Large Insurance Portfolios

Simulation, Strategy Decisions, Asset-Liability Management

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To my parents

Contents

Notation

List of abbreviations

Abbreviation	Description
AC	Average case
ALM	Asset-liability management
AMAE	Average mean absolute error
AMAPE	Average mean absolute percentage error
AMSE	Average mean squared error
AMSPE	Average mean squared percentage error
BaFin	Bundesanstalt für Finanzdienstleistungsaufsicht
	(German Federal Financial Supervisory Authority)
FS	Financing strategy
BdV	Bund der Versicherten (German Association of the Insured)
CF	Coverage fund
CM	Constant mix
CPPI	Constant proportion portfolio insurance
DAX	Deutscher Aktienindex (German stock index)
DCF	Dynamic coverage fund
EIOPA	European Insurance and Occupational Pension Authority
iid	Independent and identically distributed
IS	Investment strategy
MAE	Mean absolute error
MAPE	Mean absolute percentage error
MSE	Mean squared error
MSPE	Mean squared percentage error
NBS	New business scenario
pp	Percentage points
RCF	Relaxed coverage fund
RMSE	Root mean squared error
s	Seconds
SCF	Strict coverage fund
SCR	Solvency capital requirement
VAG	Versicherungsaufsichtsgesetz (Insurance Supervision Law)
VVG	Versicherungsvertragsgesetz (Insurance Contract Law)

List of symbols

Symbol	Description
$\mathcal{B}(p)$	Bernoulli distribution with success probability p
$\mathrm{Beta}(\alpha, \beta)$	Beta distribution with shape parameters α and β
$\mathrm{Exp}(\lambda)$	Exponential distribution with parameter λ
$\mathrm{Beta}(\alpha, \beta, a, b)$	4-parameter beta distribution with shape parameters α and β and support on $[a, b]$
$\lfloor\cdot\rfloor$	Lower Gaussian bracket, i.e. $\lfloor x\rfloor = \max\{z \in \mathbb{Z} \mid z \leq x\}$, $x \in \mathbb{R}$
$\lVert\cdot\rVert_\infty$	Maximum norm, i.e. $\lVert x\rVert_\infty = \max_i \lvert x_i\rvert$, $x \in \mathbb{R}^n$
$\mathcal{N}\left(\mu, \sigma^2\right)$	Normal distribution with expectation μ and variance σ^2
$\lVert\cdot\rVert_p$	p-norm, i.e. $\lVert x\rVert_p = \left(\sum_{i=1}^n \lvert x_i\rvert^p\right)^{\frac{1}{p}}$, $x \in \mathbb{R}^n$, $p \in \mathbb{R}_{\geq 1}$
$\mathrm{Poi}(\lambda)$	Poisson distribution with parameter λ
$(\cdot)^+$	Positive part, i.e. $(x)^+ = \max\{x, 0\}$, $x \in \mathbb{R}$
$\mathbb{R}^+$	Positive real numbers, i.e. $\mathbb{R}^+ = (0, \infty)$
$\mathcal{P}^{b_1,b_2} \succ \mathcal{P}^{c_1,c_2}$	Simulation method $\mathcal{P}^{b_1,b_2}$ performs strictly better than $\mathcal{P}^{c_1,c_2}$ regarding the considered criteria
$\mathcal{P}^{b_1,b_2} \succsim \mathcal{P}^{c_1,c_2}$	Simulation method $\mathcal{P}^{b_1,b_2}$ performs strictly better than $\mathcal{P}^{c_1,c_2}$ regarding most of the considered criteria
$\mathcal{P}^{b_1,b_2} \sim \mathcal{P}^{c_1,c_2}$	Simulation method $\mathcal{P}^{b_1,b_2}$ performs as well as $\mathcal{P}^{c_1,c_2}$ regarding the considered criteria
$\mathcal{TN}\left(\mu, \sigma^2, a, b\right)$	Truncated normal distribution with parameters μ and σ^2 and support on $[a, b]$
$\mathcal{U}(a, b)$	Uniform distribution on $[a, b]$.
$\lceil\cdot\rceil$	Upper Gaussian bracket, i.e. $\lceil x\rceil = \min\{z \in \mathbb{Z} \mid z \geq x\}$, $x \in \mathbb{R}$

Abstract

This thesis deals with the simulation of large insurance portfolios. On the one hand, we need to model the contracts' development and the insured collective's structure and dynamics. On the other hand, an important task is the forward projection of the given balance sheet. Questions that are interesting in this context, such as the question of the default probability up to a certain time or the question of whether interest rate promises can be kept in the long term, cannot be answered analytically without strong simplifications. Reasons for this are high dependencies between the insurer's assets and liabilities, interactions between existing and new contracts due to claims on a collective reserve, potential policy features such as a guaranteed interest rate, and individual surrender options of the insured. As a consequence, we need numerical calculations, and especially the volatile financial markets require stochastic simulations. Despite the fact that advances in technology with increasing computing capacities allow for faster computations, a contract-specific simulation of all policies is often an impossible task. This is due to the size and heterogeneity of insurance portfolios, long time horizons, and the number of necessary Monte Carlo simulations. Instead, suitable approximation techniques are required.

In this thesis, we therefore develop compression methods, where the insured collective is grouped into cohorts based on selected contract-related criteria and then only an enormously reduced number of representative contracts needs to be simulated. We also show how to efficiently integrate new contracts into the existing insurance portfolio. Our grouping schemes are flexible, can be applied to any insurance portfolio, and maintain the existing structure of the insured collective. Furthermore, we investigate the efficiency of the compression methods and their quality in approximating the real life insurance portfolio.

For the simulation of the insurance business, we introduce a stochastic asset-liability management (ALM) model. Starting with an initial insurance portfolio, our aim is the forward projection of a given balance sheet structure. We investigate conditions for a long-term stability or stationarity corresponding to the idea of a solid and healthy insurance company. Furthermore, a main result is the proof that our model satisfies the fundamental balance sheet equation at the end of every period, which is in line with the principle of double-entry bookkeeping. We analyze several strategies for investing in the capital market and for financing the due obligations. Motivated by observed weaknesses, we develop new, more sophisticated strategies. In extensive simulation studies, we illustrate the short- and long-term behavior of our ALM model and show impacts of different business forms, the predicted new business, and possible capital market crashes on the profitability and stability of a life insurer.

Zusammenfassung

In dieser Dissertation beschäftigen wir uns mit der Simulation von großen Lebensversicherungsbeständen. Hierzu müssen zum einen die betrachteten Verträge, die Struktur des Bestandes sowie Zu- und Abgänge modelliert werden. Zum anderen besteht eine wichtige Aufgabe darin, die gegebene Versicherungsbilanz zeitlich fortzuschreiben. In diesem Zusammenhang interessante Fragestellungen, wie die Frage nach der Höhe der Insolvenzwahrscheinlichkeit bis zu einem bestimmten Zeitpunkt oder die Frage, ob Zinsversprechen langfristig eingehalten werden können, lassen sich ohne grobe Vereinfachungen analytisch nicht beantworten. Gründe hierfür sind starke Abhängigkeiten zwischen Verbindlichkeiten und Vermögenspositionen des Versicherers, Interdependenzen zwischen Alt- und Neubestand über Ansprüche auf eine gemeinsame Reserve, bestimmte Vertragscharakteristiken wie eine garantierte Mindestverzinsung und individuelle Kündigungsrechte seitens der Versicherungsnehmer. Aus diesem Grund sind numerische Berechnungen und angesichts schwankender Finanzmärkte stochastische Simulationen unabdingbar. Eine vertragsgenaue Simulation mit akzeptabler Laufzeit ist jedoch aufgrund der Größe von Versicherungsbeständen, den langen Betrachtungszeiträumen von teilweise mehreren Jahrzehnten und der Anzahl der notwendigen (Monte-Carlo-) Simulationen selbst bei fortschreitender Technik mit wachsenden Rechenkapazitäten und Geschwindigkeiten moderner Computer eine oft unmögliche Herausforderung. Stattdessen benötigt man approximierende Simulationsmethoden.

In dieser Arbeit entwickeln wir daher ein Verfahren zur Bestandsverdichtung, bei dem aufgrund gewählter Gruppierungsmerkmale das Versicherungskollektiv in Kohorten eingeteilt und dann lediglich eine enorm verringerte Anzahl repräsentativer Verträge simuliert werden muss. Zudem zeigen wir, wie neue Verträge in bestehende Kohorten integriert werden können, wodurch Effizienz auch bei laufendem Neugeschäft gewährleistet ist. Das Verfahren ist sehr flexibel, lässt sich auf jeden Versicherungsbestand anwenden und behält die gegebene Struktur des Bestandes bei. Ferner betrachten wir verschiedene Möglichkeiten zur Erzeugung der Kohorten und überprüfen die Approximationsgenauigkeit anhand geeigneter Gütekriterien.

Für die Simulation des gesamten Versicherungsgeschäfts führen wir ein stochastisches Bilanzstrukturmodell ein. Ziel ist die Projektion der gegebenen Bilanzstruktur über den gesamten Betrachtungszeitraum. Wir untersuchen Kriterien für eine langfristige Stabilität, entsprechend der Vorstellung eines soliden und gesunden Versicherungsunternehmens. Ferner ist eines der Hauptresultate der Beweis, dass unser Modell die fundamentale Bilanzgleichung am Ende jeder Periode erfüllt. Dies steht im Einklang mit dem Prinzip der doppelten Buchführung. Wir analysieren mehrere Strategien zur Kapitalanlage und zur Finanzierung fälliger Verbindlichkeiten und nehmen beobachtete Schwächen als Anlass, neue Strategien zu entwickeln. Eine Vielzahl an Simulationsstudien illustriert das

kurz- und langfristige Verhalten unseres Bilanzstrukturmodells und zeigt Auswirkungen von verschiedenen Geschäftsmodellen, des prognostizierten Neugeschäfts und möglicher Börsencrashs auf die Rentabilität und Stabilität eines Lebensversicherers.

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1. Introduction

Insurance companies take risks on behalf of their policyholders in exchange for premium payments. Since they provide consumers and businesses with protection against negative events, insurers make a substantial contribution to economic growth, see Gründl et al. [31]. Furthermore, as financial intermediaries and large holders of government and corporate bonds, they play an important role for the stability of the whole financial system. Indeed, if insurance companies face large liquidity risks and are threatened by insolvency, they might be forced to sell assets to a large extent. This could possibly imply a decline in the concerned asset prices. In addition, they could withdraw their capital from banks and other financial institutions and thus transmit the stress to other parts of the financial system, see Deutsche Bundesbank [20].

A necessary principle for insurance to work is the pooling and sharing of risks faced by individual agents in a large collective. The building of such a large insurance portfolio requires that potential policyholders have confidence in the insurer. More specifically, the insured person must be able to rely on the insurance company meeting its due obligations at any time. In this context, life insurers play a special role since a life insurance contract may financially protect entire livelihoods, especially if there is only one family breadwinner. Moreover, as long-term agreements with contract periods of several decades in some cases, life insurance policies contrast with other types of insurance contracts and impose additional requirements on the company's financial strength such that interest rate promises can be kept in the long term.

In recent years, however, several life insurers decided to sell parts of their insurance portfolios to run-off companies, i.e. to firms that are specialized in the processing of existing contracts without issuing new policies.[1] In such cases, the contracts remain in force and the corresponding regulating authorities ensure that contractually guaranteed benefits are met. Nevertheless, the sales attracted a lot of attention and were often viewed critically in public reporting and by consumer protectors.[2] Reasons for these measures, which have certainly led to a loss of confidence among some policyholders, may have been the ongoing period of low, and in some cases even negative, interest rates on the one side and simultaneous high obligations from existing contracts on the other side. This combination makes it difficult to obtain sufficient returns on the managed funds, see Kok et al. [42], and is one of the challenges today's life insurers face. Against this background, a successful asset-liability management (ALM) seems to become more important. The application

[1]One of the largest sales was made by the Generali Group in 2019. Over 3.8 million life insurance policies were sold to the run-off company Viridium Holding, see the news release by Viridium Holding, 30 April 2019. URL: www.viridium-gruppe.com.

[2]See, e.g., the news release by the German Association of the Insured (Bund der Versicherten, BdV), 28 September 2017. URL: www.bundderversicherten.de.

of stochastic simulations can support managerial decisions by illustrating the long-term effects of potential measures.

Essentially, ALM can be seen as the goal-driven coordination of assets and liabilities of a balance sheet, see Wagner [55]. For life insurers this means that the investments in the capital market need to be reconciled with the obligations induced by the insurance products such that claims can be met when they are due. An ALM model typically combines several sub-models describing both the evolution of assets and liabilities and different external environments affecting the insurance business (e.g. the dynamics of the capital market, the interest rate environment, and the policyholders' behavior).

A lot of work has been already done within the wide-spread field of asset-liability management. In particular, stochastic ALM-modeling has become quite popular, see e.g. Bauer et al. [3], Grosen and Jørgensen [29], Hieber et al. [34], and Zaglauer and Bauer [57] and the references therein. Further ALM models are introduced in, e.g., Bohnert and Gatzert [9], Bohnert et al. [10], Burkhart et al. [13, 14], Fernández et al. [25], Gerstner et al. [27], Kling et al. [40, 41], and Kok et al. [42]. A more extensive overview of ALM models in the existing literature is provided in Section 3.1.2 below, where we also discuss some of the assumptions made.

In this thesis, we develop a general ALM model for the efficient simulation of large insurance portfolios and the preparation of the balance sheet. In the Master's thesis Diehl [22], we have already introduced a basic ALM model with a similar structure. However, the new model developed in this work is much more sophisticated and realistic, and therefore might be of greater practical relevance. At the end of this introduction, we go into more detail by elaborating on the substantial differences between the two models and by pointing out some of the new contributions.

We consider a life insurer that manages a large, heterogeneous insurance portfolio consisting of participating contracts. The holders of such policies are, in addition to the contractually guaranteed benefits, entitled to variable bonus payments which allow them to participate in the obtained surpluses. Bodie et al. [7] distinguish between defined contribution and defined benefit type of insurance products. In our case, the contracts represent hybrids of both types. Indeed, the benefit payments result from the accumulated contributions, but due to premium guarantees, interest rate promises, and entitlements to allocated surpluses, the policies may not be fully funded. Early termination of contracts is possible due to surrender options and mortality. For the latter, we use detailed life tables providing a realistic development of death probabilities. The insurance portfolio's heterogeneity is reflected by different maturities and premiums of the policies and by a wide range of biometric parameters regarding the insured collective. Starting with an initial insurance portfolio, our aim is the forward projection of a given balance sheet structure and the investigation of conditions for a long-term stability or stationarity.

Despite the fact that advances in technology with increasing computing capacities allow for faster computations, a contract-specific simulation of all policies is often an impossible task. Indeed, according to Goffard and Guerrault [28], many life insurers face run time problems. Reasons for this are the size and heterogeneity of insurance portfolios, long time horizons, and the number of necessary Monte Carlo simulations. Instead, suitable approximation techniques or improved Monte Carlo methods are required. Krah et al. [45], e.g., use the least-squares Monte Carlo method for the calculation of the solvency capital requirement (SCR) under Solvency II. For this, they need only a small number of simulations, but have to further process the output. In the recommendations of the German Association of Actuaries, an exemplary procedure for the compression of an insurance portfolio is presented, which is based on solving a restricted minimization problem for chosen target figures, cf. Deutsche Aktuarvereinigung [19]. This approach is related to cluster-based compression methods like the K-means clustering proposed by Seehafer et al. [51]. Kiermayer and Weiß [39] present a framework for the grouping of an insurance portfolio using neural networks.

In this thesis, we explicitly address the well-known run time problem and investigate impacting factors. We illustrate that the introduced model suffers from unacceptable run times, partly even for single simulation runs, if we perform naive contract-specific simulations. This shows that, in particular if considering an ongoing insurance business, where the existing insurance portfolio is permanently augmented by new policies, we need flexible, approximating simulation methods, which we provide in this thesis. We develop different methods for the compression and the simulation of a given large insurance portfolio and investigate their quality and efficiency. Since our grouping schemes are flexible and universally applicable to any insurance portfolio, they might be relevant for practical applications. The initial number of cohorts and the number of policyholders within these depend on both the size and the heterogeneity of the given insured collective. We also show how to efficiently integrate new contracts into the existing insurance portfolio. At the same time, our model provides the flexibility to consider different scenarios regarding the development of the new business and can thus be used to simulate different business forms of a life insurance company.

The general goal of the thesis is always the simulation of large insurance portfolios and the investigation of factors potentially disturbing the stability of the corresponding balance sheets. We focus on the modeling and motivate the chosen models and approaches by observations from real data. Regarding the legislation, we orient ourselves towards European and in particular to German law, e.g. by considering a lagged participation process for the surpluses, but we do not focus on the stringent adaption to all regulations or the incorporation of local accounting rules. We aim at a balance between tractability and taking into account relevant legal requirements in order to get meaningful simulation results on the long-term stability. The purpose of the extensive simulation studies is also the development of guidelines and recommendations, e.g. regarding the applied compression and simulation method. The remainder of the thesis is structured as follows.

Chapter 2 deals with the simulation of large insurance portfolios. Here, we provide a general setting for modeling the structure and the dynamics of life insurance portfolios using flexible probability distribution families. In this chapter, we propose different approaches for the efficient simulation of a given large insurance portfolio. The idea is always to group the insured collective into cohorts according to biometric, contract-related criteria and to simulate only representative contracts, which form a compressed insurance portfolio. Each of these procedures can be seen as an approximation of the real insurance portfolio by a representative, less heterogeneous portfolio of the same size where the primal general structure is maintained. To investigate the approximation quality, we introduce different measures and methods.

In Chapter 3, we motivate and describe the general framework of our basic ALM model and introduce its various, interacting parts. The interest rate environment is modeled by simulating the instantaneous risk-free interest rate. For this, we use the prominent Vasiček short rate model that allows for negative interest rates. For the controlling of the life insurance business, we introduce a management model, where decisions regarding the taken risks and the aimed profitability are made. More specifically, the management decides about the surplus participation process, the asset allocation, and the methods for financing the due obligations. Furthermore, we put emphasis on the making of the balance sheet. We do not assume that the sum of all liabilities automatically equals the sum of all assets in contrast to many models in the literature. Instead, we explicitly prove that the fundamental balance sheet equation is fulfilled at the end of every period, which is the main result of this chapter, see Theorem 3.7.3.

In Chapter 4, we perform extensive simulation studies to investigate the efficiency of the compression methods and their quality in approximating the real (uncompressed) life insurance portfolio. We then choose one method and illustrate our basic ALM model for two prominent investment strategies. We consider alternative patterns of new contract arrivals and allow for fixed and random capital market crashes. These extensions aim at both, obtaining insights on the robustness of the applied management strategies and investigating conditions for the stability or stationarity of the components of the balance sheets.

Chapter 5 builds up on Chapter 3. Here, we present several refinements, alternative modeling approaches, and further applications of the general ALM model. Motivated by the observation that the two introduced investment strategies are not well oriented to the obligations induced by the (compressed) insured collective, we develop more sophisticated strategies for investing in the capital market and for financing the periodic disbursements. Taking into account the interactions between assets and liabilities, the new strategies aim at increasing the profitability and the security of the life insurance business. Furthermore, we prove that the fundamental balance sheet equation in the refined ALM model holds at all times, too.

Chapter 6 corresponds to the second part of the simulation studies. In the presence of transaction costs, we investigate the performance and robustness of the new strategies using several numbers for illustration. In addition, we investigate the effects of further refinements and alternative modeling approaches of components of the general ALM model, which includes the modeling of the capital market, the used method for the annual interest rate declaration, and the strategy for the use of surpluses.

Chapter 7 provides our conclusions.

Differentiation from the Master's thesis

We close this introduction by elaborating on the substantial differences between the models developed in this work and the one introduced in the Master's thesis Diehl [22]. For convenience, the latter is called the old model.

The capital market is modeled identically in the old model and in the basic new model. However, motivated by observed stylized facts, we extend the latter by considering a stochastic volatility model for the stock price dynamics. In the simulation studies, we expand our capital market model by allowing for crashes in the stock and bond markets, which lead to extreme liquidity shocks and thus threaten the insurer's financial health. This setting covers the extension to corporate bond investments and provides the ability to investigate the stability of the components of the balance sheet for different asset-liability approaches and the robustness of the applied investment strategies. Like the capital market, the balance sheet is also modeled more realistically in the new model. The total amount of assets is now allocated to bonds with different times to maturity, stocks, and a cash position. Furthermore, we add a new balance sheet position, liabilities to banks, providing the opportunity to account for liabilities of all kinds, e.g. for short-positions of bonds. Regarding the asset allocation, we introduce a minimum and a maximum amount for the stock positions. While the former could be a management target, potentially leading to higher returns on the managed funds, a maximum amount for stock positions is in line with the principle of prudent business practice prescribed by the regulating authorities. We develop more evolved strategies for investing in the capital market and for financing the due obligations. In particular, some of the strategies allow for selling bonds before maturity which was excluded in the strategies considered in the Master's thesis Diehl [22]. For this, we develop an algorithm for the successive sale of bonds to reduce a defined gap in funds while taking into account a safety amount that should be kept for future disbursements. The introduced strategies are investigated in terms of chances and risks in the presence of transaction costs. Regarding the bonus declaration, we introduce a method where the interest rate declaration is directly linked to the obtained returns and still accounts for the current amount of (free) reserves. The method can potentially decrease the probability of default. The new model is in line with the principle of double-entry bookkeeping as required in accounting, too. For this, we provide new proofs that the fundamental balance sheet equation is fulfilled at all times.

In addition to the presented ALM model, all parts regarding the simulation of life insurance portfolios are new. In the Master's thesis Diehl [22], we assumed that the insurance portfolio was already represented by a (reduced) number of model points, each consisting of a predefined number of policyholders. We indicated possible criteria for the generation of cohorts, but a compression of life insurance portfolios did not take place. An ongoing insurance business was modeled by assuming that a predefined number of model points emerge in every period, where each cohort consisted of a random number of policyholders. The new cohorts were added to the existing insurance portfolio. In the simulation studies of the Master's thesis Diehl [22], we found that this method is not efficient as its run time depends over-proportionally on the number of considered periods. As a consequence, we were only able to consider the case where two cohorts of new customers emerge per period and we needed to restrict to much smaller insurance portfolios. More specifically, the simulated insurance portfolio consisted of 500 cohorts each comprising 100 policies. In contrast, the standard size of the insurance portfolios considered in this work is $500,000$ and even larger portfolios that consist of 10,000,000 policies are simulated.

2. Large Insurance Portfolios

This chapter deals with the simulation of existing large insurance portfolios. The modeling approaches are motivated by observations from real data presented in Section 2.1. A general simulation framework and suitable probability distribution families allowing for high flexibility are introduced in Section 2.2. Here, we also introduce a measure to quantify the distance between two (empirical) probability distributions. The following section (Section 2.3), is about the structure and the dynamics of life insurance portfolios. On the one hand, the simulation should be realistic to get reliable results. For this, we need to consider large, heterogeneous insurance portfolios containing policies with different characteristics and policyholders with individual biometric properties and behavior. Furthermore, the dynamic development of an existing insurance portfolio over time needs to be modeled. On the other hand, one needs efficient simulations with tolerable computer running times. Apart from the size and the heterogeneity of the insurance portfolio, and the fact that we need many simulations,[1] there exists another factor having a negative influence on the efficiency, namely the considered time horizon. Since life insurance contracts typically have long maturities (often several decades), it is important to look at larger time horizons, too. This is in addition to the short-term considerations prescribed by regulating authorities.[2] Therefore, approximation techniques are required. In Section 2.4, we propose different approaches for the efficient generation of compressed insurance portfolios and the incorporation of new contracts allowing for efficient simulations over large time horizons. Here, we also introduce measures to investigate the quality of the compression methods.

2.1. Observations from real life insurance portfolios

For the following observations, we consider Allianz Life being currently the largest life insurer in Germany.[3] The data is taken from the corresponding annual business reports in the years 2008−2021.[4]

[1] As a rule of thumb, Monte Carlo requires at least 10,000 simulated paths per parameter set.

[2] E.g. under Solvency II, capital requirements are calculated on a one-year basis taking the 99.5% value-at-risk of own funds as risk measure, see Wagner [55]. Different mathematical interpretations of the corresponding solvency capital requirement (SCR) are compared by Christiansen and Niemeyer [15] who also generalize the SCR definition to future points in time.

[3] According to the statistics regarding direct insurers provided by the German Federal Financial Supervisory Authority (BaFin), the market share in 2019 was 29.22%, see Bundesanstalt für Finanzdienstleistungsaufsicht [12].

[4] For the last nine years, the annual reports are available at www.allianz.de.

