35}$$

are the elastic and plastic work, respectively. We now construct a cyclic process whose energy dissipation has the wrong sign.

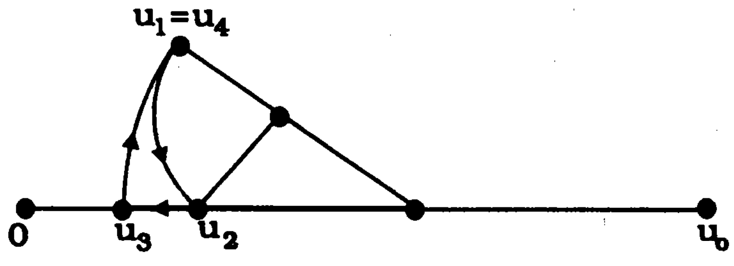

Figure 5: A cyclic process which produces energy.

Example 3.2 Let the situation be as in figure 5. We define $\sigma^{(d)}(t_i) = u_i$, $0 \le i \le 4$, for some $0 = t_0 < t_1 < \ldots < t_4$. We interpolate $\sigma^{(d)}$ linearly in $[0, t_1]$ and $[t_2, t_3]$, and by a circular path with radius equal to $a(t_1)$ and $a(t_3)$ in the intervals $[t_1, t_2]$ respectively $[t_3, t_4]$. Consequently, $a(t)$ remains constant and $< \dot\sigma^{(d)}(t), n(t) >= 0$ during the circular motion, whereas

$$< \dot\sigma^{(d)}(t), n(t) > > 0, \quad < \sigma^{(d)}(t), n(t) > < 0, \quad t \in (t_2, t_3). \tag{36}$$

We compute the total work along the cycle $u_1 \to u_2 \to u_3 \to u_4 = u_1$ from (34), (35) and (36) as

$$W(t_4) - W(t_1) = \int_{t_2}^{t_3} f'(a(t)) < \dot{\sigma}^{(d)}(t), n(t) >< \sigma^{(d)}(t), n(t) > dt. \quad (37)$$

Now if the interval $[a(t_2), a(t_3)]$ belongs to the range of plastic deformation where f' is positive, we obtain from (36) that $W(t_4) - W(t_1) < 0$, which contradicts the second law of thermodynamics. □

A standard way to overcome this problem (see e.g. [1]) is to restrict the model to situations where

$$\dot{W}^p(t) = < \dot{\epsilon}^{(p)}(t), \sigma(t) > \geq 0 \quad (38)$$

almost everywhere in t. Looking at the figures accompanying the definition of the Mróz hardening rule one notices immediately that

$$< \dot{\sigma}^{(d)}(t), n(t) > \geq 0 \quad (39)$$

holds for piecewise linear stress functions. We present a precise formulation and proof of (39) for general stress functions. This is unfortunately a bit tedious since the limit process underlying Theorem 2.6 does not have a clear relation to the weak limit of the corresponding normals $n(t)$.

Lemma 3.3 *Let $\sigma^{(d)} \in W^{1,1}(0,T;\mathbf{T_{dev}})$ and $t \in [0,T]$ with $a(t) > 0$ be given and assume that $\dot{\sigma}^{(d)}$ is continuous in $[s,t]$ for some $s < t$. Then we have*

$$< \dot{\sigma}^{(d)}(t), n(t) > \geq 0. \quad (40)$$

Proof: Assume that $d := - < \dot{\sigma}^{(d)}(t), n(t) > > 0$. We revert to the notation $u(t) = \sigma^{(d)}(t)$. We abbreviate $a = a(t)$ and choose $s < t$ such that, with $\phi = F(u)$ and $M = \max_{s \leq \tau \leq t} |\dot{u}(\tau)|$, we have

$$|t-s| < \frac{ad}{4M^2}, \quad |\phi(t,a) - \phi(s,a)| < \frac{ad}{4M}, \quad |\dot{u}(t) - \dot{u}(\rho)| < \frac{d}{4} \quad \forall \rho \in [s,t]. \quad (41)$$

We then have for every $\rho, \tau \in [s,t]$

$$\begin{aligned} < \dot{u}(\rho), u(\tau) - \phi(s,a) > &= < \dot{u}(t), u(t) - \phi(t,a) > \\ &+ < \dot{u}(\rho) - \dot{u}(t), u(t) - \phi(t,a) > + < \dot{u}(\rho), u(\tau) - u(t) > \\ &+ < \dot{u}(\rho), \phi(t,a) - \phi(s,a) > < -\frac{ad}{4}. \end{aligned} \quad (42)$$

We now want to prove that $\phi(t,a) = \phi(s,a)$. To this end, we choose an equidistant partition $s = s_0 < s_1 < \ldots < s_n = t$ and set $u_k = u(s_k)$, $\phi_0 = \phi(s,\cdot)$ and $\phi_k = G(u_k, \phi_{k-1})$. We claim that

$$\phi_k(a) = \phi_0(a) \implies |u_{k+1} - \phi_k(a)| < a \implies \phi_{k+1}(a) = \phi_k(a). \quad (43)$$

The right implication is trivial. The left one follows from the estimate (we use (42))

$$\begin{aligned} |u_{k+1} - \phi_k(a)|^2 &\leq |u_{k+1} - u_k|^2 + a^2 + 2 < u_{k+1} - u_k, u_k - \phi_k(a) > \\ &= |u_{k+1} - u_k|^2 + a^2 + 2\int_{s_k}^{s_{k+1}} < \dot{u}(\rho), u(s_k) - \phi_0(a) > d\rho \\ &< M^2(s_{k+1} - s_k)^2 + a^2 - 2(s_{k+1} - s_k)\frac{ad}{4} < a^2. \end{aligned}$$

From (43) we conclude that $\phi_n(a) = \phi_0(a)$, and passing to the limit as $n \to \infty$ we obtain $\phi(t,a) = \phi(s,a)$, so we have $|u(t) - \phi(s,a)| = a$. Since (42) also implies that

$$< u(t) - u(s), u(t) - \phi(s,a) > \leq \int_s^t < \dot{u}(\rho), u(t) - \phi(s,a) > d\rho \leq -(t-s)\frac{ad}{4},$$

we arrive at the contradiction

$$\begin{aligned} a^2 &\geq |u(s) - \phi(s,a)|^2 = |u(s) - u(t)|^2 + a^2 - 2 < u(t) - u(s), u(t) - \phi(s,a) > \\ &\geq |u(s) - u(t)|^2 + a^2 + (t-s)\frac{ad}{2} > a^2 . \end{aligned}$$

The lemma is proved. □

We now show that condition (38) is satisfied for the flow rule (27) if and only if the maximal stress does not exceed twice the value of the yield stress.

Proposition 3.4 *Let $\sigma^{(d)} \in W^{1,1}(0,T;\mathbf{T}_{\text{dev}})$ be given, and assume that $\dot{\sigma}^{(d)}$ is piecewise continuous. If $f'(r) = 0$ in the range $0 \leq r \leq \| \sigma^{(d)} \|_\infty / 2$, then condition (38) is satisfied. Conversely, if $f'(r) > 0$ for some $r > 0$, then for any $\delta > 0$ we may construct along the lines of Example 3.2 a function $\sigma^{(d)}$ with $\| \sigma^{(d)} \|_\infty < 2r + \delta$ such that the second law is violated.*

Proof: Because of Lemma 3.3, it suffices to prove that $< \sigma^{(d)}(t), n(t) > > 0$ for any t with $a(t) \geq r_0 := \| \sigma^{(d)} \|_\infty / 2$. Setting $\phi = F(\sigma^{(d)})$, we have $\phi(t, 2r_0) = \phi(t, \| \sigma^{(d)} \|_\infty) = 0$, hence $|\phi(t, r_0)| \leq r_0$ and therefore, if $a(t) \geq r_0$,

$$< \sigma^{(d)}(t), n(t) > = < \phi(t, r_0) + r_0 n(t), n(t) > \geq r_0 - |\phi(t, r_0)| \geq 0 .$$

For the converse, Example 3.2 works whenever we choose $|u_0| > 2r$ and $|u_0 - u_3| = 2r$. □

Although Proposition 3.4 suggests that the flow rule (27) might be physically correct if only it is restricted to an appropriate range, we find this approach problematic for two reasons. First, the flow rule (27) implies that the plastic work is zero in neutral motion, i.e. along circular paths in deviatoric stress space, even if the amplitudes are well within the range of plastic deformation for uniaxial loading. Second, it seems more natural to connect the flow rule to the hardening rule through a mechanism of energy dissipation rather than to develop them as separate entities. Similar to the approach in [9], we now present such a connection as a generalization of the energy dissipation mechanism of the uniaxial hysteresis model. This will lead to a flow rule different from (27). (We do not try here an embedding into the full framework of thermomechanics of, for example, [14].) We write the total mechanical power (= rate of work) in the form

$$< \dot{\epsilon}(t), \sigma(t) > = \frac{d}{dt} (P_E(t) + P_M(t)) + D(t) . \tag{44}$$

