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SENSITIVITY OF LUENBERGER OBSERVERS, ε-OBSERVABILITY AND UNCERTAINTY RELATIONS

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Abstract: In this paper noises and disturbances are treated as distributions of some general class. The problem of sensitivity minimization is considered. A design procedure for the construction of Luenberger observers which estimate the state of a system with a given rate of accuracy has been proposed. The design procedure is applied to identify the first derivatives of an oscillating signal. The constraintes on a noise and on a sampling which are necessary to estimate the derivatives to a given accuracy have been obtained.

1 Introduction

State observers play an important role in the control theory. They are applied both to output-stabilization of a control system [1,5,9] and to sate-identification [11,13]. Various practical applications of Luenberger observers inspired an interest in the investigation of sensitivity and robustness of the observers with respect to sets of noises and/or disturbances. There are many literature devoted to the problem in question (see,e.g., [2,3,6,8,10,12,15-20] and review [8]).

All these papers can be roughly divided into two big groups. The first group consists of the articles in which authors use stochastic properties of noises, e.g., their covariance matrixes. The second group is composed by the papers in which the noises are treated as unknown elements of a space of functions. Our work can be certainly placed in the second group.

In the classical book [11] the observer sensitivity minimization problem has been reduced with the help of quadratic optimization methods to the solving of Riccati equation. In the book a covariance matrix of noise has been supposed to bee given. The robustness of th's method with respect to parametric disturbances has been considered in [4].

Later more universal approach based on H^{∞}- and L^{∞}- norm sensitivity minimization has been proposed [7,8,20]. In the approach noises are supposed to be in the convex subset defined by

$$\int_{-\infty}^{\infty} | d(i\phi) |^2 \cdot | W(i\phi) |^{-2} \, d\phi \leq 1,$$

where $W(i\phi)$ is a weight function characterizing a set of disturbances, $d(i\phi)$ is the fourier image of a mapping modelling noise. $| W(s) |$ is supposed to be decreasing as $| s | \rightarrow \infty$ and $| W(s) | > \frac{C}{|s|^{\ell}}$ for $| s |$ sufficiently large, where ℓ is a natural number and C is a positive real number. The sensitivity is characterized by a linear operator which H^{∞} -norm has to be done as small as possible. In the work [14] the robustness with respect to disturbances from L_2 -ball has been concerned.

In this paper we shall minimize observer sensitivity with respect to a class of distributions. Noises will be treated as distributions of a sufficiently large class given. For example, a noise can be of the form

$$\sum_{j=1}^{N} a_j \cdot e^{i\omega_j t},$$

where $a_j \in \mathrm{C}$, $\omega_j \in \mathrm{R}$ $(j = 1, \cdots, N)$ and $\min_j | \omega_j | \geq \Omega$ with Ω being a known positive real number.

The paper is organized as follows: In Section 2 we state the problem. In Section 3 we calculate estimates of sensitivity. In Section 3 we consider ϵ- observability problem. Section 4 contains simulations results.

2 Problem statement

Consider the system Σ :

$$\dot{x} = Ax + \psi(t), \qquad (1)$$

$$y = < c, x > + \Delta y(t),$$

where $y \in \mathrm{R}$ is an output signal; $c, x \in \mathrm{R}^n$ are n-dimensional real vectors and

$$< c, x > = \sum_{i=1}^{n} c_i x_i,$$

$A \in \mathrm{R}^{n \times n}$ is a real matrix with n columns and n rows.
$(\psi(t), \Delta y(t))$ is a distribution from the set $\Re^n(\Delta\Sigma, \Omega) \times \Re(\Delta Y, \Omega)$, where

$$\underbrace{\Re^n(\Delta\Sigma, \Omega) = \Re(\Delta\Sigma, \Omega) \times \ldots \times \Re(\Delta\Sigma, \Omega)}_{n}$$

and $\Re(\Delta\Sigma, \Omega)$ consists of all distributions φ, such that the function

$$\Phi_\varphi(t, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{F(\varphi)(i\mu)}{z + i\mu} e^{-i\mu t} d\mu \qquad (2.a)$$

with

$$F(\varphi)(i\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\mu t} \varphi(t) dt,$$

is analytic on $\mathrm{C}_- = \{\lambda \in \mathrm{C} \,;\; \mathrm{Re}\lambda < 0\}$ for every $t \in \mathrm{R}$ fixed and

$$| \Phi_\varphi(t, z) | \leq \frac{\Delta\Sigma}{| z + i\Omega |} \qquad (2.b)$$

for all $(t, z) \in \mathrm{R} \times \mathrm{C}$. C and R denote the fields of complex and real numbers, respectively.
Suppose

$$rank\{c, A^* c, \ldots, (A^*)^{n-1} c\} = n$$

with A^* being the conjugated operator for A. Then for the system Σ one can design Luenberger's observer

$$\dot{z}(t) = Az(t) + \ell \cdot (< c, z(t) > - y), \qquad (3)$$

where $\ell \in \mathrm{R}^n$.
The goal of this paper is to give necessary and sufficient conditions on $\Sigma, \Delta\Sigma, \Omega, \epsilon > 0$ under which there exists $\ell \in \mathrm{R}^n$, such that

$$\overline{\lim_{t \to \infty}} \, \| z(t) - x(t) \| \leq \epsilon,$$

where $\| z \|^2 = < z, z >$.