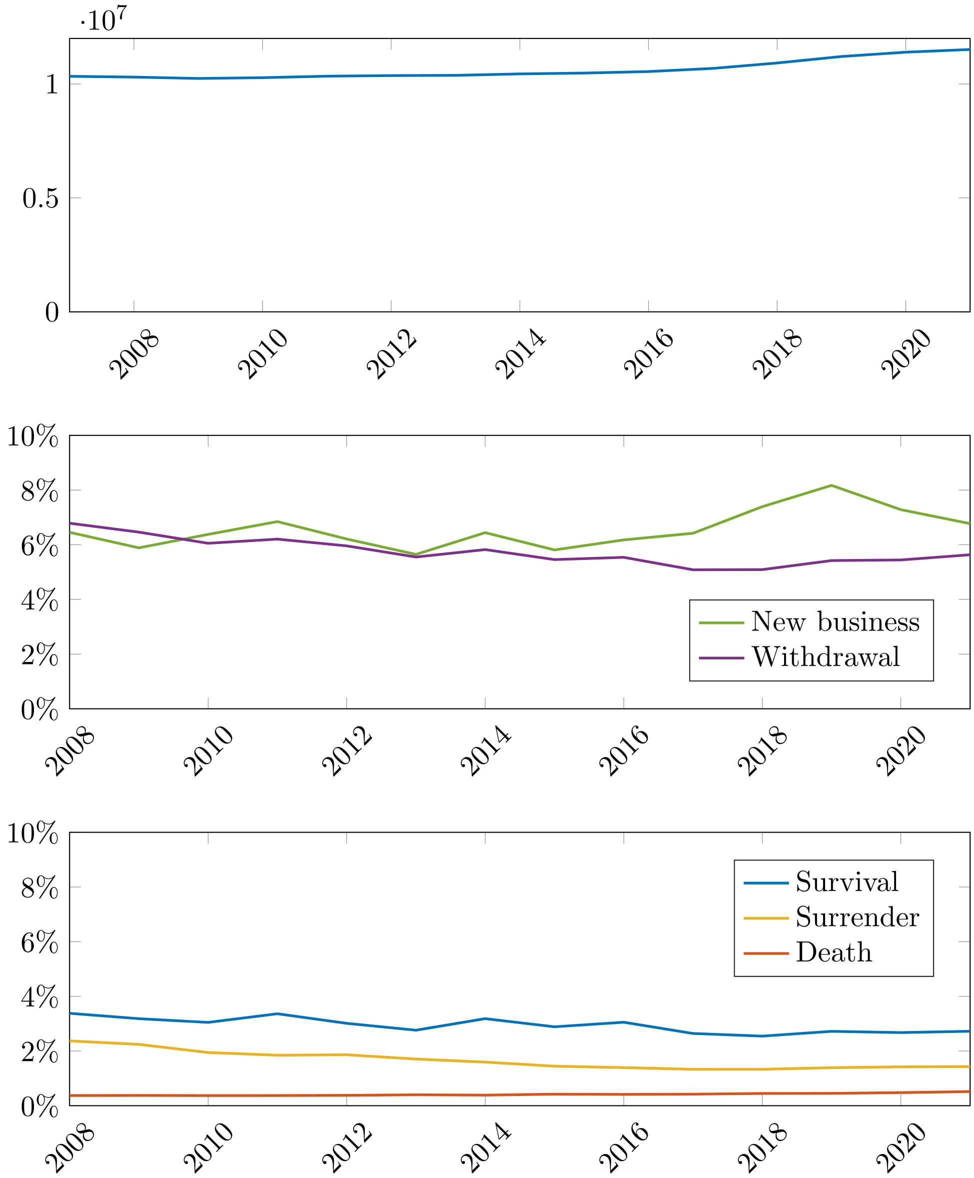

Figure 2.1.: Dynamics of the insurance portfolio of Allianz Life (Germany) in the years from 2008 to 2021. Top: insurance portfolio size. Middle and bottom: total increment and decrement and differentiated decrement during the year relative to the initial insurance portfolio size.

Figure 2.1 shows the dynamics of the insurance portfolio within the considered time horizon. From the upper part, we observe a stable development followed by a decent growth. We consider the increments (new business) and decrements (withdrawal) relative to the initial insurance portfolio size at the end of the year 2007 or the beginning of the year 2008. At that time, it consisted of 10,338,178 contracts. The relative number of new customers per year is between 5.5% and 8.5% corresponding to 1.35% and 2.06% per period if we have a quarterly discretization. We also see that the total withdrawals tend to decrease. A more differentiated consideration of the withdrawals can be found at the lower part of the figure. While the withdrawals due to expiry of policies (survival) and early cancellations (surrender) tend to decrease, the decrements due to death remain stable and increase within the last years (over 28% from 2014 to 2020). However, withdrawals due to death are still significantly smaller compared to the other two causes. Here, withdrawal due to surrender also includes the cases where the policyholders decide to set their contracts exempted from contributions. Further, withdrawal due to death also includes the cases of invalidity.

In summary, the development of all considered quantities in Figure 2.1 seems to be quite stable, which is due to the large market share of Allianz Life. Indeed, in our simulation studies, another important scenario (in addition to the run-off case motivated in the introduction) will be an ongoing insurance business with stationary new business. For smaller life insurance companies, such stationarity assumptions might not be justified since more dynamic developments can be expected. Therefore, we will also consider alternative new business scenarios.

2.2. Preliminaries

In this section, we introduce some probability distributions and mathematical concepts which we need to develop our model.

2.2.1. The four-parameter beta distribution

Modeling different scenarios regarding the age structure in the initial insurance portfolio or the future number of new contracts requires flexible distribution families. One suitable choice is given by the beta distribution introduced in the following definition.

Definition 2.2.1 (Beta distribution)**.** A random variable X is said to be beta distributed with parameters $\alpha, \beta \in \mathbb{R}_{>0}$ if it has the probability density function

$$f^X(x) = \mathbb{1}_{\{x \in [0,1]\}} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha, \beta)},$$

where $\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t$ is the beta function and $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t$ the gamma function. We write $X \sim \mathrm{Beta}\,(\alpha, \beta)$.

The expectation and the variance of a beta distributed random variable X with parameters α, β are given by

$$\mathbb{E}\,[X] = \frac{\alpha}{\alpha+\beta}$$

and

$$\mathrm{Var}\,(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

The flexibility of the beta distribution is illustrated in Figure 2.2. Some of the parameter pairs will be used in later simulation studies.

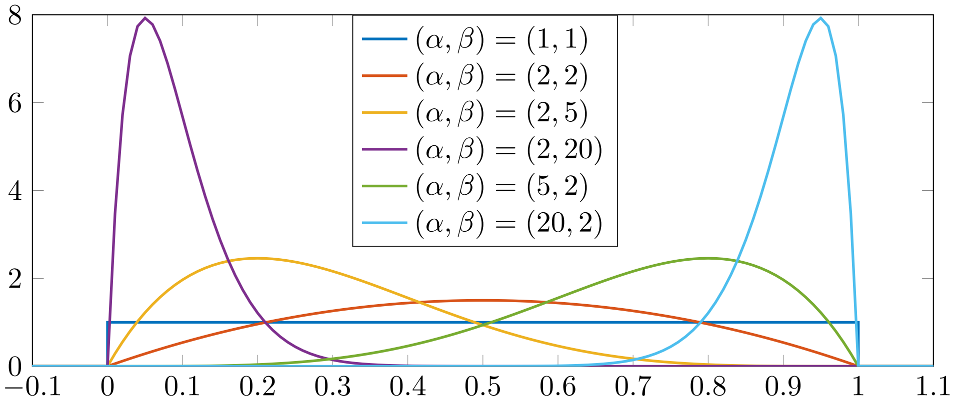

Figure 2.2.: Illustration of the probability density function of the beta distribution for different parameter specifications. For $\alpha = \beta = 1$, we get the uniform distribution on the interval $[0, 1]$.

In applications, one often needs beta distributed random variables with support on an interval $[a, b]$ with $a < b$, i.e., we like to vary the location and the scale of the beta distribution. This can be done by introducing two more parameters a and b representing the minimum and maximum values. Then, the linear transformation

$$Y = a + (b-a)X \tag{2.2.1}$$

with $X \sim \mathrm{Beta}\,(\alpha, \beta)$ yields a random variable Y following the so-called four-parameter beta distribution.

Definition 2.2.2 (Four-parameter beta distribution)**.** A random variable Y is said to be beta distributed with four parameters α, β, a, b, with $\alpha, \beta \in \mathbb{R}_{>0}$, $a, b \in \mathbb{R}$, $a < b$, if it has the probability density function

$$f^Y(y) = \mathbb{1}_{\{y\in[a,b]\}} \frac{(y-a)^{\alpha-1}(b-y)^{\beta-1}}{B(\alpha,\beta)(b-a)^{\alpha+\beta-1}}.$$

We write $Y \sim \mathrm{Beta}\,(\alpha, \beta, a, b)$.

The expectation and the variance of a four-parameter beta distributed random variable Y with parameters α, β, a, b can be calculated directly from the representation in (2.2.1) yielding

$$\mathbb{E}[Y] = a + (b-a)\frac{\alpha}{\alpha+\beta}$$

and

$$\mathrm{Var}(Y) = (b-a)^2 \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

2.2.2. The truncated normal distribution

Due to the central limit theorem, the normal distribution $\mathcal{N}(\mu,\sigma^2)$ with mean μ and variance σ^2 can be used as an approximating distribution in many applications. However, in some applications one needs a bounded support. More concretely, in this thesis we have limits regarding the policyholders' entry and exit ages. Therefore, we introduce the following conditional distribution with support on a predefined interval $[a,b]$ with $a<b$.

Definition 2.2.3 (Truncated normal distribution)**.** A random variable X is said to follow a truncated normal distribution with parameters μ,σ^2,a,b with $\mu,a,b\in\mathbb{R}$, $a<\mu<b$, $\sigma\in\mathbb{R}_{>0}$, if it has the probability density function

$$f^X(x) = \mathbb{1}_{\{x\in[a,b]\}}\frac{1}{\sigma}\frac{\varphi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)},$$

where φ and Φ are the density and the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$, respectively. We write $X\sim\mathcal{TN}(\mu,\sigma^2,a,b)$.

The expectation and the variance of a $\mathcal{TN}(\mu,\sigma^2,a,b)$-distributed random variable X are given by

$$\mathbb{E}[X] = \mu - \sigma\frac{\varphi\left(\frac{b-\mu}{\sigma}\right)-\varphi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}$$

and

$$\mathrm{Var}(X) = \sigma^2\left(1-\frac{\frac{b-\mu}{\sigma}\varphi\left(\frac{b-\mu}{\sigma}\right)-\frac{a-\mu}{\sigma}\varphi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}-\left(\frac{\varphi\left(\frac{b-\mu}{\sigma}\right)-\varphi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}\right)^2\right),$$

see, e.g. Johnson et al. [37].

Sampling from a truncated normal distribution $\mathcal{TN}(\mu,\sigma^2,a,b)$ can be reduced to sampling from a normal distribution $\mathcal{N}(\mu,\sigma^2)$ by generating normally distributed random numbers and re-sampling those lying outside the limits until all samples lie within the interval $[a,b]$.

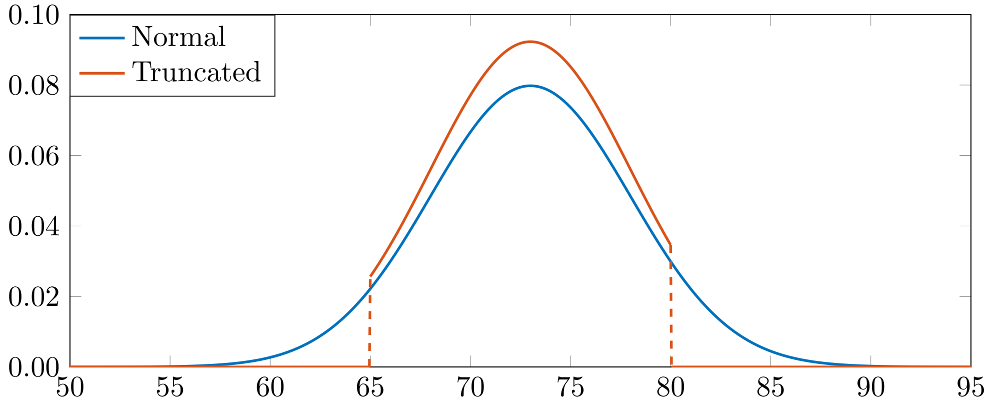

Figure 2.3.: Probability density functions of $\mathcal{N}(73, 25)$ and $\mathcal{TN}(73, 25, 65, 80)$.

Figure 2.3 shows the densities of a normal distribution and its corresponding truncated version with support on $[65, 80]$. The chosen parameters correspond to a simulation of exit ages of policyholders having policies where the minimum and the maximum exit age is set to be 65 and 80, respectively.

2.2.3. The total variation distance

An important part in this thesis consists of applying different methods for the simulation of large insurance portfolios. For this, we would like to investigate the approximation quality for which we need a measure to quantify the distance between two probability distributions. A prominent method is based on the total variation distance introduced in the following.

Let $(\Omega, \mathcal{A})$ be a measurable space and P, Q two probability measures thereon. The total variation distance $d_{TV}(P, Q)$ between P and Q is defined by

$$d_{TV}(P, Q) = \sup_{A \in \mathcal{A}} |P(A) - Q(A)|.$$

It can be interpreted as the largest possible difference between the probabilities of an event assigned by P and Q. If Ω is discrete, i.e. finite or infinite countable, we obtain the identity

$$d_{TV}(P, Q) = \frac{1}{2} \sum_{\omega \in \Omega} |P(\omega) - Q(\omega)|. \tag{2.2.2}$$

Similarly, if P and Q have densities f and g and $\Omega = \mathbb{R}$, we have

$$d_{TV}(P, Q) = \frac{1}{2} \int_{\mathbb{R}} |f(x) - g(x)| \, \mathrm{d}x. \tag{2.2.3}$$

Assume we draw a sample $X_1, \ldots, X_N$ from P^X, i.e. $X_1, \ldots, X_N$ are independent and identically distributed with $X_1 \sim P^X$. According to Devroye and Gyorfi [21], the standard empirical probability measure (empirical distribution) P_N^X is defined by

$$P_N^X(A) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{X_n \in A\}}$$

for $A \in \mathcal{A}$. Given another sample $Y_1, \ldots, Y_N$ drawn from P^Y, we are interested in measuring the distance between the empirical distributions P_N^X and P_N^Y. In the discrete case, we can directly use the definition in equation (2.2.2) with $P = P_N^X$ and $Q = P_N^Y$. However, if P^X and P^Y are continuous, which is the typical case in many applications, this is no longer possible since Ω is not countable in that case. To deal with this issue, we proceed as follows. First, the samples' support $[\min\{X_1, \ldots, X_N, Y_1, \ldots, Y_N\}, \max\{X_1, \ldots, X_N, Y_1, \ldots, Y_N\}]$ is divided into $J = J(d_{TV})$ intervals $I_1, \ldots, I_J$ of the same length. Then, we compute the total variation distance between the two empirical measures P_N^X and P_N^Y on the basis of the J chosen intervals by

$$d_{TV}\left(P_N^X, P_N^Y; J\right) = \frac{1}{2} \sum_{j=1}^{J} \left|P_N^X(I_j) - P_N^Y(I_j)\right|. \tag{2.2.4}$$

Thus, we reduced the continuous setting to a discrete one.[5]

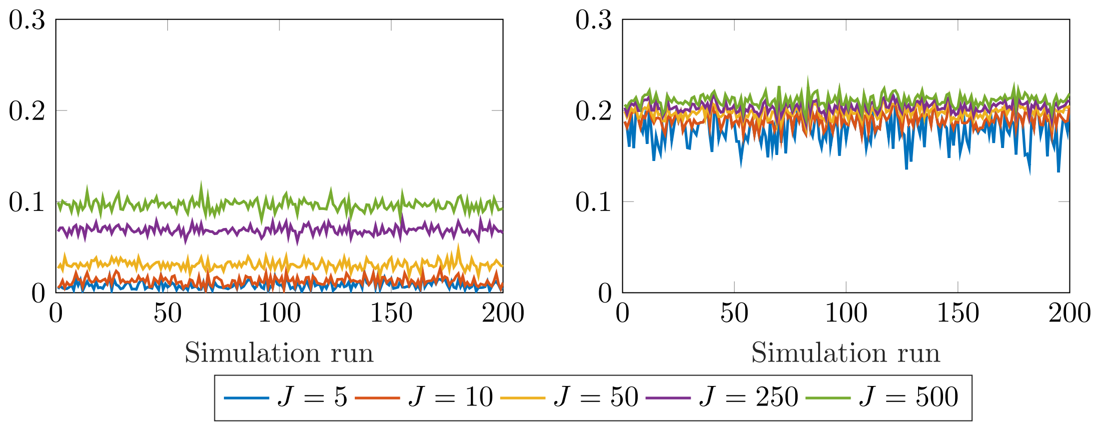

Figure 2.4.: Total variation distance for different numbers $J = J(d_{TV})$ of intervals. For each of the independent simulation runs, we generated $N = 10,000$ samples from P^X and P^Y. Left: samples drawn from the same normal distribution $P^X = P^Y = \mathcal{N}(4, 16)$. Right: samples drawn from $P^X = \mathcal{N}(4, 16)$ and $P^Y = \mathcal{N}(6, 16)$.

Note that the total variation distance computed by the introduced method may heavily depend on the number J of chosen intervals. This is illustrated on the left-hand side in Figure 2.4, where both samples are drawn from the same distribution. For a fixed

[5] Alternatively, one could estimate the functional form of the corresponding densities and apply (2.2.3).

simulation run, an increase in J yields larger values for d_{TV}. Indeed, we can always obtain the two extreme cases $d_{TV}\left(P_N^X, P_N^Y; J\right) = 0$ and $d_{TV}\left(P_N^X, P_N^Y; J\right) = 1$ by choosing $J = 1$ and J large enough such that no interval I_j contains samples from both distributions, respectively. Looking at the right-hand side in Figure 2.4, we see that the dependence on J is smaller the more the sample distributions differ. Nevertheless, whenever applying this method, one should state the chosen J since stating the total variation distance alone may not be informative at all. In general, following this method, we always have $d_{TV} \in [0, 1]$. In our applications,[6] we will always choose $J = J(d_{TV}) = 50$.

2.3. Modeling the structure and the dynamics of life insurance portfolios

In this section, we describe the characteristics of a given life insurance portfolio and its development over time. In addition to the considered product type, the biometric parameters together with the policyholders' behavior play an important role. Under biometric parameters, we understand in this thesis not only the gender or the age of the insured but also the probabilities for death and surrender.

We denote the considered time horizon throughout this thesis by $[0, \mathcal{T}]$, where $\mathcal{T}$ is the number of years to consider and 0 represents the actual time. In the following chapters, we use an equidistant time discretization of that interval, i.e.

$$0 = t_0 < t_1 < \cdots < t_K = \mathcal{T} \tag{2.3.1}$$

with $t_k = k\Delta t$, $k = 0, \ldots, K$, and a constant period length $\Delta t = \frac{\mathcal{T}}{K} = t_k - t_{k-1}$. As a consequence, the considered time horizon consists of K periods, and inventory figures like previously signed contracts are linked to past times $t_{-k} < 0$. Typically, we use a finer discretization than by calendar years.

2.3.1. Modeling the structure of life insurance portfolios

The initial number of policyholders in the insurance portfolio is denoted by δ_0. Throughout this thesis, we assume that each contract belongs to one insured person indicated by i. The auxiliary term *real* emphasizes that the corresponding quantity belongs to an actual, *real* policyholder or contract. In contrast, later we will consider *representative* customers or policies (indicated by m) to generate approximating, compressed insurance portfolios, cf. Section 2.4.

We characterize the existing insurance portfolio at time t_0 by the policyholders'

- gender $g^{\mathrm{real},i}$,
- age when signing the contract $\underline{x}^{\mathrm{real},i}$,

[6] For example, in Section 4.2 we will investigate the quality of approximating a certain distribution, in which case smaller values for d_{TV} are preferable.

- age at maximum contract term $\overline{x}^{\mathrm{real},i}$,
- current age $x_0^{\mathrm{real},i} \in \left(\underline{x}^{\mathrm{real},i}, \overline{x}^{\mathrm{real},i}\right)$,
- current premium payment $P_0^{\mathrm{real},i}$,
- current contract value $V_0^{\mathrm{real},i}$.

In practice, the listed characteristics are given and determine a snapshot of the insurance portfolio's structure. However, it is reasonable to also work with simulated data since the structure changes over time due to aging of the insured collective, withdrawals, new business, or simply by the development of the contracts. For the gender $g^{\mathrm{real},i}$, one can use the Bernoulli distribution with e.g. 0 for males and 1 for females. The entry and exit ages $\underline{x}^{\mathrm{real},i}$ and $\overline{x}^{\mathrm{real},i}$ can be modeled by truncated normal distributions, where the supports' boundaries could coincide with corresponding minimum and maximum ages. In order to simulate different age structures in the initial insurance portfolio, one can use the four-parameter beta distribution with minimum and maximum parameters $\underline{x}^{\mathrm{real},i}$ and $\overline{x}^{\mathrm{real},i}$ for modeling the current age $x_0^{\mathrm{real},i}$. This includes the case $x_0^{\mathrm{real},i} \sim \mathcal{U}\left(\underline{x}^{\mathrm{real},i}, \overline{x}^{\mathrm{real},i}\right)$. Regarding the premium payments, we use the uniform distribution on an interval defined by a minimum and a maximum premium. The exact distributional assumptions are provided in the simulation studies below, cf. Tables 4.1 and 4.2 in Section 4.1.

From the stated characteristics, we derive further needed quantities like the year of birth $y^{\mathrm{real},i}$, the elapsed, remaining, and total contract duration in periods $c_0^{\mathrm{real},i}$, $d_0^{\mathrm{real},i}$, and $e_0^{\mathrm{real},i}$, given by:

$$\begin{aligned} y^{\mathrm{real},i} &= \left\lfloor \mathcal{Y} - x_0^{\mathrm{real},i} \right\rfloor, \\ c_0^{\mathrm{real},i} &= \left\lfloor \Delta t^{-1} \left(x_0^{\mathrm{real},i} - \underline{x}^{\mathrm{real},i} \right) \right\rfloor, \\ d_0^{\mathrm{real},i} &= \left\lceil \Delta t^{-1} \left(\overline{x}^{\mathrm{real},i} - x_0^{\mathrm{real},i} \right) \right\rceil, \\ e_0^{\mathrm{real},i} &= c_0^{\mathrm{real},i} + d_0^{\mathrm{real},i}. \end{aligned} \tag{2.3.2}$$

Here, $\mathcal{Y}$ denotes the current calendar year and the ages are assumed to be (positive) real numbers. Depending on the considered product type, the contract value $V_0^{\mathrm{real},i}$ might be divided into a guaranteed part $A_0^{\mathrm{real},i}$ and a bonus part $B_0^{\mathrm{real},i}$, i.e.

$$V_0^{\mathrm{real},i} = A_0^{\mathrm{real},i} + B_0^{\mathrm{real},i}. \tag{2.3.3}$$

The account values $A_0^{\mathrm{real},i}$, $B_0^{\mathrm{real},i}$, and $V_0^{\mathrm{real},i}$ are related to the corresponding reserves and represent claims of the policyholders against the insurance company becoming due in the future. In Section 3.6, we derive representations for exemplary insurance products.

2.3.2. Modeling the dynamics of life insurance portfolios

One way to model the dynamics of a life insurance portfolio is to directly simulate a stochastic process $\delta = (\delta_k)_{k=1,\dots,K}$, where the random number δ_k denotes the amount of policies in force at time t_k. This approach aggregates withdrawals from and increments of the insurance portfolio. In order to investigate the influence of different scenarios regarding the new business and the development of withdrawals on the stability of the life insurer's balance sheets, we model decrements and increments separately. In particular, we can then distinguish between effects resulting from a decline in new business and those resulting from rising mortality or surrender rates.

Decrement of the insurance portfolio

In this thesis, withdrawals from the existing insurance portfolio can be caused by expiring contracts, premature death of the insured, and early cancellations (surrender). We assume that decrements always occur at the end of each period k. While the maturity date of the policies are known in advance, mortality and individual surrender options constitute different grades of uncertainty. To model these, there are traditionally two different approaches: a deterministic approach, where the basis could be life tables and deterministic surrender rates, and a stochastic approach, where mortality models and stochastic surrender rates are used.

Decrement of the insurance portfolio: mortality

The life table approach corresponds to the classic method in life insurance. It takes actual, observed mortality into account, where adjustments are possible to consider different scenarios regarding the demographic development. Even more flexibility is provided by (stochastic) mortality models, but they require an appropriate calibration to justify their application.

In this thesis, death of policyholders is modeled using life tables that contain annual death probabilities and which are updated from time to time. More specifically, we use the cohort life tables for Germany provided by the Federal Office of Statistics.[7] The contained annual death probabilities $q = q(x, g, y)$ depend on the age x, on the gender g, and on the year of birth y. Death probabilities $q_k^{\text{real},i}$ for time period $[t_{k-1}, t_k]$ are given by

$$q_k^{\text{real},i} = 1 - \left(1 - q\left(\min\left\{\left\lfloor x_{k-1}^{\text{real},i}\right\rfloor, x_{\max}\right\}, g^{\text{real},i}, y^{\text{real},i}\right)\right)^{\Delta t}, \tag{2.3.4}$$

where $x_{\max}$ denotes the maximum age in the life table[8] and $x_{k-1}^{\text{real},i}$ the age of the insured at time t_{k-1}. The above transformation is implied by setting $\left(1 - q_k^{\text{real},i}\right)^{\frac{1}{\Delta t}} = 1 - q$, i.e.

[7] We use the cohort life table Variant 1, which is based on a trend in the development of mortality since 2011, see Statistisches Buntdesamt [53]. More details on the derivation of the death probabilities can be found in Statistisches Bundesamt [52].

[8] In our case, we have $x_{\max} = 100$.

by assuming a constant force of mortality throughout the year.

Remark 2.3.1 (Alternative periodic death probabilities)**.** Alternatively, one could put a higher death intensity on early periods of a year[9] or just define

$$q_k = \Delta t \cdot q\left(\min\left\{\left\lfloor x_{k-1}^{\mathrm{real},i}\right\rfloor, x_{\max}\right\}, g^{\mathrm{real},i}, y^{\mathrm{real},i}\right).$$

One can show with the Bernoulli inequality that the latter generates smaller periodic death probabilities if $\Delta t \in (0,1)$. However, except for large annual death probabilities q, e.g. $q >> 0.05$, the differences $q_k^{\mathrm{real},i} - q_k$ are rather small as illustrated in Figure 2.5.

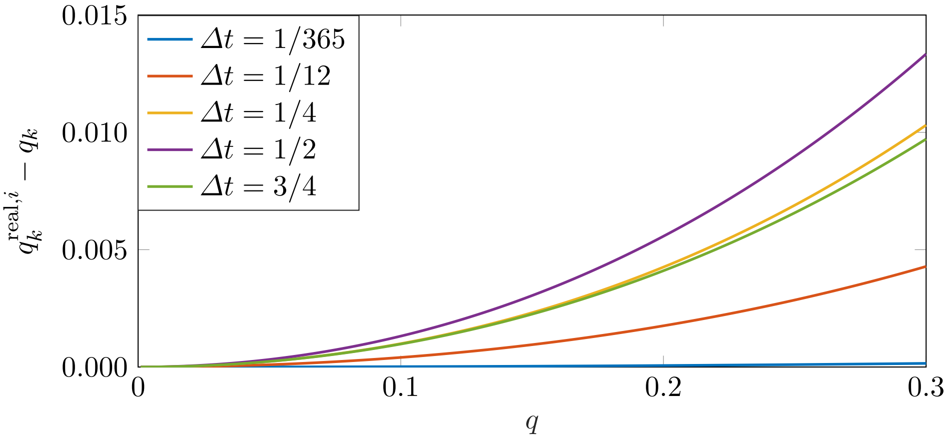

Figure 2.5.: Difference between the periodic death probabilities $q_k^{\mathrm{real},i}$ and q_k depending on the annual death probabilities q for different $\Delta t \in (0,1)$.

