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MULTIPARAMETER, POLYNOMIAL ADAPTIVE TRACKING FOR MINIMUM PHASE SYSTEMS

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Abstract:
A multiparameter, polynomial feedback strategy is introduced to solve the universal adaptive tracking problem for a class of multivariable minimum phase system and reference signals generated by a known linear time-invariant differential equation. For 2-input, 2-output, minimum phase systems (A, B, C) with $\det(CB) > 0$, a different polynomial tracking controller is given which does not invoke a spectrum unmixing set.

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Nomenclature

$\mathbb{C}_+(\mathbb{C}_-)$	open right- (left-) half complex plane
$\sigma(A)$	the spectrum of the matrix $A \in \mathbb{C}^{n\times n}$
$L_p(J)$	vector space of measurable functions $f : J \to \mathbb{R}^n$, $J \subset \mathbb{R}$ some interval, such that $\|f(\cdot)\|_{L_p(J)} < \infty$, where

$$\|f(\cdot)\|_{L_p(J)} := \begin{cases} \left[\int_J \|f(s)\|^p ds\right]^{1/p} & \text{for} \quad p \geq 1 \\ \operatorname{ess\,sup}_{s\in J} \|f(s)\| & \text{for} \quad p = \infty \end{cases}$$

1. Introduction

The problem of adaptively stabilizing a linear, minimum phase plant

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t), \end{aligned}\right\} \tag{1.1}$$

with unknown matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, unknown state dimension n, and $\det(CB) \neq 0$, by smooth controllers of the form

$$\left.\begin{aligned} u(t) &= f(y(t), k(t)), \\ \dot{k}(t) &= g(y(t), k(t)) \end{aligned}\right\} \tag{1.2}$$

and not based on any identification mechanism has been extensively studied since the mid 1980's. It was initiated by the seminal works of Mareels (1984), Mårtensson (1985), Morse (1983), Nussbaum (1983), and Willems and Byrnes (1984).

Morse (1983) conjectured that for scalar systems (1.1) with $CB \neq 0$, there are no differentiable functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ so that (1.2) stabilizes (1.1). Nussbaum (1983) proved that is valid if f, g are restricted to polynomials or rational functions in y and k. More importantly, he presented analytic functions f, g such that (1.2) is a *universal adaptive stabilizer*, i.e. the solution of the closed-loop system (1.1), (1.2) satisfies $\lim_{t\to\infty} x(t) = 0$ and $k(t)$ converges to a finite limit $k_\infty \in \mathbb{R}$. The following concept of *switching functions* was crucial in Nussbaum's approach: A piecewise right continuous function $N(\cdot) : [0, \infty) \to \mathbb{R}$ is called a *Nussbaum function* if

$$\sup_{k>0} \frac{1}{k} \int_0^k N(\tau)\, d\tau = +\infty \quad \text{and} \quad \inf_{k>0} \frac{1}{k} \int_0^k N(\tau)\, d\tau = -\infty. \tag{1.3}$$

Since then, universal adaptive stabilizers have become a major research topic in adaptive control, it has been extended to $n-$th order, multivariable systems and also to design a universal adaptive tracking controller for signals belonging to

$$\mathcal{Y}_{\text{ref}} := \left\{ y_{\text{ref}}(\cdot) \in C^\infty(\mathbb{R}, \mathbb{R}^m) \,\middle|\, \alpha(\tfrac{d}{dt}) y_{\text{ref}}(t) \equiv 0 \right\}, \tag{1.4}$$

where $\alpha(s) \in \mathbb{R}[s]$ is known. (See Ilchmann (1991) for a survey and bibliography.)

However, in almost all contributions the gain adaptation parameter $k(t)$ is one dimensional. In Nikitin and Prätzel-Wolters (1991), it was shown that Nussbaum's result does not hold true if multiple gains $k(t) \in \mathbb{R}^p$ are allowed. For single-input, single-output, minimum phase systems of the form (1.1), a universal adaptive stabilizer (1.2) with time-invariant polynomials $f : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^p$ was designed.

