TECHNISCHE UNIVERSITÄT
KAISERSLAUTERN

Fachbereich Mathematik
AG Funktionalanalysis

Convergence Rate Estimates for Degenerate Diffusions with Multiplicative Noise via (weak) Hypocoercivity Methods

Alexander Bertram

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern
zur Verleihung des akademischen Grades Doktor der Naturwissenschaften
(Doctor rerum naturalium, Dr. rer. nat.) genehmigte Dissertation

1. Gutachter: Prof. Dr. Martin Grothaus
Technische Universität Kaiserslautern

2. Gutachter: Prof. Dr. Feng-Yu Wang
Swansea University

Datum der Disputation: 29. April 2022

DE-386

Acknowledgment

As with probably every larger academic work, there are several people that had a major positive influence on the completion of this thesis, which I would therefore like to mention here.

First, I would like to thank my supervisor, Prof. Dr. Martin Grothaus, for suggesting this interesting topic to me, as well as always being available to discuss any related or semi-related questions, and pointing me towards helpful existing literature on the subject. Further, I want to express my gratitude towards Prof. Dr. Feng-Yu Wang, for agreeing to referee this dissertation.

As a nice and relaxing working environment is quite beneficial, I am grateful for the friendly atmosphere that prevails on the sixth floor of the mathematics building, induced by the past and present members of the functional analysis and financial mathematics groups, such as Dr. Torben Fattler, Claudia Korb, Andreas Nonnenmacher, Simon Wittmann, Karlo Orge, Dorothee Westphal, Konstantin Hauch, Lukas Müller, Florian Blöthner, Ricarda Rosemann, Fatlinda Avdullai, Laurena Ramadani, Bilgi Yilmaz, Magnus Wiese, Hayk Hambardzumyan, Yihua Xu, and Maximilian Diehl, among others. In particular, my thanks go to Maximilian „Torsten“ Mertin and Benedikt Eisenhuth for interesting discussions and valuable advice.

Moreover, I am indebted to the weekly "interdisciplinary project meetings" with Torsten, Dorothee, and Tobias Dietz, for allowing me to take my mind off of strictly academic topics once in a while. A similar balance was provided by the classical orchestra of the TU Kaiserslautern, as well as the SOKL, among its members I want to specifically mention Lynn, Johannes, and Thom.

In order to be perfectly honest, I also have to acknowledge the non-trivial role that coffee has played in the completion of this thesis. Moreover, I wish to thank the mathematics department of the TU Kaiserslautern and by extension the state of Rheinland-Pfalz, for providing the financial means allowing me to focus mostly on my research.

Lastly and most importantly, I am thankful for my wife Cornelia and my son Richard for always supporting me when necessary and brightening up each single day.

Contents

Introduction

In this thesis, we are concerned with the long-term behavior of solutions to stochastic differential equations on $\mathbb{R}^d$ with a variable degenerate diffusion matrix. As a starting point, consider the motion of a single particle in $\mathbb{R}^d$ and denote its position and velocity at any given time t by X_t and V_t, respectively. Then the evolution of the particle is described by the following Itô stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t &= V_t \,\mathrm{d}t \\ \mathrm{d}V_t &= -\zeta V_t \,\mathrm{d}t - \nabla\Phi(X_t)\,\mathrm{d}t + \sqrt{\frac{2\zeta}{\beta}}\,\mathrm{d}B_t, \end{aligned} \tag{0.1}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion, ζ is the friction coefficient, $-\nabla\Phi$ is an outer force field given by a potential Φ acting on the particle depending on its position, and $\beta = \frac{1}{kT}$, with T denoting the temperature of the system and k being the Boltzmann constant. This is known as the *Langevin equation*, and has been studied in this form analytically in [Con11], which serves as a reference for this thesis in terms of how to obtain solutions to the considered equations. The approach consists of applying the Itô formula to obtain the corresponding Kolmogorov backwards operator, proving that said operator generates a semigroup of contractions on an appropriate L^p-space, and then using potential theoretic tools similar to the theory of Dirichlet forms to obtain an associated stochastic process, which is shown to solve the Langevin equation.

It is easy enough to modify the above equation such that the diffusion depends on a constant positive-definite symmetric $\mathbb{R}^{d\times d}$-matrix instead of a real constant. The main change we consider here, however, is when said matrix is not constant, but instead depends on the velocity V_t. In order to be able to apply the semigroup approach as described above, an additional drift term has to be introduced. In particular, the modified equation has the form

$$\begin{aligned} \mathrm{d}X_t &= V_t \,\mathrm{d}t \\ \mathrm{d}V_t &= -\Sigma(V_t)V_t \,\mathrm{d}t - \nabla\Phi(X_t)\,\mathrm{d}t + \sum_{i,j=1}^{d} \partial_j a_{ij}(V_t)\,\mathrm{d}t + \sqrt{2}\sigma(V_t)\,\mathrm{d}B_t, \end{aligned} \tag{0.2}$$

where a_{ij} denotes the entry of Σ at position ij, and σ denotes the square root of Σ, i.e. $\Sigma = \sigma\sigma^T$. In the case of a constant Σ, we can interpret this as a stochastic perturbation of a damped Hamiltonian system: Define $H(x,v) := \frac{1}{2}|v|^2 + \Phi(x)$, then (0.2) can be written as

$$\begin{aligned} \mathrm{d}X_t &= \nabla_v H(X_t,V_t)\,\mathrm{d}t \\ \mathrm{d}V_t &= -\nabla_x \left(H(X_t,V_t)\right)\,\mathrm{d}t - \Sigma(V_t)\nabla_v H(X_t,V_t)\,\mathrm{d}t + \sqrt{2}\sigma(V_t)\,\mathrm{d}B_t. \end{aligned} \tag{0.3}$$

We can make this even more general without substantially changing the structure of the resulting operator in the following way: First, we can replace $\frac{1}{2}|v|^2$ in the definition of $H(x,v)$ by $\Psi(v)$ for some suitably differentiable $\Psi : \mathbb{R}^d \to \mathbb{R}$. Additionally, we can give up the physical interpretation as position and velocity, and instead just consider a two-component process $(X_t, Y_t)_{t\geq 0}$. Since in that case, the dimensions need not match, we let X_t be in $\mathbb{R}^{d_1}$ and Y_t be in $\mathbb{R}^{d_2}$ for possibly different d_1, $d_2 \in \mathbb{N}$. To transform between the two spaces, we introduce a constant real matrix $Q \in \mathbb{R}^{d_1 \times d_2}$ with transpose Q^*. The resulting generalized equation is of the form

$$\begin{aligned} \mathrm{d}X_t &= Q\nabla\Psi(Y_t)\,\mathrm{d}t \\ \mathrm{d}Y_t &= -\Sigma(Y_t)\nabla\Psi(Y_t)\,\mathrm{d}t - Q^*\nabla\Phi(X_t)\,\mathrm{d}t + \sum_{i,j=1}^{d_2} \partial_j a_{ij}(Y_t)\,\mathrm{d}t + \sqrt{2}\sigma(Y_t)\,\mathrm{d}B_t, \end{aligned} \tag{0.4}$$

and is the main focus of this thesis. As mentioned above, we treat it analytically by applying the Itô formula to obtain a second-order differential operator L. For sufficiently smooth functions f on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, for example $f \in C_c^\infty(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, the space of compactly supported smooth real-valued functions, Lf is given by

$$\begin{aligned} Lf(x,y) \coloneqq \operatorname{tr}[\Sigma(y)\,\mathbf{H}_y f(x,y)] - \langle \Sigma(y)\nabla\Psi(y), \nabla_y f(x,y)\rangle + \sum_{i,j=1}^{d_2} \partial_j a_{ij}(y)\partial_{y_i} f(x,y) \\ + \langle Q\nabla\Psi(y), \nabla_x f(x,y)\rangle - \langle Q^*\nabla\Phi(x), \nabla_y f(x,y)\rangle. \end{aligned} \tag{0.5}$$

Here, x denotes the first component in $\mathbb{R}^{d_1}$, y refers to the second component in $\mathbb{R}^{d_2}$, $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$, and the differential operators ∇_x, ∇_y and $\mathbf{H}_y$ are to be understood as the gradients in the first and second component, respectively, as well as the Hessian in the second component. We refer to solutions of (0.4) as *generalized Langevin dynamics*, as opposed to the special case of the second-order SDE (0.2), the solutions of which we call *Langevin dynamics*. The same naming convention will be used for the corresponding differential operators L as well.

Essential m-dissipativity

One major result of this thesis is that under relatively weak assumptions on Σ, Φ and Ψ, the operator $(L, C_c^\infty(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}))$ is essentially m-dissipative on $L^2(\mu)$, where the measure μ is defined as $\mathrm{e}^{-\Phi(x)-\Psi(y)}\,\mathrm{d}(x,y)$. This means that it is closable and its closure $(L, D(L))$ generates a strongly continuous contraction semigroup of linear operators on $L^2(\mu)$. In other words, $C_c^\infty(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ is a core for the generator $(L, D(L))$.

While often just stated as an assumption in literature, it usually requires a substantial amount of work to prove that a given subspace is indeed a generator core, even if it is known that some closed extension of the operator generates a strongly continuous semigroup. The approach we use is based on a perturbation argument, as it has been used for Langevin operators in [Con11]. First, essential self-adjointness is shown for the symmetric second-order differential operator on $\mathbb{R}^{d_2}$ induced by the first line of (0.5), which is lifted to an operator on the entire space. Then the remaining terms are added as perturbations, where the L^2-space has to be temporarily transformed in order to be able to consider each term separately as a dissipative operator. The main challenge there is that one of the terms includes a derivative along the first component, which is absent in the symmetric operator, thereby making the resulting operator non-sectorial. This is solved using a complete orthonormal family decomposing the Hilbert space into a countable family of subspaces, on which perturbation is possible. However, we require boundedness of the derivatives of both Φ and Σ for this approach to work, and relax these assumptions at the end via an involved approximation argument.

We can then reduce analytical treatment of the semigroup to considerations of the operator L on the deduced core, where one has the concrete representation from (0.5). In particular, we gain long-term convergence rate estimates for the semigroup via *hypocoercivity* methods.

Hypocoercivity

The concept of hypocoercivity was developed by Villany as a method to derive concrete rates for the convergence of degenerate dissipative operators to an equilibrium state. The systematic study of this method is collected in [Vil06] and was further developed by Dolbeault, Mouhot and Schmeiser in [DMS15] (inspired by an approach used by Hérau in [Hér05]) into the framework used in this thesis. This was further formalized including domain issues by Grothaus and Stilgenbauer in [GS14; GS16], the latter of which serves as a reference for the application of said framework to Langevin equations as considered above. The main idea is to split the generator into a symmetric and an antisymmetric part, and to introduce an orthogonal projection into the kernel of the symmetric part, which decomposes the considered Hilbert space into two subspaces. Then, under the assumption that each operator part has a spectral gap at least in the corresponding subspace, and that there is a suitably nice auxiliary operator such that the composition with any of the operators results in a bounded operator, the time derivative of an energy functional can be relatively bounded by the functional itself, resulting in convergence of the semigroup by Gronwall's Lemma. In the context of differential operators, the spectral gap condition usually requires a Poincaré inequality to hold for each measure μ_i, where $\mu = \mu_1 \otimes \mu_2$. While sufficient conditions for this are known, it still restricts the possible choices for potentials. In [GW19], Grothaus and Wang therefore developed a weaker version of the above method, which instead relies on weak Poincaré inequalities, which have been proven in [RW01] to hold under very weak assumptions. In consequence, the resulting convergence is in general no longer exponential, but concrete rates can still be computed explicitly.

The hypocoercivity method described above has been recently applied to compute convergence rate of Langevin dynamics on abstract smooth manifolds ([GM20]) and on infinite-dimensional Hilbert spaces in [EG21]. The findings in this thesis have been partially published by the author in [BG22] and [BG21]. Other approaches to derive exponential convergence include hypercontractivity as in [Wan17], where the semigroup is proven to be contractive as a mapping from L^2 to L^4, or directly using Lyapunov functions, as seen in [Wu01] or [HM17; BGH21] and the references therein. However, while for example [Wu01] theoretically allows for variable diffusion coefficients, exponential convergence for generalized Langevin dynamics with multiplicative noise as described by the equation (0.4) seems to be new independently of the method used. We emphasize again that in order to apply the hypocoercivity framework as we do here, it is absolutely necessary to have knowledge of a suitable generator core which admits a concrete representation of the operator L. So essential m-dissipativity of L is not only a separate new result, but instrumental for the estimation of the convergence rate.

The connection between the semigroup considered and solutions to the stochastic differential equation is given via generalized Dirichlet forms as developed by Stannat in [Sta99], which shows existence of a Markov process such that its transition semigroup coincides with the given contraction semigroup. This association is a special case of the more abstract result gained by Beznea, Boboc and Röckner in [BBR06] using potential theoretic methods. The resulting process is then identified as a weak solution by first proving that it solves the martingale problem corresponding to L, and then characterizing the martingales via their quadratic covariation processes. The necessary properties of the process can be inferred from the generalized Dirichlet form structure, using results by Trutnau, as seen in [Tru00] and [Tru03].

Infinite-dimensional Langevin dynamics with multiplicative noise

In the final part of this thesis, we study hypocoercivity for semigroups generated by operators similar to the definition (0.5), but for functions that are defined on a product $W = U \times V$ of two infinite-dimensional separable Hilbert spaces instead of $\mathbb{R}^d$. While the existence of an orthonormal basis allows for componentwise consideration, the first challenge is presented by the lack of a Lebesgue measure on U and V. Instead, Gaussian measures with corresponding covariance operators Q_1 and Q_2 are used as reference measures, which means that even the "potential-less" base case already includes gradient terms as an antisymmetric part of the operator. Consequently, that remains the only case we consider, as we focus instead on including suitably non-trivial variable second-order coefficients. As a motivation, we present the infinite-dimensional stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t &= K_{21}Q_2^{-1}Y_t\,\mathrm{d}t \\ \mathrm{d}Y_t &= \sum_{i=1}^{\infty} \partial_i K_{22}(Y_t)e_i - K_{22}(Y_t)Q_2^{-1}Y_t\,\mathrm{d}t - K_{12}Q_1^{-1}X_t\,\mathrm{d}t + \sqrt{2K_{22}(Y_t)}\,\mathrm{d}B_t \end{aligned} \tag{0.6}$$

for $(X_t, Y_t)_{t\geq 0}$ on W, where $K_{12} : U \to V$ is a bounded linear operator with $K_{21} = K_{12}^*$, $K_{22} : V \to \mathcal{L}(V)$ is the Fréchet-differentiable operator-valued second-order coefficient map, $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of V consisting of eigenvalues of Q_2, and $(B_t)_{t\geq 0}$ is a cylindrical Brownian motion. In a finite-dimensional setting, this would be a special case of (0.4), and so we introduce assumptions on K_{22} and K_{12} that allow us to reduce the question of essential m-dissipativity to finite-dimensional subspaces, where we can make use of our previous results.

We then prove hypocoercivity of the generated semigroup by combining the method which we use in Section 4.2 with the hypocoercivity proof from [EG21], which we can apply to the symmetric part without change. For this, we introduce further conditions to make sure that K_{22} and K_{12} are well-behaved relative to Q_2^{-1} and Q_1^{-1}. Finally, we are able to prove the existence of an associated Hunt process (X_t, Y_t) on W, for which $\sqrt{2K_{22}(Y_t)}\,\mathrm{d}B_t$ is well-defined as a stochastic integral with respect to a cylindrical Brownian motion derived from Y_t, and that provides a weak solution to the equation (0.6), at least when evaluated componentwisely. Finally, we present a concrete example that satisfies all required conditions, to show that they are feasibly verifiable and not unreasonable to assume.

Structure of the thesis

The structure is as follows: First, we introduce some necessary functional analytic and probabilistic background in Chapter 2. This includes the theory of operator semigroups and their generators, useful tools like smooth cutoff functions, as well as some condensed background on generalized Dirichlet forms and how they can be used to connect sub-Markovian semigroups with associated stochastic processes. No new results are stated, with the exception of Section 1.6, which at least does not seem to be well-known. In Chapter 3, we give a brief overview of the strong and weak hypocoercivity methods applied later, along with some comparisons and sufficient conditions to verify in practice. Again, no substantially new results are obtained, and we only include it to keep the thesis moderately self-contained. Chapter 4 sees us proving the aforementioned result on essential m-dissipativity for the operator L as defined in (0.5) on the space $L^2(\mu)$. We first provide a useful essential self-adjointness result for symmetric operators resulting from a gradient form, then we use a perturbation argument as seen in [Con11] and [Non20] to extend essential m-dissipativity progressively to the entire operator. Once the core property of smooth compactly supported functions has been shown, we can apply the two hypocoercivity methods discussed earlier to the semigroup generated by L. This happens in Chapter 5, where we impose additional assumptions on the occurring coefficients depending on the framework used, and give concrete convergence rate estimates based on those assumptions. We also discuss how this can be applied to solutions of second-order partial differential equations, as well as to weak solutions of stochastic differential equations of the form seen in (0.4). In the final Chapter, we extend the previous results to infinite-dimensional state spaces.

Notation

Although most notation used should be fairly standard, we give a brief overview here. Let $\mathbb{N} = \{1, 2, \dots\}$ denote the set of natural numbers and $\mathbb{R}$, $\mathbb{C}$ be the sets of real and complex numbers. For elements $x,\ y$ of $\mathbb{R}^d$, we write $\langle x, y\rangle$ and $|x|$ for the Euclidean inner product and norm, respectively. For a matrix $A \in \mathbb{R}^{d_1 \times d_2}$, $|A|$ denotes the Frobenius norm of A, which is consistent with the Euclidean vector norm.

Let $E \subseteq \mathbb{R}^d$ and $k \in \mathbb{N}$, then $C^k(E)$, $C_b^k(E)$ and $C_c^k(E)$ denote the spaces of k-times continuously differentiable functions from E to $\mathbb{R}$, the subset of those functions which are bounded with bounded derivatives, and the subset of those functions with compact support, respectively. For $1 \leq i, j \leq d$ and a suitably differentiable $f : E \to \mathbb{R}$, the notation $\partial_i f$ denotes the partial derivatives of f in the i-th component, $\partial_{ij} f = \partial_i \partial_j f$ and $\partial_i^2 f = \partial_{ii} f$. The symbols ∇f and $\mathbf{H} f$ are used to denote the gradient and the Hessian of f. Higher order partial derivatives are sometimes abbreviated by a multi-index notation, i.e. $\partial^\alpha f = \partial_i^{\alpha_i} \dots \partial_d^{\alpha_d} f$ for $\alpha \in \mathbb{N}^d$. For such a multi-index α, we write $|\alpha| := \sum_{i=1}^d |\alpha_i|$, $\alpha! := \prod_{i=1}^d \alpha_i!$ with corresponding binomial coefficient definition, and use the partial order $\alpha \leq \beta$ iff $\alpha_i \leq \beta_i$ for each $1 \leq i \leq d$.

If $E = E_1 \times E_2$, then we sometimes stress that fact by writing $f(x, y) := f(z)$ for each $z = (x, y) \in E$. In that case, all derivatives gain an additional index referring to the specific component, e.g. ∂_{y_i} for the partial derivative in the component y_i of (x, y), as well as ∇_x, ∇_y, $\mathbf{H}_y$, etc. If S_1, S_2 are function spaces over E_1 and E_2, then each pair $f_1 \in S_1$, $f_2 \in S_2$ admits a tensor product $f := f_1 \otimes f_2 : E \to \mathbb{R}$ defined by $f(x, y) := f_1(x) f_2(y)$. We call such an f a *pure tensor* and denote the space of all finite linear combinations of pure tensors by $S_1 \otimes S_2$.

For a topological space E, let $\mathcal{B}(E)$ denote the Borel-σ-algebra on E. If S is a function space over E, $\sigma(S)$ denotes the σ-algebra generated by pre-images of open sets in $\mathbb{R}$ under functions from S. Let $(E, \mathcal{F}, \mu)$ be a measure space, then $L^p(E; \mu)$ for $1 \leq p \leq \infty$ denotes the space of equivalence classes of $\mathcal{F} - \mathcal{B}(\mathbb{R})$-measurable functions such that $\int_E |f|^p \, \mathrm{d}\mu < \infty$ (or such that f is μ-almost everywhere bounded if $p = \infty$) under the equivalence relation of coincidence μ-a.e. These spaces are equipped with the norms $\| \cdot \|_{L^p}$ defined by $\|f\|_{L^p}^p = \int_E |f|^p \, \mathrm{d}\mu$ for $1 \leq p < \infty$ and $\|f\|_{L^\infty} = \operatorname{ess\,sup} |f|$ for $p = \infty$, where the essential supremum is to be understood with respect to μ. If $f : E \to \mathbb{R}$ is μ-integrable, then we sometimes write $\mu(f)$ for $\int_E f \, \mathrm{d}\mu$.

1 Functional analytic and probabilistic background

1.1 Linear operators on Hilbert spaces

We collect some basic facts about linear operators on Hilbert spaces that we need during the elaborations on the abstract hypocoercivity method below. While everything here is standard, we include it for the sake of completeness.

Let H be an arbitrary Hilbert space and $(T, D(T))$ be a linear operator on H.

Definition 1.1.1.

(i) Let $\widetilde{H}$ be another Hilbert space and let $L \in \mathcal{L}(H; \widetilde{H})$. Then the unique operator $L^* \in \mathcal{L}(\widetilde{H}; H)$ such that

$$(x, L^* y)_H = (Lx, y)_{\widetilde{H}} \qquad \text{for all } x \in H,\ y \in \widetilde{H}$$

is called the (Hilbert space-)*adjoint operator* to L.

(ii) Let $(T, D(T))$ be densely defined. Then the operator $(T^*, D(T^*))$ on H defined by

$$\begin{aligned} D(T^*) &\coloneqq \{y \in H \mid \exists z_y \in H \,:\, (Tx, y)_H = (x, z_y)_H \text{ for all } x \in D(T)\} \\ T^* y &\coloneqq z_y \qquad \text{for all } y \in D(T^*) \end{aligned}$$

is called the *adjoint operator* of $(T, D(T))$.

Lemma 1.1.2. *Let $(T, D(T))$ be a densely defined linear operator on a Hilbert space H and let L be a bounded linear operator with domain H.*

(i) The adjoint operator $(T^*, D(T^*))$ exists and is closed. If $D(T^*)$ is dense in H, then $(T, D(T))$ is closable and for the closure $(\overline{T}, D(\overline{T}))$ it holds $\overline{T} = T^{**}$.

(ii) L^* is bounded and $\|L^*\| = \|L\|$.

(iii) If $(T, D(T))$ is closed, then $D(T^*)$ is automatically dense in H. Consequently by (i), $T = T^{**}$.

(iv) Let $(T, D(T))$ be closed. Then the operator TL with domain

$$D(TL) = \{f \in H \mid Lf \in D(T)\}$$

is also closed.

(v) LT with domain $D(T)$ is not necessarily closed, however

$$(LT)^* = T^* L^*.$$

Proof:

(i) See [RS81, Theorem VIII.1].

(ii) Follows directly from definition: Let $x \in H$, then

$$\|L^* x\|_H^2 = (LL^* x, x)_H \leq \|L\|_{\mathcal{L}(H)} \|L^* x\|_H \|x\|_H,$$

hence $\|L^* x\|_H \leq \|L\|_{\mathcal{L}(H)} \|x\|_H$.

(iii) See again [RS81, Theorem VIII.1].

(iv) Follows directly from continuity of L and closedness of T.

(v) Note that $D(T^* L^*)$ is defined as in point (iv). Then the statement is clear since $(LTx, y)_H = (Tx, L^* y)_H$ for all $x \in D(T) = D(LT)$ and $y \in H$. □

Proposition 1.1.3. *Let $(T, D(T))$ be densely defined and either symmetric or antisymmetric. Then $(T, D(T))$ is closable.*

Proof:
Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in H such that $x_n \to 0$ and $Tx_n \to y$ for some $y \in H$. If T is symmetric, then

$$(y, z)_H = \lim_{n \to \infty} (Tx_n, z)_H = \lim_{n \to \infty} (x_n, Tz)_H = 0$$

for all $z \in D(T)$, which is dense in H. This implies $y = 0$, so T is closable. The antisymmetric case follows analogously. □

Lemma 1.1.4. *Let $(T, D(T))$ be symmetric or antisymmetric and let $P : H \to H$ be an orthogonal projection with $P(D) \subseteq D(T)$ for some dense subspace $D \subseteq D(T)$ of H. Then*

(i) $D(T) \subseteq D((TP)^*)$ with $(TP)^* = PT$ on D for symmetric T and $(TP)^* = -PT$ on $D(T)$ for antisymmetric T,

(ii) $P(TP)^* = (TP)^*$ on $D((TP)^*)$.

Proof:
See [GS14, Lemma 2.2]. □

1.2 Operator semigroups and their generators

Here we give an overview of the theory of one-parameter operator semigroups. The contents here are standard and can be found for example in [EN00, Chapters II, III] or [Paz83, Chapters 1, 3], the latter of which is to be assumed as reference unless stated otherwise.

Throughout this section, let X be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let X' denote its topological dual space.

1.2.1 Basics on operator semigroups

Definition 1.2.1. A family $(T_t)_{t\geq 0}$ in $\mathcal{L}(X)$ satisfying the conditions

(S1) $T_0 = I$,

(S2) $T_{s+t} = T_s T_t$ for $s, t \geq 0$ and

(S3) $T_t x \to x$ in X as $t \to 0$ for all $x \in X$

is called a *strongly continuous semigroup* (C_0S) of bounded linear operators on X.

If additionally $\|T_t\| \leq 1$ for all $t \geq 0$, then it is called a strongly continuous contraction semigroup (sccs).

The time-derivative at zero of such a semigroup is of great importance and defined as follows:

Definition 1.2.2. Let $(T_t)_{t\geq 0}$ be a C_0S on X. The operator

$$Lx := \lim_{t\to 0} \frac{T_t x - x}{t}, \qquad f \in D(L) \tag{1.2.1}$$

with corresponding domain

$$D(L) := \{x \in X \mid \text{the limit (1.2.1) exists in } X\}$$

is called the *generator* of $(T_t)_{t\geq 0}$, which in turn is said to be *generated* by $(L, D(L))$.

Remark 1.2.3. Let $(T_t)_{t\geq 0}$ be a C_0S with generator $(L, D(L))$. Then

(i) $T_t x \in D(L)$ for all $x \in D(L)$ and

$$\frac{\mathrm{d}T_t x}{\mathrm{d}t} = LT_t x = T_t Lx.$$

(ii) For all $x \in D(L)$ and $s, t \geq 0$, it holds that

$$T_t x - T_s x = \int_s^t T_u Lx \,\mathrm{d}u = \int_s^t LT_u x \,\mathrm{d}u.$$

(iii) $(L, D(L))$ is closed and densely defined.

(iv) Let $(S_t)_{t\geq 0}$ be a C_0S which is also generated by $(L, D(L))$. Then $T_t = S_t$ for all $t \geq 0$. In particular, C_0-semigroups are characterized by their generators.

Theorem 1.2.4. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a Hilbert space H with generator $(L, D(L))$. Then $(T_t^*)_{t\geq 0}$ is a C_0-semigroup generated by the adjoint $(L^*, D(L^*))$ of $(L, D(L))$ and satisfies $\|T_t^*\| = \|T_t\|$ for all $t \geq 0$.*

C_0-semigroups are often used to provide solutions to the following problem:

Definition 1.2.5. Given a closed operator $(L, D(L))$ on X and some $x_0 \in X$. Then the *abstract Cauchy problem* corresponding to $(L, D(L))$ with initial condition x_0 consists of finding a function $u : [0, \infty) \to X$ satisfying

$$u(0) = x_0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Lu(t) \quad \text{for all } t \geq 0. \tag{1.2.2}$$

A continuously differentiable function u is called a *classical solution*, if $u(t) \in D(L)$ for all $t > 0$ and (1.2.2) holds in the strong sense.

A continuous function u is called a *mild solution*, if

$$\int_0^t u(s)\,\mathrm{d}s \in D(L) \quad \text{and} \quad L \int_0^t u(s)\,\mathrm{d}s = u(t) - x_0.$$

Clearly every classical solution is also a mild solution. The connection to C_0-semigroups is given by the following equivalence.

Theorem 1.2.6. *Let $(L, D(L))$ be a closed operator on the Banach space X. Then the following are equivalent:*

(i) For all $x \in X$, there is a unique mild solution to the abstract Cauchy problem associated with $(L, D(L))$ with initial condition x,

(ii) The resolvent set of $(L, D(L))$ is nonempty and for all $x \in D(L)$, there is a unique classical solution to the abstract Cauchy problem associated with $(L, D(L))$ with initial condition x,

(iii) $(L, D(L))$ generates a C_0S $(T_t)_{t \geq 0}$ on X.

In that case, the solution is given by $u(t) = T_t x$.

Proof:
See [Are+01, Theorem 3.1.12]. □

Thus, it is an important question whether a given closed linear operator generates a C_0-semigroup. Since we only consider sccs in our applications later, we will focus on results for that special case, although analogous statements for general C_0-semigroups exist as well. The characterization of generators is given by the following famous theorem:

Theorem 1.2.7 (Hille-Yosida). *A linear operator $(L, D(L))$ on X is the generator of an sccs $(T_t)_{t \geq 0}$ on X if and only if*

(i) $(L, D(L))$ is closed,

(ii) $(L, D(L))$ is densely defined, i.e. $D(L)$ is dense in X, and

(iii) $(0, \infty) \subseteq \rho(L)$ and $\|\lambda(\lambda I - L)^{-1}\|_{\mathcal{L}(X)} \leq 1$ for all $\lambda > 0$.

In that case, for any $\lambda > 0$, $x \in X$, we obtain the following representation of the inverse operators:

$$(\lambda I - L)^{-1}x = \int_0^\infty \mathrm{e}^{-\lambda s} T_s x \, \mathrm{d}s.$$

Remark 1.2.8. Note that from point (iii), it already follows that $(L, D(L))$ is closed due to the closed graph theorem. So we could leave (i) out of the equivalence without changing the result.

Now we introduce a third object after the generator and the semigroup, which will later be the main connection between semigroup theory and stochastic processes associated by generalized Dirichlet forms. The following are taken from the first chapter of [MR92], which is recommended as reading material on Dirichlet forms.

Definition 1.2.9. A family $(G_\alpha)_{\alpha>0}$ of bounded linear operators on X is called a *strongly continuous contraction resolvent (sccr)*, if

(i) $\lim_{\alpha\to\infty} \alpha G_\alpha x = x$ for all $x \in H$.

(ii) $\|\alpha G_\alpha\|_{\mathcal{L}(H)} \leq 1$ for all $\alpha > 0$.

(iii) $G_\alpha - G_\beta = (\beta - \alpha) G_\alpha G_\beta$ for all $\alpha, \beta > 0$.

Lemma 1.2.10. *Let $(G_\alpha)_{\alpha>0}$ be an sccr on H, then there is exactly one linear operator $(L, D(L))$ such that $(0, \infty) \subseteq \rho(L)$ and $G_\alpha = (\alpha I - L)^{-1}$ for all $\alpha > 0$. This operator is closed, densely defined, and is called the* generator *of $(G_\alpha)_{\alpha>0}$.*

On the other hand, let $(L, D(L))$ be a densely defined operator with $(0, \infty) \subseteq \rho(L)$ and set $G_\alpha := (\alpha I - L)^{-1}$ for each $\alpha > 0$. If $\|\alpha G_\alpha\|_{\mathcal{L}(H)} \leq 1$ for all $\alpha > 0$, then $(G_\alpha)_{\alpha>0}$ is an sccr.

Proof:
See [MR92, Proposition 1.5, Proposition 1.3]. □

Immediately from this Lemma and Theorem 1.2.7, we get

Corollary 1.2.11. *A densely defined linear operator $(L, D(L))$ on H generates an sccs $(T_t)_{t\geq 0}$ on H if and only if it generates an sccr $(G_\alpha)_{\alpha>0}$ on H, and in that case it holds that $G_\alpha = \int_0^\infty \mathrm{e}^{-\alpha s} T_s \, \mathrm{d}s$.*

1.2.2 (Essential) m-dissipativity

In practice, we rarely use Hille-Yosida to prove that a concrete operator $(L, D(L))$ generates an sccs. Instead, the approach developed by Lumer and Phillips is used, which uses the notions of (maximal) dissipativity.

Definition 1.2.12. Let $x \in X$. Then its *duality set* $F(x)$ is defined by

$$F(x) := \{x' \in X' \mid x'(x) = \|x\|^2 = \|x'\|^2\} \subseteq X'.$$

Note that this set is nonempty by the Hahn-Banach theorem.

Definition 1.2.13. A linear operator $(L, D(L))$ is called *dissipative* if for each $x \in D(L)$, there is some element $x' \in F(x)$ such that $\operatorname{Re}(x'(Lx)) \leq 0$.

Remark 1.2.14. Let X be a Hilbert space with inner product $(\cdot,\cdot)_X$.

(i) A linear operator $(L, D(L))$ is dissipative if and only if $\operatorname{Re}((Lx, x)_X) \leq 0$ for all $x \in D(L)$. In particular, sums of dissipative operators with the same domain are again dissipative.

(ii) Let $(A, D(A))$ be antisymmetric. Then

$$\operatorname{Re}((Ax, x)_X) = \operatorname{Re}(-(x, Ax)_X) = -\operatorname{Re}(\overline{(Ax, x)_X}) = -\operatorname{Re}((Ax, x)_X),$$

hence $\operatorname{Re}((Ax, x)_X) = 0$ and $(A, D(A))$ is dissipative.

(iii) Let $(S, D(S))$ be symmetric and negative semi-definite. Then

$$\operatorname{Re}((Sx, x)_X) = (Sx, x)_X \leq 0$$

and therefore $(S, D(S))$ is dissipative.

A useful characterization of dissipativity is given by

Theorem 1.2.15. *A linear operator $(L, D(L))$ is dissipative iff*

$$\|(\lambda I - L)x\| \geq \lambda \|x\| \quad \text{for all } x \in D(L), \lambda > 0.$$

We collect some useful properties of dissipative operators:

Lemma 1.2.16. *Let $(L, D(L))$ be a dissipative linear operator on the Banach space X.*

(i) If $(L, D(L))$ is densely defined, then it is closable and the closure $(\overline{L}, D(\overline{L}))$ is again dissipative. Moreover, $\overline{\mathcal{R}(\lambda I - L)} = \mathcal{R}(\lambda I - \overline{L})$ for all $\lambda > 0$.

(ii) If $\mathcal{R}(\lambda_0 I - L) = X$ for some $\lambda_0 > 0$, then $(L, D(L))$ does not possess a proper dissipative extension. Moreover, $(0, \infty) \in \rho(L)$ and $\|(\lambda I - L)^{-1}\|_{\mathcal{L}(X)} \leq \lambda^{-1}$ for all $\lambda > 0$. In particular, $(L, D(L))$ is closed and $\mathcal{R}(\lambda I - L) = X$ for all $\lambda > 0$.

(iii) If X is reflexive and $\mathcal{R}(I - L) = X$, then $D(L)$ is dense in X, i.e. $(L, D(L))$ is densely defined.

This motivates the following definitions:

Definition 1.2.17. Let $(L, D(L))$ be a densely defined linear operator on X.

(i) $(L, D(L))$ is called *m-dissipative*, if it is dissipative and $\mathcal{R}(\lambda I - L) = X$ for one (hence all) $\lambda > 0$.

(ii) $(L, D(L))$ is called *essentially m-dissipative*, if it is dissipative and $\mathcal{R}(\lambda I - L)$ is dense in X for one (hence all) $\lambda > 0$.

Remark 1.2.18. The above Lemma yields the following observations:

(i) An essentially m-dissipative operator is closable and its closure is m-dissipative. This explains why that property carries over to all $\lambda > 0$.

(ii) An m-dissipative operator is dissipative and maximal with that property. As a consequence, any dissipative extension of an essentially m-dissipative operator is again essentially m-dissipative.

With these concepts defined, we can give an alternate characterization for generators of sccs:

Theorem 1.2.19 (Lumer-Phillips). *Let $(L, D(L))$ be a linear operator on the Banach space X. Then it is the generator of an sccs on X if and only if it is densely defined and m-dissipative. In that case, it follows that* $\mathrm{Re}(x'(Lx)) \leq 0$ *for all $x \in D(L)$ and* all $x' \in F(x)$.

As an immediate consequence, we gain the primary tool to find generators of sccs:

Corollary 1.2.20. *Let $(L, D(L))$ be an essentially m-dissipative operator on X, then its closure generates an sccs on X.*

1.2.3 Perturbation theory

Often enough, it is rather inconvenient to check essential m-dissipativity of a given operator by hand. Instead, one may find that the operator is in a way "close enough" to a different one, for which such a property is already known. This motivates the following perturbation arguments, which give sufficient conditions for the perturbed operator to retain essential m-dissipativity. For the sake of convenience, we only consider operators on a Hilbert space H here.

Definition 1.2.21. Let $(A, D(A))$ and $(B, D(B))$ be linear operators on H. Then B is said to be *A-bounded* if $D(A) \subseteq D(B)$ and there exist constants $a, b \in (0, \infty)$ such that

$$\|Bf\|_H \leq a\|Af\|_H + b\|f\|_H \tag{1.2.3}$$

holds for all $f \in D(A)$. The number $\inf\{a \in \mathbb{R} \mid$ (1.2.3) holds for some $b \in (0, \infty)\}$ is called the *A-bound* of B.

Theorem 1.2.22. *Let $D \subseteq H$ be a dense linear subspace. Let (A, D) be an essentially m-dissipative linear operator on H and let (B, D) be dissipative and A-bounded with A-bound strictly less than* 1. *Then $(A + B, D)$ is essentially m-dissipative and its closure is given by $(\overline{A} + \overline{B}, D(\overline{A}))$.*

Proof:
Let $(A, D(A))$ and $(B, D(B))$ denote the closures of (A, D) and (B, D), respectively. Due to (1.2.3), it follows that $D(B) \supseteq D(A)$, $(B, D(B))$ is $(A, D(A))$-bounded with relative bound strictly less than 1, and $(A, D(A))$ is m-dissipative. By [Paz83, Corollary 3.3], $(A+B, D(A))$ is m-dissipative and in particular a closed extension of $(A+B, D)$. Let $x \in D(A)$, then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in D such that $x_n \to x$, $Ax_n \to Ax$ and $Bx_n \to Bx$ in H due to (1.2.3), as $n \to \infty$. So $(A+B, D(A))$ is indeed the closure of $(A+B, D)$. □

Although we don't need it in the following, the proof of the above theorem shows that under stronger restrictions, the dense range condition of a perturbed operator can be satisfied even if the perturbation isn't dissipative itself:

Lemma 1.2.23. *Let $D \subseteq H$ be a dense linear subspace and let (A, D) be essentially m-dissipative. Assume that (B, D) is closable and A-bounded with some relative bound $a < \frac{1}{2}$ and $b \in (0, \infty)$. Then $(A+B, D)$ is closable with closure $(\overline{A}+\overline{B}, D(\overline{A}))$, which has a resolvent set which includes (λ_0, ∞), where $\lambda_0 := \frac{b}{1-2a}$. In particular, $(\lambda I - (A+B))(D)$ is dense in H for all $\lambda \in (\lambda_0, \infty)$.*

Proof:
Denote by $(A, D(A))$ and $(B, D(B))$ the closures of (A, D) and (B, D), respectively. As above, $(A, D(A))$ is m-dissipative, $D(A) \subseteq D(B)$, and (1.2.3) holds for $a < \frac{1}{2}$ and $b \in (0, \infty)$ for all $x \in D(A)$.

First, we show that $(A+B, D(A))$ is closed: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $D(A)$ such that $x_n \to x$ and $(A+B)x_n \to y$ for some $x, y \in H$ as $n \to \infty$. For $n, m \in \mathbb{N}$, we have

$$\begin{aligned}\|Ax_n - Ax_m\|_H &\le \|(A+B)(x_n - x_m)\|_H + \|B(x_n - x_m)\|_H \\ &\le \|(A+B)(x_n - x_m)\|_H + \frac{1}{2}\|Ax_n - Ax_m\|_H + b\|x_n - x_m\|_H,\end{aligned}$$

so

$$\|Ax_n - Ax_m\|_H \le 2\|(A+B)(x_n - x_m)\|_H + 2b\|x_n - x_m\|_H.$$

Since the right hand side describes a Cauchy sequence, Ax_n converges to some limit y_A, so $x \in D(A)$ by closedness of $(A, D(A))$, hence $(A+B, D(A))$ is closed. As in the proof of Theorem 1.2.22, it follows that it is the closure of $(A+B, D)$.

For any $\lambda_0 < \lambda \in \mathbb{R}$, it holds that $(\lambda I - A)(D(A)) = H$ by m-dissipativity of $(A, D(A))$, which implies

$$(\lambda I - (A+B))(D(A)) = (I - B(\lambda I - A)^{-1})(\lambda I - A)(D(A)) = (I - B(\lambda I - A)^{-1})(H).$$

For any $x \in H$, $(\lambda I - A)^{-1}x$ is in $D(A)$, so that we can use (1.2.3) to obtain

$$\begin{aligned}\|B(\lambda I - A)^{-1}x\|_H &\le a\|A(\lambda I - A)^{-1}x\|_H + b\|(\lambda I - A)^{-1}x\|_H \\ &\le a\left(\|x\|_H + \lambda\|(\lambda I - A)^{-1}x\|_H\right) + b\|(\lambda I - A)^{-1}x\|_H \\ &\le \left(2a + \frac{b}{\lambda}\right)\|x\|_H,\end{aligned}$$

where the last inequality is due to Theorem 1.2.15. Therefore, $B(\lambda I - A)^{-1}$ is a bounded operator and by definition of λ, we have $\|B(\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < 1$. This means that the inverse of $I - B(\lambda I - A)^{-1}$ exists by the Neumann series, so that

$$(\lambda I - (A + B))(D(A)) = (I - B(\lambda I - A)^{-1})(H) = H.$$

Hence $\lambda \in \rho(A + B)$ since $(A + B, D(A))$ is closed, and the claim follows. □

This result can be useful if the space changes during perturbation, so that the added operator isn't dissipative at the time of perturbation. As long as the final operator is dissipative in the final space considered, and one can transfer denseness in one space to the last one, this is sufficient to ensure essential m-dissipativity of the end result.

A useful criterion for verifying A-boundedness is given by:

Lemma 1.2.24. *Let $D \subseteq H$ be a dense linear subspace, (A, D) be an essentially m-dissipative operator, and (B, D) be dissipative on H. Assume that there exist constants $c, d \in (0, \infty)$ such that*

$$\|Bx\|_H^2 \leq c(Ax, x)_H + d\|x\|_H^2$$

holds for all $x \in D$. Then B is A-bounded with A-bound 0.

Proof:
Let $\varepsilon > 0$ be arbitrary and $x \in H$. Then by Cauchy-Bunyakovsky-Schwarz and the Young inequality,

$$(Ax, x)_H \leq \|Ax\|_H\|x\|_H = (\varepsilon\|Ax\|_H)(\tfrac{1}{\varepsilon}\|x\|_H) \leq \frac{\varepsilon^2}{2}\|Ax\|_H^2 + \frac{1}{2\varepsilon^2}\|x\|_H^2,$$

which implies

$$\|Bx\|_H^2 \leq \frac{c\varepsilon^2}{2}\|Ax\|_H^2 + \left(d + \frac{c}{2\varepsilon^2}\right)\|x\|_H^2 \leq \left(\varepsilon\sqrt{\frac{c}{2}}\|Ax\|_H + \sqrt{d + \frac{c}{2\varepsilon^2}}\|x\|_H\right)^2.$$

Taking the square root of both sides proves the claim, since ε can be chosen arbitrarily small. □

We also require the following generalization of the perturbation method:

Lemma 1.2.25. *Let $D \subseteq H$ be a dense linear subspace, (A, D) be an essentially m-dissipative operator, and (B, D) be dissipative on H. Assume that there exists a complete orthogonal family $(P_n)_{n\in\mathbb{N}}$, i.e. each P_n is an orthogonal projection, $P_nP_m = 0$ for all $n \neq m$ and $\sum_{n\in\mathbb{N}} P_n = I$ strongly, such that*

$$P_n(D) \subseteq D, \qquad P_nA = AP_n, \quad \text{and} \quad P_nB = BP_n$$

for all $n \in \mathbb{N}$. Set $A_n := AP_n$, $B_n := BP_n$, both with domain $D_n := P_n(D)$, as operators on $P_n(H)$. Assume that each B_n is A_n-bounded with A_n-bound strictly less than 1. Then $(A+B, D)$ is essentially m-dissipative.

Proof:
See [CG08, Lemma 3] □

Finally, we include the complexified setting, since one of the perturbation steps later on will require complex operators instead of real ones. Luckily, essential m-dissipativity carries over in both directions, as seen in the following.

Definition 1.2.26. Let $(L, D(L))$ be a linear operator on the real Hilbert space H. Then the *complexification* $H_{\mathbb{C}}$ of H is defined by

$$H_{\mathbb{C}} \coloneqq \{[x, y] \mid x, y \in H\} = H \times H,$$

and is equipped with the following operations:

$$\begin{aligned} [x, y] + [v, w] &\coloneqq [x + v, y + w], \\ (a + \mathrm{i}b)[x, y] &\coloneqq [ax - by, ay + bx], \\ ([x, y], [v, w])_{H_{\mathbb{C}}} &\coloneqq (x, v)_H + (y, w)_H - \mathrm{i}(x, w)_H + \mathrm{i}(y, v)_H \end{aligned}$$

for all $a, b \in \mathbb{R}$, $x, y, v, w \in H$. Then $H_{\mathbb{C}}$ is a complex Hilbert space with norm $\|[x, y]\|_{H_{\mathbb{C}}}^2 = \|x\|_H^2 + \|y\|_H^2$. The complexification $L_{\mathbb{C}}$ of L is given by

$$L_{\mathbb{C}}[x, y] \coloneqq [Lx, Ly] \qquad \text{for all } [x, y] \in D(L_{\mathbb{C}}) \coloneqq D(L) \times D(L).$$

By definition, we quickly see the following:

Lemma 1.2.27. *Let $(L, D(L))$ be a linear operator on the real Hilbert space H.*

(i) $(L, D(L))$ is dissipative iff $(L_{\mathbb{C}}, D(L_{\mathbb{C}}))$ is dissipative.

(ii) $(L, D(L))$ is essentially m-dissipative iff $(L_{\mathbb{C}}, D(L_{\mathbb{C}}))$ is dissipative.

(iii) $(L, D(L))$ is closable if and only if $(L_{\mathbb{C}}, D(L_{\mathbb{C}}))$ is closable, in which case $((\overline{L})_{\mathbb{C}}, D((\overline{L})_{\mathbb{C}})) = (\overline{L_{\mathbb{C}}}, D(\overline{L_{\mathbb{C}}}))$.

In practice, this means that we can switch to a complexified setting if needed within a proof for essential m-dissipativity. In particular, we can allow perturbation of a real essentially m-dissipative operator by a complex dissipative operator which is relatively bounded by the complexification of the first one. At the end of such a perturbation process, we can reduce the final operator back to the real setting, as long as it is a complexification of a real operator.

1.3 Sub-Markovian semigroups and generalized Dirichlet forms

1.3.1 Semigroups on L^2-spaces

Throughout this section, let $(E, \mathcal{F}, \mu)$ be a probability space. We consider C_0-semigroups and their generators on the space $L^2(E;\mu)$. Most statements here hold more generally for L^p-spaces of σ-finite measures, for which we refer to [Con11, Chapter 1].

Definition 1.3.1. Let (L, D) be a linear operator on $L^2(E;\mu)$. If

$$\int_E Lf \,\mathrm{d}\mu \le 0 \quad \text{or} \quad \int_E Lf \,\mathrm{d}\mu = 0 \quad \text{for all } f \in D,$$

then μ is said to be a *subinvariant* or *invariant* measure for (L, D), respectively.

Definition 1.3.2. Let $(T_t)_{t\ge 0}$ be a C_0-semigroup on $L^2(E;\mu)$.

(i) If

$$\int_E T_t f \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu \quad \text{for all } f \in L^2(E;\mu)$$

holds for all $t \ge 0$, then μ is said to be invariant for $(T_t)_{t\ge 0}$.

(ii) If $T_t 1 = 1$ holds for all $t \ge 0$, then $(T_t)_{t\ge 0}$ is said to be *conservative*.

The connection between these concepts is given by

Lemma 1.3.3.

(i) A C_0-semigroup $(T_t)_{t\ge 0}$ on $L^2(E;\mu)$ is conservative if and only if the adjoint semigroup $(T_t^*)_{t\ge 0}$ is μ-invariant.

(ii) A C_0-semigroup $(T_t)_{t\ge 0}$ with generator $(L, D(L))$ is conservative iff $1 \in D(L)$ with $L1 = 0$.

(iii) A C_0-semigroup $(T_t)_{t\ge 0}$ with generator $(L, D(L))$ is μ-invariant if $(L, \mathcal{C})$ is μ-invariant for a core $\mathcal{C}$.

Proof:

(i) Let μ be invariant for $(T_t^*)_{t\ge 0}$ and let $t \ge 0$. Then

$$\int_E f \,\mathrm{d}\mu = \int_E T_t^* f \,\mathrm{d}\mu = \int_E f T_t 1 \,\mathrm{d}\mu$$

for all $f \in L^2(E;\mu)$, which implies $T_t 1 = 1$. The other direction follows similarly.

(ii) This is clear by Remark 1.2.3 (ii).

(iii) Let μ be invariant for $(L, \mathcal{C})$, then it is also invariant for $(L, D(L))$. Fix some $t \geq 0$. Using Remark 1.2.3 (i) and difference quotients, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_E T_t f \,\mathrm{d}\mu = \int_E L T_t f \,\mathrm{d}\mu = 0$$

for all $f \in D(L)$, hence

$$\int_E T_t f \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu \quad \text{for all } f \in D(L).$$

By Remark 1.2.3 (iii), $D(L)$ is dense in $L^2(E;\mu)$ and we obtain invariance of μ for $(T_t)_{t\geq 0}$. The other direction is clear. □

Next we introduce some concepts which are important in the context of associated stochastic processes. For this, we assume $(T_t)_{t\geq 0}$ to be an sccs.

Definition 1.3.4.

(i) A linear operator $A \in \mathcal{L}(L^2(E;\mu))$ is called *sub-Markovian*, if $0 \leq Af \leq 1$ for all $f \in L^2(E;\mu)$ with $0 \leq f \leq 1$.

(ii) A contraction semigroup $(T_t)_{t\geq 0}$ is called *sub-Markovian*, if T_t is sub-Markovian for each $t \geq 0$.

(iii) A contraction resolvent $(G_\alpha)_{\alpha>0}$ is called *sub-Markovian*, if αG_α is sub-Markovian for all $\alpha > 0$.

(iv) A closed densely defined linear operator $(L, D(L))$ is said to be a *Dirichlet operator*, if

$$\int_E Lf((f-1)^+)\,\mathrm{d}\mu \leq 0 \quad \text{for all } f \in D(L).$$

The following Lemma is due to [MR92, p. I.4.3]:

Lemma 1.3.5. *Let $(L, D(L))$ be the generator of an sccs $(T_t)_{t\geq 0}$ and an sccr $(G_\alpha)_{\alpha>0}$. Then the following are equivalent:*

(i) $(G_\alpha)_{\alpha>0}$ is sub-Markovian.

(ii) $(T_t)_{t\geq 0}$ is sub-Markovian.

(iii) $(L, D(L))$ is a Dirichlet operator.