Here, $D(t)$ is the dissipation rate, which should turn out to be nonnegative, $P_E(t)$ denotes the standard elastic potential

$$P_E(t) = \frac{1}{2} < A\sigma(t), \sigma(t) > , \tag{45}$$

and we propose a *memory potential* $P_M(t)$ (or *hysteresis potential* in the terminology of [9]) of the form

$$P_M(t) = \frac{1}{2}\int_0^\infty |\phi(t,r)|^2 \eta(r)\, dr\,, \tag{46}$$

where, as in (31), $\eta = f'' \geq 0$. We also propose the flow rule

$$\epsilon^p(t) = \int_0^\infty \phi(t,r)\eta(r)\, dr\,. \tag{47}$$

Actually, (31) and (47) coincide in the uniaxial case. From (26), (33), (34) and (44) – (47) we compute, assuming that $\partial_t\phi(t,r)$ exists almost everywhere and is bounded for $r > 0$, which is the case if e.g. $\sigma^{(d)}$ is piecewise linear,

$$D(t) = \int_0^\infty < \partial_t\phi(t,r), \sigma^{(d)}(t) - \phi(t,r) > \eta(r)\, dr\,. \tag{48}$$

We will prove below that the Mróz hardening rule satisfies a certain *energy inequality* which in turn implies

$$< \partial_t\phi(t,r), \sigma^{(d)}(t) - \phi(t,r) > \;\geq\; 0 \tag{49}$$

almost everywhere. This leads to a satisfactory state of affairs, since then $D(t) \geq 0$ almost everywhere, and consequently no cyclic process will violate the second law of thermodynamics regardless of the form or amplitude of the loading history. This follows from the observation that the function $t \mapsto \phi(t,r)$ (and therefore, also ϵ^p, P_E and P_M as functions of t), are T-periodic in t for T-periodic piecewise linear (and hence, by continuity, for T-periodic continous) input functions $\sigma^{(d)}$ with the possible exception of the first cycle. Before we prove (49), however, we briefly note that the flow rule (47), too, yields a continuous (i.e. well-posed) stress-strain relation.

Proposition 3.5 *Let us define the modified Mróz operator* $\epsilon^p = M^*(\sigma^{(d)})$ *as the composition of the flow rule (47) and the Mróz hardening rule* $\phi = F(\sigma^{(d)})$. *Then for any* $\sigma_1, \sigma_2 \in C(0,T;\mathbf{T_{dev}})$ *we have*

$$\| M^*(\sigma_1^{(d)}) - M^*(\sigma_2^{(d)}) \|_\infty \leq \sqrt{2R} f'(R) (\| \sigma_1^{(d)} - \sigma_2^{(d)} \|_\infty)^{\frac{1}{2}}\,,$$

where $R = \max\{\| \sigma_1^{(d)} \|_\infty, \| \sigma_2^{(d)} \|_\infty\}$.

Proof: This is an immediate consequence of Theorem 2.6. □

We now present an energy inequality for the Mróz hardening rule.

Proposition 3.6 *Let* $\sigma^{(d)} \in W^{1,1}(0,T;\mathbf{T_{dev}})$ *be given, set* $\phi = F(\sigma^{(d)})$. *Then we have*

$$\frac{1}{2}\left(|\phi(t,r)|^2 - |\phi(s,r)|^2\right) - < \phi(t,r), \sigma^{(d)}(t) > + < \phi(s,r), \sigma^{(d)}(s) > + \int_s^t < \phi(\tau,r), \dot\sigma^{(d)}(\tau) > d\tau \leq 0 \tag{50}$$

for every $0 \leq s < t \leq T$ *and every* $r \geq 0$.

Dividing both sides of (50) by $t - s$ and letting s tend to t, we immediately see that (49) holds for every $t \in [0, T]$ for which $\partial_t \phi(t, r)$ exists and at which $\dot{\sigma}^{(d)}$ is continuous (actually it suffices that t is a Lebesgue point of $\dot{\sigma}^{(d)}$), so the dissipation rate $D(t)$ as defined above is always nonnegative.

The remainder of this section is devoted to the proof of Proposition 3.6. We first prove a discrete version of the energy inequality (50).

Lemma 3.7 *Let $\{u_k\}_{k \geq 0}$ be a sequence of input values in U, set $\phi_k = G(u_k, \phi_{k-1})$, $\phi_{-1} = 0$. Then we have*

$$< \phi_k(r) - \phi_{k-1}(r), \phi_k(r) - u_k > \ \leq \ 0 \tag{51}$$

for every $k \geq 0$ and every $r \geq 0$, and

$$\begin{aligned} \frac{1}{2}\left(|\phi_m(r)|^2 - |\phi_l(r)|^2\right) \quad - & \quad < \phi_m(r), u_m > \ + \ < \phi_l(s), u_l > \\ & + \sum_{k=l}^{m-1} < \phi_k(r), u_{k+1} - u_k > \ \leq \ 0 \end{aligned} \tag{52}$$

for every $m > l \geq 0$ and every $r \geq 0$.

Proof: We first prove (51). Set

$$a_k \ = \ \alpha(u_k, \phi_{k-1}) \ = \ \min\{r \geq 0 : \ |u_k - \phi_{k-1}(r)|\}\,.$$

Assume that $0 < r < a_k$, otherwise (51) holds trivially. Then

$$< \phi_k(r) - \phi_{k-1}(r), \phi_k(r) - u_k > = < \phi_k(r) - \phi_k(a_k) + \phi_{k-1}(a_k) - \phi_{k-1}(r), \phi_k(r) - u_k >$$

$$= \ -r(a_k - r) + \ < \phi_{k-1}(a_k) - \phi_{k-1}(r), \phi_k(r) - u_k > \ \leq \ 0$$

so (51) is proved. We now substitute the right hand side of

$$\phi_k(r) \ = \ \frac{1}{2}(\phi_k(r) + \phi_{k-1}(r)) \ + \ \frac{1}{2}(\phi_k(r) - \phi_{k-1}(r))$$

for the second occurrence of $\phi_k(r)$ in (51) and obtain

$$\begin{aligned} 0 \ \geq & \ \sum_{k=l+1}^{m} < \phi_k(r) - \phi_{k-1}(r), \frac{1}{2}(\phi_k(r) + \phi_{k-1}(r)) - u_k > \\ = & \ \frac{1}{2}\left(|\phi_m(r)|^2 - |\phi_l(r)|^2\right) \ - \ \sum_{k=l+1}^{m} < \phi_k(r) - \phi_{k-1}(r), u_k > \,. \end{aligned}$$

Rearranging the last sum, inequality (52) follows. □

Proof of Proposition 3.6 Let $0 \leq s < t \leq T$ be given. We approximate the function $\sigma^{(d)}$ by piecewise linear interpolants $u^n : [0, T] \to U$ on the partition $\pi^n = (t_k^n)$ with $0 = t_0^n < t_1^n < \ldots < t_{k_n}^n = T$ such that t and s belong to each partition and that $\max_k(t_{k+1}^n - t_k^n) \to 0$ as n tends to infinity. Fix $n \in N$ and set $\phi^n = F(u^n)$. We apply

(52) to the sequence $u_k = u^n(t_k^n)$ with m and l chosen such that $t = t_m^n$, $s = t_l^n$. Since $\phi_k = \phi^n(t_k^n)$, we get

$$\begin{aligned} E^n &:= \frac{1}{2}\left(|\phi^n(t,r)|^2 - |\phi^n(s,r)|^2\right) - < \phi^n(t,r), u^n(t) > + < \phi^n(s,r), u^n(s) > \\ &\quad + \int_s^t < \phi^n(\tau,r), \dot{u}^n(\tau) > d\tau \\ &\le \int_s^t < \phi^n(\tau,r), \dot{u}^n(\tau) > d\tau - \sum_{k=l}^{m-1} < \phi^n(t_k^n,r), u_{k+1} - u_k > \\ &= \sum_{k=l}^{m-1} \frac{1}{t_{k+1}^n - t_k^n} \int_{t_k^n}^{t_{k+1}^n} < \phi^n(\tau,r) - \phi^n(t_k^n,r), u_{k+1} - u_k > d\tau . \end{aligned} \tag{53}$$

From Theorem 2.6 we obtain, if $t_k^n \le \tau \le t_{k+1}^n$,

$$\begin{aligned} |\phi^n(\tau,r) - \phi^n(t_k^n,r)| &\le \sqrt{2\| u^n \|_\infty |u^n(\tau) - u^n(t_k^n)|} \\ &\le \sqrt{2\| \sigma^{(d)} \|_\infty \max_k |\sigma^{(d)}(t_{k+1}^n) - \sigma^{(d)}(t_k^n)|} =: \delta^n . \end{aligned} \tag{54}$$

From (53) and (54) we conclude that

$$E^n \le \sum_{k=l}^{m-1} \delta^n |u_{k+1} - u_k| \le \delta^n \int_0^T |\dot{\sigma}^{(d)}(\tau)| d\tau . \tag{55}$$

Since u^n converges to $\sigma^{(d)}$ in $W^{1,1}$ and ϕ^n converges to ϕ uniformly, E^n converges to the left hand side of (50). We also see from (54) that δ^n converges to 0. The proof is complete.

4 Proof of the continuity result

In this section we prove Theorem 2.6. We have to estimate differences $\| \phi - \psi \|_\infty$ in terms of differences $\| u - v \|_\infty$ of the corresponding input functions. We begin with a lemma which relates $\| \phi - \psi \|_\infty$ to the distance of the two extreme corners in a special case.