In Section 2 of the present note, we shall extend this result to multivariable minimum phase systems with spectrum of CB either in $\mathbb{C}_+$ or in $\mathbb{C}_-$, but unknown in which complex half plane. The stabilizer presented can also be used in series connection with an internal model to guarantee tracking for the same class of systems and the class of reference signal (1.4). In Section 3, the class of minimum phase, 2-input, 2-output systems is considered with the different assumption $\det(CB) > 0$ on the high-frequency gain. A different universal, polynomial, adaptive tracking controller is given. There is no need for the use of an spectrum unmixing set, cf. Mårtensson (1986). A topic of future research is to extend theses *smooth* strategies to the class of multivariable, minimum phase systems with $\det(CB) \neq 0$.

2. Eigenvalues of CB are either in $\mathbb{C}_-$ or $\mathbb{C}_+$

In this section, we consider the system class

$$\left.\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) \in \mathbb{R}^n \\ y(t) = Cx(t), \\ (A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \quad \text{is minimum phase,} \\ \text{there exists a } \tau \in \{-1,1\} \text{ such that } \quad \sigma(\tau CB) \subset \mathbb{C}_-, \end{array}\right\} \tag{2.1}$$

with state dimension n unknown, but the number of inputs resp. outputs is available to the designer.

We call the system (1.1) *minimum phase* if

$$\det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0 \quad \text{for all} \quad s \in \overline{\mathbb{C}}_+, \tag{2.2}$$

that is, (1.1) is stabilizable, detectable and the transfer function $C(sI_n - A)^{-1}B$ has no roots in the closed right-half plane $\overline{\mathbb{C}}_+$.

The proposed adaptive feedback strategy for stabilizing each system belonging to (2.1) is

$$u(t) = \langle \lambda, k(t) \rangle \cdot y(t) \tag{2.3}$$

$$\dot{k}(t) = \|y(t)\|^2 Rk(t), \qquad k(0) = k_0 \tag{2.4}$$

with $R \in \mathbb{R}^{2l\times 2l}$, $\lambda \in \mathbb{R}^{2l}$ satisfying

$$\left.\begin{array}{l} (R,\lambda) \quad \text{is observable,} \\ \sigma(R) \cap (\overline{\mathbb{C}}_- \cup \mathbb{R}) = \emptyset. \end{array}\right\} \tag{2.5}$$

The intuition for this control strategy is as follows. Although (2.4) is a time-varying differential equation, the solution can be given explicitly by

$$k(t) = e^{R \int_0^t \|y(s)\|^2 ds} \cdot k(0).$$

By (2.5), the unbounded oscillatory behaviour of the solution of $\dot{v} = Rv$ carries over to k(t), and the observability of (R, λ) ensures unbounded oscillations in $< \lambda, k(t) >$ as long as the 'stability indicator function' $\int_0^t \|y(s)\|^2 ds$ is not converging to a finite limit.

Before proving the main result of this section, a technical lemma is needed.

2.1 Lemma
If $(\lambda, R) \in \mathbb{R}^{2l} \times \mathbb{R}^{2l \times 2l}$ satisfies (2.5) and $k_0 \in \mathbb{R}^{2l}$ with $k_0 \neq 0$, then $z(t) := \langle \lambda, e^{Rt} k_0 \rangle$ is a Nussbaum function.

Proof: Since $e^{Rt}k_0$ is the solution of $\dot{v}(t) = Rv(t)$, $v(0) = k_0$, and (2.5) is satisfied, it follows that $z(t)$ is of the form

$$z(t) = \sum_{j=1}^{\mu} e^{\gamma_j t} \left[r_j(t) \cos(\omega_j t) + m_j(t) \sin(\omega_j t) \right]$$

where $\gamma_\mu > \cdots > \gamma_1 > 0$ are the real parts of the eigenvalues of R, $\omega_1, \ldots, \omega_\mu > 0$ are the positive immaginary parts of the eigenvalues of R, and $r_j(\cdot), m_j(\cdot) \in \mathbb{R}[t]$ for $j = 1, \ldots, \mu$. Using integration by parts, it is easily seen that

$$\int_0^t z(s) ds = r_0 + \sum_{j=1}^{\mu} e^{\gamma_j t} \left[\hat{r}_j(t) \cos(\omega_j t) + \hat{m}_j(t) \sin(\omega_j t) \right]$$