Decrement of the insurance portfolio: surrender

The life insurance contracts considered in this thesis are equipped with surrender options, i.e. the policyholders can actively terminate their contracts before maturity.[10] In that case, the insured person receives the cash surrender value, i.e. the contract value reduced by a surrender fee.

There are two hypotheses trying to explain what causes exercise of surrender options. Following the emergency fund hypothesis, policyholders cancel their contracts if they need money (the emergency fund) during times of financial distress or if they are unable to pay the premiums anymore. An indicator for this hypothesis would be rising surrender rates during economic recessions, where unemployment rates increase. Evidence for this hypothesis is provided in Outreville [50]. The interest rate hypothesis claims that surrender rates depend on the market interest rates since these can be seen as a representative of opportunity costs for owning a life insurance contract, see Milhaud et al. [49]. Thus,

[9] In the literature, this method is known as the Balducci's assumption, cf. Gerber [26].

[10] In Germany, for example, this right is required by the Insurance Contract Law VVG in §168.

higher interest rates would yield higher surrender rates since policyholders could cancel their life insurance to benefit from more attractive investment possibilities. Evidence for this hypothesis is provided in Kuo et al. [47]. An interest rate-driven approach for the simulation of surrender rates including macroeconomic control variables is also applied in Kubitza et al. [46]. However, statistical methodologies globally fail at giving accurate individual surrender predictions since the policyholders' behavior is complex, not always rational, and depend heterogeneously on many factors, see Barsotti et al. [2]. This motivates to model the cancellation rates directly, e.g. as a random variable as done by Biagini et al. [4].

In this thesis, we follow such a direct approach, too. We model the contracts' surrender options by introducing probabilities $u_k^{\text{real},i}$ of the insured terminating their contracts in period k. Different modeling approaches can easily be adopted depending on the life insurer's business form and the applied simulation method. This includes deterministic and stochastic surrender rates. Considering a single policy, it is reasonable to assume that the probability increases in the first periods of the contract time due to growing uncertainties. Since terminating a contract is linked with paying cancellation fees, the surrender probabilities can be expected to decrease after reaching a maximum around the run time's midterm. According to our approximating simulation method for an ongoing insurance business with new business that we will develop in Section 2.4, new contracts are sorted into existing cohorts. Thus, we have an averaging of policies including newly signed ones to form the updated cohort throughout the simulation. This balances the effects of individual contracts and makes it reasonable to assume the same surrender probability in each period. If not stated otherwise, we therefore model the surrender probability using an exponential distribution with parameter u, namely

$$u_k^{\text{real},i} = \mathbb{1}_{\left\{d_k^{\text{real},i} > 0\right\}} \cdot \left(1 - \mathrm{e}^{-u\Delta t}\right). \tag{2.3.5}$$

The indicator function $\mathbb{1}_{\{\}}$ ensures that surrender is only possible if the insurance policy is still in force in the following period, i.e. if the actual remaining contract duration $d_k^{\text{real},i}$ is positive.

Increment of the insurance portfolio: new business

Providing and selling new contracts is essential for life insurers writing long-term insurance business. Without new business, i.e. in the case of a run-off, we cannot expect to obtain any kind of stationarity or stability in the development of the balance sheets. Furthermore, new premiums might finance existing contracts with high guarantees signed at earlier times through investments in the same reference portfolio.

We model new business by assuming that there arrives a random number δ_k^{new} of new customers in every period k, independent of other periods. The policies are signed at the beginning of a period. In the later simulation studies, we assume

$$\delta_k^{\text{new}} \sim \text{Poi}\left(\Lambda_k\right),$$

where Λ_k is a random variable itself. More specifically, Λ_k is assumed to follow a four-parameter beta distribution with two period-depending shape parameters α_k and β_k and two location parameters $\Lambda_k^{\min}$ and $\Lambda_k^{\max}$ representing the minimum and maximum values, i.e.

$$\Lambda_k \sim \text{Beta}\left(\alpha_k, \beta_k, \Lambda_k^{\min}, \Lambda_k^{\max}\right).$$

The last two parameters may be chosen as fractions of the initial insurance portfolio size δ_0. Different new business scenarios can be considered by specifying the shape parameters correspondingly. We elaborate on this in Section 4.4, where we provide detailed parameterizations and investigate for ongoing insurance business the impact of varying patterns of new contract arrivals on the stability of the life insurer's balance sheets. In particular, the considered scenarios include the case where a sudden increase (decrease) in the demand of life insurance leads to a positive (negative) shock on the future number of new customers. Note that investigations are not limited to the new business scenarios considered in this thesis. Due to the high flexibility of our model, the analysis can be easily adopted to further scenarios, e.g., to a deferred run-off or to regime-switching scenarios based on Markov chains.

2.4. Compression of life insurance portfolios and simulation

In this section, we develop different methods for the compression and the simulation of a given large insurance portfolio. The idea is always to group similar contracts together in so-called model points or cohorts. We then only simulate one exemplary contract per model point over time. Those simulated contracts, whose characteristics are averaged values, belong to representative policyholders. It is then assumed that all contracts within a model point coincide with the exemplary one. In particular, they all expire at the same time. The procedure can be seen as an approximation of the real insurance portfolio with many individual contracts by another one of the same size but with less individual contracts.

Seehafer et al. [51] propose a cluster-based compression method, where the insurance portfolio is divided into a predefined number of clusters and then a K-means algorithm is applied to improve their location. The corresponding centers then form a compressed insurance portfolio. A K-means algorithm is also used by Goffard and Guerrault [28]. Typical drawbacks are the fact that one has to choose K in advance and the algorithm's run time. Furthermore, Kiermayer and Weiß [39] criticize the absence of an active control of the involved risk features and the implicitly assumed linear dependence. Accordingly, they present a grouping method using neural networks, thereby capturing non-linear dependencies. Numerical investigations for term life insurance contracts and defined contribution plans within a run-off scenario indicate that their approach is superior compared to K-means clustering. Both, K- means clustering and methods using neural networks, are not directly applicable to an ongoing insurance business with new business. Indeed, repeating the procedure in every time step where new customers arrive would not be efficient.

Neural networks are known to potentially out-perform classic approaches in detecting and re-creating structures in a given data set like an insurance portfolio. However, a poor performance is possible if the given structure substantially changes in the future and external events occur that cannot be predicted by historical developments. Examples for the latter are crashes in the capital market or extreme new business scenarios. Furthermore, neural networks only yield pointwise forecasts, i.e. information regarding the prediction uncertainty is missing. In contrast, Monte Carlo-based simulation methods always provide these information through the corresponding quantiles.

The concepts developed in this thesis are universally applicable to any insurance portfolio and any business form of a life insurance company. In particular, efficiency is maintained also in the case of an ongoing insurance business with new business. The stochastic simulation of the latter, including different scenarios regarding the expected development of new contract arrivals, incorporates uncertainty in addition to the risks implied by the capital market and individual policyholder behavior. Our simulation methods allow us to quantify these uncertainties. In the following sections, we describe the grouping procedures in more detail.

2.4.1. A general compression and simulation method

The characteristics of the initial insurance portfolio are as described in Section 2.3.1, with δ_0 denoting the initial number of policyholders. In our model, the ages correspond to (positive) real numbers. The generation of cohorts requires a selection of grouping criteria. In the following, the insured collective is divided into M_0 non-empty cohorts due to the three grouping criteria

1. gender $g^{\text{real},i}$,
2. integer current age $\left\lfloor x_0^{\text{real},i} \right\rfloor \in \mathbb{N}$, and
3. integer exit age $\left\lfloor \overline{x}^{\text{real},i} \right\rfloor \in \mathbb{N}$.

Note that the signing age $\underline{x}^{\text{real},i}$ is not a grouping criterion and the real policies within a model point can have different contract periods. Indeed, the actual signing age $\underline{x}^{\text{real},i}$ is only needed to calculate the elapsed contract period $c_0^{\text{real},i}$, see equation (2.3.2), and thus for the calculation of the initial contract value.

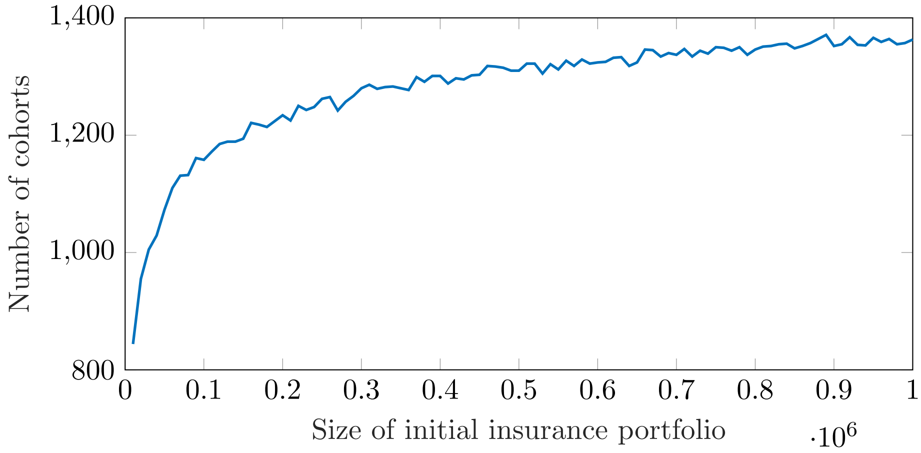

Figure 2.6.: Number of generated model points depending on the size of the initial insurance portfolio (of which each is simulated separately).

The number M_0 of generated cohorts depends on both the size and the heterogeneity of the initial insurance portfolio as illustrated in Figure 2.6. Here, each insurance portfolio is simulated separately based on the distributional assumptions of the biometric parameters described in Table 4.1. Due to the compression, the number of individual, different contracts is reduced enormously. The smallest insurance portfolio consists of $10,000$ policies and is divided into 844 model points corresponding to a reduction of approximately 92%. The number of model points increases slower for large insurance portfolios and is bounded from above due to age limits. The dependence on the heterogeneity can be seen from the non-monotonic growth. In particular, two different insurance portfolios of the same size do not necessarily lead to the same number of model points.

After the grouping, we assign numbers and randomly select[11] a representative policyholder from each cohort $m \in \{1, \dots, M_0\}$ denoting its

- gender by g^m,
- its current age by $x_0^m \in \mathbb{R}^+$, and
- its exit age by $\overline{x}^m \in \mathbb{R}^+$.

By this, we also take fractions of the year into account. We can define a surjective function $\pi_0 : \{1, \dots, \delta_0\} \to \{1, \dots, M_0\}$ such that $\pi_0(i) = m$ if policyholder i is sorted into cohort m. Each model point $m \in \{1, \dots, M_0\}$ contains δ_0^m policies, with

$$\delta_0^m = |\{i : \pi_0(i) = m\}| \, .$$

The initial size of the insurance portfolio δ_0 can thus be written as $\delta_0 = \sum_{m=1}^{M_0} \delta_0^m$.

[11]Here, we use the uniform distribution for the random selection.

The remaining contract characteristics are determined by arithmetic means, i.e., we have:

$$P_0^m = \frac{1}{\delta_0^m} \sum_{i:\pi_0(i)=m} P_0^{\mathrm{real},i}, \tag{2.4.1}$$

$$A_0^m = \frac{1}{\delta_0^m} \sum_{i:\pi_0(i)=m} A_0^{\mathrm{real},i}, \tag{2.4.2}$$

$$B_0^m = \frac{1}{\delta_0^m} \sum_{i:\pi_0(i)=m} B_0^{\mathrm{real},i}, \tag{2.4.3}$$

$$V_0^m = \frac{1}{\delta_0^m} \sum_{i:\pi_0(i)=m} V_0^{\mathrm{real},i}. \tag{2.4.4}$$

Equation (2.3.3) implies that the value V_0^m of the representative contract can be decomposed into

$$V_0^m = A_0^m + B_0^m.$$

Due to the dependence on the distribution of biometric parameters the cohorts' sizes can differ substantially, see Figure 2.7.

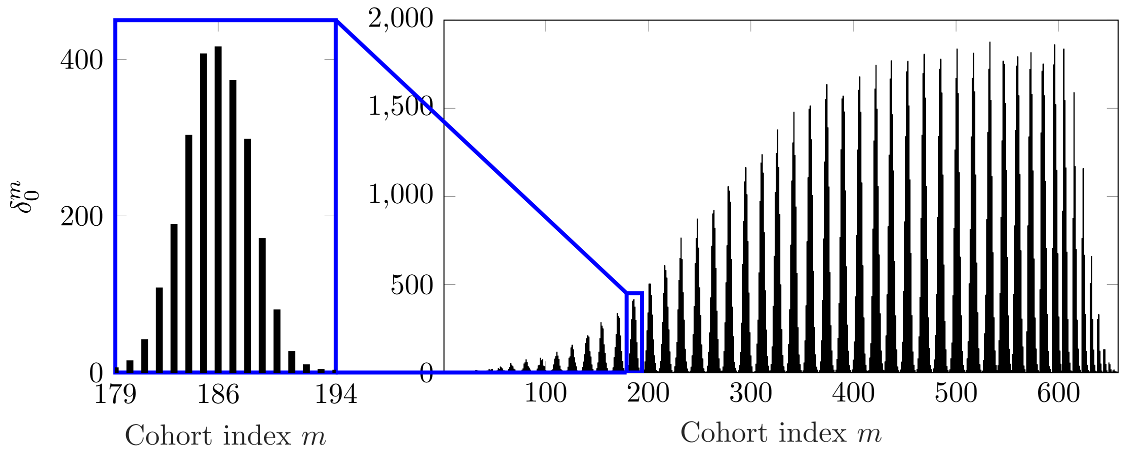

Figure 2.7.: Right: male policyholders distributed over the model points generated for an exemplary insurance portfolio of size $\delta_0 = 500,000$. Left: excerpt of those cohorts m with current age $\lfloor x_0^m \rfloor = 33$.

On the right-hand side of Figure 2.7, we show the distribution of male policyholders[12] over the cohorts generated for an exemplary insurance portfolio of size $\delta_0 = 500,000$. Here, the numbering of the cohorts reflects the corresponding age criteria in ascending order such that, for example, all 33-year-old, male policyholders with exit ages from 55

[12]The distribution of the female policyholders over the remaining cohorts is similar.

to 70 are located in cohorts 179 to 194 (blue excerpt). Note that the appearance of the shown distribution depends on the specific way of numbering the cohorts. On average, there are 383 policyholders per model point.

The development of the cohorts

After describing the cohorts and their initial state at the beginning of the simulation, we now present the subsequent development. More specifically, recursive representations of the size of the insurance portfolio δ_k, the number of cohorts M_k, and the number of policyholders in the individual cohorts δ_k^m are derived for all periods $k = 1, \dots, K$. In Figure 2.8, we illustrate the events related to a cohort $m \in \{1, \dots, M_k\}$ within the period $[t_{k-1}, t_k]$.

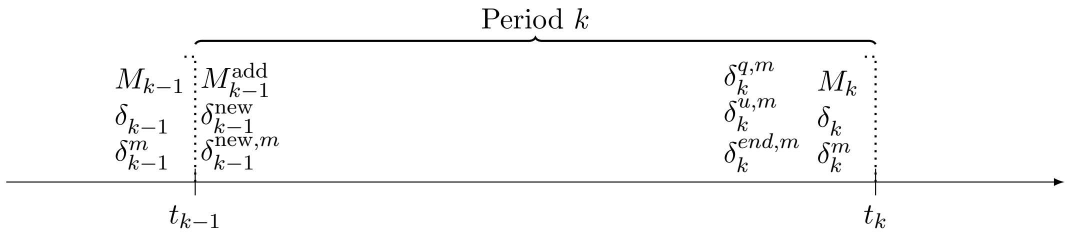

Figure 2.8.: Representation of period k regarding the insured collective.

The updated current age at time t_k and the remaining contract period are described recursively, i.e.

$$x_k^m = x_{k-1}^m + \Delta t$$

and

$$d_k^m = d_{k-1}^m - 1, \tag{2.4.5}$$

with

$$d_0^m = \left\lceil \frac{\overline{x}^m - x_0^m}{\Delta t} \right\rceil \in \mathbb{N}.$$

The updated current ages are needed for the periodic death probabilities, compare equation (2.3.4).

At the beginning of period k, i.e. at time t_{k-1}, a random number of new customers $\delta_{k-1}^{\text{new}}$ signs up. Their age structure is generated based on suitable distributions as done for the existing insurance portfolio. At first, the collective of new customers is divided into N_{k-1} cohorts based on the established criteria related with arithmetic means of the other characteristics. To avoid an uncontrolled increase in the number of cohorts, we proceed as follows. If the grouping criteria of a new cohort $n \in \{1, \dots, N_{k-1}\}$ coincide with those of one of the existing cohorts $m \in \{1, \dots, M_{k-1}\}$, i.e.

1. $g^{\text{new},n} = g^m$,
2. $\lfloor \underline{x}^{\text{new},n} \rfloor = \left[x^m_{k-1} \right]$, and
3. $\lfloor \overline{x}^{\text{new},n} \rfloor = \lfloor \overline{x}^m \rfloor$,

the new and existing cohort can be merged. Here, $[x^m_k]$ is defined by

$$[x^m_k] = \begin{cases} \left[x^m_{k-1}\right] + 1, & \text{if} \quad k \equiv 0 \left(\operatorname{mod} \frac{1}{\Delta t}\right) \\ \left[x^m_{k-1}\right], & \text{else} \end{cases} \tag{2.4.6}$$

for $k \geq 1$ and $[x^m_0] \coloneqq \lfloor x^m_0 \rfloor$ ensuring that there exists at most one cohort m that can be merged with the new cohort n. In case of merging, we denote the number of additional customers in this cohort m by $\delta^{\text{new},m}_{k-1}$ and adjust the premium payment and the account values according to the weighted average, i.e. after merging the new cohort n into existing cohort m, we get

$$\begin{aligned} P^m_{(k-1)^+} &= \frac{1}{\delta^m_{k-1} + \delta^{\text{new},m}_{k-1}} \left(\delta^m_{k-1} P^m_{k-1} + \delta^{\text{new},m}_{k-1} P^{new,n}_{k-1} \right), \\ A^m_{(k-1)^+} &= \frac{\delta^m_{k-1}}{\delta^m_{k-1} + \delta^{\text{new},m}_{k-1}} A^m_{k-1} + P^m_{(k-1)^+}, && (2.4.7) \\ B^m_{(k-1)^+} &= \frac{\delta^m_{k-1}}{\delta^m_{k-1} + \delta^{\text{new},m}_{k-1}} B^m_{k-1}, && (2.4.8) \end{aligned}$$

and

$$V^m_{(k-1)^+} = A^m_{(k-1)^+} + B^m_{(k-1)^+}. \tag{2.4.9}$$

Those cohorts who could not be merged augment the set of existing model points by M^{add}_{k-1}.

Remark 2.4.1 (Efficiency of the simulation method)**.** Note that once a cohort number is assigned to a real policyholder or contract, it remains fixed, i.e. $\pi_{k+1}(i) = m$ if $\pi_k(i) = m$. This property is important as it forms the basis for efficient simulations of life insurers with new business. Indeed, a new compression of the whole insurance portfolio at later times t_k, $k \geq 1$, is not required. Instead, only new customers need to be grouped in cohorts and then merged into existing ones if possible. By (2.4.6), we ensure that there is at most one cohort in which new policies are potentially merged into. Alternatively, one could update the current age every period implying that there might be a new cohort and several existing ones that share the same grouping criteria. In that case, we would merge all involved cohorts together.

At the end of period k, starting with the size of the cohort $\delta^m_{k-1} + \delta^{\text{new},m}_{k-1}$ and taking the survival probability $1 - q^m_k$, the surrender probability u^m_k, which is assumed to be independent of the survival probability, and the remaining contract period d^m_k into account, the number of policyholders in model point m at time t_k which remain in the insured collective is given by

$$\delta_k^m = \mathbb{1}_{\{d_k^m > 0\}} \cdot (1 - u_k^m)(1 - q_k^m)\left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right). \tag{2.4.10}$$

Note that we only consider policyholders who have survived the period for the surrender option. Therefore, the numbers of deaths and cancellations in model point m amount to

$$\delta_k^{q,m} = q_k^m \cdot (\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}) \qquad \text{and} \qquad \delta_k^{u,m} = u_k^m \cdot (1 - q_k^m)(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}).$$

Finally the new number of model points and the size of the insured collective are

$$M_k = M_{k-1} + M_{k-1}^{\text{add}} \qquad \text{and} \qquad \delta_k = \sum_{m=1}^{M_k} \delta_k^m.$$

Remark 2.4.2 (Handling of empty cohorts)**.** Empty cohorts can be deleted during a simulation and the numeration of the cohorts adjusted. Especially in the case of an ongoing insurance business with new business, this is an important step to maintain efficiency for large time horizons by avoiding a permanent increase in the number of cohorts. However, for a better readability, this step is not included in our notation.

2.4.2. Variants of the compression and simulation method

In this section, we develop alternative grouping and simulation methods of insurance portfolios. So far, two policyholders are assigned to the same cohort if they have the same gender, integer current age, and integer exit age. The same procedure applies to new contracts, and then new and existing cohorts are merged if possible. We transfer that idea and present in the following a more general framework for criteria-based compression methods.

For this, we introduce the following notation. For $b_1, b_2 \in \mathbb{N}_{\geq 1}$ let $\mathcal{P}^{b_1,b_2}$ denote the method of simulating the compressed portfolio obtained by grouping those policyholders of the same gender into the same cohort whose current age and exit ages lie within intervals of a length b_1 and b_2, respectively. The method from the last section can then be denoted by $\mathcal{P}^{1,1}$ and is thus a special case of the general class of compression and simulation methods presented in this section. The method of simulating the uncompressed insurance portfolio is denoted by $\mathcal{P}^{0,0}$ and corresponds to a naive contract-specific simulation. In the following, we describe the compression and simulation methods in more detail.

For given $b_1, b_2 \in \mathbb{N}_{\geq 1}$ and minimum entry and exit ages $\underline{x}^{\min}$ and $\overline{x}^{\min}$, we construct intervals

$$\begin{aligned} \underline{I_l} &= \left[\underline{x}^{\min} + (l-1)b_1, \underline{x}^{\min} + lb_1\right), \\ \overline{I_l} &= \left[\overline{x}^{\min} + (l-1)b_2, \overline{x}^{\min} + lb_2\right), \end{aligned}$$

for $l = 1, 2, \ldots$, until an interval $\underline{I_l}$ and $\overline{I_l}$ contains the maximum entry and exit age $\underline{x}^{\max}$

and $\overline{x}^{\max}$, respectively. As before, we need to transform the real-valued ages of the individual policyholders into ages that are suitable for the grouping and merging procedure. While in Section 2.4.1 we took the integer values, here the transformed ages are defined by the lower boundaries of the intervals that contain the corresponding real ages. For example, the transformed current age $\left[x_0^{\mathrm{real},i}\right]$ is given by

$$\left[x_0^{\mathrm{real},i}\right] := \underline{x}^{\min} + (l-1)b_1$$

if $x_0^{\mathrm{real},i}$ lies in interval $\underline{I}_l$.

Any two policyholders are then assigned to the same cohort if they have the same gender, transformed current, and transformed exit age, thereby dividing the insured collective into M_0 cohorts. As before, after the grouping, we assign numbers and randomly select a representative policyholder from each cohort $m \in \{1, \ldots, M_k\}$, denoting its gender, current age, and exit age correspondingly. The remaining contract characteristics are, analogously to before, determined by arithmetic means. The merging of new and existing cohorts is done as before, where the transformed current ages must be updated according to equation (2.4.6) to ensure that there is at most one existing cohort that can be merged with a new one. As a consequence, Remark 2.4.1 also applies to the general compression and simulation methods, and choosing $b_1, b_2 > 1$ further increases the degree of compression, thereby potentially also the efficiency. Note that the transformed ages are only needed for the grouping and merging procedure, but for the death probabilities we still use the real-valued ages of the representative policyholders.

2.4.3. Efficiency and approximation quality

Regarding the quality of a compression method, we find two competing goals. On the one hand, we would like to have a large efficiency, i.e. the compression method itself should be easily applicable and simulating the compressed insurance portfolio should require a much smaller computing time in comparison with simulating the original (uncompressed) one. On the other hand, from a good simulation method we expect a certain performance, i.e. the approximation of the real insurance portfolio and its development should be accurate.

In this section, we therefore introduce instruments to investigate the quality of the compression and simulation methods. After having constructed a suitable model for the development of a given portfolio of insurance contracts and the corresponding balance sheets in Chapter 3, we will use these to measure the efficiency and the goodness of approximation for different business forms of a life insurance company. This will help us to derive corresponding recommendations as to which simulation methods can be used for which purposes (cf. Section 4.2).