Lemma 4.1 *Let $\phi, \psi \in \Psi$ be two memory states with $\phi(r) = 0$ for $r \ge b$, $\psi(r) = 0$ for $r \ge a$, $|\psi(0)| = a$. Then for $\delta = \max\{|b - a|, |\psi(0) - \phi(0)|\}$ we have*

$$\| \phi - \psi \|_\infty \le \sqrt{2\,\delta \max\{a,b\}} , \tag{56}$$

and if moreover $\delta \le \frac{2}{3}\max\{a,b\}$, then

$$\| \phi - \psi \|_\infty \le \sqrt{\delta \max\{a,b\}} . \tag{57}$$

(The latter estimate is sharp as the example in figure 3 shows.)

Proof: Since we have $|\psi(r) - \phi(r)| \le \delta$, if r lies between a and b, and since $\delta \le a + b$, we only have to consider the case where $0 \le r \le \min\{a,b\}$. In this range, ψ has to be a straight line, so

$$\phi(r) - \psi(r) = \frac{r}{a}\phi(r) + (1 - \frac{r}{a})(\phi(r) - \psi(0)) . \tag{58}$$

By assumption, we have

$$a^2 = |\psi(0)|^2 = |\phi(r) - \psi(0)|^2 + |\phi(r)|^2 - 2 < \phi(r), \phi(r) - \psi(0) > . \tag{59}$$

Inserting (59) into the square of (58) we obtain with the aid of

$$|\phi(r)| \le b - r, \quad |\phi(r) - \psi(0)| \le |\phi(r) - \phi(0)| + \delta \le r + \delta,$$

the identity plus estimate

$$\begin{aligned} |\phi(r) - \psi(r)|^2 &= \frac{r}{a}|\phi(r)|^2 + (1 - \frac{r}{a})|\phi(r) - \psi(r)|^2 - r(a-r) \\ &\le (b-a)\frac{r}{a}(b + a - 2r) + (1 - \frac{r}{a})\delta(2r + \delta). \end{aligned} \tag{60}$$

If we introduce $\epsilon = b - a$, then (60) becomes

$$|\phi(r) - \psi(r)|^2 \le f(\epsilon, r) = \frac{2}{a}(\delta + \epsilon)r(a-r) + \frac{r}{a}\epsilon^2 + (1 - \frac{r}{a})\delta^2 .$$

Maximizing f with respect to r yields

$$g(\epsilon) := \max_{0 \le r \le a} f(\epsilon, r) = \frac{1}{2}\left[a(\delta + \epsilon) + \delta^2 + \epsilon^2 + \frac{(\delta - \epsilon)^2(\delta + \epsilon)}{4a}\right].$$

To conclude the proof, we have to consider the following cases separately.

- $b \le a$. Then $0 \ge \epsilon \ge \max\{-\delta, \delta - 2a\}$, and

  $$g'(\epsilon) = \frac{1}{8a}(2a + \delta + 3\epsilon)(2a - \delta + \epsilon) \ge 0, \; g(\epsilon) \le g(0) = \frac{\delta}{2a}(a + \frac{\delta}{2})^2,$$

  and the assertion follows easily.

- $b > a$, $\delta \le b$. Then $0 < \epsilon \le \delta$ and

  $$g(\epsilon) \le g_1(\epsilon) := \frac{1}{2}\left[b\delta + \epsilon(b - \delta) + \delta^2 + \frac{\delta^2 - \epsilon^2}{4}\right], \; g_1'(\epsilon) = \frac{1}{2}(b - \delta - \frac{\epsilon}{2}).$$

  For $\delta \le \frac{2}{3}b$ we have $g_1(\epsilon) \le g_1(\delta) = b\delta$, for $\delta \in (\frac{2}{3}b, b)$ we get $g_1(\epsilon) \le g_1(2(b-\delta)) < 2b\delta$.

- $b > a$, $\delta \in (b, 2b)$. Then $0 < \epsilon \le 2b - \delta$, hence $\delta - \epsilon \le 2(b - \epsilon) = 2a$ and

  $$g(\epsilon) \le g_2(\epsilon) := \frac{1}{2}\left[b\delta + \epsilon(b - \delta) + \delta^2 + \frac{\delta^2 - \epsilon^2}{2}\right], \; g_2'(\epsilon) = \frac{1}{2}(b - \delta - \epsilon) < 0.$$

  This implies $g(\epsilon) \le g_2(0) < 2b\delta$, and lemma 4.1 is proved. □

The following lemma, whose proof constitutes the main effort of this section, is crucial for the derivation of Theorem 2.6.

Lemma 4.2 *Let $R > 0$ and $\delta_0 > 0$ be given. Let us call* admissible pair *any pair of input sequences $\{u_k\}_{k=0}^n$, $\{v_k\}_{k=0}^n$ with $n \in \mathbb{N}$ and*

$$\max_{0 \le k \le n} \{|u_k|, |v_k|\} \le R, \quad \max_{0 \le k \le n} |u_k - v_k| \le \delta_0. \tag{61}$$

Let us assume that there exists an admissible pair such that the corresponding sequences $\{\phi_k\}_{k=0}^n$, $\{\psi_k\}_{k=0}^n$ of memory states from Definition 2.4 satisfy

$$\max_{0 \le k \le n} \| \phi_k - \psi_k \|_\infty > \sqrt{2R\delta_0}. \tag{62}$$

Let N be the total number of corners of $\{\phi_k\}_{k=0}^n$ and $\{\psi_k\}_{k=0}^n$, i.e.

$$N = \sum_{k=0}^{n} (N_c(\phi_k) + N_c(\psi_k)),$$

where, for any piecewise linear $\psi \in \Psi$, $N_c(\psi)$ denotes the number of corners of ψ. Then there exists an admissible pair $\{\hat{u}_k\}_{k=0}^{\hat{n}}$, $\{\hat{v}_k\}_{k=0}^{\hat{n}}$ satisfying (62) such that the total number of the corners of the corresponding memory states is less than N.

As an immediate consequence, the memory states corresponding to any admissible pair satisfy

$$\max_{0 \le k \le n} \| \phi_k - \psi_k \|_\infty \le \sqrt{2R\delta_0}. \tag{63}$$

Since the proof of Lemma 4.2 takes several pages, we first use it to prove the main continuity result.

Proof of Theorem 2.6. For a fixed $R > 0$, we define

$$U_R = \{ u \in U : |u| \le R \}, \quad \Psi_R = \{ \psi \in \Psi : \psi(r) = 0 \text{ for any } r \ge R \}.$$

Now let $u \in C(0,T;U_R)$ be piecewise linear. We first claim that $\phi = F(u) \in C(0,T;\Psi_R)$. To this end, we observe that Lemma 4.1 implies for any $\psi = \phi(t,\cdot)$, $t \in [0,T]$, and any $v \in U$ the estimate (see (11) and (16) for the definition of G and L)

$$\| G(v,\psi) - \psi \|_\infty \le \sqrt{2|v - \psi(0)| \max\{|v|, L(\psi)\}}, \tag{64}$$

and this in turn implies that $\phi(t,\cdot)$ depends continously upon t. Next, we consider piecewise linear inputs $u, v \in C(0,T;U_R)$ with $\delta := \| u - v \|_\infty$. We fix $t \in [0,T]$ and let $\{t_k\}_{k=0}^n$ denote a partition of $[0,T]$ which includes the point t and is such that both u and v are linear within each interval $[t_k, t_{k+1}]$. We apply Lemma 4.2 to the admissible pair $\{u_k\}_{k=0}^n$, $\{v_k\}_{k=0}^n$, where $u_k = u(t_k)$, $v_k = v(t_k)$. From (63) we obtain the estimate

$$\max_{r \ge 0} |(Fu)(t,r) - (Fv)(t,r)| \le \sqrt{2R \| u - v \|_\infty} \tag{65}$$

Therefore, the operator F is uniformly continuous on the set of piecewise linear input functions, which is dense in the space $C(0,T;U_R)$, and F has values in the space $C(0,T;\Psi_R)$. Because U_R and Ψ_R are complete metric spaces, F can be extended uniquely to an operator from $C(0,T;U_R)$ to $C(0,T;\Psi_R)$, such that (65) holds for any $u, v \in C(0,T;U_R)$. This concludes the proof of the theorem. □

The remainder of this section is devoted to the proof of Lemma 4.2. Some intermediate steps will be formulated as separate lemmas within the proof.