for some $r_0 \in \mathbb{R}$, $\hat{r}_j(\cdot), \hat{m}_j(\cdot) \in \mathbb{R}[t]$, $j = 1, \ldots, \mu$.
Since $k_0 \neq 0$, observability of (R, λ) yields $z(\cdot) \not\equiv 0$, and hence $\left(t \mapsto \int_0^t z(s) ds\right) \not\equiv r_0$, whence there exists

$$\zeta := \max_{1 \leq j \leq \mu} \{ \hat{r}_j(\cdot) \cos(\omega_j \cdot) + \hat{m}_j(\cdot) \sin(\omega_j \cdot) \not\equiv 0 \}$$

and

$$\int_0^t z(s) ds = e^{\gamma_\zeta t} \left[\hat{r}_\zeta(t) \cos(\omega_\zeta t) + \hat{m}_\zeta(t) \sin(\omega_\zeta t) + g(t) \right]$$

with

$$g(t) := e^{-\gamma_\zeta t} r_0 + \sum_{j=1}^{\zeta - 1} e^{(\gamma_j - \gamma_\zeta)t} \left[\hat{r}_j(t) \cos(\omega_j t) + \hat{m}_j(t) \sin(\omega_j t) \right].$$

Since $\gamma_j < \gamma_\zeta$ for all $j = 1, \ldots, \zeta - 1$, it follows that $\lim_{t \to \infty} g(t) = 0$.
Assuming $\hat{r}_\zeta(\cdot) \not\equiv 0$ and setting

$$\theta_k = \frac{k\pi}{\omega_\zeta}$$

yields

$$\frac{1}{\theta_k}\int_0^{\theta_k} z(s)ds = e^{\gamma_\zeta \theta_k}\left[(-1)^k \hat{r}_\zeta(\theta_k) + g(\theta_k)\right],$$

which proves that $z(\cdot)$ is a Nussbaum function. If $\hat{r}_\zeta(\cdot) \equiv 0$, then necessarily $\hat{m}_\zeta(\cdot) \not\equiv 0$, and by setting

$$\theta_k = \frac{\pi}{2} + \frac{k\pi}{\omega_\zeta}$$

we conclude, in a similar manner, $z(\cdot)$ is a Nussbaum function. This completes the proof. □

Now we are in a position to prove the main result of this sesction, that is, the adaptive feedback strategy (2.3), (2.4) is a universal adaptive stabilizer for the class (2.1).

2.2 Theorem

If $(\lambda, R) \in \mathbb{R}^{2l} \times \mathbb{R}^{2l\times 2l}$ satisfies (2.5) and (A, B, C) is belonging to the class (2.1), then the feedback strategy (2.3), (2.4) applied to (2.1), for arbitrary initial conditions $x_0 \in \mathbb{R}^n$, $k_0 \in \mathbb{R}^{2l}$, $k_0 \neq 0$, yields a closed-loop system (1.1), (2.3), (2.4) with the properties

(i) there exists a unique solution $(x(\cdot), k(\cdot)) : [0,\infty) \to \mathbb{R}^{n+2l}$,

(ii) $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}^{2l}$ exists,

(iii) $\lim_{t\to\infty} x(t) = 0$,

(iv) $k(\cdot) \in L_\infty(0,\infty)$, $x(\cdot) \in L_2(0,\infty) \cap L_\infty(0,\infty)$.

Proof: Since the right hand side of the closed-loop system (1.1), (2.3), (2.4) is locally Lipschitz in (x, k), there exists a maximal interval of existence $[0, t')$ for the solution, for some $t' > 0$.

The state space transformation $[y^T, z^T]^T = S^{-1}x$, where $S := [B(CB)^{-1}, N] \in GL_n(\mathbb{R})$ and $N \in \mathbb{R}^{n\times(n-m)}$ denotes a basis matrix of ker$C$, converts (1.1) into

$$\left.\begin{array}{rcll} \dot{y}(t) &=& A_1 y(t) + A_2 z(t) + CBu(t), & \\ \dot{z}(t) &=& A_3 y(t) + A_4 z(t), & (y(0)^T, z(0)^T) = S^{-1}x_0 \end{array}\right\} \tag{2.6}$$

with $A_1 \in \mathbb{R}^{m\times m}$, $A_2 \in \mathbb{R}^{m\times(n-m)}$, $A_3 \in \mathbb{R}^{(n-m)\times m}$, $A_4 \in \mathbb{R}^{(n-m)\times(n-m)}$. The minimum phase property yields $\sigma(A_4) \subset \mathbb{C}_-$.