Efficiency

The extent of a compression can be measured by the compression ratio CR_0 describing the ratio of original data size to the compressed data size, see Wang et al. [56]. In our setting, where the compressed insurance portfolio has the same size as the uncompressed one, the compressed data size corresponds to the number of those representative contracts that are actually modeled, and thus to the number of cohorts $M_0 = M_0\left(\mathcal{P}^{b_1,b_2}\right)$ depending on the applied compression and simulation method $\mathcal{P}^{b_1,b_2}$. Therefore, we define the compression ratio by

$$CR_0 = \frac{\delta_0}{M_0}.$$

An efficient compression would lead to a smaller number of model points M_0 and thus to a larger compression ratio CR_0. In this thesis, we use an alternative measure to describe the extent of a compression, namely the compression factor CF_0. It is defined via the inverse compression ratio, i.e. here by

$$CF_0 = CR_0^{-1} = \frac{M_0}{\delta_0}.$$

Smaller values for CF_0 are associated with a more efficient compression method. In the case of an ongoing insurance business, new contracts may increase the number of cohorts. Therefore, we need to look at its development, too. At time t_k, $k \geq 1$, the compression factor is defined by

$$CF_k = \frac{M_k^{\mathrm{sim}}}{\delta_k}, \tag{2.4.11}$$

where $M_k^{\mathrm{sim}} = M_k^{\mathrm{sim}}\left(\mathcal{P}^{b_1,b_2}\right)$ denotes the number of non-empty cohorts that are actually modeled according to $\mathcal{P}^{b_1,b_2}$. δ_k is the size of the uncompressed insurance portfolio if we apply $\mathcal{P}^{0,0}$. We have $M_0^{\mathrm{sim}} = M_0$, $M_k^{\mathrm{sim}} \leq M_k$, and typically $M_k^{\mathrm{sim}} < M_k$ for $k \geq 1$ since M_k also accounts for empty cohorts which are deleted during the simulation, cf. Remark 2.4.2. For run-off scenarios, looking at CF_k is not necessary since the number of non-empty cohorts M_k^{sim} decreases with the size of the insurance portfolio.

Remark 2.4.3 (Run time)**.** To compare the efficiency of different simulation methods, one could also directly measure the required run times. Indeed, it is also important to (pre-)estimate the run time required for a Monte Carlo simulation. However, here we need to be careful. In contrast to the compression ratio or compression factor, the run times heavily depend on the available computer equipment and the instantaneous CPU utilization. Therefore, different simulation methods should always be applied on the same computer under the same conditions. Only then it is reasonable to compare different run times.

Approximation quality

In order to measure the approximation quality, we need to simulate both the compressed and the uncompressed insurance portfolio, which will be done in Section 4.2. As a compression and simulation method, we choose $\mathcal{P}^{b_1,b_2}$ for given b_1, b_2. Regarding the uncompressed insurance portfolio, we apply the contract-specific simulation $\mathcal{P}^{0,0}$. For a selected quantity of interest, the sample distributions at a certain time point obtained by corresponding Monte Carlo simulations can be compared by the total variation distance as described in Section 2.2.3. In the following, we describe more criteria reflecting the approximation quality.

Let X_k be a quantity of interest at time t_k. For now, this could be the total value of the insurer's assets, the equity, or the obtained surplus. We denote by $X_k^{0,0,(n)}$ and $X_k^{b_1,b_2,(n)}$ the n-th simulated value of X_k applying $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$, respectively. Note that n refers to the n-th Monte Carlo path determining the new business scenario and the dynamics of the capital market. Thereby, we allow for a pathwise comparison between $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ and can thus investigate the direct impact of the chosen compression and simulation method. The average approximation error at a certain time point t_k can be obtained by the arithmetic mean of the distances between $X_k^{0,0,(n)}$ and $X_k^{b_1,b_2,(n)}$, $n = 1, \ldots, N$. The distances can be measured on the basis of different p-norms. In this thesis, we consider as absolute error measures the mean squared error (MSE) and the mean absolute error (MAE) defined by

$$\mathrm{MSE} = \frac{1}{N} \sum_{n=1}^{N} \left(X_k^{0,0,(n)} - X_k^{b_1,b_2,(n)} \right)^2$$

and

$$\mathrm{MAE} = \frac{1}{N} \sum_{n=1}^{N} \left| X_k^{0,0,(n)} - X_k^{b_1,b_2,(n)} \right|,$$

respectively. For positive quantities of interest X_k, we may also consider relative error measures, e.g. the mean squared percentage error (MSPE) and the mean absolute percentage error (MAPE) defined by

$$\mathrm{MSPE} = \frac{1}{N} \sum_{n=1}^{N} \frac{\left(X_k^{0,0,(n)} - X_k^{b_1,b_2,(n)} \right)^2}{X_k^{0,0,(n)}}$$

and

$$\mathrm{MAPE} = \frac{1}{N} \sum_{n=1}^{N} \frac{\left| X_k^{0,0,(n)} - X_k^{b_1,b_2,(n)} \right|}{X_k^{0,0,(n)}},$$

respectively. The average approximation error per period is an additional number for illustration and can be derived from the above error measures by averaging over all periods.

For example, the average mean squared error (AMSE) is defined by

$$\mathrm{AMSE} = \frac{1}{K+1} \sum_{k=0}^{K} \mathrm{MSE},$$

and AMAE, AMSPE, and AMAPE analogously.

As a graphical visualization, we will plot the average development of X_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the same window, i.e. we plot

$$\frac{1}{N} \sum_{n=1}^{N} X_k^{0,0,(n)} \qquad \text{and} \qquad \frac{1}{N} \sum_{n=1}^{N} X_k^{b_1,b_2,(n)} \tag{2.4.12}$$

for $k = 0, \ldots, K$.

To investigate the robustness of the compression and simulation methods, we look at the worst-case approximation corresponding to that simulation path $n^* = n^*\left(\mathcal{P}^{b_1,b_2}\right) \in \{1, \ldots, N\}$ for which a chosen error measure is maximal. Again, different error measures can be constructed depending on the chosen norm. Here, we use the sum of absolute errors as an error measure so that n^* is given by

$$n^* = \operatorname{argmax} \left\{ \sum_{k=0}^{K} \left| X_k^{b_1,b_2,(n)} - X_k^{0,0,(n)} \right| : n \in \{1, \ldots, N\} \right\}. \tag{2.4.13}$$

All the above methods and error measures to investigate the approximation quality will be applied in Section 4.2 after having built a corresponding ALM model in Chapter 3.

We close this section by investigating the quality of the compression methods with respect to the characteristics of a given insured collective $\mathcal{P}^{0,0}$. The simulation of the initial insurance portfolio is based on the distributional assumptions of the biometric parameters described in Table 4.2 with $\delta_0 = 10,000$ signed contracts at time t_0, including new policies from the first period.

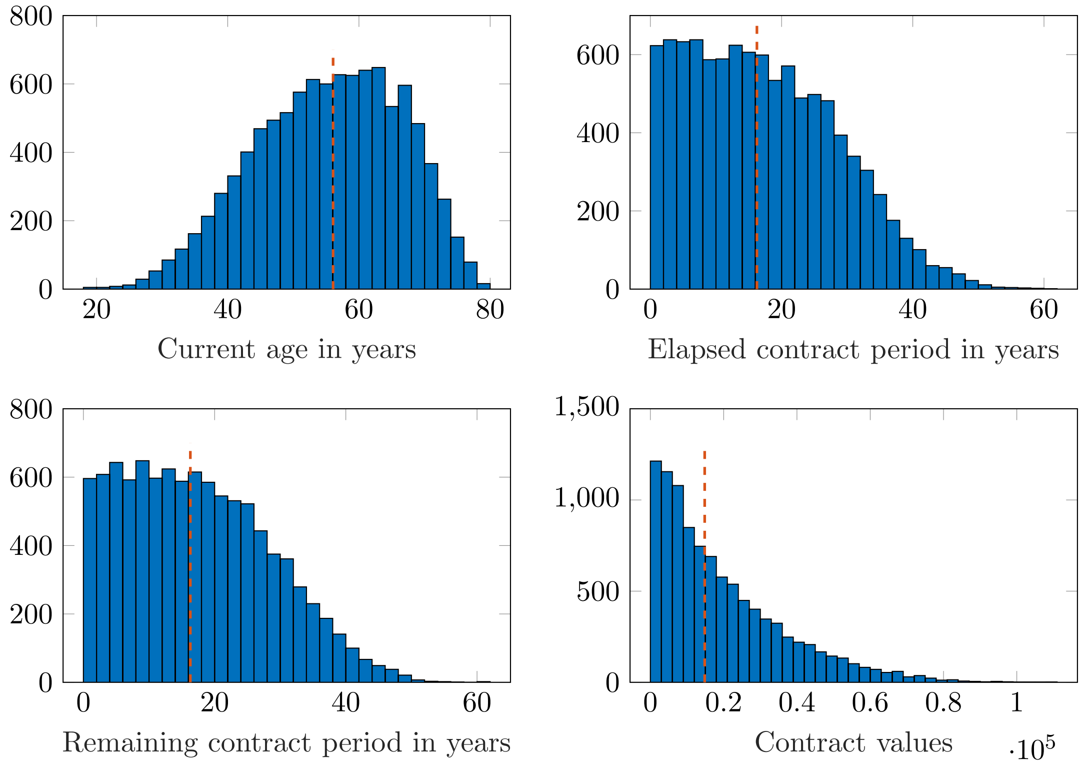

Figure 2.9.: Distribution of selected characteristics of the initial insurance portfolio $\mathcal{P}^{0,0}$ consisting of $\delta_0 = 10,000$ policies. The dashed lines correspond to the sample medians.

Some of the insurance portfolio's characteristics are illustrated in Figure 2.9, where we show the distributions of

- the current ages $x_0^{\mathrm{real},i}$,
- the elapsed contract durations $\left\lfloor x_0^{\mathrm{real},i} - \underline{x}^{\mathrm{real},i} \right\rfloor$,
- the remaining contract durations $\left\lceil \overline{x}^{\mathrm{real},i} - x_0^{\mathrm{real},i} \right\rceil$, and
- the contract values $V_0^{\mathrm{real},i}$.

The premium sizes are uniformly distributed on $[50, 500]$. The corresponding sample median is 276.05.

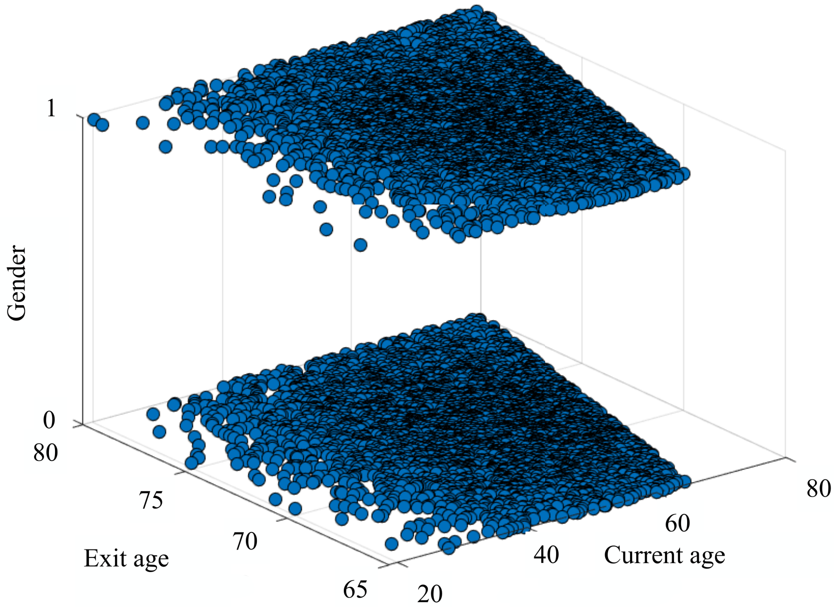

Figure 2.10.: Illustration of the uncompressed insurance portfolio regarding the grouping criteria. Each point corresponds to a contract, here characterized by the policyholder's current age $x_0^{\mathrm{real},i}$, exit age $\overline{x}^{\mathrm{real},i}$, and gender $g^{\mathrm{real},i}$.

Figure 2.10 illustrates the initial insurance portfolio with respect to the policyholders' current age $x_0^{\mathrm{real},i}$, exit age $\overline{x}^{\mathrm{real},i}$, and gender $g^{\mathrm{real},i} \in \{0,1\}$. Each point corresponds to an individual contract. Regarding the policyholders' ages, we have the constraints $\underline{x}^{\mathrm{real},i} \in [15,65]$, $\overline{x}^{\mathrm{real},i} \in [65,80]$, and $x_0^{\mathrm{real},i} \in \left[\underline{x}^{\mathrm{real},i}, \overline{x}^{\mathrm{real},i}\right]$.

In the following, the given initial insurance portfolio $\mathcal{P}^{0,0}$ is compressed applying different methods $\mathcal{P}^{b_1,b_2}$. Here, we consider

$$(b_1,b_2) \in \{(1,1),(2,2),(5,5),(1,10),(10,1),(10,10)\}\,.$$

In Figure 2.11, we illustrate the obtained compressed insurance portfolios $\mathcal{P}^{b_1,b_2}$. Each point corresponds to the representative contract from a cohort $m \in \{1,\ldots,M_0\}$. The number of contracts that needs to be simulated is substantially decreased, from $\delta_0 = 10{,}000$ to e.g. $M_0 = 1{,}339$ and $M_0 = 26$ if we apply $\mathcal{P}^{1,1}$ and $\mathcal{P}^{10,10}$ corresponding to a reduction of 86.61% and 99.74%, respectively. At the same time, the existing structure is maintained.

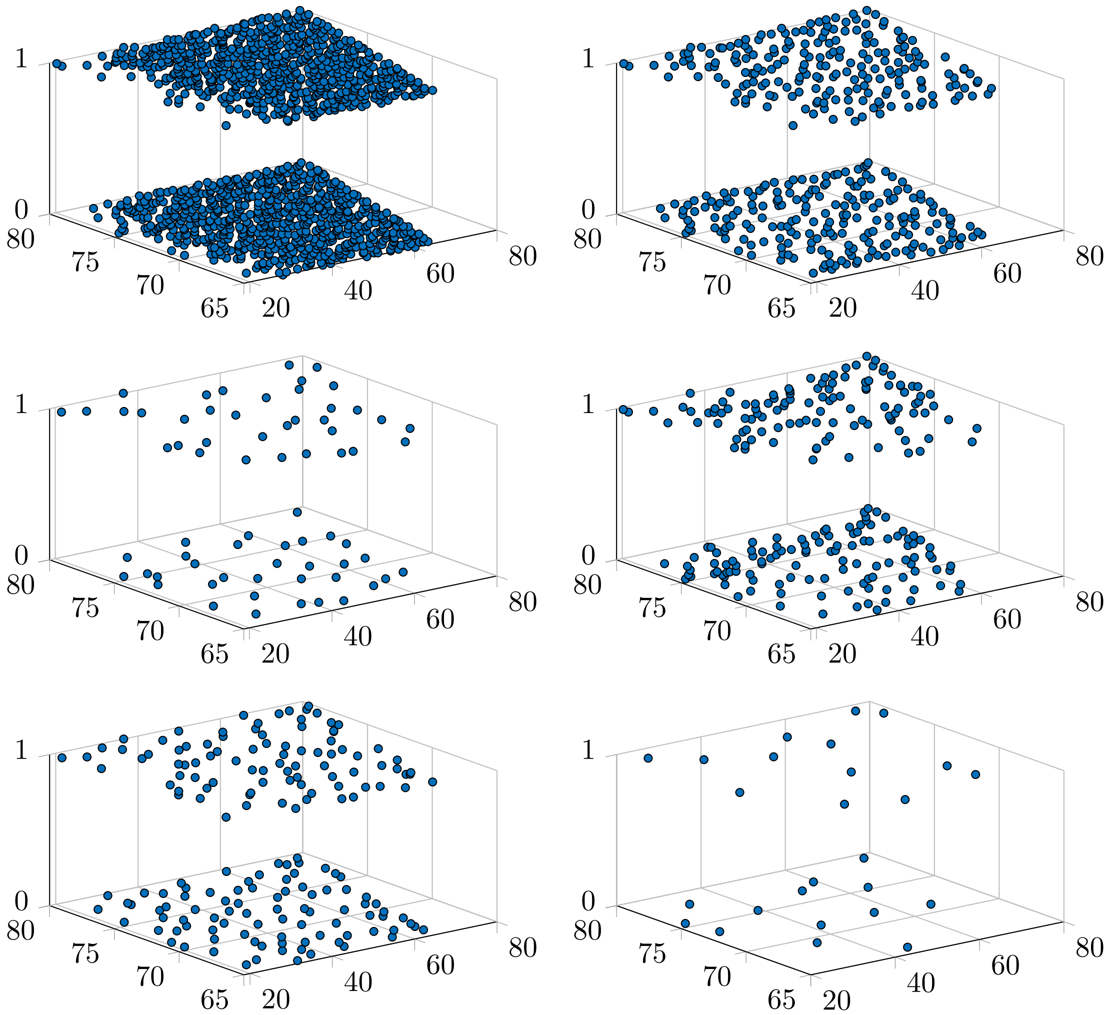

Figure 2.11.: Illustration of the insurance portfolio from Figure 2.10 if applying different compression methods $\mathcal{P}^{b_1,b_2}$. Each point corresponds to the representative contract from a cohort $m \in \{1,\ldots,M_0\}$, here characterized by the representative's current age x_0^m, exit age $\overline{x}^m$, and gender g^m. Top-left: $\mathcal{P}^{1,1}$, $M_0 = 1,339$. Top-right: $\mathcal{P}^{2,2}$, $M_0 = 406$. Middle-left: $\mathcal{P}^{5,5}$, $M_0 = 69$. Middle-right: $\mathcal{P}^{1,10}$, $M_0 = 225$. Bottom-left: $\mathcal{P}^{10,1}$, $M_0 = 176$. Bottom-right: $\mathcal{P}^{10,10}$, $M_0 = 26$.

In the following, we measure the quality of the compression methods regarding a characteristic X in terms of the root mean squared error (RMSE). It is defined by

$$\mathrm{RMSE} = \sqrt{\frac{1}{\delta_0}\sum_{i=1}^{\delta_0}\left(X^{\mathrm{real},i} - X^{\pi_0(i)}\right)^2},$$

where π_0 denotes the surjective function from Section 2.4.1 with $\pi_0(i) = m$ if policyholder i is sorted into cohort m according to a compression method $\mathcal{P}^{b_1,b_2}$. Recall that it is assumed that all contracts within a model point coincide with the exemplary one, i.e. we have $X^{\text{real},i} = X^m$ for all i with $\pi_0(i) = m$.

X	$\mathcal{P}^{1,1}$	$\mathcal{P}^{2,2}$	$\mathcal{P}^{5,5}$	$\mathcal{P}^{1,10}$	$\mathcal{P}^{10,1}$	$\mathcal{P}^{10,10}$
Premium size $P_0^{\text{real},i}$	84.71	105.68	120.88	111.41	114.15	124.26
Entry age $\underline{x}^{\text{real},i}$	4.11	4.97	5.42	5.13	5.27	5.57
Current age $x_0^{\text{real},i}$	0.09	0.23	0.74	0.22	0.69	2.13
Exit age $\overline{x}^{\text{real},i}$	0.04	0.12	0.78	0.50	0.15	1.58
Remaining contract period	0.11	0.26	1.09	0.74	0.67	2.70
Contract value $V_0^{\text{real},i}$	6659	7579	8015	7664	8341	8821

Table 2.1.: RMSE of selected characteristics of the insurance portfolio for different compression methods $\mathcal{P}^{b_1,b_2}$.

Table 2.1 summarizes the results for several characteristics. Clearly, the method $\mathcal{P}^{1,1}$ performs best. We can also see that the approximation error is much smaller for the grouping criteria. Nevertheless, also for the remaining characteristics, the RMSE is small compared to the corresponding (sample) medians.

Note that for all grouping schemes $\mathcal{P}^{b_1,b_2}$, there is no approximation error regarding the aggregated premium payments and contract values at time t_0, i.e. we have

$$\sum_{m=1}^{M_0} \delta_0^m \cdot P_0^m = \sum_{i=1}^{\delta_0} P_0^{\text{real},i}$$

and

$$\sum_{m=1}^{M_0} \delta_0^m \cdot V_0^m = \sum_{i=1}^{\delta_0} V_0^{\text{real},i}.$$

Throughout the simulation, i.e. at times t_k with $k \geq 1$, this will not be the case as we will see in Section 4.2.

3. A General ALM Model

In this chapter, we introduce a general asset-liability management (ALM) model which will also be the basis for many simulation studies later. In Section 3.1, we give a short introduction to the wide-spread field of asset-liability management and discuss ALM models applied in the existing literature. The structure of our ALM model is displayed in Section 3.2, where we also describe relevant risk drivers we take into account. The model consists of different, interacting parts which can be characterized as internal and external sub-models. An example for the latter is the applied capital market model which we introduce in Section 3.3. Sections 3.5, 3.6, and 3.7 correspond to the internal models, where the (management of the) life insurance company decides over the applied strategies for investing in the capital market and for financing the due obligations, over the use of surpluses, over the product design, and, to a certain extent, over the rules of accounting for preparing the balance sheet introduced in Section 3.4. In this context, modeling the structure and the dynamics of life insurance portfolios (cf. Chapter 2) would correspond to an external sub-model.

Regarding the notation, we use an upper index m referring to a quantity of model point m. For example in Section 3.6, P_0^m denotes the premium size of the representative of cohort m for the first period. However, our model can also be applied to uncompressed insurance portfolios, for which the individual premium size of policyholder $i \in \{1, \dots, \delta_0\}$ for the first period could then be denoted by $P_0^{\mathrm{real},i}$ as done in Section 2.3.1.

Some parts of this chapter are already published in Diehl et al. [23], where we developed and illustrated a reduced version of the following ALM model. In addition to the consideration of more general capital market models and the development of an alternative financing strategy, another substantial extension is the incorporation of a minimum stock share that might be part of the management's targets. Accordingly, many quantities are now defined differently and corresponding main results required new proofs. In this regard, the model presented in Diehl et al. [23] can be seen as a special case of the general ALM model introduced in this chapter.

3.1. ALM and ALM modeling

3.1.1. Introduction

Following the definition of Wagner [55], asset-liability management (ALM) can be seen as the goal-driven coordination of assets and liabilities of a balance sheet. The investments in the capital market need to be reconciled with the obligations induced by the insurance products such that claims can be met when they are due. An important goal consists of managing the financial stability by controlling the taken risk positions. Based on that, a second goal is the managing of the profitability by optimizing the taken risks and the resulting returns. The controlled risk-return positions involve levels of the contracts, the investments, and the whole management of the insurance company.

While the importance of a successful ALM is similar for both, banks and insurance companies, the requirements and the implementations differ as their business models and risk profiles do, cf. Insurance Europe [35]. Since the core activity of insurers is risk pooling and risk transfer, the insurer is exposed to risks depending on both assets and liabilities and the way they interact. Indeed, there are strong interactions between assets and liabilities of a life insurer's balance sheet. For example, the size of the guaranteed interest rate and the applied surplus participation scheme impose requirements on the risk-return profiles of the investments in the capital market. On the other hand, the financing and profitability of the signed contracts depend on the development of the capital market and the applied investment strategies. In this thesis, we elaborate on these interactions. An important task will be the derivation of robust and well-performing investment and financing strategies which take into account the obligations induced by the insured collective.

An appropriate ALM will result in an insurer controlling its assets and liabilities in such a way that e.g. the default probability for a certain period of time stays below a predefined threshold, see Gründl et al. [31]. Here, the application of stochastic simulations can support managerial decisions by illustrating the long-term effects of potential measures. This requires the development of suitable (ALM) models.

3.1.2. Literature overview

In the following, we give a short overview of the work that has been already done within the wide-spread field of asset-liability management and stochastic ALM-modeling. We also point out some of the typical assumptions.

Many papers focus on the valuation of insurance contracts, see e.g. Bauer et al. [3], Grosen and Jørgensen [29], Hieber et al. [34], or Zaglauer and Bauer [57] and the references therein. Conditions for fair prices are derived by calculating discounted expectations of the final benefit payments under a risk-neutral measure Q. For doing so, one typically needs strong simplifications. Examples are the consideration of single insurance contracts or one cohort of identical policies, the restriction to one lump-sum premium payment at the contract's inception, and the negligence of surrender.

Further ALM models are introduced in, e.g., Bohnert and Gatzert [9], Bohnert et al. [10], Burkhart et al. [13, 14], Fernández et al. [25], Gerstner et al. [27], Kling et al. [40, 41], and Kok et al. [42]. Most of them restrict to run-off companies, i.e. to firms that are specialized in the processing of existing contracts without issuing new policies. Exceptions are Kling et al. [40, 41], looking at a life insurer in a "steady state" where a constant fraction of the liabilities is paid out as benefit payments every year. However, they consider neither mortality nor surrender effects, justifying this by the assumption that new business roughly compensates for withdrawals and surrender thus does not influence the amount of assets and liabilities. This appears to be a rather strong assumption, as individual contract characteristics, biometric parameters, and the policyholders' behavior are not taken into account. Another exception is Burkhart et al. [13] who analyze the impact of new business on the liabilities within a risk-neutral setting. Their insurance portfolio consists of different cohorts of identical policies. However, it is rather homogeneous than heterogeneous as each cohort represents all contracts concluded within a specific year. Such a grouping scheme disregards the diverse nature of an insured collective. Hieber et al. [34] emphasize the importance of the heterogeneity aspect within life insurance portfolios. Indeed, the assumption of a single contract or cohort structure neglects the fact that the joint management of different contracts leads to interactions between existing and newly signed policies, e.g., due to management rules, claims on the same bonus reserve, or individual surrender options. Gerstner et al. [27] present a quite general ALM model where a run-off company manages the processing of a heterogeneous insurance portfolio represented by a (reduced) number of model points. It is not shown how these are generated but the authors sketch possible grouping criteria. For their numerical investigations, they simulate the needed data for the 500 cohorts each consisting of 100 identical policies.

3.2. Introduction of our ALM model

As stated by Albrecher et al. [1], it is crucial to address the imperfectness of an introduced ALM model and its sensitivity. Like for any mathematical model, it requires the right balance between simplicity and accuracy of the applied models. In this work, we aim at a balance between tractability and taking into account relevant legal requirements in order to get meaningful simulation results on the long-term stability of the insurance business. Before presenting the overall structure of our ALM model, we shortly describe risk drivers we found relevant for an appropriate ALM model. We also state how these are taken into account in this thesis.

3.2.1. Considered risk factors

Insurance companies take risks on behalf of their policyholders in exchange for premium payments, and can thus be seen as liability-driven financial intermediaries, see Albrecher et al. [1]. Accordingly, insurers are mainly exposed to actuarial risk, market risk, and the risk of mismatch between assets and liabilities, see Insurance Europe [35]. The latter is especially critical for life insurers writing long-term business. We take that risk into account by developing investment and financing strategies that are better orientated to the obligations induced by the insurance contracts than other prominent strategies. Indeed, Albrecher et al. [1] find that the liability structure is the benchmark for a successful asset-liability management. In the following, we address the other two main risk factors.