Proof of Lemma 4.2 Let $\{u_k\}_{k=0}^n$, $\{v_k\}_{k=0}^n$ be an admissible pair such that the corresponding memory states $\{\phi_k\}_{k=0}^n$, $\{\psi_k\}_{k=0}^n$ satisfy (62). Let us call $\{u_k\}$, $\{v_k\}$ *reducible*, if the conclusion of the lemma holds. We set $\delta = \max\{|u_k - v_k| : 0 \leq k \leq n\} \leq \delta_0$. We obviously have

$$|L(\phi_k) - L(\psi_k)| \leq \delta \quad \text{for any } 0 \leq k \leq n\,. \tag{66}$$

We may further assume that

$$M := \|\phi_n - \psi_n\|_\infty > \|\phi_k - \psi_k\|_\infty \quad \text{for any} \quad 0 \leq k \leq n-1\,, \tag{67}$$

otherwise $\{u_k\}$, $\{v_k\}$ is obviously reducible. Since we have to analyze the structure of ϕ_n and ψ_n in great detail, we will simply write ϕ and ψ instead. We denote the corners of ϕ and ψ by $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$, where we count from the end points $P_0 := \phi(0) = u_n$ and $Q_0 := \psi(0) = v_n$; the end points P_0, Q_0 are not counted as corners. Both ϕ and ψ must have at least one corner; otherwise (62) could not hold. Let us further define the unit vectors (we will use the primed ones only after we have established that the corners referred to exist)

$$e_\phi = \frac{P_1 - P_0}{|P_1 - P_0|}\,,\ e'_\phi = \frac{P_2 - P_1}{|P_2 - P_1|}\,,\ e_\psi = \frac{Q_1 - Q_0}{|Q_1 - Q_0|}\,,\ e'_\psi = \frac{Q_2 - Q_1}{|Q_2 - Q_1|}\,, \tag{68}$$

which point along the first two line segments of ϕ and ψ. Let us moreover define

$$p = \max\{\, r \geq 0 : |\phi(r) - \psi(r)| = M\,\}\,. \tag{69}$$

Because of (67) and Lemma 2.5(i), at least one of the memory states ϕ and ψ, let us say ψ, has no corners in the interval $[0,p]$, so $Q_1 = \psi(s)$ for some $s > p$. The definition of p then implies that $\phi(p)$ is a corner of ϕ. The line

$$X = \{\, \psi(s) + \lambda x : \lambda \in \mathbf{R}\,\}\,, \quad x = \phi(p) - \psi(s)\,, \tag{70}$$

which passes through the points $\psi(s)$ and $\phi(p)$, will also play a central role.

Lemma 4.3 *(i) The memory state ψ has at least two corners, so $Q_2 = \psi(s')$ for some $s' > s$.*
(ii) We have $< e_\psi, \phi(p) - \psi(p) > < 0$ and $< e_\psi, x > < p - s < 0$.
(iii) We have

$$|x|^2 > M^2 + (s-p)^2\,. \tag{71}$$

(iv) The memory state ϕ has at least two corners, so $P_2 = \phi(p')$ for some $p' > 0$.

Proof: (i) If ψ has only one corner, then Lemma 4.1 together with (66) implies that $\|\phi - \psi\|_\infty \leq \sqrt{2R\delta}$ in contradiction to (62).
(ii) We have $\psi(p) - v_n = pe_\psi$ and

$$\begin{aligned}(p+\delta)^2 &\geq |\phi(p) - v_n|^2 \\ &= |\phi(p) - \psi(p)|^2 + |\psi(p) - v_n|^2 + 2 < \phi(p) - \psi(p), \psi(p) - v_n > \\ &= M^2 + p^2 + 2p < e_\psi, \phi(p) - \psi(p) > \,.\end{aligned} \tag{72}$$

On the other hand, we have $\phi(R) = \psi(R) = 0$, hence

$$\sqrt{2R\delta} \;<\; M \;=\; |\,\phi(p) - \psi(p)\,| \;\leq\; 2(R-p)\,.$$

This yields $2(R-p)^2 > R\delta > (R-p)\delta$, hence $2R > 2p+\delta$ and $2R\delta + p^2 > (p+\delta)^2$. Comparing the last inequality to (72) we obtain the first inequality and also the second, since $< e_\psi, \psi(p) - \psi(s) >= p - s$.
(iii) The assertion follows from (ii), since

$$|x|^2 \;=\; |\phi(p) - \psi(p)|^2 \;+\; (s-p)^2 \;-\; 2(s-p)\; < e_\psi, \phi(p) - \psi(p) > \,.$$

(iv) Assume that $\phi(p)$ is the only corner of ϕ. Then we have $\phi(p) = 0$ and $L(\phi) = p$, and from (iii) we conclude that

$$L(\psi) \;\geq\; s + |\psi(s) - \phi(p)| \;\geq\; s + M \;>\; s + \delta \;>\; p + \delta\,.$$

which contradicts (66). □

Lemma 4.4 *If* $< x, e_\psi - e'_\psi > < 0$ *does not hold, then the pair* $\{u_k\}$, $\{v_k\}$ *is reducible.*

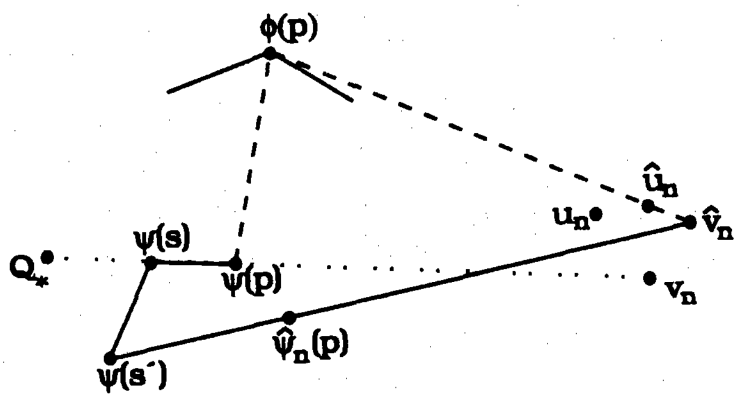

Figure 6: Illustration of the proof of Lemma 4.4

Proof: Let us assume that $< x, e_\psi > \;\geq\; < x, e'_\psi >$. Introducing an auxiliary point $Q_* = v_n + s' e_\psi$, compare figure 6, we obtain $|Q_* - \psi(s)| = s' - s$ and

$$\begin{aligned} |\phi(p) - Q_*|^2 \;&=\; |\phi(p) - \psi(s) + \psi(s) - Q_*|^2 \;=\; |x|^2 + (s'-s)^2 \\ &\quad - 2(s'-s) < x, e_\psi > \;\leq\; |x|^2 + (s'-s)^2 - 2(s'-s) < x, e'_\psi > \\ &=\; |\phi(p) - \psi(s) + \psi(s) - \psi(s')|^2 \;=\; |\phi(p) - \psi(s')|^2\,. \end{aligned} \qquad (73)$$

We now choose some $\hat{v}_n \in U$ with $|\psi(s') - \hat{v}_n| = s'$ and $|\phi(p) - \hat{v}_n| = p + \delta$; this is possible since (73) and (ii) imply that $|s' - (p+\delta)| \leq |\psi(s') - \phi(p)| \leq s' + p + \delta$. We define

$$\hat{u}_n = \frac{\delta}{p+\delta}\phi(p) + \frac{p}{p+\delta}\hat{v}_n\,,\; \hat{\phi}_n = G(\hat{u}_n, \phi)\,,\; \hat{\psi}_n = G(\hat{v}_n, \psi)\,, \hat{e}_\psi = \frac{1}{s'}(\psi(s') - \hat{v}_n)\,. \quad (74)$$

The pair $\{u_0, \ldots, u_{n-1}, \hat{u}_n\}, \{v_0, \ldots, v_{n-1}, \hat{v}_n\}$ is admissible, since $|\hat{v}_n| \le L(\hat{\psi}_n) = L(\psi) \le R$. From Lemma 2.5 (ii) we conclude that $\{\phi_0, \ldots, \phi_{n-1}, \hat{\phi}_n\}, \{\psi_0, \ldots, \psi_{n-1}, \hat{\psi}_n\}$ are the corresponding memory states. Since $\hat{\psi}_n$ has fewer corners than ψ, the proof will be complete if we can show that $|\hat{\phi}_n(p) - \hat{\psi}_n(p)| \ge |\phi(p) - \psi(p)|$. To this end, from the identities $\hat{\phi}_n(p) = \phi(p)$ and

$$
\begin{aligned}
|\phi(p) - Q_*|^2 &= |\phi(p) - v_n|^2 + s'^2 - 2s' < \phi(p) - v_n, e_\psi >, \\
|\phi(p) - \psi(s')|^2 &= (p+\delta)^2 + s'^2 - 2s' < \phi(p) - \hat{v}_n, \hat{e}_\psi >, \\
|\phi(p) - \psi(p)|^2 &= |\phi(p) - v_n|^2 + p^2 - 2p < \phi(p) - v_n, e_\psi >, \\
|\hat{\phi}_n(p) - \hat{\psi}_n(p)|^2 &= (p+\delta)^2 + p^2 - 2p < \phi(p) - \hat{v}_n, \hat{e}_\psi >,
\end{aligned}
$$

we get with the aid of (73) that

$$
\begin{aligned}
|\hat{\phi}_n(p) - \hat{\psi}_n(p)|^2 - |\phi(p) - \psi(p)|^2 &= (1 - \frac{p}{s'}) \left[(p+\delta)^2 - |\phi(p) - v_n|^2\right] + \\
&\quad + \frac{p}{s'} \left(|\phi(p) - \psi(s')|^2 - |\phi(p) - Q_*|^2\right) \quad \ge \quad 0. \quad \square
\end{aligned}
$$

We now focus our attention upon the memory state ϕ.