For sub-Markovian sccs, we can show conservativity more directly:

Lemma 1.3.6.

(i) If $(T_t)_{t\geq 0}$ is sub-Markovian, then so is $(T_t^*)_{t\geq 0}$.

(ii) *Let $(T_t)_{t\geq 0}$ be a sub-Markovian sccs. Then conservativity and μ-invariance are equivalent.*

Proof:

(i) Since $\|T_t f\|_{L^1(E;\mu)} \leq \|T_t f\|_{L^2(E;\mu)}$ for all $f \in L^2(E;\mu)$, the semigroup $(T_t)_{t\geq 0}$ is also L^1-contractive. Then for $0 \leq f \leq 1$ and all $0 \leq g \in L^2(E;\mu)$, we get

$$0 \leq \int_E T_t^* f g \,\mathrm{d}\mu = \int_E f T_t g \,\mathrm{d}\mu \leq \|f\|_\infty \|T_t g\|_{L^1(E;\mu)} \leq \int_E g \,\mathrm{d}\mu.$$

This implies $0 \leq T_t^* f \leq 1$, so $(T_t^*)_{t\geq 0}$ is sub-Markovian.

(ii) Due to Lemma 1.3.3 (i), we only have to show that conservativity of $(T_t)_{t\geq 0}$ and $(T_t^*)_{t\geq 0}$ are equivalent. Let therefore $T_t 1 = 1$ for all $t \geq 0$. Then

$$\int_E (1 - T_t^* 1) \,\mathrm{d}\mu = \int_E 1 \,\mathrm{d}\mu - \int_E 1 T_t 1 \,\mathrm{d}\mu = \int_E (1 - T_t 1) \,\mathrm{d}\mu = 0,$$

so $T_t^* 1 = 1$ μ-a.e. since $(1 - T_t^* 1) \geq 0$ due to part (i). □

Now we introduce an interesting class of operators that are more easily verifiable.

Definition 1.3.7. Let (L, D) be a densely defined linear operator on $L^2(E;\mu)$. The *Carré du champ operator* of L is the bilinear operator $\Gamma : D \times D \to L(E;\mu)$ given by

$$\Gamma(u, v) := \tfrac{1}{2}\left(L(uv) - uL(v) - vL(u)\right) \qquad \text{for } u, v \in D.$$

Here $L(E;\mu)$ refers to the space of all μ-classes of functions on E.

(L, D) is called an *abstract diffusion operator*, iff it satisfies the following:

(i) For any $m \in \mathbb{N}$, $u_1, \dots, u_m \in D$ and $\varphi \in C^\infty(\mathbb{R}^m)$ with $\varphi(0) = 0$, it holds that $\varphi(u_1, \dots, u_m) \in D$ and

$$L\varphi(u_1, \dots, u_m) = \sum_{k=1}^m \frac{\partial \varphi}{\partial x_k}(u_1, \dots, u_m) L(u_k) + \sum_{k,\ell=1}^m \frac{\partial^2 \varphi}{\partial x_\ell \partial x_k}(u_1, \dots, u_m) \Gamma(u_k, u_\ell).$$

(ii) $\Gamma(u, u) \geq 0$ for all $u \in D$.

We fix some $\alpha \geq 0$ and assume the following:

Assumption (SI). For all $f \in D$ with $f \geq 0$ μ-a.e., both f and Lf are in $L^1(E;\mu)$ and μ is sub-invariant for $(L - \alpha I, D)$, i.e.

$$\int_E Lf \,\mathrm{d}\mu \leq \alpha \int_E f \,\mathrm{d}\mu.$$

Then we obtain the following result to find Dirichlet operators, which can be found in [Ebe99, Lemma 1.9].

Lemma 1.3.8. *Let (SI) hold and assume that the closure $(L, D(L))$ of the abstract diffusion operator (L, D) generates an sccs $(T_t)_{t\geq 0}$. Then $(T_t)_{t\geq 0}$ is sub-Markovian, so in particular $(L, D(L))$ is a Dirichlet operator.*

1.3.2 Generalized Dirichlet forms and associated Markov processes

Throughout this section, let E be a topological Hausdorff space such that its Borel-σ-algebra $\sigma(E)$ is generated by the continuous real-valued functions on E. Let further μ be a probability measure on $(E, \mathcal{B}(E))$ such that $H := L^2(E;\mu)$ is a real separable Hilbert space.

Definition 1.3.9. Let $(\mathcal{A}, D(\mathcal{A}))$ be a densely defined positive semidefinite bilinear form, i.e. $D(\mathcal{A}) \subseteq H$ is a dense linear subspace and $\mathcal{A}$ is a bilinear map with $\mathcal{A}(u,u) \geq 0$ for all $u \in D(\mathcal{A})$. The *symmetric part* $(\tilde{\mathcal{A}}, D(\mathcal{A}))$ of $(\mathcal{A}, D(\mathcal{A}))$ is given by

$$\tilde{\mathcal{A}}(u,v) := \frac{1}{2}(\mathcal{A}(u,v) + \mathcal{A}(v,u)),$$

and the *antisymmetric part* $(\check{\mathcal{A}}, D(\mathcal{A}))$ is given by

$$\check{\mathcal{A}}(u,v) := \frac{1}{2}(\mathcal{A}(u,v) - \mathcal{A}(v,u)).$$

Further set $(\mathcal{A}_\alpha, D(\mathcal{A}))$ for $\alpha > 0$ as

$$\mathcal{A}_\alpha(u,v) := \mathcal{A}(u,v) + \alpha(u,v),$$

and define $(\tilde{\mathcal{A}}_\alpha, D(\mathcal{A}))$ analogously.

$(\tilde{\mathcal{A}}, D(\mathcal{A}))$ is called a *symmetric closed form*, if $(D(\mathcal{A}), \tilde{\mathcal{A}}_1)$ is a Hilbert space. $(\mathcal{A}, D(\mathcal{A}))$ is called a *coercive closed form*, if $(\tilde{\mathcal{A}}, D(\mathcal{A}))$ is a symmetric closed form and the following *weak sector condition* holds:

Assumption (WSC). There exists a so-called *continuity constant* $K > 0$ such that

$$|\mathcal{A}_1(u,v)| \leq K \mathcal{A}_1(u,u)^{\frac{1}{2}} \mathcal{A}_1(v,v)^{\frac{1}{2}}$$

for all $u, v \in D(\mathcal{A})$.

Note that a symmetric closed form satisfies (WSC) with $K = 1$ due to the Cauchy-Bunyakovsky-Schwarz inequality. Since $\mathcal{A}(u,u) = \tilde{\mathcal{A}}(u,u)$ for all $u \in D(\mathcal{A})$, the weak sector condition implies that $\mathcal{A}$ is a continuous bilinear form on the Hilbert space $(D(\mathcal{A}), \tilde{\mathcal{A}}_1)$, which we will naturally consider when referring to the Hilbert space $D(\mathcal{A})$.

Definition 1.3.10. Let $(\mathcal{A}, D(\mathcal{A}))$ be a coercive closed form. Define the linear operator $(L, D(L))$ via

$$D(L) := \{u \in D(\mathcal{A}) \mid \exists\, Lu \in H \,:\, (-Lu, v)_H = \mathcal{A}(u,v) \text{ for all } v \in D(\mathcal{A})\}.$$

Then $(L, D(L))$ is negative semidefinite, closed and generates an sccs $(T_t)_{t\geq 0}$ as well as an sccr $(G_\alpha)_{\alpha>0}$ on H (see [MR92, Section I.2]). It is called the *generator* of $(\mathcal{A}, D(\mathcal{A}))$, and all four objects $(L, D(L))$, $(T_t)_{t\geq 0}$, $(G_\alpha)_{\alpha>0}$ and $(\mathcal{A}, D(\mathcal{A}))$ are associated with each other.

Definition 1.3.11. A coercive closed form $(\mathcal{A}, D(\mathcal{A}))$ on H is called a *semi-Dirichlet form*, if

$$u \in D(\mathcal{A}) \implies u^+ \wedge 1 \in D(\mathcal{A}) \text{ and } \mathcal{A}(u + (u^+ \wedge 1), u - (u^+ \wedge 1)) \geq 0. \tag{1.3.1}$$

If additionally

$$\mathcal{A}(u - (u^+ \wedge 1), u + (u^+ \wedge 1)) \geq 0 \quad \text{for all } u \in D(\mathcal{A}), \tag{1.3.2}$$

then $(\mathcal{A}, D(\mathcal{A}))$ is called a *Dirichlet form*.

Theorem 1.3.12. *Let $(\mathcal{A}, D(\mathcal{A}))$ be a coercive closed form on H with associated infinitesimal generator $(L, D(L))$, sccs $(T_t)_{t\geq 0}$ and sccr $(G_\alpha)_{\alpha>0}$. Then the following are equivalent:*

(i) $(\mathcal{A}, D(\mathcal{A}))$ *is a semi-Dirichlet form.*

(ii) $(G_\alpha)_{\alpha>0}$ *is sub-Markovian.*

(iii) $(T_t)_{t\geq 0}$ *is sub-Markovian.*

(iv) $(L, D(L))$ *is a Dirichlet operator.*

Proof:
See [MR92, Theorem I.4.4] together with Lemma 1.3.8. □

The theory of Dirichlet forms provides an associated Markov process, so that important properties of the process can be verified analytically for the Dirichlet form. This correspondence between probabilistic and analytic concepts has proven very successful, and has been summarized in the book [MR92]. However, it is not suitable for our applications, since we do not satisfy the weak sector condition, due to the degenerate nature of the symmetric part of the bilinear form associated with our Dirichlet operator. Instead, we turn to the theory of generalized Dirichlet forms, as developed and summarized in [Sta99]. Since the complete definition of such forms would take too much space, we only consider the special case that we use later in applications, and refer to the mentioned book for intermediary definitions and proofs.

Theorem 1.3.13. *Let $(L, D(L))$ be a Dirichlet operator that generates an sccs $(T_t)_{t\geq 0}$ on H and set $\mathcal{A} = 0$ with $D(\mathcal{A}) = H$. Then the bilinear form $\mathcal{E}$ associated with $(\mathcal{A}, D(\mathcal{A}))$ and $(\Lambda, D(\Lambda))$ as described in [Sta99, Definition 2.9] is a generalized Dirichlet form which is given by*

$$\mathcal{E}(u, v) = \begin{cases} -(Lu, v) & \text{if } u \in D(L),\ v \in H \\ -(u, L^* v) & \text{if } u \in H,\ v \in D(L^*) \end{cases}.$$

Moreover, the C_0-resolvents $(G_\alpha)_{\alpha>0}$ and $(G^*_\alpha)_{\alpha>0}$ of contractions on H associated to the bilinear form $\mathcal{E}$ as defined in [Sta99, Section I.3] correspond exactly to the resolvents generated by $(L, D(L))$ and $(L^*, D(L^*))$, respectively.

Proof:
See [Sta99, Section I.4, Example 4.9 (ii)]. □

Now we introduce a few more definitions related to generalized Dirichlet forms, in order to formulate the main theorem that yields an associated Markov process. In particular, we define what exactly we refer to as a Markov process within this thesis. We fix some Dirichlet operator $(L, D(L))$ generating an sccs $(T_t)_{t\geq 0}$ and an sccr $(G_\alpha)_{\alpha>0}$ on H and the associated generalized Dirichlet form $\mathcal{E}$ as obtained from Theorem 1.3.13.

Definition 1.3.14. Let $\alpha > 0$. An element $u \in H$ is called *α-excessive* if $\beta G_{\beta+\alpha} u \leq u$ for all $\beta \geq 0$. The set of all α-excessive elements is denoted by $\mathcal{P}_\alpha$.

Definition 1.3.15. For an element $h \in H$ let $\mathcal{L}_h := \{v \in H : v \geq h\}$ and $\beta_h : H \to H$ be defined by $\beta_h(v) := (v-h)^-$.

For $h \in H$ and $\alpha > 0$, let h_α be the unique element of $D(L)$ such that

$$\mathcal{E}_1(h_\alpha, v) = \alpha(\beta_h(h_\alpha), v)_H \quad \text{for all } v \in H.$$

Let h satisfy $\mathcal{L}_h \cap D(L) \neq \emptyset$. An element $e_h \in \mathcal{L}_h \cap \mathcal{P}_1$ is called a *1-reduced function* of h, if

(i) $\lim_{\alpha\to\infty} h_\alpha = e_h$ in H,

(ii) $e_h \leq u$ for all $u \in \mathcal{L}_h \cap \mathcal{P}_1$,

(iii) $\mathcal{E}_1(e_h, v) \geq \|e_h\|_H^2$ for all $v \in \mathcal{L}_h \cap D(L^*)$.

Remark 1.3.16. The existence and uniqueness of h_α follows from [Sta99, Prop. III.1.6], and existence of 1-reduced functions is shown in [Sta99, Proposition III.1.7].

Definition 1.3.17. For an open subset $U \subseteq E$ and an element $f \in H$ which satisfies $\mathcal{L}_{\mathbb{1}_U f} \cap D(L) \neq \emptyset$, we define $f_U := e_{\mathbb{1}_U f}$ as the 1-reduced function of $\mathbb{1}_U f$.

Definition 1.3.18.

(i) An increasing sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of E is called an *$\mathcal{E}$-nest*, if for every element $u \in D(L) \cap \mathcal{P}_1$, it holds that $\lim_{n\to\infty} u_{E\setminus F_n} = 0$ in H.

(ii) An $\mathcal{E}$-nest $(F_n)_{n\in\mathbb{N}}$ is called *regular*, if for all $n \in \mathbb{N}$, $U \subseteq E$ open, it holds that $\mu(U \cap F_n) = 0$ implies $U \subseteq E \setminus F_n$.

(iii) A subset $N \subseteq E$ is called *$\mathcal{E}$-exceptional*, if there is an $\mathcal{E}$-nest $(F_n)_{n\in\mathbb{N}}$ such that $N \subseteq \bigcap_{n\in\mathbb{N}} E \setminus F_n$.

(iv) A property which is satisfied for all points outside some $\mathcal{E}$-exceptional set is said to hold *$\mathcal{E}$-quasi everywhere*.

(v) A function f defined outside of some $\mathcal{E}$-exceptional set $N \subseteq E$ is called *$\mathcal{E}$-quasi-continuous*, if there is some $\mathcal{E}$-nest $(F_n)_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} F_n \subseteq E \setminus N$ and $f|_{F_n}$ is continuous for all $n \in \mathbb{N}$.

(vi) A sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{E}$-quasi-everywhere defined real-valued functions is said to converge *$\mathcal{E}$-quasi-uniformly* to some $\mathcal{E}$-quasi-everywhere defined real-valued function f, if there is some $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that f_k, f are defined on $\bigcup_{k\in\mathbb{N}} F_k$ and f_k converges uniformly to f on each F_k.

Proposition 1.3.19. *Let $0 < \varphi \in H$ be arbitrary and $h := G_1\varphi$. Then the map $\operatorname{Cap}_\varphi$ from the subsets of E to $\mathbb{R}$ defined by*

$$\begin{aligned} \operatorname{Cap}_\varphi(U) &:= (h_U, \varphi)_H && \text{for open } U \subseteq E, \\ \operatorname{Cap}_\varphi(A) &:= \inf_{A\subseteq U \text{open}} \operatorname{Cap}_\varphi(U) && \text{for arbitrary } A \subseteq E, \end{aligned}$$

is a Choquet capacity, i.e.

(i) $\operatorname{Cap}_\varphi(\bigcup_{n\in\mathbb{N}} A_n) = \sup_{n\in\mathbb{N}} \operatorname{Cap}_\varphi(A_n)$ *for any increasing sequence of arbitrary subsets A_n of E.*

(ii) $\operatorname{Cap}_\varphi(\bigcap_{n\in\mathbb{N}} K_n) = \inf_{n\in\mathbb{N}} \operatorname{Cap}_\varphi(K_n)$ *for any decreasing sequence of compact subsets K_n of E.*

(iii) $\operatorname{Cap}_\varphi(\bigcup_{n\in\mathbb{N}} A_n) \le \sum_{n\in\mathbb{N}} \operatorname{Cap}_\varphi(A_n)$ *for any sequence of arbitrary subsets A_n of E.*

Moreover, it characterizes $\mathcal{E}$-nests in the sense that any increasing sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of E is an $\mathcal{E}$-nest if and only if $\lim_{n\to\infty} \operatorname{Cap}_\varphi(F_n^c) = 0$.

Proof:
This is shown in [Sta99, Propositions 2.8, 2.10]. □

Lemma 1.3.20. *Let $(N_k)_{k\in\mathbb{N}}$ be a sequence of $\mathcal{E}$-exceptional sets. Then $N := \bigcup_{k\in\mathbb{N}} N_k$ is also $\mathcal{E}$-exceptional.*

Proof:
For each N_k, there is some $\mathcal{E}$-nest $(F_n^k)_{n\in\mathbb{N}}$ such that $N_k \subseteq E \setminus F_n^k$ for all $n \in \mathbb{N}$. Due to Proposition 1.3.19, for each $n, k \in \mathbb{N}$, there is some $m(n,k) \in \mathbb{N}$ such that $\operatorname{Cap}_\varphi((F_{m(n,k)}^k)^c) \le \frac{1}{n2^k}$. Now set $F_n^\infty := \bigcap_{k\in\mathbb{N}} F_{m(n,k)}^k$ for each $n \in \mathbb{N}$. Clearly the F_n^∞ are closed and form an increasing sequence. Since

$$\operatorname{Cap}_\varphi((F_n^\infty)^c) = \operatorname{Cap}_\varphi(\bigcup_{k\in\mathbb{N}} (F_{m(n,k)}^k)^c) \le \sum_{k\in\mathbb{N}} \operatorname{Cap}_\varphi((F_{m(n,k)}^k)^c) = \frac{1}{n},$$

the sequence $(F_n^\infty)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-nest due to Proposition 1.3.19. Since clearly $N \subseteq E \setminus F_n^\infty$ for all $n \in \mathbb{N}$ by construction, this shows that N is $\mathcal{E}$-exceptional. □

The following is due to [Sta99, Corollary 3.8]:

Lemma 1.3.21. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $D(L)$ such that each u_n has an $\mathcal{E}$-quasi-continuous μ-version $\tilde{u}_n$. Assume that $u_n \to u$ in $D(L)$ for some $u \in D(L)$. Then u has an $\mathcal{E}$-quasi-continuous μ-version $\tilde{u}$ and there is a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $\tilde{u}_{n_k} \to \tilde{u}$ $\mathcal{E}$-quasi-uniformly.*

Instead of verifying the definition of $\mathcal{E}$-nests by hand, we rely on the following useful criterion, which is due to [Sta99, Remark III.2.11]:

Proposition 1.3.22. *Let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of closed sets such that*

$$D := \{u \in D(L) \mid u = 0 \ \mu\text{-a.e. on } E \setminus F_n \text{ for some } n \in \mathbb{N}\}$$

is dense in $D(L)$, i.e. with respect to the graph norm induced by $(L, D(L))$. Then $(F_n)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-nest.

Definition 1.3.23. The generalized Dirichlet form $\mathcal{E}$ is called *quasi-regular*, if

(i) There exists an $\mathcal{E}$-nest consisting of compact sets.

(ii) There is a dense subset of $D \subseteq D(L)$ such that each $f \in D$ has an $\mathcal{E}$-quasi-continuous μ-version.

(iii) There is a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(L)$ such that for each $n \in \mathbb{N}$, u_n has a $\mathcal{E}$-quasi-continuous μ-version $\tilde{u}_n$, and that $\{\tilde{u}_n : n \in \mathbb{N}\}$ separates the points of $E \setminus N$ for some $\mathcal{E}$-exceptional set $N \subseteq E$.

Proposition 1.3.24. *Let $\mathcal{E}$ be quasi-regular. Then each $f \in D(L)$ has an $\mathcal{E}$-quasi-continuous μ-version.*

Proof:
This follows directly from Lemma 1.3.21 and (ii) in the definition of quasi-regularity. □

Now we have defined all the necessary concepts to state the theorem giving the association to a Markov process. First, we specify how such a process is to be understood in this context.

Definition 1.3.25. The tuple $\mathbf{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ is called a time homogeneous *Markov process* with state space E, life time ζ and corresponding filtration $(\mathcal{M}_t)_{t\geq 0}$ if

(M1) $X_t : \Omega \to E_\Delta$ is $\mathcal{M}_t - \mathcal{B}(E_\Delta)$-measurable for all $t \geq 0$ and $X_t(\omega) = \Delta$ if and only if $t \geq \zeta(\omega)$ for all $\omega \in \Omega$.

(M2) For all $t \geq 0$ there is a map $\theta_t : \Omega \to \Omega$ such that $X_s \circ \theta_t = X_{s+t}$ for all $s \geq 0$.

(M3) $(P_x)_{x\in E_\Delta}, \xi)$ is a family of probability measures on $(\Omega, \mathcal{M})$ such that $x \mapsto P_x(B)$ is $\mathcal{B}(E_\Delta)^*$-measurable for all $B \in \mathcal{F}$ and $\mathcal{B}(E_\Delta)$-measurable for all $B \in \sigma(X_t : t \geq 0)$. Furthermore, it holds that $P_\Delta(X_0 = \Delta) = 1$.

(M4) For all $A \in \mathcal{B}(E_\Delta)$, $s, t \geq 0$, and $x \in E_\Delta$, it holds that

$$P_x(X_{t+s} \in A \mid \mathcal{M}_t) = P_{X_t}(X_s \in A) \qquad P_x\text{-a.s.}$$

Here $\mathcal{B}(E_\Delta)^* := \bigcap_{\nu\in\mathcal{P}(E_\Delta)} \mathcal{B}(E_\delta)^\nu$ is the σ-algebra of universally measurable sets, where $\mathcal{P}(E_\Delta)$ denotes the set of all probability measures on E_Δ and $\mathcal{B}(E_\delta)^\nu$ is the ν-completion of $\mathcal{B}(E_\delta)$.

For a measure μ on $(E_\Delta, \mathcal{B}(E_\Delta))$, we define the measure P_μ via $P_\mu(A) := \int_{E_\Delta} P_x(A)\,\mu(\mathrm{d}x)$ for all $A \in \mathcal{F}$.

Definition 1.3.26. A Markov process $\mathbf{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ with life time ζ is called a *right process* if

(M5) $P_x(X_0 = x) = 1$ for all $x \in E_\Delta$.

(M6) The map $t \mapsto X_t(\omega)$ is right-continuous on $[0, \infty)$ for all $\omega \in \Omega$.

(M7) The filtration $(\mathcal{M}_t)_{t\geq 0}$ is right-continuous. Moreover, for any probability measure μ on $(E_\Delta, \mathcal{B}(E_\delta))$ and any $(\mathcal{M}_t)_{t\geq 0}$-stopping time τ, it holds that

$$P_\mu(X_{\tau+s} \in A \mid \mathcal{M}_\tau) = P_{X_\tau}(X_s \in A) \qquad P_\mu\text{-a.s.}$$

for all $A \in \mathcal{B}(E_\Delta)$ and $s \geq 0$.

As in [Con11, Definition 2.2.3], we only consider right processes with

$$\mathcal{M}_t = \mathcal{F}_t := \bigcup_{\nu\in\mathcal{P}(E_\Delta)} (\mathcal{F}_t^0)^{P_\nu|_{\mathcal{F}_\infty^0}}$$

and

$$\mathcal{M} = \mathcal{F} := \bigcap_{\nu\in\mathcal{P}(E_\Delta)} (\mathcal{F}_\infty^0)^{P_\nu|_{\mathcal{F}_\infty^0}}$$

from now on, where $\mathcal{F}_t^0 := \sigma(X_s \,:\, s \in [0,t])$, since that change of filtration retains all properties of an existing right process.

Definition 1.3.27. For a subset $A \in \mathcal{B}(E_\Delta)$ of E, let $\sigma_A := \inf\{t > 0 \,:\, X_t \in A\}$ be the first hitting time with respect to $\mathbf{M}$.

Definition 1.3.28. Let $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ be a right process with life time ζ and let μ be a finite measure on $(E_\Delta, \mathcal{B}(E_\Delta))$.

(i) $\mathbf{M}$ is called *μ-tight*, if there is some increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact metrizable sets in E such that

$$P_\mu\left(\lim_{n\to\infty} \sigma_{E\setminus K_n} < \zeta\right) = 0.$$

(ii) $\mathbf{M}$ is called *μ-special standard*, if

(M8) $X_{t-} := \lim_{s\uparrow t} X_s$ exists in E for all $t \in (0,\zeta)$ P_μ-a.s.

(M9) $\lim_{n\to\infty} X_{\tau_n} = X_\tau$ P_μ-a.s. on $\{\tau < \zeta\}$ and X_τ is $\bigvee_{n\in\mathbb{N}} \mathcal{F}_{\tau_n}^{P_\mu}$-measurable for every increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of $(\mathcal{F}_t^{P_\mu})_{t\geq 0}$-stopping times with limit τ.

(iii) $\mathbf{M}$ is called *special standard*, if it is μ-special standard for all probability measures μ on $(E_\Delta, \mathcal{B}(E_\Delta))$.

(iv) $\mathbf{M}$ is called a *Hunt process*, if (M8) and (M9) hold with ζ replaced by ∞ and E replaced by E_Δ.

Definition 1.3.29. Let $\mathbf{M}$ be a right process and define the families $(p_t)_{t\geq 0}$ and $(R_\alpha)_{\alpha>0}$ via

$$p_t f(x) := \mathrm{E}_x[f(X_t)] \quad \text{ and } R_\alpha f(x) := \int_0^\infty \mathrm{e}^{-\alpha t} p_t f(x)\,\mathrm{d}t$$

for all $x \in E$ and $f \in \mathcal{B}_b(E)$, where E_x denotes the expectation with respect to P_x, and $\mathcal{B}_b(E)$ denotes the set of bounded Borel-measurable real-valued functions on E. Then $(p_t)_{t\geq 0}$ and $(R_\alpha)_{\alpha>0}$ are called the *transition semigroup* and *resolvent* of $\mathbf{M}$, respectively.

$\mathbf{M}$ is called *associated with* $\mathcal{E}$, if $R_\alpha f$ is a μ-version of $G_\alpha f$ for all $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap H$. $\mathbf{M}$ is called *properly associated with* $\mathcal{E}$ *in the resolvent sense*, if additionally $R_\alpha f$ is $\mathcal{E}$-quasi-continuous for all $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap H$.

Remark 1.3.30. For a right process $\mathbf{M}$, the transition semigroup is a sub-Markovian semigroup of kernels on $(E, \mathcal{B}(E))$ (cf [MR92, Section II.4]) and the resolvent is a sub-Markovian resolvent of kernels. If $\mathbf{M}$ is properly associated with $\mathcal{E}$ in the resolvent sense, then $R_\alpha f$ is a $\mathcal{E}$-q.c. μ-version of $G_\alpha f$ for all $\alpha > 0$ and even all $f \in H$. Moreover, it holds that $p_t f$ is a μ-version of $T_t f$ for all $f \in H$. This resembles [MR92, Exercises IV.2.9, IV.2.7] and can be seen for example in [CG05, Lemma 3.38]. Note that as seen in [Sta99, Example IV.1.5], we do not obtain $\mathcal{E}$-quasi-continuity of $p_t f$ in general.

Finally, we state the theorem for existence of associated Markov processes:

Theorem 1.3.31. *Let $\mathcal{E}$ be a quasi-regular generalized Dirichlet form associated with $\mathcal{A} = 0$ and a Dirichlet operator $(L, D(L))$ generating an sccs $(T_t)_{t\geq 0}$ on H. Let further $\mathcal{Y} \subseteq D(L) \cap L^\infty(E;\mu)$ be a core for $(L, D(L))$ which is an algebra, i.e. closed under multiplication. Then there exists a μ-tight special standard process $\mathbf{M}$ which is properly associated in the resolvent sense with $\mathcal{E}$.*

Proof:
This is a special case of [Sta99, Theorem 2.2]. □

Lemma 1.3.32. *Let $\mathcal{E}$ be properly associated in the resolvent sense with a μ-tight special standard process $\mathbf{M}$. If for every open subset $U \subseteq E$, there exists an increasing sequence $(u_n)_{n\in\mathbb{N}}$ of non-negative continuous elements of $D(L)$ such that*

(i) $\operatorname{supp}(u_n) \subseteq U$ *for all* $n \in \mathbb{N}$, $\sup_{n\in\mathbb{N}} u_n \leq \mathbb{1}_U$ *and* $\sup_{n\in\mathbb{N}} u_n > 0$ *on* U.

(ii) $Lu_n = 0$ *μ-a.e. on* $E \setminus \operatorname{supp}(u_n)$ *for all* $n \in \mathbb{N}$.

Then it holds that

$$P_x\left(t \mapsto X_t \text{ is continuous on } [0,\zeta)\right) = 1$$

for $\mathcal{E}$-quasi all $x \in E$.

Proof:
This is a special case of [Tru03, Theorem 3.3]. □

Remark 1.3.33. Let $\mathcal{E}$ be quasi-regular, properly associated to some μ-tight special standard process $\mathbf{M}$, and let a property ρ hold for $(X_t)_{t\geq 0}$ P_x-a.s. for $\mathcal{E}$-quasi all $x \in E$, i.e. for all $x \in E \setminus N$ with N being $\mathcal{E}$-exceptional. Due to the definition of exceptional sets, there is some exceptional $N_2 \in \mathcal{B}(E)$ such that $N_2 \supseteq N$. Then, by restriction and trivial extension as in [MR92, Chapter IV.3], we can obtain a modified μ-tight special standard process $\mathbf{M}'$ which is also properly associated in the resolvent sense with $\mathcal{E}$ such that ρ holds for $(X_t')_{t\geq 0}$ P_x'-a.s. for all $x \in E$.

1.3.3 Sub-Markovian semigroups and associated probability measures on path spaces

In this section, we consider probability measures $\mathbb{P}$ on the space $D([0,\infty);E_\Delta)$ of càdlàg paths on the state space E. Again, there are no new results here, as we mainly wish to collate useful results from [Con11] in the special case of our L^2-setting with respect to a probability measure μ. We refer to Section 2.1 of the named source for more general and detailed results on general L^p-spaces, as well as to [EK86] for more details on path spaces. We summarize the setting and some notation in the following:

Definition 1.3.34. Let E be a Polish space such that $\mathcal{B}(E)$ is generated by the continuous real-valued functions on E, and let E_Δ be E with the cemetery Δ adjoined, where every function on E is extended trivially to E_Δ. Let μ be a probability measure on $(E, \mathcal{B}(E))$. By $D([0,\infty);E_\Delta)$, we denote the space of càdlàg paths on E_Δ, i.e. the set of all maps $[0,\infty) \ni t \mapsto Z_t \in E_\Delta$ which are right-continuous and have finite left limits for each $t > 0$. Let $D([0,\infty);E_\Delta)$ be equipped with the Skorokhod topology (see [EK86, Chapter 3.5]), so that it is a Polish space, and denote the corresponding Borel-σ-algebra by $\mathcal{B}_D$. Then the space of continuous paths $C([0,\infty),E_\Delta)$ is a closed and hence measurable subset of $D([0,\infty);E_\Delta)$, with the induced topology corresponding to uniform convergence on compact sets (see [EK86, Problem 3.11.25]), and corresponding Borel-σ-algebra denoted by $\mathcal{B}_C$. Moreover, the sets $D([0,\infty);E)$ and $C([0,\infty);E)$ are measurable subsets of $D([0,\infty);E_\Delta)$ and $C([0,\infty);E_\Delta)$, respectively.

We call a probability measure $\mathbb{P}$ on $(D([0,\infty);E_\Delta), \mathcal{B}_D)$ or another path space a *probability law*, if the set $\mathcal{Z}$ of *Zombie paths* is a subset of a $\mathbb{P}$-null set, i.e. $\mathbb{P}$-a.s. $Z_t = \Delta$ for any $t \geq 0$ implies $Z_s = \Delta$ for all $s \geq 0$. In particular, this is the case if $\mathbb{P}$ is the image measure of a probability measure P on $(\Omega, \mathcal{F})$ under a measurable mapping ϕ such that $\phi(\Omega) \cap \mathcal{Z} = \emptyset$, see [Con11, Remark 2.1.2]. Here and later within this section, $(Z_s)_{s\geq 0}$ refers to a generic path from the path space considered and Z_t refers to the projection $\pi_t((Z_s)_{s\geq 0})$ of the path to its state at time $t \geq 0$. In particular, we use this notation for the expectation $\mathbb{E}$, such that $\mathbb{E}[f(Z_t)]$ refers to

$$\int_{D([0,\infty);E_\Delta)} f(\pi_t((Z_s)_{s\geq 0}))\, \mathbb{P}(\mathrm{d}(Z_s)_{s\geq 0}).$$

For a probability law $\mathbb{P}$, we define its *initial distribution* to be the image measure of $\mathbb{P}$ under the random variable Z_0. The *life-time* $\zeta : D([0,\infty); E_\Delta) \to [0,\infty]$ is defined by $\zeta := \inf\{t \geq 0 : Z_t = \Delta\}$.

Definition 1.3.35. Let $\mathbb{P}$ be a probability law on $D([0,\infty); E_\Delta)$ with initial distribution $h\mu$, where $0 \leq h \in L^2(E;\mu)$ is a probability density with respect to μ. Let $(T_t)_{t\geq 0}$ be a sub-Markovian sccs on $L^2(E;\mu)$, then it is said to be *associated with* $\mathbb{P}$ if for all non-negative $f_1, \dots, f_k \in L^\infty(E;\mu)$, $0 \leq t_1, < \cdots < t_k < \infty$, $k \in \mathbb{N}$, it holds that

$$\mathbb{E}\left[\prod_{1\leq i\leq k} f_i(Z_{t_i})\right] = \big(h, T_{t_1}(f_1 T_{t_2-t_1}(f_2 \dots T_{t_{k-1}-t_{k-2}}(f_{k-1} T_{t_k-t_{k-1}} f_k)))\big)_H \tag{1.3.3}$$

In order to make this concept relevant for consideration, we state the following Lemma which connects it to Markov processes associated with semigroups as in the previous section.

Lemma 1.3.36. *Let* $\mathbf{M}$ *be a μ-special standard process associated with a sub-Markovian sccs* $(T_t)_{t\geq 0}$ *as in Remark 1.3.30, and let $h \in L^2(E;\mu)$ be a non-negative probability density with respect to μ. Define the measure $P_{h\mu}$ on $(\Omega, \mathcal{F})$ as in Definition 1.3.25 and let $\mathbb{P}_{h\mu}$ be the image measure of $P_{h\mu}$ under the map $\Omega : \omega \mapsto (X_t(\omega))_{t\geq 0} \in D([0,\infty); E_\Delta)$. Then $\mathbb{P}_{h\mu}$ is associated with $(T_t)_{t\geq 0}$ as in Definition 1.3.35.*

Proof:
Consider $\Omega_D := \{\omega \in \Omega : (X_t(\omega))_{t\geq 0} \in D([0,\infty); E_\Delta)$ and the map

$$\Omega_D : \omega \mapsto (X_t(\omega))_{t\geq 0} \in D([0,\infty); E_\Delta),$$

which we denote by $(\tilde{X}_t)_{t\geq 0}$ after extending it to Ω via

$$(\tilde{X}_t)_{t\geq 0}(\omega) := (x_0)_{t\geq 0} \quad \text{for some } x_0 \in E_\Delta.$$

Due to property (M8), the Ω_D is the complement of a P_μ-null set and therefore $(\tilde{X}_t)_{t\geq 0}$ is $\mathcal{F}^{P_{h\mu}}-\mathcal{B}_D$-measurable, since $\mathcal{B}_D$ is generated by the projection maps $\pi_s : (Z_t)_{t\geq 0} \mapsto Z_s$. Therefore, we may define $\mathbb{P}_{h\mu}$ as the image measure of $P_{h\mu}$ under $(\tilde{X}_t)_{t\geq 0}$. Now let $k \in \mathbb{N}$, $f_1, \dots, f_k \in L^\infty(E;\mu)$ be non-negative and $0 \leq t_1 < \dots t_k < \infty$. Then for any $x \in E$, it holds that

$$\mathrm{E}_x\left[\prod_{1\leq i\leq k} f_i(X_{t_i})\right] = p_{t_1}(f_1 p_{t_2-t_1}(f_2 \dots p_{t_k-t_{k-1}} f_k))(x). \tag{1.3.4}$$

This can be seen since due to the tower property of conditional expectation together with the Markov property (M4),

$$\begin{aligned}\mathrm{E}_x[f_1(X_{t_1})f_2(X_{t_2})] &= \mathrm{E}_x\left[\mathrm{E}_x[f_1(X_{t_1})f_2(X_{t_2})|\mathcal{F}_{t_1}]\right] = \mathrm{E}_x\left[f_1(X_{t_1})\,\mathrm{E}_x[f_2(X_{t_2})|\mathcal{F}_{t_1}]\right]\\ &= \mathrm{E}_x\left[f_1(X_{t_1})\,\mathrm{E}_{X_{t_1}}[f_2(X_{t_2-t_1})]\right] = \mathrm{E}_x[f_1(X_{t_1})p_{t_2-t_1}f_2(X_{t_1})]\\ &= p_{t_1}(f_1 p_{t_2-t_1}f_2)(x).\end{aligned}$$

Since the right hand side of (1.3.4) is a μ-version of $T_{t_1}(f_1 T_{t_2-t_1}(f_2 \dots T_{t_k-t_{k-1}} f_k))$, integration with respect to $h\mu$ yields (1.3.3), since

$$\mathbb{E}_{h\mu}\left[\prod_{1\le i\le k} f_i(Z_{t_i})\right] = \mathrm{E}_{h\mu}\left[\prod_{1\le i\le k} f_i(\tilde{X}_{t_i})\right] = \mathrm{E}_{h\mu}\left[\prod_{1\le i\le k} f_i(X_{t_i})\right]$$

□

Definition 1.3.37. Let $\mathbb{P}$ be a probability law on $D([0,\infty);E_\Delta)$ and let (L,D) be a linear operator on $L^2(E;\mu)$. Then $\mathbb{P}$ is said to solve the *Martingale problem* for (L,D), if

(i) For every $f\in D$ and $t\ge 0$ it holds $\int_0^t |Lf(Z_s)|\,\mathrm{d}s<\infty$ $\mathbb{P}$-a.s. and the random variables $f(Z_t)$ and $\int_0^t Lf(Z_s)\,\mathrm{d}s$ on $D([0,\infty);E_\Delta)$ are $\mathbb{P}$-a.s. well-defined, i.e. independent of the chosen μ-version of f and Lf.

(ii) For all $f\in D$ and $t\ge 0$, the random variable $M_t^{[f],L}$ defined by

$$M_t^{[f],L} := f(Z_t) - f(Z_0) - \int_0^t Lf(Z_s)\,\mathrm{d}s$$

is $\mathbb{P}$-integrable and the corresponding process $(M_t^{[f],L})_{t\ge0}$ is an $(\mathcal{F}_t^0)_{t\ge0}$-martingale.

Here $\mathcal{F}_t^0 := \sigma\{Z_s : s\in[0,t]\}$ and can be replaced by $\mathcal{F}_{t,+}^0 := \bigcap_{s>t}\mathcal{F}_s^0$ and its $\mathbb{P}$-completion when f is continuous, see [Con11, Remark 2.1.7].

Lemma 1.3.38. *Let $\mathbb{P}$ be a probability law on $D([0,\infty);E_\Delta)$ with initial distribution $h\mu$ for some non-negative probability density $h\in L^2(E;\mu)$. Assume that $\mathbb{P}$ is associated to some sub-Markovian sccs $(T_t)_{t\ge0}$ with generator $(L,D(L))$. Then $\mathbb{P}$ solves the martingale problem for $(L,D(L))$. Moreover, if $f\in D(L)$ with $f^2\in D(L)$ and $Lf\in L^4(E;\mu)$, then*

$$N_t^{[f],L} := (M_t^{[f],L})^2 - \int_0^t L(f^2)(Z_s) - (2fLf)(Z_s)\,\mathrm{d}s,\quad t\ge0,$$

also defines an $(\mathcal{F}_t^0)_{t\ge0}$-martingale.

Proof:
See [Con11, Lemma 2.1.8].

□

Lemma 1.3.39. *Let $(L, \mathcal{C})$ be an abstract diffusion operator on H as defined in Definition 1.3.7. Assume further that $\mathcal{C}$ consists of bounded continuous functions, and that $(L, \mathcal{C})$ has an extension which generates a sub-Markovian sccs $(T_t)_{t\geq 0}$ for which μ is invariant. Let $\mathbb{P}$ be a probability law on $D([0,\infty); E_\Delta)$ associated with $(T_t)_{t\geq 0}$ with initial distribution μ, then $(f(Z_t))_{t\geq 0}$ is $\mathbb{P}$-a.s. continuous for every $f \in \mathcal{C}$.*

If there is a countable subset $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ that separates the points of E, then $(Z_t)_{t\geq 0}$ is $\mathbb{P}$-a.s. continuous on $[0, \zeta)$, and if E is locally compact, it holds that $\mathbb{P}(C([0,\infty); E_\Delta)) = 1$.

Proof:
This is a special case of [Con11, Lemma 2.1.10, Corollary 2.1.11]. □

We now state some more properties of probability laws on path spaces which can be verified by their associated sub-Markovian semigroups. As in the source [Con11], we only consider continuous paths from now on.

Definition 1.3.40. Let $\mathbb{P}$ be a probability law on $C([0,\infty), E_\Delta)$ with initial distribution μ.

(i) $\mathbb{P}$ is said to be *conservative*, if $\zeta = \infty$ holds $\mathbb{P}$-a.s.

(ii) $\mathbb{P}$ is said to have *invariant measure* μ if $\mathbb{P} \circ Z_t^{-1} = \mu$ for all $t \geq 0$.

Lemma 1.3.41. *Let $\mathbb{P}$ be a probability law on $C([0,\infty), E_\Delta)$ with initial distribution μ, which is associated with a sub-Markovian sccs $(T_t)_{t\geq 0}$ on $L^2(E;\mu)$. Then*

(i) $\mathbb{P}$ is conservative if and only if $(T_t)_{t\geq 0}$ is conservative.

(ii) μ is invariant for $\mathbb{P}$ if and only if μ is invariant for $(T_t)_{t\geq 0}$.

Definition 1.3.42. Let $\mathbb{P}$ be a probability law on $C([0,\infty), E_\Delta)$ with invariant measure μ. For any $A \in \mathcal{B}_C$ and $t \geq 0$, define

$$\varphi_t A := \{(Z_s)_{s\geq 0} \mid (Z_{t+s})_{s\geq 0} \in A\}.$$

(i) $\mathbb{P}$ is said to be *ergodic*, if for $A_1, A_2 \in \mathcal{B}_C$, it holds that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{P}(\varphi_t A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2).$$

(ii) $\mathbb{P}$ is said to be *weakly mixing*, if there is some set $I \subseteq [0,\infty)$ with relative measure 1, that is

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}_I(t)\,\mathrm{d}t = 1,$$

such that for any A_1, A_2 in $\mathcal{B}_C$, it holds that

$$\mathbb{P}(\varphi_t A_1 \cap A_2) \to \mathbb{P}(A_1)\mathbb{P}(A_2) \quad \text{as } t\to\infty,\ t \in I.$$

$\mathbb{P}$ is said to be *strongly mixing*, if $I = [0,\infty)$.

Clearly, strong mixing implies weak mixing, which in turn implies ergodicity.

Lemma 1.3.43. *Let $\mathbb{P}$ be a probability law on $C([0,\infty), E_\Delta)$ with invariant measure μ, which is associated with a sub-Markovian sccs $(T_t)_{t\geq 0}$ on $L^2(E;\mu)$. Then $\mathbb{P}$ is weakly mixing if and only if for all $f, g \in L^\infty(E;\mu)$, there is some set $I \subseteq [0,\infty)$ of relative measure 1 such that*

$$(g, T_t f)_{L^2(E;\mu)} \to \mu(f)\mu(g) \quad \text{as } t \to \infty,\ t \in I.$$

Proof:
See [Con11, Remark 2.1.13], where the the statement is proved with $f, g \in L^\infty(E;\mu)$ replaced by $f, g \in L^2(E;\mu)$. Since in our case, all bounded functions are in $L^2(E;\mu)$ and the proof only uses indicator functions and T_t applied to indicator functions, which are all bounded due to the sub-Markov property of $(T_t)_{t\geq 0}$, our formulation is valid. □

1.4 Mollifiers and cutoffs

Definition 1.4.1. Let $\varphi \in L^1(\mathbb{R}^d)$ and $f \in L^p(\mathbb{R}^d)$ with $p \in [1,\infty]$. Then the *convolution* of φ with f is defined as

$$(\varphi * f)(x) := \int_{\mathbb{R}^d} \varphi(x-y) f(y)\,\mathrm{d}y \int_{\mathbb{R}^d} \varphi(y) f(x-y)\,\mathrm{d}y$$

and is an element of $L^p(\mathbb{R}^d)$ with

$$\|\varphi * f\|_{L^p} \leq \|\varphi\|_{L^1} \|f\|_{L^p}.$$

Lemma 1.4.2. *Let the same assumptions hold as above.*

(i) Let $\varphi \in C_c^k(\mathbb{R}^d)$ for some $k \in \mathbb{N}$, then $\varphi * f \in C^k(\mathbb{R}^d)$ with $\partial^s(\varphi * f) = (\partial^s \varphi) * f$ for all $|s| \leq k$.

(ii) It holds that

$$\operatorname{supp}(\varphi * f) \subseteq \overline{\{x + y \mid x \in \operatorname{supp}(\varphi),\ y \in \operatorname{supp}(f)\}},$$

so in particular $\varphi * f \in C_c^\infty(\mathbb{R}^d)$ if $\varphi \in C_c^\infty(\mathbb{R}^d)$ and f has compact support.

This also works in a way for weak derivatives, which we prove here since finding a reference proved difficult:

Lemma 1.4.3. *Let $\varphi \in L^1(\mathbb{R}^d)$, $1 \leq p \leq \infty$ and $m \in \mathbb{N}$. Then, for each $f \in H^{m,p}(\mathbb{R}^d)$, the convolution $\varphi * f$ is in $H^{m,p}(\mathbb{R}^d)$ as well, and it holds that $\partial^s(\varphi * f) = \varphi * \partial^s f$ for all $|s| \leq m$, where ∂^s denotes the weak derivative in both cases.*

If $\varphi \in C^k(\mathbb{R}^d)$, then the weak derivative $\partial^s(\varphi * f)$ coincides with the strong derivative $\partial^s(\varphi * f) = (\partial^s \varphi) * f$ for all $|s| \leq \min\{m, k\}$.

Proof:
First, let $p \in [1, p)$ and $f \in H^{m,p}(\mathbb{R}^d)$. For each $|s| \leq m$, the convolutions $\varphi * \partial^s f$ are well-defined as elements of $L^p(\mathbb{R}^d)$. We can approximate by a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ with respect to the $H^{m,p}$-norm. In particular, each $f_n \in L^1(\mathbb{R}^d)$, so that $\varphi * f_n = f_n * \varphi$. Then the previous Lemma yields

$$\partial^s(f_n * \varphi) = (\partial^s f_n) * \varphi = \varphi * \partial^s f_n.$$

Since

$$\|\varphi * \partial^s f - \varphi * \partial^s f_n\|_{L^p} = \|\varphi * (\partial^s f - \partial^s f_n)\|_{L^p} \leq \|\varphi\|_{L^1}\|\partial^s f - \partial^s f_n\|_{L^p} \to 0$$

as $n \to \infty$, it follows that $\partial^s(f_n * \varphi)$ converges to $\varphi * \partial^s f$ in $L^p(\mathbb{R}^d)$ for each $|s| \leq m$, which implies $\varphi * f \in H^{m,p}(\mathbb{R}^d)$ with $\partial^s(\varphi * f) = \varphi * \partial^s f$ for all $|s| \leq m$.

Now consider $p = \infty$ and let $f \in H^{m,\infty}(\mathbb{R}^d)$ for some $m \in \mathbb{N}$. Let further $g \in C_c^\infty(\mathbb{R}^d)$. Then application of Fubini-Tonelli and the definition of weak differentiability yields

$$\begin{aligned}
\int_{\mathbb{R}^d} \partial^s g(x)(\varphi * f)(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} \partial^s g(x) \int_{\mathbb{R}^d} \varphi(x-y)f(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \varphi(y) \int_{\mathbb{R}^d} \partial^s g(x) f(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&= (-1)^{|s|} \int_{\mathbb{R}^d} \varphi(y) \int_{\mathbb{R}^d} g(x)\partial^s f(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&= (-1)^{|s|} \int_{\mathbb{R}^d} g(x)(\varphi * \partial^s f)(x)\,\mathrm{d}x
\end{aligned}$$

for all $|s| \leq m$, so indeed $\varphi * f \in H^{m,\infty}(\mathbb{R}^d)$ with $\partial^s(\varphi * f) = \varphi * \partial^s f$.

The last statement is immediate, since strong derivatives always coincide with weak ones if they exist. □

There is one special class of functions for which convolutions are useful:

Definition 1.4.4. A sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $L^1(\mathbb{R}^d)$ is said to be an *approximate identity* or *Dirac sequence*, if $\varphi_n \geq 0$ and $\|\varphi_n\|_{L^1} = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^d \setminus B_r(0)} \varphi_n(x)\,\mathrm{d}x = 0 \quad \text{for all } r > 0.$$

Remark 1.4.5. Define $\widetilde{\varphi} \in C_c^\infty(\mathbb{R}^d)$ via

$$\widetilde{\varphi}(x) := \begin{cases} \exp\left(-\frac{1}{1-(2|x|)^2}\right) & \text{for } |x| < \frac{1}{2}, \\ 0 & \text{else,} \end{cases}$$

and set $\varphi := \|\widetilde{\varphi}\|_{L^1}^{-1}\widetilde{\varphi}$. For each $\varepsilon > 0$ and $x \in \mathbb{R}^d$, set $\varphi_\varepsilon(x) := \varepsilon^{-d}\varphi(\frac{x}{\varepsilon})$. Then $\varphi_\varepsilon \geq 0$, $\|\varphi_\varepsilon\|_{L^1} = 1$ and

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d \setminus B_r(0)} \varphi_\varepsilon(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d \setminus B_{\frac{r}{\varepsilon}}(0)} \varphi(x)\,\mathrm{d}x = 0 \quad \text{for all } r > 0.$$

Moreover, since $\operatorname{supp}(\varphi) \subseteq B_1(0)$, we get $\operatorname{supp}(\varphi_\varepsilon) \subseteq B_\varepsilon(0)$. Such a sequence is called *standard approximate identity*, and φ is called a *mollifier*.

A useful application of such convolutions are the cutoff functions:

Lemma 1.4.6. *Let $\Omega \subseteq \mathbb{R}^d$ be open and $K \subseteq \Omega$ be compact. Then, for all $\delta \in (0,\infty)$ with $B_\delta(K) \subseteq \Omega$, there is a smooth cutoff function $\eta \in C_c^\infty(\Omega)$ for K with*

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } K, \quad |\partial^s \eta| \le C_{d,s}\delta^{-|s|} \text{ for all } s \in \mathbb{N}^d,$$

where $C_{d,s}$ is independent of δ and K.

Indeed, if $(\varphi_\varepsilon)_{\varepsilon>0}$ is a standard approximate identity, then $\eta := \varphi_{\frac{\delta}{4}} * \mathbb{1}_{B_{\frac{\delta}{2}}(K)}$ has all required properties.