We will develop also a numerical procedure for the calculation of $\ell \in \mathrm{K_q}$, where

$$\mathrm{K_q} = \{z \in \mathrm{C^n}\ ;\ z = rq, r \in \mathrm{R_+}\},$$

$q_j \neq q_i$ for $i \neq j$ and

$$q \in \mathrm{RC^n_-} = \{z \in \mathrm{C^n}\ ;\ \mathrm{Re}\, z_i\ < 0\ \ (i = 1, 2, \dots, n),\ \ X_j(z) \in \mathrm{R}\ \ (j = 1, \dots, n)\},$$

where $X_j(z)_{j=1}^n$ are the elementary symmetric polinomials in the n variables,i.e.,

$$X_1(z) = \sum_{i=1}^{n} z_i, X_2(z) = -\sum_{i<j} z_i z_j, \dots, X_n(z) = (-1)^{n+1} \prod_{j=1}^{n} z_j$$

and $z_1 \cdot \ldots \cdot z_n$ is denoted by $\prod_{j=1}^n z_j$. In the other words, the problem is to construct $\ell \in \mathrm{R^n}$ which minimizes the sensitivity of the system Σ with respect to disturbances from $\Re^n(\Delta\Sigma, \Omega) \times \Re(\Delta Y, \Omega)$.

3 Estimates of sensitivity

Sensitivity of the system Σ with respect to a set of disturbances D is defined as follows.

Definition 1. Let D be a subset in $\Re^n(\Delta\Sigma, \Omega) \times \Re(\Delta Y, \Omega)$.Then sensitivity $S_D^\Sigma(\lambda)$ of the system Σ with respect to D is defined by

$$S_D^\Sigma(\lambda) = \sup_{(\psi, \Delta y) \in D} \overline{\lim_{t \to \infty}} \| \int_{-\infty}^{\infty} e^{\bar{A}(t-\tau)}(\psi(\tau) + \ell \Delta y(\tau)) d\tau \|, \tag{4}$$

where $\bar{A} = A + \ell c^T$ and eigenvalues $\lambda = \{\lambda_i\}_{i=1}^n$ of $\bar{A}$ are supposed to be in $\mathrm{RC^n_-}$.

Since $rank\{c, Ac^T, \dots, A^{n-1}c^T\} = n$ the system Σ has the form

$$\dot{x} = J_n x + a x_1 + T\psi(t),$$

$$\tag{5}$$

$$y = x_1 + \Delta y(t)$$

under the linear coordinate transformation

$$x_{k+1} = < c, A^k z > - \sum_{j=1}^{k} a_k < c, A^{k-j} z > \quad k = 0, 1, \dots, n-1 \tag{6}$$

where z and x are the old and the new coordinates, respectively. $A^n = \sum_{j=1}^n a_j A^{n-j}$, J_n is Jordan $n \times n$ matrix, such that

$$J_n e_1 = 0, \quad J_n e_k = e_{k-1} \quad \text{for}\ \ k > 1,$$

$$e_k = \{\delta_j^k\}_{j=1}^n \quad \text{with} \quad \delta_k^k = 1 \text{ and } \delta_j^k = 0 \text{ for } j \neq k.$$

Under the coordinate transformation $x = Tz$ the Lyenberger's observer (3) has the form

$$\dot{\bar{x}}(t) = J_n\bar{x}(t) + a\bar{x}_1(t) + (\alpha - a)(x_1(t) - y(t)),$$

where entries of the vector α are the elementary symmetric polynomials, i.e.,

$$\alpha_1 = X_1(\lambda), \ \alpha_2 = X_2(\lambda), \ldots, \alpha_n = X_n(\lambda),$$

where $\lambda \in \mathrm{RC}^n_-$ is the vector with entries being eigenvalues of $A + \ell c^T$.
Consider the following matrix

$$\Im = \begin{pmatrix} \alpha_{n-1} & \alpha_{n-2} & \ldots & \alpha_1 & 1 \\ \alpha_{n-2} & \alpha_{n-3} & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & & \\ \alpha_1 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Then the next Lemma gives us the expression of $\nu \in \mathrm{R}^n$ as a linear combination of the eigenvectors of the operator $J_n + \alpha P_1$, where P_1 is the orthogonal projection $P_1 x = x_1$ for all $x \in \mathrm{R}_n$.

Lemma. Let $\lambda = \{\lambda_i\}_{i=1}^n$ be the eigenvalues of $J_n + \alpha P_1$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Then every $\nu \in \mathrm{R}^n$ admits the representation

$$\nu = \sum_{i=1}^n \frac{< \nu, \Im\xi_i >}{\prod_{j \neq i}(\lambda_i - \lambda_j)}\xi_j,$$

where $(J_n + \alpha P_1)\xi_j = \lambda_j\xi_j$ for $j = 1, 2, \ldots, n$.

Proof. The eigenvector ξ_i of the operator $J_n + \alpha P_1$ is of the form

$$\xi_i = \{1, \lambda_i - \alpha_1, \lambda_i^2 - \alpha_1\lambda_i - \alpha_2, \ldots, \lambda_i^{n-1} - \alpha_1\lambda_1 - \ldots - \alpha_{n-1}\}.$$

Every $\nu \in \mathrm{R}^n$ admits the representation $\nu = \sum_{i=1}^n \omega_i\xi_i$, where

$$\omega_i = \frac{det(\xi_1, \ldots, \xi_{i-1}, \nu, \xi_{i+1}, \ldots, \xi_n)}{det(\xi_1, \ldots, \xi_n)} \qquad (7).$$