We prove $y(\cdot) \in L_2(0, t')$. Define the positive-definite (Lyapunov-like) function

$$V(y, z) := \frac{1}{2}\langle y, Py\rangle + \frac{1}{2}\langle y, Qy\rangle,$$

where $P = P^T \in \mathbb{R}^{m\times m}$ resp. $Q = Q^T \in \mathbb{R}^{(n-m)\times(n-m)}$ are the positive-definite solutions of

$$PCB + (CB)^T P = 2\alpha I_m, \quad \text{resp.} \quad QA_4 + A_4^T Q = -2I_{n-m} \tag{2.7}$$

for some $\alpha \in \{-1,+1\}$. Differentiation of $V(y(s),z(s))$, for $s \in [0,t')$, and using (2.7), yields

$$\begin{aligned}\frac{d}{ds}V(y(s),z(s)) &= \langle y(s), P\dot{y}(s))\rangle + \langle z(s), Q\dot{z}(s)\rangle \\ &= \langle y(s), P[A_1y(s)+A_2z(s)]\rangle + \langle y(s), PCBu(s)\rangle \\ &\quad +2\langle z(s), Q[A_3y(s)+A_4z(s)]\rangle \\ &\leq \|PA_1\|\cdot\|y(s)\|^2 + [\|PA_2\|+\|QA_3\|]\|y(s)\|\|z(s)\| \\ &\quad +\alpha\langle\lambda, k(s)\rangle\|y(s)\|^2 - \|z(s)\|^2.\end{aligned}$$

Therefore, for $M := \|PA_1\| + \frac{1}{2}[\|PA_2\|+\|QA_3\|]^2$, we have

$$\frac{d}{ds}V(y(s),z(s)) \leq (M+\alpha\langle\lambda,k(s)\rangle)\|y(s)\|^2 - \frac{1}{2}\|z(s)\|^2$$

and integration over $[0,t)$ yields

$$V(y(t),z(t)) \leq V(y(0),z(0)) + \int_0^t (M+\alpha\langle\lambda,k(s)\rangle)\|y(s)\|^2 ds.$$

In view of $k(t) = e^{\theta(t)\cdot R}k(0)$ for $\theta(t) := \int_0^t \|y(s)\|^2 ds$, we obtain

$$\begin{aligned}V(y(t),z(t)) &\leq V(y(0),z(0)) + \int_0^t (M+\alpha\langle\lambda, e^{\theta(s)R}k(0)\rangle)\dot{\theta}(s)ds \\ &= V(y(0),z(0)) + \int_{\theta(0)}^{\theta(t)} (M+\alpha\langle\lambda, e^{\mu R}k(0)\rangle)d\mu.\end{aligned}$$

Suppose $y(\cdot) \notin L_2(0,t')$, then for t such that $\theta(t) > \theta(0)$ we obtain

$$V(y(t),z(t)) \leq V(y(0),z(0)) + [\theta(t)-\theta(0)]\left[M + \frac{\alpha}{\theta(t)-\theta(0)}\int_{\theta(0)}^{\theta(t)} \langle\lambda, e^{\mu R}k(0)\rangle d\mu\right]. \quad (2.8)$$

Since, by Lemma 2.1, $e^{\mu R}k(0)$ is a Nussbaum function, the right hand side of (2.8) takes negative values, thus contradicting the positiveness of $V(y(t),z(t))$, and hence $\theta(\cdot) \in L_\infty(0,t')$ which is equivalent to $y(\cdot) \in L_2(0,t')$.