Actuarial risk

The actuarial risk contains the threat of the insurer's technical ruin resulting from the combination of predetermined premiums and the inherent randomness regarding the future obligations, see Wagner [55]. It can be divided into the underwriting risk, which represents the uncertainty of the total claim amount, and the timing risk, which reflects the uncertainty regarding the time at which benefits are due. In this thesis, we estimate the default probability representing a measure for the overall risk and investigate impacting factors. We follow different approaches to reduce the life insurer's risk exposure and to increase the profitability. Even though the benefit payments of the policies considered in this thesis result from the accumulated contributions, they are random and not fully funded. Reasons for this are premium guarantees, interest rate promises, and entitlements to the allocated surpluses. Furthermore, by not restricting to run-off companies but considering ongoing insurance business with different new business scenarios, we increase the uncertainty of the future obligations and thus attach greater value to the underwriting risk. As parts of the actuarial risk, further included risk drivers are the mortality, the longevity, and the surrender risk described in the following. These biometric risks are also considered by Christiansen et al. [16] in a Solvency II framework.

Actuarial risk: mortality and longevity

Mortality is a natural reason for an early termination of the insurance contract. Depending on the insurance type and the specific policy design, early death of a substantial number of policyholders may threaten the insurers liquidity and expected profitability. As already written in Section 2.3.2, we take mortality into account. In addition, the model's flexibility allows for incorporating mortality shocks and the application of stochastic mortality models. These approaches cover the modeling of longevity, i.e. the risk that a substantial part of the insured collective lives much longer than expected. The latter is especially critical for insurance products with a (life-long) pension phase. Such policies are not considered here, but could easily be incorporated.

Actuarial risk: surrender

As already indicated in Section 2.3.2, the life insurance contracts considered in this thesis are equipped with surrender options. If exercised, the insured person receives the cash surrender value, i.e. the contract value reduced by a surrender fee. Depending on the specific contract features and the current situation of the capital market, cancellations may lead to the insurer earning money, e.g. due to surrender fees or since policyholders renounce to (parts of) the surplus participation. Nevertheless, cancellations trigger unexpected cash flows and can thus also negatively impact the insurer's profit and asset-liability management. To meet the due surrender benefit payments, the insurer might be forced to borrow money at potential higher costs or to sell assets at bad times. Especially the event of mass cancellations can cause severe liquidity strains and thus may threaten the stability or even the solvency of the life insurance business. Empirical evidence for such *insurance runs* in the past regarding the U.S. and South Korean life insurance markets are provided in Kubitza et al. [46]. Actual concerns can be found in the latest review of the stability of the German financial system, see Deutsche Bundesbank [20]. Furthermore, the surrender risk is also a risk factor in the stress tests developed by the European Insurance and Occupational Pension Authority (EIOPA). Milhaud and Dutang [48] even state that, nowadays, the surrender risk is one of the most important risk factors life insurers need to consider.

Remark 3.2.1 (Exemption of premium payments)**.** Another prominent contract feature related to surrender options is exemption of premium payments. Here, policyholders decide to stop paying premiums but the contract remains active in a frozen state with reduced entitlements (due to missing premium payments). In the future, the policyholder might also cancel her decision and restart paying premiums. Surrendering and stopping premium payments both belong to the behavioral risk but the consequences of the former are more severe. Therefore, in the model developed in this thesis, we focus on surrender options. The extension of incorporating exemption of premium payments as an additional contract feature is straightforward.

Market risk

Under market risk we understand the risk of price changes which lead to the insurer making losses from the investments. We take the market risk into account by choosing stochastic models for the simulation of the capital market. In addition, we apply a stochastic volatility model for the stock price dynamics and we allow for crashes in the stock and bond markets. The latter covers the default risk for asset holdings which is an important risk factor according to Deutsche Bundesbank [20] and which potentially arises from an increased search for yield by accepting higher risks. A substantial part of the market risk is the interest rate risk.

Market risk: interest rate risk

As long-term investors, life insurers are especially vulnerable to changes in interest rates. In times of low interest rates, it is difficult for them to obtain sufficient returns on the managed funds in order to meet obligations resulting from (the partly high) guarantees promised to the insured. Rising interest rates would therefore ease pressures in the long term and improve solvency ratios, see Deutsche Bundesbank [20]. However, a very sharp rise in interest rates could impose liquidity risks for life insurers in the short term, since the market value of their assets could decease substantially. As a result, capital requirements might not be fulfilled anymore. Furthermore, according to the interest rate hypothesis (see the discussion in Section 2.3.2), surrender rates are positively correlated with interest rates. A sudden increase of interest rates could therefore significantly raise surrender rates, potentially yielding to insurance runs in the most extreme case, cf. Kubitza et al. [46]. In the simulation studies, we consider different scenarios regarding the future development of interest rates and measure their impact on the default probability and the surplus participation.

3.2.2. Structure of our ALM model

The model consists of different, interacting parts introduced and illustrated in Figure 3.1. The modular framework enables the realization of alternative modeling approaches or the adaption to different insurance products. In this work, we especially consider different investment and financing strategies, alternative patterns of new contract arrivals, and varying capital market models where we also allow for crashes.

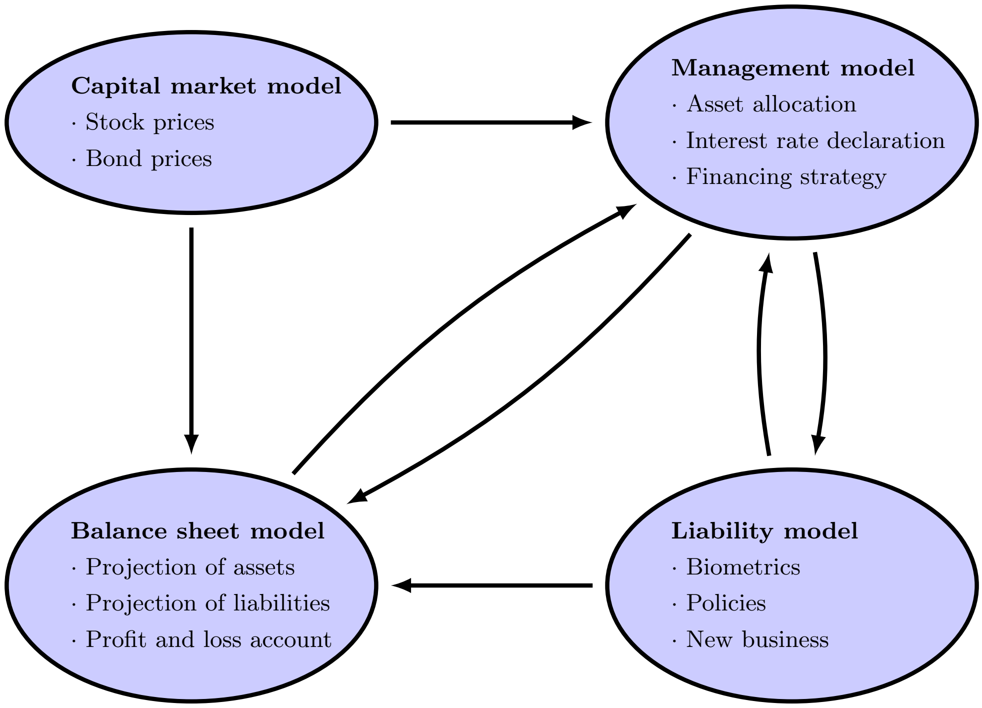

Figure 3.1.: Overall structure of the ALM model.

The capital market is modeled by short rate and stock price processes and the insurer's financial assets are presented in simplified form by stocks and bonds. On the counterpart, we have liabilities formed by the insurance contracts comprising existing and new business. In particular, our model is not confined to run-off companies. The management model describes the insurance company's decisions regarding the declaration of the bonus payments and of the interest rate for the next year, the asset allocation, and the applied methods to finance the periodic disbursements consisting of due obligations and expiring credits. The arrows in Figure 3.1 illustrate the corresponding dependencies. For example, the obtained surpluses, which are calculated within the profit and loss account, have an impact on the future interest rate declaration and thus on the development of the policyholders' contracts. To give another example, the new business has a substantial influence on the balance sheet's structure and is necessary for the objective of a long-term stability.

3.3. Capital market model

This section deals with the simulation of the capital market providing investment opportunities for the strategies developed in Section 3.5.1. We first introduce a basic capital market model, where stock prices follow a geometric Brownian motion as in the Black-Scholes model. However, since life insurance products are typically long-term agreements and contract periods of 50 years or more are possible, the assumption of a constant interest rate is not reasonable. Instead, the instantaneous risk-free interest rate, also referred to as short rate, needs to be modeled by a (correlated) stochastic process. Motivated by empirical observations of financial markets, we then introduce more sophisticated capital market models. In particular, we consider a stochastic volatility model for stock prices and allow for crashes in the bond and stock markets in our simulation studies.

Since we want to model the actual development of the capital market, all stochastic processes are simulated under the physical measure P.

3.3.1. Basis capital market model

For life insurers, the two most important markets to invest in are the bond and the stock markets. Especially for long running contracts like pension products, bond investments are dominant, see Korn and Wagner [44]. A special class of bonds are zero-coupon bonds. For the following definitions and the formula for pricing zero-coupon bonds, we refer to Brigo and Mercurio [11].

Definition 3.3.1 (Zero-coupon bond)**.** A zero-coupon bond with maturity T is a contract that guarantees its holder the payment of one unit of money at time T. Before maturity there are no intermediate payments. The price at time t is denoted by $p\,(t,T)$.

We say the bond is in default if the issuer cannot fulfill the obligation. If not stated otherwise, we consider default-free zero-coupon bonds and just call them bonds. By no-arbitrage arguments, we have $p\,(T,T)=1$. For $t<T$, the time-t-price $p\,(t,T)$ is a random variable depending on the short rate $r(t)$ defined in the following.

Definition 3.3.2 (Short rate)**.** Let $t \leq S < T$.

1. The time-t-forward rate $F\,(t,S,T)$ on $[S,T]$ is given by

$$\exp\left\{F\,(t,S,T)\,(T-S)\right\} = \frac{p\,(t,S)}{p\,(t,T)}.$$

2. The time-t-instantaneous forward rate for investment at T is

$$f\,(t,T) = \lim_{\Delta t \downarrow 0} F\,(t,T,T+\Delta t)$$

and $r\,(t) = f\,(t,t)$ the short rate.

Assuming the existence of a risk-neutral measure Q, the time-t price $p\,(t,T)$ of a zero-coupon bond with maturity T is given by the conditional expectation of the final payoff $p\,(T,T)=1$ under that measure, i.e.

$$p\,(t,T) = \mathbb{E}^Q\left[\exp\left\{-\int_t^T r(s)\mathrm{d}s\right\}\middle|\,\mathcal{F}\,(t)\right],$$

where $\mathcal{F}(t)$ models the information up to time t. In order to compute the conditional expectation, we need to choose a model for the dynamics of r characterizing the distribution of $\exp\left\{-\int_t^T r(s)\mathrm{d}s\right\}$. Here, we choose the Vasiček model introduced in Vasicek [54], where the short rate dynamics are given by

$$\mathrm{d}r(t) = (b - ar(t))\,\mathrm{d}t + \sigma_r \mathrm{d}W_r(t)\,, \tag{3.3.1}$$

for constants $b \in \mathbb{R}$, $a > 0$, $\sigma_r > 0$, and a Brownian motion W_r under the real-world measure P. The stochastic process solving (3.3.1) is an Ornstein-Uhlenbeck process. With Itô's product rule, we get

$$\begin{aligned}\mathrm{d}\left(\mathrm{e}^{at}r(t)\right) &= \mathrm{e}^{at}\mathrm{d}r(t) + r(t)\mathrm{d}\mathrm{e}^{at} \\ &= \mathrm{e}^{at}\left(b\mathrm{d}t - ar(t)\mathrm{d}t + \sigma_r\mathrm{d}W_r(t) + ar(t)\mathrm{d}t\right),\end{aligned}$$

implying

$$\mathrm{e}^{at}r(t) = r(0) + \int_0^t b\mathrm{e}^{au}\mathrm{d}u + \int_0^t \mathrm{e}^{au}\sigma_r\mathrm{d}W_r(u)$$

and thus

$$r(t) = \frac{b}{a} + \left(r(0) - \frac{b}{a}\right)\mathrm{e}^{-at} + \sigma_r\int_0^t \mathrm{e}^{-a(t-u)}\mathrm{d}W_r(u)\,.$$

For $s \le t$, we can rewrite the last equation to

$$r(t) = \frac{b}{a} + \left(r(s) - \frac{b}{a}\right)\mathrm{e}^{-a(t-s)} + \sigma_r\int_s^t \mathrm{e}^{-a(t-u)}\mathrm{d}W_r(u)\,. \tag{3.3.2}$$

Therefore, the short rate $r(t)$, conditioned on $\mathcal{F}(s)$, $s \le t$, is normally distributed satisfying

$$\mathbb{E}\left[r(t)|\,\mathcal{F}(s)\right] = \frac{b}{a} + \left(r(s) - \frac{b}{a}\right)\mathrm{e}^{-a(t-s)} \tag{3.3.3}$$

and

$$\begin{aligned}\mathrm{Var}\left(r(t)|\,\mathcal{F}(s)\right) &= \sigma_r^2\mathbb{E}\left[\left.\left(\int_s^t \mathrm{e}^{-a(t-u)}\mathrm{d}W_r(u)\right)^2\right|\mathcal{F}(s)\right] \\ &= \sigma_r^2\mathbb{E}\left[\left.\int_s^t \mathrm{e}^{-2a(t-u)}\mathrm{d}u\right|\mathcal{F}(s)\right] = \frac{\sigma_r^2}{2a}\left(1 - \mathrm{e}^{-2a(t-s)}\right),\end{aligned}$$

where we used the Itô isometry for calculating the conditional variance. Since the short rate is normally distributed, the Vasiček model allows for negative interest rates. In the current period of very low interest rates, this can be seen as an advantage. From equation (3.3.3), we conclude that the short rate is mean reverting with a reflecting the pace of the mean reversion and $\frac{b}{a}$ representing the long-term mean. In the Vasiček model, bond prices can be computed according to

$$p\left(t,T\right)=\exp\left\{A(t,T)-B(t,T)r(t)\right\},$$

for deterministic functions

$$\begin{aligned}B(t,T)&=\frac{1}{\tilde{a}}\left(1-\mathrm{e}^{-\tilde{a}(T-t)}\right)\\ A(t,T)&=\left(B\left(t,T\right)-\left(T-t\right)\right)\left(\frac{\tilde{b}}{\tilde{a}}-\frac{\sigma_r^2}{2\tilde{a}^2}\right)-\frac{\sigma_r^2}{4\tilde{a}}B(t,T)^2.\end{aligned}$$

Here, $\tilde{a}$ and $\tilde{b}$ are the short rate parameters under the equivalent martingale measure Q defined by choosing a specific functional form of the market price of interest rate risk process $\lambda\left(t\right)$ which characterizes the Radon-Nikodym density

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}(t)}=\exp\left\{-\frac{1}{2}\int_0^t\left(\lambda\left(s\right)\right)^2\mathrm{d}s-\int_0^t\lambda\left(s\right)\mathrm{d}W_r\left(s\right)\right\}.$$

By the Girsanov theorem, $W_r^Q\left(t\right)=W_r\left(t\right)+\int_0^t\lambda\left(s\right)\mathrm{d}s$ is then a standard Brownian motion under Q. There is no answer to the question on how to choose the specific functional form of $\lambda\left(t\right)$ in general. However, choosing $\lambda\left(t\right)$ to be any affine transformation of the short rate $r(t)$ leads to a stochastic differential equation for the short rate under Q with the same structure as (3.3.1). In particular, $r(t)$ is normally distributed and has the mean-reversion property also under Q. Brigo and Mercurio [11] and Kok et al. [42] follow a proportional approach. In this work, we choose

$$\lambda\left(t\right)=\lambda_0.$$

The parameters of the short rate under Q are then given by $\tilde{b}=b-\lambda_0\sigma_r$ and $\tilde{a}=a$. Therefore, the mean-reversion speed is maintained which motivates our choice. Note that the short rate's volatility is not affected by the change of measure.

Stock prices $s(t)$ follow a geometric Brownian motion,

$$\mathrm{d}s(t)=s(t)\left(\mu_s\mathrm{d}t+\sigma_s\mathrm{d}W_s(t)\right),\tag{3.3.4}$$

with drift $\mu_s\in\mathbb{R}$, volatility $\sigma_s>0$, and $W_s(t)=\rho W_r\left(t\right)+\sqrt{1-\rho^2}Z(t)$ for a Brownian motion Z independent of W_r. Thus, $W_s(t)$ and $W_r\left(t\right)$ have correlation ρ. The stochastic differential equation has the well-known solution

$$s(t)=s(0)\exp\left\{\left(\mu_s-\frac{1}{2}\sigma_s^2\right)t+\sigma_sW_s(t)\right\}.\tag{3.3.5}$$

As a consequence, stock prices are log-normally distributed. In other terms, the logarithmic returns $\ln\left(\frac{s(t)}{s(s)}\right)$, $s<t$, are normally distributed.

Note that s could be a stock market index or the value of a whole stock portfolio. For simplicity, we just speak of one stock in the following.

Simulation of the basic capital market model

Since we know the exact solutions of the stochastic differential equations for the short rate and stock price dynamics, we simulate the short rate and the stock prices directly by equations (3.3.2) and (3.3.5). For this, we apply the Euler-Maruyama method using the discretization of time (2.3.1) introduced in the previous chapter. Writing $r_k = r(t_k)$, $s_k = s(t_k)$, the applied recursions are given by

$$\begin{aligned} r_k &= \frac{b}{a} + \left(r_{k-1} - \frac{b}{a}\right) \mathrm{e}^{-a\Delta t} + \sigma_r \sqrt{\frac{1-\mathrm{e}^{-2a\Delta t}}{2a\Delta t}} \Delta_k W_r, \\ s_k &= s_{k-1} \exp\left\{ \left(\mu_s - \frac{1}{2}\sigma_s^2\right) \Delta t + \sigma_s \Delta_k W_s \right\}, \end{aligned}$$

with $\Delta_k W_s = \rho \Delta_k W_r + \sqrt{1-\rho^2} \Delta_k Z$ and independent $\Delta_k W_r, \Delta_k Z \sim \mathcal{N}(0, \Delta t)$. The values at times $t \neq t_k$ are obtained by corresponding linear interpolations.

3.3.2. More general capital market models

In the Black-Scholes model, stock prices follow a geometric Brownian motion as in our basic capital market model. An essential drawback of this modeling approach is the assumption of a constant volatility which is not consistent with observed market values of traded options. To see this, one can compute the implied volatility, i.e. the volatility which would be needed to obtain observed market prices of European call options with different maturities and strikes according to the Black-Scholes formula established by Black and Scholes [6]. If stock prices would follow the Black-Scholes model, the implied volatility would be constant over different strikes. But often it is not (*smile effect*). Empirical studies strongly contradict the assumption of a constant volatility, see e.g. Dumas et al. [24] or Desmettre et al. [18]. Furthermore, observed characteristics of financial time series, which are known as stylized facts, are not consistent with the assumption of normally distributed log-returns of stock prices. Typically, negative news have a stronger influence on the volatility than positive ones which is called the leverage effect. Thus, changes in the price and volatility are most often negative correlated. In contrast to the symmetric normal distribution, observed logarithmic returns are often asymmetric distributed. As a consequence, the skewness being defined by

$$\gamma(X) = \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^3\right]}{(\mathrm{Var}(X))^{3/2}},$$

is non-zero. For a normally distributed random variable X, we have $\gamma(X) = 0$. Moreover, historical log-returns tend to have a leptokurtic distribution, i.e. the kurtosis defined by

$$\kappa(X) = \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^4\right]}{\mathrm{Var}(X)^2},$$

is larger than 3 which would be the value if X is normally distributed. This means, empirical returns have larger tails and more probability mass in the center.

In Figure 3.2, we display the daily closing prices s_k of the German stock index (DAX) from January 2017 to December 2021. The corresponding qq plot of the $K = 1,262$ daily log-returns

$$R_k = \ln\left(\frac{s_k}{s_{k-1}}\right), \; k = 1, \dots, K,$$

on the right indicates a leptokurtic distribution. If the log-returns would follow a normal distribution, the blue crosses would lie (approximately) on the red-dashed line.

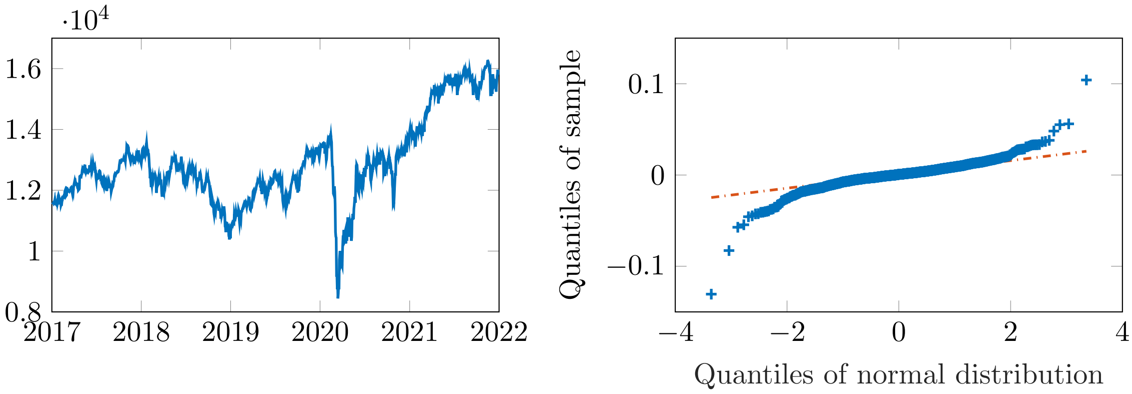

Figure 3.2.: Performance of DAX between January 2017 and December 2021 (left) and the corresponding qq plot of the daily log-returns (right).

The sample skewness $\widehat{\gamma}_K$ and the sample kurtosis $\widehat{\kappa}_K$ are defined by

$$\widehat{\gamma}_K = \frac{1}{\widehat{\mathrm{Var}}^{3/2}} \frac{1}{K} \sum_{k=1}^{K} \left(R_k - \bar{R}_K\right)^3 \qquad \text{and} \qquad \widehat{\kappa}_K = \frac{1}{\widehat{\mathrm{Var}}^{2}} \frac{1}{K} \sum_{k=1}^{K} \left(R_k - \bar{R}_K\right)^4,$$

where

$$\bar{R}_K = \frac{1}{K} \sum_{k=1}^{K} R_k \qquad \text{and} \qquad \widehat{\mathrm{Var}} = \frac{1}{K-1} \sum_{k=1}^{K} \left(R_k - \bar{R}_K\right)^2$$

denote the sample mean and sample variance, respectively. For the above 5-year time horizon, we have $\widehat{\gamma}_K = -1.0$ and $\widehat{\kappa}_K = 21.3$. For a time horizon of January 1988 to December 2021, we compute an estimated value of -0.3 for the skewness and 9.8 for the kurtosis.

In the following, we introduce a popular stochastic volatility stock market model which respects the above stylized facts, see Heston [33], Desmettre et al. [18], or Desmettre and Korn [17].

Heston model

In the Heston model (cf. Heston [33]), the stock price and variance processes are given by the coupled system of stochastic differential equations

$$\mathrm{d}s(t) = s(t)\left(\mu_s \mathrm{d}t + \sqrt{\nu(t)}\mathrm{d}W_s(t)\right), \qquad s(0) = s_0, \tag{3.3.6}$$

$$\mathrm{d}\nu(t) = \kappa\left(\theta - \nu(t)\right)\mathrm{d}t + \sigma_\nu\sqrt{\nu(t)}\mathrm{d}W_\nu(t), \qquad \nu(0) = \nu_0, \tag{3.3.7}$$

with $\mathrm{Corr}\left(W_s(t), W_\nu(t)\right) = \rho_{s\nu}$. In this representation, μ_s is the drift of the stock prices, κ the pace of the reversion to the long-term mean $\theta > 0$ of the variance, and $\sigma_\nu > 0$ its volatility.

Remark 3.3.3 (Properties of the Heston model)**.** The process $\nu(t)$ solving equation (3.3.7) is a Cox-Ingersoll-Ross process being non-negative almost surely and having a non-central Chi-Squared distribution. It is strictly positive if the Feller condition

$$2\kappa\theta \geq \sigma_\nu^2$$

is satisfied. The variance process has the mean reversion property and the leverage effect is reflected by the correlation between the two driving Brownian motions which is typically negative. Sometimes we even have $\rho_{s\nu} \approx -1$.

Simulation of the Heston model

For the simulation of the Heston model, we discretize (3.3.7) and the stochastic differential equation of the log-stock price $X(t) = \ln(s(t))$ obtained by Itô's formula:

$$\mathrm{d}X(t) = \left(\mu_s - \frac{1}{2}\nu(t)\right)\mathrm{d}t + \sqrt{\nu(t)}\mathrm{d}W_s(t).$$

By this, we reduce the discretization error and stock prices are then given by $s(t) = \mathrm{e}^{X(t)}$. While the continuous-time process $\nu(t)$ is always non-negative under the Feller condition, yet simulated approximating paths can get negative. In that case, the root expressions become complex and useless for the following iterations. Desmettre et al. [18] apply different methods dealing with this issue in the context of pricing an European call option. In this thesis, we use the reflection principle, i.e. we use the absolute value of the variance terms in the recursions:

$$X_k = X_{k-1} + \left(\mu_s - \frac{1}{2}\left|\nu_{k-1}\right|\right)\varDelta t + \sqrt{|\nu_{k-1}|}\Delta_k W_s,$$
$$\nu_k = |\nu_{k-1}| + \kappa\left(\theta - |\nu_{k-1}|\right)\varDelta t + \sigma_\nu\sqrt{|\nu_{k-1}|}\Delta_k W_\nu,$$

with $\Delta_k W_\nu = \rho_{s\nu}\Delta_k W_s + \sqrt{1-\rho_{s\nu}^2}\Delta_k Z_\nu$ and independent $\Delta_k W_s, \Delta_k Z_\nu \sim \mathcal{N}(0, \varDelta t)$. The stock price s_k at time t_k is then given by $s_k = \exp\{X_k\}$.