$det(*)$ denotes the determinant of a matrix $(*)$, $(\xi_1, \ldots, \xi_n)$ is the matrix with columns $\xi_1, \ldots, \xi_n$. It is easy to see that

$$det(\xi_1, \ldots, \xi_n) = \prod_{i>j}(\lambda_i - \lambda_j).$$

Taking the vector $\zeta = (\zeta_1, \ldots, \zeta_n)$ defined by

$$\zeta_1 = \nu_1, \quad \zeta_j = \nu_j + \sum_{i=1}^{j-1} \alpha_i\nu_{j-i} \quad (j = 2, \ldots, n) \qquad (8).$$

we obtain

$$det(\xi_1,\ldots,\xi_{i-1},\nu,\xi_{i+1},\ldots,\xi_n) = (-1)^{n-1}\prod_{k>m,k\neq i}(\lambda_k-\lambda_m)\eta_n,$$

where η_n is the n-th entry of the vector

$$\eta = \prod_{j=1}^{i-1}(I-\lambda_j J_n^*)\prod_{j=i+1}^{n}(I-\lambda_j J_n^*)\zeta, \tag{9}$$

where I is the identity matrix. If follows fromy (8),(9) that

$$\omega_i = \frac{<\nu,\Im\xi_i>}{\prod_{j\neq i}(\lambda_i-\lambda_j)}$$

and the proof is completed.

Now using the Lemma we will calculate the estimates for $S_D^{\Sigma}(\lambda)$.

Theorem 1 Let $(T\psi,\Delta y)\in\Re^n(\Delta\Sigma,\Omega)\times\Re(\Delta Y,\Omega)$ and $\lambda_k\neq\lambda_j$ for $k\neq j$, where $T:\mathrm{R}^n\to\mathrm{R}^n$ is defined in (4.6) and $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of $J_n+\alpha P_1$, where P_1 is the orthogonal projection $P_1x=x_1$ for all $x\in R^n$. Then $S_D^{\Sigma}(\lambda)$ satisfies the inequality

$$\| T \|^{-1} V(\lambda,\Delta\Sigma,\Delta Y,\Omega)\leq S_D^{\Sigma}(\lambda)\leq \| T^{-1} \| U(\lambda,\Delta\Sigma,\Delta Y,\Omega),$$

where $(J_n+\alpha P_1)\xi_j=\lambda_j\xi_j$ for $j=1,2,\ldots,n$,

$$V(\lambda,\Delta\Sigma,\Delta Y,\Omega)=\sup_{(\beta,\chi)\in\wp}\|\sum_{j=1}^{n}\frac{<\beta+\chi(\alpha-a),\Im\xi_j>}{\prod_{k\neq j}(\lambda_j-\lambda_k)(\lambda_j+i\Omega)}\xi_j\|$$

with

$$\wp=\{(\beta,\chi)\in \mathrm{C}^{n+1};\quad \|\chi\|\leq\Delta\mathrm{Y},\quad |\beta_\mathrm{j}|\leq\Delta\Sigma\ \ (\mathrm{j}=1,\ldots,\mathrm{n})\},$$

$$U(\lambda,\Delta\Sigma,\Delta Y,\Omega)=\sup_{(\varphi,\gamma)\in\Gamma}\|\sum_{j=1}^{n}\frac{<\varphi_j+\gamma_j(\alpha-a),\Im\xi_j>}{\prod_{k\neq j}(\lambda_j-\lambda_k)\mid\lambda_j+i\Omega\mid}\xi_j\|,$$

where

$$\Gamma=\{(\varphi,\gamma)\in\mathrm{C}^{\mathrm{n}\times(\mathrm{n}+1)};\ \mid\gamma_\mathrm{j}\mid\leq\Delta\mathrm{Y},\ \mid\varphi_{\mathrm{jk}}\mid\leq\Delta\Sigma\quad(\mathrm{j}=1,\ldots,\mathrm{n}\ ;\ \mathrm{k}=1,\ldots,\mathrm{n})\}.$$

Proof. The integral in(4) is equal to

$$\{e^{\bar{A}t}\theta(t)\}\ *\ (\Psi(t)+\ell\Delta y(t)),$$

where $\theta(t)$ is the step function

$$\theta(t)=0\quad for\quad t<0,$$

$$\theta(t)=1\quad for\quad t\geq 0,$$

$v * w$ is the convolution of the functions v and w ,i.e.,

$$v * w = \int_{-\infty}^{\infty} v(t-\tau)w(\tau)d\tau.$$

Due to $F(v * w) = Fv \cdot Fw$ we have

$$\{e^{\overline{A}t}\theta(t)\} * (\Psi(t) + \ell\Delta y(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\overline{A} + i\mu)^{-1}(F(\Psi) + \ell F(\Delta y))e^{-i\mu t}d\mu.$$

Hence according to the Lemma we obtain

$$\{e^{\overline{A}t}\theta(t)\} * (\Psi(t) + \ell\Delta y(t)) = \sum_{j=1}^{n} \frac{< \varphi_j + \gamma_j(\alpha - a), \Im\xi_j >}{\prod_{k\neq j}(\lambda_j - \lambda_k) \mid \lambda_j + i\Omega \mid} T^{-1}\xi_j, \tag{10}$$

where

$$\gamma_j = \frac{\mid \lambda_j + i\Omega \mid}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{F(\Delta y)}{(\lambda_j + i\mu)} e^{-i\mu t} d\mu \quad (j = 1, \ldots, n),$$

$$\varphi_{jk} = \frac{\mid \lambda_j + i\Omega \mid}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{F(\Psi_k)}{(\lambda_j + i\mu)} e^{-i\mu t} d\mu \quad (j, k = 1, \ldots, n).$$

Thus $(\varphi, \gamma) \in \Gamma$, where Γ is defined in the conditions of this theorem.