Lemma 4.5 *If $\phi(p)$ is not the first corner of ϕ, then the pair $\{u_k\}$, $\{v_k\}$ is reducible.*

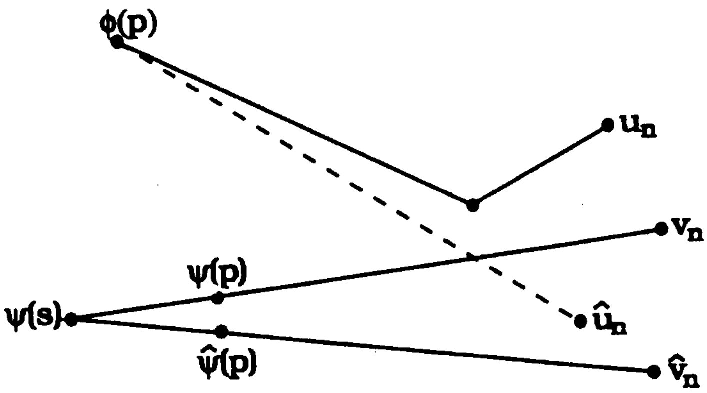

Figure 7: Illustration of Lemma 4.5

Proof: We choose $\hat{v}_n \in U$ with $|\psi(s) - \hat{v}_n| = s$ and $|\phi(p) - \hat{v}_n| = p + \delta$, see figure 7; this is possible since obviously $|s - (p+\delta)| \le |\psi(s) - \phi(p)| \le s + p + \delta$. We define $\hat{u}_n, \hat{\phi}_n, \hat{\psi}_n, \hat{e}_\psi$ as in (74). Again, the pair $\{u_0, \ldots, u_{n-1}, \hat{u}_n\}, \{v_0, \ldots, v_{n-1}, \hat{v}_n\}$ is admissible, and $\{\phi_0, \ldots, \phi_{n-1}, \hat{\phi}_n\}, \{\psi_0, \ldots, \psi_{n-1}, \hat{\psi}_n\}$ are its corresponding memory states. Since $\hat{\phi}_n$ has less corners than ϕ, it is again sufficient to prove that $|\hat{\phi}_n(p) - \hat{\psi}_n(p)| \ge |\phi(p) - \psi(p)|$. We have

$$
|\phi(p) - \psi(s) + \psi(s) - \hat{v}_n|^2 = (p+\delta)^2 > |\phi(p) - \psi(s) + \psi(s) - v_n|^2,
$$

hence $< x, \hat{e}_\psi - e_\psi > > 0$ and therefore

$$
|\hat{\phi}_n(p) - \hat{\psi}_n(p)|^2 - |\phi(p) - \psi(p)|^2 = 2(s-p) < x, \hat{e}_\psi - e_\psi > \quad > \quad 0. \quad \square
$$

The next lemma shows that the corner $Q_1 = \psi(s)$ is formed earlier than the corner $\phi(p)$.

Lemma 4.6 *Let us define*

$$m = \min\{j \le n : \psi_k(s) = \psi(s) \; for \; k = j, \dots, n\} ,$$
$$i = \min\{j \le n : \phi_k(p) = \phi(p) \; for \; k = j, \dots, n\} . \tag{75}$$

Then we may assume in the following that $m < i$; otherwise, the pair $\{u_k\}$, $\{v_k\}$ is reducible.

Proof: Let us assume that $i \le m$. From the definition of m and (12) we infer that $m < n$ and that $\psi_m(r) = v_m + re'_\psi$ for any $r \in [0, s']$. We may also assume that $< x, e_\psi - e'_\psi > < 0$; otherwise we are done because of Lemma 4.4. We then have for any $r \in [0, s']$

$$|\phi(p) - \psi_m(r)|^2 = |\phi(p) - \psi(s) + \psi(s) - \psi_m(r)|^2 = \tag{76}$$
$$= |x|^2 + (s-r)^2 + 2(s-r) < x, e'_\psi > \; > \; |x + (s-r)e_\psi|^2 = |\phi(p) - \psi(r)|^2 .$$

We have $|\phi(p) - v_m| = |\phi_m(p) - v_m| \le p + \delta$ by definition of m. Since moreover $|\phi(p) - v_m| > |\phi(p) - v_n| \ge p - \delta$ by (76) with $r = 0$, we may choose an $\hat{u}_m \in U$ such that $|\phi(p) - \hat{u}_m| = p$ and $|\hat{u}_m - v_m| \le \delta$, see figure 8. Consequently, the pair $\{u_0, \dots, u_m, \hat{u}_m\}$,

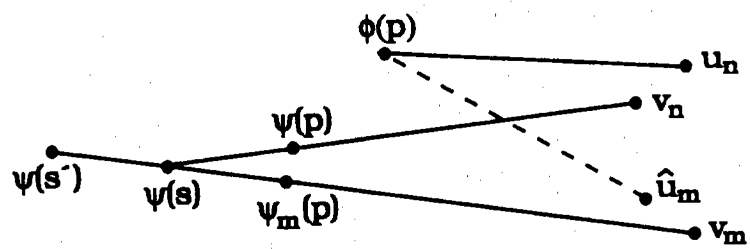

Figure 8: Illustration of the proof of Lemma 4.6

$\{v_0, \dots, v_m, v_m\}$ is admissible. Its memory states have a smaller total number of corners, because $N_c(\psi_m) = N_c(\psi) - 1$ and $N_c(\hat{\phi}_m) \le N_c(\phi)$, where $\hat{\phi}_m = G(\hat{u}_m, \phi_m) = G(\hat{u}_m, \phi)$. On the other hand, if we set $r = p$ in (76) we conclude that $|\phi(p) - \psi_m(p)| > M$, so the lemma is proved. □

To eliminate certain possible directions e'_ϕ of the second arc of ϕ, from consideration, we construct comparison inputs $\hat{v}_n$ as rotations of the input value v_n with respect to the line X. The orthogonal projection of v_n on X is given by

$$z = \psi(s) - s < e_x, e_\psi > e_x , \quad e_x = \frac{x}{|x|} . \tag{77}$$

Lemma 4.7 *Let $y \in U$ with $|y| = 1$ and $< y, x >= 0$ be given. Set*

$$\hat{v}_n = z + |v_n - z| y , \; \hat{\psi}_n = G(\hat{v}_n, \psi) , \; \hat{e}_\psi = \frac{1}{s}(\psi(s) - \hat{v}_n) . \tag{78}$$

Then, if

$$< e'_\phi - \hat{e}_\psi, \phi(p) - \hat{\psi}_n(p) > \; \ge 0 , \tag{79}$$

the pair $\{u_k\}$, $\{v_k\}$ is reducible.

Proof: Since the vectors $z - \hat{v}_n$ and $z - v_n$ have the same length and are both orthogonal to x, we have $|\psi(s) - \hat{v}_n| = |\psi(s) - v_n| = s$, so we obtain $\hat{\psi}_n$ from ψ if we replace the arc from $\psi(s)$ to v_n by an arc from $\psi(s)$ to $\hat{v}_n$. Equation (78) then implies that

$$| \xi - \hat{\psi}_n(r) | = | \xi - \psi(r) | \quad \text{for any } \xi \in X \quad \text{and any } 0 \leq r \leq s \,. \tag{80}$$

We have in particular

$$p - \delta \leq |\phi(p) - \hat{v}_n| = |\phi(p) - v_n| \leq p + \delta \,,$$

and we can therefore choose $\hat{u}_n \in U$ with $|\phi(p) - \hat{u}_n| = p$ and $|\hat{u}_n - \hat{v}_n| \leq \delta$. Setting $\hat{\phi}_n = G(\hat{u}_n, \phi)$ we see that the pair $\{u_0, \dots, u_{n-1}, \hat{u}_n\}, \{v_0, \dots, v_{n-1}, \hat{v}_n\}$ is admissible and that the corresponding pair $\{\phi_0, \dots, \phi_{n-1}, \hat{\phi}_n\}, \{\psi_0, \dots, \psi_{n-1}, \hat{\psi}_n\}$ of memory states has a total number of at most N corners. From (80) we conclude that

$$|\hat{\phi}_n(p) - \hat{\psi}_n(p)| = |\phi(p) - \psi(p)| = M \,.$$

This enables us to compute for $q = \min\{p', s\} > p$

$$\begin{aligned} |\hat{\phi}_n(q) - \hat{\psi}_n(q)|^2 - M^2 &= \int_p^q \frac{d}{dt} |\hat{\phi}_n(r) - \hat{\psi}_n(r)|^2 \, dr \\ &= 2 \int_p^q < e'_\phi - \hat{e}_\psi, \phi(r) - \hat{\psi}_n(r) > \, dr \\ &= (q-p)^2 \, |e'_\phi - \hat{e}_\psi|^2 + < e'_\phi - \hat{e}_\psi, \phi(p) - \hat{\psi}_n(p) > \; \geq \; 0 \,, \end{aligned} \tag{81}$$

where we have used the assumption (79) in the last inequality. Now $q = s$ is impossible since then (81) contradicts (67), so $q = p' < s$. We are then in the situation of figure 7 if we replace there ϕ, ψ and $\phi(p), \psi(s)$ by $\hat{\phi}_n, \hat{\psi}_n$ and $\hat{\phi}_n(p'), \hat{\psi}_n(s)$. If $\hat{\phi}_n(p') = \phi(p')$ is the first corner of $\hat{\phi}_n$, then $\hat{\phi}_n$ has less corners than ϕ and we are done. Otherwise, the maximum distance $\| \hat{\phi}_n - \hat{\psi}_n \|_\infty = \hat{\phi}_n(\hat{p}) - \hat{\psi}_n(\hat{p})$ is realized at some corner $\hat{\phi}_n(\hat{p})$ with $p' \leq \hat{p} < s$, and we conclude from Lemma 4.5 that the pair $\{u_0, \dots, u_{n-1}, \hat{u}_n\}, \{v_0, \dots, v_{n-1}, \hat{v}_n\}$ and therefore also the original pair $\{u_k\}, \{v_k\}$ is reducible. □

At this point, let us look at figure 9 for a moment. Lemma 4.3(ii) states that the angle α is greater than $\pi/2$, and lemma 4.4 means that e'_ψ cannot point towards directions in the cone indicated by β. We now show that the pair $\{u_k\}, \{v_k\}$ is reducible if e'_ϕ does not lie in the range described by γ.