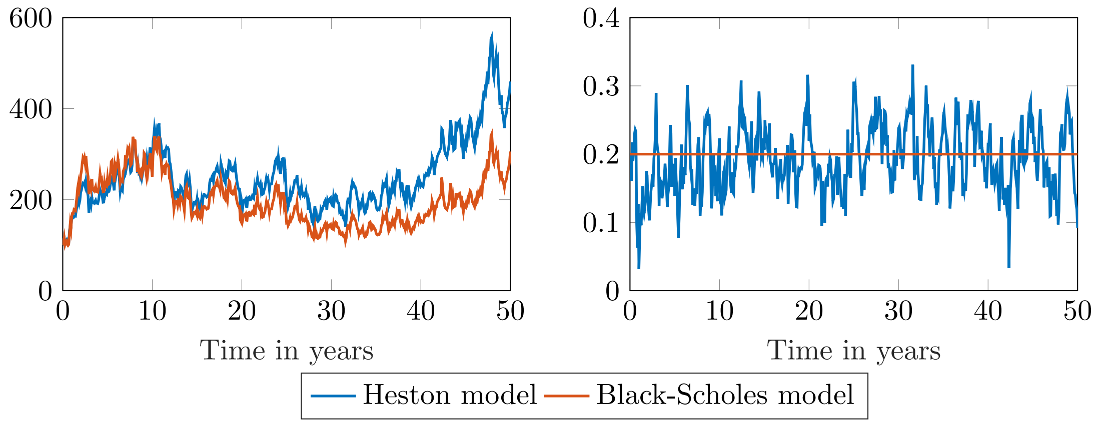

Figure 3.3.: Simulated stock prices (left) and volatility (right) in the Heston model and the Black-Scholes model. Parameters: $\mathcal{T} = 50$, $\varDelta t = \frac{1}{12}$, $s_0 = 100$, $\mu_s = 0.05$, $\sigma_s = 0.2$, $\nu_0 = 0.04$, $\kappa = 2$, $\theta = 0.04$, $\sigma_\nu = 0.2$, and $\rho_{s\nu} = -0.9$.

Figure 3.3 displays simulated stock prices in the Black-Scholes and in the Heston model (left-hand side) and the corresponding volatility (right-hand side), where we used the same parameters and the same sequence of generated random numbers for simulating the Brownian motion W_s. Therefore, a direct comparison is possible. In the Heston model, the volatility oscillates around the long-term mean $\sqrt{\theta} = \sqrt{0.04}$ which coincides with the constant volatility $\sigma_s = 0.2$ in the Black-Scholes model.

Model	Sample skewness	Sample kurtosis
Black-Scholes	0.0 (0.1)	3.0 (0.1)
Heston	0.1 (0.1)	4.4 (1.0)

Table 3.1.: Estimated mean and standard deviation (in brackets) of the sample skewness and the sample kurtosis in the Black-Scholes and in the Heston model. The estimations are based on $N = 10,000$ simulated paths of stock prices.

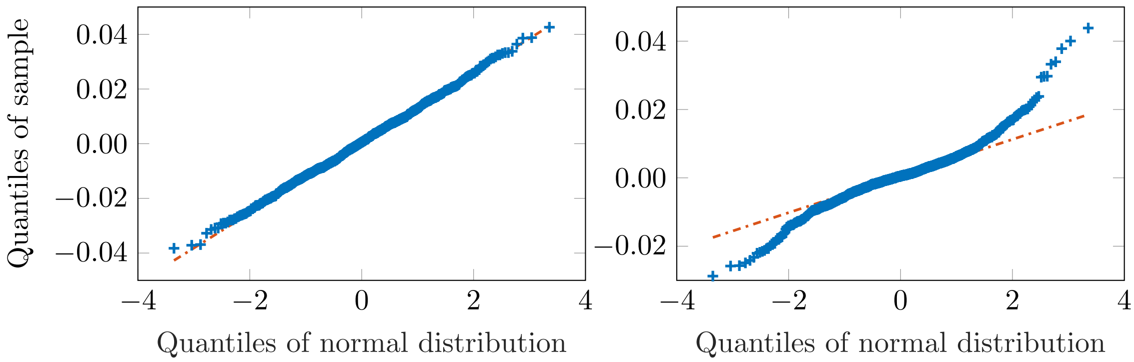

Figure 3.4.: QQ plots of log-returns of a simulated path of stock prices in the Black-Scholes model (left) and in the Heston model (right). Parameters: $\mathcal{T} = 5$, $\Delta t = \frac{1}{252}$, $s_0 = 100$, $\mu_s = 0.05$, $\sigma_s = 0.2$, $\nu_0 = 0.04$, $\kappa = 0.5$, $\theta = 0.04$, $\sigma_\nu = 0.2$, and $\rho_{s\nu} = -0.9$.

Figure 3.4 shows the qq plots of the log-returns of simulated stock prices in the Black-Scholes model (left-hand side) and in the Heston model (right-hand side). Due to the normally distributed log-returns in the Black-Scholes model, the empirical quantiles approximately coincide with the quantiles of a normal distribution. In contrast, the right-hand side looks similar to the qq plot of the daily DAX returns in Figure 3.2, which indicates that the Heston model allows for a leptokurtic distribution of log-returns. This is also indicated in Table 3.1, where we estimate the mean and the standard deviation of the sample skewness and the sample kurtosis in the considered stock price models with the same parameters on the basis of $N = 10,000$ simulated paths.

Remark 3.3.4 (Crashes in the capital market)**.** Looking again at the DAX performance in Figure 3.2, we see another important characteristic missing in the capital market models so far. Due to the COVID-19 pandemic, the DAX lost over 34% within just four weeks. Such extreme events are referred to as crashes and will be considered in later simulation studies, where we investigate the robustness of the introduced investment strategies in the presence of crashes in both stock and bond markets. This approach covers the extension to corporate bond investments.

Remark 3.3.5 (Alternative short rate models)**.** As done for modeling the stock price, we could also apply any other short rate model due to the flexibility of our model. For example, the Cox–Ingersoll–Ross model, where short rates have dynamics as the variance in the Heston model in (3.3.7), also generates short rates with the mean-reversion property but does not allow for negative interest rates, see Brigo and Mercurio [11]. The Hull-White model extends the Vasiček model by allowing for non-constant parameters but the short rate is not mean-reverting anymore.

3.4. Balance sheet

We now turn to the internal components of our ALM model. We use the established discretization scheme, see (2.3.1), dividing the simulation period into K periods $[t_{k-1}, t_k]$, $k = 1, \ldots, K$, of equal length Δt.

Legal regulations require insurance companies to draw up a comparison between assets and liabilities, e.g., for the purpose of determining the available own funds.[1] This is done by preparing a balance sheet, which requires a cash flow statement and a profit and loss account in addition to evaluating all asset and liability positions. In this thesis, the balance sheet as displayed in Table 3.2 for time t_k forms the reference for the simulation of the asset-liability management.

Assets		Liabilities	
Bonds	C_k^b	Equity	Q_k
Stocks	C_k^s	Free reserve	F_k
Cash	C_k^c	Actuarial reserve	A_k
		Bonus reserve	B_k
		Liabilities to banks	L_k
Total	C_k	Total	C_k

Table 3.2.: Considered balance sheet at time t_k.

The total capital C_k of the assets is allocated to bonds of different times to maturity, stocks, and a cash position with market values C_k^b, C_k^s, and C_k^c, respectively. On the opposite side of the balance sheet the liabilities comprise the equity Q_k, the free reserve F_k, the technical reserve V_k consisting of the actuarial reserve and the bonus reserve,

$$V_k = A_k + B_k, \tag{3.4.1}$$

and the liabilities to banks L_k. The actuarial reserve A_k represents the obligations towards the policyholders arising from the guarantees embedded in the life insurance contracts. Its development over time is independent of the capital market's variations. Surpluses that have been credited to individual contracts become part of the guarantees and are accounted for by the bonus reserve B_k. In contrast, policyholders are not entitled to unappropriated and unallocated surpluses being registered in the free reserve. As the free reserve is not assigned to individual insured, it can be used to cover future losses under strict conditions.[2] Therefore, in addition to the shareholders' equities, it is part of the own funds $F_k + Q_k$ according to Solvency II. Finally, the insurance company can take loans which must be registered as liabilities to banks. Explicit representations for all balance sheet positions are derived in Section 3.7.

[1] In Germany, e.g., this is prescribed by the Insurance Supervision Law VAG in §74.

[2] In Germany, e.g., this possibility is permitted by the Insurance Supervision Law VAG in §140, and requires the approval of the corresponding regulating authorities.

Remark 3.4.1 (Choice of the period length)**.** In practice, the balance sheet is prepared annually implying $\Delta t = 1$. By law, an insurance company must have appropriate methods to detect a potential deterioration in its financial condition.[3] In order to detect earlier how potential measures could negatively impact the insurer's financial strength, we typically choose smaller values for the period length, e.g. $\Delta t = \frac{1}{12}$ or $\Delta t = \frac{3}{12}$.

Remark 3.4.2 (Fundamental balance sheet equation)**.** At the end of an accounting year, the sum of all liabilities needs to equal the sum of all assets. Due to the complexity of bookkeeping, simplified approaches to handle that fundamental balance sheet equation are applied in the literature. They are often associated with the principle of single-entry bookkeeping. By defining the equity (sometimes also called the reserve or buffer account) residually by the difference between assets (left side of the balance sheet) and liabilities (remaining accounts on the right side), the balance sheet equation is automatically fulfilled. In comparison, we aim at providing an even balance sheet model without assuming that this relationship holds by default. We explicitly prove that the fundamental balance sheet equation is fulfilled at the end of every period. This is in line with the principle of double-entry bookkeeping as required in accounting.

3.5. Management model

In our ALM model, the management of the life insurance company decides about the asset allocation, the surplus participation process, and the strategy of financing the periodic disbursements. The two former management tasks are typically taken into account in the corresponding literature (e.g. in most of the cited papers in Section 3.1.2) by specifying the investment strategy and the applied surplus participation scheme. However, to the best of our knowledge, there is less importance attached to the other part, i.e. it is often not clear what measures are taken by the management to meet the due obligations and in which order. In this thesis, we put more emphasis on this issue and introduce several strategies to finance the periodic disbursements.

Let us now fix a period $k \in \{1, \ldots, K\}$, i.e. we consider the time interval $[t_{k-1}, t_k]$.

3.5.1. Asset allocation

Life insurers are financial intermediaries that invest the obtained premium revenues in the capital market. The investments must be in conformity with the principle of prudent business such that the taken risk positions can be controlled and due obligations can be met. We take this requirement into account by introducing a maximum stock ratio $\pi_k^{s,\max}$ for the upcoming period and by prohibiting short-selling of stocks.[4] The value $\pi_k^{s,\max}$ might be determined and prescribed by the corresponding regulating authorities and typically

[3] For European insurers, the corresponding requirements are provided by Solvency II. For German insurers, e.g., this is additionally prescribed by VAG, §132.

[4] For German life insurers, e.g., the requirement additionally means that the use of financial derivatives is only permitted if they contribute to reducing risks, see VAG §124.

depends on the country and the specific insurance product, and might be adjusted over time.

Due to the cash flows from the insurance business (new premium payments, benefit payments, credit repayments) and the price changes of the financial products (stocks, bonds), the management regularly needs to adjust the asset allocation. Such decisions are made at the beginning of a period, i.e. at time t_{k-1}. Throughout the interval (t_{k-1}, t_k), there are no changes nor adjustments possible implying a constant number of held assets until the period's end t_k. The reallocation depends on the chosen investment strategy, and requires the calculation of the tied up capital C^B_{k-1} and the position of liquid funds C^L_{k-1}. At the beginning of the period, the calculation steps follow a strict order:

1. derivation of the updated value of the life insurer's total assets $C_{(k-1)^+}$,
2. calculation of the value of the tied up capital C^B_{k-1},
3. computation of the liquid funds C^L_{k-1}, and
4. reallocation of the assets according to the chosen investment strategy (IS).

We add a '+' to the time index to highlight that a quantity's value is being updated at a given time point. For example, the total capital at time t_{k-1} is denoted by C_{k-1} and corresponds to the sum of the balance sheet prepared at the end of period $k-1$. New premiums obtained at the beginning of period k are part of the updated capital $C_{(k-1)^+}$ but not of C_{k-1}. However, as indicated in Figure 3.5, only single prices for bonds and stocks are quoted at each time point t_i.

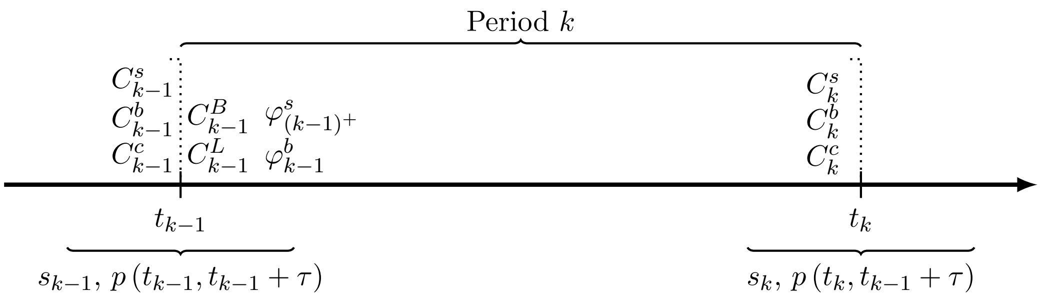

Figure 3.5.: Representation of period k regarding the asset allocation.

Here, $\varphi^s_{(k-1)^+}$ and φ^b_{k-1} denote the new numbers of stocks held and bonds with duration τ purchased at time t_{k-1}. These quantities are determined by the chosen investment strategy.

The starting point is the previous balance sheet with the life insurer's total assets C_{k-1} consisting of a cash position C^c_{k-1}, stocks C^s_{k-1}, and bonds C^b_{k-1} with different times to maturity, i.e.

$$C_{k-1} = C^c_{k-1} + C^s_{k-1} + C^b_{k-1}.$$

On the accounting date, the life insurance company holds

$$\varphi^s_{k-1} = \frac{C^s_{k-1}}{s_{k-1}}$$

stocks corresponding to a stock ratio

$$\pi^s_{k-1} = \frac{C^s_{k-1}}{C_{k-1}}.$$

Depending on the applied strategy for financing the periodic disbursements introduced in Section 3.5.3, π^s_{k-1} may exceed $\pi^{s,\max}_k$. This can especially be the case if funds are first taken from expired bonds. The updated capital $C_{(k-1)^+}$ is given by

$$C_{(k-1)^+} = C_{k-1} - \xi^{s,l}_{(k-1)^+} + \left(P_{k-1} - L^+_{k-1}\right)^+, \tag{3.5.1}$$

where $\xi^{s,l}_{(k-1)^+}$ denotes the amount of stocks sold at the beginning of period k and L^+_{k-1} the bridging loan according to the applied financing strategy. Both quantities are specified in Section 3.5.3. The premium P_{k-1} is in fact aggregated taking all cohorts, including the new ones, into account. This leads to

$$P_{k-1} = \sum_{m=1}^{M_k} \mathbb{1}_{\left\{d^m_{k-1}>0\right\}} \left(\delta^m_{k-1} + \delta^{\text{new},m}_{k-1}\right) \cdot P^m_{(k-1)^+} \tag{3.5.2}$$

with $M_k = M_{k-1} + M^{\text{add}}_{k-1}$. Regarding the bond investments, we assume that the maturity falls on a period's end implying $\frac{\tau}{\Delta t} \in \mathbb{N}$ and, in this chapter, that bonds are held until maturity.[5] Furthermore, the management strives to keep a minimum share $\pi^{s,\min}_{k-1} \in [0, \pi^{s,\max}_k]$ of the stocks. As a consequence, the tied up capital C^B_{k-1} comprises all previously purchased bonds having positive remaining residual terms, i.e. C^b_{k-1}, and parts of C^s_{k-1}. Taking into account the maximum stock ratio $\pi^{s,\max}_k$ for the upcoming period, we get

$$\begin{aligned} C^B_{k-1} &= C^b_{k-1} + \pi^{s,\min}_{k-1} C^s_{k-1} - \left(\pi^{s,\min}_{k-1} C^s_{k-1} - \pi^{s,\max}_k C_{(k-1)^+}\right)^+ \\ &= C^b_{k-1} + \min\left\{\pi^{s,\min}_{k-1} C^s_{k-1}, \pi^{s,\max}_k C_{(k-1)^+}\right\} \end{aligned}$$

representing the part of the total capital which is not available for new investments. Here,

[5] The latter assumption is relaxed in Chapter 5, where we develop investment strategies explicitly allowing for prior selling of held bonds.

we used that for any $x, y \in \mathbb{R}$ it holds

$$x - (x-y)^+ = \min\{x, y\}.$$

The cash flows are summarized in a position of liquid funds C^L_{k-1} forming the basis of the investment strategies. It is determined by premium income, liquid assets, and the previous demand for credits L^+_{k-1}, i.e.,

$$\begin{aligned} C^L_{k-1} = &\left(1 - \pi^{s,\min}_{k-1}\right) C^s_{k-1} - \xi^{s,l}_{(k-1)^+} + C^c_{k-1} + \left(P_{k-1} - L^+_{k-1}\right)^+ \\ &+ \left(\pi^{s,\min}_{k-1} C^s_{k-1} - \pi^{s,\max}_k C_{(k-1)^+}\right)^+. \end{aligned}$$

The last term reflects the potential liquidation of those parts of $\pi^{s,\min}_{k-1} C^s_{k-1}$ that exceed the maximum stock amount induced by $\pi^{s,\max}_k$. The updated capital $C_{(k-1)^+}$ can now be written as

$$C_{(k-1)^+} = C^L_{k-1} + C^B_{k-1}.$$

As all funds of the life insurance company, including the cash C^c_{k-1}, are completely invested in the capital market, $C_{(k-1)^+}$ comprises only stocks and bonds with different times to maturity after the reallocation, i.e. we then have

$$C_{(k-1)^+} = C^s_{(k-1)^+} + C^b_{(k-1)^+},$$

where $C^s_{(k-1)^+}$ represents the management's target for the stock position taking the available liquid funds and the maximum stock ratio into account. $C^b_{(k-1)^+}$ consists of C^b_{k-1} and newly purchased bonds with duration τ, i.e.

$$C^b_{(k-1)^+} = C^b_{k-1} + \varphi^b_{k-1} p\left(t_{k-1}, t_{k-1} + \tau\right),$$

where

$$\varphi^b_{k-1} = \frac{C^L_{k-1} - \left(C^s_{(k-1)^+} - \min\left\{\pi^{s,\min}_{k-1} C^s_{k-1}, \pi^{s,\max}_k C_{(k-1)^+}\right\}\right)}{p\left(t_{k-1}, t_{k-1} + \tau\right)}.$$

After the reallocation, the life insurer holds

$$\varphi^s_{(k-1)^+} = \frac{C^s_{(k-1)^+}}{s_{k-1}} \tag{3.5.3}$$

stocks corresponding to a stock ratio

$$\pi^s_{(k-1)^+} = \frac{C^s_{(k-1)^+}}{C_{(k-1)^+}}, \tag{3.5.4}$$

which respects the maximum stock ratio $\pi_k^{s,\max}$, i.e. we always have $\pi^s_{(k-1)^+} \leq \pi_k^{s,\max}$.

We close this section by introducing two prominent investment strategies we will apply in the later simulation studies. In Diehl et al. [23], we used these strategies with $\pi_k^{s,\min} = 0$ for all k.

CM strategy (constant mix)

The life insurance company intends to have a fixed share $\pi_k^{s,\text{tar}} \in [\pi_{k-1}^{s,\min}, \pi_k^{s,\max}]$ of its capital for period k invested in stocks. The adjusted stock position can be written as

$$C^s_{(k-1)^+} = \min\left\{C^L_{k-1} + \pi^{s,\min}_{k-1} C^s_{k-1}, \pi_k^{s,\text{tar}} C_{(k-1)^+}\right\}.$$

This strategy has a simple structure and is often used in the corresponding literature, e.g. in Burkhart et al. [14], Fernández et al. [25], and Gerstner et al. [27].

CPPI strategy (constant proportion portfolio insurance)

The amount of funds invested in stocks $C^s_{(k-1)^+}$ is linked to the current value of the own funds consisting of equity Q_{k-1} and free reserve F_{k-1}. The idea is that the insurer can accept a higher risk in case of sufficiently large buffers. Here, the free reserve F_{k-1} is taken into account since it contributes to the insurer's financial buffer, see Section 3.4. The strategy is characterized by a constant multiplier λ_k^{CPPI}.[6] The money invested in stocks can be represented as

$$C^s_{(k-1)^+} = \min\left\{\min\left\{C^L_{k-1}, \lambda_k^{\text{CPPI}} \cdot (Q_{k-1} + F_{k-1})^+\right\} + \pi^{s,\min}_{k-1} C^s_{k-1}, \pi_k^{s,\max} C_{(k-1)^+}\right\}.$$

The positive part of the own funds ensures that no short-selling of stocks occurs. A CPPI strategy is also considered in Bohnert et al. [10] and, in the context of dynamic hybrid products, in Bohnert [8] and Hambardzumyan and Korn [32].

3.5.2. Interest rate declaration

Insurance companies are obliged to run their businesses carefully, which is why a number of protection features are incorporated in practice. For example, prolonged life tables reduce the longevity risk for the insurance company or the investments in the financial market must be carried out cautiously. All of this usually leads to the insurance company making surpluses which belong to the policyholders and are distributed to the insured collective. In several European countries, particularly in Germany, for this a lagged participation process is applied, where surpluses are first accumulated in a collective reserve and are then allocated to the individual contracts in subsequent years, see e.g. Burkhart et al. [13, 14]. Such a procedure is typically used for participating life insurance contracts[7] and

[6] In practice, we typically have $\lambda_k^{\text{CPPI}} > 1$.

[7] In contrast, unit-linked policies make use of the process of individual saving, see Bohnert et al. [10].

can yield a stable surplus participation by smoothing potentially high-volatile obtained market returns.

In the presented model, the collective surpluses are first accounted by the free reserve F_{k-1}. They are later distributed to the individual policies depending on the type of insurance contract. This is done by declaring an annual interest rate

$$\widehat{i}_k = \begin{cases} \max\left\{\widehat{i}_G, \omega \cdot (\gamma_{k-1} - \gamma)\right\}, & \text{if} \quad k \equiv 1 \left(\operatorname{mod} \frac{1}{\Delta t}\right) \\ \widehat{i}_{k-1}, & \text{else} \end{cases} \tag{3.5.5}$$

at the beginning of the year, i.e. at times t_{k-1} with $k \equiv 1 \left(\operatorname{mod} \frac{1}{\Delta t}\right)$. Here, a guaranteed interest rate $\widehat{i}_G$ is taken into account, which might be a feature of the insurance contract.[8] The distribution ratio ω is controlled by the management and weights the deviation between the current reserve rate

$$\gamma_{k-1} = \frac{F_{k-1}}{F_{k-1} + V_{k-1}} \tag{3.5.6}$$

and an target value $\gamma \in [0, 1]$, which is kept as buffer against future losses. The interest rate declaration is binding for the whole year and thus constitutes a year-by-year or cliquet-style guarantee. Large values of $\widehat{i}_k$ are preferable from a policyholders' point of view and might increase the life insurer's competitiveness, but they are also associated with a greater strain on the free reserve and thus may threaten the future solvency.

If surpluses are credited to the policyholders' accounts every period, the annual interest rates $\widehat{i}_k$ and $\widehat{i}_G$ are transformed according to

$$i_k = \left(1 + \widehat{i}_k\right)^{\Delta t} - 1 \tag{3.5.7}$$

and

$$i_G = \left(1 + \widehat{i}_G\right)^{\Delta t} - 1 \tag{3.5.8}$$

to those for each period k.

Remark 3.5.1 (Alternative interest rate declaration)**.** The above procedure is mainly based on the current reserve rate γ_{k-1}. Alternatively, we could link the declared interest rate directly to the obtained portfolio return R_{k-1} from the last period.[9] To ensure a reasonable reduction of the free reserve F_k, we include a reserve rate-depending adjustment term i_k^{adjust} if γ_{k-1} is outside a specified interval $[\gamma_{k-1}^{\min}, \gamma_{k-1}^{\max}]$ that contains the target

[8] In practice, existing insurance portfolios typically consist of contracts equipped with varying guaranteed interest rates. In 2017, e.g., the technical interest rates used to compute the reserves of Allianz Life (Germany) ranged from 0.0% to 4.0%, see Hieber et al. [34]. An extension to such a setting is possible. Alternatively, one could divide the insurance portfolio into sub-portfolios corresponding to different product lines associated with varying guarantees as done by Goffard and Guerrault [28].

[9] At time t_0, we may obtain R_{-1} by prior simulations.

value γ. According to that alternative method, the annual declared interest rate is

$$\widehat{i}_k = \begin{cases} \max\left\{\widehat{i}_G, \omega \cdot R_{k-1} + i_k^{\text{adjust}}\right\}, & \text{if} \quad k \equiv 1 \left(\bmod \frac{1}{\Delta t}\right) \\ \widehat{i}_{k-1}, & \text{else}, \end{cases} \tag{3.5.9}$$

where

$$i_k^{\text{adjust}} = \begin{cases} -\left(\gamma_{k-1}^{\min} - \gamma_{k-1}\right), & \text{if} \quad \gamma_{k-1} < \gamma_{k-1}^{\min} \\ \left(\gamma_{k-1} - \gamma_{k-1}^{\max}\right), & \text{if} \quad \gamma_{k-1} > \gamma_{k-1}^{\max} \\ 0, & \text{else}. \end{cases}$$

If not stated otherwise, the annual interest rate is declared according to (3.5.5) throughout this thesis.