Lemma 1.4.7. *Let $U \subseteq \mathbb{R}^d$ be open. Then there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ such that $0 \le f_n \uparrow \mathbb{1}_U$ as $n \to \infty$.*

Proof:
Define the closed sets $A_n := \{x \in \mathbb{R}^d : d(x,U^c) \ge \frac{1}{n}\}$ and $K_n := A_n \cap \overline{B_n(0)}$ for each $n \in \mathbb{N}$, as well as the interior U_n of K_n. Then $(K_n)_{n\in\mathbb{N}}$ is an exhausting sequence of U consisting of compact sets. Choose ε_n such that $B_{\varepsilon_n}(K_n) \subseteq U_{n+1}$. Then due to Lemma 1.4.6, there exist $f_n \in C_c^\infty(U_{n+1})$ such that $0 \le f_n \le 1$ and $f_n \equiv 1$ on K_n. Since f_n has support in $U_{n+1} \subseteq K_{n+1}$, the sequence $(f_n)_{n\in\mathbb{N}}$ is non-negative and increasing. Let $x \in U$ with $\|x\| \le m \in \mathbb{N}$, then there is some $n \in \mathbb{N}$ such that $x \in A_n$, hence $x \in K_{n\vee m}$, so $(f_n)_{n\in\mathbb{N}}$ converges from below to $\mathbb{1}_U$. □

In the sequel, we often require an exhausting sequence for $\mathbb{R}^d$ with nice properties, which we now obtain as an application:

Remark 1.4.8. In the setting of the above Lemma, let $\Omega = \mathbb{R}^d$, choose $K_n := \overline{B_n(0)}$ and $\delta_n := n$ for each $n \in \mathbb{N}$. This results in a sequence $(\eta_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ which satisfies $0 \le \eta_n \le 1$, $\eta_n \equiv 1$ on $B_n(0)$, $\partial^s\eta_n \equiv 0$ on $B_n(0)$ for all $s \in \mathbb{N}^d$, and $|\partial^s \eta_n| \le \frac{1}{n} M_s$ for all $n \in \mathbb{N}$, where $M_s \in (0,\infty)$ is a constant depending only on s.

Another nice application of convolutions with approximate identities results in the following construction:

Lemma 1.4.9. *Let $0 < \delta < \varepsilon$ and let $[a,b]$ be a real interval. Then there exists a smooth function $h \in C^\infty(\mathbb{R})$ with the following properties:*

(i) $h(r) = r$ *for all* $r \in [a,b]$,

(ii) $h(r) = a - \delta$ *for all* $r \in (-\infty, a-\varepsilon]$,

(iii) $h(r) = b + \delta$ *for all* $r \in [b+\varepsilon, \infty)$,

(iv) $0 \leq h' \leq 1$.

Proof:
First, we define a function $\tilde{h} \in C^0(\mathbb{R})$ via

$$\tilde{h}(r) := \begin{cases} a-\delta & \text{for } r \leq a-\delta, \\ r & \text{for } r \in [a-\delta, b+\delta], \\ b+\delta & \text{for } r \geq b+\delta. \end{cases}$$

Then $\tilde{h}$ is Lipschitz-continuous with constant 1, and has a weak derivative $\tilde{h}' \in L^\infty(\mathbb{R})$ which satisfies $\tilde{h}'(r) = 1$ on (a,b) and $\tilde{h}'(r) = 0$ on $\mathbb{R} \setminus [a,b]$. Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a standard approximate identity, let $\gamma := \frac{\varepsilon-\delta}{2}$ and define $h := \varphi_\gamma * \tilde{h} \in C^\infty(\mathbb{R})$. Then clearly h satisfies (ii) and (iii). Since φ_γ is an even function, we obtain for $r \in [a,b]$ that

$$r - h(r) = \int_{\mathbb{R}} \varphi_\gamma(r-y)(r-y)\,\mathrm{d}y = \int_{-\gamma}^{\gamma} \varphi_\gamma(z) z\,\mathrm{d}z = 0,$$

so property (i) also holds. Finally, (iv) is also fulfilled, since due to Lemma 1.4.3,

$$0 \leq \int_{\mathbb{R}} \tilde{h}'(x-y)\varphi_\gamma(y)\,\mathrm{d}y = (\varphi_\gamma * \tilde{h}')(x) = (\varphi_\gamma * \tilde{h})'(x)$$

and

$$|(\varphi_\gamma * \tilde{h}')(x)| \leq \|\varphi_\gamma * \tilde{h}'\|_{L^\infty} \leq \|\varphi_\gamma\|_{L^1} \|\tilde{h}'\|_{L^\infty} \leq 1$$

for all $x \in \mathbb{R}$. □

1.5 Separability

Here we prove separability of the space of compactly supported k times continuously differentiable functions on $\mathbb{R}^d$, since we require this fact later, see Proposition 4.3.2. Although well-known, the author was not able to find a satisfactory reference, so we include the proof here.

Lemma 1.5.1. *Let $d, k \in \mathbb{N}$. Then the space $C_c^k(\mathbb{R}^d)$ equipped with the usual norm is separable.*

Proof:
Set $K_n := \overline{B_n(0)}$ for all $n \in \mathbb{N}$. Due to the Stone-Weierstraß theorem, the space $(C^0(K_n;\mathbb{R}), \|\cdot\|_{\sup})$ is separable for any $n \in \mathbb{N}$. Now set

$$C_{c,n}^0(\mathbb{R}^d) := \{f \in C_c^0(\mathbb{R}^d) : \operatorname{supp}(f) \subseteq B_n(0)\},$$

again equipped with the supremum norm, which we can consider as a subspace of $C^0(K_n;\mathbb{R})$. Since subsets of separable metric spaces are again separable wrt. the induced metric, we obtain that the countable union $C_c^0(\mathbb{R}^d) = \bigcup_{n\in\mathbb{N}} C_{c,n}^0(\mathbb{R}^d)$ is also separable.

Let $m(k)$ denote the amount of multi-indices $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$. Consider the product space $C_c^{0,m(k)}(\mathbb{R}^d) := \times_{i\in\{1,\dots,m(k)\}} C_c^0(\mathbb{R}^d)$ with the corresponding product topology, which is generated by the norm $\|\cdot\|_{\infty,m(k)}$ given by

$$\|(f_1,\dots,f_{m(k)})\|_{\infty,m(k)} := \max\{\|f_i\|_{\sup} \mid i \in \{1,\dots,m(k)\}\}.$$

As a finite product of separable spaces, $C_c^{0,m(k)}(\mathbb{R}^d)$ is separable, and since its norm is equivalent to $\|\cdot\|_k$ defined via $\|(f_1,\dots,f_{m(k)})\|_k := \sum_{i\in\{1,\dots,m(k)\}} \|f_i\|_{\sup}$, that property transfers to $(C_c^{0,m(k)}(\mathbb{R}^d), \|\cdot\|_k)$.

Finally, consider $C_c^k(\mathbb{R}^d)$ with its norm $\|f\|_{C^k} = \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{\sup}$. Clearly, we can embed $C_c^k(\mathbb{R}^d)$ isometrically into $(C_c^{0,m(k)}(\mathbb{R}^d), \|\cdot\|_k)$, so that we can consider it as a subspace of a separable metric space, making it separable as well. □

The proof implies the following:

Corollary 1.5.2. *Let $d, k \in \mathbb{N}$. Then there is a dense sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^k(\mathbb{R}^d)$ such that for every $f \in C_c^k(\mathbb{R}^d)$, there is a subsequence $(f_k)_{k\in\mathbb{N}}$ such that the supports of all f_{n_k} and f are contained in some compact set $K \subseteq \mathbb{R}^d$.*

1.6 Weighted Sobolev spaces

Here we consider L^p-spaces and more generally Sobolev spaces with respect to a weighted Lebesgue measure, where the weight function has the form e^{-V}.

Let $V \in C^0(\mathbb{R}^d)$ and define the measure $\mu := \mathrm{e}^{-V(x)}\,\mathrm{d}x$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. We assume the measure μ to be finite, and for simplicity, to be a probability measure.

Definition 1.6.1. For $m \in \mathbb{N}$, $p \in [1,\infty)$ and $\Omega \subseteq \mathbb{R}^d$ open, we define the *weighted Sobolev space*

$$W^{m,p}(\Omega;\mu) := \left\{ f \in H^{m,1}_{\mathrm{loc}}(\mathbb{R}^d) \mid \partial^s f \in L^p(\Omega;\mu) \text{ for all } |s| \leq m \right\}$$

with corresponding norm

$$\|f\|_{W^{m,p}(\Omega;\mu)} := \left(\sum_{|s|\leq m} \|\partial^s f\|^p_{L^p(\Omega;\mu)} \right)^{\frac{1}{p}}.$$

For notational convenience, we set $W^{m,p}(\mu) := W^{m,p}(\mathbb{R}^d;\mu)$.

Remark 1.6.2. Since V and therefore e^{-V} is locally bounded, μ is locally equivalent to the Lebesgue measure. In particular, $W^{m,p}_{\mathrm{loc}}(\mu) = H^{m,p}_{\mathrm{loc}}(\mathbb{R}^d)$ and therefore $W^{m,p}_c(\mu) = H^{m,p}_c(\mathbb{R}^d)$, which denotes functions with compact support.

Theorem 1.6.3. *The space $C_c^\infty(\mathbb{R}^d)$ of smooth compactly supported functions is dense in $W^{m,p}(\mu)$ for all $m \in \mathbb{N}$, $p \in [1,\infty)$. In particular, with the classical naming conventions, it follows that $W = H$, i.e. $W^{m,p}(\mu) = H^{m,p}(\mu)$.*

We prove this in a similar way as the well-known result that $C_c^\infty(\mathbb{R}^d)$ is dense in $H^{m,p}(\mathbb{R}^d)$, by approximation with compactly supported functions.

Proof:
Let $f \in W^{m,p}(\mu)$. Choose $(\eta_n)_{n\in\mathbb{N}}$ as in Remark 1.4.8. Then there is some constant $M < \infty$ such that $|\partial^s \eta_n| \le M(\frac{1}{n})^{|s|} \le M$ for all $|s| \le m$. Define $f_n := \eta_n \cdot f \in W_c^{m,p}(\mu) = H_c^{m,p}(\mathbb{R}^d)$, then we obtain

$$|\partial^s f_n| \le M \sum_{r \le s} \binom{s}{r} |\partial^r f|,$$

so in particular there is some constant $C < \infty$ such that $\|f_n\|_{W^{m,p}(\Omega;\mu)} \le C\|f\|_{W^{m,p}(\Omega;\mu)}$ for all $n \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^d$ open. Set $\Omega_n := \mathbb{R}^d \setminus \overline{B_n(0)}$, then

$$\|f - f_n\|_{W^{m,p}(\mu)} = \|(f - f_n)\|_{W^{m,p}(\Omega_n;\mu)} \le (C+1)\|f\|_{W^{m,p}(\Omega_n;\mu)} \tag{1.6.1}$$

Now let $\varepsilon > 0$. Then we can choose $N_0 \in \mathbb{N}$ such that $\|f\|_{W^{m,p}(\Omega_{N_0};\mu)} \le \frac{\varepsilon}{2(C+1)}$. Since $f_{N_0} \in H_c^{m,p}(\mathbb{R}^d)$, there is a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ such that $\|f_{N_0} - \varphi_k\|_{H^{m,p}} \to 0$ as $k \to \infty$. Without loss of generality, we assume that $\operatorname{supp}(f_{N_0}), \operatorname{supp}(\varphi_k) \subseteq U \subseteq K$ for some open U and compact $K \subseteq \mathbb{R}^d$. Due to local equivalence, there is some constant $C_K < \infty$ such that

$$\|f_{N_0} - \varphi_k\|_{W^{m,p}(\mu)} = \|f_{N_0} - \varphi_k\|_{W^{m,p}(U;\mu)} \le C_K\|f_{N_0} - \varphi_k\|_{H^{m,p}(\mathbb{R}^d)}. \tag{1.6.2}$$

Choose $k_0 \in \mathbb{N}$ such that $\|f_{N_0} - \varphi_{k_0}\|_{H^{m,p}} \le \frac{\varepsilon}{2C_K}$, then combining (1.6.1) and (1.6.2) yields $\|f - \varphi_{k_0}\|_{W^{m,p}(\mu)} \le \varepsilon$, so the denseness result follows. □

Due to this result, we refer to these Sobolev spaces by $H^{m,p}(\mu)$ in the following. We now state an integrability result for derivatives of V, provided we satisfy the following condition:

Assumption (A1). V is in $C^1(\mathbb{R}^d) \cap H^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ and there are constants $K < \infty$ and $\alpha \in [1,2)$ such that

$$|\nabla^2 V(x)| \le K(1 + |\nabla V(x)|^\alpha) \quad \mu\text{-a.e.},$$

where $\nabla^2 V$ denotes the Hessian matrix of V.

Theorem 1.6.4. *Let V satisfy (A1) and let $k \in \mathbb{N}$. Then there is some constant $C_k < \infty$ such that for all $g \in H^{1,2k}(\mu)$, the following inequality holds:*

$$\int_{\mathbb{R}^d} |\nabla V|^{2k} g^{2k} \mathrm{e}^{-V} \, \mathrm{d}x \le C_k \left(\int_{\mathbb{R}^d} g^{2k} \mathrm{e}^{-V} \, \mathrm{d}x + \int_{\mathbb{R}^d} |\nabla g|^{2k} \mathrm{e}^{-V} \, \mathrm{d}x \right). \tag{1.6.3}$$

This is a generalization of [Vil06, Lemma A.24], which proves the case where $\alpha = 1$ and $k = 1$. In order to facilitate the proof, we first show the following intermediate estimate:

Lemma 1.6.5. *Let V satisfy* (A1) *and let $k \in \mathbb{N}$, $i \in \{1,\dots,d\}$. Then there is some constant $M \in (0,\infty)$ such that*

$$\int_{\mathbb{R}^d} (\partial_i V)^{2k} g^{2k} \,\mathrm{d}\mu \le M \left(\int_{\mathbb{R}^d} g^{2k} \,\mathrm{d}\mu + \int_{\mathbb{R}^d} (\partial_i g)^{2k} \,\mathrm{d}\mu + \int_{\mathbb{R}^d} |\nabla V|^{k\alpha} g^{2k} \,\mathrm{d}\mu \right)$$

holds for all $g \in C_c^\infty(\mathbb{R}^d)$.

Proof:
Let $g \in C_c^\infty(\mathbb{R}^d)$, $k \in \mathbb{N}$ and $1 \le i \le d$. Then integration by parts yields

$$\begin{aligned}\int_{\mathbb{R}^d} (\partial_i V)^{2k} g^{2k} \,\mathrm{d}\mu &= -\int_{\mathbb{R}^d} (\partial_i V)^{2k-1}(x) g^{2k}(x) \partial_i(\mathrm{e}^{-V(x)}) \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} (2k g^{2k-1} \partial_i g (\partial_i V)^{2k-1} + (2k-1) g^{2k} (\partial_i V)^{2k-2} \partial_i^2 V) \,\mathrm{e}^{-V} \,\mathrm{d}x.\end{aligned} \tag{1.6.4}$$

Let $C \in (0,\infty)$ be arbitrary, then due to the Hölder and Young inequalities for $p = \frac{2k}{2k-1}$ and $q = 2k$, we obtain the following estimate for the first summand:

$$\begin{aligned}\int_{\mathbb{R}^d} (2k g^{2k-1} \partial_i g (\partial_i V)^{2k-1} \,\mathrm{d}\mu &\le 2k \frac{1}{C} \|g^{2k-1}(\partial_i V)^{2k-1}\|_{L^p} \cdot C \|\partial_i g\|_{L^q} \\ &\le 2k \left(\frac{1}{pC^p} \|g^{2k-1}(\partial_i V)^{2k-1}\|_{L^p}^p + \frac{C^q}{q} \|\partial_i g\|_{L^q}^q \right) \\ &= (2k-1) \frac{1}{C^p} \int_{\mathbb{R}^d} g^{2k} (\partial_i V)^{2k} \,\mathrm{d}\mu + C^q \int_{\mathbb{R}^d} (\partial_i g)^{2k} \,\mathrm{d}\mu,\end{aligned} \tag{1.6.5}$$

where L^p denotes $L^p(\mu)$. For the second summand, note that due to (A1) we have $|\partial_i^2 V| \le K(1 + |\nabla V|^\alpha)$. This implies that

$$\begin{aligned}(2k-1) \int_{\mathbb{R}^d} g^{2k} (\partial_i V)^{2k-2} \partial_i^2 V \,\mathrm{d}\mu \le K(2k-1) &\int_{\mathbb{R}^d} g^{2k} (\partial_i V)^{2k-2} \,\mathrm{d}\mu \\ + K(2k-1) &\int_{\mathbb{R}^d} g^{2k} (\partial_i V)^{2k-2} |\nabla V|^\alpha \,\mathrm{d}\mu.\end{aligned} \tag{1.6.6}$$

Using the same technique as before for $p' = \frac{2k}{2k-2}$ and $q' = k$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} g^{2k} (\partial_i V)^{2k-2} \,\mathrm{d}\mu &\le \frac{1}{C} \|g^{2k-2}(\partial_i V)^{2k-2}\|_{L^{p'}} \cdot C \|g^2\|_{L^{q'}} \\ &\le \frac{1}{p' C^{p'}} \|g^{2k-2}(\partial_i V)^{2k-2}\|_{L^{p'}}^{p'} + \frac{C^{q'}}{q'} \|g^2\|_{L^{q'}}^{q'} \\ &= \frac{1}{p' C^{p'}} \int_{\mathbb{R}^d} g^{2k} (\partial_i V)^{2k} \,\mathrm{d}\mu + \frac{C^{q'}}{q'} \int_{\mathbb{R}^d} g^{2k} \,\mathrm{d}\mu\end{aligned} \tag{1.6.7}$$

as well as

$$\begin{aligned}\int_{\mathbb{R}^d} g^{2k}(\partial_i V)^{2k-2}|\nabla V|^\alpha \,\mathrm{d}\mu &\leq \frac{1}{C}\|g^{2k-2}(\partial_i V)^{2k-2}\|_{L^{p'}} \cdot C\|g^2|\nabla V|^\alpha\|_{L^{q'}} \\ &\leq \frac{1}{p'C^{p'}}\|g^{2k-2}(\partial_i V)^{2k-2}\|_{L^{p'}}^{p'} + \frac{C^{q'}}{q'}\|g^2|\nabla V|^\alpha\|_{L^{q'}}^{q'} \\ &= \frac{1}{p'C^{p'}}\int_{\mathbb{R}^d} g^{2k}(\partial_i V)^{2k}\,\mathrm{d}\mu + \frac{C^{q'}}{q'}\int_{\mathbb{R}^d} g^{2k}|\nabla V|^{k\alpha}\,\mathrm{d}\mu.\end{aligned} \tag{1.6.8}$$

Note that for $k = 1$, these steps are not required, so p' is well-defined. Choosing C large enough such that $\frac{2k-1}{C^p}, \frac{K(2k-1)}{p'C^{p'}} \leq \frac{1}{6}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}(\partial_i V)^{2k} g^{2k}\,\mathrm{d}\mu \leq \frac{1}{2}\int_{\mathbb{R}^d}(\partial_i V)^{2k} g^{2k}\,\mathrm{d}\mu + C^q\int_{\mathbb{R}^d}(\partial_i g)^{2k}\,\mathrm{d}\mu \\ + \frac{K(2k-1)C^{q'}}{q'}\int_{\mathbb{R}^d} g^{2k}\,\mathrm{d}\mu + \frac{K(2k-1)C^{q'}}{q'}\int_{\mathbb{R}^d} g^{2k}|\nabla V|^{k\alpha}\,\mathrm{d}\mu\end{aligned}$$

by combining the inequalities in eqs. (1.6.5) to (1.6.8) with the original integration by parts result. By subtracting the first summand from both sides, we obtain the desired claim for

$$M \coloneqq \max\left\{2C^q, \frac{2K(2k-1)C^{q'}}{q'}\right\}.$$

□

Proof (of Theorem 1.6.4):
First, we assume $g \in C_c^\infty(\mathbb{R}^d)$. Since

$$|\nabla V|^{2k} = \left(\sum_{i=1}^d (\partial_i V)^2\right)^k \leq d^{k-1}\sum_{i=1}^d (\partial_i V)^{2k},$$

we have

$$\begin{aligned}\int_{\mathbb{R}^d}|\nabla V|^{2k} g^{2k}\,\mathrm{d}\mu &\leq d^{k-1}\sum_{i=1}^d\int_{\mathbb{R}^d}(\partial_i V)^{2k} g^{2k}\,\mathrm{d}\mu \\ &\leq d^k M\left(\int_{\mathbb{R}^d} g^{2k}\,\mathrm{d}\mu + \int_{\mathbb{R}^d}|\nabla g|^{2k}\,\mathrm{d}\mu + \int_{\mathbb{R}^d}|\nabla V|^{k\alpha} g^{2k}\,\mathrm{d}\mu\right)\end{aligned} \tag{1.6.9}$$

for some $M \in (0, \infty)$ due to Lemma 1.6.5 (wlog we assume $M \geq 1$). The last summand can be estimated by

$$\begin{aligned}\int_{\mathbb{R}^d}|\nabla V|^{k\alpha} g^{2k}\,\mathrm{d}\mu &= \int_{\mathbb{R}^d}|g|^k|\nabla V|^k|g|^k|\nabla V|^{(\alpha-1)k}\,\mathrm{d}\mu \\ &\leq \frac{1}{C}\|g^k|\nabla V|^k\|_{L^2(\mu)} \cdot C\|g^k|\nabla V|^{(\alpha-1)k}\|_{L^2(\mu)} \\ &\leq \frac{1}{2C^2}\int_{\mathbb{R}^d} g^{2k}|\nabla V|^{2k}\,\mathrm{d}\mu + \frac{C^2}{2}\int_{\mathbb{R}^d} g^{2k}|\nabla V|^{2k(\alpha-1)}\,\mathrm{d}\mu,\end{aligned} \tag{1.6.10}$$

where $C \in (0, \infty)$ is arbitrary. Plugging (1.6.10) into (1.6.9) and choosing $C = \sqrt{d^k M}$, we can subtract half the left side from both sides and obtain

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla V|^{2k} g^{2k} \, \mathrm{d}\mu \le 2d^k M \left(\int_{\mathbb{R}^d} g^{2k} \, \mathrm{d}\mu + \int_{\mathbb{R}^d} |\nabla g|^{2k} \, \mathrm{d}\mu \right) \\ + d^{2k} M^2 \int_{\mathbb{R}^d} g^{2k} |\nabla V|^{2k(\alpha-1)} \, \mathrm{d}\mu. \end{aligned} \tag{1.6.11}$$

If $\alpha = 1$, then $|\nabla V|^{2k(\alpha-1)} = 1$ and the desired inequality (1.6.3) holds for $C_k = 3(d^{2k}M^2)$.

If $0 < 2(\alpha - 1) \le 1$, then $|\nabla V|^{2k(\alpha-1)} \le 1 + |\nabla V|^k$, and therefore

$$d^{2k} M^2 \int_{\mathbb{R}^d} g^{2k} |\nabla V|^{2k(\alpha-1)} \, \mathrm{d}\mu \le d^{2k} M^2 \int_{\mathbb{R}^d} g^{2k} \, \mathrm{d}\mu + d^{2k} M^2 \int_{\mathbb{R}^d} |g|^k |\nabla V|^k |g|^k \, \mathrm{d}\mu$$

with

$$d^{2k} M^2 \int_{\mathbb{R}^d} |g|^k |\nabla V|^k |g|^k \, \mathrm{d}\mu \le d^{2k} M^2 \left(\frac{1}{2D^2} \int_{\mathbb{R}^d} g^{2k} |\nabla V|^{2k} \, \mathrm{d}\mu + \frac{D^2}{2} \int_{\mathbb{R}^d} |g|^{2k} \, \mathrm{d}\mu \right),$$

where we can choose $D = d^k M$. Then (1.6.11) implies (1.6.3) for $C_k = 5d^{4k} M^4$.

In the remaining case that $2(\alpha - 1) \in (1, 2)$, we obtain instead

$$\begin{aligned} \int_{\mathbb{R}^d} g^{2k} |\nabla V|^{2k(\alpha-1)} \, \mathrm{d}\mu &= \int_{\mathbb{R}^d} |g|^k |\nabla V|^k |g|^k |\nabla V|^{k(2(\alpha-1)-1)} \, \mathrm{d}\mu \\ &\le \frac{1}{2D^2} \int_{\mathbb{R}^d} g^{2k} |\nabla V|^{2k} \, \mathrm{d}\mu + \frac{D^2}{2} \int_{\mathbb{R}^d} g^{2k} |\nabla V|^{2k(2(\alpha-1)-1)} \, \mathrm{d}\mu, \end{aligned}$$

where we again choose $D = d^k M$. Using this in (1.6.11) and repeating this procedure iteratively provides us with estimates of the form

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla V|^{2k} g^{2k} \, \mathrm{d}\mu \le 2^m d^k M \left(\int_{\mathbb{R}^d} g^{2k} \, \mathrm{d}\mu + \int_{\mathbb{R}^d} |\nabla g|^{2k} \, \mathrm{d}\mu \right) \\ + (d^k M)^{2^m} \int_{\mathbb{R}^d} g^{2k} |\nabla V|^{k\gamma_m(\alpha)} \, \mathrm{d}\mu. \end{aligned}$$

where $\gamma_m(\alpha) = 2^m(\alpha - 2) + 2$. Since there is some $\varepsilon > 0$ such that $\alpha = 2 - \varepsilon$, we can write $\gamma_m(\alpha) = 2 - 2^m \varepsilon$, so there is some $m_0 \in \mathbb{N}$ such that $0 < \gamma_{m_0}(\alpha) \le 1$. Then estimate (1.6.3) follows as above for the case $0 < 2(\alpha - 1) \le 1$. Thus the claim holds for all $g \in C_c^\infty(\mathbb{R}^d)$.

Now it remains to generalize the statement to all $g \in H^{1,2k}(\mu)$. Let g be such a function. Then due to Theorem 1.6.3, there is a sequence $(g_n)_{n \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ such that $g_n \to g$ in $H^{1,2k}(\mu)$-norm as $n \to \infty$. Each g_n satisfies (1.6.3) and since

$$\int_{\mathbb{R}^d} f^{2k} \, \mathrm{d}\mu + \int_{\mathbb{R}^d} |\nabla f|^{2k} \, \mathrm{d}\mu \le d^{k-1} \left(\int_{\mathbb{R}^d} f^{2k} \, \mathrm{d}\mu + \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_i f^{2k} \, \mathrm{d}\mu \right) = d^{k-1} \|f\|_{H^{1,2k}(\mu)}^{2k}$$

and

$$\|f\|^{2k}_{H^{1,2k}(\mu)} = \int_{\mathbb{R}^d} f^{2k}\,\mathrm{d}\mu + \sum_{i=1}^{d} \int_{\mathbb{R}^d} \partial_i f^{2k}\,\mathrm{d}\mu \leq d\left(\int_{\mathbb{R}^d} f^{2k}\,\mathrm{d}\mu + \int_{\mathbb{R}^d} |\nabla f|^{2k}\,\mathrm{d}\mu\right)$$

for all $f \in H^{1,2k}(\mu)$, the right hand side of (1.6.3) converges to

$$C_k\left(\int_{\mathbb{R}^d} g^{2k}\mathrm{e}^{-V}\,\mathrm{d}x + \int_{\mathbb{R}^d} |\nabla g|^{2k}\mathrm{e}^{-V}\,\mathrm{d}x\right)$$

as $n \to \infty$. For a subsequence $(g_{n_\ell})_{\ell\in\mathbb{N}}$, we get that $|\nabla V|^{2k} g_{n_\ell}^{2k}$ converges μ-almost everywhere to $|\nabla V|^{2k} g^{2k}$, so Fatou's Lemma yields the desired claim. □

Remark 1.6.6. Since μ is a finite measure, applying Theorem 1.6.4 to $g \equiv 1$ shows that $|\nabla V| \in L^p(\mu)$ for any $p \in [1, \infty)$. Moreover, due to **(A1)**, the same holds for $|\nabla^2 V|$.

2 Abstract hypocoercivity frameworks

Here we introduce the abstract hypocoercivity frameworks which we later apply to concrete generators. First, we give some data conditions, i.e. properties of the underlying space and compatibility with the considered operators. Then we present the hypocoercivity method developed by Dolbeault, Mouhot and Schmeiser ([DMS09]), in the rigorous form elaborated by Grothaus and Stilgenbauer ([GS14]) including domain issues. Afterwards, we give the necessary assumptions and the convergence result of the weak hypocoercivity method, which was worked out by Grothaus and Wang ([GW19]) by reworking the proof and assuming weaker lower bounds for the operators. In practice, these are verified by assuming weak Poincaré inequalities of the type seen in [RW01].

In order to compare these methods to each other, we refer to the original hypocoercivity as strong hypocoercivity, and rearrange the individual assumptions so that they can be matched more easily to their counterparts. In particular, we refrain from imposing a setting of L^2-spaces with invariant measure, since we can modify the considered space later in our application so that the proofs in [GS14] can be applied regardless. For a more detailed explanation, see Remark 2.4.1.

2.1 Data conditions

Let H be a separable Hilbert space with inner product $(\cdot,\cdot)_H$ and induced norm $\|\cdot\|_H$, which has an orthogonal decomposition $H = H_1 \oplus H_2$ with corresponding orthogonal projections $P : H \to H_1$, $(I-P) : H \to H_2$. Let further $(L, D(L))$ be a densely defined linear operator that generates a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on H. We assume the following structure on L:

Assumption (D1). $L = S - A$ on $\mathcal{D}$, where S is symmetric, A is antisymmetric, and $\mathcal{D} \subseteq D(L)$ is a core for $(L, D(L))$.

As seen by Proposition 1.1.3, both $(S, \mathcal{D})$ and $(A, \mathcal{D})$ are closable, and we denote their closures by $(S, D(S))$ and $(A, D(A))$, respectively. These two operators are linked to the decomposition of H in the following way:

Assumption (D2). $H_1 \subseteq D(S)$ and $S \equiv 0$ on H_1.

Assumption (D3). $P(\mathcal{D}) \subseteq D(A)$, $AP(\mathcal{D}) \subseteq D((AP)^*)$ and $PAP \equiv 0$ on $\mathcal{D}$. Here $(AP)^*$ is the adjoint of the densely defined closed operator $(AP, D(AP))$ with

$$D(AP) = \{x \in H \mid Px \in D(A)\}.$$

Remark 2.1.1.

(i) Assume (D2), then $SP = 0$ and $(I-P)(\mathcal{D}) \subseteq D(S)$, since $\mathcal{D} \subseteq D(S)$. Similarly, $(I-P)(\mathcal{D}) \subseteq D(A)$ if (D3) holds.

(ii) In (D3), the first assumption implies $\mathcal{D} \subseteq D(AP)$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $D(AP)$ such that $x_n \to x$ and $APx_n \to y$ for some $x, y \in H$. Then $Px_n \to Px$ and $Px_n \in D(A)$ for all $n \in \mathbb{N}$, so closedness of $(A, D(A))$ implies $Px \in D(A)$ with $APx = y$. Hence $(AP, D(AP))$ is indeed densely defined and closed, so that $(AP)^*$ is well-defined. Furthermore, it holds that $(AP)^* = -PA$ on $\mathcal{D}$, hence on $D(A)$.

Definition 2.1.2. We define the operator $(G, D(G))$ by

$$G := -(AP)^* AP, \qquad D(G) := \{x \in D(AP) \mid APx \in D((AP)^*)\}.$$

Remark 2.1.3. Due to von Neumann's theorem ([Ped89, Theorem 5.1.9]), $(G, D(G))$ is self-adjoint and $I - G : D(G) \to H$ is bijective with bounded inverse. Since G is dissipative, it generates an sccs on H, which we denote by $(\mathrm{e}^{tG})_{t\geq 0}$.

Note that due to (D3), we have $\mathcal{D} \subseteq D(G)$. If additionally, $AP(\mathcal{D}) \subseteq D(A)$, then $G = PA^2P$ on $\mathcal{D}$.

This allows us to define the following operator, which is bounded with operator norm less than 1 due again to [Ped89, Theorem 5.1.9]:

Definition 2.1.4. Define the operator $(B, D(B))$ as

$$B := (I - G)^{-1}(AP)^*, \qquad D(B) := D((AP)^*).$$

Due to boundedness, it extends uniquely to a bounded operator $B : H \to H$.

For $0 \leq \varepsilon < 1$, the modified energy functional H_ε is then defined as

$$H_\varepsilon[x] := \frac{1}{2}\|x\|_H^2 + \varepsilon(Bx, x)_H \qquad \text{for } x \in H,$$

and satisfies

$$\frac{1-\varepsilon}{2}\|x\|^2 \leq H_\varepsilon[x] \leq \frac{1+\varepsilon}{2}\|x\|^2 \qquad \text{for all } x \in H \tag{2.1.1}$$

due to (D3) and [GS14, Lemma 2.4].

Both following hypocoercivity methods provide sufficient conditions such that $\frac{\mathrm{d}}{\mathrm{d}t}H_\varepsilon[T_t x]$ can be bounded by a term depending only on $H_\varepsilon[T_t x]$ in the strong setting, or additionally depending on f via a carefully chosen functional in the weak setting, where $x \in \mathcal{D}$. This extends to $x \in D(L)$, and Gronwall's Lemma yields a convergence rate estimate for $(T_t x)_{t\geq 0}$ for all such x. Depending on the setting, this can then be further extended to a larger subspace of H.

2.2 Strong hypocoercivity

Additionally to the above stated data conditions, assume the following:

Assumption (H1). *Boundedness of auxiliary operators:* The operators $(BS, \mathcal{D})$ and $(BA(I-P), \mathcal{D})$ are bounded and there exist constants $c_1, c_2 < \infty$ such that

$$\|BSx\| \le c_1 \|(I-P)x\| \quad \text{and} \quad \|BA(I-P)x\| \le c_2 \|(I-P)x\|$$

hold for all $x \in \mathcal{D}$.

Assumption (H2). *Microscopic coercivity:* There exists some $\Lambda_m > 0$ such that

$$-(Sx, x)_H \ge \Lambda_m \|(I-P)x\|^2 \qquad \text{for all } x \in \mathcal{D}.$$

Assumption (H3). *Macroscopic coercivity:* There is some $\Lambda_M > 0$ such that

$$\|APx\|^2 \ge \Lambda_M \|Px\|^2 \qquad \text{for all } x \in D(G). \tag{2.2.1}$$

Then the following hypocoercivity theorem follows, compare also [GS16, Theorem 2.2]:

Theorem 2.2.1. *Assume that* (D1)–(D3) *and* (H1)–(H3) *hold. Then there exist strictly positive constants $\kappa_1, \kappa_2 < \infty$ which are explicitly computable in terms of Λ_m, Λ_M, c_1 and c_2 such that for all $x \in H$ we have*

$$\|T_t x\|_H \le \kappa_1 \mathrm{e}^{-\kappa_2 t} \|x\|_H \quad \textit{for all } t \ge 0.$$

More specifically, for $\delta > 0$, $\varepsilon \in (0,1)$ and $0 < \kappa < \infty$ satisfying

$$\begin{aligned} \kappa \|T_t x\|^2 \le &\left(\Lambda_m - \varepsilon(1 + c_1 + c_2)\left(1 + \frac{1}{2\delta}\right)\right) \|(I-P)T_t x\|^2 \\ &+ \varepsilon \left(\frac{\Lambda_M}{1+\Lambda_M} - (1 + c_1 + c_2)\frac{\delta}{2}\right) \|PT_t x\|^2 \end{aligned} \tag{2.2.2}$$

for all $x \in D(L)$, $t \ge 0$, the constants κ_1 and κ_2 can be given as

$$\kappa_1 = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}, \qquad \kappa_2 = \frac{\kappa}{1+\varepsilon}.$$

Remark 2.2.2. The estimate (2.2.2) can always be satisfied: First, choose $\delta > 0$ such that

$$\frac{\Lambda_M}{1+\Lambda_M} - (1 + c_1 + c_2)\frac{\delta}{2} > 0.$$

Then, choose $\varepsilon > 0$ small enough such that

$$\Lambda_m - \varepsilon(1 + c_1 + c_2)\left(1 + \frac{1}{2\delta}\right) > 0$$

as well. Since $\|T_t x\|^2 = \|(I-P)T_t x\|^2 + \|PT_t x\|^2$, κ can be chosen as

$$\min\left\{ \frac{\Lambda_M}{1+\Lambda_M} - (1 + c_1 + c_2)\frac{\delta}{2},\ \Lambda_m - \varepsilon(1 + c_1 + c_2)\left(1 + \frac{1}{2\delta}\right)\right\}.$$

2.3 Weak hypocoercivity

Additionally to the data conditions (D1)–(D3), assume the following:

Assumption (WH1). There is a constant $N < \infty$ such that

$$\begin{aligned} (BS(I-P)x, Px)_H &\leq \frac{N}{2}\|(I-P)x\|_H\|Px\|_H \quad \text{and} \\ -(BA(I-P)x, Px)_H &\leq \frac{N}{2}\|(I-P)x\|_H\|Px\|_H \quad \text{for all } x \in \mathcal{D}. \end{aligned}$$

Assumption (WH2). There is a functional $\Theta : H \to [0, \infty]$ with the following properties:

(i) The set $\{x \in H : \Theta(x) < \infty\}$ is dense in H.

(ii) $\Theta(Px) \leq \Theta(x)$ for all $x \in H$.

(iii) $\Theta(T_t x) \leq \Theta(x)$ for all $x \in H, t \geq 0$.

(iv) $\Theta(\mathrm{e}^{tG}x) \leq \Theta(x)$ for all $x \in H, t \geq 0$, where $(\mathrm{e}^{tG})_{t\geq 0}$ denotes the C_0-semigroup generated by $(G, D(G))$.

Assumption (WH3). For any $x \in D(L)$ there is some sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ such that $x_n \to x$ in H and

$$\limsup_{n\to\infty}(-Lx_n, x_n)_H \leq (-Lx, x)_H, \qquad \limsup_{n\to\infty}(\Theta x_n) \leq \Theta(x)$$

as $n \to \infty$.

Assumption (WH4). There exist decreasing functions $\alpha_i : (0, \infty) \to [1, \infty)$, $i = 1, 2$, such that

$$\|Px\|^2 \leq \alpha_1(r)\|APx\|^2 + r\Theta(Px), \quad r > 0,\ x \in D(AP) \tag{2.3.1}$$

and

$$\|(I-P)x\|^2 \leq \alpha_2(r)(-Sx, x)_H + r\Theta(x), \quad r > 0,\ x \in \mathcal{D}. \tag{2.3.2}$$

This allows us to state the main weak hypocoercivity result:

Theorem 2.3.1. *Let (D1)–(D3) and (WH1)–(WH4) be satisfied. Then there exist constants $c_1, c_2 > 0$ such that*

$$\|T_t x\|^2 \leq \xi(t)(\|x\|^2 + \Theta(x)), \quad t \geq 0,\ x \in D(L) \tag{2.3.3}$$

holds for

$$\xi(t) := c_1 \inf\left\{ r > 0 : c_2 t \geq \alpha_1(r)^2 \alpha_2\left(\frac{r}{\alpha_1(r)^2}\right)\log\left(\tfrac{1}{r}\right)\right\}, \tag{2.3.4}$$

which goes to 0 as $t \to \infty$.

2.4 Remarks and sufficient conditions

Remark 2.4.1 (Difference to cited sources).

(i) In the original source [GS14], the hypocoercivity conditions were named to stay consistent with the corresponding conditions in [DMS09], and the data conditions were added to ensure well-definedness by considering domain issues. Since we only repeat the results for application later, we take the liberty to rearrange and rename the conditions in favor of a clearer structure. The same applies to the weak hypocoercivity conditions.

(ii) Moreover, in [GS14], it was assumed that H is an L^2-space with respect to a probability measure which is invariant for $(L, D(L))$, and that $(T_t)_{t\geq 0}$ is conservative. In return, a second projection P_S is defined via $P_S f = Pf + (f, 1)_H$, which is used instead of P in **(H2)**. Furthermore, the right hand sides of the inequalities in **(H1)** allow either projection each, independently. The proofs carry over to our assumptions without change, and our formulation corresponds to the generality in which the results in [GW19] were stated.

Since the conditions **(H2)** and **(H3)** are verified in practice by assuming Poincaré inequalities, the projection P_S appears to be a more natural choice. However, in our applications, we restrict ourselves to subspaces of L^2-spaces where each element f satisfies $(f, 1)_H = 0$, so that both projections coincide. The convergence result is then lifted to the entire space afterwards.

Remark 2.4.2 (Comparison).

(i) Condition **(H1)** is stronger than **(WH1)**. However, since $Sx = S(I - P)x$ for all $x \in H$, [GS14, Remark 2.17] shows that Theorem 2.2.1 holds under **(WH1)** as well. In applications, we verify **(H1)** in any case.

(ii) If the α_i are constant in **(WH4)**, then **(H2)** and **(H3)** are satisfied. In that case, Theorem 2.3.1 also yields exponential convergence, since then

$$\xi(t) = \exp\left(-\frac{c_2}{\alpha_1\alpha_2} t\right).$$

(iii) In our applications, **(H2)** and **(H3)** are verified by assuming Poincaré inequalities, which translates to additional assumptions on the Lebesgue-density of the considered probability measure. In contrast, **(WH4)** only requires the validity of weak Poincaré inequalities, which do not require additional assumptions.

If $(G, \mathcal{D})$ is already essentially self-adjoint, then we can reduce some of the above conditions to more easily verifiable ones, which is shown in the two following Lemmas, the first of which corresponds to [GS14, Corollary 2.13].

Lemma 2.4.3. *If $\mathcal{D}$ is a core for $(G, D(G))$, then* **(H3)** *is satisfied if* (2.2.1) *holds for all $x \in \mathcal{D}$.*

Lemma 2.4.4. *Let $\mathcal{D}$ be a core for $(G, D(G))$. Let $(T, D(T))$ be a linear operator with $\mathcal{D} \subseteq D(T)$ and assume $AP(\mathcal{D}) \subseteq D(T^*)$. Then*

$$(I-G)(\mathcal{D}) \subseteq D((BT)^*) \quad \text{with} \quad (BT)^*(I-G)x = T^*APx, \quad x \in \mathcal{D}.$$

If there exists some $C < \infty$ such that

$$\|(BT)^*y\| \leq C\|y\| \qquad \text{for all } y = (I-G)x, \quad x \in \mathcal{D}, \tag{2.4.1}$$

then $(BT, D(T))$ is bounded and its closure $(\overline{BT})$ is a continuous operator on H with $\|\overline{BT}\| = \|(BT)^*\| \leq C$.

In particular, if $(S, D(S))$ and $(A, D(A))$ satisfy these assumptions with constant C_S and C_A, respectively, then **(H1)** *is satisfied with $c_1 = C_S$, $c_2 = C_A$, and* **(WH1)** *is satisfied with $N = \frac{1}{2}\max\{C_S, C_A\}$.*

Proof:
Let $z \in D((AP)^*)$ and $x \in \mathcal{D}$. Set $y = (I-G)x$. By the representation of B on $D((AP)^*)$ together with self-adjointness of $(I-G)^{-1}$ and $\mathcal{D} \subseteq D(AP)$, we get

$$(z, B^*y)_H = (Bz, (I-G)x)_H = ((AP)^*z, x)_H = (z, APx)_H.$$

Since $D(AP)$ is dense in H, this implies $B^*y = APx \in D(T^*)$. By Lemma 1.1.2 (v), we obtain $(BT)^*y = T^*B^*y = T^*APx$.

By essential self-adjointness and hence essential m-dissipativity of G, $(I-G)(\mathcal{D})$ is dense in H. Therefore by (2.4.1), the closed operator $((BT)^*, D((BT)^*))$ is a bounded operator with domain H. Since $(BT, D(T))$ is densely defined, by Lemma 1.1.2 (i) and (ii), it is closable with $\overline{BT} = (BT)^{**}$, which is a bounded operator on H with the stated norm.

The last part follows immediately since $Sx = S(I-P)x$ for all $x \in \mathcal{D}$. □

3 Essential m-dissipativity for generalized Langevin operators

3.1 Introduction

In this chapter, we concern ourselves with showing essential m-dissipativity for a class of second-order differential operators on an L^2-space, where they are defined on a set $\mathcal{C}$ of compactly supported smooth functions. These operators are of the form $L = S - A$, where S is a symmetric negative-semidefinite and A is an antisymmetric operator, and are associated in a natural way via integration by parts to a bilinear gradient form. This structure already guarantees the existence of a maximally dissipative extension of $(L, \mathcal{C})$ which generates a strongly continuous contraction semigroup on the appropriate Hilbert space. However, it is still important to show that $\mathcal{C}$ is suitably large to form a core of this generator, so that we may reduce the analytic treatment of the operator to that set. In particular, both Hypocoercivity frameworks presented above require sufficient knowledge of such a core.

The notations introduced in this part will be used for the remainder of the section without further mention. Let $d_1, d_2 \in \mathbb{N}$ and set the state space as $E = \mathbb{R}^{d_1+d_2}$ with corresponding Borel-σ-algebra $\mathcal{F} = \mathcal{B}(\mathbb{R}^{d_1+d_2})$. For a measurable function $V : \mathbb{R}^{d_i} \to \mathbb{R}$, denote by $Z(V)$ the integral $\int_{\mathbb{R}^{d_i}} \mathrm{e}^{-V(x)}\,\mathrm{d}x$. In the following, the first d_1 components of E will be written as x, the latter d_2 components as y. We denote the standard Euclidean inner product of two elements $u, v \in \mathbb{R}^{d_i}$ by $\langle u, v \rangle$. By $\nabla_x f$ and $\nabla_y f$, we denote the gradient with respect to the first and second variable of a sufficiently differentiable function $f : E \to \mathbb{R}$, respectively. Analogously, we define the Hessian matrices $H_x f$ and $H_y f$.

The approach here will be as follows: First we introduce the general shape of differential operators considered, along with the corresponding invariant measures that will fix the L^2-Hilbert space on which the generated semigroups shall act. Note that later on (see Section 4.3.1), we will transform this setting into the corresponding Fokker-Planck formulation, which might be more familiar to the reader. Secondly, we specify a set of conditions on the first and second order coefficients that we require to obtain our result. Then, we prove essential self-adjointness of the symmetric part $(S, \mathcal{C})$ with Lipschitz-continuous coefficients, and use perturbation theory to obtain essential m-dissipativity of the entire operator $(L, \mathcal{C})$, albeit still with Lipschitz coefficients. Lastly, we use the dense range condition together with approximation by Lipschitz coefficients to obtain the result in full generality.

3.2 The differential operators

We now introduce the differential operators S, A and L as mentioned above.

Definition 3.2.1. Let $\Sigma = (a_{ij})_{1\le i,j\le d_2}$ be a variable symmetric matrix with $a_{ij} : \mathbb{R}^{d_2} \to \mathbb{R}$ being locally weakly differentiable. Let further $\Phi : \mathbb{R}^{d_1} \to \mathbb{R}$ and $\Psi : \mathbb{R}^{d_2} \to \mathbb{R}$ also be locally weakly differentiable, and Q be a real $d_1 \times d_2$-matrix with transpose Q^*.

We consider operators of the form

$$\begin{aligned} Sf &= \operatorname{tr}[\Sigma H_y f] + \sum_{i=1}^{d_2} b_i \partial_{y_i} f = \sum_{i,j=1}^{d_2} a_{ij}\partial_{y_j}\partial_{y_i} f + \sum_{i=1}^{d_2} b_i \partial_{y_i} f, && (3.2.1)\\ &\qquad \text{where } b_i(y) = \sum_{j=1}^{d_2} (\partial_j a_{ij}(y) - a_{ij}(y)\partial_j \Psi(y)), && (3.2.2)\\ Af &= Q^*\nabla\Phi \cdot \nabla_y f - Q\nabla\Psi \cdot \nabla_x f, && (3.2.3)\\ Lf &= (S - A)f, && (3.2.4)\\ \hat{L}f &= (S + A)f, \qquad \text{for } f \in \mathcal{C}, && (3.2.5) \end{aligned}$$

where we set $\mathcal{C} := C_c^\infty(\mathbb{R}^{d_1}) \otimes C_c^\infty(\mathbb{R}^{d_2})$ as the tensor product space of smooth compactly supported functions on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$.

First, we specify the potentials Φ and Ψ:

Assumption (Φ1). The potential $\Phi : \mathbb{R}^{d_1} \to \mathbb{R}$ is assumed to be bounded from below and locally Lipschitz-continuous.

Assumption (Ψ1). The potential Ψ is a measurable, locally bounded function and satisfies $Z(\Psi) < \infty$.

This allows the introduction of the following measures:

Definition 3.2.2. Denote by μ_1 and μ_2 the measure $\mathrm{e}^{-\Phi(x)}\,\mathrm{d}x$ on $(\mathbb{R}^{d_1}, \mathcal{B}(\mathbb{R}^{d_1}))$ and the probability measure $Z(\Psi)^{-1}\mathrm{e}^{-\Psi(y)}\,\mathrm{d}y$ on $(\mathbb{R}^{d_2}, \mathcal{B}(\mathbb{R}^{d_2}))$, respectively.

Further define the product measure $\mu = \mu_1 \otimes \mu_2$ on $(E, \mathcal{F})$ and set $X := L^2(E;\mu)$, where E is the state space $\mathbb{R}^{d_1+d_2}$ and $\mathcal{F}$ the Borel-σ-algebra $\mathcal{B}(E)$.

To ensure well-definedness of the operators on X, we further assume

Assumption (Ψ2). It holds that $\Psi \in H^{1,1}_{\mathrm{loc}}(\mathbb{R}^{d_2})$ as well as $\partial_j \Psi \in L^2_{\mathrm{loc}}(\mathbb{R}^{d_2})$ for $1 \le j \le d_2$.

We also assume the following about $\Sigma = (a_{ij})_{1\le i,j\le d}$ with $a_{ij} : \mathbb{R}^{d_2} \to \mathbb{R}$:

Assumption (Σ1). Σ is symmetric and uniformly strictly elliptic, i.e. there is some $0 < c_\Sigma < \infty$ such that

$$\langle v, \Sigma(y)v\rangle \geq c_\Sigma^{-1} \cdot |v|^2 \quad \text{for all } v \in \mathbb{R}^{d_2} \text{ and } \mu_2\text{-almost all } y \in \mathbb{R}^{d_2}.$$

Assumption (Σ2). For each $1 \leq i, j \leq d_2$, a_{ij} is bounded and locally Lipschitz-continuous.

Remark 3.2.3. We remark the following to explain how the above considerations yield well-definedness of all terms:

(i) Due to local boundedness of Φ and Ψ, the measures μ_1 and μ_2 are locally equivalent to the Lebesgue measures on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. Therefore, the notation $H^{n,p}_{\mathrm{loc}}(\mathbb{R}^{d_i})$ does not depend on the specific measure used.

(ii) Since locally Lipschitz functions are locally absolutely continuous, they are almost everywhere differentiable such that the gradient coincides almost everywhere with the weak derivative and is bounded. In particular, the above assumptions imply that $\Phi \in H^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^{d_1})$ and $a_{ij} \in H^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^{d_2})$ for all $1 \leq i, j \leq d_2$.