Lemma 4.8 *We may assume that* $x = \phi(p) - \psi(s)$ *satisfies*

$$< x, e'_\phi > \; < \; < x, e_\psi > \,; \tag{82}$$

otherwise, the pair $\{u_k\}, \{v_k\}$ *is reducible.*

Proof: We fix any $y \in U$ with $|y| = 1$, $< y, x > = 0$ and $< y, e'_\phi > \leq 0$; moreover, if x and e'_ϕ are not parallel, we require y to be a linear combination of x and e'_ϕ. According to lemma 4.7, we may assume that $\hat{\psi}_n, \hat{e}_\psi$ as defined in (78) satisfy

$$< e'_\phi - \hat{e}_\psi, \phi(p) - \hat{\psi}_n(p) > \; < \; 0 \,. \tag{83}$$

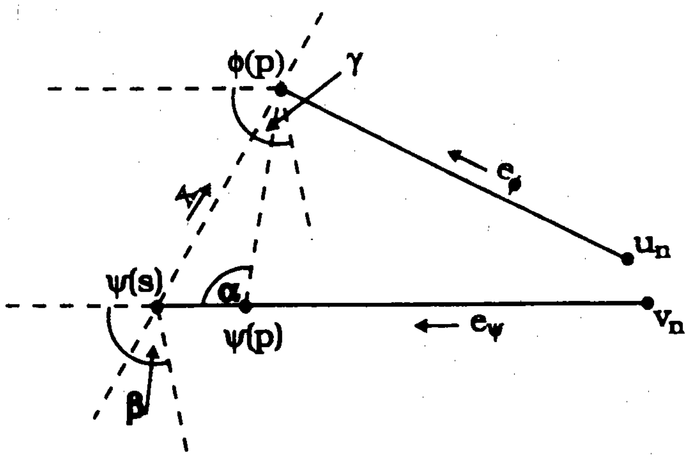

Figure 9: Restrictions on the second arcs.

The construction of (77) and (78) yields the orthogonal decomposition

$$\begin{aligned} e'_\phi &= ae_x - \sqrt{1-a^2}y, \quad a = < e_x, e'_\phi >, \\ \hat{e}_\psi &= ce_x - \sqrt{1-c^2}y, \quad c = < e_x, \hat{e}_\psi > = < e_x, e_\psi > . \end{aligned} \tag{84}$$

as well as the formula

$$\phi(p) - \hat{\psi}_n(p) = x + (s-p)\hat{e}_\psi . \tag{85}$$

With the aid of (84) and (85), the estimate (83) becomes

$$\begin{aligned} 0 &> < e'_\phi - \hat{e}_\psi, x + (s-p)\hat{e}_\psi > \\ &= a(|x| + (s-p)c) + (s-p)\sqrt{1-c^2}\sqrt{1-a^2} - (c|x| + (s-p)) \\ &= |x|\left(a(1-c^2) - c\sqrt{1-a^2}\sqrt{1-c^2}\right) - \\ &\quad - ((s-p) + c|x|)\left(1 - ac - \sqrt{1-a^2}\sqrt{1-c^2}\right) \end{aligned} \tag{86}$$

Since $s - p + c|x| < 0$ by lemma 4.3(ii) and

$$1 - ac - \sqrt{1-a^2}\sqrt{1-c^2} = \frac{1}{2}\left(\left(\sqrt{1-a^2} - \sqrt{1-c^2}\right)^2 + (a-c)^2\right) \geq 0,$$

we conclude from (86) that $a\sqrt{1-c^2} < c\sqrt{1-a^2}$ and therefore $a < c$, as the function $t \to t/\sqrt{1-t^2}$ is increasing on $(-1, 1)$. Lemma 4.8 is proved. □

To finish the proof of lemma 4.2, we show that we already have exhausted all possible cases. To this end, we assume that the pair $\{u_k\}$, $\{v_k\}$ is not reducible and derive a contradiction to (82). With m and i as defined in (75), we know that $i > m$ from lemma 4.6, so $\psi_i(s) = \psi(s)$. From (15) we see that

$$\phi_i(r) = u_i + re'_\phi \quad \text{for any } 0 \leq r \leq p' . \tag{87}$$

The following computation formally expresses that u_i is "too far out". Assumption (82) and lemma 4.3(ii) imply $s - p < - < x, e'_\phi >$, so we get from (87), (71)

$$(s+\delta)^2 \geq |\psi_i(s) - u_i|^2 = |\psi(s) - \phi(p) + \phi(p) - u_i|^2$$

$$= \quad |x|^2 + p^2 - 2p < x, e'_\varphi > \; > \; M^2 + (s-p)^2 + p^2 + 2p(s-p)$$
$$= \quad M^2 + s^2 \; > \; s^2 + 2R\delta \,. \qquad (88)$$

On the other hand, with $\varphi(R) = \psi(R) = 0$ we obtain from (71)

$$((R-s) + (R-p))^2 \geq |x|^2 \; > \; M^2 + (s-p)^2 \; > \; 2R\delta + (s-p)^2 \,.$$

Moving the term $(s-p)^2$ to the left and simplifying we get

$$2(R-s)(R-p) \; > \; R\delta \,. \qquad (89)$$

We put (88) and (89) together and obtain

$$\delta^2 \; > \; 2(R-s)\delta \; > \; \frac{R}{R-p}\delta^2 \,,$$

which is a contradiction since $p < p' \leq R$. This completes the proof of lemma 4.2.

5 Proof of the bounded variation result

This section is devoted to the proof of Theorem 2.8. Due to the memory buildup, it may happen that $\int |\partial_t \phi(t,r)| dt$ becomes large for some time period although $|u'(t)|$ is small during that time. (It does not happen if dim(U) = 1, i.e. in the memory structure of the scalar Preisach operator.) This constitutes the essential difficulty, and we have to introduce several intermediate quantities attached to the corners of the evolving memory state to overcome it.

For the whole of this section, let us fix a piecewise linear input $u : [0,T] \to U$ together with the corresponding piecewise linear memory state $\phi(t,r) = (Fu)(t,r)$, and let us fix a number $r > 0$. To estimate $\int |\partial_t \phi(t,r)| dt$, we have to study the actual movement $t \mapsto \phi(t,r)$. Let $P_i(t)$, $0 \leq i \leq N(t)$ denote the corners of $\phi(t,\cdot)$ counted from the end $P_0(t) = 0$, so $\phi(t,\cdot)$ has $N(t)+1$ corners. The other end $\phi(t,0) = u(t)$ is not counted as a corner, but we use the convention $P_{N(t)+1}(t) = u(t)$. (We have reversed the numbering of section 4 since we now have to do a forward instead of a backward analysis.) For each corner $P_i(t)$, we define its r-coordinate $r_i(t)$ and the unit vector $e_i(t)$ pointing towards it from $P_{i+1}(t)$ by the formulas

$$e_i(t) = \frac{P_i(t) - P_{i+1}(t)}{|P_i(t) - P_{i+1}(t)|} \,, \quad P_i(t) = \phi(t, r_i(t)) \,, \quad 0 \leq i \leq N(t) \,, \qquad (90)$$

Actually, $r_0(t)$ is not uniquely specified by (90), so we set

$$r_0(t) = \max_{0 \leq s \leq t} |u(s)| \,. \qquad (91)$$

The last corner $P_{N(t)}(t)$, which represents the midpoint of the largest currently active yield surface, plays a central role in our analysis. We therefore introduce the abbreviations

$$P(t) = P_{N(t)}(t) \,, \quad a(t) = r_{N(t)}(t) \,, \quad e(t) = e_{N(t)}(t) \,, \qquad (92)$$

so $a(t)$ denotes the radius of the largest currently active yield surface and $e(t)$ the inward normal common to all active yield surfaces. We obviously have

$$P(t) \;=\; u(t) \;+\; a(t)e(t)\,. \tag{93}$$

We choose a partition $0 = t_0 < t_1 < \ldots < t_K = T$ of the interval $[0, T]$ such that $u'(t)$ is constant in each subinterval (t_j, t_{j+1}). We may obviously assume that $u' \neq 0$ in each subinterval (otherwise we just drop such an interval). Passing to a suitable refinement, if necessary, we may also assume that, in each subinterval $[t_j, t_{j+1}]$, one of the following five cases occurs.

Case (E) (Enlarge) For $t \in [t_j, t_{j+1}]$ we have

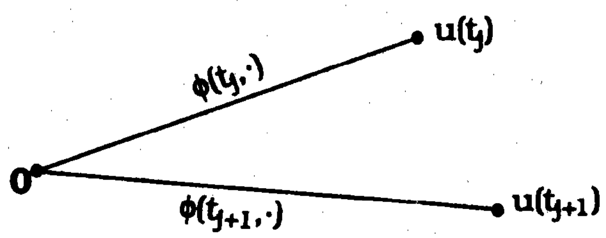

Figure 10: Enlarge

$$a(t) \;=\; r_0(t) \;=\; |u(t)|\,, \quad N(t) \;=\; 0\,, \tag{94}$$

and $a'(t)$ is a positive constant.