3.5.3. Financing of the disbursements

In addition to the asset allocation, the use of surpluses, and the interest rate declaration, another important task of the management of a life insurance company is to ensure sufficient liquidity such that obligations can be met when they are due, i.e. the life insurer needs to be able to create the payments promised to its customers at all times. However, uncertainty regarding the size of the total claim amount and the time at which benefit payments are due on the one side and the fact that a substantial part of the capital is tied up on the other side, can lead to a situation where the amount of available cash is not sufficient. In that case, the management needs to choose a priority order of potential measures that are taken to increase the liquidity and to create the promised payments. We call the exact procedure of such measures a *financing strategy* (FS). In this chapter, we consider the following potential measures:[10]

- take funds from expired bonds,
- sell stocks,
- raise short-term credits,
- take funds from premium income, and
- raise long-term credits.

The short-term credit is a bridging loan, e.g. an over-night credit, from the end of period k to the beginning of period $k+1$ and represents the demand for credits at time t_k. Due to the short maturity, it is assumed that no interests need to be paid. Note that premium revenues P_k at time t_k are assigned to the following period $k+1$. As a result, they are not initially used to cover the benefits. By this assumption, we respect the fact that, in

[10] In Chapter 5, we additionally consider the possibility of selling held bonds.

practice, the end of a period does not coincide with the beginning of the following one.[11] Thus, new premiums of period $k+1$ are not available to pay the benefits due at the end of period k, but they can be used to repay the bridging loan at the beginning of period $k+1$. If the bridging loan cannot be repaid completely, the life insurer decides to raise a long-term credit with a fixed duration and variable interest rate. This can be interpreted as emitting or short-selling of bonds.

As written in the previous section, the management strives to keep a minimum stock share $\pi_k^{s,\min} \in [0, \pi_{k+1}^{s,\max}]$ implying that stocks are only sold from the remaining part $\left(1-\pi_k^{s,\min}\right)\varphi^s_{(k-1)^+}s_k$ to increase the required liquidity. In the case $\pi_k^{s,\min} = 0$, the management might sell all stocks, while $\pi_k^{s,\min} = 1$ leads to the insurer selling no stocks at all.[12] In practice, there might be even more constraints that the management needs to take into account.

According to the priority order, the life insurer makes full use of a measure before applying another one. As indicated in Figure 3.6, stocks can be sold at the end of period k or at the beginning of the next one. The corresponding market values of sold stocks are denoted by $\xi_k^{s,l}$ and $\xi_{k^+}^{s,l}$, respectively. Especially in the presence of transaction costs, it might be more attractive to borrow money via the bridging loan than to sell stock shares. In contrast, the other measures can either be taken only before or only after the preparation of the balance sheet at time t_k.

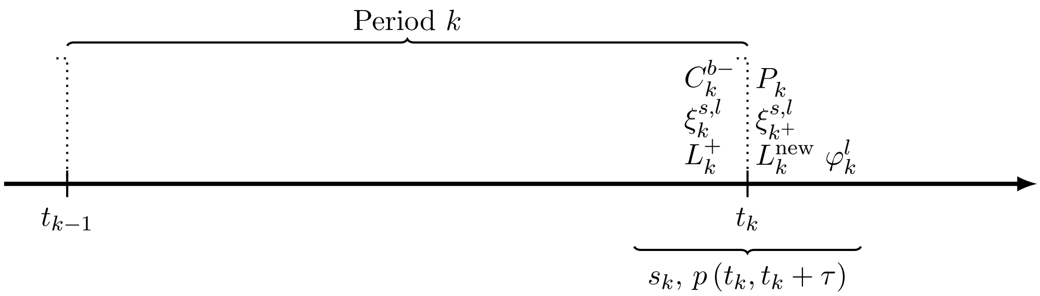

Figure 3.6.: Measures for financing the disbursements D_k assigned to the corresponding periods.

We always assume that funds are first taken from expired bonds C_k^{b-} as this measure can be realized most easily and without costs. All bonds with duration τ that were purchased

[11]The corresponding time distance, however, is small which motivates our assumption of single asset prices at each time point t_i, see Figures 3.5 and 3.6.

[12]In any case, stocks might be sold during the asset reallocation at the beginning of the following period due to the maximum stock ratio $\pi_{k+1}^{s,\max}$.

at time $t_{k-\frac{\tau}{\Delta t}}$ are expiring, i.e. we have

$$C_k^{b-} = \varphi^b_{k-\frac{\tau}{\Delta t}} p\left(t_k, t_k\right) = \varphi^b_{k-\frac{\tau}{\Delta t}}$$

with $p\left(t_k, t_{k-\frac{\tau}{\Delta t}} + \tau\right) = p\left(t_k, t_k\right) = 1$. Taking a bridging loan L_k^+ is always the last measure before preparing the balance sheet at time t_k. At the beginning of period $k+1$, the life insurer can use the premium income, sell stocks, and raise a long-term credit L_k^{new} by emitting φ_k^l bonds with duration τ, i.e.

$$\varphi_k^l = \frac{L_k^{\text{new}}}{p\left(t_k, t_k + \tau\right)}. \tag{3.5.10}$$

The exact representations of the involved quantities ($\xi_k^{s,l}$, L_k^+, $\xi_{k+}^{s,l}$, and L_k^{new}) depend on the chosen strategy for financing the disbursements D_k consisting of benefit payments $\mathcal{B}_k$ and expiring (long-term) credits L_k^-, i.e.

$$D_k = \mathcal{B}_k + L_k^- \tag{3.5.11}$$

with

$$L_k^- = \varphi^l_{k-\frac{\tau}{\Delta t}} p\left(t_k, t_k\right) = \varphi^l_{k-\frac{\tau}{\Delta t}}. \tag{3.5.12}$$

In the following, we present two examples.

Financing strategy 1 (FS 1)

This method was also used in Diehl et al. [23] but with $\pi_k^{s,\min} = 0$. Following this method, the order of the individual measures are:

1. take funds from expired bonds C_k^{b-},
2. sell stocks $\xi_k^{s,l}$,
3. raise short-term credits L_k^+,
4. take funds from premium income P_k, and
5. raise long-term credits L_k^{new}.

As already indicated, funds are first taken from expired bonds C_k^{b-}. It is possible that these payouts do not cover the disbursements D_k, in which case the life insurer sells stocks in the amount of

$$\xi_k^{s,l} = \min\left\{\left(D_k - C_k^{b-}\right)^+, \left(1 - \pi_k^{s,\min}\right) \varphi^s_{(k-1)^+} s_k\right\}.$$

If these two measures are not sufficient to meet the due obligations, there remains a demand for credits L_k^+ in the amount of

$$\begin{aligned} L_k^+ &= \left(\left(D_k - C_k^{b-} \right)^+ - \left(1 - \pi_k^{s,\min} \right) \varphi_{(k-1)^+}^s s_k \right)^+ \\ &= \left(D_k - C_k^{b-} - \left(1 - \pi_k^{s,\min} \right) \varphi_{(k-1)^+}^s s_k \right)^+ , \end{aligned}$$

which is satisfied by raising a short-term credit. At the beginning of period $k+1$, premium income P_k is used to repay the bridging loan, but no stocks are sold, i.e. we have $\xi_{k^+}^{s,l} = 0$. The long-term credits to enter newly thus amount to

$$L_k^{\text{new}} = \left(L_k^+ - P_k \right)^+ .$$

Financing strategy 2 (FS 2)

Note that for the first method it is possible that stocks are sold at the end of period k and directly repurchased at the beginning of the following period at the same price s_k. Such a procedure might not be desirable if there exist transaction costs for buying and selling. Therefore, we introduce the following method.

1. take funds from expired bonds C_k^{b-},
2. raise short-term credits L_k^+,
3. take funds from premium income P_k,
4. sell stocks $\xi_{k^+}^{s,l}$, and
5. raise long-term credits L_k^{new}.

Again, we first take funds from expired bonds C_k^{b-}. However, now we directly raise a short-term credit L_k^+ in the amount of

$$L_k^+ = \left(D_k - C_k^{b-} \right)^+$$

if necessary instead of selling stocks, i.e. we have $\xi_k^{s,l} = 0$. At the beginning of the following period, premium income P_k is used to repay the bridging loan, and only if this is not sufficient, the life insurer sells stocks worth

$$\xi_{k^+}^{s,l} = \min \left\{ \left(L_k^+ - P_k \right)^+ , \left(1 - \pi_k^{s,\min} \right) C_k^s \right\} .$$

As a result, the amount of sold stocks due to financing the periodic disbursements is potentially much smaller compared to applying the first method. Finally, the credits to enter newly amount to

$$L_k^{\text{new}} = \left(L_k^+ - P_k - \left(1 - \pi_k^{s,\min} \right) C_k^s \right)^+ .$$

Note that the amount of raised long-term credits coincide applying both methods, that is, we can write

$$L_k^{\text{new}} = \left(L_k^+ - P_k\right)^+ - \xi_{k^+}^{s,l} \tag{3.5.13}$$

for both strategies. However, the amount of raised short-term credits and correspondingly the balance sheet position "liabilities to banks" may substantially differ.

3.6. Liability model

The insurance company's liabilities consist of the commitments entered into by the conclusion of insurance contracts of different types. The biometric parameters and the behavior of the insured was already described in Section 2.3. In addition, the policies heavily depend on the structural characteristics. The benefit spectrum of an insurance product specifies the potential sizes of the benefit payments to the individual policyholder in different scenarios, e.g. at maturity or in case of prior death, see Kahlenberg [38]. In this section, we generally introduce the guaranteed and the bonus part of the benefit spectrum covering both, participating and non-participating contracts. We conclude the section by deriving explicit representations of the policyholders' accounts for a prominent class of insurance contracts.

3.6.1. The benefit spectrum

In contrast to premiums which are typically paid in advance, i.e. at the beginning of a period $[t_{k-1}, t_k]$, benefits due in that period are paid out at time t_k. As indicated at the beginning of Chapter 3 regarding the notation, we assume that we apply one of the methods developed in Section 2.4 for the compression and the simulation of the insurance portfolio. As a result, the cash flows are associated with the representative contracts of the individual cohorts. The benefit spectrum of an insurance product can thus be characterized by listing the premium and benefit payments for all cohorts $m \in \{1, \ldots, M_k\}$ and all periods $k \in \{1, \ldots, K\}$.

The total amount of due benefits $\mathcal{B}_k$ introduced in the last section, can be decomposed into

$$\mathcal{B}_k = E_k + T_k + S_k, \tag{3.6.1}$$

where the survival, death, and surrender benefit payments E_k, T_k, and S_k are calculated by

$$E_k = \sum_{m=1}^{M_k} (1 - u_k^m)(1 - q_k^m)\left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) E_k^m, \tag{3.6.2}$$

$$T_k = \sum_{m=1}^{M_k} q_k^m \left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) T_k^m, \tag{3.6.3}$$

$$S_k = \sum_{m=1}^{M_k} u_k^m (1 - q_k^m)\left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) S_k^m. \tag{3.6.4}$$

Here, q_k^m and u_k^m are the death and surrender probabilities introduced in Section 2.3.2 for period k which are associated with the representative of cohort m. E_k^m, T_k^m, and S_k^m correspond to the payments of a single policyholder of model point m at time t_k and are determined by the specific insurance contract via the benefit spectrum. For participating contracts, these payments can each be divided into a guaranteed and a bonus part denoted by a superscript G and B, respectively. We can then write

$$\begin{aligned} E_k^m &= E_k^{G,m} + E_k^{B,m}, \\ T_k^m &= T_k^{G,m} + T_k^{B,m}, \\ S_k^m &= S_k^{G,m} + S_k^{B,m}, \end{aligned}$$

whereby equations (3.6.2), (3.6.3), and (3.6.4) can be written as

$$\begin{aligned} E_k &= E_k^G + E_k^B, \\ T_k &= T_k^G + T_k^B, \\ S_k &= S_k^G + S_k^B, \end{aligned} \tag{3.6.5}$$

with, e.g.,

$$E_k^G = \sum_{m=1}^{M_k} (1 - u_k^m)(1 - q_k^m)\left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) E_k^{G,m}, \tag{3.6.6}$$

$$T_k^G = \sum_{m=1}^{M_k} q_k^m \left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) T_k^{G,m}, \tag{3.6.7}$$

$$S_k^G = \sum_{m=1}^{M_k} u_k^m (1 - q_k^m)\left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) S_k^{G,m} \tag{3.6.8}$$

representing the aggregated guaranteed survival, death, and surrender benefit payments due at time t_k.

The introduced general framework regarding the benefit spectrum covers many types of life insurance products. For example, a pure term-life insurance without surrender options can be considered by choosing $u_k^m = 0$ and $S_k^m = E_k^m = 0$ for all m and all k implying $S_k = E_k = 0$. In the following, we introduce a prominent type of participating contracts incorporating all parts of the above benefit spectrum.

3.6.2. Development of the policyholders' accounts

As this thesis aims at presenting the fundamental relations in simulating the asset-liability management, we assume that the insurance portfolio consists of one contract type, namely a classic endowment insurance. It is equipped with a guaranteed interest rate for the invested premiums, a surrender option, and time-depending death benefits. Moreover, we consider a lump-sum benefit payment and not a pension phase. The policyholders are, in addition to the contractually guaranteed benefits, entitled to variable bonus payments which allow them to participate in the obtained surpluses. Once surpluses are allocated to the policyholders, i.e. credited to the corresponding accounts, they become part of the guarantees and are then periodically at least compounded with the guaranteed interest rate. The benefit payments are determined by the actual value of the policyholders' accounts consisting of both accumulated and compounded contributions and bonus payments. These account values represent the individual policyholders' entitlements and are not fully funded. Indeed, as described in Sections 3.5.1 and 3.5.3, the insurer's assets are completely invested in the capital market and it is possible that assets need to be liquidated or loans must be raised to finance the due benefit payments.

In our model, the benefit spectrum of the classic endowment insurance can be represented as

$$\begin{aligned} E_k^m &= E_k^{G,m} + E_k^{B,m} = \mathbb{1}_{\{d_k^m = 0\}} V_k^m, \\ T_k^m &= T_k^{G,m} + T_k^{B,m} = \mathbb{1}_{\{d_k^m \geq 0\}} V_k^m, \\ S_k^m &= S_k^{G,m} + S_k^{B,m} = \mathbb{1}_{\{d_k^m > 0\}} \vartheta V_k^m \end{aligned}$$

with the contract value V_k^m and the parameter $\vartheta \in [0, 1]$ being the so-called surrender factor. As introduced in Section 2.4, the remaining contract period of the policies in cohort m at time t_k is denoted by d_k^m. The guaranteed and variable parts of the benefit spectrum are related to the actuarial account A_k^m and the bonus account B_k^m being specified to be

$$\begin{aligned} E_k^{G,m} &= \mathbb{1}_{\{d_k^m = 0\}} A_k^m \quad && \text{and} \quad & E_k^{B,m} &= \mathbb{1}_{\{d_k^m = 0\}} B_k^m, \\ T_k^{G,m} &= \mathbb{1}_{\{d_k^m \geq 0\}} A_k^m \quad && \text{and} \quad & T_k^{B,m} &= \mathbb{1}_{\{d_k^m \geq 0\}} B_k^m, \\ S_k^{G,m} &= \mathbb{1}_{\{d_k^m > 0\}} \vartheta A_k^m \quad && \text{and} \quad & S_k^{B,m} &= \mathbb{1}_{\{d_k^m > 0\}} \vartheta B_k^m. \end{aligned} \tag{3.6.9}$$

Therefore, the value V_k^m of the total account of model point m at time t_k is composed by the actuarial account A_k^m and the bonus account B_k^m reflecting the assured guaranteed part and the variable bonus part of the entitlements, respectively.

The actuarial account includes all premium payments periodically compounded with the guaranteed interest rate i_G from equation (3.5.8). The bonus account covers the excess of periodically compounding the total account with the transformed declared interest rate i_k from equation (3.5.7) surpassing the actuarial account. Therefore, in the uncompressed insurance portfolio, the actuarial account's value $A_0^{\mathrm{real},j}$ of policyholder $j \in \{1, \dots, \delta_0\}$ at time t_0 equals

$$A_0^{\mathrm{real},j} = \sum_{k=1}^{c_0^{\mathrm{real},j}} (1 + i_G)^k \, P_{-k}^{\mathrm{real},j},$$

where $c_0^{\mathrm{real},j}$ is the elapsed contract term in periods from equation (2.3.2). In the case of constant premiums, i.e. if we have $P_k^{\mathrm{real},j} = P^{\mathrm{real},j}$ for all k, we get the closed-form

$$A_0^{\mathrm{real},j} = \begin{cases} P^{\mathrm{real},j} \, (1 + i_G) \, \frac{(1+i_G)^{c_0^{\mathrm{real},j}} - 1}{i_G}, & \text{if} \quad i_G \neq 0 \\ P^{\mathrm{real},j} c_0^{\mathrm{real},j}, & \text{else.} \end{cases}$$

Regarding the bonus account, $B_0^{\mathrm{real},j}$ equals the amount of past surpluses that have been credited to policyholder j, periodically compounded up to time t_0.

According to the applied grouping procedure, the representative's account values A_0^m and B_0^m correspond to the arithmetic means of the real contracts grouped into cohort $m \in \{1, \dots, M_0\}$, see equation (2.4.2). As the new business potentially changes the size of the cohorts, the representative accounts must be adjusted as described in Section 2.4.1. Furthermore, the policyholders' accounts for the classic endowment insurance are updated at the end of every period $[t_{k-1}, t_k]$. Then the value of all actuarial and bonus accounts in model point $m \in \{1, \dots, M_k\}$ at times t_k with $k \geq 1$ equal

$$A_k^m = (1 + i_G) \, A_{(k-1)^+}^m \tag{3.6.10}$$

and

$$B_k^m = (1 + i_k) \, B_{(k-1)^+}^m + (i_k - i_G) \, A_{(k-1)^+}^m, \tag{3.6.11}$$

where $A_{(k-1)^+}^m$ and $B_{(k-1)^+}^m$ are the adjusted values specified in equations (2.4.7) and (2.4.8). Combining these two accounts yields the contract value

$$V_k^m = A_k^m + B_k^m = (1 + i_k) \, V_{(k-1)^+}^m$$

with $V_{(k-1)^+}^m$ from equation (2.4.9). In the following Section 3.7, the representative accounts are linked to the corresponding reserves of the balance sheet.

3.7. Balance sheet model

After having introduced the relevant components, we now describe the modeling of the balance sheet. This comprises the development of the assets and the projection of the liabilities introduced in Table 3.2. The starting point of the later simulations is always a given balance sheet at time t_0. The development over time depends on a variety of factors including the considered product type, the applied strategies for investing in the capital market and financing the due obligations, the used surplus participation scheme, and biometric parameters of the insured collective. Further external impacts are the developments of the capital market and the new business. In the following, we derive explicit representations of all balance sheet positions. As a main result, we prove that the fundamental balance sheet equation holds at all times.

3.7.1. Projection of the assets

The balance sheet summarizes the business performance at the end of each period $[t_{k-1}, t_k]$ meaning that the values of the capital positions for cash C_k^c, stocks C_k^s, and bonds C_k^b at time t_k are calculated. For this, the following business steps occur:

1. reevaluation of the assets,

2. financing the disbursements according to the chosen strategy FS, and

3. preparation of the balance sheet.

The actual value of the assets is given by the prices s_k and $p\,(t_k, t_i + \tau)$. The bonds were purchased at times t_i. The number $\varphi^s_{(k-1)^+}$ of held stocks is given by equation (3.5.3) and depends on the chosen investment strategy regarding the asset allocation, see Section 3.5.1.

According to the assumption from Section 3.5.3 that funds are first taken from expired bonds C_k^{b-}, the position of cash C_k^c equals

$$C_k^c = \left(C_k^{b-} - D_k\right)^+ \tag{3.7.1}$$

with the disbursements D_k from equation (3.5.11). The stock position C_k^s can be generally represented by

$$\begin{aligned} C_k^s &= \varphi^s_{(k-1)^+} s_k - \left(D_k - C_k^{b-}\right)^+ + L_k^+ \\ &= \varphi^s_{(k-1)^+} s_k - \xi_k^{s,l}, \end{aligned}$$

where the precise amount $\xi_k^{s,l}$ of sold stocks and the demand for credits L_k^+ are determined by the chosen strategy for financing the disbursements. Note that applying strategy FS 2 from Section 3.5.3 yields $\xi_k^{s,l} = 0$. Finally, C_k^b comprises all previously purchased bonds having positive remaining residual terms, i.e.

$$C_k^b = \sum_{i=k-\frac{\tau}{\Delta t}+1}^{k-1} \varphi_i^b p\left(t_k, t_i + \tau\right).$$

Recall that $\frac{\tau}{\Delta t} \in \mathbb{N}$ since we assume that the maturity falls on a period's end. Furthermore, as bonds are held until maturity in this chapter, the number φ_i^b of bonds with duration τ purchased at time t_i remains constant.

The total capital C_k is defined by the sum of all assets, i.e. by

$$C_k = C_k^b + C_k^s + C_k^c. \tag{3.7.2}$$

3.7.2. Projection of the liabilities

Turning to the balance sheet's liabilities, we first describe the evolution of the actuarial reserve A_k, the bonus reserve B_k, and the technical reserve $V_k = A_k + B_k$. The policyholders' accounts A_k^m, B_k^m, and V_k^m are linked to these reserves via

$$A_k = \sum_{m=1}^{M_k} \delta_k^m A_k^m, \tag{3.7.3}$$

$$B_k = \sum_{m=1}^{M_k} \delta_k^m B_k^m, \tag{3.7.4}$$

$$V_k = \sum_{m=1}^{M_k} \delta_k^m V_k^m, \tag{3.7.5}$$

where δ_k^m denotes the size of cohort m at time t_k from equation (2.4.10) in Section 2.4. The following proposition shows the relation to the previous balance sheet at time t_{k-1}.

Proposition 3.7.1 (Recursive schemes of the reserves)**.** *Consider the endowment insurance with surrender factor $\vartheta > 0$ from Section 3.6.2. Then, it holds for all $k = 1, \ldots, K$:*

(i) $A_k = (1 + i_G)\left(A_{k-1} + P_{k-1}\right) - \left(E_k^G + T_k^G + \frac{1}{\vartheta} S_k^G\right)$,

(ii) $B_k = (1 + i_k) B_{k-1} + (i_k - i_G)\left(A_{k-1} + P_{k-1}\right) - \left(E_k^B + T_k^B + \frac{1}{\vartheta} S_k^B\right)$,

(iii) $V_k = (1 + i_k)\left(V_{k-1} + P_{k-1}\right) - \left(E_k + T_k + \frac{1}{\vartheta} S_k\right)$.

Proof. To prove the first statement (i), we use the definitions (3.7.3) and (2.4.10) of A_k and δ_k^m and the equation $\mathbb{1}_{\{d_k^m > 0\}} = \mathbb{1}_{\{d_k^m \geq 0\}} - \mathbb{1}_{\{d_k^m = 0\}}$, yielding

$$A_k = \sum_{m=1}^{M_k} \left(\mathbb{1}_{\{d_k^m \geq 0\}} - \mathbb{1}_{\{d_k^m = 0\}}\right)(1 - u_k^m)(1 - q_k^m)\left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) A_k^m.$$

Writing

$$\begin{aligned}
A_k = &\sum_{m=1}^{M_k} \mathbb{1}_{\{d_k^m \geq 0\}} \cdot \left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) A_k^m \\
&- \sum_{m=1}^{M_k} \mathbb{1}_{\{d_k^m \geq 0\}} q_k^m \left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) A_k^m \\
&- \sum_{m=1}^{M_k} \mathbb{1}_{\{d_k^m \geq 0\}} u_k^m \left(1 - q_k^m\right) \left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) A_k^m \\
&- \sum_{m=1}^{M_k} \mathbb{1}_{\{d_k^m = 0\}} \cdot \left(1 - u_k^m\right) \left(1 - q_k^m\right) \left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) A_k^m
\end{aligned}$$

we can use the specified guaranteed parts of the benefit spectrum of the endowment insurance (3.6.9) and equations (3.6.6), (3.6.7), and (3.6.8) to obtain

$$A_k = \sum_{m=1}^{M_k} \mathbb{1}_{\{d_k^m \geq 0\}} \cdot \left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) A_k^m - T_k^G - \frac{1}{\vartheta} S_k^G - E_k^G.$$

Note that $\mathbb{1}_{\{d_k^m \geq 0\}} u_k^m = \mathbb{1}_{\{d_k^m > 0\}} u_k^m$ due to the definition of u_k^m, compare (2.3.5). Inserting the representation (3.6.10) of A_k^m, using equations (3.5.2) and (3.7.3) for P_{k-1}^m and A_{k-1} and taking into account that $\mathbb{1}_{\{d_k^m \geq 0\}} = \mathbb{1}_{\{d_{k-1}^m > 0\}}$ due to (2.4.5) leads to statement (i). Note that $A_{k-1}^m = 0$ for all new cohorts from period k, i.e. for all $m \in \{M_{k-1}+1, \ldots, M_k\}$, since the account values represent the policyholders' entitlements, and benefits are only paid at the end of a period, i.e. for new customers from period k at earliest at time t_k.

To prove the second statement, we derive

$$B_k = \sum_{m=1}^{M_k} \mathbb{1}_{\{d_k^m \geq 0\}} \cdot \left(\delta_{k-1}^m + \delta_{k-1}^{\text{new},m}\right) B_k^m - E_k^B - T_k^B - \frac{1}{\vartheta} S_k^B$$

in the same manner as above using now the specified bonus parts of the benefit spectrum. Continuing with the same argumentation and inserting the representation (3.6.11) of B_k^m leads to statement (ii).

For the last statement, we add equations (i) and (ii) to get

$$A_k + B_k = (1 + i_k) \left(A_{k-1} + P_{k-1} + B_{k-1}\right) - \left(E_k + T_k + \frac{1}{\vartheta} S_k\right),$$

where we used the decompositions in (3.6.5) of the aggregated benefit payments into guaranteed and bonus parts. The definition of the technical reserve in (3.4.1) shows (iii). □

As described in Section 3.5.3, the life insurer can raise short-term and long-term credits. While the former represent bridging loans (over-night credits) without interest rate payments, the latter are equipped with a fixed duration τ and a variable interest rate and can thus be interpreted as emitting of bonds. Therefore, the balance sheet position liabilities to banks L_k consists of the bridging loan L_k^+ at time t_k and long-term credits that have not been repaid yet, i.e.

$$L_k = L_k^+ + \sum_{i=k-\frac{\tau}{\Delta t}+1}^{k-1} \varphi_i^l p\left(t_k, t_i + \tau\right), \tag{3.7.6}$$

where φ_i^l is the number of bonds emitted at time t_i defined by equation (3.5.10).