Now estimating the norm in (4) and taking into account (10), we obtain the upper estimate of sensitivity $S_D^{\Sigma}(\lambda)$.
It remains to calculate the low estimate of $S_D^{\Sigma}(\lambda)$. Notice that a distribution $\beta e^{i\omega t} \in \Re(\Delta\Sigma, \Omega)$ for all $\omega \in R, \beta \in C$, such that $\mid \beta \mid \leq \Delta\Sigma$ and $\mid \omega \mid \geq \Omega$. Let $(\Psi, \Delta y) = (\beta e^{i\Omega t}, \chi e^{i\Omega t})$, i.e., $\Psi_j = \beta_j e^{i\Omega t}$ $(j = 1, \ldots, n)$, $\Delta y = \chi e^{i\Omega t}$, where $\mid \chi \mid \leq \Delta Y$, $\mid \beta_j \mid \leq \Delta\Sigma$ $(j = 1, \ldots, n)$.
Evidently $(\beta e^{i\Omega t}, \chi e^{i\Omega t}) \in \Re^n(\Delta\Sigma, \Omega) \times \Re(\Delta x, \Omega)$ and after the substitution of the paire $(\beta e^{i\Omega t}, \chi e^{i\Omega t})$ in (10) the resulted expression is

$$\{e^{\overline{A}t}\theta(t)\} * (\beta e^{i\Omega t} + \ell\chi e^{i\Omega t}) = e^{i\Omega t} \sum_{j=1}^{n} \frac{< \beta + (\alpha - a)\chi, \Im\xi_j >}{\prod_{k\neq j}(\lambda_j - \lambda_k)(\lambda_j + i\Omega)} T^{-1}\xi_j.$$

Therefore

$$\| T \|^{-1} V(\lambda, \Delta\Sigma, \Delta Y, \Omega) \leq \sup_{(\beta,\chi)\in\wp} \lim_{t\to\infty} \| (e^{\overline{A}t}\theta(t)) * (\beta + \ell\chi)e^{i\Omega t} \| \leq S_D^{\Sigma}(\lambda),$$

where $\wp$ is defined in the conditions of this theorem.The proof is completed.

Necessary and sufficient conditions for $S_D^{\Sigma}(\lambda) \leq \epsilon$ with $\epsilon > 0$ fixed follow from Theorem 1. Indeed, if $D \subset \Re^n(\Delta\Sigma, \Omega) \times \Re(\Delta Y, \Omega)$ and $(\beta e^{i\Omega t}, \chi e^{i\Omega t}) \subset D$ for some $\beta \in C^n$, $\chi \in C$, $\Omega \in R$, then $S_D^{\Sigma}(\lambda) \leq \epsilon$ yields

$$V(\lambda, \Delta\Sigma, \Delta Y, \Omega) \leq \| T \| \cdot \epsilon$$

which is a necessary condition for $\epsilon-$ observability. A sufficient condition for $\epsilon-$ observability follows from

$$S_D^{\Sigma}(\lambda) \leq \| T^{-1} \| U(\lambda, \Delta\Sigma, \Delta Y, \Omega),$$

i.e., if $U(\lambda, \Delta\Sigma, \Delta Y, \Omega) \leq \frac{\epsilon}{\|T^{-1}\|}$, then $S_D^{\Sigma}(\lambda) \leq \epsilon$.

4 ϵ-observability and uncertainty relations

Here necessary and safficient conditions for ϵ-observability will be obtained.

Definition 2. A system Σ is said to be ϵ-observable (wiht respect to disturbances from D) iff there existens $\lambda \in RC_{-}^{n}$, such that $S_D^{\Sigma}(\lambda) \leq \epsilon$.

A simple sufficient condition for ϵ-observability follows directly from Theorem 1.

Proposition. For all $\lambda \in RC_{-}^{n}$ with $\lambda_k \neq \lambda_j$ for $k \neq j$ and for all $\epsilon > 0, \Delta\Sigma > 0$, $\Delta Y > 0$ one can find a positive real number H, such that for all $\Omega > H$ the inequality

$$S_D^{\Sigma}(\lambda) \leq \epsilon,$$

where $D \subset \Re^n(\Delta\Sigma, \Omega) \times \Re(\Delta Y, \Omega)$, holds.

Thus noises of higher frequencies ($\Omega >> 1$) whose absolute values are small in compearing with their frequencies, can be easily filtered by Luenberger's observers. If $\Omega << 1$ or $\Omega = 0$, than $\Re^n(\Delta\Sigma, \Omega) \times \Re(\Delta Y, \Omega)$ containes also noises of a low frequency. The noises of low frequencies give the largest contribution in observation errors. The following necessary condition for ϵ-observability does not depend on a noise frequency.

Theorem 2 (uncertainty relations). Let

$$D = \Re^n(\Delta\Sigma, 0) \times \Re^n(\Delta Y, 0), \ \ \Delta\Sigma > \Delta Y \sqrt{\sum_{i=1}^{n}(a_i^2)}$$

and observer eigenvalues be $\lambda_k = rq_k$ ($k = 1, \dots, n$), where $q \in RC_{-}^{n}$ fixed and $q_k \neq q_j$ for $k \neq j$, $\{a_k\}_{k=1}^{n}$ be coefficients of the characteristic polyonmial of the matrix A. Then it is necessary for ϵ-observability that the following uncertainty relation hold:

$$| \gamma_j(q) |^{1-\frac{j}{n}} | \varphi_j(q) |^{\frac{j}{n}} \sigma_j^{1-\frac{j}{n}} (\Delta Y)^{\frac{j}{n}} \leq \| T \| (j)^{\frac{j}{n}} \frac{(\delta_j^n + n - j)^{1-\frac{j}{n}}}{n} \epsilon \quad j = 1, \dots, n, \qquad (11)$$

where $\delta_j^n = 0$ for $j \neq n$ and $\delta_n^n = 1$,

$$\sigma_j = (\Delta\Sigma)^2 - (\Delta Y)^2 \sum_{j=1}^{n-1}(a_j)^2 - \text{sgn}(\gamma_j(q)\varphi_j(q)) a_n \Delta Y$$

$$\gamma_j(q) = \sum_{k=1}^{n} \frac{\lambda_k^{n-j} - \alpha_1 \lambda_k^{n-j-1} - \ldots - \alpha_{n-j}}{r^{n-j} q_k \prod_{\nu \neq k}(q_k - q_\nu)},$$

$$\varphi_j(q) = \sum_{k=1}^{n} \frac{< \alpha, \Im\xi_k > (\lambda_k^{n-j} - \alpha_1 \lambda_k^{n-j-1} - \ldots - \alpha_{n-j})}{r^{2n-j} q_k \prod_{\nu \neq k}(q_k - q_\nu)}, \quad (j = 1, \ldots, n).$$

Proof. Let as take $\lambda_1 = rq_1, \ldots, \lambda_n = rq_n$. Then making use of Theorem 1, we obtain that $\epsilon-$ observability yields

$$V(\lambda, \Delta\Sigma, \Delta Y, 0) \leq \| T \| \epsilon.$$

Therefore

$$\sum_{j=1}^{n} \frac{< \rho + \mu(\alpha - a), \Im\xi_j >}{\prod_{m \neq j}(\lambda_j - \lambda_m)\lambda_j} (\lambda_j^{n-k} - \alpha_1 \lambda_j^{n-k-1} - \ldots - \alpha_{n-k}) \mid \leq \| T \| \epsilon \tag{12}$$

$$(k = 1, \ldots, n)$$

for all $\rho \in \mathrm{R}^n, \mu \in \mathrm{R}$, such that $| \rho | \leq \Delta\Sigma, | \mu | \leq \Delta Y$. Take $| \mu | = \Delta Y, \rho_j = a_j \mu$ for $j = 1, \ldots, n-1$ and

$$\rho_n = \sqrt{(\Delta\Sigma)^2 - (\Delta Y)^2 \sum_{j=1}^{n-1} (a_j)^2}.$$

The signes of ρ_n and μ can be chosen so that (12) has the form

$$\frac{\sigma_i}{r^i} \mid \gamma_i(q) \mid + r^{n-1} \Delta Y \mid \varphi_i(q) \mid \leq \| T \| \epsilon \tag{13}$$

$$(i = 1, \ldots, n)$$

The function of r in the left hand said of (13) has the minimum equal to

$$(\Delta Y)^{\frac{1}{n}} (\sigma_j)^{1-\frac{1}{n}} \mid \varphi_j(q) \mid^{\frac{1}{n}} \mid \gamma_j(q) \mid^{1-\frac{1}{n}} \frac{n}{(\delta_j^n + (n-j))^{1-\frac{1}{n}} \cdot (j)^{\frac{1}{n}}}.$$

Hence the uncertainty relations (11) are necessary for (13). This completes the proof.

Since $\Omega' > \Omega$ implies $\Re(\Delta\Sigma, \Omega') \subset \Re(\Delta\Sigma, \Omega)$, Theorem 2 qives us a necessary condition for ϵ - differentiability of a signal disturbed by a noise of any frequency.
The uncertaint relations (11) have simpler form whenever

$$a_i = 0, \quad < c, A^i \psi > = 0 \text{ for all } i = 1, \ldots, n-1.$$

Under these conditions Lyenberger's observer is a differentiator of the signal $y(t)$

$$\max_t \mid \frac{d^{n+1} y(t)}{dt^{n+1}} \mid \leq \Delta\Sigma \tag{14}$$

and measured with error $\Delta y(t)$. Thus ϵ-observability in this case means ϵ-differentiability and ϵ-observer is called ϵ-differentiator.

The next corollary of Theorem 2 gives us a necessary condition for ϵ-differentiability.

Corollary. Let $y(t)$ be a $(n+1)$-differentiable function of time satisfying the inequality (14). Suppose further $y(t)$ is measured with an error $\Delta y(t)$ such that $\max_t \mid \Delta y(t) \mid \leq \Delta Y$, where ΔY is a positive real number given. Then for ϵ-differentiability of order n of the signal $y(t)$ it is necessary that uncertainty relations

$$\mid \gamma_j(q) \mid^{1-\frac{j}{n}} \mid \varphi_j(q) \mid^{\frac{j}{n}} (\Delta\Sigma)^{1-\frac{j}{n}} (\Delta Y)^{\frac{j}{n}} < \frac{(\delta_n^j + (n-j))^{1-\frac{j}{n}}}{n} \cdot j^{\frac{j}{n}} \cdot \epsilon \quad (j = 1, \ldots, n)$$

hold, where $q \in RC^n_-$ and the functions $\gamma_j(q), \quad \varphi_j(q) \quad (j = 1, \ldots, n)$ are defined in Theorem 2.

Proof. ϵ-differentiability of order n of the signal $y(t)$ is reduced to ϵ-observability of a state of the system

$$\dot{x} = J_n x + \psi, \qquad y = x_1 + \Delta y, \tag{15}$$

where J_n is Jordan matrix and $\psi_j = 0$ for $j = 1, \ldots, n-1$. Now the application of the uncertainty relations (11) completes the proof.