Case (CM) (Create and Move) Here, $r_i(t)$ and $P_i(t)$ are constant for $i \leq N(t_j)$, and

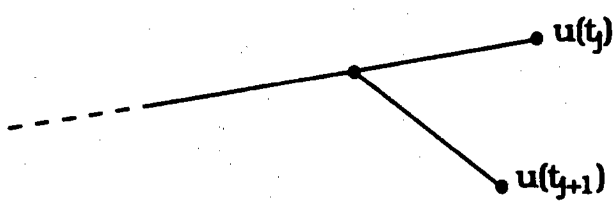

Figure 11: Create and Move

$$N(t) \;= N(t_j) \;+\; 1\,, \quad a(t) \;=\; \frac{1}{2}(t - t_j)\frac{|u'|^2}{< e(t_j), u' >}\,, \quad t \in (t_j, t_{j+1}]\,. \tag{95}$$

Since $a(t_j) > 0$, the function a has a downward jump at $t = t_j$.

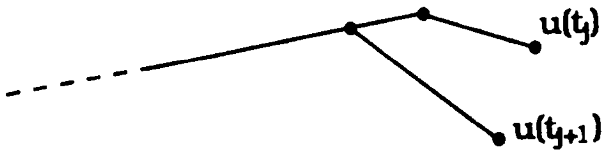

Figure 12: Move

Case (M) (Move) For $t \in [t_j, t_{j+1}]$ we have $N(t) = N(t_j) \geq 1$, and $r_i(t)$ and $P_i(t)$ are constant for $i \leq N(t_j) - 1$. The values $a(t)$ and $e(t)$ are implicitly determined by $|e(t)| = 1$ and by

$$P(t) \;=\; u(t) + a(t)e(t) \;=\; (a(t) - a(t_j))e_* \;+\; a(t_j)e(t_j) \;+\; u(t_j), \qquad (96)$$

where we have abbreviated $e_* := e_{N(t_j)-1}(t_j)$. An elementary computation involving the implicit function theorem shows that the function $a = a(t)$ is continous and strictly increasing in $[t_j, t_{j+1}]$.

Case (MM) (Move and Merge) This is the same as case (M) except for the modification $N(t_{j+1}) = N(t_j) - 1$ which takes into account the merge at $t = t_{j+1}$.

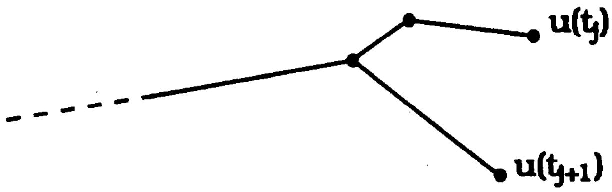

Figure 13: Move and Merge

Case (MDM) (Move and Double Merge) Here, both corners vanish in the merge. The description of case (M) remains valid for $t < t_{j+1}$, but we have $N(t_{j+1}) = N(t_j) - 2 \geq 0$ and $a(t_{j+1}) > \lim_{t \uparrow t_{j+1}} a(t)$, so a has an upward jump at $t = t_{j+1}$.

Finally, we may assume that on every partition interval (t_j, t_{j+1}), either $a(t) < r$ for all t, or $a(t) > r$ for all t.

A partition with all the properties above will be called *regular*. It is easy to see that any partition can be refined to a regular partition, and that any refinement of a regular partition is again a regular partition.

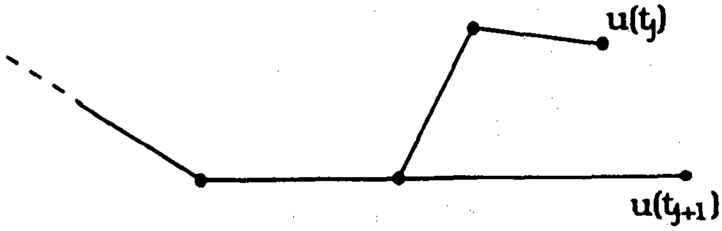

Figure 14: Move and Double Merge

Next, for any regular partition $\{t_j\}$ we want to define the *activity period* $I_i(t_j)$ of the corner $P_i(t_j)$ as the time period prior to t_j during which P_i or a corner merged into P_i has moved. This is achieved as follows. Set $I_i(t_0) = I_i(0) = \emptyset$ for any $i \geq 0$ and define recursively for $j = 1, \ldots, M$

$$\begin{aligned} I_i(t_{j+1}) &= I_i(t_j), & i &< N(t_{j+1}) \\ I_{N(t_{j+1})}(t_{j+1}) &= (t_j, t_{j+1}) \cup \bigcup_{k \geq N(t_{j+1})} I_k(t_j), & & \\ I_i(t_{j+1}) &= \emptyset, & i &> N(t_{j+1}). \end{aligned} \tag{97}$$

We see in particular that in the cases (MM) and (MDM), the activity period of the last corner swallows up the activity periods of the corners which merged into it. It is easy to see that

$$\bigcup_{i \geq 0} \overline{I_i(t_j)} = [0, t_j], \qquad 0 \leq j \leq M, \tag{98}$$

and that, for any $i \neq k$ and any j,

$$I_i(t_j) \cup I_k(t_j) = \emptyset. \tag{99}$$

Next, we denote by $V_i(t_j)$ the *input variation during the activity period* $I_i(t_j)$, namely

$$V_i(t_j) = \int_{I_i(t_j)} |u'(t)|\, dt \quad . \tag{100}$$

Because of (98) and (99), we have

$$\sum_{i \geq 0} V_i(t_j) = \int_0^{t_j} |u'(t)|\, dt, \qquad 0 \leq j \leq M. \tag{101}$$

It turns out useful to extract from the memory state $\phi(t, \cdot)$ the numbers

$$\begin{aligned} M_i(t) &= r_i(t)\, |\, e_i(t) - e_{i-1}(t)\, |, & 1 &\leq i \leq N(t), \\ M_i(t) &= 0, \quad \text{otherwise.} \end{aligned} \tag{102}$$

which represent the smallest input variation capable of producing the corner $P_i(t)$.

Finally, we want to define the contribution to the output variation related to the movement of the corner $P_i(t)$. Since $\partial_t\phi(t,r) = 0$ if $r > a(t)$ and

$$\phi(t,r) \;=\; u(t) \;+\; r\,e(t), \qquad \text{if } 0 \le r \le a(t)\,, \tag{103}$$

we are interested in the variation of $e(t)$ in the time period where $r \le a(t)$. We therefore define the index set $J_i^j(r)$ by

$$J_i^j(r) \;=\; \{\, k : (t_k, t_{k+1}) \subset I_i(t_j),\; a(t) \ge r \text{ on } (t_k, t_{k+1}) \,\}\,, \tag{104}$$

and the contribution to the output variation $d_i(t_j)$ by

$$d_i(t_j) \;=\; \sum_{k \in J_i^j(r)} |e(t_{k+1}) - e(t_k)|\,. \tag{105}$$

The following lemma, which relates the various quantities just defined, constitutes the key to the proof of Theorem 2.8.

Lemma 5.1 *For every regular partition we have*

$$M_i(t_j) \;\le\; V_i(t_j)\,, \tag{106}$$

$$J_i^j(r) \;=\; \emptyset\,, \qquad \text{if } r_i(t_j) \;\le\; r \tag{107}$$

$$d_i(t_j) \;\le\; \left(\frac{2}{r} - \frac{1}{r_i(t_j)}\right) V_i(t_j)\,, \qquad \text{if } r_i(t_j) \;>\; r \tag{108}$$

for any $j = 1, \ldots, K$ *and any* $i \ge 0$.

Proof: We use induction over j. It is easy to see that (106) - (108) hold for $j = 0$ (or $j = 1$ if $u(0) = 0$). Let us suppose now that (106) - (108) hold for some $j \ge 0$ and all $i \ge 0$. For $i > N(t_{j+1})$ we have from (97) that $M_i(t_{j+1}) = V_i(t_{j+1}) = 0$ and $J_i^{j+1}(r) = \emptyset$, hence (106) - (108) hold for $j+1$ in place of j. For $i < N(t_{j+1})$ we have $I_i(t_{j+1}) = I_i(t_j)$, $r_i(t_{j+1}) = r_i(t_j)$ and $e_i(t_{j+1}) = e_i(t_j)$, hence (106) - (108) with j replaced by $j+1$ follow from the induction hypothesis. It remains to perform the induction step for $i = N(t_{j+1})$. We consider the five cases (E), (CM), (M), (MM) and (MDM) separately.

(E) We have $i = 0$, so $M_i(t_{j+1}) = 0$ and there is nothing to prove for (106). If $r_0(t_{j+1}) \le r$, then $j \notin J_0^{j+1}(r)$ and $r_0(t_j) < r$, so $J_0^{j+1}(r) = J_0^j(r) = \emptyset$. If $r_0(t_{j+1}) > r$, then $J_0^{j+1}(r) = J_0^j(r) \cup \{j\}$. In this case we have

$$e(t_{j+1}) \;=\; \frac{u(t_{j+1})}{r_0(t_{j+1})}\,, \qquad e(t_j) \;=\; \frac{u(t_j)}{r_0(t_j)}\,,$$

hence

$$|e(t_{j+1}) - e(t_j)| \;\le\; \frac{1}{r_0(t_j)}|u(t_{j+1}) - u(t_j)|\,.$$

Formula (105) then gives

$$d_0(t_{j+1}) \;\le\; d_0(t_j) \;+\; \frac{1}{r_0(t_j)}|u(t_{j+1}) - u(t_j)|\,.$$

Together with the identity $V_0(t_{j+1}) = V_0(t_j) + |u(t_{j+1}) - u(t_j)|$ and the induction hypothesis, we obtain the induction step for (108) in both cases $r_0(t_j) = r$ and $r_0(t_j) > r$.