Boundedness of $\theta(\cdot)$ yields boundedness of $k(\cdot)$. Going back to the closed-loop system (1.1), (2.3), (2.4), classical results of the theory of ordinary differential equations yield $t' = \infty$. This proves assertions (i) and (ii).
Since $y(\cdot) \in L_2(0,\infty)$ and $\sigma(A_4) \subset \mathbb{C}_-$, it follows from the second equation in (2.6) that $z(\cdot) \in L_2(0,\infty)$. Therefore, (iv) holds true.
Now boundedness of $k(\cdot)$ implies, using again (2.6), that $\dot{y}(\cdot), \dot{z}(\cdot) \in L_2(0,\infty)$. This finally

yields assertion (iii) of the theorem and the proof is complete. □

Theorem 2.2 can be used to design a universal adaptive tracking controller by connecting the universal adaptive stabilizer in series with an internal model $\frac{\beta(\cdot)}{\alpha(\cdot)} I_m$ which reduplicates the reference dynamics. This idea has been introduced for single-input, single-output systems by Mareels (1984) and Helmke et al. (1990). For multivariable systems it was independently extended by Townley and Owens (1991), Miller and Davison (1991), and Logemann and Ilchmann (1991). The internal model is constructed as follows. Let $\beta(\cdot) \in \mathbb{R}[s]$ be Hurwitz with $\deg \beta(\cdot) = \deg \alpha(\cdot)$, $\alpha(\cdot) \in \mathbb{R}[s]$ with zeros only in $\overline{\mathbb{C}}_+$ and determining (1.4). Let $(\hat{A}, \hat{B}, \hat{C}, d^*) \in \mathbb{R}^{m \times m} \times \mathbb{R}^m \times \mathbb{R}^{m \times 1} \times \mathbb{R}$ be a minimal state space realization of $\frac{\beta(s)}{\alpha(s)}$, $d^* := \lim_{s \to \infty} \frac{\beta(s)}{\alpha(s)}$. Thus, a minimal state space realization of $\frac{\beta(s)}{\alpha(s)} I_m$ is given by

$$\left.\begin{aligned} \dot{\xi}(t) &= \hat{A}^* \xi(t) + \hat{B}^* v(t), \quad \xi(0) = \xi_0 \\ u(t) &= \hat{C}^* \xi(t) + d^* I_m v(t) \end{aligned}\right\} \qquad (2.9)$$

with

$$\hat{A}^* = diag\{\hat{A}, \ldots, \hat{A}\} \in \mathbb{R}^{mp \times mp}, \quad \hat{B}^* = diag\{\hat{B}, \ldots, \hat{B}\} \in \mathbb{R}^{mp \times m}, \\ \hat{C}^* = diag\{\hat{C}, \ldots, \hat{C}\} \in \mathbb{R}^{m \times mp},$$

and a straightforward calculation yields that a stabilizable and detectable state space realization of $\bar{G}(s) = \frac{\beta(s)}{\alpha(s)} I_m G(s)$ is

$$\left.\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}v(t), \quad \bar{x}(0) = (x_0^T, \xi_0^T)^T \\ y(t) &= \bar{C}\bar{x}(t) \end{aligned}\right\} \qquad (2.10)$$

with

$$\bar{A} = \begin{bmatrix} A & B\hat{C}^* \\ 0 & \hat{A}^* \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} d^* B \\ \hat{B}^* \end{bmatrix}, \quad \bar{C} = [C, 0], \quad \bar{x} = (x^T, \xi^T)^T.$$

The main ingredient for designing a tracking controller is the following lemma, for a proof see Miller and Davison (1991).

2.3 Lemma

If $y_{ref}(\cdot) \in \mathcal{Y}_{ref}$, see (1.4), $\alpha(\cdot) \in \mathbb{R}[s]$ with zeros only in $\overline{\mathbb{C}}_+$, (1.1) is minimum phase, and $\bar{A}$, $\bar{C}$ are given as in (2.10), then there exists a $\tilde{x}_0 \in \mathbb{R}^{n+mp}$ and $M > 0$ such that

$$\left.\begin{aligned} \dot{\tilde{x}}(t) &= \bar{A}\tilde{x}(t), \quad \tilde{x}(0) = \tilde{x}_0 \\ y_{ref}(t) &= \bar{C}\tilde{x}(t) \end{aligned}\right\} \qquad (2.11)$$

and

$$\|\tilde{x}(t)\| \leq M \left[1 + \max_{s \in [0,t]} \|y_{ref}(s)\|\right] \quad \text{for all} \quad t \geq 0. \qquad (2.12)$$