Proposition 3.2.4. *For $f, g \in \mathcal{C}$, it holds that*

$$(Sf, g)_X = -\int_E \langle \nabla_y f, \Sigma \nabla_y g\rangle \,\mathrm{d}\mu \quad \textit{and} \quad (Af, g)_X = \int_E \langle \nabla_y f, Q^* \nabla_x g\rangle - \langle \nabla_x f, Q \nabla_y g\rangle \,\mathrm{d}\mu.$$

In particular, $(S, \mathcal{C})$ is symmetric negative-semidefinite, $(A, \mathcal{C})$ is antisymmetric, and $(L, \mathcal{C})$, $(\hat{L}, \mathcal{C})$ are dissipative and induce a gradient form via

$$\begin{aligned} (Lf, g)_X &= -\int_E \left\langle \nabla f, \begin{pmatrix} 0 & -Q \\ Q^* & \Sigma \end{pmatrix} \nabla g \right\rangle \,\mathrm{d}\mu, \\ (\hat{L}f, g)_X &= -\int_E \left\langle \nabla f, \begin{pmatrix} 0 & Q \\ -Q^* & \Sigma \end{pmatrix} \nabla g \right\rangle \,\mathrm{d}\mu, \quad \textit{for } f \in \mathcal{C}. \end{aligned} \tag{3.2.6}$$

Proof:
Let $f, g \in \mathcal{C}$. Then integration by parts yields

$$\int_E a_{ij}(\partial_{y_j}\partial_{y_i} f) g \,\mathrm{e}^{-\Phi-\Psi} \,\mathrm{d}(x,y) = -\int_E \partial_{y_i} f((\partial_j a_{ij} - a_{ij}\partial_j \Psi) g + a_{ij}\partial_{y_j} g) \,\mathrm{e}^{-\Phi-\Psi} \,\mathrm{d}(x,y)$$

for all $1 \leq i, j \leq d_2$, which implies

$$\begin{aligned} (Sf, g)_X &= -\sum_{i,j=1}^{d_2} Z(\Psi)^{-1} \int_E a_{ij}(\partial_{y_i} f)(\partial_{y_j} g) \mathrm{e}^{-\Phi-\Psi} \,\mathrm{d}(x,y) \\ &= -\int_E \langle \nabla_y f, \Sigma \nabla_y g\rangle \,\mathrm{d}\mu. \end{aligned}$$

In particular, using the property (Σ1), we obtain

$$(Sf, f)_X = -\int_E \langle \nabla_y f, \Sigma \nabla_y f\rangle \,\mathrm{d}\mu \leq -c_\Sigma^{-1} \|\nabla_y f\|_X^2 \leq 0,$$

so $(S,\mathcal{C})$ is indeed symmetric negative-semidefinite. Using the fact that $\partial_{x_i}\mathrm{e}^{-\Phi-\Psi} = \partial_i\Phi\,\mathrm{e}^{-\Phi-\Psi}$, another set of integration by parts gives

$$\int_E Q_{ij}(\partial_{y_j}f)g(\partial_i\Phi)\,\mathrm{e}^{-\Phi-\Psi}\,\mathrm{d}(x,y) = -\int_E Q_{ij}((\partial_{x_i}\partial_{y_j}f)g + \partial_{y_j}f\partial_{x_i}g)\,\mathrm{e}^{-\Phi-\Psi}\,\mathrm{d}(x,y)$$

and

$$\int_E -Q_{ij}(\partial_{x_i}f)g(\partial_j\Psi)\,\mathrm{e}^{-\Phi-\Psi}\,\mathrm{d}(x,y) = \int_E Q_{ij}((\partial_{y_j}\partial_{x_i}f)g + \partial_{x_i}f\partial_{y_j}g)\,\mathrm{e}^{-\Phi-\Psi}\,\mathrm{d}(x,y).$$

for all $1\le i\le d_1$, $1\le j\le d_2$. Adding these equalities yields after summation over i and j:

$$\begin{aligned}(Af,g)_X &= Z(\Psi)^{-1}\int_E(\langle Q^*\nabla\Phi,\nabla_y f\rangle - \langle Q\nabla\Psi,\nabla_x f\rangle)g\,\mathrm{e}^{-\Phi-\Psi}\,\mathrm{d}(x,y)\\ &= \int_E\langle\nabla_y f, Q^*\nabla_x g\rangle - \langle\nabla_x f, Q\nabla_y g\rangle\,\mathrm{d}\mu.\end{aligned}$$

In particular,

$$(Af,f)_X = \int_E\langle Q\nabla_y f,\nabla_x f\rangle - \langle\nabla_x f, Q\nabla_y f\rangle\,\mathrm{d}\mu = 0,$$

hence $(A,\mathcal{C})$ is antisymmetric. By Remark 1.2.14, both S and A as well as their sum $\hat{L}$ and difference L are dissipative operators with domain $\mathcal{C}$, with the representation (3.2.6) following directly from the above calculations. □

By Lemma 1.2.16 (i), all four operators are closable, and we denote their closures respectively by $(S,D(S))$, $(A,D(A))$, $(L,D(L))$ and $(\hat{L},D(\hat{L}))$.

3.3 Preliminary results for symmetric operators

We first prove essential self-adjointness, equivalently essential m-dissipativity, for a wide class of symmetric differential operators on L^2-spaces. This is essentially a combination of two results by Bogachev, Krylov, and Röckner, namely [BKR01, Corollary 2.10] and [BKR97, Theorem 7], however, the combined statement does not seem to be well known and might hold interest as the basis for similar m-dissipativity proofs. We use the slightly more general statement from [BGS13, Theorem 5.1] in order to relax the assumptions.

Theorem 3.3.1. *Let $d\ge 2$ and consider $H = L^2(\mathbb{R}^d,\mu)$ where $\mu = \rho\,\mathrm{d}x$, $\rho = \varphi^2$ for some $\varphi\in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^d)$ such that $\frac{1}{\rho}\in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. Let $A = (a_{ij})_{1\le i,j\le d}:\mathbb{R}^d\to\mathbb{R}^{d\times d}$ be symmetric and locally strictly elliptic with $a_{ij}\in L^\infty(\mathbb{R}^d)$ for all $1\le i,j\le d$. Assume there is some $p>d$ such that $a_{ij}\in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ for all $1\le i,j\le d$ and that $|\nabla\rho|\in L^p_{\mathrm{loc}}(\mathbb{R}^d)$. Consider the bilinear form (B,D) given by $D = C_c^\infty(\mathbb{R}^d)$ and*

$$B(f,g) \coloneqq (\nabla f, A\nabla g)_H = \int_{\mathbb{R}^d}(\nabla f(x), A(x)\nabla g(x))_{\mathrm{euc}}\,\rho(x)\,\mathrm{d}x,\qquad f,g\in D.$$

Define further the linear operator (S, D) via

$$Sf := \sum_{i,j=1}^{d} a_{ij}\partial_j\partial_i f + \sum_{i=1}^{d} b_i\partial_i f, \qquad f \in D,$$

where $b_i = \sum_{j=1}^{d}(\partial_j a_{ij} + a_{ij}\frac{\partial_j \rho}{\rho}) \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$, so that $B(f,g) = (-Sf, g)_H$. Then (S, D) is essentially self-adjoint on H.

Proof:
Analogously to the proof of [BKR97, Theorem 7], it can be shown that ρ is continuous, hence locally bounded. Assume that there is some $g \in H$ such that

$$\int_{\mathbb{R}^d} (S - I)f(x) \cdot g(x) \cdot \rho(x) \, \mathrm{d}x = 0 \quad \text{for all } f \in D. \tag{3.3.1}$$

Define the locally finite signed Borel measure ν via $\nu = g\rho \, \mathrm{d}x$, which is then absolutely continuous with respect to the Lebesgue measure. By definition it holds that

$$\int_{\mathbb{R}^d} \left(\sum_{i,j=1}^{d} a_{ij}\partial_j\partial_i f + \sum_{i=1}^{d} b_i\partial_i f - f \right) \mathrm{d}\nu = 0 \quad \text{for all } f \in D,$$

so by [BGS13, Theorem 5.1], the density $g \cdot \rho$ of ν is in $H^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ and locally Hölder continuous, hence locally bounded. This implies $g = g\rho \cdot \frac{1}{\rho} \in L^p_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ and $\nabla g = \nabla(g\rho) \cdot \frac{1}{\rho} - (g\rho)\frac{\nabla\rho}{\rho^2} \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$. Hence $g \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$, is locally bounded, and $g \cdot b_i \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for all $1 \le i \le d$. Therefore, we can apply integration by parts to (3.3.1) and get for every $f \in D$:

$$\begin{aligned} 0 &= -\sum_{i,j=1}^{d} (a_{ij}\partial_i f, \partial_j g)_H - \sum_{i=1}^{d} (\partial_i f, b_i g)_H + \sum_{i=1}^{d} (\partial_i f, b_i g)_H - (f, g)_H \\ &= -\int_{\mathbb{R}^d} (\nabla f, A\nabla g)_{\mathrm{euc}} \, \mathrm{d}\mu - (f, g)_H. \end{aligned} \tag{3.3.2}$$

Note that this equation can then be extended to all $f \in H^{1,2}(\mathbb{R}^d)$ with compact support, since $p > 2$ by definition. Now let $\psi \in C_c^\infty(\mathbb{R}^d)$ and set $\eta = \psi g \in H^{1,2}(\mathbb{R}^d)$, which has compact support. The same then holds for $f := \psi\eta \in H^{1,2}(\mathbb{R}^d)$. Elementary application of the product rule yields

$$(\nabla\eta, A\nabla(\psi g))_{\mathrm{euc}} = (\nabla f, A\nabla g)_{\mathrm{euc}} - \eta(\nabla\psi, A\nabla g)_{\mathrm{euc}} + g(\nabla\eta, A\nabla\psi)_{\mathrm{euc}}. \tag{3.3.3}$$

From now on, for $a, b : \mathbb{R}^d \to \mathbb{R}^d$, let (a, b) always denote the evaluation of the Euclidean inner product $(a,b)_{\mathrm{euc}}$. By using (3.3.3) and applying (3.3.2) to f, we get

$$\begin{aligned}\int_{\mathbb{R}^d} (\nabla(\psi g), A\nabla(\psi g))\,\mathrm{d}\mu + \int_{\mathbb{R}^d} (\psi g)^2\,\mathrm{d}\mu &= \int_{\mathbb{R}^d} (\nabla\eta, A\nabla(\psi g))\,\mathrm{d}\mu + \int_{\mathbb{R}^d} \eta\psi g\,\mathrm{d}\mu \\ &= \int_{\mathbb{R}^d} (\nabla f, A\nabla g)\,\mathrm{d}\mu - \int_{\mathbb{R}^d} \eta(\nabla\psi, A\nabla g)\,\mathrm{d}\mu + \int_{\mathbb{R}^d} g(\nabla\eta, A\nabla\psi)\,\mathrm{d}\mu + \int_{\mathbb{R}^d} fg\,\mathrm{d}\mu \\ &= -\int_{\mathbb{R}^d} \psi g(\nabla\psi, A\nabla g)\,\mathrm{d}\mu + \int_{\mathbb{R}^d} g(\nabla(\psi g), A\nabla\psi)\,\mathrm{d}\mu \\ &= \int_{\mathbb{R}^d} g^2(\nabla\psi, A\nabla\psi)\,\mathrm{d}\mu,\end{aligned}$$

where the last step follows from the product rule and symmetry of A. Since A is locally strictly elliptic, there is some $c > 0$ such that

$$0 \le \int_{\mathbb{R}^d} c(\nabla(\psi g), \nabla(\psi g))\,\mathrm{d}\mu \le \int_{\mathbb{R}^d} (\nabla(\psi g), A\nabla(\psi g))\,\mathrm{d}\mu$$

and therefore it follows that

$$\int_{\mathbb{R}^d} (\psi g)^2\,\mathrm{d}\mu \le \int_{\mathbb{R}^d} g^2(\nabla\psi, A\nabla\psi)\,\mathrm{d}\mu. \tag{3.3.4}$$

Let $(\eta_n)_{n\in\mathbb{N}}$ be as in Remark 1.4.8. Then (3.3.4) holds for all $\psi = \eta_n$. By dominated convergence, the left part converges to $\|g\|_H^2$ as $n \to \infty$. The integrand of the right hand side term is dominated by $d^2C^2M \cdot g^2 \in L^1(\mu)$, where C corresponds to $\max_{|s|=1} M_s$ from Remark 1.4.8 and $M := \max_{1\le i,j\le d} \|a_{ij}\|_\infty$. By definition of the η_n, that integrand converges pointwisely to zero as $n \to \infty$, so again by dominated convergence it follows that $g = 0$ in H.

This implies that $(S - I)(D)$ is dense in H and therefore that (S, D) is essentially self-adjoint.□

Remark 3.3.2. The above theorem also holds for $d = 1$, as long as $p \ge 2$. Indeed, continuity of ρ follows from similar regularity estimates, see [BKR97, Remark 2]. The proof of [BGS13, Theorem 5.1] mirrors the proof of [BKR01, Theorem 2.8], where $d \ge 2$ is used to apply [BKR01, Theorem 2.7]. However, in the cases where it is applied, this distinction is not necessary (since $p' < q$ always holds). Finally, the extension of (3.3.2) requires $p \ge 2$.

3.4 Essential m-dissipativity for Lipschitz coefficients and potential

Throughout this section, we assume **(Φ1)**, **(Ψ1)**–**(Ψ2)** and **(Σ1)**–**(Σ2)** to be satisfied.

Additionally, we require the following restrictions:

Assumption (Ψ3). It holds that $\Psi \in H^{2,1}_{\mathrm{loc}}(\mathbb{R}^{d_2};\mu_2)$ and there are constants $K < \infty$ and $\alpha \in [1,2)$ such that

$$|\nabla^2\Psi| \leq K(1 + |\nabla\Psi|^\alpha),$$

where $\nabla^2\Psi$ denotes the Hessian matrix of Ψ.

Assumption (Σ3). There are constants $0 \leq M < \infty$, $0 \leq \beta < 1$ such that for all $1 \leq i,j,k \leq d_2$

$$|\partial_k a_{ij}(y)| \leq M(\mathbb{1}_{B_1(0)}(y) + |y|^\beta) \qquad \text{for } \mu_2\text{-almost all } y \in \mathbb{R}^{d_2}.$$

For notational simplicity, we define the following constants:

Definition 3.4.1. Let Σ satisfy (Σ2). Then we set

$$M_\Sigma := \max\{\|a_{ij}\|_\infty \,:\, 1 \leq i,j \leq d_2\} \qquad \text{and}$$
$$B_\Sigma := \max\{|\partial_j a_{ij}(y)| \,:\, y \in \overline{B_1(0)},\ 1 \leq i,j \leq d_2\}.$$

If Σ additionally satisfies (Σ3), then we define

$$N_\Sigma := \sqrt{M_\Sigma^2 + B_\Sigma^2 + d_2 M^2}.$$

Remark 3.4.2. Using Sobolev embedding and interpolation theory, it can be shown that all three assumptions on Ψ together imply that Ψ is continuously differentiable and that $\nabla\Psi$ is locally Lipschitz-continuous. The proof is analogous to the proof of [Con11, Lemma A6.2]. In particular, this means that all coefficients appearing in the definition of the operator L are locally bounded.

Moreover, due to Remark 1.6.6, both $|\nabla\Psi|$ and $|\nabla^2\Psi|$ are in $\bigcap_{1\leq p<\infty} L^p(\mu_2)$.

This allows us to formulate

Proposition 3.4.3. *The equality* (3.2.6) *also holds for all* $f \in C$, $g \in H^{1,2}_{\mathrm{loc}}(E)$. *In particular, this applies to* $g \equiv 1$, *which yields* $\int_E Lf \,\mathrm{d}\mu = 0$ *for all* $f \in C$, *so that* μ *is an invariant measure for* (L, C), *see Definition 1.3.1.*

Proof:
Let $f \in C$, $g \in H^{1,2}_{\mathrm{loc}}(E)$. Then there is some $m \in \mathbb{N}$ such that $\operatorname{supp}(f) \subseteq B_m(0)$. Let $\varphi_m : E \to \mathbb{R}$ be a cutoff function as defined in Remark 1.4.8 and set $\tilde{g} := \varphi_m \cdot g$. Then $\tilde{g} \in H^{1,2}(E; \mathrm{d}(x,y))$ and can be approximated in Sobolev norm by functions $g_n \in C_c^\infty(E)$. Due to Remark 3.4.2, the expression $Lf \cdot \mathrm{e}^{-\Phi-\Psi}$ is bounded and therefore

$$(Lf, g_n)_X = Z(\Psi)^{-1} \int_E (Lf \cdot \mathrm{e}^{-\Phi-\Psi}) g \,\mathrm{d}(x,y) \to (Lf, \tilde{g})_X = (Lf, g)_X$$

as $n \to \infty$ via Hölder inequality. A similar argument for the right hand side yields

$$-\int_E \left\langle \nabla f, \begin{pmatrix} 0 & -Q \\ Q^* & \Sigma \end{pmatrix} \nabla g_n \right\rangle \mathrm{d}\mu \to \int_E \left\langle \nabla f, \begin{pmatrix} 0 & -Q \\ Q^* & \Sigma \end{pmatrix} \nabla g \right\rangle \mathrm{d}\mu$$

as $n \to \infty$, which proves the claim. □

Under the assumptions so far, we obtain the following essential m-dissipativity result:

Lemma 3.4.4. *The differential operator* $(\widetilde{S}, C_c^\infty(\mathbb{R}^{d_2}))$*, where*

$$\widetilde{S}f = \sum_{i,j=1}^{d} a_{ij}\partial_j\partial_i f + \sum_{i=1}^{d}\sum_{j=1}^{d}(\partial_j a_{ij} - a_{ij}\partial_j\Psi)\partial_i f \quad \text{for } f \in C_c^\infty(\mathbb{R}^{d_2}),$$

is essentially self-adjoint on $L^2(\mathbb{R}^{d_2}, \mu_2)$*.*

Proof:
We apply Theorem 3.3.1 with $A = \Sigma$ and $\rho = \mathrm{e}^{-\Psi}$ to show essential self-adjointness. We show that the assumptions there are satisfied for $p = \infty$: Due to **(Ψ1)**, the function $\mathrm{e}^{\Psi} = \frac{1}{\rho}$ is locally bounded. Remark 3.4.2 further yields that $\nabla\Psi$ is continuous and therefore $|\nabla\rho| = |\nabla\Psi \mathrm{e}^{-\Psi}|$ is locally bounded. The conditions on A are satisfied due to **(Σ1)** and **(Σ2)**.

This now directly implies the following:

Corollary 3.4.5. *The operator* $(S, \mathcal{C})$ *as in* (3.2.1) *is essentially self-adjoint, hence essentially m-dissipative on* X*.*

Proof:
Let $g = g_1 \otimes g_2 \in \mathcal{C}$ be a pure tensor. Then by Lemma 3.4.4, there is a sequence $(\tilde{f}_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^{d_2})$ such that $(I - \widetilde{S})\tilde{f}_n \to g_2$ in $L^2(\mathbb{R}^{d_2}, \mu_2)$ as $n \to \infty$. Define $f_n \in \mathcal{C}$ for each $n \in \mathbb{N}$ as $f_n \coloneqq g_1 \otimes \tilde{f}_n$. Then

$$\|(I - S)f_n - g\|_X = \|g_1 \otimes ((I - \widetilde{S})\tilde{f}_n - g_2)\|_X = \|g_1\|_{L^2(\mu_1)} \cdot \|(I - \widetilde{S})\tilde{f}_n - g_2\|_{L^2(\mu_2)},$$

which converges to zero as $n \to \infty$. By taking linear combinations, this shows that $(I - S)(\mathcal{C})$ is dense in $\mathcal{C}$ wrt. the X-norm. Since $\mathcal{C}$ is dense in X, $(S, \mathcal{C})$ is essentially m-dissipative and its closure $(S, D(S))$ generates a strongly continuous contraction semigroup. □

Since S is dissipative on $D_0 \coloneqq L_c^2(\mu_1) \otimes C_c^\infty(\mathbb{R}^{d_2}) \supseteq \mathcal{C}$, the operator (S, D_0) is essentially m-dissipative as well. We introduce the unitary transformations

$$U : H \to L^2(E, \mathrm{d}(x,y)), \qquad f \mapsto \sqrt{Z(\Psi)^{-1}}\, \mathrm{e}^{-\frac{1}{2}(\Phi+\Psi)} \qquad \text{and} \tag{3.4.1}$$

$$U_\Psi : L^2(\mu_2) \to L^2(\mathbb{R}^{d_2}, \mathrm{d}y), \qquad f \mapsto \sqrt{Z(\Psi)^{-1}}\, \mathrm{e}^{-\frac{1}{2}\Psi} \tag{3.4.2}$$

as well as the subspace $D_1 \coloneqq L_c^2(\mathbb{R}^{d_1}, \mathrm{d}x) \otimes U_\Psi C_c^\infty(\mathbb{R}^{d_2})$ of $L^2(E, \mathrm{d}(x,y))$. Note that due to **(Φ1)**, $\mathrm{e}^{-\Phi}$ is strictly positive and locally bounded, which implies that $L_c^2(\mathbb{R}^{d_1}, \mathrm{d}x)$ and $L_c^2(\mu_1)$ coincide. Hence we obtain $D_1 = UD_0$ and essential m-dissipativity of (L_0, D_1), where $L_0 = USU^{-1}$. For $f \in D_1$, we obtain the representation

$$L_0 f = \sum_{i,j=1}^{d} a_{ij}\partial_{y_j}\partial_{y_i} f - \frac{1}{4}(\nabla\Psi, \Sigma\nabla\Psi)f + \frac{1}{2}\sum_{i,j=1}^{d_2} a_{ij}\partial_j\partial_i\Psi f + \sum_{i,j=1}^{d} \partial_j a_{ij}(\tfrac{1}{2}\partial_i\Psi f + \partial_{y_i} f),$$

where the differential operators ∇, ∂_j and ∂_i are understood in the distributional sense.

For the next step, we have to move over to the complexified setting, where $((L_0)_{\mathbb{C}}, D_1 \times D_1)$ is still essentially m-dissipative by Lemma 1.2.27. Then we can perturb that complexified operator by the multiplication operator $(A_1, D_1 \times D_1)$ given by

$$A_1[f,g](x,y) := \mathrm{i}[\langle Q\nabla\Psi, x\rangle f(x,y), \langle Q\nabla\Psi, x\rangle g(x,y)] \qquad \text{for all } f,g \in D_1.$$

Clearly $(A_1, D_1 \times D_1)$ is well-defined in $L^2_{\mathbb{C}}(E, \mathrm{d}(x,y))$ and dissipative. For the sake of convenience, we omit the complexified notation from now on, since Lemma 1.2.27 allows us to retain any m-dissipativity result obtained by perturbation also for the real case, as long as the operator at the end of the perturbation process is a complexification of a real operator.

Proposition 3.4.6. *The operator (L_1, D_1) defined by $L_1 := L_0 + A_1$ is essentially m-dissipative on $L^2(E, \mathrm{d}(x,y))$.*

Proof:
We apply the perturbation argument Lemma 1.2.25. To this end, we introduce the complete orthogonal family of projections P_n defined via $P_n f := \xi_n f$, where ξ_n is given by $\xi_n(x,y) = \mathbb{1}_{[n-1,n)}(|x|)$. Each P_n leaves D_1 invariant and commutes with both L_0 and A_1. We have to show that each $P_n A_1$ is $P_n L_0$-bounded with relative bound zero. Let $f \in P_n D_1$. Then it holds that

$$\|\mathrm{i}\langle Q\nabla\Psi, x\rangle f\|_{L^2}^2 \leq n^2 |Q|_2^2 \int_E |\nabla\Psi|^2 f^2 \,\mathrm{d}(x,y).$$

Hence, it is enough to show that there are finite constants a, b such that

$$\int_E |\nabla\Psi|^2 f^2 \,\mathrm{d}(x,y) \leq a(L_0 f, f) + b\|f\|_{L^2} \qquad \text{for all } f \in D_1. \tag{3.4.3}$$

We get

$$\begin{aligned}
\int_E |\nabla\Psi|^2 f^2 \,\mathrm{d}(x,y) &\leq 4c_\Sigma \int_E \frac{1}{4}\langle \nabla\Psi, \Sigma\nabla\Psi\rangle f^2 \,\mathrm{d}(x,y) \\
&\leq 4c_\Sigma \left(\int_E \frac{1}{4}\langle \nabla\Psi, \Sigma\nabla\Psi\rangle f^2 \,\mathrm{d}(x,y) + \int_E \langle \nabla_y f, \Sigma\nabla_y f\rangle \,\mathrm{d}(x,y) \right) \\
&= 4c_\Sigma \left((-L_0 f, f)_{L^2} + \int_E \frac{1}{2} \sum_{i,j=1}^{d_2} (a_{ij}\partial_j\partial_i\Psi + \partial_j a_{ij}\partial_i\Psi) f^2 \,\mathrm{d}(x,y) \right)
\end{aligned}$$

Let $R_1 := 4c_\Sigma$ and recall that due to (Σ3) with $\beta = 0$, it holds that $|\partial_j a_{ij}| \leq 2M$. Using the Hölder and Young inequalities for $p = q = 2$, it follows that

$$\begin{aligned}
\frac{R_1}{2} \left| \int_E \sum_{i,j=1}^{d_2} \partial_j a_{ij} \partial_i \Psi f^2 \,\mathrm{d}(x,y) \right| &\leq \frac{R_1}{2} \sum_{i=1}^{d_2} \left\| \left(\sum_{j=1}^{d_2} \partial_j a_{ij} \right) f R_1^{1/2} \right\|_{L^2} \cdot \|R_1^{-1/2} \partial_i \Psi f\|_{L^2} \\
&\leq \frac{1}{4} \sum_{i=1}^{d_2} \int_E \left(R_1^2 \left(\sum_{j=1}^{d_2} \partial_j a_{ij} \right)^2 + (\partial_i \Psi)^2 \right) f^2 \,\mathrm{d}(x,y) \\
&\leq 16 c_\Sigma^2 M^2 d_2^3 \|f\|_{L^2}^2 + \frac{1}{4} \int_E |\nabla\Psi|^2 f^2 \,\mathrm{d}(x,y).
\end{aligned}$$

Now recall (Ψ3) and set $R_2 \coloneqq 8c_\Sigma M_\Sigma K$; then again with Hölder and Young, but for $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$, we get

$$\begin{aligned}
\frac{R_1}{2}\left|\int_E \sum_{i,j=1}^{d_2} a_{ij}\partial_j\partial_i\Psi f^2 \,\mathrm{d}(x,y)\right| &\le \frac{R_1}{2}\int_E |\Sigma|_2 \cdot |\nabla^2\Psi|_2 f^2 \,\mathrm{d}(x,y)\\
&\le 2c_\Sigma M_\Sigma K \int_E (1+|\nabla\Psi|^\alpha) R_2^{-\frac{\alpha}{2}} f^\alpha R_2^{\frac{\alpha}{2}} f^{2-\alpha}\,\mathrm{d}(x,y)\\
&\le \frac{R_2}{4}\left(\|f\|_{L^2}^2 + \||\nabla\Psi|^\alpha R_2^{-\frac{\alpha}{2}} f^\alpha\|_{L^{\frac{2}{\alpha}}} \cdot \|R_2^{\frac{\alpha}{2}} f^{2-\alpha}\|_{L^{\frac{2}{2-\alpha}}}\right)\\
&\le \frac{R_2}{4}\|f\|_{L^2}^2 + \frac{\alpha R_2}{8R_2}\||\nabla\Psi| f\|_{L^2}^2 + \frac{(2-\alpha)R_2^{\frac{2}{2-\alpha}}}{8}\|f\|_{L^2}^2\\
&\le \left(\frac{R_2}{4} + \frac{R_2^{\frac{2}{2-\alpha}}}{8}\right)\|f\|_{L^2}^2 + \frac{1}{4}\int_E |\nabla\Psi|^2 f^2\,\mathrm{d}(x,y).
\end{aligned}$$

Combining these three inequalities yields (3.4.3) with

$$a = -8c_\Sigma \quad \text{and} \quad b = 2R_1^2 M^2 d_2^3 + \frac{R_2}{2} + \frac{R_2^{\frac{2}{2-\alpha}}}{4}. \qquad \square$$

Since $C_c^\infty(\mathbb{R}^{d_1}) \otimes U_\Psi C_c^\infty(\mathbb{R}^{d_2})$ is dense in D_1 wrt. the graph norm of L_1, we obtain essential m-dissipativity of $(L_1, C_c^\infty(\mathbb{R}^{d_1}) \otimes U_\Psi C_c^\infty(\mathbb{R}^{d_2}))$ and therefore also of its dissipative extension (L_1, D_2) with $D_2 \coloneqq \mathcal{S}(\mathbb{R}^{d_1}) \otimes U_\Psi C_c^\infty(\mathbb{R}^{d_2}))$, where $\mathcal{S}(\mathbb{R}^{d_1})$ denotes the set of smooth functions of rapid decrease on $\mathbb{R}^{d_1}$. Applying Fourier transform in the x-component leaves D_2 invariant and shows that (L_2, D_2) is essentially m-dissipative, where $L_2 f = L_0 f + \langle Q\nabla\Psi, \nabla_x f\rangle$. At this point, L_2 is clearly a complexified real operator again, since the imaginary factor does not occur anymore, and we can therefore move back to the real setting.

Now we add the part depending on the potential Φ.

Proposition 3.4.7. *Let Σ satisfy (Σ3) with $\beta = 0$ and let Φ be Lipschitz-continuous. Then the operator (L', D_2) with $L' f = L_2 f - \langle Q^*\nabla\Phi, \nabla_y f\rangle$ is essentially m-dissipative on $L^2(E, \mathrm{d}(x,y))$.*

Proof:
It holds due to antisymmetry of $\langle Q\nabla\Psi, \nabla_x\rangle$ that

$$\begin{aligned}
\|\langle Q^*\nabla\Phi, \nabla_y f\rangle\|_{L^2}^2 \le \|Q^*\nabla\Phi\|_\infty^2 c_\Sigma \bigg(&(\nabla_y f, \Sigma\nabla_y f)_{L^2}\\
&+ \left(\frac{\langle\nabla\Psi, \Sigma\nabla\Psi\rangle}{4} f - \langle Q\nabla\Psi, \nabla_x f\rangle, f\right)_{L^2}\bigg),
\end{aligned}$$

which analogously to the proof of Proposition 3.4.6 again implies that the antisymmetric, hence dissipative operator $(\nabla\Phi\nabla_y, D_2)$ is L_2-bounded with bound zero. This shows the claim. $\square$

Now that the perturbation process is completed on $L^2(E; \mathrm{d}(x, y))$, we transform the resulting operator back into an operator on X. This yields:

Theorem 3.4.8. *Let Σ satisfy (Σ3) with $\beta = 0$ and Φ be Lipschitz-continuous. Then $(L, \mathcal{C})$ is essentially m-dissipative on H.*

Proof:
Denote by $H_c^{1,\infty}(\mathbb{R}^{d_1})$ the space of functions in $H^{1,\infty}(\mathbb{R}^{d_1}, \mathrm{d}x)$ with compact support and set $D' := H_c^{1,\infty}(\mathbb{R}^{d_1}) \otimes U_\Psi C_c^\infty(\mathbb{R}^{d_2})$. As (L', D') is dissipative and its closure extends (L', D_2), it is itself essentially m-dissipative. The unitary transformation U as defined in (3.4.1) satisfies $U^{-1}D' = D_3 := H_c^{1,\infty}(\mathbb{R}^{d_1}) \otimes C_c^\infty(\mathbb{R}^{d_2})$, and explicit calculation shows that $U^{-1}L'U = L$ on D_3. Using again the fact that unitary transformations preserve essential m-dissipativity, this means that (L, D_3) is essentially m-dissipative. It remains to approximate the first component: Let $f = g \otimes h \in D_3$ be a pure tensor. Since $g \in H_c^{1,\infty}(\mathbb{R}^{d_1}) \subseteq H^{1,2}(\mathbb{R}^{d_1}, \mathrm{d}x)$, there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^{d_1})$ such that $g_n \to g$ in $H^{1,2}(\mathbb{R}^{d_1}, \mathrm{d}x)$. Setting $f_n := g_n \otimes h$, we obtain

$$\|f - f_n\|_X^2 = Z(\Phi)^{-1}Z(\Psi)^{-1}\left(\int_{\mathbb{R}^{d_2}} h(y)^2\, \mathrm{e}^{-\Psi(y)}\, \mathrm{d}y\right)\int_{\mathbb{R}^{d_1}} (g(x) - g_n(x))^2 \mathrm{e}^{-\Phi(x)}\, \mathrm{d}x.$$

Since $\mathrm{e}^{-\Phi}$ is bounded due to **(Φ1)**, it follows that $f_n \to f$ in X. Furthermore, we have the representation

$$\begin{aligned} L(f - f_n) &= (g - g_n) \otimes (\mathrm{tr}[\Sigma\nabla^2 h] + \sum_{i=1}^{d_2} b_i\partial_i h) \\ &\quad - \sum_{j=1}^{d_2}(Q^*\nabla\Phi)_j(g - g_n) \otimes \partial_j h + \sum_{i=1}^{d_1} \partial_i(g - g_n) \otimes (Q\nabla\Psi)_i h. \end{aligned}$$

The first part vanishes in X as $n \to \infty$ with the same argument as before, and the last part also vanishes since $\nabla(g - g_n) \to 0$ in X and $(Q\nabla\Psi)_i \in L_c^2(\mathbb{R}^{d_2})$ for all $1 \le i \le d_1$. Since $\nabla\Phi$ is bounded as Φ is Lipschitz-continuous, we also obtain

$$\left\|\sum_{j=1}^{d_2}(Q^*\nabla\Phi)_j(g - g_n) \otimes \partial_j h\right\|_X \to 0$$

as $n \to \infty$. This together shows that f can be approximated in L-graph norm by functions from $\mathcal{C}$, which implies essential m-dissipativity of $(L, \mathcal{C})$. □

3.5 Essential m-dissipativity for locally Lipschitz coefficients

The goal of this section is to extend the previous result Theorem 3.4.8 to a more general coefficient matrix Σ as well as only locally Lipschitz-continuous potential Φ. In our approach, we find that relaxing assumptions on one of these objects will force slightly more restrictive conditions on the other up to some boundary cases as a trade-off of sorts. This relationship between Σ and Φ together with the necessary assumptions on Ψ is summarized in the following condition:

Assumption (C). Let (Ψ1)–(Ψ3) be fulfilled, and let Σ satisfy conditions (Σ1)–(Σ3) with constants $M \in [0,\infty)$, $\beta \in [0,1)$. Let Φ satisfy (Φ1). If $\beta > 0$, assume additionally the existence of constants $N < \infty$ and $\gamma \in (0, \frac{1}{\beta})$ such that

$$|\nabla\Phi(x)| \le N(1+|x|^\gamma) \qquad \text{for } \mu_1\text{-almost all } x \in \mathbb{R}^{d_1}.$$

Under these assumptions, we end up with the following result, which we prove afterwards by approximation through Lipschitz-continuous coefficients:

Theorem 3.5.1. *Let condition (C) be fulfilled. Then the linear operator (L, C) as defined in Definition 3.2.1 is essentially m-dissipative on X.*

Note that this is a strict generalization of Theorem 3.4.8, since Lipschitz-continuity of Φ implies that $|\nabla\Phi|$ is bounded and $\gamma = 1$ is always admissible independently of the value of β.

From now on, we will assume condition (C) to be satisfied. For the proof, we first need to find appropriate approximations for Σ and Φ. Without loss of generality, we assume $\Phi \ge 0$ (otherwise simply consider $\tilde{\Phi} := \Phi + a$, where $a \le \Phi(x)$ for all $x \in \mathbb{R}^{d_1}$).

Definition 3.5.2. For $n \in \mathbb{N}$ we define Σ_n via

$$\Sigma_n = (a_{ij,n})_{1\le i,j\le d_2}, \quad a_{ij,n}(y) := a_{ij}\left(\left(\frac{n}{|y|} \wedge 1\right) y\right).$$

For each $m \in \mathbb{N}$ choose some $\eta_m \in C_c^\infty(\mathbb{R}^{d_1})$ such that $\eta = 1$ on the ball $B_m(0)$ and set $\Phi_m = \eta_m\Phi$. Then set $\mu_{1,m} := \mathrm{e}^{-\Phi_m}\,\mathrm{d}x$, $X_m := L^2(E, \mu_{1,m} \otimes \mu_2)$ and define the linear operator $(L_{n,m}, C)$ on X_m via

$$L_{n,m}f = \operatorname{tr}[\Sigma_n H_y f] + \sum_{i=1}^{d}\sum_{j=1}^{d}(\partial_j a_{ij,n} - a_{ij,n}\partial_j\Psi)\partial_{y_i} f + \langle Q\nabla\Psi, \nabla_x f\rangle - \langle Q^*\nabla\Phi_m, \nabla_y f\rangle.$$

For these modifications, we obtain

Lemma 3.5.3. *Let $n, m \in \mathbb{N}$. Then Σ_n satisfies (Σ1)–(Σ3) with constants $\beta_n = 0$ and $M_n = (\sqrt{d_2}+1)Mn^\beta$. Φ_m fulfills (Φ1) and is Lipschitz-continuous. In particular, the operator $(L_{n,m}, C)$ is essentially m-dissipative on X_m, coincides with (L, C) on $B_m(0)\times B_n(0)$, and it holds that $\|\cdot\|_X \le \|\cdot\|_{X_m}$.*

Proof:
By definition, Σ_n coincides with Σ on $B_n(0)$, while for $y \notin B_n(0)$, the value is constant on the ray starting at 0 and passing through y, i.e. $\Sigma_n(y) = \Sigma(\frac{y}{|y|})$. Clearly, this implies that Σ_n is symmetric with bounded continuous coefficients and uniformly strictly elliptic with $c_{\Sigma_n} = c_\Sigma$. For $y \in B_n(0)$, (Σ3) implies $|\partial_k a_{ij,n}| \le 2Mn^\beta$ for all $1 \le k \le d_2$. For $y \in \mathbb{R}^{d_2} \setminus \overline{B_n(0)}$, the chain rule suggests

$$|\partial_k a_{ij,n}| = \left|\partial_k a_{ij}\left(\frac{ny}{|y|}\right) - \sum_{\ell=1}^{d_2} \partial_\ell a_{ij}\left(\frac{ny}{|y|}\right)\frac{ny_k y_\ell}{|y|^3}\right| \le (\sqrt{d_2}+1)Mn^\beta.$$

Hence, Σ_n satisfies (Σ3) with $\beta_n = 0$ and $M_n := (\sqrt{d_2}+1)Mn^\beta$, which means the a_{ij} are Lipschitz-continuous.

Clearly $0 \le \Phi_m \le \Phi$ and $\partial_i\Phi_m = \partial_i\eta_m\Phi + \eta_m\partial_i\Phi$ is bounded for all $1 \le i \le d_1$ since Φ is locally Lipschitz and η_m is smooth with compact support. Moreover, $\mathrm{e}^{-\Phi} \le \mathrm{e}^{-\Phi_m}$ implies $\|\cdot\|_X \le \|\cdot\|_{X_m}$. Now by Theorem 3.4.8, $(L_{n,m}, \mathcal{C})$ is essentially m-dissipative on X_m, and since $\Sigma_n = \Sigma$ on $B_n(0)$ and $\Phi_m = \Phi$ on $B_m(0)$, it holds that $L_{n,m}f = Lf$ for all $f \in \mathcal{C}$ on $B_m(0)\times B_n(0)$. □

Next we show that certain occurring terms can be relatively bounded in X_m by $(I - L_{n,m})$, where the bound is independent from the choice of n and m. This allows us to use dissipativity to perform the necessary estimates to verify the dense range condition for $(L, \mathcal{C})$ on X.

Lemma 3.5.4. *There is a constant $D_1 < \infty$ such that for all $n, m \in \mathbb{N}$ and Σ_n, Φ_m as defined above, the following holds for all $f \in \mathcal{C}$ and $1 \le j \le d_2$:*

$$\|\partial_j\Psi f\|_{X_m} \le D_1 n^\beta \|(I - L_{n,m})f\|_{X_m},$$
$$\|\partial_{y_j} f\|_{X_m} \le D_1 n^\beta \|(I - L_{n,m})f\|_{X_m}.$$

Proof:
Define the unitary transformations $U_m : X_m \to L^2(E, \mathrm{d}(x,v))$ analogously to (3.4.1), as well as the operator $L'_{n,m} = U_m L_{n,m} U_m^{-1}$, and let $f \in U_m\mathcal{C} = C_c^\infty(\mathbb{R}^{d_1}) \otimes U_\Psi C_c^\infty(\mathbb{R}^{d_2})$. Then

$$\begin{aligned} L'_{n,m}f &= \sum_{i,j=1}^{d} a_{ij,n}\partial_{y_j}\partial_{y_i} f - \frac{1}{4}(\nabla\Psi, \Sigma_n\nabla\Psi)f + \frac{1}{2}\sum_{i,j=1}^{d_2} a_{ij,n}\partial_j\partial_i\Psi f \\ &\quad + \sum_{i,j=1}^{d} \partial_j a_{ij,n}(\tfrac{1}{2}\partial_i\Psi f + \partial_{y_i} f) - \langle Q\nabla\Psi, \nabla_x f\rangle + \langle Q^*\nabla\Phi_m, \nabla_y f\rangle. \end{aligned}$$

Analogously to the proof of Proposition 3.4.6 and due to antisymmetry of $\langle Q\nabla\Psi, \nabla_x\rangle$ and $\langle Q^*\nabla\Phi_m, \nabla_y\rangle$ on $L^2(\mathrm{d}(x,y))$, it holds that

$$\begin{aligned} \|\partial_j\Psi U_m^{-1} f\|_{X_m}^2 = \|\partial_j\Psi f\|_{L^2(\mathrm{d}(x,v))}^2 &\le 4c_\Sigma \int_E \frac{1}{4}\langle\nabla\Psi, \Sigma_n\nabla\Psi\rangle f^2 + \langle\nabla_y f, \Sigma_n\nabla_y f\rangle \,\mathrm{d}(x,v) \\ &\le a(-L'_{n,m}f, f)_{L^2} + b_n\|f\|_{L^2}^2, \end{aligned} \tag{3.5.1}$$

where $a = -8c_\Sigma$ and

$$b_n = 2R_1^2(M_n)^2 d_2^3 + \frac{R_2}{2} + \frac{R_2^{\frac{2}{2-\alpha}}}{4} \le 8R_1^2M^2n^{2\beta}d_2^4 + \frac{R_2}{2} + \frac{R_2^{\frac{2}{2-\alpha}}}{4}.$$

Since by the Hölder and Young inequalities, together with dissipativity of $(L'_{n,m}, U_m\mathcal{C})$, it holds that

$$\begin{aligned} (-L'_{n,m}f, f)_{L^2} + \|f\|_{L^2}^2 = ((I - L'_{n,m})f, f)_{L^2} &\le \frac{1}{4}\left(\|(I - L'_{n,m})f\|_{L^2} + \|f\|_{L^2}\right)^2 \\ &\le \frac{1}{4}\left(2\|(I - L'_{n,m})f\|_{L^2}\right)^2 = \|(I - L'_{n,m})f\|_{L^2}^2, \end{aligned}$$

the estimate (3.5.1) implies the existence of some $D_1 < \infty$ such that

$$\|\partial_j \Psi U_m^{-1} f\|_{X_m} \leq D_1 n^\beta \|(I - L'_{n,m}) f\|_{L^2} = D_1 n^\beta \|(I - L_{n,m}) U_m^{-1} f\|_{X_m}.$$

For the second part, note that $\partial_{y_j} U_m^{-1} f = U_m^{-1} \partial_{y_j} f + \frac{1}{2} \partial_j \Psi U_m^{-1} f$ and that

$$\begin{aligned}\|U_m^{-1} \partial_{y_j} f\|_{X_m}^2 = (\partial_{y_j} f, \partial_{y_j} f)_{L^2}^2 &\leq c_\Sigma \int_E \langle \nabla_y f, \Sigma_n \nabla_y f\rangle + \frac{1}{4} \langle \nabla \Psi, \Sigma_n \nabla \Psi\rangle f^2 \,\mathrm{d}(x,y) \\ &\leq \frac{1}{4} \left(a(-L'_{n,m} f, f)_{L^2} + b_n \|f\|_{L^2}^2 \right) \leq \frac{1}{4} D_1^2 n^{2\beta} \|(I - L_{n,m}) U_m^{-1} f\|_{X_m}^2. \end{aligned}$$

□

Finally we verify essential m-dissipativity of $(L, \mathcal{C})$ on X by checking the dense range condition. More specifically, we show that every pure tensor $g \in \mathcal{C}$ can be approximated in X via $(I - L) f_n$, where $f_n \in \mathcal{C}$ for all $n \in \mathbb{N}$. This then easily extends to the entirety of $\mathcal{C}$ via linear combinations, and therefore proves that $(I - L, \mathcal{C})$ has dense range in X, since $\mathcal{C}$ is dense in X. Since $(L, \mathcal{C})$ is dissipative, it is thereby essentially m-dissipative.

In order to prove this approximation result, we introduce cutoff functions tailored to our coefficients and the specific tensor g that we wish to approximate:

Definition 3.5.5. Assume condition (C) and let $g \in \mathcal{C}$ be a pure tensor. Denote the support of g by $K_x \times K_y$, where K_x and K_y are compact sets in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. Fix some α satisfying $\beta < \alpha < \frac{1}{\gamma}$. For any $\delta > 0$, we define $\delta_x \coloneqq \delta^\alpha$ and $\delta_y = \delta$.

By Lemma 1.4.6, for each $\delta > 0$, define smooth cutoff functions $\phi_\delta \in C_c^\infty(\mathbb{R}^{d_1})$, $\psi_\delta \in C_c^\infty(\mathbb{R}^{d_2})$ with $0 \leq \phi_\delta, \psi_\delta \leq 1$, $\operatorname{supp}(\phi_\delta) \subseteq B_{\delta_x}(K_x)$, $\operatorname{supp}(\psi_\delta) \subseteq B_{\delta_y}(K_y)$, $\phi_\delta = 1$ on K_x and $\psi_\delta = 1$ on K_y, which satisfy

$$\|\partial^s \phi_\delta\|_\infty \leq C_\phi \delta_x^{-|s|} \quad \text{and} \quad \|\partial^w \psi_\delta\|_\infty \leq C_\psi \delta_y^{-|s|}$$

for all multi-indices $s \in \mathbb{N}^{d_1}$, $w \in \mathbb{N}^{d_2}$ with $|s|, |w| \leq 2$. The constants C_ϕ, C_ψ are independent of δ.

Further define $\chi_\delta \in \mathcal{C}$ via

$$\chi_\delta(x,y) \coloneqq \phi_\delta(x) \psi_\delta(y).$$

Finally, for any $f \in \mathcal{C}$, set $f_\delta \in \mathcal{C}$ as $f_\delta \coloneqq \chi_\delta f$.

Without loss of generality, we consider δ and hence δ^α sufficiently large such that $\operatorname{supp}(\phi_\delta) \subseteq B_{2\delta^\alpha}(0)$, $\operatorname{supp}(\psi_\delta) \subseteq B_{2\delta}(0)$ and that there are $n, m \in \mathbb{N}$ that satisfy

$$\operatorname{supp}(\phi_\delta) \times \operatorname{supp}(\psi_\delta) \subseteq B_m(0) \times B_n(0) \subseteq B_{2\delta^\alpha}(0) \times B_{2\delta}(0). \tag{3.5.2}$$

The following then holds:

Lemma 3.5.6. *Let $g \in C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\mathbb{R}^d)$ and ϕ, ψ as above. Then there is a constant $D_2 < \infty$ and a function $\rho : \mathbb{R} \to \mathbb{R}$ satisfying $\rho(s) \to 0$ as $s \to \infty$, such that for any δ, n and m satisfying* (3.5.2),

$$\|(I-L)f_\delta - g\|_X \leq \|(I-L_{n,m})f - g\|_{X_m} + D_2 \cdot \rho(\delta)\|(I-L_{n,m})f\|_{X_m}$$

holds for all $f \in \mathcal{C}$.

Proof:
By the product rule,

$$\begin{aligned}
\|(I-L)f_\delta - g\|_X \leq \|\chi_\delta((I-L)f-g)\|_X &+ \sum_{i,j=1}^{d_2} \|\phi_\delta(x)a_{ij}(y)\partial_j\partial_i\psi_\delta(y)f\|_X \\
&+ 2\sum_{i,j=1}^{d_2} \|\phi_\delta(x)a_{ij}(y)\partial_i\psi_\delta(y)\partial_{y_j}f\|_X + \sum_{i,j=1}^{d_2} \|\phi_\delta(x)\partial_j a_{ij}(y)\partial_i\psi_\delta(y)f\|_X \\
&+ \sum_{i,j=1}^{d_2} \|\phi_\delta(x)a_{ij}(y)\partial_j\Psi(y)\partial_i\psi_\delta(y)f\|_X \\
&+ \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} \|Q_{ij}\partial_i\phi_\delta(x)\partial_j\Psi(y)\psi_\delta(y)f\|_X \\
&+ \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} \|Q_{ij}\partial_i\Phi(x)\phi_\delta(x)\partial_i\psi_\delta(y)f\|_X.
\end{aligned}$$

Due to the choice of n and m, every $\|\cdot\|_X$ on the right hand side can be replaced with $\|\cdot\|_{X_m}$, a_{ij} by $a_{ij,n}$, and Φ by Φ_m, hence L by $L_{n,m}$.

Using this, we now give estimates for each summand of the right hand side, in their order of appearance:

(1) $\|\chi_\delta((I-L)f-g)\|_X \leq \|(I-L_{n,m})f-g\|_{X_m}$,

(2) $\|\phi_\delta(x)a_{ij}(y)\partial_j\partial_i\psi_\delta(y)f\|_{X_m} \leq M_\Sigma C_\psi \delta^{-2}\|f\|_{X_m}$,

(3) $\|\phi_\delta(x)a_{ij}(y)\partial_i\psi_\delta(y)\partial_{y_j}f\|_{X_m} \leq M_\Sigma C_\psi \delta^{-1}\|\partial_{y_j}f\|_{X_m}$,

(4) $\|\phi_\delta(x)\partial_j a_{ij,n}(y)\partial_i\psi_\delta(y)f\|_{X_m} \leq \max\{B_\Sigma, 2\sqrt{d_2}M \cdot (2\delta)^{\beta\vee 0}\}C_\psi\delta^{-1}\|f\|_{X_m}$,

(5) $\|\phi_\delta(x)a_{ij}(y)\partial_j\Psi(y)\partial_i\psi_\delta(y)f\|_{X_m} \leq M_\Sigma C_\psi\delta^{-1}\|\partial_j\Psi(y)f\|_{X_m}$,

(6) $\|Q_{ij}\partial_i\phi_\delta(x)\partial_j\Psi(y)\psi_\delta(y)f\|_{X_m} \leq |Q_{ij}|C_\phi\delta^{-\alpha}\|\partial_j\Psi(y)f\|_{X_m}$,

(7) $\|Q_{ij}\partial_i\Phi(x)\phi_\delta(x)\partial_i\psi_\delta(y)f\|_{X_m} \leq |Q_{ij}|N(1+(2\delta^\alpha)^\gamma)C_\psi\delta^{-1}\|f\|_{X_m}$,

where the last inequality is due to $|\partial_i\Phi(x)| \leq N(1+|x|^\gamma)$ for all $x \in \mathbb{R}^d$ and the support of the cutoff as in (3.5.2).