The uncertainty relations (11) are simple to apply. However, they do not give any recommendation how to design an observer. Moreover they are far from sufficient cinditions when a higher frequency noise is under consideration.

In order to obtain ϵ-observability sufficient conditions depending both on $\Delta\Sigma, \Delta Y$ and on frequency Ω we will use the inequality

$$S_D^{\Sigma}(\lambda) \leq \| T^{-1} \| V(\lambda, \Delta\Sigma, \Delta Y, \Omega)$$

from Theorem 1.

Proposition. If

$$\inf_{\lambda \in RC^n_-} U(\lambda, \Delta\Sigma, \Delta Y, \Omega) < \frac{\epsilon}{\| T^{-1} \|},$$

then Σ is ϵ-observable.

This proporition can be successively applied only to systems in low dimensions. Difficulties connected with the calculation of the infinum make the application impossible to higher dimensional systems.

Here we will solve simpler problem, i.e., we calculate a value of $r \geq 0$, such that a function being an upper bound for $U(rq, \Delta\Sigma, \Delta Y, \Omega)$ reaches minimum at r. Thereby we will obtain ϵ-observability sufficient conditions. The value of $r > 0$ can be numerically calculated as follows.

Procedure (choice of r)

Step 1. Let $\lambda_i = rq_i$ $(i = 1, \ldots, n)$, $r > 0$, $q \in \mathrm{RC}^n_-$ and $q_i \neq q_k$ for $j \neq k$ $(j, k = 1, \ldots, n)$. Then k-th entry of the vector

$$\sum_{j=1}^{n} \frac{< \varphi_j + \gamma_j(a - a), \Im\xi_j >}{\prod_{k \neq j}(\lambda_j - \lambda_k) \mid \lambda_j + i\Omega \mid}\xi_j$$

has the form

$$\sum_{j=1}^{n} \frac{1}{\mid rq_j + i\Omega \mid} \sum_{\mu=-(n-k)}^{k} \omega_{kj\mu}(\varphi_j, \gamma_j) r^{\mu} \quad (k = 1, 2, \ldots, n).$$

The mapping $\omega_{kj\mu}(\varphi_j, \gamma_j)$ is linear with respect to φ_j and γ_j. Calculate the functions

$$W_k(r) = \sum_{j=1}^{n} \frac{1}{\mid rq_j + i\Omega \mid} \sum_{\mu=-(n-k)}^{k} b_{kj\mu}(\Delta\Sigma, \Delta Y) r^{\mu} \quad (k = 1, 2, \ldots, n),$$

where

$$b_{kj\mu}(\Delta\Sigma, \Delta Y) = \max_{\mu=1,\ldots,n} \mid \omega_{kj\mu}(\varphi_j, \gamma_j) \mid,$$

$$\mid \gamma_j \mid \leq \Delta Y, \quad \mid \varphi_{j\mu} \mid < \Delta\Sigma \quad (\mu = 1, 2, \ldots, n).$$

Then we obtain

$$U(rq, \Delta\Sigma, \Delta Y, \Omega) \leq W(r),$$

where $W(r) = \sum_{i=1}^{n} W_i(r)$.

Step 2. Calculate minimum of the function $W(r)$ for $r > 0$. Then the point $r^* > 0$, such that $W(r^*) = \min_{r>0} W(r)$ is the number which we are looking for.

The next theorem contains sufficient condition for the existencs of $r^* > 0$.

Theorem 3. Let $q \in \mathrm{RC}^n_-$ and $q_j \neq q_k$ for $j \neq k$. Suppose further

$$\{\sum_{j=1}^{n} \frac{b_{njn}}{\mid rq_j + i\Omega \mid}\}\{\sum_{j=1}^{n} \frac{b_{1j1-n}}{\mid r \cdot q_j + i\Omega \mid}\} \neq 0.$$

Then $r^* > 0$ being calculated in Step 2 exists.

Proof. If the conditions are met, then $W(r)$ has the pole of order $(n - 1)$ in the origin and the pole of order n at infinity. Therefore one can find $0 < r^* < \infty$, such that $\min_{r>0} W(r) = W(r^*)$.

Corollary. Let all conditions of Theorem 3 be met and $r^* > 0$ be number obtained in Step 2 of the procedure. Then $W(r^*) < \epsilon$ yields ϵ - observability of Σ.

The proposed numerical procedure does not give us optimal Luenberger's observer. However we can design an observer solving the problem of ϵ - observability for $\epsilon > W(r^*)$. The value of $W(r^*)$ depends on ΔY, $\Delta\Sigma$, Ω, $q \in RC^n_-$ and can decrease after variations of $q \in RC^n_-$. The problem of an optimal choice for q is beyond the scope of this work.

Finishing this section we notice that the choice of $r^* > 0$ can be improved after having used an asymptotic of $U(rq, \Delta\Sigma, \Delta Y, \Omega)$ either for $\Omega \to \infty$ (high frequency noise) or for $\Omega \to 0$ (low frequency noise).

5 Examples (identification of oscillators)

Here the results obtained above will be illustrated by examples.

Example 1. Let us estimate the precision of a calculation of the first three derivatives of the signal $x(t)$ by means of the differentiator (3) with the information $y(t) = x([\frac{t}{h}]\cdot h)$ available, where $x(t)$ is governed by the differential equation

$$(\frac{d}{dt})^2 x(t) = -\omega^2 x(t) \quad \omega \in R \tag{16}$$

and $[\frac{t}{h}]$ denotes $\max_{n\leq\frac{t}{h}}\{n;\quad n \in Z\}$, $Z = \{0, \pm1, \pm2, \ldots\}$.