2.4 Theorem

The internal model (2.9) and the feedback strategy

$$v(t) = \langle \lambda, k(t) \rangle [y(t) - y_{\text{ref}}(t)] \quad (2.13)$$

$$\dot{k}(t) = \|y(t) - y_{\text{ref}}(t)\|^2 Rk(t), \qquad k(0) = k_0 \quad (2.14)$$

applied to any system (A, B, C) belonging to (2.1), for arbitrary initial conditions $x_0 \in \mathbb{R}^n$, $k_0 \in \mathbb{R}^{2l}$, $k_0 \neq 0$, yields a closed-loop system (1.1), (2.9), (2.13), (2.14) with the properties

(i) there exists a unique solution $(x(\cdot),, \xi(\cdot), k(\cdot)) : [0, \infty) \to \mathbb{R}^{n+mp+2l}$,

(ii) $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}^{2l}$ exists,

(iii) $\|x(t)\| + \|\xi(t)\| \leq M[1 + \|y_{\text{ref}}(t)\|]$ for all $t \geq 0$ and some $M > 0$,

(iv) $\lim_{t\to\infty} \|y(t) - y_{\text{ref}}(t)\| = 0$.

Proof: We only sketch the proof. Using Lemma 2.3, the universal adaptive tracking problem can be converted into a stabilization problem for

$$\left.\begin{array}{rcl} \dot{x}_e(t) &=& \bar{A}x_e(t) + \bar{B}v(t) \\ y(t) - y_{\text{ref}}(t)) &=& \bar{C}x_e(t), \end{array}\right\} \quad (2.15)$$

where $x_e(t) := \bar{x}(t) - \tilde{x}(t)$. Since $\bar{C}\bar{B} = d^*CB$, Theorem 2.2 can be applied to (2.15) and (i), (ii), (iv) follow. (iii) is also a simple consequence of Lemma 2.3. □

2.5 Remark

(i) If

$$sgn(CB) := \begin{cases} +1 & , \quad \text{if} \quad \sigma(CB) \subset \mathbb{C}_+ \\ -1 & , \quad \text{if} \quad \sigma(CB) \subset \mathbb{C}_- \end{cases}$$

is known, then the transient behaviour can be improved by setting

$$u(t) = -sgn(CB) \cdot |\langle \lambda, k(t) \rangle| \cdot y(t).$$

This avoids unnecessary switching.

(ii) The adaptive controller is capable of tolerating certain state and input disturbances. Let

$$h : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^m, \quad g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$$

be continuous functions of $t \in \mathbb{R}_+$, locally Lipschitz with respect to $x \in \mathbb{R}^n$ and $\|h(t,x)\| \leq \hat{h} \cdot \|x\|$ and $\|g(t,x)\| \leq \hat{g} \cdot \|x\|$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, where $\hat{h} > 0$ and $\hat{g} > 0$ are some real constants. Then it can be shown that the results of Theorem 2.2 and 2.4 remain valid if the control strategy is applied to

$$\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + g(t, x(t)) + B[u(t) + h(t, x(t))], \quad x(0) \in \mathbb{R}^n \\ y(t) &=& Cx(t), \end{array}$$

provided $\hat{g}$ depending on (A, B, C) is small enough.

3. Smooth adaptive tracking for two-input, two-output systems

In this section, we consider the system class

$$\left.\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) \in \mathbb{R}^n \\ y(t) = Cx(t), \\ (A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 2} \times \mathbb{R}^{2\times m} \quad \text{is minimum phase,} \\ \det(CB) > 0. \end{array}\right\} \tag{3.1}$$

This class deserves our attention because multiparameter, polynomial adaptive stabilizers can be designed for it in a natural way, which is different from what was proposed in Section 2. Indeed, for the class (3.1) we can propose the following adaptive feedback controller:

$$\left.\begin{array}{rcl} u(t) & = & K(t)y(t), \\ \dot{K}(t) & = & \|y(t)\|^2 K(t)R, \quad K(0) = K_0 \in \mathbb{R}^{2\times 2} \end{array}\right\} \tag{3.2}$$

with

$$\left.\begin{array}{l} R \in \mathbb{R}^{2l\times 2l} \quad \text{satisfying} \quad \sigma(R) \cap (\overline{\mathbb{C}}_- \cup \mathbb{R}) = \emptyset, \\ \det(K_0) > 0. \end{array}\right\} \tag{3.3}$$

If $\det(CB) < 0$ instead of the last condition in (3.1), then the results of this section are also valid if $\det(K_0) < 0$ is required in (3.3).