Note that due to dissipativity of $(L_{n,m}, C)$ on X_m, $\|f\|_{X_m} \leq \|(I-L_{n,m})f\|_{X_m}$ holds. By application of Lemma 3.5.4, the terms $\|\partial_{y_j} f\|_{X_m}$ and $\|\partial_j \Psi f\|_{X_m}$ can also be bounded by a factor of $\|(I-L_{n,m})f\|_{X_m}$. Together, this shows the existence of D_2 independent of n, m, such that

$$\|(I-L)f_\delta - g\|_X \leq \|(I-L_{n,m})f - g\|_{X_m} + D_2 \cdot \rho(\delta)\|(I-L_{n,m})f\|_{X_m}$$

where

$$\rho(\delta) := \delta^{-2} + 2 \cdot 2^\beta \delta^{\beta-1} + 2^{\beta\vee 0}\delta^{(\beta\vee 0)-1} + 2^\beta\delta^{\beta-\alpha} + \delta^{-1} + 2^\gamma \delta^{\alpha\gamma-1}.$$

Clearly $\rho(\delta) \to 0$ as $\delta \to \infty$ due to $\beta < 1$ and the definition of α. □

Finally, we are able to prove Theorem 3.5.1 completely as stated:

Proof (of Theorem 3.5.1):
Fix some pure tensor $0 \neq g \in C$. We show that for each $\varepsilon > 0$, we can find some $f_\delta \in C$ such that

$$\|(I-L)f_\delta - g\|_X < \varepsilon.$$

Choose $\delta > 0$ large enough such that $\rho(\delta) < \frac{\varepsilon}{4D_2\|g\|_X}$ (where ρ, D_2 are provided by Lemma 3.5.6), and that there exist n, m satisfying (3.5.2), which we fix.

Due to Theorem 3.4.8, there is an $f \in C$ such that $\|(I-L_{n,m})f - g\|_{X_m} < \min\{\frac{\varepsilon}{2}, \|g\|_X\}$. For this f and the chosen δ, define f_δ as in Definition 3.5.5. Note that due to the choice of the cutoffs, it holds that $\|g\|_X = \|g\|_{X_m}$, therefore application of Lemma 3.5.6 yields

$$\begin{aligned}\|(I-L)f_\delta - g\|_X &\leq \|(I-L_{n,m})f - g\|_{X_m} + D_2\rho(\delta)\|(I-L_{n,m})f\|_{X_m} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4\|g\|_X}(\|(I-L_{n,m})f - g\|_{X_m} + \|g\|_{X_m}) \\ &< \frac{\varepsilon}{2} + \frac{2\varepsilon\|g\|_X}{4\|g\|_X} = \varepsilon.\end{aligned}$$

Since C is dense in X, this proves that the dissipative operator (L, C) has dense range on X and is therefore essentially m-dissipative. □

Corollary 3.5.7. *Let (C) hold. Then $(\widehat{L}, C)$ is essentially m-dissipative on X and its closure $(\widehat{L}, D(\widehat{L}))$ generates the adjoint semigroup $(T_t^*)_{t\geq 0}$ of $(T_t)_{t\geq 0}$.*

Proof:
The first statement is immediate since we obtain $\widehat{L}$ from L by considering $-Q$ instead of Q. By definition, it holds that $(Lf, f)_X = (f, \widehat{L}g)_X$ for all $f, g \in C$, and since C is dense in $D(L)$ with respect to L-graph norm, the same holds for all $f \in D(L), g \in C$. This means that on C, $\widehat{L}$ coincides with the adjoint operator $(L^*, D(L^*))$ of $(L, D(L))$, which generates the adjoint semigroup $(T_t^*)_{t\geq 0}$ and is therefore m-dissipative. We therefore obtain that $(L^*, D(L^*))$ is the closure of $(\widehat{L}, C)$. □

3.6 Properties of the associated semigroup

In this section, we consider the semigroup generated by $(L, D(L))$ in the special case that μ_1 is also a probability measure, so that X becomes a probability space. First, we show that $(L, \mathcal{C})$ is an abstract diffusion operator on X (see Definition 1.3.7):

Lemma 3.6.1. *Let (C) be satisfied. Then $(L, \mathcal{C})$ is an abstract diffusion operator on X.*

Proof:
First, we have to compute the Carré du champ operator Γ. Since for $u, v \in \mathcal{C}$,

$$\begin{aligned} L(uv) = &\sum_{i,j=1}^{d_2} a_{ij}(u\partial_{y_j}\partial_{y_i}v + \partial_{y_j}u\partial_{y_i}v + \partial_{y_j}v\partial_{y_i}u + v\partial_{y_j}\partial_{y_i}u) \\ &+ \sum_{i=1}^{d_2} b_i(u\partial_{y_i}v + v\partial_{y_i}u) + \langle Q\nabla\Psi, u\nabla_x v + v\nabla_x u\rangle - \langle Q^*\nabla\Phi, u\nabla_y v + v\nabla_y u\rangle, \end{aligned}$$

it is clear that

$$\Gamma(u,v) = \langle \nabla_y u, \Sigma\nabla_y v\rangle \qquad \text{for } u, v \in \mathcal{C}.$$

Due to positive definiteness of Σ, this implies the second condition in Definition 1.3.7. Now let $m \in \mathbb{N}$, $u_1, \dots, u_m \in \mathcal{C}$ and $\varphi \in C^\infty(\mathbb{R}^m)$ with $\varphi(0) = 0$. We set $g \coloneqq \varphi \circ (u_1, \dots, u_m) : E \to \mathbb{R}$ and $K \coloneqq \bigcup_{k=1,\dots,m} \operatorname{supp}(u_k)$. Then $u_k = 0$ on K^c for all $k = 1, \dots, m$ and therefore $g = 0$ on K^c, so $g \in \mathcal{C}$. The chain rule implies

$$\begin{aligned} \partial_{y_i} g &= \sum_{k=1}^m \partial_k\varphi(u_1,\dots,u_m)\frac{\partial u_k}{\partial y_i} \qquad \text{and} \\ \partial_{y_j}\partial_{y_i} g &= \sum_{k,\ell=1}^m \partial_\ell\partial_k\varphi(u_1,\dots,u_m)\frac{\partial u_\ell}{\partial y_j}\frac{\partial u_k}{\partial y_i} + \sum_{k=1}^m \partial_k\varphi(u_1,\dots,u_m)\frac{\partial^2 u_k}{\partial y_j\partial y_i}, \end{aligned}$$

which yields

$$\begin{aligned} Lg &= \sum_{k=1}^m \partial_k\varphi(u_1,\dots,u_m)L(u_k) + \sum_{k,\ell=1}^m \partial_\ell\partial_k\varphi(u_1,\dots,u_m)\sum_{i,j=1}^{d_2} a_{ij}\frac{\partial u_\ell}{\partial y_j}\frac{\partial u_k}{\partial y_i} \\ &= \sum_{k=1}^m \partial_k\varphi(u_1,\dots,u_m)L(u_k) + \sum_{k,\ell=1}^m \partial_\ell\partial_k\varphi(u_1,\dots,u_m)\langle\nabla_y u_k, \Sigma\nabla_y u_\ell\rangle. \end{aligned}$$

This shows that $(L, \mathcal{C})$ is indeed an abstract diffusion operator on X. □

We now immediately obtain the following:

Theorem 3.6.2. *Let (C) hold and μ be a probability measure on E. Then the semigroup $(T_t)_{t\geq 0}$ generated by the closure $(L, D(L))$ of $(L, \mathcal{C})$ is contractive, sub-Markovian, μ-invariant and conservative.*

Proof:
Lumer-Phillips (Theorem 1.2.19) implies that $(L, D(L))$ generates an sccs $(T_t)_{t\geq 0}$ on X. Since μ is an invariant measure for $(L, \mathcal{C})$ (see Proposition 3.4.3), Lemma 1.3.3 (iii) implies that it is invariant for $(T_t)_{t\geq 0}$. Due to Lemma 3.6.1, we can therefore apply Lemma 1.3.8 to show the sub-Markov property. By the same argument, $(T_t^*)_{t\geq 0}$ is also μ-invariant, so by Lemma 1.3.3 (i) we obtain conservativity of $(T_t)_{t\geq 0}$. □

4 Hypocoercivity for degenerate diffusion semigroups

We now embed the differential operator L as defined in Definition 3.2.1 and its associated semigroup into the abstract hypocoercivity framework. Recall the setting defined in Definition 3.2.2. We assume the following throughout the remaining considerations:

Assumption (H). Condition (C) holds and μ_1 is a probability measure. Moreover, QQ^* is an invertible $d_1 \times d_1$-matrix.

Then $(L, \mathcal{C})$ is essentially m-dissipative and we can use the results from Section 3.6.

Definition 4.0.1. We define

$$\begin{aligned} H &:= \{f \in L^2(E;\mu) \,:\, \mu(f) = 0\} \subseteq X, \\ H_1 &:= \{f \in H \,:\, f(x,y) \text{ does not depend on } y\}, \\ H_2 &:= H_1^{\perp}, \\ Pf &:= \int_{\mathbb{R}^{d_2}} f(x,y)\,\mu_2(\mathrm{d}y) \quad \text{ for all } f \in H, \text{ and} \\ \mathcal{D} &:= \{f \in C^\infty(E) \,:\, \nabla f \text{ has compact support},\ \mu(f) = 0\} \subseteq H. \end{aligned}$$

H is a separable Hilbert space which inherits its inner product and norm from X. We may consider any $f \in H_1$ as an element of $L^2(\mu_1)$ with $\mu_1(f) = 0$.

We have the following properties:

Lemma 4.0.2.

(i) $P : H \to H_1$ *is an orthogonal projection with range* H_1.

(ii) Each $f \in \mathcal{D}$ *can be expressed as* $g - \mu(g)$ *for some* $g \in C_c^\infty(E)$.

(iii) $\mathcal{D}$ *is dense in* H *and* $P(\mathcal{D}) \subseteq \mathcal{D}$.

(iv) $(L, \mathcal{D})$ *is well-defined and essentially m-dissipative on* H.

Proof:

(i) It is clear that $P^2 = P$, and Jensen's inequality implies

$$\|Pf\|^2 = \int_E \left(\int_{\mathbb{R}^{d_2}} f(x,y)\, \mu_2(\mathrm{d}y) \right)^2 \mathrm{d}\mu \le \int_E \int_{\mathbb{R}^{d_2}} f(x,y)^2\, \mu_2(\mathrm{d}y)\, \mathrm{d}\mu = \|f\|^2$$

for all $f \in H$, so P is an orthogonal projection and therefore so is $I - P$. Clearly $\mathcal{R}(P) \subseteq H_1$, and for all $f \in H_1$, $g \in H$ we get

$$(f, (I-P)g)_H = \int_{\mathbb{R}^{d_1}} f(x) \left(\int_{\mathbb{R}^{d_2}} g(x,y) - Pg(x)\, \mu_2(\mathrm{d}y) \right) \mu_1(\mathrm{d}x) = 0,$$

which implies $\mathcal{R}(I-P) \subseteq H_2$ and therefore $\mathcal{R}(P) = H_1$.

(ii) Let $f \in \mathcal{D}$. Since ∇f has compact support, f is constant outside of a compact set with value $c \in \mathbb{R}$. Set $g \coloneqq f - c$, then $g \in C_c^\infty(E)$ and $\mu(g) = \mu(f) - c = -c$, which means $f = g - \mu(g)$.

(iii) Let $h \in H$, then there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ such that $g_n \to h$ in X. Set $f_n \coloneqq g_n - \mu(g_n)$, then $f_n \in \mathcal{D}$ and $\|h - f_n\|_H \le \|h - g_n\|_X + |\mu(g_n)|$ for each $n \in \mathbb{N}$. Since

$$|\mu(g_n)| = |\mu(g_n) - \mu(h)| \le \|g_n - h\|_{L^1(\mu)} \le \|g_n - h\|_X,$$

$(f_n)_{n\in\mathbb{N}}$ converges to h in H as $n \to \infty$.

Now let $f \in \mathcal{D}$. Since integration preserves smoothness, $Pf \in C^\infty(E)$, and clearly $\mu(Pf) = 0$. Since f is constant outside of a compact set, the same holds for Pf, which implies that ∇Pf has compact support.

(iv) Approximation by cutoff functions as in Remark 1.4.8 shows that 1 is in the domain of $(L, D(L))$, $(S, D(S))$ and $(A, D(A))$, and that all three operators act trivially on constants. For $f \in \mathcal{D}$, point (ii) then shows $Lf = Lg$, and the same holds for S and A, which shows dissipativity of these operators on $\mathcal{D}$. Moreover, as in Proposition 3.4.3, it follows that $\mu(Sf) = (f, S1)_X = (f, 0)_X = 0$ as well as $\mu(Af) = (Af, 1)_X = (f, A1) = 0$, and thus $\mu(Lf) = 0$, so $(L, \mathcal{D})$ is indeed an operator on H.

Let $h \in H \subseteq X$. Then due to Theorem 3.5.1, there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ such that $(I - L)g_n \to h$ in X as $n \to \infty$. Define $f_n \coloneqq g - \mu(g)$, then $f_n \in \mathcal{D}$ for all $n \in \mathbb{N}$ and

$$(I - L)f_n = g_n - \mu(g_n) - Lg_n \to h - \mu(h) = h \qquad \text{as } n \to \infty.$$

This proves essential m-dissipativity of $(L, \mathcal{D})$ on H. □

Definition 4.0.3. From now on, we define the operators $(L, D(L))$, $(S, D(S))$ as well as $(A, D(A))$ on H as the closures of $(L, \mathcal{D})$, $(S, \mathcal{D})$ and $(A, \mathcal{D})$, respectively. If we need to distinguish them from their counterparts on X, we denote them with superscript H, e.g. $(L^H, D(L^H))$.

For any $f \in H$, we write f_P for Pf considered as an element of $L^2(\mu_1)$.

Now we are in the setting described in the start of Section 2.1, and we immediately obtain that **(D1)** is satisfied. Also, we get the following without additional constraints:

Proposition 4.0.4. *Condition* **(D2)** *is satisfied.*

Proof:
Let $f \in H$, then $f_P \in L^2(\mu_1)$ with $\mu_1(f_P) = 0$. As in the proof of Lemma 4.0.2, there is a sequence $(h_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{R}^{d_1})$ with $h_n \to f_P$ in $L^2(\mu_1)$ such that ∇h_n has compact support and $\mu_1(h_n) = 0$ for all $n \in \mathbb{N}$.

Define $g_n \in \mathcal{D}$ by $g_n(x, y) := h_n(x)$ for all $(x, y) \in E$. Then $Sg_n = 0$ for all $n \in \mathbb{N}$ and $g_n \to Pf$ in H as $n \to \infty$. Since $(S, D(S))$ is closed, this proves $Pf \in D(S)$ with $SPf = 0$. □

Proposition 4.0.5. *Condition* **(D3)** *is satisfied. Moreover, even $AP(\mathcal{D}) \subseteq D(A)$ and therefore $(AP)^*AP = -PA^2P$ on $\mathcal{D}$.*

Proof:
Let $f \in \mathcal{D}$, then $Pf \subseteq \mathcal{D} \subseteq D(A)$ due to Lemma 4.0.2 (iii). Since Pf only depends on the first component, we get

$$APf(x, y) = -\langle Q\nabla\Psi(y), \nabla_x(Pf)(x)\rangle = -\sum_{i=1}^{d_1}\sum_{j=1}^{d_2} Q_{ij}\partial_j\Psi(y)\partial_{x_i}(Pf)(x). \tag{4.0.1}$$

Fix i and j. Since $\partial_{x_i}(Pf)$ has compact support, we can apply integration by parts to obtain

$$\begin{aligned} P(\partial_j\Psi\partial_{x_i}(Pf))(x) &= \int_{\mathbb{R}^{d_2}} \partial_j\Psi\partial_{x_i}(Pf)(x)\,\mathrm{d}\mu_2 = -\int_{\mathbb{R}^{d_2}} \partial_{x_i}(Pf)(x)\partial_j(\mathrm{e}^{-\Psi(y)})\,\mathrm{d}y \\ &= \int_{\mathbb{R}^{d_2}} \partial_{y_j}\partial_{x_i}(Pf)(x)\,\mathrm{d}\mu_2 = 0 \end{aligned}$$

for all $x \in \mathbb{R}^{d_1}$, which together with (4.0.1) shows $PAP \equiv 0$ on $\mathcal{D}$.

Set $h = \partial_{x_i}(Pf) \otimes \partial_j\Psi \in H$. Due to Remark 3.4.2, $\partial_j\Psi$ is in $H^{1,4}(\mathbb{R}^{d_2}; \mu_2)$. Therefore, due to Theorem 1.6.3, there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^{d_2})$ which converges to $\partial_j\Psi$ in $H^{1,4}(\mu_2)$-norm. The same holds for the sequence $(\varphi_n - \mu_2(\varphi_n))_{n\in\mathbb{N}}$, since $\mu_2(\varphi_n)$ converges to $\mu_2(\partial_j\Psi) = 0$. Now define

$$g_n(x, y) := (\varphi_n(y) - \mu_2(\varphi_n)) \cdot \partial_{x_i}(Pf)(x) \qquad \text{for each } n \in \mathbb{N},$$

then $(g_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}$ and approximates h in H. Moreover, we can apply A to get

$$Ag_n = \langle Q^*\nabla_x\Phi, \nabla_y(\varphi_n)\partial_{x_i}(Pf)\rangle - \langle Q\nabla_y\Psi, (\varphi_n - \mu_2(\varphi_n))\nabla_x(\partial_{x_i}(Pf))\rangle.$$

Let $k \in \{1, \dots, d_2\}$. By construction, $\partial_{y_k}\varphi_n$ converges to $\partial_k\partial_j\Psi$ in $L^4(\mu_2)$, hence also in $L^2(\mu_2)$. Since $\partial_k\Psi$ is in $L^4(\mu_2)$ and $(\varphi_n - \mu_2(\varphi_n))$ converges to $\partial_j\Psi$ in $L^4(\mu_2)$, we obtain that Ag_n converges in H to

$$\langle Q^*\nabla_x\Phi, \nabla_y(\partial_j\Psi)\partial_{x_i}(Pf)\rangle - \langle Q\nabla_y\Psi, (\partial_j\Psi)\nabla_x(\partial_{x_i}(Pf))\rangle.$$

Note that this expression is well-defined, since $\nabla_x\Phi$ is locally bounded and $\partial_{x_i}(Pf)$ has compact support on $\mathbb{R}^{d_1}$.

Since $(A, D(A))$ is closed, this shows that $h \in D(A)$, which implies $APf \in D(A)$ due to (4.0.1). Since A is antisymmetric, Lemma 1.1.4 implies $APf \in D((AP)^*)$ with $(AP)^*APf = -PA^2Pf$. □

So far, no new assumptions had to be made to ensure compliance with either hypocoercivity framework. This changes now that we have verified all required data conditions. As seen in the above proof, the general case can get quite technical, so we split the remaining work into two parts. In the first part, we fix a popular choice for Ψ, which greatly simplifies many expressions and leads to strong hypocoercivity. In the second part, we keep the setting as general as possible, and prove weak hypocoercivity.

To finish this preliminary part, we introduce a final condition on Φ which is require in either case:

Assumption (Φ3). We assume that $\Phi \in C^2(\mathbb{R}^{d_1})$ and that there is a constant $C < \infty$ such that

$$|\nabla^2\Phi(x)| \le C(1 + |\nabla\Phi(x)|) \quad \text{for all } x \in \mathbb{R}^{d_1}.$$

Remark 4.0.6. Note that if **(Φ3)** holds, then Remark 1.6.6 implies that both $|\nabla\Phi|$ and $|\nabla^2\Phi|$ are in $L^2(\mu_1)$.

4.1 Weak hypocoercivity for generalized Langevin dynamics with multiplicative noise

While we don't assume that Ψ is the standard Gaussian measure, we still require a bit more structure:

Assumption (Ψ4). There is some $\psi \in C^2(\mathbb{R})$ such that $\Psi(y) = \psi(|y|^2)$.

Remark 4.1.1. If $\Psi(y) = \psi(|\Lambda y - a|^2)$ for some invertible $\Lambda \in \mathbb{R}^{d_2 \times d_2}$ and $a \in \mathbb{R}^{d_2}$, then we can use transformations to modify Ψ to suffice **(Ψ4)**. Indeed, we consider the new potential $\overline{\Psi}(y) := \Psi(\Lambda^{-1}(y + a)) = \psi(|y|^2)$, with $\overline{\mu_2}$ and $\overline{\mu}$ defined accordingly. Let $J : L^2(\mu) \to L^2(\overline{\mu})$ be given by $Jf(x, y) := \sqrt{|\det(\Lambda)|}^{-1} f(x, \Lambda^{-1}(y + a))$. Then J is a unitary transformation which leaves $\mathcal{D}$ invariant, and we define the operator $(\overline{L}, \mathcal{D})$ by $\overline{L} := JLJ^{-1}$. This operator has the same structure as L, except with $\overline{Q} = Q\Lambda^*$ and $\overline{\Sigma}(y) = \Lambda\Sigma(\Lambda^{-1}(y + a))\Lambda^*$. So wlog we can assume that already $\Psi(y) = \psi(|y|^2)$.

We can now summarize the results for the antisymmetric part A proved in [GW19]:

Lemma 4.1.2. *If* (Φ3) *and* (Ψ4) *are satisfied, then* $(G, \mathcal{D}) := (PA^2P, \mathcal{D})$*, which is given for all* $f \in \mathcal{D}$ *by*

$$Gf = \frac{\mu_2(|\nabla\Psi|^2)}{d_2} \sum_{i,j=1}^{d_1} (QQ^*)_{ij}(\partial_{x_j}\partial_{x_i} - \partial_{x_j}\Phi\,\partial_{x_i})Pf,$$

is essentially self-adjoint and its closure $(G, D(G))$ *generates a sub-Markovian strongly continuous semigroup on* H*. Moreover, there is a constant* c_A *only depending on the choice of* Φ *and* Ψ *such that*

$$\|(BA)^* g\|_H \le c_A \|g\|_H \qquad \text{for all } g \in (I-G)\mathcal{D}.$$

This allows us to use Lemma 2.4.4 to prove **(WH1)**. Indeed, the inequality for the antisymmetric part follows immediately from the previous Lemma, so it remains to show the first inequality. For this, we introduce the final assumption on Ψ:

Assumption (Ψ5). Ψ is three times weakly differentiable and for any $1 \le i, j, k \le d_2$, it holds that $\partial_i\partial_j\partial_k\Psi \in L^2(\mu_2)$.

Note that in light of Remark 3.4.2, this holds in particular if the third order partial derivatives are dominated by a multiple of $(1 + |\nabla^2\Psi|^\beta)$ for any $0 < \beta < \infty$. We also need to assume some additional integrability on $\nabla\Sigma$:

Assumption (Σ4). There is some $1 < p_\Sigma \le \infty$ such that $|\nabla\Sigma| \in L^{2p_\Sigma}(\mu_2)$.

Lemma 4.1.3. *Let* (Ψ4), (Ψ5) *and* (Σ4) *be satisfied. Then the operator* $T : \mathcal{D} \to L^2(\mu)$ *with*

$$Tf(x,y) := \sum_{i=1}^{d_1} w_i(y) \cdot (\partial_{x_i}Pf)(x) \qquad \text{for } w_i : \mathbb{R}^{d_2} \to \mathbb{R} \quad \text{defined by}$$

$$w_i := \sum_{j,k,\ell=1}^{d_2} Q_{ij}\left(\partial_k a_{k\ell}(\partial_\ell\partial_j\Psi) + a_{k\ell}(\partial_k\partial_\ell\partial_j\Psi) - a_{k\ell}(\partial_\ell\partial_j\Psi)(\partial_k\Psi)\right)$$

is well-defined and there is some $R < \infty$ *such that* $\|Tf\|^2_{L^2(\mu)} \le R\|\nabla f_P\|^2_{L^2(\mu_1)}$ *for all* $f \in \mathcal{D}$.

Proof:
Let $f \in \mathcal{D}$. Since

$$|Tf| \le \sum_{i=1}^{d_1} |w_i| \cdot |(\partial_{x_i}Pf)| \le \left(\sum_{i=1}^{d_1} |w_i|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{d_1} |\partial_{x_i}(Pf)|^2\right)^{\frac{1}{2}},$$

it follows that

$$\|Tf\|^2_{L^2(\mu)} \le \int_{\mathbb{R}^{d_2}} \sum_{i=1}^{d_1} |w_i|^2 \,\mathrm{d}\mu_2 \cdot \int_{\mathbb{R}^{d_1}} \sum_{i=1}^{d_1} |\partial_{x_i}(Pf)|^2 \,\mathrm{d}\mu_1 = \sum_{i=1}^{d_1} \|w_i\|^2_{L^2(\mu_2)} \|\nabla f_P\|^2_{L^2(\mu_1)}$$

For each $1 \le i \le d_1$, we get

$$\|w_i\|_{L^2(\mu_2)} \le |Q| \sum_{j,k,\ell=1}^{d_2} \|\partial_k a_{k\ell}(\partial_\ell \partial_j \Psi)\|_{L^2} + \|a_{k\ell}(\partial_k \partial_\ell \partial_j \Psi)\|_{L^2} + \|a_{k\ell}(\partial_\ell \partial_j \Psi)(\partial_k \Psi)\|_{L^2}.$$

Let $1 \le q_\Sigma < \infty$ be such that $\frac{1}{p_\Sigma} + \frac{1}{q_\Sigma} = 1$. Due to Remark 3.4.2, we have $|\nabla^2 \Psi| \in L^{2q_\Sigma}(\mu_2)$. Then by (Σ4), it holds that

$$\|\partial_k a_{k\ell}(\partial_\ell \partial_j \Psi)\|_{L^2} \le \|(\partial_k a_{k\ell})^2\|_{L^{p_\Sigma}} \|(\partial_\ell \partial_j \Psi)^2\|_{L^{q_\Sigma}} < \infty.$$

Boundedness of Σ together with (Ψ5) yields

$$\|a_{k\ell}(\partial_k \partial_\ell \partial_j \Psi)\|_{L^2} \le M_\Sigma \|\partial_k \partial_\ell \partial_j \Psi\|_{L^2} < \infty,$$

and by (Ψ3) and again Remark 3.4.2, we also get

$$\|a_{k\ell}(\partial_\ell \partial_j \Psi)(\partial_k \Psi)\|_{L^2} \le K M_\Sigma \left(\|\partial_k \Psi\|_{L^2} + \||\nabla \Psi|^{1+\alpha}\|_{L^2} \right) < \infty.$$

This shows that each w_i is in $L^2(\mu_2)$ and therefore $R := \sum_{i=1}^{d_1} \|w_i\|^2_{L^2(\mu_2)} < \infty$. □

Proposition 4.1.4. *Let (Ψ4), (Ψ5) and (Σ4) be satisfied. Then $AP(\mathcal{D}) \subseteq D(S^*)$ and $S^* APf = Tf$ for all $f \in \mathcal{D}$.*

Proof:
Let $f \in \mathcal{D}$. We have to show that $Tf \in H$ and $(Sh, APf)_H = (h, Tf)_H$ for all $h \in D(S)$. Fix one such h, then by definition of $(S, D(S))$ there is a sequence $(h_n)_{n \in \mathbb{N}}$ in $\mathcal{D}$ such that $h_n \to h$ and $Sh_n \to Sh$ in H. For each $n \in \mathbb{N}$, Proposition 3.2.4 generalized as in Proposition 3.4.3 yields

$$\begin{aligned}(Sh_n, APf)_H &= (Sh_n, -Q\nabla\Psi \cdot \nabla(Pf))_H = \mu(\langle \nabla_y h_n, \Sigma \nabla_y (Q\nabla\Psi \cdot \nabla_x(Pf))\rangle) \\ &= \sum_{i=1}^{d_1} \sum_{j,k,\ell=1}^{d_2} \int_E \partial_{y_k} h_n(x,y) a_{k\ell}(y) Q_{ij} \partial_\ell \partial_j \Psi(y) \partial_{x_i}(Pf)(x) \, \mathrm{d}\mu.\end{aligned}$$

Now integration by parts in the coordinate y_k for each $k \in \{1, \dots, d_2\}$ gives

$$(Sh_n, APf)_H = (h_n, Tf)_{L^2(\mu)}.$$

Letting $n \to \infty$, we obtain $(Sh, APf)_H = (h, Tf)_{L^2(\mu)}$. In particular, this holds for $h \equiv 1$, which implies

$$\mu(Tf) = (1, Tf)_{L^2(\mu)} = (S1, APf) = 0.$$

So $Tf \in H$ and the claim follows. □

Finally, we are able to verify (WH1):

Proposition 4.1.5. *Let (Φ3), (Ψ4), (Ψ5) and (Σ4) hold. Then (WH1) is satisfied.*

Proof:
Due to Lemma 4.1.2, $\mathcal{D}$ is a core for $(G, D(G)) = (-(AP)^*AP, D((AP)^*AP))$. Proposition 4.0.5 shows $AP(\mathcal{D}) \subseteq D(A) \subseteq D(A^*)$, and Proposition 4.1.4 proves $AP(\mathcal{D}) \subseteq D(S^*)$. Therefore, we can apply Lemma 2.4.4 to both $(A, D(A))$ and $(S, D(S))$, and obtain boundedness of BA with $\|BA\| \leq c_A$ due to Lemma 4.1.2.

Let $f \in \mathcal{D}$, then it follows that $(BS)^*(I-G)f = S^*APf = Tf$ and

$$\begin{aligned}\|Tf\|_H^2 \leq R\|\nabla f_P\|^2_{L^2(\mu_1)} &= \frac{d_2 R}{\mu_2(|\nabla\Psi|^2)}\frac{\mu_2(|\nabla\Psi|^2)}{d_2}\|(QQ^*)^{-1}QQ^*\nabla f_P\|^2_{L^2(\mu_1)} \\ &\leq \frac{d_2 R|(QQ^*)^{-1}Q|^2}{\mu_2(|\nabla\Psi|^2)}\frac{\mu_2(|\nabla\Psi|^2)}{d_2}\|Q^*\nabla f_P\|^2_{L^2(\mu_1)}.\end{aligned}$$

□

Further, since via integration by parts and Lemma 4.1.2

$$\begin{aligned}\|Q^*\nabla f_P\|^2_{L^2(\mu_1)} &= -\int_{\mathbb{R}^{d_1}} \sum_{i,j=1}^{d_1} (QQ^*)_{ij}\, f_P\,(\partial_j\partial_i f_P - \partial_j\Phi\partial_i f_P)\,\mathrm{d}\mu_1 \\ &= -\int_{\mathbb{R}^{d_1+d_2}} \sum_{i,j=1}^{d_1} (QQ^*)_{ij}\, Pf\,(\partial_{x_j}\partial_{x_i}Pf - \partial_j\Phi\partial_{x_i}Pf)\,\mathrm{d}\mu \\ &= -\frac{d_2}{\mu_2(|\nabla\Psi|^2)}\int_E Pf\cdot Gf\,\mathrm{d}\mu,\end{aligned}$$

we obtain for $g := (I-G)f$ due to dissipativity of $(G, \mathcal{D})$ that

$$\begin{aligned}\|(BS)^*g\|_H^2 = \|Tf\|_H^2 &\leq \frac{d_2 R|(QQ^*)^{-1}Q|^2}{\mu_2(|\nabla\Psi|^2)}\int_E Pf\,(-G)f\,\mathrm{d}\mu \\ &\leq \frac{d_2 R|(QQ^*)^{-1}Q|^2}{\mu_2(|\nabla\Psi|^2)}\|Pf\|_H(\|(I-G)f\|_H + \|f\|_H) \\ &\leq \frac{2d_2 R|(QQ^*)^{-1}Q|^2}{\mu_2(|\nabla\Psi|^2)}\|(I-G)f\|_H^2 = \frac{2d_2 R|(QQ^*)^{-1}Q|^2}{\mu_2(|\nabla\Psi|^2)}\|g\|_H^2.\end{aligned}$$

Using the second part of Lemma 2.4.4, this shows that $(BS, D(S))$ is bounded and hence **(WH1)** is fulfilled.

Definition 4.1.6. Let the functional $\Theta : H \to [0, \infty]$ be defined by

$$\Theta f := \|f\|^2_{\mathrm{osc}} = (\operatorname{ess\,sup}(f) - \operatorname{ess\,inf}(f))^2$$

for all $f \in H$.

Proposition 4.1.7. *The functional Θ satisfies Condition* **(WH2)**.

Proof:
Since any bounded function f fulfills $\Theta f < \infty$, the set $\{f \in H \mid \Theta(f) < \infty\}$ is clearly dense in H. Moreover, we have

$$\begin{aligned}\operatorname{ess\,inf}(f) &= \int_{\mathbb{R}^{d_2}} \operatorname{ess\,inf}(f)\,\mu_2(\mathrm{d}y) \le \operatorname{ess\,inf}\left(\int_{\mathbb{R}^{d_2}} f(x,y)\,\mu_2(\mathrm{d}y)\right) = \operatorname{ess\,inf}(Pf)\\ &\le \operatorname{ess\,sup}(Pf) = \operatorname{ess\,sup}\left(\int_{\mathbb{R}^{d_2}} f(x,y)\,\mu_2(\mathrm{d}y)\right) \le \operatorname{ess\,sup}(f),\end{aligned}$$

so $\Theta(Pf) \le \Theta(f)$ for all $f \in H$.

Let $f \in H$. If f is not bounded, then $\Theta(f) = \infty$ and the remaining inequalities are trivial. So let f be bounded with $l_f := \operatorname{ess\,inf}(f)$, $u_f := \operatorname{ess\,sup}(f)$. If $l_f = u_f$, then f is constant and so are $T_t f$ and $\mathrm{e}^{tG} f$ due to conservativity, which can be seen for $(\mathrm{e}^{tG})_{t\ge 0}$ since $1 \in \mathcal{D} \subseteq D(G)$ and $G1 = 0$, compare Lemma 1.3.3, and for $(T_t)_{t\ge 0}$ due to Theorem 3.6.2.

So we assume $l_f < u_f$ and define $g := \frac{1}{u_f - l_f}(f - l_f)$. Then $\operatorname{ess\,inf}(g) = 0$, $\operatorname{ess\,sup}(g) = 1$ and therefore $\Theta(g) = 1$. Moreover, conservativity and sub-Markov property of $(T_t)_{t\ge 0}$ implies

$$\Theta(T_t f) = \Theta((u_f - l_f)T_t g + l_f) = (u_f - l_f)^2 \Theta(T_t g) \le (u_f - l_f)^2 = \Theta(f)$$

for all $t \ge 0$, and the same holds for the semigroup generated by $(G, D(G))$. □

Proposition 4.1.8. *Θ also satisfies Condition* **(WH3)**.

Proof:
We use the same construction as in [GW19]: Fix some $f \in D(L)$ and set $\gamma_1 := \operatorname{ess\,inf} f$, $\gamma_2 := \operatorname{ess\,sup} f$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ such that $g_n \to f$ and $Lg_n \to Lf$ in H as $n \to \infty$. Then set $f_n := h_n \circ g_n$, where $h_n \in C^\infty(\mathbb{R})$ satisfies $0 \le h_n' \le 1$ and

$$h_n(r) = \begin{cases} r & \text{for } r \in [\gamma_1, \gamma_2] \\ \gamma_1 - \frac{1}{2n} & \text{for } r \le \gamma_1 - \frac{1}{n} \\ \gamma_2 + \frac{1}{2n} & \text{for } r \ge \gamma_2 + \frac{1}{n} \end{cases},$$

which is possible due to Lemma 1.4.9.

Then $f_n \to f$ in H as $n \to \infty$, $\limsup_{n\to\infty} \|f_n\|_{\mathrm{osc}} \le \|f\|_{\mathrm{osc}}$, and

$$\begin{aligned}\limsup_{n\to\infty}(-Lf_n, f_n) &= \limsup_{n\to\infty} \mu(\langle \nabla_y f_n, \Sigma \nabla_y f_n\rangle) = \limsup_{n\to\infty} \mu((h_n'(g_n))^2 \langle \nabla_y g_n, \Sigma \nabla_y g_n\rangle)\\ &\le \limsup_{n\to\infty} \mu(\langle \nabla_y g_n, \Sigma \nabla_y g_n\rangle) = \limsup_{n\to\infty}(-Lg_n, g_n) = (-Lf, f).\end{aligned}$$

□

Finally, it remains to show the weak Poincaré inequalities are satisfied. For this, we use [RW01, Theorem 3.1], which for our purposes states that, given a probability measure $\mu_V = \mathrm{e}^V \mathrm{d}x$ on $\mathbb{R}^d$ with locally bounded V, there exists a decreasing function $\alpha : (0,\infty) \to (0,\infty)$ such that

$$\mu_V(f^2) - (\mu_V(f))^2 \le \alpha(r)\mu_V(|\nabla f|^2) + r\|f\|_{\mathrm{osc}}^2, \quad r > 0, f \in C_b^1(\mathbb{R}^d). \tag{4.1.1}$$

Without loss of generality, we may assume that $\alpha \geq 1$, which facilitates some estimates regarding the convergence rate later. This allows us to verify the last necessary assumption:

Proposition 4.1.9. *Let* **(Φ3)**, **(Ψ4)** *and* **(Ψ5)** *be fulfilled. Then Condition* **(WH4)** *is satisfied.*

Proof:
The proof of [GW19, Theorem 1.1], under the assumptions given, implies that there is some constant $0 < c < \infty$ such that $f_P \in H^{1,2}(\mu_1)$ with $\mu_1(|\nabla_x(f_P)|^2) \leq c\|APf\|^2$ for all $f \in D(AP)$. Inequality (4.1.1) for $V = \Phi$ shows the existence of some decreasing $\alpha_\Phi : (0,\infty) \to [1,\infty)$ such that

$$\|Pf\|_H^2 = \mu_1(f_P^2) \leq c\alpha_\Phi(r)\|APf\|_H^2 + r\Theta(Pf)$$

for all $f \in D(AP)$. This proves the first inequality in **(WH4)** for $\alpha_1 = c\alpha_\Phi$, and we can assume that $\alpha_1 \geq \alpha_\Phi$.

That proof also gives a procedure to verify the second inequality: Let $f \in \mathcal{D}$ and $x \in \mathbb{R}^{d_1}$, set $f_x := f(x,\cdot) - Pf(x) \in C^\infty(\mathbb{R}^{d_2})$. Then ∇f_x has compact support, so that f_x is bounded; $\mu_2(f_x) = 0$ and $\|f_x\|_{\mathrm{osc}} \leq \|f\|_{\mathrm{osc}}$. Therefore, (4.1.1) is applicable and yields the existence of a decreasing function α_Ψ such that

$$\begin{aligned}\mu_2(f_x^2) &\leq \alpha_\Psi(r)\mu_2(|\nabla_y f(x,\cdot)|^2) + r\|f_x\|_{\mathrm{osc}}^2 \\ &\leq c_\Sigma\alpha_\Psi(r)\mu_2(\langle\nabla_y f(x,\cdot), \Sigma\nabla_y f(x,\cdot)\rangle) + r\|f\|_{\mathrm{osc}}^2\end{aligned}$$

for all $r > 0$, $f \in \mathcal{D}$ and $x \in \mathbb{R}^{d_1}$. Integrating that expression wrt. μ_1 gives

$$\begin{aligned}\|(I-P)f\|_H^2 = \int_{\mathbb{R}^{d_1}} \mu_2(f_x^2)\,\mu_1(\mathrm{d}x) &\leq c_\Sigma\alpha_\Psi(r)\int_E \langle\nabla_y f, \Sigma\nabla_y f\rangle\,\mu(\mathrm{d}(x,y)) + r\|f\|_{\mathrm{osc}}^2 \\ &= c_\Sigma\alpha_\Psi(r)(-Sf,f)_H + r\Theta(f)\end{aligned}$$

for all $f \in \mathcal{D}$, $r > 0$, so the second inequality in **(WH4)** is satisfied with $\alpha_2 := c_\Sigma\alpha_\Psi$, and again we may assume $\alpha_2 \geq \alpha_\Psi$ without loss of generality. □

Now that we have found sufficient assumptions in order for the weak hypocoercivity conditions to hold, we are ready to state the final result:

Theorem 4.1.10. *Recall the setting as defined in Definition 3.2.2 and Definition 3.2.1. Let QQ^* be invertible and let Σ satisfy* **(Σ1)**–**(Σ4)**.

Let $\Phi \in C^2(\mathbb{R}^{d_1})$ be bounded from below with μ_1 being a probability measure, and let there be a constant $C < \infty$ such that

$$|\nabla^2\Phi| \leq C(1 + |\nabla\Phi|).$$

If β from **(Σ3)** *is strictly positive, let further $N < \infty$, $0 \leq \gamma < \frac{1}{\beta}$ such that $|\nabla\Phi(x)| \leq N(1 + |x|^\gamma)$.*

Let $\psi \in C^3([0,\infty))$, $K < \infty$, $1 \leq \alpha < 2$, such that

$$\Psi(y) = \psi(|y|^2), \quad Z(\Psi) < \infty, \quad \text{and} \quad |\nabla^2\Psi| \leq K(1 + |\nabla\Psi|^\alpha).$$

Assume further that $\partial_i\partial_j\partial_k\Psi \in L^2(\mu_2)$ for all $1 \le i,j,k \le d_2$.

Then there exist decreasing functions $\alpha_\Phi, \alpha_\Psi : (0,\infty) \to [1,\infty)$ and constants $0 < c_1, c_2 < \infty$ such that

$$\mu_1(f^2) - \mu_1(f)^2 \le \alpha_\Phi(r)\mu_1(|\nabla_x f|^2) + r\|f\|_{\mathrm{osc}}^2, \quad r > 0, f \in C_b^1(\mathbb{R}^{d_1}),$$
$$\mu_2(f^2) - \mu_2(f)^2 \le \alpha_\Psi(r)\mu_2(|\nabla_y f|^2) + r\|f\|_{\mathrm{osc}}^2, \quad r > 0, f \in C_b^1(\mathbb{R}^{d_2}),$$

and

$$\mu((T_t f)^2) - \mu(T_t f)^2 \le \xi(t)\|f\|_{\mathrm{osc}}^2, \qquad \text{for all } t \ge 0,\ f \in L^\infty(\mu), \tag{4.1.2}$$

where $(T_t)_{t\ge0}$ is the sccs generated by the closure $(L, D(L))$ of $(L, \mathcal{C})$ on X and

$$\xi(t) := c_1 \inf\left\{ r > 0 \,:\, c_2 t \ge \alpha_\Phi(r)^2 \alpha_\Psi\left(\frac{r}{\alpha_\Phi(r)^2}\right)\log\left(\frac{1}{r}\right)\right\}. \tag{4.1.3}$$

Proof:
It is easy to see that the assumptions here imply **(H)** as well as **(Σ4)**, **(Φ3)**, **(Ψ4)** and **(Ψ5)**. Then Theorem 3.5.1 applies to show that $(L, D(L))$ is essentially m-dissipative on X, and we obtain all the results from Theorem 3.6.2 for the generated semigroup $(T_t)_{t\ge0}$ on X.

Now we restrict our considerations to the subspace $H \subseteq X$. Due to Lemma 4.0.2, the closure $(L^H, D(L^H))$ of $(L, \mathcal{D})$ on H is m-dissipative and generates an sccs $(T_t^H)_{t\ge0}$ on H. Then all data conditions **(D1)**–**(D3)** are satisfied, and Propositions 4.1.5 and 4.1.7 to 4.1.9 show that Theorem 2.3.1 is applicable. This yields the estimate

$$\|T_t^H f\|^2 \le \xi(t)(\|f\|_H^2 + \Theta(f)), \quad t \ge 0,\ f \in D(L^H), \tag{4.1.4}$$

where

$$\tilde{\xi}(t) := c_1 \inf\left\{ r > 0 \,:\, c_2 t \ge \alpha_1(r)^2 \alpha_2\left(\frac{r}{\alpha_1(r)^2}\right)\log\left(\tfrac{1}{r}\right)\right\}, \tag{4.1.5}$$

for some constants $c_1, c_2 \in (0,\infty)$ and α_1, α_2 as in the proof of Proposition 4.1.9. That proof also provided the decreasing functions α_Φ and α_Ψ as required for the claim.

Due to μ-invariance of $(T_t)_{t\ge0}$, the restriction of $(T_t)_{t\ge0}$ to H is an sccs on H, and its generator coincides with $(L^H, D(L^H))$ on $\mathcal{D}$, so $T_t|_H = T_t^H$ for all $t \ge 0$. Moreover, conservativity of $(T_t)_{t\ge0}$ further shows

$$\mu(T_t f) = \mu(f) = \mu(f)T_t(1) = T_t(\mu(f)),$$

hence

$$\mu((T_t f)^2) - \mu(T_t f)^2 = \mu\left((T_t f)^2 - 2T_t f T_t(\mu(f)) + (T_t\mu(f))^2\right) = \mu((T_t^H(f - \mu(f)))^2).$$

Let $f \in D(L)$. Since $f - \mu(f) \in D(L) \cap H$, a simple approximation argument shows $f - \mu(f) \in D(L^H)$, so (4.1.4) together with

$$\|f - \mu(f)\|_H^2 = \mu((f - \mu(f))^2) \le \|f - \mu(f)\|_{L^\infty}^2 \le \|f\|_{\mathrm{osc}}^2 = \Theta(f)$$

implies

$$\mu((T_t f)^2) - \mu(T_t f)^2 \leq \tilde{\xi}(t)\|f\|_{\mathrm{osc}}^2, \quad t \geq 0,\ f \in D(L). \tag{4.1.6}$$

Let $f \in L^\infty(\mu) \subseteq L^2(\mu)$. Then as in the proof of Proposition 4.1.8, we can find a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{C} \subseteq D(L)$ such that $f_n \to f$ in X and $\limsup_{n\to\infty} \|f_n\|_{\mathrm{osc}} \leq \|f\|_{\mathrm{osc}}$. This shows that (4.1.6) extends to all $f \in L^\infty(\mu)$.

Finally, define

$$\xi(t) \coloneqq c_1 \inf\left\{ r > 0 \,:\, c_2 t \geq \alpha_\Phi(r)^2 \alpha_\Psi\left(\frac{r}{\alpha_\Phi(r)^2}\right)\log\left(\frac{1}{r}\right)\right\}. \tag{4.1.7}$$

Let $r > 0$ satisfy the inequality in (4.1.5). If $r \geq 1$, then $\log(r^{-1}) \leq 0$ and r clearly satisfies the inequality in (4.1.7), so we assume $r < 1$. Since we assumed $\alpha_1 = c\alpha_\Phi$ and $\alpha_2 = c_\Sigma \alpha_\Psi$ with $c, c_\Sigma \geq 1$, we get $\alpha_1 \geq \alpha_\Phi$, $\alpha_2 \geq \alpha_\Psi$ and

$$\frac{r}{\alpha_1(r)^2} \leq \frac{r}{\alpha_\Phi(r)^2}, \qquad \text{hence } \alpha_\Psi\left(\frac{r}{\alpha_1(r)^2}\right) \geq \alpha_\Psi\left(\frac{r}{\alpha_\Phi(r)^2}\right),$$

since α_Ψ is decreasing. So r satisfies the inequality in (4.1.7), which shows that $\tilde{\xi}(t) \leq \xi(t)$, so the claim is proven. □

Concrete examples of possible potentials

Here we give some examples for combinations of potentials Φ and Ψ along with the resulting convergence rate. The examples are taken from [GW19], where they were chosen since corresponding weak Poincaré inequalities were shown in [RW01]. We use the following notation for all occurring choices:

Definition 4.1.11. Let φ and ψ be real-valued function on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. We write $\Phi \sim \varphi$, if there is some $h_1 \in C_b^2(\mathbb{R}^{d_1})$ such that $\Phi = \varphi + h_1$. On the other hand, we write $\Psi \sim \psi$ if there is some $h_2 \in C_b^3(\mathbb{R}^{d_2})$ with $\Psi = \psi + h_2$, where it is assumed that there is some $\eta \in C^3([0,\infty))$ with $h_2(y) = \eta(|y|^2)$.

This relation is explained by the following:

Lemma 4.1.12. *Let φ satisfy (Φ1)–(Φ3), as well as the inequality from (C). If $\Phi \sim \varphi$, then the same holds for Φ. Similarly, if ψ satisfies (Ψ1)–(Ψ5), then so does Ψ in the case that $\Psi \sim \psi$.*

Proof:
We only prove the second statement, as the rest follows analogously. Let ψ be as stated, and let $h_2 \in C_b^3(\mathbb{R}^{d_2})$ with $\Psi = \psi + h_2$. Let M_h denote the bound of h_2 and all its derivatives. We show each of the required conditions:

(Ψ1) Clearly Ψ is measurable and locally bounded, and it holds that

$$\int_{\mathbb{R}^{d_2}} \mathrm{e}^{-\psi(y)-h_2(|y|)}\,\mathrm{d}y \le \mathrm{e}^{M_h} \int_{\mathbb{R}^{d_2}} \mathrm{e}^{-\psi(y)}\,\mathrm{d}y,$$

so that $Z(\Psi) < \infty$.

(Ψ2) Since all derivatives of h_2 are bounded, they are locally L^p-integrable for any $1 \le p \le \infty$.

(Ψ3) Due to the previous point, we only need to show the inequality. It holds that

$$|\nabla^2 \Psi| \le M_h + |\nabla^2 \psi| \le M_h + K(1 + |\nabla \psi|^\alpha) \le M_h + K(1 + |\nabla \Psi|^\alpha + M_h^\alpha),$$

which implies the existence of a suitable constant K_Φ such that ψ satisfies **(Ψ3)**.

(Ψ4) This is clear due to definition of h_2.

(Ψ5) This is also immediate, since h_2 is three times continuously differentiable with bounded derivatives, and by using the same kind of estimate as in point (Ψ1). □

Now we introduce the considered example functions, along with their weak Poincaré inequalities. The next Lemma follows from the proof of [RW01, Example 1.4] and is stated in [GW19, Lemma 3.3]:

Lemma 4.1.13. *Let $\mu_V := \mathrm{e}^{-V}\,\mathrm{d}x$ be a probability measure on $\mathbb{R}^d$. Then there exists a decreasing $\alpha_V : (0,\infty) \to (0,\infty)$ such that the weak Poincaré inequality (4.1.1) holds. In particular, α_V can be specified for the following examples:*

(i) If $V \sim k|x|^\delta$ or $V \sim k(1+|x|^2)^{\frac{\delta}{2}}$ for some $k, \delta \in (0,\infty)$, then

$$\alpha_V(r) = c\left(\log(1 + r^{-1})\right)^{\frac{4(1-\delta)^+}{\delta}}$$

for some constant $c \in (0,\infty)$ is a valid choice.

(ii) If $V \sim \frac{d+p}{2}\log(1+|x|^2)$ for some $p \in (0,\infty)$, then

$$\alpha_V(r) = c r^{-\theta(p)}$$

for some constant $c \in (0,\infty)$ is a valid choice, where

$$\theta(p) := \min\left(\frac{d+p+2}{p}, \frac{4p+4+2d}{(p^2-4-2d-2p)^+}\right).$$

(iii) If $V \sim \frac{d}{2}\log(1+|x|^2) + p\log(\log(\mathrm{e}+|x|^2))$ for some $p \in (1,\infty)$, then

$$\alpha_V(r) = c_1 \mathrm{e}^{c_2 r^{-\frac{1}{p-1}}}$$

for some constants $c_1, c_2 \in (0,\infty)$ is a valid choice.

Here it doesn't matter which definition of $\sim$ is used.