We assume that it is known that ω^2 and $(x(0), \frac{d}{dt}x(t)|_{t=0})$ satisfy the inequalities

$$\begin{gathered} |\ \omega^4 - 1\ | \leq 0.1, \\ (\frac{d}{dt}x(0) - 1)^2 + (x(0) - 1)^2 < 0.1. \end{gathered} \tag{17}$$

Applying the differentiator (3), we can estimate the derivatives

$$\{(\frac{d}{dt})^j x(t)\}_{j=1}^3$$

with an error bounded by $\epsilon > 0$. We will investigate the relation between $\epsilon > 0$ and $h > 0$.

The system (16) has the first integral, i.e.,

$$(\omega x(t))^2 + (\frac{d}{dt}x(t))^2 = (\omega\cdot x(0))^2 + (\frac{d}{dt}x(0))^2.$$

Hence

$$\frac{d}{dt}x(t) \leq \sqrt{\omega(x(0))^2 + (\frac{d}{dt}x(0))^2}$$

and taking into account

$$\max_t \mid \Delta y_t \mid \leq \max_t \mid \frac{d}{dt} x(t) \mid \cdot h$$

we obtain $\Delta Y = 1.77 \cdot h$. All entries of the disturbance $\psi(t)$ equal to zero except the last one satisfying

$$\mid \psi_4 \mid \leq (1 - \omega)^4 \mid x(t) \mid .$$

Therefore making use of the first integral and taking into account (17), we have $\Delta\Sigma = 0.182$. If the eigenvalues of the differentiator are chosen to be $\lambda_j = r \cdot q_j \quad (j = 1, 2, 3, 4)$, $q_1 = \bar{q}_2$, $q_3 = \bar{q}_4$ and $q_1 = -\frac{1}{2} + i\frac{1}{2}$, $q_3 = -\frac{1}{\sqrt{3}} + i\frac{1}{\sqrt{6}}$, then the uncertainty relations from Corollary of Theprem 2 give us the following relation between h and ϵ.

$$h \leq \min\{4.75 \cdot \epsilon, 9.01 \cdot 10^{-4} \cdot \epsilon^{1+\frac{1}{3}}, 3.42 \cdot \epsilon^2, 1.02 \cdot 10^{-3} \cdot \epsilon^4\}.$$

This inequality is a necessary condition for ϵ - differentiability of $x(t)$ measured in discrete moments of time $\{t_0 + jh\}_{j=0}^{\infty}$.

Example 2. Consider the problem of ϵ - observability of the state $(x(t), \frac{d}{dt}x(t))$ of the system

$$(\frac{d}{dt})^2 x(t) = -5x(t) + \Delta\sigma_t,$$

$$y(t) = x(t) + \Delta y_t,$$

where $(\Delta\sigma_t, \Delta y_t) \in \Re(\Delta\Sigma, \Omega) \times \Re(\Delta Y, \Omega)$. We will look for Luenberger's observer having the form

$$\dot{z}_1 = z_2 + \alpha_1(z_1 - y),$$

$$\dot{z}_2 = -5z_1 + \alpha_2(z_1 - y).$$

The parameters $\alpha_1, \quad \alpha_2 \quad \in R$ are the functions of the desired eigenvalues $\{\lambda_i\}_{i=1}^2$ which are supposed to be of the form

$$\lambda_i = rq_i \quad i = 1, 2,$$

$$q_1 = q_2 \ , \ q_1 = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}},$$

where r is a real positive number to be calculated. Following Step 1 of the procedure (i.e., choice of r) we obtain

$$W(r) = (\frac{\Delta\Sigma + 5\Delta Y}{r} + \Delta Y \cdot r) \cdot \frac{1+r}{\sqrt{2}} \cdot (\frac{1}{\sqrt{\frac{r^2}{2} + (\Omega + \frac{r}{\sqrt{2}})^2}} + \frac{1}{\sqrt{\frac{r^2}{2} + (\Omega - \frac{r}{\sqrt{2}})^2}}).$$

If $\Delta\Sigma = 100, \Omega = 200, \Delta Y = 0.1$, then $W(r)$ reaches the minimum at the point $r^* = 7.77$ and $W(r^*) = 0.85$. Thus having taken the eigenvalues $\lambda_1 = r^*(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})$, $\lambda_2 = \bar{\lambda}_1$

we obtain $\epsilon \sim 0.85$. For computer simulations we take $\Delta\sigma_t = \Delta\Sigma \cdot \sin(\Omega t)$, $\Delta y_t = \Delta Y \cdot \cos(\Omega t)$ and using the discrete scheme

$$x(j+1) = 2 \cdot x(j) - x(j-1) - 5 \cdot x(j) \cdot h^2 + \Delta\Sigma \sin(\Omega \cdot jh) \cdot h^2$$

for the modelling of $x(t)$ and the dicsrete scheme

$$y(j) = x(j) + \Delta Y \cdot \cos(\Omega \cdot j \cdot h),$$

$$z_1(j) = z_1(j-1) + h \cdot z_2(j-1) + \alpha_1 \cdot (z_1(j-1) - y(j-1)) \cdot h,$$

$$z_2(j) = z_2(j-1) - 5 \cdot z_1(j-1) \cdot h + \alpha_2(z_1(j-1) - y(j-1)) \cdot h$$

for the differentiator we obtain the results of simulations shown on Fig.1. It remaines to notice that $h = 0.005$ and $j = 1, \ldots, 850$.