Note that the control strategy (3.2) is polynomial and therefore smooth, as opposed to the known piecewise continuous approach using a spectrum unmixing set. See Byrnes and Willems (1984), Mårtensson (1987, 1991), and Ilchmann and Logemann (1992).

Before we state the main theorem, a technical lemma is needed.

3.1 Lemma
If $\alpha, \omega, \varphi \in \mathbb{R}$, $\alpha > 0, \omega \neq 0$, then the function

$$\nu : \mathbb{R} \to \mathbb{R}, \quad t \mapsto e^{\alpha t}\cos(\omega t + \varphi)$$

is a Nussbaum function.

Proof: The statement follows from Lemma 2.1 since

$$e^{\alpha t}\cos(\omega t + \varphi) = z(t) = \langle \lambda, e^{Rt}k_0 \rangle,$$

where

$$\begin{array}{rcl} R & = & \begin{bmatrix} 0 & 1 \\ -\omega^2 - \alpha^2, & 2\alpha \end{bmatrix}, \\ \lambda & = & (1,0)^T \qquad , k_0 = (\cos\varphi, \alpha\cos\varphi - \omega\sin\varphi)^T. \end{array}$$

This completes the proof.
Note, that also a simple direct proof goes as follows: Define

$$\theta_{2k} := \frac{(2k+\frac{1}{2})\pi}{|\omega|} - \frac{\varphi}{\omega}, \qquad \theta_{2k+1} := \frac{2(k+1)+\frac{1}{2}}{|\omega|}\pi - \frac{\varphi}{\omega}, \quad \text{for} \quad k \in \mathbb{N}.$$

It is easy to see that

$$\frac{1}{\theta_{2k+1}} \int_0^{\theta_{2k+1}} \nu(t)dt \leq \frac{1}{\theta_{2k+1}} \left[\theta_{2k} + \frac{2}{|\omega|} e^{\alpha\theta_{2k}}\right] = \frac{\theta_{2k}}{\theta_{2k} + \frac{3\pi}{2\omega}} + \frac{2}{|\omega|} \frac{e^{\alpha\theta_{2k}}}{\theta_{2k} + \frac{3\pi}{2\omega}},$$

and hence, using a similar argument for θ_{2k},

$$\lim_{k\to\infty} \frac{1}{\theta_{2k+1}} \int_0^{\theta_{2k+1}} \nu(t)dt = +\infty, \quad \lim_{k\to\infty} \frac{1}{\theta_{2k}} \int_0^{\theta_{2k}} \nu(t)dt = -\infty.$$

□

3.2 Theorem

The adaptive controller (3.2) is a universal adaptive stabilizer for the class (3.1), i.e. for every (A, B, C) belonging to (3.1) and every initial condition $x(0) = x_0$, the closed-loop system (1.1), (3.2) satisfies

(i) there exists a unique solution $(x(\cdot), K(\cdot)) : [0,\infty) \to \mathbb{R}^n \times \mathbb{R}^{2\times 2}$,

(ii) $\lim_{t\to\infty} K(t) = K_\infty \in \mathbb{R}^{2\times 2}$ exists,

(iii) $\lim_{t\to\infty} x(t) = 0$,

(iv) $k(\cdot) \in L_\infty(0,\infty), \quad x(\cdot) \in L_2(0,\infty) \cap L_\infty(0,\infty)$.