Lemma 4.1.14. *Let V and μ_V be any of the options in Lemma 4.1.13. Then both $\Phi = V$ and $\Psi = V$ satisfy all requirements from Theorem 4.1.10, at least if* **(Σ3)** *holds for $\beta = 0$.*

Proof:
By definition, the measure μ_V is a probability measure, and clearly V is locally bounded, bounded from below, and in $C^3(\mathbb{R}^d)$ for any choice. In order to verify **(Ψ3)** and **(Φ3)**, we need to compute the gradient and the Hessian. We only do this for the first choice, as it only requires tedious calculations. So consider $g(x) := k(1+|x|^2)^{\frac{\delta}{2}}$ for $k, \delta \in (0,\infty)$. We obtain $\nabla g(x) = k\delta x(1+|x|^2)^{\frac{\delta}{2}-1}$, so $|\nabla g(x)|^2 = k^2\delta^2|x|^2(1+|x|^2)^{\delta-2}$. In particular, the inequality for Φ from **(C)** holds for g with $\gamma = \delta - 1$. For the second derivatives, we get

$$\partial_j\partial_i g(x) = 2k\delta(\tfrac{\delta}{2}-1)x_ix_j(1+|x|^2)^{\frac{\delta}{2}-2} + \delta_{ij}k\delta(1+|x|^2)^{\frac{\delta}{2}-1}$$

for all $1 \le i, j \le d$. This implies

$$\begin{aligned}|\nabla^2 g(x)|^2 = k^2\delta^2\Big(4(\tfrac{\delta}{2}-1)^2(1+|x|^2)^{\delta-4}|x|^4 &+ 4(\tfrac{\delta}{2}-1)(1+|x|^2)^{\delta-3}|x|^2 \\ &+ d(1+|x|^2)^{\delta-2}\Big).\end{aligned}$$

It is clear that for $|x| \ge 1$, this expression can be bounded by a multiple of $|\nabla g(x)|^2$, so the inequality from **(A1)** holds for g. By Lemma 4.1.12, all these properties of g carry over to V, where we can also apply Remark 1.6.6 to verify **(Ψ5)** for V, since the third derivatives can also be relatively bounded by $|\nabla V|$. To summarize, all requirements for Theorem 4.1.10 are met, as long as **(Σ3)** holds for some $\beta < \frac{1}{(\delta-1)^+}$.

For the other two choices of V, the same can be verified by analogous calculations, and since in those cases $|\nabla V|$ is even bounded, we don't need to assume any restriction on β. □

This means that the examples studied in [GW19, Example 1.1] also fit our assumptions, and we can therefore carry over the convergence rates that were given in that reference. However, we note that we are able to choose $\Psi \sim V$ instead of $\Psi = V$ for such a function V, since our assumption **(Ψ5)** is easier to check in this context than the assumption

$$\sup_{r\ge 0}\left|\psi'(r) + 2r\psi''(r) - \frac{2r\psi'''(r) + (d_2+2)\psi''(r)}{\psi'(r)}\right| < \infty$$

which was required in the given reference, where ψ is to be understood as in **(Ψ4)**. For illustration purposes, we include two of the considered cases here and give the concrete convergence rate, but since we don't add anything new here, we skip the remainder of the cases.

(i) Let $\Phi \sim k(1+|x|^2)^{\delta/2}$ and $\Psi \sim \kappa(1+|y|^2)^{\varepsilon/2}$, where $k, \kappa, \delta, \varepsilon \in (0,\infty)$ are constant. Then, (4.1.2) holds with

$$\xi(t) = \exp(-c_2 t^{\omega(\delta,\varepsilon)}), \text{ where } \omega(\delta,\varepsilon) = \frac{\delta\varepsilon}{\delta\varepsilon + 8\varepsilon(1-\delta)^+ + 4\delta(1-\varepsilon)^+},$$

for some constants $0 < c_1, c_2 < \infty$. In particular, if $\delta, \varepsilon \geq 1$, then the decay rate is exponential.

(ii) Let $d \coloneqq d_1 = d_2$, $\Phi \sim \frac{q+d}{2}\log(1+|x|^2)$ for some $q > 0$ and $\Psi \sim \frac{p+d}{2}\log(1+|y|^2)$ for some $p > 0$. Then, (4.1.2) holds with

$$\begin{aligned} \xi(t) &= c_2(1+t)^{-\omega(p,q)}(\log(\mathrm{e}+t))^{\omega(p,q)}, && \text{where} \\ \omega(p,q) &= \frac{1}{2\theta(q)+\theta(p)+2\theta(q)\theta(p)} && \text{and} \\ \theta(r) &= \min\left(\frac{d+r+2}{r}, \frac{4r+4+2d}{(r^2-4-2d-2r)^+}\right). \end{aligned}$$

for some constants $0 < c_1, c_2 < \infty$.

4.2 Strong hypocoercivity for Langevin dynamics with multiplicative noise

In this part we apply the strong hypocoercivity framework as described in Section 2.2. While we can reuse some considerations from the weak case above, we require more assumptions on the potentials. Motivated by the application to the Langevin equation (4.3.5), we fix $\Psi(y) = \frac{1}{2}|y|^2$ and otherwise assume the conditions **(H)** and **(Φ3)**. This immediately implies the following result:

Proposition 4.2.1. *Ψ defined by $\Psi(y) \coloneqq \frac{1}{2}|y|^2$ satisfies all conditions* **(Ψ1)–(Ψ5)***. Moreover $\mu_2(|\nabla\Psi|^2) = d_2$.*

Proof:
Clearly $\Psi \in C^\infty(\mathbb{R}^{d_2})$ with $\partial_k\Psi(y) = y_k$, $\partial_j\partial_k\Psi(y) = \delta_{kj}$ and therefore $\partial_i\partial_j\partial_k\Psi(y) = 0$ for all $1 \leq i, j, k \leq d_2$. It is well-known that $\frac{1}{2\pi}\mathrm{e}^{-\Psi(y)}$ is the probability density function for the standard normal distribution on $\mathbb{R}^{d_2}$, so $Z(\Psi) = 2\pi$. This implies all wanted conditions.

Moreover, integration by parts yields

$$\begin{aligned} \mu_2(|\nabla\Psi|^2) &= Z(\Psi)^{-1}\int_{\mathbb{R}^{d_2}} |y|^2\mathrm{e}^{-\frac{1}{2}|y|^2}\,\mathrm{d}y = Z(\Psi)^{-1}\sum_{i=1}^{d_2}(-1)\int_{\mathbb{R}^{d_2}} y\partial_i\left(\mathrm{e}^{-\frac{1}{2}|y|^2}\right)\mathrm{d}y \\ &= Z(\Psi)^{-1}\sum_{i=1}^{d_2}\int_{\mathbb{R}^{d_2}} \mathrm{e}^{-\frac{1}{2}|y|^2}\,\mathrm{d}y = d_2 Z(\Psi)^{-1}Z(\Psi) = d_2. \end{aligned}$$

□

Proposition 4.2.2. *Σ satisfies* (Σ4) *for all $p_\Sigma \geq 1$.*

Proof:
The Hölder-inequality implies

$$\||\nabla\Sigma|\|_{L^{2p}(\mu_2)}^{2p} = \int_{\mathbb{R}^{d_2}} \left(\sum_{i,j,k=1}^{d_2} |\partial_k a_{ij}(y)|^2 \right)^p \,\mathrm{d}\mu_2$$
$$\leq (d_2^3)^{p-1} \sum_{i,j,k=1}^{d_2} \int_{\mathbb{R}^{d_2}} |\partial_k a_{ij}(y)|^{2p} \,\mathrm{d}\mu_2$$

for all $p \in \mathbb{N}$. Due to (Σ3), we know that $|\partial_k a_{ij}(y)| \leq M(1+|y|) = M(1+|\nabla\Psi|)$, so $|\partial_k a_{ij}| \leq 2M$ on $B_1(0)$ and $|\partial_k a_{ij}| \leq 2M|y|$ on $\mathbb{R}^{d_2} \setminus B_1(0)$ for all $i, j, k \in \{1, \dots, d_2\}$. This means

$$\int_{\mathbb{R}^{d_2}} |\partial_k a_{ij}(y)|^{2p} \,\mathrm{d}\mu_2 \leq \int_{\mathbb{R}^{d_2}} (2M)^{2p} \,\mathrm{d}\mu_2 + \int_{\mathbb{R}^{d_2}} (2M)^{2p}|y|^{2p} \,\mathrm{d}\mu_2$$

Iterating the integration by parts as above, we can see that $\mu_2(|y|^{2p}) < \infty$ for any $p \in \mathbb{N}$. □

This shows that we can apply all results obtained by the weak hypocoercivity method for our setting. In particular, as a consequence of Lemma 4.1.2, we obtain

Corollary 4.2.3. *The operator $(G, \mathcal{D}) := (PA^2P, \mathcal{D})$, which is given for all $f \in \mathcal{D}$ by*

$$Gf(x,y) = \sum_{i,j=1}^{d_1} (QQ^*)_{ij}(\partial_{x_j}\partial_{x_i} - \partial_j\Phi(x)\partial_{x_i})Pf(x,y),$$

is essentially self-adjoint and its closure generates a sub-Markovian sccs $(\mathrm{e}^{tG})_{t\geq 0}$ on H. Moreover, there is a constant c_Φ depending only on the choice of Φ such that

$$\|BA(I-P)f\|_H \leq c_\phi \|(I-P)f\|_H \qquad \text{for all } f \in \mathcal{D}.$$

However, we would like to specify the convergence rate in more detail than before. Recall the constants set in Definition 3.4.1. Then we get

Lemma 4.2.4. *For all $k, j \in \{1, \dots, d_2\}$, it holds that*

$$\|\partial_k a_{jk} - a_{jk} y_k\|_{L^2(\mu_2)} \leq N_\Sigma.$$

Let the operator $(T, \mathcal{D})$ be defined as in Lemma 4.1.3. Then $w_i = \sum_{j,k=1}^{d_2} Q_{ij}(\partial_k a_{jk} - a_{jk} y_k)$ and R can be chosen as $(d_2)^4 |Q|^2 N_\Sigma^2$

Proof:
Due to integration by parts, it holds that

$$\int_{\mathbb{R}^{d_2}} a_{jk}^2 y_k^2 \,\mathrm{d}\mu_2 = \int_{\mathbb{R}^{d_2}} a_{jk}^2 + 2a_{jk} y_k \partial_k a_{jk} \,\mathrm{d}\mu_2.$$

Hence we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{d_2}} (\partial_k a_{jk} - a_{jk} y_k)^2 \,\mathrm{d}\mu_2 &= \int_{\mathbb{R}^{d_2}} (\partial_k a_{jk})^2 + a_{jk}^2 \,\mathrm{d}\mu_2 \\
&\leq \int_{B_1(0)} (\partial_k a_{jk})^2 \,\mathrm{d}\mu_2 + \int_{\mathbb{R}^{d_2}\setminus B_1(0)} (\partial_k a_{jk})^2 \,\mathrm{d}\mu_2 + M_\Sigma^2 \\
&\leq B_\Sigma^2 + \int_{\mathbb{R}^{d_2}\setminus B_1(0)} (M|y|^\beta)^2 \,\mathrm{d}\mu_2 + M_\Sigma^2 \\
&\leq B_\Sigma^2 + M_\Sigma^2 + \sum_{k=1}^{d_2} M^2 \int_{\mathbb{R}^{d_2}} y_k^2 \,\mathrm{d}\mu_2 \\
&= B_\Sigma^2 + M_\Sigma^2 + d_2 M^2 = N_\Sigma^2.
\end{aligned}$$

The representation of the w_i follows directly since Σ is symmetric, and as in Lemma 4.1.3 we have

$$\begin{aligned}
R &:= \sum_{i=1}^{d_1} \|w_i\|_{L^2(\mu_2)}^2 \leq d_2^2 \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} \|Q_{ij}(\partial_k a_{jk} - a_{jk} y_k)\|_{L^2(\mu_2)}^2 \leq d_2^3 \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} |Q_{ij}|^2 N_\Sigma^2 \\
&= d_2^3 |Q|^2 N_\Sigma^2.
\end{aligned}$$

□

Proposition 4.2.5. *Condition (H1) is satisfied with* $c_1 = \sqrt{2d_2^3} N_\Sigma |Q| \cdot |(QQ^*)^{-1} Q|$ *and* $c_2 = c_\Phi$.

Proof:
The statement for c_2 follows directly from Corollary 4.2.3, and the one for c_1 results by plugging Proposition 4.2.1 and Lemma 4.2.4 into the proof of Proposition 4.1.5. □

It remains to verify microscopic and macroscopic coercivity. This is done via classical Poincaré inequalities for the measures μ_1 and μ_2. Since we don't know enough about Φ, we have to introduce a new assumption.

Assumption (Φ4). The probability measure μ_1 on $\mathbb{R}^{d_1}$ satisfies a Poincaré inequality of the form

$$\Lambda_\Phi \|f\|_{L^2(\mu_1)}^2 \leq \int_{\mathbb{R}^{d_1}} |\nabla f|^2 \,\mathrm{d}\mu_1 \qquad \text{for all } f \in \mathcal{D},$$

where $\Lambda_\Phi \in (0, \infty)$.

A sufficient condition for **(Φ4)** to hold is given by the following:

Theorem 4.2.6. *Let* $V \in C^2(\mathbb{R}^d)$ *be bounded from below,* $\mu_V := \mathrm{e}^{-V(x)}\,\mathrm{d}x$ *be a probability measure, and let one of the following assumptions hold:*

(i) There exist $\alpha > 0$ *and* $R \geq 0$ *such that*

$$\langle x, \nabla V(x) \rangle \geq \alpha |x| \qquad \textit{for all } |x| \geq R.$$

(ii) There exist $a \in (0,1)$, $c > 0$ and $R \geq 0$ such that

$$a|\nabla V(x)|^2 - \Delta V(x) \geq c \qquad \text{for all } |x| \geq R.$$

Then there is some $C_P \in (0,\infty)$ such that μ_V satisfies the Poincaré inequality

$$\mathrm{Var}_{\mu_V}(f) := \int_{\mathbb{R}^d} (f - \mu_V(f))^2 \,\mathrm{d}\mu_V \leq C_P \int_{\mathbb{R}^d} |\nabla f|^2 \,\mathrm{d}\mu_V \tag{4.2.1}$$

for all $f \in C^\infty(\mathbb{R}^d)$. In particular, this holds for convex V.

Proof:
See [Bak+08, Corollary 1.4]. □

Remark 4.2.7. Note that due to Theorem 1.6.3, this Poincaré inequality extends to all $f \in H^{1,2}(\mu_V)$.

Now we can verify the last two hypocoercivity conditions:

Proposition 4.2.8. *Condition* **(H3)** *holds for $\Lambda_M := \Lambda_\Phi |(QQ^*)^{-1}Q|^{-2}$ and* **(H2)** *is satisfied for $\Lambda_m := c_\Sigma^{-1}$.*

Proof:
Due to [Bec89], the probability measure μ_2 fulfills (4.2.1) with $C_P = 1$. Let $f \in \mathcal{D}$ and set $f_x := f(x,\cdot) - Pf(x)$ for any $x \in \mathbb{R}^{d_1}$. Clearly $f_x \in C^\infty(\mathbb{R}^{d_2})$ with $\nabla_y f_x = \nabla_y f(x,\cdot)$ and $\mu_2(f_x) = 0$ for all $x \in \mathbb{R}^{d_1}$.

$$\begin{aligned}
-(Sf,f)_H &= \int_E \langle \nabla_y f, \Sigma \nabla_y f\rangle \,\mathrm{d}\mu \geq c_\Sigma^{-1} \int_E |\nabla_y f|^2 \,\mathrm{d}\mu \\
&= c_\Sigma^{-1} \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} |\nabla_y f_x|^2 \,\mathrm{d}\mu_2 \,\mathrm{d}\mu_1 \geq c_\Sigma^{-1} \int_{\mathbb{R}^{d_1}} \mathrm{Var}_{\mu_2}(f_x)\, \mu_1(\mathrm{d}x) \\
&= c_\Sigma^{-1} \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} f_x^2 \,\mathrm{d}\mu_2 \,\mu_1(\mathrm{d}x) = c_\Sigma^{-1} \|(I-P)f\|_H^2
\end{aligned}$$

for all $f \in \mathcal{D}$, so **(H2)** is indeed satisfied.

Let $f \in \mathcal{D}$. Then, as in (4.0.1), we get $APf = -\langle Qy, \nabla_x f_P\rangle$ and therefore

$$\|APf\|_H^2 = \sum_{i,k=1}^{d_1} \sum_{k,\ell=1}^{d_2} \int_{\mathbb{R}^{d_1}} \partial_i(f_P)\partial_k(f_P)\,\mathrm{d}\mu_1 \int_{\mathbb{R}^{d_2}} (Q_{ij}y_j)(Q_{k\ell}y_\ell)\,\mathrm{d}\mu_2.$$

Integration by parts shows

$$\int_{\mathbb{R}^{d_2}} (Q_{ij}y_j)(Q_{k\ell}y_\ell)\,\mathrm{d}\mu_2 = Q_{ij}Q_{k\ell}\delta_{j\ell},$$

so we obtain

$$\|APf\|_H^2 = \sum_{i,k=1}^{d_1} (QQ^*)_{ik} \int_{\mathbb{R}^{d_1}} \partial_i(f_P)\partial_k(f_P)\, \mathrm{d}\mu_1 = \int_{\mathbb{R}^{d_1}} |Q^*\nabla(f_P)|^2\, \mathrm{d}\mu_1.$$

Assumption (Φ4) together with invertibility of QQ^* now implies

$$\begin{aligned}\Lambda_\Phi \|f_P\|_{L^2(\mu_1)}^2 &\le \|\nabla_x f_P\|_{L^2(\mu_1)}^2 = \|(QQ^*)^{-1}QQ^*\nabla_x f_P\|_{L^2(\mu_1)}^2 \\ &\le |(QQ^*)^{-1}Q|^2 \|Q^*\nabla_x f_P\|_{L^2(\mu_1)}^2 = |(QQ^*)^{-1}Q|^2 \|APf\|_H^2.\end{aligned}$$

Since $\|APf\|_H^2 = ((AP)^*APf, f)_H = (-Gf, f)_H$ and $\mathcal{D}$ is a core for $(G, D(G))$ by Lemma 4.1.2, this inequality extends to all $f \in D(G)$. □

Corollary 4.2.9. *In the case that $d_1 = d_2 =: d$ and $Q = I$, we satisfy (H1) with $c_1 = \sqrt{2d^3}N_\Sigma$ and (H3) with $\Lambda_M = \Lambda_\Phi$.*

Proof:
The latter follows from the proof of Proposition 4.2.8, since $|\nabla_x f_P| = |Q^*\nabla_x f_P|$.

For the other part, note that in the proof of Lemma 4.2.4, we get $\|w_i\|_{L^2(\mu_2)}^2 \le d^2 N_\Sigma^2$, and hence $R \le d^3 N_\Sigma^2$. In the proof of Proposition 4.1.5, we use again $|\nabla_x f_P| = |Q^*\nabla_x f_P|$ and Proposition 4.2.1 to obtain $\|(BS)^*g\|_H^2 \le 2R\|g\|_H^2$. Application of the second part of Lemma 2.4.4 then yields $c_1 = \sqrt{2R} = \sqrt{2d^3}N_\Sigma$ as wanted. □

Now that all necessary conditions are verified with concrete estimates for each required constant, we can state our strong hypocoercivity result for the case where the second component measure is standard Gaussian:

Theorem 4.2.10. *Recall the setting as defined in Definition 3.2.2 and Definition 3.2.1. Let QQ^* be invertible, $\Psi(y) := \frac{1}{2}|y|^2$ and let Σ satisfy (Σ1)–(Σ3).*

Let $\Phi \in C^2(\mathbb{R}^{d_1})$ be bounded from below with μ_1 being a probability measure, and let there be constants $C, \Lambda_\Phi \in (0,\infty)$ such that

$$|\nabla^2\Phi| \le C(1 + |\nabla\Phi|) \quad \text{and} \quad \Lambda_\Phi \|f - \mu_1(f)\|_{L^2(\mu_1)}^2 \le \|\nabla f\|_{L^2(\mu_1)}^2$$

for all $f \in C_c^\infty(\mathbb{R}^{d_1})$. If β from (Σ3) is strictly positive, let further $N < \infty$, $0 \le \gamma < \frac{1}{\beta}$ such that $|\nabla\Phi(x)| \le N(1 + |x|^\gamma)$.

Define N_Σ as in Definition 3.4.1 and let $(T_t)_{t\ge 0}$ be the sccs generated by the closure $(L, D(L))$ of $(L, \mathcal{C})$ on X.

Then it holds that for each $\theta_1 \in (1,\infty)$, there is some $\theta_2 \in (0,\infty)$ such that

$$\|T_t f - \mu(f)\|_X \le \theta_1 \mathrm{e}^{-\theta_2 t} \|f - \mu(f)\|_X$$

for all $f \in X$ and all $t \geq 0$. In particular, θ_2 can be specified as

$$\theta_2 = \frac{\theta_1 - 1}{\theta_1} \frac{c_\Sigma^{-1}}{n_1 + n_2 N_\Sigma + n_3 N_\Sigma^2},$$

and the coefficients $n_i \in (0, \infty)$ only depend on the choice of Φ and Q.

Proof:
Under the given assumptions, we can apply Theorem 3.5.1 due to Proposition 4.2.1, which shows that $(L, \mathcal{C})$ is essentially m-dissipative, so that its closure $(L, D(L))$ generates an sccs $(T_t)_{t\geq 0}$ on X. Since μ is a probability measure, we also obtain the properties from Theorem 3.6.2.

As in the proof of Theorem 4.1.10, the closure $(L^H, D(L^H))$ of $(L, \mathcal{D})$ generates an sccs $(T_t^H)_{t\geq 0}$ on H which coincides with $(T_t)_{t\geq 0}$ on H. By Propositions 4.2.5 and 4.2.8, we can apply Theorem 2.2.1, which yields the existence of constants $\kappa_1, \kappa_2 \in (0, \infty)$ such that

$$\|T_t^H f\|_H \leq \kappa_1 \mathrm{e}^{-\kappa_2 t} \|f\|_H \quad \text{for all } t \geq 0,\ f \in H,$$

which immediately generalizes to

$$\|T_t f - \mu(f)\|_X \leq \kappa_1 \mathrm{e}^{-\kappa_2 t} \|f - \mu(f)\|_X \quad \text{for all } t \geq 0,\ f \in X,$$

by conservativity of $(T_t)_{t\geq 0}$.

It remains to prove the convergence rate estimate, which follows the idea in Remark 2.2.2 and uses the same technique as the analogue proof in [GS16]. Note that in the context of that Theorem, the mentioned Propositions give us the following constants:

$$\begin{aligned} c_1 &= c_Q N_\Sigma \quad \text{with } c_Q := \sqrt{2d_2^3} |Q| \cdot |(QQ^*)^{-1} Q|, \\ c_2 &= c_\Phi, \\ \Lambda_m &= c_\Sigma^{-1} \quad \text{and} \\ \Lambda_M &= \Lambda_Q(\Phi) := \Lambda_\Phi |(QQ^*)^{-1} Q|^{-2}. \end{aligned}$$

In Equation (2.2.2), set

$$\delta := \frac{\Lambda_M}{1 + \Lambda_M} \frac{1}{1 + c_1 + c_2} = \frac{\Lambda_Q(\Phi)}{1 + \Lambda_Q(\Phi)} \frac{1}{1 + c_Q N_\Sigma + c_\Phi}.$$

Then the coefficients on the right hand side can be written as $c_\Sigma^{-1} - \varepsilon r_{Q,\Phi}(N_\Sigma)$ and $\varepsilon s_{Q,\Phi}$ respectively, where

$$\begin{aligned} r_{Q,\Phi}(N_\Sigma) &:= (1 + c_\Phi + c_Q N_\Sigma) \left(1 + \frac{1 + \Lambda_Q(\Phi)}{2\Lambda_Q(\Phi)} (1 + c_\Phi + c_Q N_\Sigma) \right), \\ s_{Q,\Phi} &:= \frac{1}{2} \cdot \frac{\Lambda_Q(\Phi)}{1 + \Lambda_Q(\Phi)}, \end{aligned}$$

and $\varepsilon = \varepsilon_\Phi(\Sigma) \in (0,1)$ still needs to be determined. Write $r_{Q,\Phi}(N_\Sigma) + s_{Q,\Phi}$ as the polynomial

$$r_{Q,\Phi}(N_\Sigma) + s_{Q,\Phi} = a_1 + a_2 N_\Sigma + a_3 N_\Sigma^2,$$

where all $a_i \in (0,\infty)$, $i = 1,\dots,3$ depend only on Q and Φ. Then define

$$\tilde{\varepsilon}_{Q,\Phi}(N_\Sigma) := \frac{N_\Sigma}{r_{Q,\Phi}(N_\Sigma) + s_{Q,\Phi}} = \frac{N_\Sigma}{a_1 + a_2 N_\Sigma + a_3 N_\Sigma^2}.$$

Some rough estimates show $\tilde{\varepsilon}_\Phi(N_\Sigma) \in (0,1)$. Now let $v > 0$ be arbitrary and set

$$\varepsilon := \frac{v}{1+v}\frac{c_\Sigma^{-1}}{N_\Sigma}\tilde{\varepsilon}_{Q,\Phi}(N_\Sigma) \in (0,1).$$

This holds since $c_\Sigma^{-1} \le M_\Sigma \le N_\Sigma$, which follows from (Σ1) for some unit vector. Then $\varepsilon r_{Q,\Phi}(N_\Sigma) + \varepsilon s_{Q,\Phi} = \frac{v}{1+v} c_\Sigma^{-1} < c_\Sigma^{-1}$, hence we get the estimate

$$c_\Sigma^{-1} - \varepsilon r_{Q,\Phi}(N_\Sigma) > \varepsilon s_{Q,\Phi} = \frac{v}{1+v}\frac{2c_\Sigma}{n_1 + n_2 N_\Sigma + n_3 N_\Sigma^2} =: \kappa,$$

where all $n_i \in (0,\infty)$ depend on Q and Φ, and are given by

$$n_i := \frac{2}{s_{Q,\Phi}} a_i, \qquad \text{for each } i = 1,\dots,3.$$

This means that κ is smaller than both coefficients in (2.2.2), so that this inequality holds as seen in Remark 2.2.2. The convergence rate coefficients κ_1, κ_2 are then given by the second part of Theorem 2.2.1 as

$$\kappa_1 = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} = \sqrt{\frac{1+v+\frac{c_\Sigma^{-1}}{N_\Sigma}\tilde{\varepsilon}_{Q,\Phi}(N_\Sigma)v}{1+v-\frac{c_\Sigma^{-1}}{N_\Sigma}\tilde{\varepsilon}_{Q,\Phi}(N_\Sigma)v}} \le \sqrt{1+2v+v^2} = 1+v \quad \text{and}$$

$$\kappa_2 = \frac{\kappa}{1+\varepsilon} > \frac{1}{2}\kappa$$

Hence, by choosing $\theta_1 = 1+v$ and $\theta_2 = \frac{1}{2}\kappa = \frac{\theta_1 - 1}{\theta_1}\frac{c_\Sigma^{-1}}{n_1 + n_2 N_\Sigma + n_3 N_\Sigma^2}$, the rate of convergence claimed in the theorem is shown. □

4.3 Application to partial differential equations and stochastic differential equations

The aim of this section is to apply the previously obtained convergence rate results for operator semigroups generated by Kolmogorov operators to solutions of different differential equations. We do this in order of immediacy, so we start with the Cauchy problem associated to the generator, move on to a different formulation as a Fokker-Planck equation in gradient form, and finish with stochastic differential equations. The latter then motivate the terms “stochastic Hamiltonian systems” and “Langevin dynamics” in the titles of Section 4.1 and Section 4.2, respectively.

4.3.1 Second-order partial differential equations

Assume the context of Theorem 3.5.1. We consider the following partial differential equation for $u : [0,\infty) \to X$:

$$\begin{aligned} u(0) &= u_0 \\ \frac{\mathrm{d}}{\mathrm{d}t}u(t) &= \operatorname{tr}[\Sigma\, \mathbf{H}_y\, u(t)] - \langle \Sigma\nabla\Psi, \nabla_y u(t)\rangle + \sum_{i=1}^{d_2}\sum_{j=1}^{d_2} \partial_j a_{ij}\partial_{y_i} u(t) \\ &\quad + \langle Q\nabla\Psi, \nabla_x u(t)\rangle - \langle Q^*\nabla\Phi, \nabla_y u(t)\rangle. \end{aligned} \tag{4.3.1}$$

Due to Theorem 1.2.6, the semigroup $(T_t)_{t\geq 0}$ on X generated by $(L, D(L))$ solves this Cauchy problem in the sense that $u(t) \coloneqq T_t u_0$ is the unique mild solution for all $u_0 \in X$ and the unique classical solution for $u_0 \in D(L)$.

If $\Psi(y) = \frac{1}{2}|y|^2$ and Q, Σ as well as Φ satisfy the conditions assumed in Theorem 4.2.10, then we obtain directly that the solution u to (4.3.1) with initial condition u_0 converges to the integral of u_0 under μ with the given convergence rate.

Similarly, if u_0 is bounded and Q, Σ, Φ and Ψ satisfy the assumptions of Theorem 4.1.10, then we also get convergence for the solution of (4.3.1) to $\mu(u_0)$ with the convergence rate described, depending on the functions given by the weak Poincaré inequalities.

Now consider the following partial differential equation in divergence form:

$$\begin{aligned} u(0) &= u_0 \\ \frac{\mathrm{d}}{\mathrm{d}t}u(t) &= \sum_{i,j=1}^{d_2} \partial_{y_i}(a_{ij}\partial_{y_j} u(t) + a_{ij}\partial_j\Psi u(t)) - \langle Q\nabla\Psi, \nabla_x u(t)\rangle + \langle Q^*\nabla\Phi, \nabla_y u(t)\rangle. \end{aligned} \tag{4.3.2}$$

It is easy to see that on $\mathcal{C}$, the right hand side term corresponds to the formal adjoint $\widetilde{L}$ of $(L, D(L))$ on $L^2(\mathbb{R}^{d_1+d_2}, \mathrm{d}(x,y))$ applied to $u(t)$. This means that (4.3.2) describes the Fokker-Planck equation corresponding to the Kolmogorov backwards equation associated with L. We define the Hilbert space

$$Y \coloneqq L^2(\mathbb{R}^{d_1+d_2}; \nu), \qquad \text{where } \nu = Z(\Psi)^{-1}\exp(\Phi(x) + \Psi(y))\,\mathrm{d}(x,y).$$

Then the mapping T defined by

$$T : Y \to X, \quad Tf = \rho^{-1} f \quad \text{with} \quad \rho(x,y) \coloneqq \mathrm{e}^{-(\Phi(x)+\Psi(y))},$$

is a unitary transformation between Y and X. Moreover, for $f \in \mathcal{C}$, we see that $\widetilde{L}f = T^{-1}L^*Tf$, where $(L^*, D(L^*))$ is the adjoint of $(L, D(L))$ on X. As seen in Corollary 3.5.7, $(L^*, \mathcal{C})$ results from $(L, \mathcal{C})$ by using $-Q$ instead of Q, and is therefore essentially m-dissipative on X, which implies the same for its dissipative extension $(L^*, C_c^2(E))$. In both hypocoercivity methods, we assume that the potentials are at least C^2, so $C_c^2(E)$ is a T-invariant domain. This means that $(\widetilde{L}, C_c^2(E))$ is essentially m-dissipative and generates the sccs $(\widetilde{T}_t)_{t\geq 0} = (T^{-1}T_t^*T)_{t\geq 0}$ on Y.

Let $\Psi(y) = \frac{1}{2}|y|^2$ and let Q, Σ, Φ satisfy the assumptions of Theorem 4.2.10. Fix some $u_0 \in Y$ and set $u(t) \coloneqq \widetilde{T}_t u_0$, which is the unique mild or classical solution of (4.3.2). We get

$$\begin{aligned}\|u(t) - \nu(u_0\rho)\rho\|_Y &= \|T^{-1}T_t^*Tu_0 - \nu(u_0\rho)T^{-1}(1)\|_Y = \|T_t^*Tu_0 - \mu(Tu_0)\|_X \\ &\leq \theta_1 \mathrm{e}^{-\theta_2 t}\|Tu_0 - \mu(Tu_0)\|_X = \theta_1 \mathrm{e}^{-\theta_2 t}\|u_0 - \nu(u_0\rho)\rho\|_Y\end{aligned}$$

for all $t \geq 0$, $\theta_1 \in (1,\infty)$ and θ_2 chosen accordingly. This shows that the solution to (4.3.2) converges to a multiple of the stationary solution ρ with rate $\theta_1\mathrm{e}^{-\theta_2 t}$. Stationary solution here means that $u(t) = \rho$ solves (4.3.2) with initial condition $u_0 = \rho$, which follows from conservativity of $(T_t^*)_{t\geq 0}$.

Analogously to the first Cauchy problem, we can also apply Theorem 4.1.10 under its assumptions to obtain

$$\|u(t) - \nu(u_0\rho)\rho\|_Y^2 \leq \xi(t)\left\|\frac{u_0}{\rho}\right\|_{\mathrm{osc}}^2$$

for all $t \geq 0$ and all $u_0 \in Y$ such that Tu_0 is bounded.

4.3.2 Generalized stochastic Hamiltonian systems

Consider the operator $(L, D(L))$ as in Definition 3.2.1 and assume **(H)**. Due to Theorem 3.6.2, $(L, D(L))$ is a Dirichlet operator, so we can apply Theorem 1.3.13 to obtain the associated generalized Dirichlet form $\mathcal{E}$ on X.

Proposition 4.3.1. *There is a special standard Hunt process* $\mathbf{M}$ *as in Definition 1.3.28, where* $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t, Y_t)_{t\geq 0}, (P_{(x,y)})_{(x,y)\in E})$*, which is properly associated in the resolvent sense with* $\mathcal{E}$*. Moreover,* $\mathbf{M}$ *has continuous paths and infinite life time* $P_{(x,y)}$*-a.s. for all* $(x,y) \in E$*.*

Proof:
In order to assure the existence of an associated right process, we need to find an $\mathcal{E}$-nest first. Consider the core $\mathcal{C}$ of $(L, D(L))$ consisting of smooth functions with compact support, and set $F_n \coloneqq \overline{B_n(0)}$, which is compact since E is finite-dimensional. Clearly each $f \in \mathcal{C}$ satisfies $f = 0$ outside of some F_n, so by Proposition 1.3.22 the sequence $(F_n)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-nest of compact sets.

With this, we are left with one property left to show so that $\mathcal{E}$ is quasi-regular, as defined in Definition 1.3.23. Since $C_c^2(E)$ is separable (see Lemma 1.5.1) and separates the points of E, property (iii) of Definition 1.3.23 is fulfilled since $C_c^2(E)$ is a subset of $D(L)$, as $(L, C_c^2(E))$ is a well-defined dissipative extension of $(L, \mathcal{C})$. Furthermore, $\mathcal{C}$ is an algebra of bounded functions, so we can apply Theorem 1.3.31, which yields the existence of a μ-tight special standard process $\mathbf{M}$ which is properly associated in the resolvent sense with $\mathcal{E}$.

Due to being a differential operator without constant part, it holds that $Lf = 0$ on $E \setminus \mathrm{supp}(f)$ for all $f \in \mathcal{C}$. Moreover, due to Lemma 1.4.7, for each open $U \subseteq E$ there is a sequence $(f_n)_{n\in\mathbb{N}}$ in

$\mathcal{C}$ such that $0 \le f_n \uparrow \mathbb{1}_U$. Therefore, Lemma 1.3.32 shows that $\mathbf{M}$ has $P_{(x,y)}$-a.s. continuous paths up to the life time ζ for $\mathcal{E}$-quasi all $(x, y) \in E$.

Since $(T_t)_{t\ge0}$ is conservative as seen in Theorem 3.6.2, we have that $T_t 1 = 1$ for all $t \ge 0$ and therefore $G_1 1 = 1$. Due to proper association with $\mathbf{M}$, this implies that $R_1 1 = 1$ $\mathcal{E}$-quasi everywhere, where $(R_\alpha)_{\alpha>0}$ is the resolvent of $\mathbf{M}$, since $R_1 1$ is quasi-continuous. Since

$$R_1 1(x, y) = \mathrm{E}_{(x,y)} \left[\int_0^\infty \mathrm{e}^{-t} 1(X_t, Y_t)\,\mathrm{d}t \right],$$

this shows that $\zeta = \infty$ $P_{(x,y)}$-a.s. for $\mathcal{E}$-quasi all $(x, y) \in E$. Due to Remark 1.3.33, we may assume that $\mathbf{M}$ has continuous paths and infinite life time $P_{(x,y)}$-a.s. for all $(x, y) \in E$, which also implies that it is a Hunt process, and that we do not need to adjoin the cemetery Δ to the state space.□

Proposition 4.3.2. *Fix locally bounded μ-versions of Σ, Φ and Ψ. For $\mathcal{E}$-quasi-all $(x, y) \in E$, $P_{(x,y)}$ solves the martingale problem for $(L, C_c^2(E))$ in the sense that*

$$M_t^{[f],L} := f((X_t, Y_t)) - f((X_0, Y_0)) - \int_0^t Lf((X_s, Y_s))\,\mathrm{d}s \tag{4.3.3}$$

is an $(\mathcal{F}_t)_{t\ge0}$-martingale with respect to $P_{(x,y)}$ for all $f \in C_c^2(E)$.

Proof:
First, for any $f \in D(L)$ with quasi-continuous μ-version $\tilde{f}$ (which exists due to Proposition 1.3.24), $M_t^{[\tilde{f}],L}$ is an $(\mathcal{F}_t)_{t\ge0}$-martingale with respect to $P_{(x,y)}$ for $\mathcal{E}$-quasi all $(x, y) \in E$, where the exceptional set depends on f. This follows as in [CG08, Theorem 3 (iii)] via the Fukushima decomposition, see [Tru00, Theorem 4.5]. Let $(f_n)_{n\in\mathbb{N}}$ be a countable dense subset of $C_c^2(E)$, such that for any $f \in C_c^2(E)$, there is a subsequence $(f_k)_{k\in\mathbb{N}}$ which converges to f in C^2-norm and such that the supports of f_{n_k}, $k \in \mathbb{N}$ and f are included in one compact set $K \subseteq E$, see Corollary 1.5.2. Due to Lemma 1.3.20, the martingale property is satisfied wrt. $P_{(x,y)}$ for all f_n simultaneously, for $\mathcal{E}$-quasi-all $(x, y) \in E$. Let $f \in C_c^2(E)$ be arbitrary, then there is some subsequence $(f_k)_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ and $Lf_{n_k} \to Lf$ uniformly on E as $k \to \infty$. The claim then follows as in [CG08, Corollary 1]. □

The first part of this proof implies the following, see also [CG08, Theorem 3 (iv)].

Corollary 4.3.3. *Let $0 \le h \in L^2(E;\mu)$ be a probability density with respect to μ, for example $h \equiv 1$, and define $P_{h\mu}$ as in Definition 1.3.25. Then $P_{h\mu}$ solves the martingale problem for $(L, D(L))$ in the sense that $M_t^{[f],L}$ as in (4.3.3) is an $(\mathcal{F}_t)_{t\ge0}$-martingale with respect to $P_{h\mu}$ for all $f \in D(L)$.*

Proposition 4.3.4. *The process $(X_t, Y_t)_{t\ge0}$ with initial distribution P_μ is a weak solution of the Itô stochastic differential equation*

$$\begin{aligned} \mathrm{d}X_t &= Q\nabla\Psi(Y_t)\,\mathrm{d}t \\ \mathrm{d}Y_t &= \sqrt{2}\sigma(Y_t)\mathrm{d}B_t - (Q^*\nabla\Phi(X_t) - b(Y_t))\,\mathrm{d}t, \end{aligned} \tag{4.3.4}$$

which is the original SDE (0.4) that motivated the definition of L.

Proof:
Using cutoffs as in Remark 1.4.8 and stopping times like $\sigma_{E\setminus\overline{B_n(0))}}$, see Definition 1.3.27, it follows that $M_t^{[f],L}$ is a continuous local martingale for all $f\in C^2(E)$, where Lf is interpreted pointwisely. By setting $f(x,y):=z_i$ and $f(x,y):=z_iz_j$, where z_i is either x_i or y_i, we get the following:

(i) $f(x,y)=x_i$ implies $X_t^i-X_0^i-\int_0^t(Q\nabla\Psi(Y_s))_i\,\mathrm{d}s$ is a local martingale, denoted by $M_t^{x_i}$.

(ii) If $f(x,y)=x_ix_j$, then $Lf=0$, and hence $[M^{x_i},M^{x_j}]_t=0$, as in the proof of [KS98, Proposition 5.4.6]. Together with (i), this means $X_t^i-X_0^i=\int_0^t(Q\nabla\Psi(Y_s))_i\,\mathrm{d}s$.

(iii) $f(x,y)=y_i$ implies $Y_t^i-Y_0^i-\int_0^t b_i(Y_s)\,\mathrm{d}s+\int_0^t(Q^*\nabla\Phi(X_s))_i\,\mathrm{d}s$ is a local martingale, which we call $M_t^{y_i}$.

(iv) If $f(x,y)=y_iy_j$, then again as in the proof of [KS98, Proposition 5.4.6], we obtain that $[M^{y_i},M^{y_j}]_t=2\int_0^t a_{ij}(Y_s)\,\mathrm{d}s$.

As already mentioned in (ii), we get $\mathrm{d}X_t=Q\nabla\Psi(Y_t)\mathrm{d}t$. Define $M_t^y:=(M_t^{y_1},\dots,M_t^{y_{d_2}})$, $b(y):=(b_1(y),\dots,b_{d_2}(y))$ and let $\sigma(y)$ be the positive square root of $\Sigma(y)$, which is invertible everywhere since Σ is. Set $B_t:=\frac{1}{\sqrt{2}}\int_0^t\sigma^{-1}(Y_s)\mathrm{d}M_s^y$, which is a continuous local martingale since σ^{-1} is locally bounded due to uniform strict ellipticity of Σ. Then

$$\mathrm{d}Y_t=\mathrm{d}M_t^y+b(Y_t)\mathrm{d}t-Q^*\nabla\Phi(X_t)\mathrm{d}t=\sqrt{2}\sigma(Y_t)\mathrm{d}B_t+b(Y_t)\mathrm{d}t-Q^*\nabla\Phi(X_t)\mathrm{d}t.$$

By basic properties of the Itô integral (e.g. [KS98, 3.(2.19)]), we see that

$$\begin{aligned}
[B^i,B^j]_t&=\frac{1}{2}\int_0^t\sum_{k,\ell=1}^{d_2}\sigma_{ik}^{-1}\sigma_{j\ell}^{-1}(Y_s)\,\mathrm{d}[M^{y_k},M^{y_\ell}]_s=\int_0^t\sum_{k,\ell=1}^{d_2}\sigma_{ik}^{-1}\sigma_{j\ell}^{-1}a_{k\ell}(Y_s)\,\mathrm{d}s\\
&=\int_0^t\sum_{k,\ell=1}^{d_2}\sigma_{ik}^{-1}a_{k\ell}(\sigma^{-1})^*_{\ell j}(Y_s)\,\mathrm{d}s=\int_0^t(\sigma^{-1}\Sigma(\sigma^{-1})^*)_{ij}(Y_s)\,\mathrm{d}s\\
&=\delta_{ij}t,
\end{aligned}$$

so $(B_t)_{t\geq0}$ is a Brownian motion on $\mathbb{R}^{d_2}$ due to Lévy's characterization, which completes the proof. $\square$

Remark 4.3.5. Alternatively, we can find a weak solution to (4.3.4) by considering the probability law $\mathbb{P}_{h\mu}$ on $D([0,\infty),E_\Delta)$ induced by $P_{h\mu}$ as in Lemma 1.3.36. By Lemma 1.3.39 and Lemma 1.3.41, it holds that $\mathbb{P}_{h\mu}(C([0,\infty),E))=1$, so that we can consider $\mathbb{P}_{h\mu}$ as a measure on the space of continuous paths with infinite life time. Application of Lemma 1.3.38 and identifying the quadratic covariations of $M_t^{[f],L}$ via $N_t^{[f],L}$ for cutoffs of $f(x,y)=z_i$ as above, it follows analogously that the canonical coordinate process $(Z_t)_{t\geq0}$ of $\mathbb{P}$ is a weak solution to (4.3.4).

Now assume the setting of Theorem 4.1.10. Then as a result, we obtain for the transition semigroup $(p_t)_{t\geq0}$ of $\mathbf{M}$, that

$$\operatorname{Var}_\mu(p_tf)=\mu((p_tf)^2)-(\mu(p_tf))^2\leq\xi(t)\|f\|_{\mathrm{osc}}^2\qquad\text{for all }t\geq0,f\in L^\infty(\mu),$$

with ξ given by the above theorem, since $p_t f$ is a μ-version of $T_t f$ for all $f \in L^\infty(\mu)$ due to Remark 1.3.30. Moreover, the measure $\mathbb{P}_\mu$ is strongly mixing, hence ergodic, due to Lemma 1.3.43.

Similarly, under the assumptions of Theorem 4.2.10, for any $\theta_1 \in (1, \infty)$, one can choose θ_2 as stated to get

$$\operatorname{Var}_\mu(p_t f) = \|p_t f - \mu(p_t f)\|_H^2 = \|p_t f - \mu(f)\|_H^2 \leq \theta_1^2 \mathrm{e}^{-2\theta_2 t} \|f - \mu(f)\|_H^2$$

for all $t \geq 0$ and $f \in H$, which again yields strong mixing for $\mathbb{P}_\mu$. In the special case $d \coloneqq d_1 = d_2$ and $Q = I$, the stochastic differential equation (4.3.4) reduces to

$$\begin{aligned} \mathrm{d}X_t &= Y_t \,\mathrm{d}t \\ \mathrm{d}Y_t &= \sqrt{2}\sigma(Y_t)\mathrm{d}B_t - \left(\nabla\Phi(X_t) + \Sigma(Y_t)Y_t - \sum_{i,j=1}^{d} \partial_j a_{ij}(Y_t) \right) \mathrm{d}t, \end{aligned} \tag{4.3.5}$$

which in the case of constant Σ describes a Langevin equation with diffusion coefficient Σ. This justifies our title of Section 4.2, since the dependence of Σ on Y_t introduces multiplicative noise.

5 Langevin dynamics with multiplicative noise on infinite-dimensional Hilbert spaces

In this final chapter, we extend the essential m-dissipativity and hypocoercivity results gained earlier to infinite-dimensional Hilbert spaces. Due to the lack of a Lebesgue measure, we use a non-degenerate Gaussian measure as reference measure, which already is a probability measure. This corresponds to the case where in the definition of L in Definition 3.2.1, the potentials are both chosen to be Gaussian. For existence and properties of such Gaussian measures, we refer to [Pra06, Chapter 1]. While we can carry over many of the assumptions worked out to ensure the existence of a sensible generator core and to satisfy the hypocoercivity conditions, we have to be careful in how to choose the infinite-dimensional analogues. For example, we cannot assume the second-order coefficients to be uniformly strictly elliptic as in **(Σ1)**, since we require them to be of trace class, which implies that the sequence of eigenvalues converge to 0 due to the spectral theorem.

5.1 Preliminaries

Let X be a real separable Hilbert space with inner product $(\cdot,\cdot)_X$, $\mathcal{B}(X)$ be the corresponding Borel-σ-algebra, and let μ be a centered non-degenerate Gaussian measure on $(X, \mathcal{B}(X))$. We denote the set of all linear bounded operators on X by $\mathcal{L}(X)$, the subset of positive semi-definite symmetric operators by $\mathcal{L}^+(X)$, and the set of operators additionally being of trace class by $\mathcal{L}_1^+(X)$. The set of Hilbert-Schmidt operators on X is denoted by $\mathcal{L}_2(X)$.

The covariance operator corresponding to μ is denoted by Q and is an element of $\mathcal{L}_1^+(X)$, which is injective and hence positive-definite since μ is non-degenerate. Since it is symmetric and of trace class, there is a complete orthonormal system $B_X = (e_n)_{n\in\mathbb{N}}$ in X consisting of eigenvectors of Q to the positive eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$, which we can assume to be decreasing to zero. Due to positivity of Q, there exists an inverse operator defined on $Q(X)$ that satisfies $Q^{-1}e_k = \frac{1}{\lambda_k}e_k$ and therefore a square root $Q^{-\frac{1}{2}}$ defined on $\operatorname{span}\{e_k : k \in \mathbb{N}\}$ characterized by $Q^{-\frac{1}{2}}e_k = \frac{1}{\sqrt{\lambda_k}}e_k$.

Definition 5.1.1. For each $n \in \mathbb{N}$, define $X_n := \operatorname{span}\{e_1, \dots, e_n\}$ and denote the orthogonal projection from X to X_n by P_n, with the corresponding coordinate map $p_n : X \to \mathbb{R}^n$. This means

$$P_n(x) := \sum_{k=1}^{n} (x, e_k)_X e_k \quad \text{and} \quad p_n(x) := ((x, e_1)_X, \dots, (x, e_n)_X)$$

for all $x \in X$.

Let $C_b^\infty(\mathbb{R}^n)$ denote the space of bounded smooth real-valued functions on $\mathbb{R}^n$ with bounded derivatives and let μ^n be the image measure of μ under p_n, for each $n \in \mathbb{N}$. Then define

$$\mathcal{F}C_b^\infty(B_X, n) := \{f : X \to \mathbb{R} \mid f(x) = \varphi(p_n(x)) \text{ for some } \varphi \in C_b^\infty(\mathbb{R}^n)\}, \tag{5.1.1}$$

$$\mathcal{F}C_b^\infty(B_X) := \bigcup_{n\in\mathbb{N}} \mathcal{F}C_b^\infty(X, n), \tag{5.1.2}$$

$$L_n^2(X, \mu) := \{f : X \to \mathbb{R} \mid f = g \circ p_n \text{ for some } g \in L^2(\mathbb{R}^n, \mu^n)\}. \tag{5.1.3}$$

In order to deal with $L_n^2(X, \mu)$ in a practical way, we need to accurately know the measure μ^n. [Pra06, Corollary 1.19] yields the following characterization:

Lemma 5.1.2. *Let $n \in \mathbb{N}$, and $x_1, \dots, x_n \in X$. The image measure ν^n of μ under the map*

$$X \ni x \mapsto ((x, x_1)_X, \dots, (x, x_n)_X) \in \mathbb{R}^n$$

is the centered n-dimensional Gaussian measure with covariance matrix

$$Q_{\nu^n} := ((Qx_i, x_j)_X)_{1\le i,j\le n}.$$

In particular, the covariance matrix of μ^n is just $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

This directly implies the following:

Corollary 5.1.3. *For any $x_1, x_2 \in X$, it holds that*

$$\int_X (x, x_1)_X (x, x_2)_X \, \mu(\mathrm{d}x) = (Qx_1, x_2)_X.$$

This also implies that $\|x\|_X^2$ and therefore $\|x\|_X$ is μ-integrable, since

$$\int_X \|x\|_X^2 \, \mathrm{d}\mu = \int_X \sum_{n\in\mathbb{N}} (x, e_n)_X^2 \, \mathrm{d}\mu = \sum_{n\in\mathbb{N}} (Qe_n, e_n)_X = \sum_{n\in\mathbb{N}} \lambda_n < \infty$$

due to monotone convergence and the fact that Q is trace class.