(CM) One immediately checks from figure 11 and the definitions that $I_i(t_{j+1}) = (t_j, t_{j+1})$, $M_i(t_j) = V_i(t_j) = |u(t_{j+1}) - u(t_j)|$ and $J_i^{j+1}(r) = \emptyset$, so (106) - (108) hold for $j+1$.

(M) Since $N(t_{j+1}) = N(t_j)$ and $a(t) = r_i(t)$ in $[t_j, t_{j+1}]$, the basic identity (97) can be rewritten for $t = t_{j+1}$ as

$$r_i(t_{j+1})(e_i(t_{j+1}) - e_{i-1}(t_{j+1})) = r_i(t_j)(e_i(t_j) - e_{i-1}(t_j)) - (u(t_{j+1}) - u(t_j)),$$

and (102) yields

$$M_i(t_{j+1}) \leq M_i(t_j) + |u(t_{j+1}) - u(t_j)|. \tag{109}$$

We have $I_i(t_{j+1}) = I_i(t_j) \cup ((t_j), (t_{j+1}))$, so

$$V_i(t_{j+1}) = V_i(t_j) + |u(t_{j+1}) - u(t_j)|, \tag{110}$$

and the induction step for (106) follows easily. In the case $r_i(t_{j+1}) \leq r$, the induction step for (107) and (108) is trivial, since $J_i^{j+1}(r) = J_i^j(r) = \emptyset$. If, on the other hand, $r_i(t_{j+1}) > r$, then $J_i^{j+1}(r) = J_i^j(r) \cup \{j\}$, hence

$$d_i(t_{j+1}) = d_i(t_j) + |e(t_{j+1}) - e(t_j)|. \tag{111}$$

Another reformulation of (97) at $t = t_{j+1}$ gives

$$r_i(t_{j+1})(e(t_{j+1}) - e(t_j)) = (r_i(t_{j+1}) - r_i(t_j))(e_{i-1}(t_j) - e_i(t_j)) - (u(t_{j+1}) - u(t_j)), \tag{112}$$

which implies

$$|e(t_{j+1}) - e(t_j)| \leq \left(\frac{1}{r_i(t_j)} - \frac{1}{r_i(t_{j+1})}\right) M_i(t_j) + \frac{1}{r_i(t_{j+1})}|u(t_{j+1}) - u(t_j)|.$$

The induction hypothesis for $r_i(t_j) > r$ (the other case is analogous) and (110) and (111) yield

$$\begin{aligned} d_i(t_{j+1}) &\leq d_i(t_j) + \left(\frac{1}{r_i(t_j)} - \frac{1}{r_i(t_{j+1})}\right) V_i(t_j) + \frac{1}{r_i(t_{j+1})}|u(t_{j+1}) - u(t_j)| \\ &\leq \left(\frac{2}{r} - \frac{1}{r_i(t_{j+1})}\right) V_i(t_{j+1}) - \left(\frac{2}{r} - \frac{2}{r_i(t_{j+1})}\right) |u(t_{j+1}) - u(t_j)|, \end{aligned}$$

which completes the induction step in the case (M).

(MM) Now we have

$$a(t_{j+1}) = r_i(t_{j+1}) = r_i(t_j), \quad e(t_j) = e_{i+1}(t_j), \quad e(t_{j+1}) = e_i(t_{j+1}).$$

The basic identity (97) becomes at $t = t_{j+1}$

$$u(t_{j+1}) - u(t_j) + r_i(t_{j+1})e_i(t_{j+1}) = (r_i(t_j) - r_{i+1}(t_j))e_i(t_j) + r_{i+1}(t_j)e_{i+1}(t_j). \tag{113}$$

Using $r_i(t_{j+1}) = r_i(t_j)$, $e_{i-1}(t_{j+1}) = e_{i-1}(t_j)$, we easily obtain from (113) that

$$M_i(t_{j+1}) \ \le\ M_i(t_j) \ +\ M_{i+1}(t_j) \ +\ |u(t_{j+1}) - u(t_j)|\,. \tag{114}$$

We have by definition that $I_i(t_{j+1}) = I_i(t_j) \cup I_{i+1}(t_j) \cup (t_j, t_{j+1})$, hence

$$V_i(t_{j+1}) \ =\ V_i(t_j) \ +\ V_{i+1}(t_j) \ +\ |u(t_{j+1}) - u(t_j)|\,, \tag{115}$$

and the induction step for (106) follows easily from (115) and the induction hypothesis. Concerning (107) and (108), the case $r_i(t_j) \le r$ is again trivial, assume now that $r_i(t_j) > r$. We have

$$\begin{aligned} J_i^{j+1}(r) &= J_i^j(r) \cup J_{i+1}^j(r) \cup \{j\} \\ d_i(t_{j+1}) &= d_i(t_j) \ +\ d_{i+1}(t_j) \ +\ |e(t_{j+1}) - e(t_j)|\,. \end{aligned} \tag{116}$$

We rewrite (113) as

$$r_i(t_{j+1})(e(t_{j+1}) - e(t_j)) \ =\ (r_i(t_j) - r_{i+1}(t_j))(e_i(t_j) - e_{i+1}(t_j)) \ -\ (u(t_{j+1}) - u(t_j))\,,$$

therefore

$$|e(t_{j+1}) - e(t_j)| \ \le\ \left(\frac{1}{r_{i+1}(t_j)} - \frac{1}{r_i(t_j)}\right) M_{i+1}(t_j) \ +\ \frac{1}{r_i(t_{j+1})}|u(t_{j+1}) - u(t_j)|\,.$$

the induction hypothesis together with (115) and (116) now yields the induction step similarly as in the case (M).

(MDM) We have obviously $M_i(t_{j+1}) = M_i(t_j)$ and

$$I_i(t_{j+1}) \ =\ (t_j, t_{j+1}) \cup \bigcup_{k=0}^{2} I_{i+k}(t_j) \tag{117}$$

hence

$$V_i(t_{j+1}) \ =\ |u(t_{j+1}) - u(t_j)| \ +\ \sum_{k=0}^{2} V_{i+k}(t_j)\,, \tag{118}$$

and the induction step for (106) follows easily. For (107) and (108), the case $r_i(t_{j+1}) \le r$ is as simple as in the previous situations (M) and (MM). If $r_{i+1}(t_j) \le r \le r_i(t_{j+1})$, then $J_i^{j+1}(r) = J_i^j(r)$, hence $d_i(t_{j+1}) = d_i(t_j)$, and the assertion follows immediately from the induction hypothesis and (118). The last case to be considered arises when $r \le r_{i+2}(t_j) < r_{i+1}(t_j)$. From (117) we get

$$\begin{aligned} J_i^{j+1}(r) &= \{j\} \cup \bigcup_{k=0}^{2} J_{i+k}^j(r)\,, \\ d_i(t_{j+1}) &= |e(t_{j+1}) - e(t_j)| \ +\ \sum_{k=0}^{2} d_{i+k}(t_j)\,. \end{aligned} \tag{119}$$

The basic vector identity (97) at $t = t_{j+1}$ becomes

$$\begin{aligned} r_{i+1}(t_j)(e(t_{j+1}) - e(t_j)) \ =\ & (r_{i+1}(t_j) - r_{i+2}(t_j))(e_{i+1}(t_j) - e_{i+2}(t_j)) \\ & - (u(t_{j+1}) - u(t_j))\,, \end{aligned}$$

therefore

$$|\epsilon(t_{j+1}) - \epsilon(t_j)| \le \left(\frac{1}{r_{i+2}(t_j)} - \frac{1}{r_{i+1}(t_j)}\right) M_{i+2}(t_j) + \frac{1}{r_{i+1}(t_j)}|u(t_{j+1}) - u(t_j)|.$$

The induction hypothesis together with (118) and (119) now completes the induction step similarly as in the cases (M) and (MM).

Lemma 5.1 is proved. □

Proof of Theorem 2.8 First, from the description of the five cases above it is easy to see that $t \mapsto \phi(t,r)$ is absolutely continous in each partition interval of a regular partition. Next, we note that Lemma 5.1 implies that

$$d_i(t_j) \le \frac{2}{r} V_i(t_j) \tag{120}$$

for any $i \ge 0$ and any $j \ge 1$. From (103) - (105) and (120) we obtain for any regular partition

$$\begin{aligned}\sum_{j=0}^{K-1} |\phi(t_{j+1},r) - \phi(t_j,r)| &= \sum_{i\ge 0} \sum_{k \in J_i^{K-1}(r)} |\phi(t_{k+1},r) - \phi(t_k,r)| \\ &\le \int_0^T |u'(t)|\,dt + r\sum_{i\ge 0} d_i(T) \le 3\int_0^T |u'(t)|\,dt.\end{aligned}$$

Since we may arbitrarily refine the partition, the assertion of Theorem 2.8 is proved.

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