Proof: Since the right hand side of the closed-loop system is locally Lipschitz in (x, k), there exists a maximal interval of existence $[0, t')$ for the solution, for some $t' > 0$.
Similar to the proof of Theorem 2.2, (1.1) is converted into the form (2.6) with $A_1 \in \mathbb{R}^{2\times 2}$, $A_2 \in \mathbb{R}^{2\times(n-2)}$, $A_3 \in \mathbb{R}^{(n-2)\times 2}$, $A_4 \in \mathbb{R}^{(n-2)\times(n-2)}$, and $\sigma(A_4) \subset \mathbb{C}_-$.
We prove $y(\cdot) \in L_2(0, t')$. Define the positive-definite (Lyapunov like) function

$$V(y,z) := \frac{1}{2}\langle y, Wy\rangle + \frac{1}{2}\langle y, Qy\rangle,$$

where

$$QA_4 + A_4^T Q = -2I_{n-2} \quad \text{and} \quad W = \left[\sqrt{(CBK(0))(CBK(0))^T}\right]^{-1}.$$

Differentiation of $V(y(s), z(s))$, for $s \in [0, t')$, along the solution of the closed-loop system yields, in a similar manner as in the proof of Theorem 2.2,

$$\frac{d}{ds}V(y(s), z(s)) \leq M\|y(s)\|^2 - \frac{1}{2}\|z(s)\|^2 + \langle y(s), WCBu(s)\rangle$$

for $M := \|WA_1\| + \frac{1}{2}[\|WA_2\| + \|QA_3\|]^2$. Since

$$K(t) = K(0)e^{\theta(t)R} \quad \text{for} \quad \theta(t) := \int_0^t \|y(\tau)\|^2 d\tau,$$

we obtain, for $L := WCBK(0)$,

$$\langle y(t), WCBu(t)\rangle = \langle y(t), Le^{\theta(t)R}y(t)\rangle = \|y(t)\| \cdot \|Le^{\theta(t)R}y(t)\| \cdot \cos\zeta(t),$$

where $\zeta(t)$ is the angle between $y(t)$ and $L^{\theta(t)R}y(t)$. Since L is orthogonal with $\det(L) = 1$, L is a rotation through some angle $\varphi > 0$. Since $\sigma(R) = \{\alpha \pm i\omega\}$ for some $\alpha > 0$, $\omega \neq 0$, there exists a unitary matrix $S \in \mathbb{C}^{2\times 2}$ such that

$$\bar{S}^T e^{\theta(t)R}\bar{S} = e^{\alpha\theta(t)} exp\left\{\theta(t)\begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix}\right\},$$

and hence $e^{\theta(t)R}$ is a combination of a rotation through the angle $\theta(t)\omega$ and a lenghten $e^{\alpha\theta(t)}$. This yields

$$\langle y(t), WCBu(t)\rangle = e^{\alpha\theta(t)}\|y(t)\|^2\cos(\theta(t)\omega + \varphi),$$

and so

$$\frac{d}{ds}V(y(s), z(s)) \leq \left[M + e^{\alpha\theta(t)}\cos(\theta(t)\omega + \varphi)\right]\dot{\theta}(t),$$

whence integration over $[0, t')$ gives

$$V(y(t), z(t)) \leq V(y(0), z(0)) + \int_{\theta(0)}^{\theta(t)} [M + e^{\alpha\eta}\cos(\eta\omega + \varphi)]\, d\eta.$$

Suppose $y(\cdot) \notin L_2(0, t')$, then $\lim_{t\to t'}\theta(t) = \infty$ and, for $\theta(t) > \theta(0) = 0$, we obtain

$$V(y(t), z(t)) \leq V(y(0), z(0)) + \theta(t)\left[M + \frac{1}{\theta(t)}\int_{\theta(0)}^{\theta(t)} e^{\alpha\eta}\cos(\eta\omega + \varphi)d\eta\right].$$

Since $t \mapsto e^{\alpha t}\cos(t\omega + \varphi)$ is a Nussbaum function, see Lemma 3.1, the right hand side of the above inequality takes negative values, contradicting the positiveness of the left hand side, and therefore $y(\cdot) \in L_2(0, t')$.
The remainder of the proof follows in a similar manner as the proof of Theorem 2.2, it is

omitted. This completes the proof. □

3.3 Remark

If the Hurwitz polynomial $\beta(\cdot)$ for the internal model is chosen so that $\hat{d} > 0$, see (2.9), then the controller (3.2) in series with the internal model (2.9) is a universal adaptive tracking controller for the class (3.1) and the class of reference signals (1.4). The proof is analogous to Theorem 2.4.

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