We require the following:

Lemma 5.1.4. *$\mathcal{F}C_b^\infty(B_X)$ is dense in $L^2(X, \mu)$ and $\mathcal{F}C_b^\infty(B_X, n)$ is dense in $L_n^2(X, \mu)$ for all $n \in \mathbb{N}$.*

Proof:
See [PL14, Lemma 2.2]. □

Definition 5.1.5. Let $f : X \to \mathbb{R}$ be Fréchet-differentiable, then we denote its derivative at point x by $Df(x) \in X'$, where X' denotes the topological dual space of X. If f is twice Fréchet-differentiable, then its second order Fréchet derivative at point x is denoted by $D^2 f(x) \in \mathcal{L}(X; X')$. By identifying X with its dual via the Riesz isomorphism, we can interpret $Df(x)$ as an element of X and $D^2 f(x)$ as an element of $\mathcal{L}(X)$. Then, for $i, j \in \mathbb{N}$, we denote the partial derivative in direction e_i at point x by $\partial_i f(x) = (Df(x), e_i)$, and the second order partial derivative in directions e_i and e_j by $\partial_{ij} f(x) = \partial_{ji} f(x) = (D^2 f(x) e_i, e_j)$, since $D^2 f(x)$ is symmetric, see for example [Die69, (8.12.2)].

Remark 5.1.6. By definition, it holds that $Df(x) = \sum_{n\in\mathbb{N}} \partial_i f(x) e_i$ for all Fréchet-differentiable $f : X \to \mathbb{R}$. If $f = \varphi \circ p_n$ for some $n \in \mathbb{N}$, $\varphi \in C_b^\infty(\mathbb{R}^n)$, then the chain rule implies $Df(x) = \sum_{i=1}^n \partial_i \varphi(p_n(x)) e_i \in X_n$ for all $x \in X$.

With that notation, the following integration by parts formula follows, as seen by using Lemma 5.1.2 and the classical integration by parts formula:

Lemma 5.1.7. *Let $f, g \in \mathcal{F}C_b^1(B_X)$, which is defined analogously to $\mathcal{F}C_b^\infty(B_X)$. Then*

$$\int_X \partial_i f g \,\mathrm{d}\mu = -\int_X f \partial_i g \,\mathrm{d}\mu + \int_X (x, Q^{-1} e_i)_X f(x) g(x) \,\mu(\mathrm{d}x). \tag{5.1.4}$$

As can be seen for example in [AFP19, Proposition 4.5], the following Poincaré inequality holds:

Lemma 5.1.8. *Let λ_1 denote the largest eigenvalue of Q, then*

$$\int_X (QDf, Df)_X \,\mathrm{d}\mu \geq \lambda_1 \int_X (f - \mu(f))^2 \,\mathrm{d}\mu$$

holds for all $f \in \mathcal{F}C_b^\infty(B_X)$.

5.2 Setting and Operators

Now let U and V be Hilbert spaces like X with inner products $(\cdot,\cdot)_U$ and $(\cdot,\cdot)_V$ and Gaussian measures μ_1, μ_2 respectively. Let Q_i denote the covariance operators of μ_i, and let $B_U = (d_n)_{n\in\mathbb{N}}$ and $B_V = (e_n)_{n\in\mathbb{N}}$ be the induced orthonormal bases of U and V, respectively. The respective projections to induced subspaces are denoted by P_n^U, p_n^U, P_n^V and p_n^V.

Definition 5.2.1. Define the real separable Hilbert space $W = U \times V$ with the canonically defined inner product $(\cdot,\cdot)_W$, Borel-σ-algebra $\mathcal{B}(W) = \mathcal{B}(U) \otimes \mathcal{B}(V)$ and product measure $\mu := \mu_1 \otimes \mu_2$. Due to [Pra06, Theorem 1.12], μ is a centered Gaussian measure with covariance Q defined by $Q(u, v) = (Q_1 u, Q_2 v)$.

Set

$$B_W := \{(d_n, 0) \mid n \in \mathbb{N}\} \cup \{(0, e_n) \mid n \in \mathbb{N}\} \subseteq W.$$

as well as

$$\mathcal{F}C_b^\infty(B_W, n) := \{f : W \to \mathbb{R} \mid f(u,v) = \varphi(p_n^U(u), p_n^V(v)) \text{ for some } \varphi \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)\},$$

and define $\mathcal{F}C_b^\infty(B_W)$, $L_n^2(W,\mu)$ analogously, where $\mu^n := \mu_1^n \otimes \mu_2^n$, with μ_i being centered Gaussian measures on $\mathbb{R}^n$ with covariance matrices $Q_{i,n}$.

Note that $\mathcal{F}C_b^\infty(B_W, n)$ defined this way coincides with the definition in Definition 5.1.1 for $X = W$ by trivially extending φ. In particular, we can use the result of Lemma 5.1.4 for $X = W$.

Definition 5.2.2. For sufficiently differentiable $f : W \to \mathbb{R}$ and all $w = (u,v) \in W$, set

$$\begin{aligned}
D_1 f(w) &:= \sum_{n\in\mathbb{N}} (Df(w), (d_n, 0))_W (d_n, 0) \in U, \\
D_2 f(w) &:= \sum_{n\in\mathbb{N}} (Df(w), (0, e_n))_W (0, e_n) \in V, \\
\partial_{i,1} f(w) &:= (D_1 f(w), d_i)_U \\
\partial_{i,2} f(w) &:= (D_2 f(w), e_i)_V.
\end{aligned}$$

The second order derivatives and partial derivatives are named analogously.

Now that all necessary derivatives are defined, we can define differential operators on $\mathcal{F}C_b^\infty(B_W)$ analogously to Definition 3.2.1.

Definition 5.2.3. Let $K_{12} \in \mathcal{L}(U;V)$ be a bounded linear operator from U to V and set $K_{21} = K_{12}^* \in \mathcal{L}(V;U)$. Further assume the invariance properties $K_{12}(U_n) \subseteq V_n$ and $K_{21}(V_n) \subseteq U_n$ for all $n \in \mathbb{N}$.

Let the map $K_{22} : V \to \mathcal{L}^+(V)$ be Fréchet-differentiable with $DK_{22}(v) \in \mathcal{L}(V; \mathcal{L}(V))$ and partial derivatives $\partial_i K_{22}(v) = (DK_{22}(v))(e_i) \in \mathcal{L}(V)$ for each $v \in V$. Assume that for each $v \in V$, it holds that $K_{22}(v)(V_n) \subseteq V_n$ for all $n \in \mathbb{N}$.

Moreover, assume there is a strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that for each $n \le m_k$, it holds that $K_{22}(v)|_{V_n} = K_{22,m_k}(P_{m_k}^V(v))|_{V_n}$ for all $v \in V$, where $K_{22,m_k} : V_{m_k} \to \mathcal{L}^+(V_{m_k})$ is bounded and continuously Fréchet-differentiable. Assume that for each $k \in \mathbb{N}$, there is a constant $M_k \in (0,\infty)$ such that

$$\begin{aligned}
\sup_{v\in V_{m_k}} \|K_{22,m_k}(v)\|_{\mathcal{L}(V_{m_k})} &\le M_k \quad \text{and} \\
\|\partial_i K_{22,m_k}(v)\|_{\mathcal{L}(V_{m_k})} &\le M_k(1 + \|v\|_{V_{m_k}}) \quad \text{for all } v \in V_{m_k}, 1 \le i \le m_k,
\end{aligned}$$

and set $m^K(n) := \min_{k\in\mathbb{N}}\{m_k : m_k \ge n\}$.

Then the differential operators $(S, \mathcal{F}C_b^\infty(B_W))$ and $(A, \mathcal{F}C_b^\infty(B_W))$ on $H \coloneqq L^2(W,\mu)$ are defined by

$$\begin{aligned} Sf(u,v) \coloneqq \operatorname{tr}\left[K_{22}(v) \circ D_2^2 f(u,v)\right] &+ \sum_{i,j\in\mathbb{N}} (\partial_j K_{22}(v)e_i, e_j)_V \partial_{i,2} f(u,v) \\ &- (v, Q_2^{-1}K_{22}(v)D_2 f(u,v))_V \end{aligned}$$

and

$$Af(u,v) \coloneqq (u, Q_1^{-1}K_{21}D_2 f(u,v))_U - (v, Q_2^{-1}K_{12}D_1 f(u,v))_V,$$

respectively, for all $(u,v) \in W$. Finally, $(L, \mathcal{F}C_b^\infty(B_W))$ is defined via $L \coloneqq S - A$.

Remark 5.2.4. The invariance assumptions made on K_{12}, K_{21} and K_{22} ensure that S and A are well-defined on $\mathcal{F}C_b^\infty(B_W)$ and still yield finitely based functions. Indeed, let $f \in \mathcal{F}C_b^\infty(B_W, n)$ for some $n \in \mathbb{N}$ with corresponding $\varphi \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. By trivially extending φ if necessary, we can assume $m^K(n) = n$. Note that $Q_1^{-1}K_{21}D_2 f(u,v) \in U_n$, $Q_2^{-1}K_{12}D_1 f(u,v) \in V_n$, and $Q_2^{-1}K_{22}(v)D_2 f(u,v) \in V_n$ for all $(u,v) \in W$. Therefore, these maps are bounded in (u,v) due to uniform boundedness of $K_{22,n}$ and the fact that all derivatives of f are bounded. Together with the observation that all sums appearing in the definition of S are finite, as well as Corollary 5.1.3, it follows that $Sf(u,v) = Sf(P_n^U u, P_n^V v)$ and $Af(u,v) = Af(P_n^U u, P_n^V)$ as well as $Sf,\ Af \in H$.

Lemma 5.2.5. *On $H = L^2(W;\mu)$, the operator $(S, \mathcal{F}C_b^\infty(B_W))$ is symmetric and negative semi-definite; $(A, \mathcal{F}C_b^\infty(B_W))$ is antisymmetric, and therefore $(L, \mathcal{F}C_b^\infty(B_W))$ is dissipative. For $f, g \in \mathcal{F}C_b^\infty(B_W)$, we have the representation*

$$(Lf, g)_H = -\int_W (D_2 f, K_{22}D_2 g)_V - (D_1 f, K_{21}D_2 g)_U + (D_2 f, K_{12}D_1 g)_V \,\mathrm{d}\mu.$$

Proof:
Let $f, g \in \mathcal{F}C_b^\infty(B_W)$. As in Remark 5.2.4, we can assume $f, g \in \mathcal{F}C_b^\infty(B_W, n)$ for some $n \in \mathbb{N}$ with $m^K(n) = n$. For any $(u,v) \in W$, it holds that

$$Q_1^{-1}K_{21}D_2 f(u,v) = \sum_{k=1}^n \partial_{k,2} f(u,v) Q_1^{-1}K_{21}e_k = \sum_{k,\ell=1}^n \partial_{k,2} f(u,v)(K_{21}e_k, d_\ell) Q_1^{-1} d_\ell.$$

Using Lemma 5.1.7, we obtain

$$\int_W (u, Q_1^{-1}d_\ell)_U \partial_k f(u,v) g(u,v)\mu(\mathrm{d}(u,v)) = \int_W (g\partial_{\ell,1}\partial_{k,2} f + \partial_{k,2} f \partial_{\ell,1} g)\mathrm{d}\mu,$$

which shows that

$$\left((u, Q_1^{-1}K_{21}D_2 f)_U, g\right)_H = \int_W (K_{21}D_2 f, D_1 g)_U \,\mathrm{d}\mu + \sum_{k,\ell=1}^n (K_{21}e_k, d_\ell)_U (g, \partial_{\ell,1}\partial_{k,2} f)_H.$$

Similarly, it holds that

$$\left((v, Q_2^{-1}K_{12}D_1 f)_V, g\right)_H = \int_W (K_{12}D_1 f, D_2 g)_V \,\mathrm{d}\mu + \sum_{k,\ell=1}^n (K_{12}d_\ell, e_k)_V (g, \partial_{k,2}\partial_{\ell,1} f)_H.$$

Using $K_{12}^* = K_{21}$, this implies that

$$(Af, g)_H = \int_W (D_2 f, K_{12} D_1 g)_V - (D_1 f, K_{21} D_2 g)_U \,\mathrm{d}\mu,$$

so in particular $(Af, f)_H = 0$. Now consider the operator S. As before, we have

$$\begin{aligned} Q_2^{-1} K_{22}(v) D_2 f(u, v) &= \sum_{i=1}^n \partial_{i,2} f(u, v) Q_2^{-1} K_{22}(v) e_i \\ &= \sum_{i,j=1}^n \partial_{i,2} f(u, v)(K_{22}(v) e_i, e_j)_V Q_2^{-1} e_j \end{aligned}$$

for all $(u, v) \in W$. Due to the assumptions on K_{22}, the maps

$$(u, v) \mapsto \partial_{i,2} f(u, v)(K_{22}(v) e_i, e_j)_V$$

are elements of $\mathcal{F}C_b^1(B_W, n)$ for all $1 \le i, j \le n$, see Remark 5.2.4. Therefore, integration by parts is possible and yields

$$\begin{aligned} \int_W (v, Q_2^{-1} e_j) \partial_{i,2} f(K_{22} e_i, e_j)_V g \,\mathrm{d}\mu = &\int_W \partial_{j,2} \partial_{i,2} f(K_{22} e_i, e_j)_V g \,\mathrm{d}\mu \\ &+ \int_W \partial_{i,2} f(\partial_j K_{22} e_i, e_j)_V g \,\mathrm{d}\mu \\ &+ \int_W \partial_{i,2} f(K_{22} e_i, e_j)_V \partial_{j,2} g \,\mathrm{d}\mu \end{aligned}$$

Summing up over i and j, we get

$$\sum_{i,j=1}^n \int_W \partial_{i,2} f(K_{22} e_i, e_j)_V \partial_{j,2} g \,\mathrm{d}\mu = \int_W (K_{22} D_2 f, D_2 g)_V \,\mathrm{d}\mu = \int_W (D_2 f, K_{22} D_2 g)_V \,\mathrm{d}\mu$$

and

$$\begin{aligned} \sum_{i,j=1}^n \int_W \partial_{j,2} \partial_{i,2} f(K_{22} e_i, e_j)_V g \,\mathrm{d}\mu &= \sum_{i=1}^n \int_W (K_{22} e_i, (D_2^2 f)(e_i))_V g \,\mathrm{d}\mu \\ &= \int_W \operatorname{tr}\left[K_{22} D_2^2 f\right] g \,\mathrm{d}\mu \end{aligned}$$

due to pointwise symmetry of K_{22}. Therefore, we indeed get

$$(Sf, g)_H = -\sum_{i,j=1}^n \int_W \partial_{i,2} f(K_{22} e_i, e_j)_V \partial_{j,2} g \,\mathrm{d}\mu = -\int_W (D_2 f, K_{22} D_2 g)_V \,\mathrm{d}\mu,$$

which shows that S is symmetric and negative semi-definite since K_{22} is positive semi-definite. By Remark 1.2.14, all three operators are dissipative with domain $\mathcal{F}C_b^\infty(B_W)$. □

5.3 Essential m-dissipativity

Now we prove that the operator $(L, \mathcal{F}C_b^\infty(B_W))$ on H is essentially m-dissipative. Since dissipativity was shown in Lemma 5.2.5, it remains to show that $(I-L)(\mathcal{F}C_b^\infty(B_W))$ is dense in H, see Definition 1.2.17. Since $\mathcal{F}C_b^\infty(B_W)$ is dense in H, it suffices to approximate all such functions by $I-L$. The main idea here is to fix a finite dimension based on the target function, and to interpret L as an operator on the finite-dimensional subspace, which is possible due to Remark 5.2.4. Then we use our finite-dimensional m-dissipativity result from Theorem 3.5.1.

Definition 5.3.1. Fix $n \in \mathbb{N}$ such that $m^K(n) = n$. Then we define

$$K_{12,n} := ((K_{12}d_i, e_j)_V)_{ij}, \quad K_{21,n} := (K_{12,n})^* \quad \text{and}$$

$$\Sigma_n(y) := \left(\left(K_{22,n}\left(\sum_{k=1}^n y_k e_k \right) e_i, e_j \right)_V \right)_{ij} \qquad \text{for all } y \in \mathbb{R}^n.$$

Moreover, define the operators S_n, A_n and L_n on $H_n := L^2(\mu^n)$ as in Definition 3.2.1 for $\Sigma = \Sigma_n$, $\Phi(x) = \frac{1}{2}\langle x, Q_{1,n}^{-1}x\rangle$, $\Psi(y) = \frac{1}{2}\langle y, Q_{2,n}^{-1}y\rangle$, and $Q = K_{21,n}$, but with the domain $C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

Lemma 5.3.2. *Let $f \in \mathcal{F}C_b^\infty(B_W, n)$ for some $n \in \mathbb{N}$ with $m^K(n) = n$, with $f(u,v) = \varphi(p_n^U u, p_n^V v)$ for $\varphi \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Then $Sf(u,v) = S_n\varphi(p_n^U u, p_n^V v)$, and analogue statements hold for A and L.*

Proof:
Follows directly from Remark 5.2.4 together with $\nabla\Phi(x) = Q_{1,n}^{-1}x$ and $\nabla\Psi(y) = Q_{2,n}^{-1}y$. □

Next, we fix some assumptions on K_{22} such that they imply sufficient conditions on each Σ_n to ensure that the dense range condition can be verified. In particular, we need infinite-dimensional analogues to **(Σ1)**–**(Σ3)**.

Assumption (K1). Assume that there is some positive-definite $K_{22}^0 \in \mathcal{L}^+(V)$ which leaves each V_n invariant, such that

$$(v, K_{22}(y)v)_V \geq (v, K_{22}^0 v)_V \qquad \text{for all } v, y \in V.$$

Assumption (K2). Let $(m_k)_{k\in\mathbb{N}}$ be the sequence from the definition of K_{22} in Definition 5.2.3 and, for each $n \in \mathbb{N}$, let $k(n)$ be the k such that $m_{k(n)} = m^K(n)$. Assume that there are sequences $(\beta_k)_{k\in\mathbb{N}}$ in $[0,1)$ and $(N_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ such that, for all $n \in \mathbb{N}$,

$$|(\partial_i K_{22,m^K(n)}(v)e_n, e_j)_V| \leq N_{k(n)}(1 + \|v\|_{V_{m^K(n)}}^{\beta_{k(n)}})$$

for all $v \in V_{m_k}$, $1 \leq i \leq m^K(n)$ and $1 \leq j \leq n$.

For $n \in \mathbb{N}$, set $N^K(n) := 2\max\{N_{k(j)} : 1 \leq j \leq n\}$ and $\beta^K(n) := \max\{\beta_{k(n)} : 1 \leq j \leq n\}$.

Lemma 5.3.3. *Let $n \in \mathbb{N}$ with $m^K(n) = n$ and define Σ_n as in Definition 5.3.1. If K_{22} satisfies* **(K1)** *and* **(K2)***, then Σ_n satisfies* **(Σ1)**–**(Σ3)**.

Proof:
Set $\Sigma_n^0 \in \mathbb{R}^{n\times n}$ analogously to Σ_n for $K_{22,n}^0$. Since K_{22}^0 is positive-definite, all eigenvalues $\lambda_1^0, \dots, \lambda_n^0$ of Σ_n^0 are positive, and we define $c_n := \min_{i\in\{1,\dots,n\}} \lambda_i > 0$. For any $y \in \mathbb{R}^n$, set $\tilde{y} := \sum_{i=1}^n y_i e_i \in V_n$. Then

$$\langle y, \Sigma_n(v)y\rangle = (\tilde{y}, K_{22,n}(\tilde{v})\tilde{y})_V \geq (\tilde{y}, K_{22,n}^0\tilde{y})_V = \langle y, \Sigma_n^0 y\rangle \geq c_n|y|^2$$

for all $y, v \in \mathbb{R}^n$. Hence, (Σ1) holds with $c_{\Sigma_n} := c_n^{-1}$. Due to definition of K_{22}, we already have that all entries of Σ_n are bounded and continuously differentiable, hence in particular locally Lipschitz. Moreover, let $a_{ij,n}$ denote the entry of Σ_n at position (i,j), where we may assume that $j \leq i$, and let $k \in \{1,\dots,n\}$. Then

$$\begin{aligned}|\partial_k a_{ij,n}(y)| &= |\partial_k (K_{22,n}(\tilde{y})e_i, e_j)_V| = |(\partial_k K_{22,n}(\tilde{y})e_i, e_j)_V| \leq N_{k(i)}(1 + \|\tilde{y}\|_{V_n}^{\beta_n}) \\ &\leq 2N_{k(i)}(1 + \|\tilde{y}\|_{V_n}^{\beta^K(n)}) \leq N^K(n)(1 + \|\tilde{y}\|_{V_n}^{\beta^K(n)})\end{aligned}$$

by (K2), so Σ_n satisfies (Σ3) with constants $M = N^K(n)$ and $\beta = \beta^K(n)$. □

Remark 5.3.4. From the invariance properties of K_{12}, K_{21} and K_{22} in Definition 5.2.3, it follows quickly that they are all diagonal in the sense that $K_{12}d_i = \alpha_i e_i$ for some real α_i and $K_{22}(v)e_i = \lambda_{22,i}(v)e_i$ for some non-negative continuously differentiable $\lambda_{22,i} : V \to \mathbb{R}$. We haven't used that fact concretely so far, since it seems natural to allow at least “block-diagonal” operators with bounded block size, such that they agree with the blocks induced by the sequence $(m_k)_{k\in\mathbb{N}}$. However, this makes some assumptions now and later harder to verify, so we keep the assumptions in this form. In that case, (K1) just means that each $\lambda_{22,i}$ is bounded from below by a positive constant $\lambda_i^0 \in \mathbb{R}$, and (K2) reduces to $|\partial_i \lambda_{22,n}(v)| = |\partial_i \lambda_{22,n}(P_{m^K(n)}^V v)| \leq N_{k(n)}(1 + \|P_{m^K(n)}^V v\|_V^{\beta_{k(n)}})$ for all $1 \leq i \leq m^K(n)$ and $n \in \mathbb{N}$.

Proposition 5.3.5. *Let $n \in \mathbb{N}$ such that $m^K(n) = n$ and let K_{22} satisfy* (K1) *and* (K2)*. Then $(L_n, C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ is essentially m-dissipative on $L^2(\mathbb{R}^n \times \mathbb{R}^n, \mu^n)$.*

Proof:
We use Theorem 3.5.1, so we have to verify condition (C). Due to Lemma 5.3.3, the conditions on Σ_n are satisfied. By definition of Φ and Ψ, it follows that $\nabla\Phi(x) = Q_{1,n}^{-1}x$, $\nabla\Psi(y) = Q_{2,n}^{-1}$ and $\nabla^2\Phi(x) = 0 = \nabla^2\Psi(y)$ for all $x, y \in \mathbb{R}^n$. In particular, (Ψ1)–(Ψ3) and (Φ1) are satisfied. Moreover, for any $x \in \mathbb{R}^n$, it holds that

$$|\nabla\Phi(x)|^2 = \sum_{i=1}^n \frac{1}{\lambda_{1,i}^2} x_i^2 \leq \frac{1}{\lambda_{1,n}^2}|x|^2,$$

since $Q_{1,n} = \operatorname{diag}(\lambda_{1,1}, \dots, \lambda_{1,n})$, where $(\lambda_{1,i})_{i\in\mathbb{N}}$ is the decreasing sequence of eigenvalues of Q_1. Therefore, Φ satisfies the last condition of (C) for $N = \lambda_{1,n}^{-1}$ and $\gamma = 1 < (\beta^K(n))^{-1}$. As a result of Theorem 3.5.1, it follows that $(L_n, C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ is essentially m-dissipative on $L^2(\mu^n)$. Since $C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ extends that domain, and $(L_n, C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ is dissipative on $L^2(\mu^n)$ due to Lemma 5.2.5, the claim follows. □

Now finally, we are able to prove the central result of this section:

Theorem 5.3.6. *Let K_{22} satisfy* **(K1)** *and* **(K2)**. *Then $(L, \mathcal{F}C_b^\infty(B_W))$ is essentially m-dissipative on $H = L^2(\mu)$.*

Proof:
As mentioned above, we only need to show that $(I - L)(\mathcal{F}C_b^\infty(B_W))$ is dense in H, since Lemma 5.2.5 provides dissipativity of $(L, \mathcal{F}C_b^\infty(B_W))$. Let $g \in \mathcal{F}C_b^\infty(B_W)$, then there is some $n \in \mathbb{N}$ such that $g \in \mathcal{F}C_b^\infty(B_W, n)$. As before, we extend g trivially to $\mathcal{F}C_b^\infty(B_W, m^K(n))$, so that we can assume $n = m^K(n)$. Let $\varphi_g \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $g(u, v) = \varphi(p_n^U u, p_n^V v)$ for all $(u, v) \in W$ and let $\varepsilon > 0$. Then

$$\begin{aligned}
\|(I - L)f - g\|_{L^2(\mu)}^2 &= \int_W \left((I - L)f(P_n^U u, P_n^V v) - g(P_n^U u, P_n^V v)\right)^2 \mu(\mathrm{d}(u, v)) \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} \left((I - L_n)\varphi_f(x, y) - \varphi_g(x, y)\right)^2 \mu^n(\mathrm{d}(x, y)) \\
&= \left\|(I - L_n)\varphi_f - \varphi_g\right\|_{L^2(\mu^n)}^2
\end{aligned}$$

for all $f \in \mathcal{F}C_b^\infty(B_W, n)$ with corresponding $\varphi_f \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Due to Proposition 5.3.5, there is some $h \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\|(I - L_n)h - \varphi_g\|_{L^2(\mu^n)} < \varepsilon$. Setting $f_h(u, v) := h(p_n^U u, p_n^V v)$ yields $f_h \in \mathcal{F}C_b^\infty(B_W, n)$ with $\|(I - L)f_h - g\|_{L^2(\mu)} < \varepsilon$. Since $\mathcal{F}C_b^\infty(B_W)$ is dense in H, this proves that $(I - L)(\mathcal{F}C_b^\infty(B_W))$ is dense in H. □

5.4 Hypocoercivity

Throughout the remainder of this chapter, we assume **(K1)** and **(K2)** unless specifically stated otherwise. In that case, the operator $(L, \mathcal{F}C_b^\infty(B_W))$ as defined in Definition 5.2.3 is dissipative and therefore closable on $H = L^2(\mu)$ (see Lemma 1.2.16), and its closure $(L, D(L))$ generates an sccs $(T_t)_{t \geq 0}$ on H by Theorem 5.3.6.

As in Chapter 4, we restrict the setting to the Hilbert space $H^0 := \{f \in H : \mu(f) = 0\}$ and operator domain $\mathcal{D} := \mathcal{F}C_b^\infty(B_W) \cap H^0$.

Proposition 5.4.1. *The operator $(L, \mathcal{D})$ is essentially m-dissipative on H^0 and its closure, denoted by $(L_0, D(L_0))$, generates a sub-Markovian sccs $(T_t^0)_{t \geq 0}$ on H^0.*

Proof:
Similar to the proof of Lemma 3.6.1, it can be shown that $(L, \mathcal{F}C_b^\infty(B_W))$ is an abstract diffusion operator, which implies that it is a Dirichlet operator and that $(T_t)_{t \geq 0}$ is sub-Markovian. Due to Lemma 1.3.3, $(T_t)_{t \geq 0}$ is conservative and μ-invariant, since $1 \in \mathcal{F}C_b^\infty(B_W)$ and $L1 = 0$ by Lemma 5.2.5. The latter further implies that $Lf \in H^0$ for all $\mathcal{F}C_b^\infty(B_W)$. All this together means that $(L, \mathcal{D})$ and the restriction $(T_t^0)_{t \geq 0}$ of $(T_t)_{t \geq 0}$ to H^0 are well-defined as operators on H^0.

Dissipativity of $(L,\mathcal{D})$ is inherited from $(L,\mathcal{F}C_b^\infty(B_W))$, and the dense range condition can be verified as follows: Let $f\in H^0$, then there is a sequence $(f_n)_{n\in\mathbb{N}}$ in H such that $(I-L)f_n\to f$ in H. In particular, $\mu(f_n)\to\mu(f)=0$, so by setting $g_n:=f_n-\mu(f_n)$, it follows that $(I-L)g_n\to f$ since L acts trivially on constants. Since $g_n\in\mathcal{D}$ for all $n\in\mathbb{N}$, it follows that $(L,\mathcal{D})$ is essentially m-dissipative. As in the proof of Theorem 4.1.10, its closure $(L_0,D(L_0))$ is the generator of $(T_t^0)_{t\geq0}$, which is sub-Markovian since $(T_t)_{t\geq0}$ is. □

Definition 5.4.2. Let $H^0=H_1^0\oplus H_2^0$, where H_1^0 is provided by the orthogonal projection

$$P\,:\,H^0\to H_1^0,\quad f\mapsto Pf:=\int_W f(u,v)\,\mu_2(\mathrm{d}v).$$

For any $f\in H$, we can interpret Pf as an element of $L^2(\mu_1)$, in which case we denote it by f_P. Further let $(S_0,D(S_0))$ and $(A_0,D(A_0))$ be the closures in H^0 of $(S,\mathcal{D})$ and $(A,\mathcal{D})$, respectively.

Since we want to use the results from [EG21], we also define the following operators analogously to the cited source:

Definition 5.4.3. The operators $(C,D(C))$ and $(Q_1^{-1}C,D(Q_1^{-1}C))$ on H^0 are defined by

$$\begin{aligned} C&:=K_{21}Q_2^{-1}K_{12}, & D(C)&:=\{u\in U\mid K_{12}u\in D(Q_2^{-1})\}\\ Q_1^{-1}C&:=Q_1^{-1}K_{21}Q_2^{-1}K_{12}, & D(Q_1^{-1}C)&:=\{u\in D(C)\mid Cu\in D(Q_1^{-1})\}, \end{aligned}$$

respectively. In particular, $\mathcal{F}C_b^\infty(B_U)$ is a subset of the domain of both operators.

For all $f\in\mathcal{F}C_b^\infty(B_U)$, define $Nf(u):=\mathrm{tr}[CD^2f(u)]-(u,Q_1^{-1}CDf(u))_U$ for all $u\in U$.

In order to gain useful properties of N, we need the following:

Assumption (K3). The operator $K_{21}K_{12}=K_{12}^*K_{12}$ is positive-definite on U.

We collect a few properties of the newly defined operators:

Proposition 5.4.4. *Let $(C,D(C))$ and $(Q_1^{-1}C,D(Q_1^{-1}C))$ be defined as above. Then:*

(i) $(C,D(C))$ is symmetric and positive semi-definite on U.

(ii) $(C,D(C))$ leaves U_n invariant for each $n\in\mathbb{N}$.

(iii) If (K3) holds, then the operator $(N,\mathcal{F}C_b^\infty(B_U))$ is essentially self-adjoint on $L^2(U;\mu_1)$.

Proof:

(i) Let $u_1,u_2\in D(C)$. Then

$$(Cu_1,u_2)_U=(Q_2^{-1}K_{12}u_1,K_{12}u_2)_V=(K_{12}u_1,Q_2^{-1}K_{12}u_2)_V=(u_1,Cu_2)$$

due to symmetry of Q_2^{-1} and definition of K_{21}. Since Q_2^{-1} is positive-definite, positive semi-definiteness of C follows immediately.

(ii) Since K_{12} maps U_n to V_n, K_{21} maps V_n to U_n, and Q_2^{-1} leaves V_n invariant, the claim follows directly.

(iii) This can be proven analogously to Theorem 5.3.6, since the matrices in $\mathbb{R}^{n\times n}$ induced by C are constant with positive eigenvalues, which allows usage of Theorem 3.3.1 by point (i). Even though N might be unbounded, it is well-defined on $\mathcal{F}C_b^\infty(B_U)$ within $L^2(U;\mu_1)$, and dissipativity is implied by the integration by parts formula from Lemma 5.1.7. □

As a consequence, we gain the following estimates, see [EG21, Theorem 2]:

Lemma 5.4.5. *Let* **(K3)** *be satisfied, then*

$$\int_U (CDf, Df)_U \,\mathrm{d}\mu_1 \le \int_U ((I-N)f)^2 \,\mathrm{d}\mu_1 \quad \textit{and} \tag{5.4.1}$$

$$\int_U (Q_1^{-1}CDf, CDf)_U \,\mathrm{d}\mu_1 \le 4\int_U ((I-N)f)^2 \,\mathrm{d}\mu_1 \tag{5.4.2}$$

for all $f \in \mathcal{F}C_b^\infty(B_U)$.

The following is now a collection of results from [EG21]:

Proposition 5.4.6. *Let* **(K3)** *be satisfied. Then:*

(i) $H_1^0 \subseteq D(S_0)$ *with* $S_0 \circ P = 0$.

(ii) $P(\mathcal{D}) \subseteq D(A_0)$ *and* $A_0Pf(u,v) = (-v, Q_2^{-1}K_{12}D_1(Pf)(u,v))$ *for all* $f \in \mathcal{D}$, $(u,v) \in W$.

(iii) $PA_0Pf = 0$ *for all* $f \in \mathcal{D}$.

(iv) $A_0P(\mathcal{D}) \subseteq D(A_0)$ *with*

$$A_0^2Pf = \sum_{i,j\in\mathbb{N}} (v, Q_2^{-1}K_{12}d_i)_V (v, Q_2^{-1}K_{12}d_j)\partial_{1,ij}(Pf) - (u, Q_1^{-1}CD_1(Pf))_U$$

for all $f \in \mathcal{D}$.

(v) $Gf := PA_0^2Pf = \mathrm{tr}[CD_1^2 f_P] - (u, Q_1^{-1}CD_1 f_P)_U$ *for all* $f \in \mathcal{D}$, *and* $(G,\mathcal{D})$ *is essentially m-dissipative on* H^0.

In particular, the data conditions **(D1)**–**(D3)** *are satisfied, and* $\mathcal{D}$ *is a core for the operator* $(G, D(G))$ *as defined in Definition 2.1.2.*

Proof:

(i) Let $f \in H^0$. Then there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{F}C_b^\infty(B_W)$ such that $f_n \to f$, therefore $f_n - \mu(f_n) \to f$ in H^0. For each $n \in \mathbb{N}$, $f_n - \mu(f_n)$ is in $\mathcal{D}$, and clearly $P(f_n - \mu(f_n)) \in \mathcal{D}$ as well. Since $\mathcal{D} \subseteq D(S_0)$ and $S_0P(f_n - \mu(f_n)) = 0$ for all $n \in \mathbb{N}$, we can use that $P(f_n - \mu(f_n)) \to Pf$ to obtain that $Pf \in D(S_0)$ with $S_0Pf = 0$, since $(S_0, D(S_0))$ is closed.

(ii) see [EG21, Lemma 6 (i)].

(iii) see [EG21, Lemma 7].

(iv) see [EG21, Lemma 6 (ii)].

(v) see [EG21, Lemma 6 (iii), Proposition 3]. □

In particular, we can define the bounded operator B on H^0 as in Definition 2.1.4, which acts as $(I-G)^{-1}(A_0P^*)$ on $D(A_0P^*)$. Now we can start verifying the hypocoercivity conditions, where we start with boundedness of the auxiliary operators $BA_0(I-P)$ and BS_0. For the first part, we directly obtain from [EG21, Proposition 6]:

Proposition 5.4.7. *The operator $(BA_0(I-P), \mathcal{D})$ is bounded, so that the second inequality in* **(H1)** *holds with $c_2 = 8$.*

Now we introduce two new assumptions, which combine with **(K3)** to yield boundedness of BS_0.

Assumption (K4). Recall **(K1)** and let $K_{22}^v : V \to \mathcal{L}^+(V)$ be such that $K_{22}(v) = K_{22}^0 + K_{22}^v(v)$ for each $v \in V$. Further, let the following hold:

(i) There is some $C_{22}^0 \in (0,\infty)$ such that $\|Q_2^{-\frac{1}{2}} K_{22}^0 Q_2^{-\frac{1}{2}}\|_{\mathcal{L}(V)} \le C_{22}^0$.

(ii) There exists some $C_{22}^v \in (0,\infty)$ fulfilling

$$\|Q_2^{-1} K_{22}^v(v) Q_2^{-\frac{1}{2}}\|_{\mathcal{L}(V)} \le C_{22}^v$$

for all $v \in V$.

In particular, this implies that

$$\|Q_2^{-\frac{1}{2}} K_{22}(v) Q_2^{-\frac{1}{2}}\|_{\mathcal{L}(V)} \le C_{22} := C_{22}^0 + \sqrt{\lambda_{2,1}}C_{22}^v$$

for all $v \in V$, where $\lambda_{2,1}$ denotes the largest eigenvalue of Q_2.

Assumption (K5). Let $(\lambda_{2,n})_{n\in\mathbb{N}}$ denote the sequence of eigenvalues of Q_2, and recall the assumptions on $\partial_i K_{22}$ from **(K2)**. Assume that the sequence $(\alpha_n^{22})_{n\in\mathbb{N}}$ with $\alpha_n^{22} := N^K(n)(\lambda_{2,n})^{-\frac{1}{2}}$ is an element of $\ell^2(\mathbb{R})$, and set $M_{22} := \|(\alpha_n^{22})_{n\in\mathbb{N}}\|_{\ell^2}$.

Proposition 5.4.8. *Let* **(K3)**, **(K4)** *and* **(K5)** *hold. Then $(BS_0, \mathcal{D})$ is a bounded operator on H^0, and the first inequality in* **(H1)** *is satisfied for*

$$c_1 := C_{22}^0 + C_{22}^v\sqrt{\mathrm{tr}[Q_2]} + M_{22}\sqrt{8(1+\mathrm{tr}[Q_2])}.$$

Proof:
We prove this analogously to Proposition 4.1.5. So let $f \in \mathcal{D}$ and $h \in D(S_0)$ be arbitrary. Then by definition of $D(S_0)$, there is a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ such that $h_n \to h$ and $S_0 h_n \to S_0 h$ in H^0 as $n \to \infty$. Fix some $n \in \mathbb{N}$, then

$$\begin{aligned} (S_0 h_n, A_0 P f)_{H^0} &= \int_W (D_2 h_n(u,v), K_{22}(v) D_2 A_0 P f(u,v))_V \, \mu(\mathrm{d}(u,v)) \\ &= -\int_U \int_V (D_2 h_n(u,v), K_{22}(v) Q_2^{-1} K_{12} D_1 f_P(u))_V \, \mu(\mathrm{d}(u,v)) \\ &= -\sum_{k\in\mathbb{N}} \int_U \int_V \partial_{2,k} h_n(u,v) (K_{22}(v) e_k, Q_2^{-1} K_{12} D_1 f_P(u))_V \, \mu(\mathrm{d}(u,v)). \end{aligned} \tag{5.4.3}$$

Here, the first equality follows from the representation of S in Lemma 5.2.5, the second equality follows from Proposition 5.4.6 (ii), and the last line is due to symmetry of $K_{22}(v)$ for any $v \in V$. Note that the sum there is finite due to invariance properties of K_{22} and K_{12}. Applying integration by parts (see Lemma 5.1.7), we obtain

$$\begin{aligned} (S_0 h_n, A_0 P f)_{H^0} = &\sum_{k\in\mathbb{N}} \int_U \int_V h_n (\partial_k K_{22}(v) e_k, Q_2^{-1} K_{12} D_1 f_P(u))_V \, \mathrm{d}\mu \\ &- \sum_{k\in\mathbb{N}} \int_U \int_V (v, Q_2^{-1} e_k)_V h_n (e_k, K_{22}(v) Q_2^{-1} K_{12} D_1 f_P(u))_V \, \mathrm{d}\mu \\ = &(h_n, Tf)_H, \end{aligned}$$

where $T : \mathcal{D} \to H$ is defined by

$$Tf := \sum_{k\in\mathbb{N}} (\partial_k K_{22}(v) e_k, Q_2^{-1} K_{12} D_1 f_P(u))_V - (v, Q_2^{-1} K_{22}(v) Q_2^{-1} K_{12} D_1 f_P(u))_V$$

Note that Tf is indeed in H, since all appearing sums are finite and $\|v\|_V \in L^2(\mu_2)$, together with the properties of K_{22}. Moreover, since $1 \in \mathcal{F}C_b^\infty(B_W)$, it follows analogously to (5.4.3) that $\mu(Tf) = (1, Tf)_H = (S1, A_0 P f)_H = 0$, so $Tf \in H^0$. Now letting $n \to \infty$, we see that $A_0 P f \in D(S_0^*)$ with $S_0^* A_0 P f = Tf$.

This means that we are able to apply Lemma 2.4.4, so we set $g := (I - G) f$. We need to show that there is some $C < \infty$ such that

$$\|(BS_0)^* g\|_H = \|S_0^* A_0 P f\|_H = \|Tf\|_H \le C \|g\|_H \tag{5.4.4}$$

holds for any choice of f.

Due to Corollary 5.1.3 and **(K4)**, we have that

$$\begin{aligned}
\|(v, Q_2^{-1}K_{22}^0 Q_2^{-1}K_{12}D_1 f_P)_V\|_H^2 &= \int_U (K_{22}^0 Q_2^{-1}K_{12}D_1 f_P(u), Q_2^{-1}K_{22}^0 Q_2^{-1}K_{12}D_1 f_P(u))_V \,\mu_1(\mathrm{d}u) \\
&= \int_U ((Q_2^{-\frac{1}{2}}K_{22}^0 Q_2^{-\frac{1}{2}})^2 Q_2^{-\frac{1}{2}}K_{12}D_1 f_P, Q_2^{-\frac{1}{2}}K_{12}D_1 f_P)_V \,\mathrm{d}\mu_1 \\
&\leq (C_{22}^0)^2 \int_U (CD_1 f_P, D_1 f_P)_V \,\mathrm{d}\mu_1 \leq (C_{22}^0)^2 \int_U ((I-N)f_P)^2 \,\mathrm{d}\mu_1 \\
&= (C_{22}^0)^2 \int_U \left(\int_V (I-G)f \,\mathrm{d}\mu_2\right)^2 \mathrm{d}\mu_1 \leq (C_{22}^0)^2 \|g\|_H^2,
\end{aligned}$$

where we applied the estimate from (5.4.1). On the other hand,

$$\begin{aligned}
\|(v, Q_2^{-1}K_{22}^v(v) Q_2^{-1}K_{12}D_1 f_P(u))_V\|_H^2 &\leq \int_W \|v\|_V^2 \|(Q_2^{-1}K_{22}^v(v)Q_2^{-\frac{1}{2}})Q_2^{-\frac{1}{2}}K_{12}D_1 f_P(u)\|_V^2 \,\mathrm{d}\mu \\
&\leq \int_W \|v\|_V^2 (C_{22}^v)^2 \,\|Q_2^{-\frac{1}{2}}K_{12}D_1 f_P(u)\|_V^2 \,\mathrm{d}\mu \\
&= (C_{22}^v)^2 \int_V \|v\|_V^2 \,\mathrm{d}\mu_2 \int_U (CD_1 f_P, D_1 f_P)_V \,\mathrm{d}\mu_1 \\
&\leq (C_{22}^v)^2 \,\mathrm{tr}[Q_2]\, \|g\|_H^2,
\end{aligned} \tag{5.4.5}$$

again by **(K4)**, Corollary 5.1.3, and (5.4.1).

This shows that the second summand of Tf can be bounded relatively to g.

For the first summand, note that we can find some $\alpha_{22}^{(k)}(v) \in (0,\infty)$ such that

$$Q_2^{-\frac{1}{2}}\partial_k K_{22}(v)e_k = \alpha_{22}^{(k)}(v)e_k$$

for all $k \in \mathbb{N}$. Then

$$(\partial_k K_{22}(v)e_k, Q_2^{-1}K_{12}D_1 f_P(u))_V = \alpha_{22}^{(k)}(v)(e_k, Q_2^{-\frac{1}{2}}K_{12}D_1 f_P(u))_V,$$

so since $\alpha_{22}^{(k)}(v) \leq \alpha_k^{22}(1+\|v\|^{\beta^K(k)}) \leq 2\alpha_k^{22}(1+\|v\|)$ by **(K5)**, it follows that

$$\left(\sum_{k\in\mathbb{N}} (\partial_k K_{22}(v)e_k, Q_2^{-1}K_{12}D_1 f_P(u))_V\right)^2 \leq 4M_{22}^2(1+\|v\|)^2 \sum_{k\in\mathbb{N}} (e_k, Q_2^{-\frac{1}{2}}K_{12}D_1 f_P(u))_V^2.$$

Note that the right hand side factorizes into the u- and v-dependent components, so integration over μ yields a product of integrals with respect to μ_1 and μ_2.

Since $(1+\|v\|)^2 \leq 2(1+\|v\|^2)$, we obtain

$$\begin{aligned}
&\int_W \left(\sum_{k\in\mathbb{N}} (\partial_k K_{22}(v)e_k, Q_2^{-1}K_{12}D_1 f_P(u))_V\right)^2 \mathrm{d}\mu \\
&\qquad \leq 8M_{22}^2(1+\mathrm{tr}[Q_2]) \int_U \|Q_2^{-\frac{1}{2}}K_{12}D_1 f_P\|_V^2 \,\mathrm{d}\mu_1.
\end{aligned}$$

As in (5.4.5), this shows that the first summand of Tf can also be bounded relative to g. Overall, we can see

$$\|Tf\|_H \leq \left(C_{22}^0 + C_{22}^v\sqrt{\operatorname{tr}[Q_2]} + M_{22}\sqrt{8(1+\operatorname{tr}[Q_2])}\right)\|g\|_H.$$

This means that (5.4.4) holds for $C = c_1$ as claimed, and Lemma 2.4.4 proves that c_1 is indeed an upper bound for the operator BS_0. □

Remark 5.4.9. In the proof, we have always used $(CD_1 f_p, D_1 f_p)_U$ as a bounding term, in order to apply the first inequality from Lemma 5.4.5. It seems clear that by involving eigenvalues of Q_1 into the assumptions **(K4)** and **(K5)**, we can leverage all the invariance properties across finite-dimensional subspaces to instead use $(Q_1^{-1}CDf_P, CDf_P)_U$ as a bound. In either case, the aforementioned Lemma enables us to the bound all terms relatively to g. However, since this would make K_{22} also dependent of Q_1, it doesn't feel natural to assume, and would introduce more confusing notation, hence we skip it here.

Now that the first hypocoercivity condition is proven, we are left to show **(H2)** and **(H3)**. For this, we assume modified Poincaré inequalities based on K_{22} and K_{12}.

Assumption (K6). Assume that there is some $c_S \in (0,\infty)$ such that

$$\int_V (K_{22}(v)D_2 f, D_2 f)_V \,\mathrm{d}\mu_2 \geq c_S \int_V (f - \mu_2(f))^2 \,\mathrm{d}\mu_2$$

for all $f \in \mathcal{F}C_b^\infty(B_V)$.

Assumption (K7). Assume that there is some $c_A \in (0,\infty)$ such that

$$\int_U (Q_2^{-1}K_{12}D_1 f, K_{12}D_1 f)_V \,\mathrm{d}\mu_1 \geq c_A \int_U (f - \mu_1(f))^2 \,\mathrm{d}\mu_1$$

for all $f \in \mathcal{F}C_b^\infty(B_U)$.

Remark 5.4.10.

(i) Recall K_{22}^0 from **(K1)**, and let $\lambda_{2,i}$ denote the i-th eigenvalue of Q_2. Then due to Lemma 5.1.8, we have

$$\int_V (Q_2 D_2 f, D_2 f)_V \,\mathrm{d}\mu_2 \geq \lambda_{2,1} \int_V (f - \mu_2(f))^2 \,\mathrm{d}\mu_2$$

for all $f \in \mathcal{F}C_b^\infty(B_V)$. So if there is some $\omega_{22} \in (0,\infty)$ such that $\lambda_k^0 \geq \omega_{22}\lambda_{2,k}$ for each $k \in \mathbb{N}$, where λ_k^0 denotes the eigenvalue of K_{22}^0 to the eigenvector e_k, then **(K6)** holds with $c_S = \frac{\lambda_{2,1}}{\omega_{22}}$.

(ii) Similarly, if there is some $\omega_{12} \in (0,\infty)$ such that $\lambda_{12,k}^2 \geq \omega_{12}\lambda_{1,k}\lambda_{2,k}$ for all $k \in \mathbb{N}$, where $\lambda_{12,k}$ denotes the singular value of K_{12} to d_k, i.e. $K_{12}d_k = \lambda_{12,k}e_k$, then **(K7)** holds with $c_A = \frac{\lambda_{1,1}}{\omega_{12}}$.

(iii) In the case of (i), together with **(K4)**, we can see that in the sense of eigenvalues, K_{22} is "equivalent" to Q_2. This is not surprising, since in the case $K_{22}(v) = Q_2$ for all $v \in V$, the symmetric operator S interpreted on $L^2(\mu_2)$ is the well-known Ornstein-Uhlenbeck operator from Malliavin calculus, for which hypercontractivity results are known, see for example [LP20] for an overview. As can be seen in [Wan17], hypercontractivity of the semigroup yields exponential convergence to the equilibrium measure.

Under these conditions, we can easily verify macroscopic and microscopic coercivity:

Proposition 5.4.11. *Let* **(K6)** *hold. Then* S_0 *satisfies* **(H2)** *with* $\Lambda_m = c_S$.

Proof:
Let $f \in \mathcal{D}$ and set $f_u := f(u,\cdot) - Pf(u) \in \mathcal{F}C_b^\infty(B_V)$ for any $u \in U$. Then $\mu_2(f_u) = 0$ and $D_2 f_u(v) = D_2 f(u,v)$ for all $u \in U, v \in V$. By **(K6)** and Lemma 5.2.5, it then holds that

$$\begin{aligned} c_S\|(I-P)f\|_H^2 &= c_S \int_U \int_V f_u^2 \,\mathrm{d}\mu_2\,\mathrm{d}\mu_1 \leq \int_U \int_V (K_{22}(v)D_2 f_u, D_2 f_u)_V \,\mathrm{d}\mu_2\,\mathrm{d}\mu_1 \\ &= \int_W (K_{22}D_2 f, D_2 f)_V \,\mathrm{d}\mu. = -(S_0 f, f)_{H^0} \end{aligned}$$

□

Proposition 5.4.12. *Let* **(K7)** *hold. Then* A_0 *satisfies* **(H3)** *with* $\Lambda_M = c_A$.

Proof:
Let $f \in \mathcal{D}$, then $f_P \in \mathcal{F}C_b^\infty(B_U)$ with $\mu_1(f_P) = 0$. By using **(K7)**, then Corollary 5.1.3, and finally Proposition 5.4.6 (ii), it follows that

$$\begin{aligned} c_A\|Pf\|_H^2 &= c_A \int_U f_P^2 \,\mathrm{d}\mu_1 \leq \int_U (Q_2^{-1}K_{12}D_1 f_P, K_{12}D_1 f_P)_V \,\mathrm{d}\mu_1 \\ &= \int_U \int_V (v, Q_2^{-1}K_{12}D_1 f_P)_V^2 \,\mathrm{d}\mu_2\,\mathrm{d}\mu_1 = \|A_0 P f\|_{H^0}^2. \end{aligned}$$

□

The main result of this section is now immediate:

Theorem 5.4.13. *Let the conditions* **(K1)**–**(K7)** *hold. Then the semigroup* $(T_t)_{t\geq 0}$ *on* $H = L^2(\mu)$ *generated by the closure* $(L, D(L))$ *of* $(L, \mathcal{F}C_b^\infty(B_W))$ *is hypocoercive in the sense that for each* $\theta_1 \in (1,\infty)$*, there is some* $\theta_2 \in (0,\infty)$*, which can be explicitly computed from* c_S*,* c_A*, and the bounds* M_{22} *and* C_{22}*, such that*

$$\|T_t f - \mu(f)\|_H \leq \theta_1 \mathrm{e}^{-\theta_2 t}\,\|f - \mu(f)\|_H$$

for all $f \in H$ *and all* $t \geq 0$.