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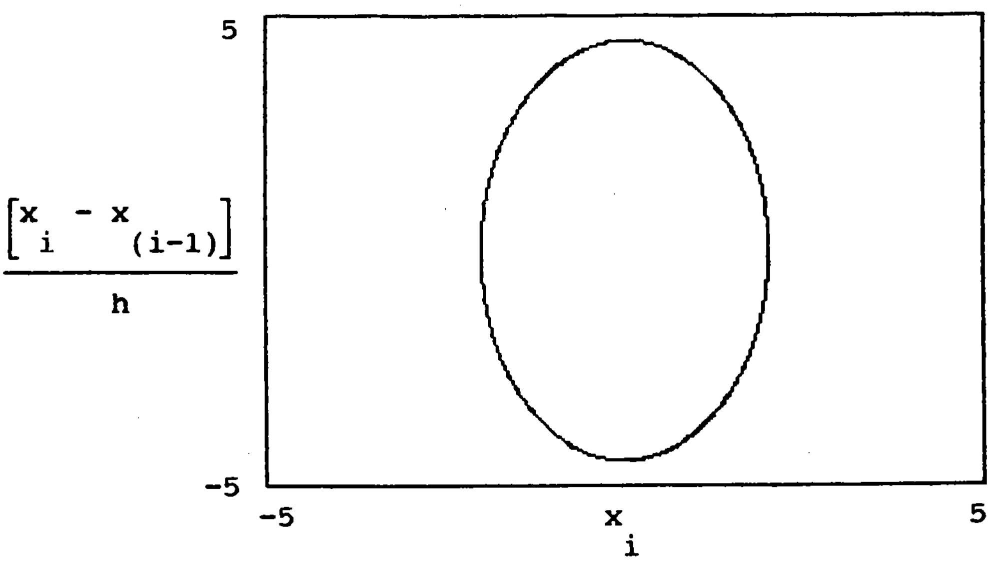

Fig.(1.a). The trajectory of the system without disturbances. The initial conditions x(0)=2, dx/dt=0.

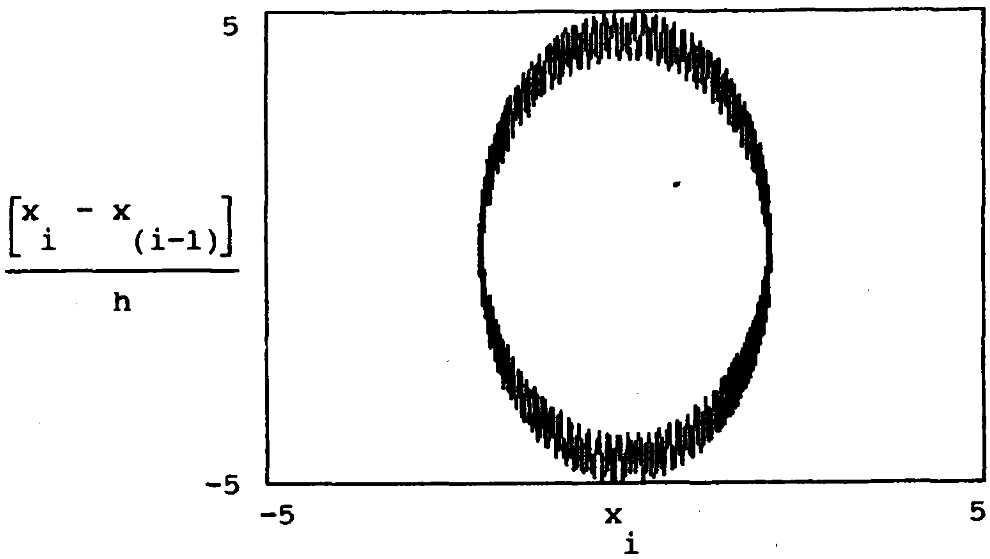

Fig.(1.b). The same trajectory as on Fig.(1.a) but under the disturbance 100 sin(200t) .

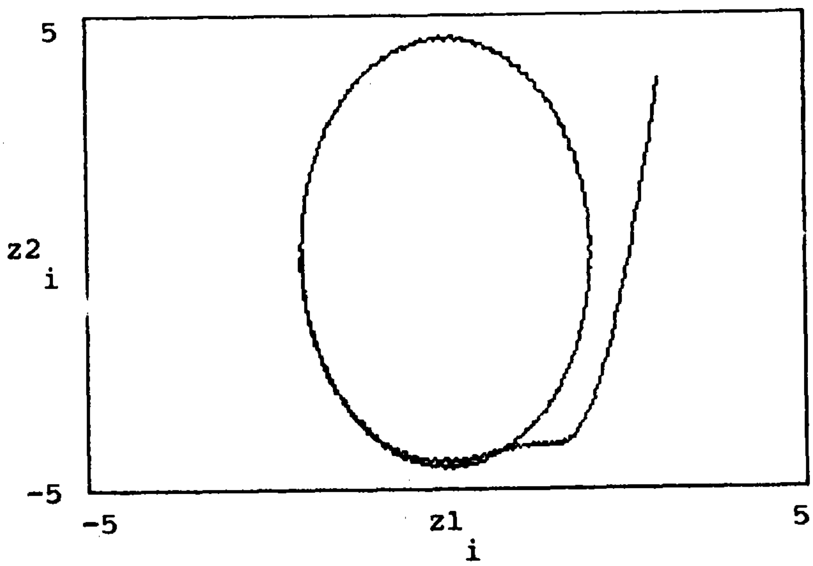

Fig.(1.c). The dynamic of the observation process.