Proof:
Follows as in the proof of Theorem 4.2.10 from Theorem 2.2.1. □

5.5 The associated stochastic process

As in the finite-dimensional case, we want to find an associated stochastic process by using generalized Dirichlet forms. However, in order to prove quasi-regularity of the form, we need an $\mathcal{E}$-nest of compact sets. In $\mathbb{R}^d$, we used the ascending sequence of closed balls around the origin with radius $n \in \mathbb{N}$, the analogue of which are not compact in W by Heine-Borel. So we instead consider the weak topology $\mathcal{T}$ on W, i.e. the topology corresponding to weak convergence. The Banach-Alaoglu theorem then implies that $B_n(0) \subseteq (W, \|\cdot\|_W)$ are compact with respect to $\mathcal{T}$. We need to make sure that this change in topology still satisfies the restrictions on the considered spaces to apply Theorem 1.3.31.

Definition 5.5.1. Let $\mathcal{T}$ denote the weak topology on W. Then $\mathcal{B}(W_{\mathcal{T}})$ denotes the corresponding Borel-σ-algebra and the space of continuous functions from $(W, \mathcal{T})$ to $\mathbb{R}$ is denoted by $C_{\mathcal{T}}(W)$.

Lemma 5.5.2. *The topological space $(W, \mathcal{T})$ is a Lusin space, and in particular Hausdorff. Moreover, $\mathcal{B}(W) = \mathcal{B}(W_{\mathcal{T}}) = \sigma(C_{\mathcal{T}}(W))$, and $f \in C_{\mathcal{T}}(W)$ implies $f \in C^0(W)$. In particular, $H = L^2(W, \mathcal{B}(W), \mu) = L^2(W, \mathcal{B}(W_{\mathcal{T}}), \mu)$ is separable.*

Proof:
Let $w, z \in W$ be distinct points, then $(w, v)_W \neq (z, v)_W$ for $v = w - z$. Let U_w, U_z be open neighborhoods of (w, v) and (z, v) in $\mathbb{R}$ that separate these points. Clearly the map $(\cdot, v) : W \to \mathbb{R}$ is continuous with respect to $\mathcal{T}$, so the pre-images of U_w and U_z are open in $\mathcal{T}$ and disjoint, so $(W, \mathcal{T})$ is Hausdorff.

Let $I : (W, \|\cdot\|_W) \to (W, \mathcal{T})$ denote the identity map. Since a norm-convergent sequence in W is also weakly convergent, we get that I is continuous, so $(W, \mathcal{T})$ is the image of a Polish space under a continuous map, hence Lusin.

Clearly the weak topology is a weaker (smaller) topology than the original one, since every weakly closed set is also strongly closed. This then implies $\mathcal{B}(W_{\mathcal{T}}) \subseteq \mathcal{B}(W)$. For the other direction, we note that as above, the closed ε-balls are weakly closed sets, and therefore in $\mathcal{B}(W_{\mathcal{T}})$. Since then $B_\varepsilon(w) = \bigcup_{n\in\mathbb{N}} \overline{B_{\varepsilon - \frac{1}{n}}(w)} \in \mathcal{B}(W_{\mathcal{T}})$, we get $\mathcal{B}(W_{\mathcal{T}}) \supseteq \mathcal{B}(W)$ since $\mathcal{B}(W)$ is generated by the open ε-balls, as $(W, \|\cdot\|_W)$ is separable.

Clearly $\mathcal{B}(W) = \sigma(C^0(W))$, since the norm is continuous. Since $\mathcal{T}$ is weaker than the standard topology, the notion of continuity for real-valued functions is stronger, so that $\sigma(C_{\mathcal{T}}(W)) \subseteq \mathcal{B}(W)$. Since the maps $(\cdot, w_i)_W$ are continuous with respect to $\mathcal{T}$ for all $w_i \in B_W$, the same holds for the compositions

$$N_n(w) = N_n(u, v) := \sum_{i=1}^{n} (u, d_i)_U^2 + \sum_{i=1}^{n} (v, e_i)_V^2 = \|(P_n^U u, P_n^V v)\|_W^2 \quad \text{for } w = (u, v) \in W$$

for all $n \in \mathbb{N}$. This implies that for any $w \in W$, $\varepsilon > 0$ and $n \in \mathbb{N}$, the set

$$B_{\varepsilon,n}(w) := \{z = (u, v) \in W \mid \|(P_n^U u, P_n^V v)\|_W^2 < \varepsilon\}$$

is in $\mathcal{T}$, which shows that

$$\widetilde{B_\varepsilon(w)} := \bigcap_{n\in\mathbb{N}} B_{\varepsilon,n}(w) \in \mathcal{B}(W) \qquad \text{for all } w \in W,\ \varepsilon > 0.$$

One can observe that $B_\varepsilon(w) \subseteq \widetilde{B_\varepsilon(w)} \subseteq \overline{B_\varepsilon(w)}$ for any $\varepsilon > 0$, $w \in W$, which therefore finally yields

$$B_\varepsilon(w) = \bigcup_{n\in\mathbb{N}} \widetilde{B_{\varepsilon-\frac{1}{n}}(w)} \subseteq \mathcal{B}(W) \qquad \text{for all } \varepsilon > 0,\ w \in W,$$

which proves the claim as above. Lastly, separability of H follows from denseness of $\mathcal{F}C_b^\infty(B_W)$, since that implies denseness of $\mathcal{F}C_c^\infty(B_W)$, so that Lemma 1.5.1 can be applied to $\mathcal{F}C_c^\infty(B_W, n)$ for each $n \in \mathbb{N}$. □

This means that we are in the appropriate setting of Section 1.3.2 while still being able to use the results obtained earlier about the operator $(L, \mathcal{F}C_b^\infty(B_W))$ and the semigroup $(T_t)_{t\geq 0}$ on H generated by its closure $(L, D(L))$. We only need to pay attention to the topology whenever we refer to continuity of functions defined on W.

Remark 5.5.3. Due to the proof of Proposition 5.4.1, we know that $(L, D(L))$ is a Dirichlet operator, so that $\mathcal{E}$ as defined in Theorem 1.3.13 yields a generalized Dirichlet form on H. Since the generator core $\mathcal{F}C_b^\infty(B_W)$ consists of finitely based continuous bounded functions, which are in particular $\mathcal{T}$-continuous, we can see that $\mathcal{F}C_b^\infty(B_W)$ satisfies the conditions on $\mathcal{Y}$ in Theorem 1.3.31 as well as those on D in Definition 1.3.23 (ii). For point (iii) of quasi-regularity, we can use the fact that $C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is separable with respect to the supremum norm for each $n \in \mathbb{N}$, so there is a countable subset A_n that separates the points of $\mathbb{R}^n \times \mathbb{R}^n$. The set $A_n \circ (p_n^U, p_n^V)$ of elements from A_n composed with the projections then separates points of W_n. Taking the union over all $n \in \mathbb{N}$, we end up with a countable set in $\mathcal{F}C_c^\infty(B_W) \subseteq D(L)$ consisting of $\mathcal{T}$-continuous functions that separates points of W.

Therefore, all that remains to show quasi-regularity of $\mathcal{E}$ is to prove that $F_n := \overline{B_n(0)}$ defines an $\mathcal{E}$-nest. For this, we employ the strategy used in [EG21, Lemma 3], and consequently assume the following:

Assumption (K8). There is a function $\rho \in L^1(W;\mu)$ such that for each $n \in \mathbb{N}$, ρ_n defined via $\rho_n(u,v) := \rho(P_n^U u, P_n^V v)$ is also in $L^1(E;\mu)$ and converges to ρ there as $n \to \infty$, and such that

$$(P_n^U u, Q_1^{-1} K_{21} P_n^V v)_U - (Q_2^{-1} K_{12} P_n^U u, P_n^V v)_V \leq \rho_n(u,v)$$

for all $n \in \mathbb{N}$ and $(u,v) \in W$.

Remark 5.5.4. In the special case that $U = V$ and $Q_1 = Q_2$, this is satisfied for $\rho = 0$. Otherwise, the invariance properties of K_{12} imply that

$$(P_n^U u, Q_1^{-1} K_{21} P_n^V v)_U - (Q_2^{-1} K_{12} P_n^U u, P_n^V v)_V = (P_n^U u, (Q_1^{-1} K_{21} - K_{21} Q_2^{-1}) P_n^V v)_U.$$

In the notation of Remark 5.4.10, if $\lambda_{12,k}(\lambda_{1,k}^{-1} - \lambda_{2,k}^{-1})$ is bounded by a uniform constant C for all $k \in \mathbb{N}$, then $\rho = \frac{C}{2}\|\cdot\|_W^2$ satisfies (K8). This holds in particular if $\lambda_{12,k} \leq \frac{C}{2}\min\{\lambda_{1,k}, \lambda_{2,k}\}$ for all $k \in \mathbb{N}$.

Proposition 5.5.5. *Let (K1)–(K8) hold. Then the generalized Dirichlet form $\mathcal{E}$ is quasi-regular and there is a μ-tight special standard process $\mathbf{M}$ on $(W, \mathcal{T})$ which is properly associated with $\mathcal{E}$ in the resolvent sense.*

Proof:
Let $F_n := \{w \in W \ : \ \|w\|_W \leq n\}$ for each $n \in \mathbb{N}$. We have to prove that $(F_n)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-nest. For notation purposes, we write $N(u,v) := \|(u,v)\|_W^2$ and $N_n(u,v) = N(P_n^U u, P_n^V v)$ for $n \in \mathbb{N}$. Moreover, we only consider those $n \in \mathbb{N}$ that satisfy $n = m^K(n)$, which provide an increasing sequence. Using a sequence of cutoff functions as in Remark 1.4.8, we see that $N_n \in D(L)$ with

$$\begin{aligned}\frac{1}{2}LN_n(u,v) &= \mathrm{tr}[K_{22,n}(P_n^V v)] + \sum_{j=1}^n (\partial_j K_{22}(v)e_j, P_n^V v)_V - (P_n^V v, Q_2^{-1}K_{22}(v)P_n^V v)_V \\ &\qquad - (P_n^U u, Q_1^{-1}K_{21}P_n^V v)_U + (P_n^V v, Q_2^{-1}K_{12}P_n^U u)_V \\ &\geq \sum_{j=1}^n (\partial_j K_{22}(v)e_j, P_n^V v)_V - (P_n^V v, Q_2^{-1}K_{22}(v)P_n^V v)_V - \rho_n(u,v)\end{aligned}$$

By (K4), it follows that

$$|(P_n^V v, Q_2^{-1}K_{22}(v)P_n^V v)_V| \leq C_{22}\|P_n^V v\|_V^2,$$

and since $\partial_j K_{22}(v)e_j = \alpha_j(P_n^V v)e_j$ for some suitable $\alpha_j(P_n^V v)$ and all $1 \leq j \leq n$, (K5) implies that

$$\begin{aligned}\left|\sum_{j=1}^n (\partial_j K_{22}(v)e_j, P_n^V v)_V\right| &\leq \sum_{j=1}^n |\alpha_j(P_n^V v)||(e_j, P_n^V v)_V| \\ &\leq \sqrt{\sum_{j=1}^n |\alpha_j(P_n^V v)|^2}\sqrt{\sum_{j=1}^n (e_j, P_n^V v)_V^2} \\ &\leq 2(1+\|P_n^V v\|_V)\|P_n^V v\|_V \sqrt{\sum_{j=1}^n N^K(j)^2} \\ &\leq 2\sqrt{\lambda_{2,1}}M_{22}(1+\|P_n^V v\|_V)^2 \\ &\leq 4\sqrt{\lambda_{2,1}}M_{22}(1+\|P_n^V v\|_V^2),\end{aligned}$$

where $\lambda_{2,1}$ denotes the first (and largest) eigenvalue of Q_2. While being a crude bound, this means that, when setting

$$g(u,v) := N(u,v) + 2\rho(u,v) + 8\sqrt{\lambda_{2,1}}M_{22}(1+\|v\|_V^2) + 2C_{22}\|v\|_V^2,$$

and defining $g_n(u,v)$ as $g(P_n^U u, P_n^V v)$, we get

$$(I - L)N_n(u,v) \leq g_n(u,v) \tag{5.5.1}$$

for all $(u, v) \in W$ and $n \in \mathbb{N}$ with $n = m^K(n)$. Clearly $g \in L^1(\mu)$ and g_n converges to g in $L^1(\mu)$. As seen in [Con11, Lemma 1.3.11], since μ is a probability measure, the operator $(L, D(L))$ is essentially m-dissipative in $L^1(\mu)$ and generates a sub-Markovian sccs $(T^1_t)_{t\geq 0}$ on $L^1(\mu)$ with corresponding sub-Markovian resolvent $(G^1_\alpha)_{\alpha>0}$, which coincides with the L^2-resolvent $(G_\alpha)_{\alpha>0}$ on $L^2(\mu)$. Applying this resolvent to both sides of (5.5.1), we obtain $N_n(u, v) \leq G^1_1 g_n(u, v)$, and by convergence as $n \to \infty$, we see that $N(u, v) \leq G^1_1 g(u, v)$ for all $(u, v) \in W$.

As in the proof of [BBR06, Proposition 5.5], this implies that

$$\alpha G_{\alpha+1} g^{\frac{1}{2}} = \alpha G^1_{\alpha+1} g^{\frac{1}{2}} \leq g^{\frac{1}{2}} \qquad \text{for all } \alpha > 0,$$

so that $\tilde{g} := g^{\frac{1}{2}}$ is a 1-excessive function in H as defined in Definition 1.3.14, which dominates N. In particular, for any $n \in \mathbb{N}$, we see that $\mathbb{1}_{F_n^c} \leq \frac{1}{n} N \leq \frac{1}{n}\tilde{g}$.

In the context of Proposition 1.3.19, choose $\varphi \equiv 1 \in H$ and therefore $h = G_1\varphi = 1$. For any open set U, it follows that $h_U = e_{\mathbb{1}_U}$. Then for the corresponding capacity, we get

$$\mathrm{Cap}_\varphi(F_n^c) = (h_{F_n^c}, \varphi)_H \leq \frac{1}{n}(\tilde{g}, 1)_H \qquad \text{for all } n \in \mathbb{N},$$

since $e_{\mathbb{1}_U} \leq \frac{1}{n}\tilde{g}$ by definition of the 1-reduced function. Clearly, this means that $\mathrm{Cap}_\varphi(F_n^c) \to 0$ as $n \to \infty$, so that $(F_n)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-nest of $\mathcal{T}$-compact sets by the last statement of Proposition 1.3.19.

This together with Remark 5.5.3 shows that all requirements to apply Theorem 1.3.31 are satisfied, which yields the associated process as claimed. □

Next, we verify some properties of the paths.

Proposition 5.5.6. *Let $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t, Y_t)_{t\geq 0}, (P_w)_{w\in W_\Delta})$ be the process obtained by Proposition 5.5.5 with corresponding life time ζ. As in Definition 1.3.25, we set $P_\mu := \int_{W_\Delta} P_w \,\mu(\mathrm{d}w)$. Then $(X_t, Y_t)_{t\geq 0}$ has infinite life time and weakly continuous paths P_μ-a.s. In particular, $(X_t, Y_t)_{t\geq 0}$ is a μ-invariant Hunt process.*

Proof:
Let $\mathbb{P}_\mu$ be the probability law on $D([0, \infty); W_\Delta)$ obtained as the image measure under the orbit maps from the process, as seen in Lemma 1.3.36. Then $\mathbb{P}_\mu$ is associated with the semigroup $(T_t)_{t\geq 0}$ in the sense of Definition 1.3.35. By Lemma 1.3.39, it follows that the coordinate process $(Z_t)_{t\geq 0}$ has $\mathbb{P}_\mu$-almost surely $\mathcal{T}$-continuous paths up to its life time. Since $(T_t)_{t\geq 0}$ is conservative and μ-invariant, the same holds for $\mathbb{P}_\mu$ via Lemma 1.3.41. Since $\mathbb{P}_\mu$ describes the distribution of paths of $(X_t, Y_t)_{t\geq 0}$ with initial distribution P_μ, the claim follows. □

Remark 5.5.7. As a direct consequence of Lemma 1.3.38 (see also [BBR06, Proposition 1.4]), we see that for any $f \in D(L)$,

$$M_t^{[f],L} := f(X_t, Y_t) - f(X_0, Y_0) - \int_0^t Lf(X_s, Y_s)\,\mathrm{d}s$$

is P_μ-integrable and describes an $(\mathcal{F}_t)_{t\geq 0}$-martingale. Moreover, if $f^2 \in D(L)$ with $Lf \in L^4(\mu)$, then

$$N_t^{[f],L} := (M_t^{[f],L})^2 - \int_0^t L(f^2)(X_s, Y_s) - (2fLf)(X_s, Y_s)\,\mathrm{d}s, \quad t \geq 0,$$

describes a martingale as well.

We use the following functions to evaluate our process:

Lemma 5.5.8. *For any $i \in \mathbb{N}$, define f_i, g_i via*

$$W \ni (u,v) \mapsto f_i(u,v) := (u, d_i)_U \in \mathbb{R},$$
$$W \ni (u,v) \mapsto g_i(u,v) := (v, e_i)_V \in \mathbb{R}.$$

Then f_i, g_i, f_i^2, g_i^2 are in $D(L)$, $L(f_i^2), L(g_i^2) \in L^4(\mu)$ and we have that

$$\begin{aligned} Lf_i(u,v) &= (v, Q_2^{-1}K_{12}d_i), \\ L(f_i^2)(u,v) &= 2f_i(u,v)Lf_i(u,v), \\ Lg_i(u,v) &= (\partial_i K_{22}(v)e_i, e_i)_V - (v, Q_2^{-1}K_{22}(v)e_i)_V - (u, Q_1^{-1}K_{21}e_i)_U, \\ L(g_ig_j)(u,v) &= 2\delta_{ij}(e_i, K_{22}(v)e_i)_V + g_i(u,v)Lg_i(u,v) + g_jLg_i(u,v) \end{aligned}$$

for all $(u,v) \in W$.

Proof:
This follows by using a sequence of cutoff functions for each f_i or g_i, and using that since $(v, Q_2^{-1}K_{22}e_i) \leq C_{22}(v, e_i)$, all occurring coefficients are bounded, and we can integrate in the respective finite-dimensional spaces over μ^i, where the identities are in any L^p. □

We summarize the implications of the above in the following:

Proposition 5.5.9. *Let (K1)–(K8) hold. Then the process $(X_t, Y_t)_{t\geq 0}$ associated with $\mathcal{E}$ solves the martingale problem for $(L, D(L))$. Moreover, for any $i \in \mathbb{N}$, we have that the real-valued processes $(X_t^i)_{t\geq}$ and $(Y_t^i)_{t\geq 0}$ defined by $X_t^i = (X_t, d_i)_U$ and $Y_t^i = (Y_t, e_i)_V$ have continuous paths P_μ-almost surely and satisfy*

$$\begin{aligned} X_t^i - X_0^i &= \int_0^t (Y_s, Q_2^{-1}K_{12}d_i)_V\,\mathrm{d}s \qquad \textit{and} \\ Y_t^i - Y_0^i &= \int_0^t (\partial_i K_{22}(Y_s)e_i, e_i)_V - (Y_s, Q_2^{-1}K_{22}(Y_s)e_i)_V - (X_s, Q_1^{-1}K_{21}e_i)_U\,\mathrm{d}s + M_t^{[g_i],L} \end{aligned} \tag{5.5.2}$$

with $(M_t^{[g_i],L})_{t\geq 0}$ being a continuous martingale such that for $i, j \in \mathbb{N}$, we have

$$[M^{[g_i],L}, M^{[g_j],L}]_t = 2\delta_{ij}(e_i, K_{22}(Y_t)e_i)_V.$$

Proof:
The statement about the martingale problem was already mentioned in Remark 5.5.7. From Proposition 5.5.6, we know that $(X_t, Y_t)_{t\geq 0}$ has P_μ-a.s. weakly continuous paths, which implies continuity of the coordinate processes $(X_t^i)_{t\geq 0}$ and $(Y_t^i)_{t\geq 0}$ as defined.

With the definitions from Lemma 5.5.8, we see that $X_t^i = f_i(X_t, Y_t)$ and $Y_t^i = g_i(X_t, Y_t)$. From the statement of the Lemma itself, we see that

$$M_t^{[f_i],L} = X_t^i - X_0^i - \int_0^t (Y_s, Q_2^{-1} K_{12} d_i)\,\mathrm{d}s$$

and $N_t^{[f_i],L} = (M_t^{[f_i],L})^2$, which implies $[M^{[f_i],L}]_t = 0$, hence $M_t^{[f_i],L} = 0$. This proves the first line in (5.5.2). The second line follows analogously, and the representation of the quadratic covariations follows by evaluating $\frac{1}{2}(N_t^{[g_i+g_j],L} - N_t^{[g_i],L} - N_t^{[g_j],L})$. Note that in the definition of $M_t^{[g_i],L}$, it holds that all terms featuring $K_{22}(Y_s)$ can be replaced with $K_{22}(P_n^V Y_s)$, where $n = m^K(i)$, which ensures continuity of the martingales. □

Next, we want to prove that the process is also a weak solution in some sense of an infinite-dimensional stochastic differential equation. For this, we need to construct a suitable cylindrical Brownian motion on V, such that we can express the process described by $M_t^V := \sum_{i\in\mathbb{N}} M_t^{[g_i],L} e_i$ as a stochastic integral of $\sqrt{K_{22}}$ similarly to Proposition 4.3.4.

Lemma 5.5.10. *For each $k \in \mathbb{N}$, define the real-valued stochastic process $\beta^{(k)}$ via*

$$\beta_t^{(k)} := \frac{1}{\sqrt{2}} \int_0^t \lambda_{22,k}^{-\frac{1}{2}}(Y_s)\,\mathrm{d}M_s^{[g_k],L}.$$

Then $(\beta^{(k)})_{k\in\mathbb{N}}$ is an independent sequence of one-dimensional Brownian motions.

Proof:
Let $k \in \mathbb{N}$ and choose $n \geq k$ such that $n = m^K(n)$. We set $M_t^{(n)} := (M_t^{[g_1],L}, \dots, M_t^{[g_n],L})$ and $\Sigma_t^{(n)} := \operatorname{diag}(\lambda_{22,1}^{-\frac{1}{2}}(P_n^V Y_s), \dots, \lambda_{22,n}^{-\frac{1}{2}}(P_n^V Y_s))$. As in the proof of Proposition 4.3.4, it follows that

$$B_t^{(n)} := \int_0^t \Sigma_s^{(n)}\,\mathrm{d}M_s^{(n)}$$

is an n-dimensional Brownian motion, since $(K_{22} e_i, e_i)_V = \lambda_{22,i}$ for all $i \in \mathbb{N}$. Clearly, the k-th component of $B_t^{(n)}$ is just $\beta_t^{(k)}$, independently of n. In particular, $\{\beta^{(1)}, \dots, \beta^{(n)}\}$ is independent for any $n \in \mathbb{N}$. □

Now we fix some $T \in (0, \infty)$ and define the process $(B_t)_{t\in[0,T]}$ on V via

$$B_t := \sum_{k=1}^{\infty} \beta_t^{(k)} e_k, \qquad t \in [0, T].$$

This is a cylindrical Brownian motion on V as defined in [PR07, Proposition 2.5.2], as can be seen by choosing $J : V \to V$, $J := Q_2^{\frac{1}{2}}$, since then

$$B_t^{Q_2} := \sum_{k=1}^{\infty} \beta_t^{(k)} J e_k$$

defines a Q_2-Wiener process on V.

Definition 5.5.11. Set $V_0 := Q_2^{\frac{1}{2}} V$ and equip it with the inner product

$$(a, b)_{V_0} := (Q_2^{-\frac{1}{2}} a, Q_2^{-\frac{1}{2}} b)_V \qquad \text{for all } a, b \in V_0,$$

which makes V_0 a separable Hilbert space with orthonormal basis $(Q_2^{\frac{1}{2}} e_i)_{i \in \mathbb{N}}$. Define $\mathcal{L}_2^0 := \mathcal{L}_2(V_0; V)$ as the Hilbert space of Hilbert-Schmidt operators from V_0 to V. Further let

$$\mathcal{A}_T := \sigma\{Y : [0, T] \times \Omega \to \mathbb{R} \mid Y \text{ is left-continuous and } (\mathcal{F}_t)_{t \in [0,T]}\text{-adapted.}\}$$

If a process $X : [0, T] \times \Omega \to \mathcal{L}_2^0$ is $\mathcal{A}_T$-$\mathcal{B}(\mathcal{L}_2^0)$-measurable, it is called *predictable.*

Lemma 5.5.12. *The process $\sqrt{K_{22}(Y_t)} J^{-1}$, $t \in [0, T]$, is $\mathcal{L}_2^0$-valued and predictable.*

Proof:
For each $v \in V$, we have

$$\sum_{i \in \mathbb{N}} (\sqrt{K_{22}(v)} J^{-1} Q_2^{\frac{1}{2}} e_i, \sqrt{K_{22}(v)} J^{-1} Q_2^{\frac{1}{2}} e_i)_V \leq C_{22} \operatorname{tr}[Q_2]$$

due to (K4), which implies that $\sqrt{K_{22}(v)} J^{-1} \in \mathcal{L}_2(V_0; V)$ for any $v \in V$. Moreover,

$$A_i := (\sqrt{K_{22}}(Y_t) J^{-1} Q_2^{\frac{1}{2}} e_i, \sqrt{K_{22}}(Y_t) J^{-1} Q_2^{\frac{1}{2}} e_i)_V = (K_{22}(P^V_{m^K(i)} Y_t) e_i, e_i)_V$$

is continuous and $(\mathcal{F}_t)_{t \in [0,T]}$-adapted for any $i \in \mathbb{N}$. Fix some $\varepsilon > 0$ and set

$$B := \{(t, \omega) \in [0, T] \times \Omega \mid \| \sqrt{K_{22}}(Y_t(\omega)) J^{-1} \|_{\mathcal{L}_2^0} \leq \varepsilon\},$$

as well as $B_k := \{\sum_{i=1}^k A_i \leq \varepsilon\} \in \mathcal{A}_T$ for each $k \in \mathbb{N}$. Then $B = \bigcap_{k \in \mathbb{N}} B_k \in \mathcal{A}_T$ as well. It is easily seen that similarly, all pre-images of closed ε-balls in $\mathcal{L}_2^0$ under $\sqrt{K_{22}(Y_\cdot)} J^{-1}$ are in $\mathcal{A}_T$, so that the process is indeed predictable, since $\mathcal{L}_2^0$ is separable. □

By [PR07, Section 2.3], this means that $\sqrt{K_{22}(Y_t)} J^{-1}$ is integrable with respect to the Q_2-Wiener process $(B_t^{Q_2})_{t \in [0,T]}$, which implies that $\sqrt{K_{22}(Y_t)}$ is integrable with respect to $(B_t)_{t \in [0,T]}$ with

$$I(\sqrt{K_{22}(Y_t)}) := \int_0^t \sqrt{K_{22}(Y_s)} \, \mathrm{d}B_s := \int_0^t \sqrt{K_{22}(Y_s)} J^{-1} \, \mathrm{d}B_s^{Q_2}$$

for all $t \in [0,T]$. By applying [PR07, Lemma 2.4.1] for the operators $p_i : V \to \mathbb{R}$, $p_i(v) := (v, e_i)_V$, we see that

$$\sqrt{2}(I(\sqrt{K_{22}(Y_t)}), e_i)_V = \sqrt{2}\int_0^t (\sqrt{K_{22}(Y_s)}J^{-1}\cdot, e_i)_V \,\mathrm{d}B_s^{Q_2} = \sqrt{2}\int_0^t \lambda_{22,i}^{\frac{1}{2}}(Y_s)\,\mathrm{d}\beta_s^{(i)}$$

due to the invariance properties of $K_{22}(v)$ for any $v \in V$, which evaluates to $M_t^{[g_i],L}$ by definition of $\beta^{(i)}$. Together, we obtain

$$M_t^V := \sum_{i\in\mathbb{N}} M_t^{[g_i],L} e_i = \int_0^t \sqrt{K_{22}(Y_s)}\,\mathrm{d}B_s. \tag{5.5.3}$$

As a result, we get the following:

Proposition 5.5.13. *Let (K1)–(K8) hold and consider the following Itô stochastic differential equation for $(X_t, Y_t)_{t\geq 0}$ on W:*

$$\begin{aligned}
\mathrm{d}X_t &= K_{21}Q_2^{-1}Y_t\,\mathrm{d}t \\
\mathrm{d}Y_t &= \sum_{i=1}^{\infty} \partial_i K_{22}(Y_t)e_i - K_{22}(Y_t)Q_2^{-1}Y_t\,\mathrm{d}t - K_{12}Q_1^{-1}X_t\,\mathrm{d}t + \sqrt{2K_{22}(Y_t)}\,\mathrm{d}B_t,
\end{aligned} \tag{5.5.4}$$

where $(B_t)_{t\geq 0}$ is a cylindrical Brownian motion on V.

Then the process $(X_t, Y_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P_\mu)$ associated with $\mathcal{E}$ is a weak solution of (5.5.4) in the sense that there is a cylindrical Brownian motion $(B_t)_{t\geq 0}$ on V such that (5.5.4) holds in each component, i.e. evaluated by any $(\cdot, (d_i, 0))_W$ or $(\cdot, (0, e_i))_W$ for $i \in \mathbb{N}$. Moreover, the transition semigroup $(p_t)_{t\geq 0}$ of $(X_t, Y_t)_{t\geq 0}$ is hypocoercive with the rate computed in Theorem 5.4.13, which shows exponential convergence of the weak solution to the equilibrium described by the invariant measure μ.

Proof:
The componentwise statement follows from Proposition 5.5.9 since for example

$$(X_t, d_i) - (X_0, d_i) = X_t^i - X_0^i = \int_0^t (Y_s, Q_2^{-1}K_{12}d_i)_V\,\mathrm{d}s = \int_0^t (K_{21}Q_2^{-1}Y_s, d_i)_U\,\mathrm{d}s,$$

together with the representation of $M_t^{[g_i],L}$ as $(M_t^V, e_i) = (\int_0^t \sqrt{K_{22}(Y_s)}\,\mathrm{d}B_s, e_i)_V$ from Equation (5.5.3). By association, $(p_t)_{t\geq 0}$ is a μ-version of $(T_t)_{t\geq 0}$, and therefore also satisfies the estimate from Theorem 5.4.13. □

Corollary 5.5.14. *The componentwise statement above can be extended in the following way: It holds P_μ-a.s. for any $t \in [0,\infty)$, $\vartheta \in D(Q_2^{-1}K_{12})$ and $\theta \in D(Q_1^{-1}K_{21})$ that*

$$\begin{aligned}
(X_t, \vartheta) &= (X_0, \vartheta) + \int_{[0,t]} (Y_s, Q_2^{-1}K_{12}\vartheta)\,\mathrm{d}s, \\
(Y_t, \theta) &= (Y_0, \theta) + \int_{[0,t]} \left(\sum_{i=1}^{\infty} \partial_i K_{22}(Y_s)e_i, \theta \right) - (Y_s, Q_2^{-1}K_{22}\theta) - (X_s, Q_1^{-1}K_{21}\theta)\,\mathrm{d}s \\
&\quad + (\sqrt{2K_{22}(Y_t)}B_t, \theta).
\end{aligned}$$

The proof follows analogously to the corresponding proof of [EG21, Theorem 5], since all occurring terms are well-defined due to **(K4)** *and* **(K5)***. Note that in particular, we need not assume that* $\theta \in D(Q_2^{-1} K_{22}(v))$*, since* $\|Q_2^{-1} K_{22}(v)\|_{\mathcal{L}(V)} \leq C_{22}$ *for all* $v \in V$*.*

5.6 A short example

Here we give a quick example for a specific choice of $U = V$, $Q_1 = Q_2$, K_{12} and K_{22}, such that all conditions **(K1)**–**(K8)** are satisfied, so that all previously obtained results can be applied. This specific choice is motivated by the example used in [EG21].

Let $U = V = X := L^2((0,1); \mathbb{R}; dx)$ and consider the negative Dirichlet Laplacian $-\Delta$ on X, which is defined by

$$D(\Delta) := H_0^{1,2}((0,1)) \cap H^{2,2}((0,1)), \qquad -\Delta f := -f'' \quad \text{for all } f \in D(\Delta).$$

Then we choose $Q_1 = Q_2 := (-\Delta)^{-1}$ with the corresponding orthonormal eigenvectors $d_k = e_k := \sqrt{2} \sin(\pi k \cdot)$ to the eigenvalues $\lambda_{1,k} = \lambda_{2,k} = \frac{1}{k^2 \pi^2}$. Evidently, Q_1 and Q_2 are symmetric and of trace class, so that there exist corresponding Gaussian measures μ_1 and μ_2 on $(U, \mathcal{B}(U))$ and $(V, \mathcal{B}(V))$, respectively.

Now we choose K_{22} by specifying its eigenvalue functions $\lambda_{22,k} : V \to \mathbb{R}$. Let $c_0, c_v \in (0, \infty)$ be constant. For each $k \in \mathbb{N}$, let $\beta_k \in (0,1)$, $\varphi_k \in C_b^1(\mathbb{R}; [0,\infty))$ and $\psi_k \in C_b^1(\mathbb{R}^k; [0,\infty))$, and define

$$\lambda_{22,k}(v) := c_0 \lambda_{2,k} + \gamma_k \left(\varphi_k(\lambda_{2,k}^{-\frac{1}{2}} \, |p_k^V v|^{\beta_k + 1}) + \psi_k(\lambda_{2,k}^{-\frac{1}{2}} \, p_k^V v) \right),$$

where

$$\gamma_k := \frac{c_v \lambda_{2,k}^{\frac{3}{2}}}{\|\varphi_k\|_{C^1} + \|\psi_k\|_{C^1}}.$$

Then we see clearly that $c_0 \lambda_{2,k} \leq \lambda_{22,k}(v) = \lambda_{22,k}(P_k^V v) \leq (c_0 + \sqrt{\lambda_{2,1}} c_v) \lambda_{2,k}$ for all $k \in \mathbb{N}$, $v \in V$. Moreover, for $i > k$, we have $\partial_i \lambda_{22,k}(v) = 0$, and for $1 \leq i \leq k$, it holds for all $v \in V$ that

$$\begin{aligned} |\partial_i \lambda_{22,k}(v)| &= \lambda_{2,k}^{-\frac{1}{2}} \gamma_k \left| \varphi_k'(\lambda_{2,k}^{-\frac{1}{2}} |p_k^V v|^{\beta_k + 1})(\beta_k + 1) |p_k^V v|^{\beta_k - 1} (v, e_i) + \partial_i \psi(\lambda_{2,k}^{-\frac{1}{2}} p_k^V v) \right| \\ &\leq \lambda_{2,k}^{-\frac{1}{2}} (\beta_k + 1) \gamma_k (\|\varphi_k'\|_{C^0} + \|\psi_k\|_{C^1})(1 + |p_k^V v|^{\beta_k}) \\ &\leq 2 c_v \lambda_{2,k} (1 + \|P_k^V v\|_V^{\beta_k}). \end{aligned}$$

Now we simply set $K_{22}(v) e_i := \lambda_{22,k}(v) e_i$, which describes a symmetric positive-definite bounded linear operator on V as required for Definition 5.2.3. Moreover, **(K1)** holds for $K_{22}^0 = c_0 Q_2$, **(K2)** is satisfied for $m_k(n) = n$ and $N_k := c_v \lambda_{2,k}$, **(K4)** holds for $C_{22}^0 = c_0$ and $C_{22}^v = c_v$, and in **(K5)** we get $\alpha_n^{22} \leq c_v \lambda_{2,n}^{\frac{1}{2}}$, which describes an ℓ^2-sequence since Q_2 is of trace class. Finally, the estimate in Remark 5.4.10 holds for $\omega_{22} = c_0^{-1}$, which implies that **(K6)** is satisfied for $c_S := c_0 \lambda_{2,1}$. Finally, we note that at least for the choices $K_{12} = I$ or $K_{12} = Q_2 = Q_1$, the remaining assumptions are satisfied. Indeed, **(K3)** is obvious, **(K7)** follows from Remark 5.4.10 (ii) since $\lambda_{1,k}$ decreases to zero, and **(K8)** holds for $\rho = 0$ due to Remark 5.5.4.

5.7 Outlook

While the result attained in this chapter is overall well-rounded, there are still some unsatisfactory assumptions that were made, as well as potential for generalization.

We start with the statement on essential m-dissipativity: The proof provided above seems more like a first step as opposed to a fully generalized argument, as we simply use the finite-dimensional result from Theorem 3.5.1 without having to modify or approximate the operator. This is due in one part to the invariance properties of $K_{22}(v)$ and K_{12} for any $v \in V$ but also due to the *other* invariance, namely the dependence of K_{22} only on $P_n^V v$ when considered on V_n, at least for $n = m^K(n)$. It seems plausible that this could be generalized at least to depending in a "very small way" on the remaining components as well. As long as the resulting operator L is dissipative, the dense range condition could be verified via approximation of finitely-dependent operators L_n. Since the difference $(L - L_n)f$ would be estimated by scaled-down second-order partial derivatives of f, we would need a way to estimate those by $(I - L_n)f$ again, in order to choose the appropriate n only depending on the finitely-based function g we wish to approximate. In the finite-dimensional case, this was done via Lemma 3.5.4, which allowed the approximating operators to be chosen a priori for each g. A promising approach seems to be to use the L^p-regularity estimates provided in the recent paper [Bra+13], which provide relative bounds of $\partial_{ij} f$ by $L_n f$ depending, in particular, on the dimension. If we make the assumption that the influence of higher-dimensional components is bounded by a suitably small factor of this relative bound, then at least the second-order terms could be controlled a priori. However, the derivatives of K_{22} would also have to be treated, and in order to formulate adequate conditions, it would be helpful to provide concrete estimates of the relative bounds, which seems even harder noting the erratum appended to [Bra+13].

Another way to generalize the operator would be to include an additional potential $\Phi : U \to \mathbb{R}$, which changes the measure μ_1 by introducing a density $\mathrm{e}^{-\Phi}$. This was treated for constant K_{22} in [EG21], where L^2-regularity estimates (see Theorem 2 of the mentioned reference) were instrumental for retaining essential m-dissipativity. These regularity estimates do not hold in our case, as the derivatives of K_{22} add terms which are not easily bounded by the remaining ones. However, in the finite-dimensional setting, similar regularity estimates are proven and used in [DG01, Proposition 4.2], so there might be potential for a clever Hölder-Young argument.

Finally, as mentioned in Remark 5.3.4, most statements should still be valid when assuming a "block-diagonal" structure for K_{12} and $K_{22}(v)$, as long as the block size is bounded. While it would certainly be interesting to consider coefficients without such invariance properties, most arguments regarding well-definedness and boundedness of occurring sums, as well as the construction of the cylindrical Brownian motion, would have to change fundamentally.

Bibliography

[AFP19] L. Angiuli, S. Ferrari, and D. Pallara. "Gradient estimates for perturbed Ornstein–Uhlenbeck semigroups on infinite-dimensional convex domains". In: *Journal of Evolution Equations* 19.3 (Sept. 2019), pp. 677–715. ISSN: 1424-3202. DOI: 10.1007/s00028-019-00491-y.

[Are+01] Wolfgang Arendt et al. *Vector-Valued Laplace Transforms and Cauchy Problems*. Vol. 96. Springer, Basel, Jan. 2001. ISBN: 978-3-0348-0086-0. DOI: 10.1007/978-3-0348-0087-7.

[Bak+08] Dominique Bakry et al. "A simple proof of the Poincaré inequality for a large class of probability measures". In: *Electronic Communications in Probability* 13 (2008), pp. 60–66. DOI: 10.1214/ECP.v13-1352.

[BBR06] Lucian Beznea, Nicu Boboc, and Michael Röckner. "Markov processes associated with L^p-resolvents and applications to stochastic differential equations on Hilbert space". In: *Journal of Evolution Equations* 6.4 (Dec. 2006), pp. 745–772. ISSN: 1424-3202. DOI: 10.1007/s00028-006-0287-2.

[Bec89] William Beckner. "A Generalized Poincaré Inequality for Gaussian Measures". In: *Proceedings of the American Mathematical Society* 105.2 (1989), pp. 397–400. ISSN: 00029939, 10886826. DOI: 10.2307/2046956.

[BG21] Alexander Bertram and Martin Grothaus. *Convergence Rate for Degenerate Partial and Stochastic Differential Equations via weak Poincaré Inequalities*. 2021. arXiv: 2110.05536 [math.PR].

[BG22] Alexander Bertram and Martin Grothaus. "Essential m-dissipativity and hypocoercivity of Langevin dynamics with multiplicative noise". In: *Journal of Evolution Equations* 22.1 (Mar. 2022), p. 11. ISSN: 1424-3202. DOI: 10.1007/s00028-022-00773-y.

[BGH21] Fabrice Baudoin, Maria Gordina, and David Herzog. "Gamma Calculus Beyond Villani and Explicit Convergence Estimates for Langevin Dynamics with Singular Potentials". In: *Archive for Rational Mechanics and Analysis* 241 (Aug. 2021). DOI: 10.1007/s00205-021-01664-1.

[BGS13] Benedict Baur, M. Grothaus, and Patrik Stilgenbauer. "Construction of L^p-strong Feller Processes via Dirichlet Forms and Applications to Elliptic Diffusions". In: *Potential Analysis* 38 (2013), pp. 1233–1258. DOI: https://doi.org/10.1007/s11118-012-9314-9.

[BKR01] V. I. Bogachev, N. V. Krylov, and M. Röckner. "On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions". English (US). In: *Communications in Partial Differential Equations* 26.11-12 (Jan. 2001), pp. 2037–2080. eprint: https://www.tandfonline.com/doi/pdf/10.1081/PDE-100107815.

[BKR97] Vladimir I. Bogachev, Nicolai V. Krylov, and Michael Röckner. "Elliptic regularity and essential self-adjointness of Dirichlet operators on $\mathbb{R}^n$". en. In: *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze* Ser. 4, 24.3 (1997), pp. 451–461. eprint: http://www.numdam.org/item/ASNSP_1997_4_24_3_451_0.pdf.

[Bra+13] M. Bramanti et al. "Global Lp estimates for degenerate Ornstein-Uhlenbeck operators with variable coefficients". In: *Mathematische Nachrichten* 286.11-12 (2013), pp. 1087–1101. DOI: https://doi.org/10.1002/mana.201200189.

[CG05] Florian Conrad and Martin Grothaus. "Non-sectorial diffusions and an application to continuous N-particle Langevin dynamics for a general class of interaction potentials". MA thesis. TU Kaiserslautern, 2005.

[CG08] Florian Conrad and Martin Grothaus. "Construction of N-Particle Langevin Dynamics for $H^{1,\infty}$-Potentials via Generalized Dirichlet Forms". In: *Potential Analysis* 28.3 (May 2008), pp. 261–282. DOI: 10.1007/s11118-008-9076-6.

[Con11] Florian Conrad. "Construction and analysis of Langevin dynamics in continuous particle systems". PhD thesis. TU Kaiserslautern, 2011. ISBN: 978-3-86853-996-7.

[DG01] Giuseppe Da Prato and Beniamin Goldys. "Elliptic Operators on Rd with Unbounded Coefficients". In: *Journal of Differential Equations* 172.2 (2001), pp. 333–358. ISSN: 0022-0396. DOI: https://doi.org/10.1006/jdeq.2000.3866.

[Die69] "Chapter VIII: Differential Calculus". In: *Foundations of Modern Analysis*. Ed. by J. Dieudonneé. Vol. 10. Pure and Applied Mathematics. Elsevier, 1969, pp. 147–196. DOI: https://doi.org/10.1016/S0079-8169(08)60170-1.

[DMS09] Jean Dolbeault, Clément Mouhot, and Christian Schmeiser. "Hypocoercivity for kinetic equations with linear relaxation terms". In: *Comptes Rendus Mathematique* 347.9 (2009), pp. 511–516. ISSN: 1631-073X. DOI: 10.1016/j.crma.2009.02.025.

[DMS15] Jean Dolbeault, Clément Mouhot, and Christian Schmeiser. "Hypocoercivity for linear kinetic equations conserving mass". In: *Trans. Amer. Math. Soc.* 367th ser. (2015), pp. 3807–3828. DOI: 10.1090/S0002-9947-2015-06012-7.

[Ebe99] Andreas Eberle. "Motivation and basic definitions: Uniqueness problems in various contexts". In: *Uniqueness and Non-Uniqueness of Semigroups Generated by Singular Diffusion Operators*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1999, pp. 9–40. ISBN: 978-3-540-48076-1. DOI: 10.1007/BFb0103047.

[EG21] Benedikt Eisenhuth and Martin Grothaus. *Hypocoercivity for non-linear infinite-dimensional degenerate stochastic differential equations*. 2021. arXiv: 2109.09999 [math.PR].

[EK86] Stewart N. Ethier and Thomas G. Kurtz. *Markov Processes: Characterization and Convergence.* John Wiley & Sons, Ltd, 1986. Chap. 3, pp. 95–154. ISBN: 9780470316658. DOI: 10.1002/9780470316658.

[EN00] Klaus-Jochen Engel and Rainer Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* New York, NY: Springer New York, 2000. ISBN: 978-0-387-22642-2. DOI: 10.1007/0-387-22642-7.

[GM20] Martin Grothaus and Maximilian Mertin. *Hypocoercivity of Langevin-type dynamics on abstract smooth manifolds.* 2020. arXiv: 2006.11567 [math.PR].

[GS14] Martin Grothaus and Patrik Stilgenbauer. "Hypocoercivity for Kolmogorov backward evolution equations and applications". In: *Journal of Functional Analysis* 267.10 (2014), pp. 3515–3556. ISSN: 0022-1236. DOI: https://doi.org/10.1016/j.jfa.2014.08.019.

[GS16] Martin Grothaus and Patrik Stilgenbauer. "Hilbert space hypocoercivity for the Langevin dynamics revisited". In: *Methods Funct. Anal. Topology* 22.2 (2016), pp. 152–168. ISSN: 1029-3531. eprint: http://mfat.imath.kiev.ua/article/2016/02/mfat_2016_02_847.pdf.

[GW19] Martin Grothaus and Feng-Yu Wang. "Weak Poincaré inequalities for convergence rate of degenerate diffusion processes". In: *The Annals of Probability* 47.5 (2019), pp. 2930–2952. DOI: 10.1214/18-AOP1328.

[Hér05] Frédéric Hérau. "Hypocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation". In: *Asymptotic Analysis* 46 (Apr. 2005). eprint: https://arxiv.org/pdf/math/0503351.

[HM17] David Herzog and Jonathan Mattingly. "Ergodicity and Lyapunov Functions for Langevin Dynamics with Singular Potentials". In: *Communications on Pure and Applied Mathematics* 72 (Nov. 2017). DOI: 10.1002/cpa.21862.

[KS98] Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus.* New York, NY: Springer New York, 1998. ISBN: 978-1-4612-0949-2. DOI: 10.1007/978-1-4612-0949-2.

[LP20] A. Lunardi and D. Pallara. "Ornstein-Uhlenbeck semigroups in infinite dimension". In: 378.2185 (Oct. 2020), p. 20190620. DOI: 10.1098/rsta.2019.0620.

[MR92] Zhi-Ming Ma and Michael Röckner. *Markov Processes and Dirichlet Forms.* Berlin, Heidelberg: Springer Berlin Heidelberg, 1992. ISBN: 978-3-642-77739-4. DOI: 10.1007/978-3-642-77739-4.

[Non20] Andreas Nonnenmacher. "Operator semigroups and infinite dimensional analysis applied to problems from mathematical physics". PhD thesis. Technische Universität Kaiserslautern, 2020. eprint: https://kluedo.ub.uni-kl.de/files/6128/Dissertation+Nonnenmacher+301020.pdf.

[Paz83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. New York, NY: Springer New York, 1983. ISBN: 978-1-4612-5561-1. DOI: 10.1007/978-1-4612-5561-1.

[Ped89] Gert K. Pedersen. *Analysis Now*. New York, NY: Springer New York, 1989. ISBN: 978-1-4612-1007-8. DOI: 10.1007/978-1-4612-1007-8.

[PL14] Giuseppe Da Prato and Alessandra Lunardi. “Sobolev regularity for a class of second order elliptic PDE’s in infinite dimension”. In: *The Annals of Probability* 42.5 (2014), pp. 2113–2160. DOI: 10.1214/14-AOP936.

[PR07] Claudia Prévôt and Michael Röckner. *A Concise Course on Stochastic Partial Differential Equations*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007. ISBN: 978-3-540-70781-3. DOI: 10.1007/978-3-540-70781-3.

[Pra06] Giuseppe Da Prato. *An Introduction to Infinite-Dimensional Analysis*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006. ISBN: 978-3-540-29021-6. DOI: 10.1007/3-540-29021-4.

[RS81] M. Reed and B. Simon. *I: Functional Analysis*. Methods of Modern Mathematical Physics. Elsevier Science, 1981. ISBN: 9780080570488.

[RW01] M. Röckner and F.-Y. Wang. “Weak Poincaré Inequalities and L^2-Convergence Rates of Markov Semigroups”. In: *Journal of Functional Analysis* 185 (Oct. 2001), pp. 564–603. DOI: 10.1006/jfan.2001.3776.

[Sta99] Wilhelm Stannat. “The Theory Of Generalized Dirichlet Forms And Its Applications In Analysis And Stochastics”. In: *Mem. Amer. Math. Soc* 142.678 (1999). DOI: 10.1090/memo/0678.

[Tru00] Gerald Trutnau. “Stochastic calculus of generalized Dirichlet forms and applications to stochastic differential equations in infinite dimensions”. In: *Osaka journal of Mathematics* 37 (2000), pp. 315–343. eprint: https://dlisv03.media.osaka-cu.ac.jp/contents/osakacu/sugaku/111F0000002-03702-4.pdf.

[Tru03] Gerald Trutnau. “On a class of non-symmetric diffusions containing fully nonsymmetric distorted Brownian motions”. In: *Forum Mathematicum* 15.3 (2003), pp. 409–437. DOI: doi:10.1515/form.2003.022.

[Vil06] Cédric Villani. “Hypocoercivity”. In: *Memoirs of the American Mathematical Society* 202 (Oct. 2006). DOI: 10.1090/S0065-9266-09-00567-5.

[Wan17] Feng-Yu Wang. “Hypercontractivity and applications for stochastic Hamiltonian systems”. In: *Journal of Functional Analysis* 272.12 (2017), pp. 5360–5383. DOI: 10.1016/j.jfa.2017.03.015.

[Wu01] Liming Wu. “Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems”. In: *Stochastic Processes and their Applications* 91.2 (2001), pp. 205–238. ISSN: 0304-4149. DOI: 10.1016/S0304-4149(00)00061-2.

Academic background

since December 2017	PhD student at TU Kaiserslautern
September 2017	Master thesis “Feynman Formulae for Semigroups in L^p-spaces”
2015 – 2017	student in the “Master Mathematics International” programme at TU Kaiserslautern with semester abroad at the National University of Singapore
July 2015	Bachelor thesis “A Dirichlet Form Approach to the Construction of Stochastic Dynamics in a Predator-Prey Model”
2012 – 2015	student in the “Bachelor Mathematik” programme with minor in computer science at TU Kaiserslautern
March 2012	Abitur at Burggymnasium Kaiserslautern

Akademischer Werdegang

seit Dezember 2017	Promotionsstudent an der TU Kaiserslautern
September 2017	Masterarbeit „Feynman Formulae for Semigroups in L^p-spaces“
2015 – 2017	Masterstudium „Mathematics International“ an der TU Kaiserslautern mit Auslandssemester an der National University of Singapore
Juli 2015	Bachelorarbeit „A Dirichlet Form Approach to the Construction of Stochastic Dynamics in a Predator-Prey Model“
2012 – 2015	Bachelorstudium Mathematik mit Nebenfach Informatik an der TU Kaiserslautern
März 2012	Abitur am Burggymnasium Kaiserslautern