The last two positions, the free reserve F_k and the equity Q_k, depend on the generated surplus G_k. This arises at time t_k from the investments in the financial market and in practice also from conservative estimates for interest rates, death probabilities, and expenses used for the calculation of premiums.[13] In our model, the total surplus G_k is divided into an interest and a surrender component, i.e.

$$G_k = G_k^I + G_k^S. \tag{3.7.7}$$

The interest surplus G_k^I is given by the difference between the total capital market return on the one side and the total interests deposited in the policyholders' accounts and the credits on the other side, namely

$$G_k^I = \varphi^s_{(k-1)^+} \Delta s_k + \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi_i^b \Delta p_{k,i} - i_k \left(V_{k-1} + P_{k-1}\right) - \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi_i^l \Delta p_{k,i}, \tag{3.7.8}$$

with $\Delta s_k = s_k - s_{k-1}$ and $\Delta p_{k,i} = p\left(t_k, t_i + \tau\right) - p\left(t_{k-1}, t_i + \tau\right)$. The surrender surplus for the classic endowment insurance is given by

$$G_k^S = \left(\frac{1}{\vartheta} - 1\right) S_k \tag{3.7.9}$$

and is non-negative for the surrender factor $\vartheta \in (0, 1]$. Here, S_k are the aggregated surrender benefit payments from equation (3.6.4) and $\frac{1}{\vartheta} S_k$ corresponds to the reduction of the technical reserve due to cancellations, see Proposition 3.7.1 (*iii*).

The surpluses are distributed between the insured collective and the shareholders. Due to legal requirements, most of a positive raw surplus belongs to the insured collective while shareholders participate to a small extent through dividend payments. In the presented allocation, a fixed portion αG_k is deposited in the free reserve F_k and the remaining amount is credited to the equity Q_k. The parameter $\alpha \in [0, 1]$ is referred to as participation

[13] Indeed, reporting standards for insurers require a surplus decomposition regarding both contributions of individual policyholders and different risk sources. This is typically a non-trivial task and may become technically challenging. Jetses and Christiansen [36], e.g., propose an axiomatic approach that leads to the so-called infinitesimal sequential updating decomposition principle.

rate.[14] More difficult and controversial is the question of how losses should be shared between shareholders and policyholders since there are hardly any generally applicable legal regulations in this regard, see Burkhart et al. [14]. In Chapter 5, we elaborate on this issue and discuss potential fair splits. For now we assume that the free reserve fully absorbs losses $G_k < 0$ while the withdrawals are limited to the available funds, i.e. we have

$$F_k = \left(F_{k-1} + \min\left\{\alpha G_k, G_k\right\}\right)^+ . \tag{3.7.10}$$

If the free reserve does not suffice, i.e. $|G_k| > F_{k-1}$, the shareholders absorb the remaining loss $F_{k-1} + G_k$. As a result, the equity Q_k can be represented as

$$Q_k = Q_{k-1} + \min\left\{\left(\left(1-\alpha\right) G_k\right)^+ , F_{k-1} + G_k\right\} . \tag{3.7.11}$$

Remark 3.7.2. By construction, surpluses are completely allocated every period, i.e. we have for all $k = 1, \ldots, K$:

$$F_k + Q_k = F_{k-1} + Q_{k-1} + G_k.$$

Indeed, the latter equation will be needed to prove that the fundamental balance sheet equation is respected at any time, see Theorem 3.7.3.

3.7.3. The fundamental balance sheet equation

As we finished the development of our model, we now present the central theorem showing that the balance sheet equation is fulfilled at each point in time $t_k = k\Delta t$. That is the business activities lead to equal sums of assets and the liabilities as displayed in Table 3.2.

Theorem 3.7.3 (Verification of the model)**.** *Consider the endowment insurance with surrender factor $\vartheta > 0$ and suppose that the sum of all assets equals the sum of all liabilities at the start of the simulation, i.e., $C_0 = A_0 + B_0 + F_0 + Q_0 + L_0$. Then, the fundamental balance sheet equation is fulfilled at any time, i.e. it holds*

$$C_k = A_k + B_k + F_k + Q_k + L_k$$

for all $k = 0, \ldots, K$.

Proof. We prove the statement by induction over k. By assumption, the equality holds at time t_0. As induction hypothesis, we assume that the equality

$$C_{k-1} = A_{k-1} + B_{k-1} + F_{k-1} + Q_{k-1} + L_{k-1}$$

holds for all times t_i with $i \leq k-1 < K$. For the induction step, we first decompose the

[14] Under German legislation, a typical value would be $\alpha \in [0.9, 1]$ meaning that at least 90% of the risk surplus has to be credited to the policyholders' accounts, see Wagner [55].

total capital C_k into

$$C_k = C_k^b + C_k^s + C_k^c.$$

Taking price changes from period k, the demand for credits L_k^+, and the disbursements D_k at time t_k into account, the latter equation is linked to the stock and bond part after the reallocation of assets at time t_{k-1} via

$$C_k = C^b_{(k-1)^+} + C^s_{(k-1)^+} + \varphi^s_{(k-1)^+}\Delta s_k + \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi_i^b \Delta p_{k,i} + L_k^+ - D_k.$$

Using

$$\begin{aligned} D_k &= \mathcal{B}_k + L_k^-, \\ \mathcal{B}_k &= E_k + T_k + S_k, \\ C_{(k-1)^+} &= C^b_{(k-1)^+} + C^s_{(k-1)^+}, \end{aligned}$$

and plugging in the representation of $C_{(k-1)^+}$ from equation (3.5.1), we get

$$\begin{aligned} C_k = C_{k-1} &- \xi^{s,l}_{(k-1)^+} + \left(P_{k-1} - L^+_{k-1}\right)^+ + \varphi^s_{(k-1)^+}\Delta s_k + \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi_i^b \Delta p_{k,i} \\ &+ L_k^+ - E_k - T_k - S_k - L_k^-. \end{aligned}$$

According to the specific representation of the surplus G_k from equations (3.7.7), (3.7.8), and (3.7.9), we can write

$$\begin{aligned} C_k = C_{k-1} &- \xi^{s,l}_{(k-1)^+} + \left(P_{k-1} - L^+_{k-1}\right)^+ + G_k + i_k\left(V_{k-1} + P_{k-1}\right) - \frac{1}{\vartheta} S_k \\ &+ \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi_i^l \Delta p_{k,i} + L_k^+ - E_k - T_k - L_k^-. \end{aligned}$$

Using the recursive scheme of the technical reserve V_k from Proposition 3.7.1 (*iii*) and the induction hypothesis, we get

$$\begin{aligned} C_k = A_{k-1} &+ B_{k-1} + F_{k-1} + Q_{k-1} + L_{k-1} - \xi^{s,l}_{(k-1)^+} + \left(P_{k-1} - L^+_{k-1}\right)^+ + G_k + V_k \\ &- \left(V_{k-1} + P_{k-1}\right) + \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi_i^l \Delta p_{k,i} + L_k^+ - L_k^-. \end{aligned}$$

The relation between own funds and surplus in Remark 3.7.2 and $V_{k-1} = A_{k-1} + B_{k-1}$

imply

$$\begin{aligned}C_k &= F_k + Q_k + L_{k-1} - \xi^{s,l}_{(k-1)^+} + \left(P_{k-1} - L^+_{k-1}\right)^+ + V_k - P_{k-1} \\ &\quad + \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi^l_i \Delta p_{k,i} + L^+_k - L^-_k .\end{aligned}$$

Using equation (3.7.6) for L_{k-1} and observing the equality

$$L^{\text{new}}_{k-1} = \left(P_{k-1} - L^+_{k-1}\right)^+ - \xi^{s,l}_{(k-1)^+} - P_{k-1} + L^+_{k-1}$$

for both financing strategies, see also equation (3.5.13), we can write

$$C_k = F_k + Q_k + V_k + L^{\text{new}}_{k-1} + \sum_{i=k-\frac{\tau}{\Delta t}}^{k-2} \varphi^l_i p\left(t_{k-1}, t_i + \tau\right) + \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi^l_i \Delta p_{k,i} + L^+_k - L^-_k .$$

Since

$$\begin{aligned}L^{\text{new}}_{k-1} &= \varphi^l_{k-1} p\left(t_{k-1}, t_{k-1} + \tau\right), \\ L^-_k &= \varphi^l_{k-\frac{\tau}{\Delta t}} p\left(t_k, t_{k-\frac{\tau}{\Delta t}} + \tau\right), \\ \Delta p_{k,i} &= p\left(t_k, t_i + \tau\right) - p\left(t_{k-1}, t_i + \tau\right),\end{aligned}$$

the latter equation becomes

$$\begin{aligned}C_k &= F_k + Q_k + V_k + \sum_{i=k-\frac{\tau}{\Delta t}+1}^{k-1} \varphi^l_i p\left(t_k, t_i + \tau\right) + L^+_k \\ &= A_k + B_k + F_k + Q_k + L_k\end{aligned}$$

which completes the proof. □

4. Simulation Studies: Part I

In this chapter, we perform several simulation studies to illustrate our ALM model. To this end, we consider a classic endowment insurance equipped with the specifications from Section 3.6.2. The survival, death, and surrender benefit payments are given by (3.6.9). If not stated otherwise, we consider a life insurance company providing new business. Furthermore, we use the basic capital market model from Section 3.3.1 and apply financing strategy FS1 from Section 3.5.3.

After the initialization and the parameter specification in Section 4.1, we investigate in Section 4.2 the performance of some of the introduced compression and simulation methods regarding efficiency and approximation quality. A lot of the existing literature mainly focuses on run-off scenarios, see the literature overview in 3.1.2. In Section 4.3, we therefore compare in our ALM model the impact of the incorporation of stationary new business with the corresponding run-off scenario. The effects of non-stationary contract arrivals on the balance sheet structure is studied in Section 4.4. It follows a comparison study of CM and CPPI strategies in Section 4.5, where we take into account both the life insurer's and the policyholders' point of view. Finally, we perform a sensitivity analysis to investigate the influence of selected parameters, see Section 4.6. Some of the results in this chapter are already published in Diehl et al. [23].

4.1. Initialization and parameter specification

Initialization

We start with the liability side of the balance sheet. The initial values of actuarial, bonus, and technical reserve, A_0, B_0, and V_0, are calculated according to equations (3.7.3), (3.7.4), and (3.7.5). The initial values of the respective policyholders' accounts, A_0^m, B_0^m, and V_0^m, and the initial premium P_0^m are determined by the arithmetic means of the actual contracts belonging to one cohort, see equations (2.4.2), (2.4.3), (2.4.4), and (2.4.1). Likewise, the initial number of model points M_0, the sizes of the cohorts δ_0^m, and the representatives' characteristics (e.g. the premium size P_k^m) depend on the applied method for compressing and simulating the insurance portfolio, thereby on the distribution of the actual biometric parameters, see Section 2.4.

Equation (3.5.6) implies for given initial reserve rate $\gamma_0 \in [0, 1)$ the value of the free reserve F_0 by

$$F_0 = \frac{\gamma_0 V_0}{1 - \gamma_0}.$$

Similarly, the amount of equity Q_0 is determined by the initial fraction $\psi_0 \in [0,1)$ of own funds $F_0 + Q_0$,

$$\psi_0 = \frac{F_0 + Q_0}{A_0 + B_0 + F_0 + Q_0 + L_0},$$

through

$$Q_0 = \frac{(\psi_0 - \gamma_0)\, V_0}{(1-\psi_0)\,(1-\gamma_0)} + \frac{\psi_0 L_0}{1-\psi_0}$$

for given initial liabilities to banks L_0 and $V_0 = A_0 + B_0$. The choice $\psi_0 \geq \gamma_0$ ensures non-negative initial equity. Respecting the fundamental balance sheet equation at time t_0 yields total assets

$$C_0 = A_0 + B_0 + Q_0 + F_0 + L_0.$$

The initial values C_0^s and C_0^c of the stock and cash position are specified by initial fractions π_0^s and π_0^c of C_0, i.e.,

$$\begin{aligned} C_0^s &= \pi_0^s C_0, \\ C_0^c &= \pi_0^c C_0, \end{aligned}$$

with $\pi_0^s + \pi_0^c \leq 1$. Regarding the bond part

$$C_0^b = (1 - \pi_0^s - \pi_0^c)\, C_0,$$

we assume a uniform allocation, i.e. the numbers $\varphi^b_{1-\frac{\tau}{\Delta t}}, \ldots, \varphi^b_{-1}$ of bonds purchased at past times $t_{1-\frac{\tau}{\Delta t}}, \ldots, t_{-1}$ coincide with

$$\varphi^b_{-1} = \frac{C_0^b}{\sum_{i=1-\frac{\tau}{\Delta t}}^{-1} p\,(t_0, t_i + \tau)}.$$

Parameter specification

In the following, we specify two exemplary parameter configurations used for the numerical investigations in this thesis, while modifications are made in some places to consider different scenarios regarding the insurer's business form, the interest rate environment, and to allow for possible capital market crashes.

Tables 4.1 and 4.2 refer to the calendar years 2021 and 2022, respectively. The annual guaranteed interest rate for the endowment insurance is set to the maximum technical interest rate for German insurers which was reduced by the German Federal Ministry of Finance from 0.9% in 2021 to 0.25% in 2022. Furthermore, a few other parameters are changed, e.g. the minimum stock ratio and the distribution parameters of the ages. Regarding the initial balance sheet, there is no liquidity gap at the beginning of the first period, i.e. $L_0^+ = L_0 = 0$, and the initial amount of cash equals the average excessive value of expiring bonds obtained by prior simulations yielding $\pi_0^c = 6.03\%$.

Parameter	**Description**	**Value**
	General setting and balance sheet	
$\mathcal{Y}$	Current calendar year	2021
$\mathcal{T}$	Time horizon	50 years
Δt	Period length	0.25 years
γ_0	Initial reserve rate	10%
ψ_0	Initial fraction of own funds	12%
π_0^s	Initial ratio of stocks	10%
π_0^c	Initial ratio of cash	6.03%
L_0^+	Initial bridging credit	0
L_0	Initial value of liabilities to banks	0
$\pi_k^{s,\max}$	Maximum stock ratio	35%
	Capital market model	
a	Mean-reversion speed of the short rate process	0.5
$\frac{b}{a}$	Long-term mean of the short rate process	0.7%
λ_0	Market price of interest rate risk parameter	2%
σ_r	Volatility of the short rate process	3%
r_0	Initial value of the short rate process	0.5%
μ_s	Drift of the stock price process	4%
σ_s	Volatility of the stock price process	20%
s_0	Initial value of the stock price process	100
ρ	Correlation between short rate and stock	-10%
	Management model	
$\pi_k^{s,\min}$	Minimum stock ratio	0%
$\pi_k^{s,\text{tar}}$	Target stock ratio in the CM strategy	10%
λ_k^{CPPI}	Multiplier in the CPPI strategy	2
τ	Maturity of bonds	3 years
$\widehat{i}_G$	Annual guaranteed interest rate	0.9%
γ	Target reserve rate	10%
ω	Distribution ratio	0.3
α	Participation rate	0.9
ϑ	Surrender factor	0.9
	Structure and dynamics of insurance portfolio	
$g^{\text{real},i}$	Gender	$\sim \mathcal{B}\,(0.5)$
$\underline{x}^{\text{real},i}$	Entry age	$\sim \mathcal{TN}\,(36, 36, 15, 55)$
$\overline{x}^{\text{real},i}$	Exit age	$\sim \mathcal{TN}\,(62, 5, 55, 70)$
$x_0^{\text{real},i}$	Current age	$\sim \mathcal{U}\left(\underline{x}^{\text{real},i}, \overline{x}^{\text{real},i}\right)$
u	Surrender probability parameter	3%
$P_0^{\text{real},i}$	Constant, periodic premium payments	$\sim \mathcal{U}\,(50, 500)$
$B_0^{\text{real},i}$	Initial value of the bonus account	0
δ_0	Initial size of the insurance portfolio	500,000
δ_k^{new}	Number of new customers in period k	$\sim \text{Poi}\,(\Lambda_k)$
Λ_k	Poisson parameter in δ_k^{new}	$\sim \text{Beta}\,(\alpha_k, \beta_k, 0.5\% \cdot \delta_0, 2.2\% \cdot \delta_0)$

Table 4.1.: Exemplary scenario for calendar year 2021.

Parameter	Description	Value
	General setting and balance sheet	
$\mathcal{Y}$	Current calendar year	2022
$\mathcal{T}$	Time horizon	50 years
Δt	Period length	0.25 years
γ_0	Initial reserve rate	10%
ψ_0	Initial fraction of own funds	12%
π_0^s	Initial ratio of stocks	10%
π_0^c	Initial ratio of cash	6.03%
L_0^+	Initial bridging credit	0
L_0	Initial value of liabilities to banks	0
$\pi_k^{s,\max}$	Maximum stock ratio	35%
	Capital market model	
a	Mean-reversion speed of the short rate process	0.5
b/a	Long-term mean of the short rate process	0.7%
λ_0	Market price of interest rate risk parameter	2%
σ_r	Volatility of the short rate process	3%
r_0	Initial value of the short rate process	0.1%
μ_s	Drift of the stock price process	4%
σ_s	Volatility of the stock price process	20%
s_0	Initial value of the stock price process	100
ρ	Correlation between short rate and stock	-10%
	Management model	
$\pi_k^{s,\min}$	Minimum stock ratio	5%
$\pi_k^{s,\text{tar}}$	Target stock ratio in the CM strategy	10%
λ_k^{CPPI}	Multiplier in the CPPI strategy	2
τ	Maturity of bonds	5 years
$\widehat{i}_G$	Annual guaranteed interest rate	0.25%
γ	Target reserve rate	10%
ω	Distribution ratio	0.3
α	Participation rate	0.9
ϑ	Surrender factor	0.9
	Structure and dynamics of insurance portfolio	
$g^{\text{real},i}$	Gender	$\sim \mathcal{B}(0.5)$
$\underline{x}^{\text{real},i}$	Entry age	$\sim \mathcal{TN}(38, 64, 15, 65)$
$\overline{x}^{\text{real},i}$	Exit age	$\sim \mathcal{TN}(73, 25, 65, 80)$
$x_0^{\text{real},i}$	Current age	$\sim \mathcal{U}\left(\underline{x}^{\text{real},i}, \overline{x}^{\text{real},i}\right)$
u	Surrender probability parameter	3%
$P_0^{\text{real},i}$	Constant, periodic premium payments	$\sim \mathcal{U}(50, 500)$
$B_0^{\text{real},i}$	Initial value of the bonus account	0
δ_0	Initial size of the insurance portfolio	500,000
δ_k^{new}	Number of new customers in period k	$\sim \text{Poi}(\Lambda_k)$
Λ_k	Poisson parameter in δ_k^{new}	$\sim \text{Beta}(\alpha_k, \beta_k, 0.5\% \cdot \delta_0, 2.2\% \cdot \delta_0)$

Table 4.2.: Exemplary scenario for calendar year 2022.

Genders, entry ages, exit ages, and premium payments are assumed to be independent, while current ages are assumed to be conditionally independent. As motivated in Chapter 2, we use the four-parameter beta distribution for the average number Λ_k of new customers per period k. Alternative new business scenarios can be considered by choosing different shape parameters α_k and β_k, while the remaining two parameters represent the minimum and maximum amount of newly issued contracts per period. Their values are inspired from the observations from real life insurance portfolios, see Section 2.1.

The parameters within the capital market model represent a low interest rate environment. We assume that the two Brownian motions driving the short rate and the stock prices have a negative correlation ρ. This implies that bond prices and stocks are assumed to be positively correlated as there is an inverse relationship between bond prices and interest rates. The assumption is consistent with the rationale that rising bond yields (i.e. decreasing bond prices) might cause investors to move out of stocks into bonds. The decreasing demand in stocks then leads to falling stock prices.

4.2. Compression and simulation methods: efficiency and approximation quality

In this section, we perform several simulation studies to investigate the performance of some of the compression and simulation methods $\mathcal{P}^{b_1,b_2}$ introduced in Sections 2.4.1 and 2.4.2. We analyze their efficiency and quality in approximating the development of the uncompressed insurance portfolio and the corresponding balance sheets. For this, we use methods and error measures derived in Section 2.4.3, which requires to perform naive contract-specific simulations, i.e. we need to apply $\mathcal{P}^{0,0}$, too.

If not stated otherwise, the input parameters and the distributional assumptions are taken from Table 4.2, except for the minimum stock ratio and the maturity of newly purchased bonds which are set to $\pi_k^{s,\min} = 0$ and $\tau = 3$, respectively. Regarding the new business, we choose $(\alpha_k, \beta_k) = (1, 1)$ yielding $\Lambda_k \sim \mathcal{U}\left(0.5\% \cdot \delta_0, 2.2\% \cdot \delta_0\right)$.

4.2.1. Efficiency: run time and impacting factors

Before performing the simulation studies, where each Monte Carlo simulation will consist of $N = 10,000$ simulated paths, we investigate the dependence of the required run time for a single simulation run on the size δ_0 of the initial insurance portfolio (thus also on the number δ_k^{new} of new customers in each period) and on the number of considered periods K. Since $K = \frac{\mathcal{T}}{\Delta t}$, increasing the time horizon while maintaining the period length is equivalent to decreasing Δt correspondingly while keeping $\mathcal{T}$ constant. Note that the run times were obtained by the built-in stopwatch timer functions `tic` and `toc` in Matlab on the same computer under the same conditions (e.g. same instantaneous CPU utilization). Furthermore, for a good comparability of the run times required by the different simulation methods, the initial (uncompressed) insurance portfolio and the corresponding balance sheet are fixed. We want to emphasize that the only purpose in displaying the required run times lies in the comparison of the applied simulation methods. A single value of a run time without a comparative number is not very meaningful since it depends on the available computer equipment.

Impacting factor: the size of the insurance portfolio

We begin with investigating the dependence of the run time on the size δ_0 of the initial insurance portfolio with a fixed time horizon $\mathcal{T} = 50$ years and $\Delta t = 0.25$ yielding a total number of $K = 200$ considered periods.

In the case of a run-off scenario without new business, the contract-specific simulation $\mathcal{P}^{0,0}$ requires 3.8 seconds per path, i.e. 633 minutes for a Monte Carlo simulation with $N = 10,000$ simulated paths if the initial insurance portfolio consists of $\delta_0 = 500,000$ policies. Instead, the approximating simulation method $\mathcal{P}^{1,1}$ only requires 0.01 seconds per path or 1.7 minutes for a Monte Carlo simulation, corresponding to a run time reduction of 99.7%. Simulating insurance portfolios obtained by coarser simulation methods, i.e. $\mathcal{P}^{b_1,b_2}$ with $\max\{b_1, b_2\} > 1$, are even faster.

In the case of an ongoing insurance business with new customer arrivals in every period k, $k \in \{1, \dots, K\}$, the required run times are much higher even for smaller insurance portfolios as displayed in Table 4.3.

δ_0	Run time [s] $\mathcal{P}^{0,0}$	Run time [s] $\mathcal{P}^{1,1}$
10,000	8.5	1.2
50,000	33	2.8
100,000	63	3.8
250,000	157	5.0
500,000	300	6.1
1,000,000	600 (estimated)	7.3
10,000,000	6000 (estimated)	16.0

Table 4.3.: Average required run times in seconds for simulating a single Monte Carlo path applying $\mathcal{P}^{0,0}$ and $\mathcal{P}^{1,1}$ for insurance portfolios of different sizes δ_0 in the case of an ongoing insurance business. The number of considered periods is $K = 200$.

The contract-specific simulation $\mathcal{P}^{0,0}$ requires now 300 seconds per path, i.e. 833 hours for a Monte Carlo simulation if the initial insurance portfolio consists of $\delta_0 = 500,000$ policies. The average run time of $\mathcal{P}^{1,1}$ is 6.1 and thus nearly 50 times smaller.[1] We also see that the dependence on δ_0 is stronger if we apply $\mathcal{P}^{0,0}$, i.e. here the run time is more exposed to the size of the insurance portfolio.

[1] If we fix a path of new customers, the required run time is only 0.02 seconds per path even for $\delta_0 = 10,000,000$, i.e. about three minutes for a Monte Carlo simulation. A contract-specific simulation is eight times slower for $\delta_0 = 500,000$ and may cause memory problems for larger insurance portfolios.

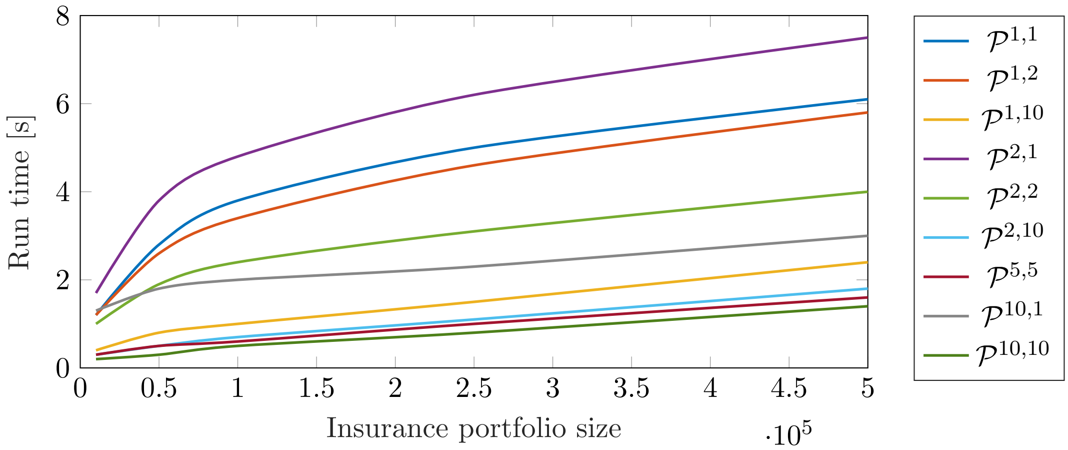

Figure 4.1.: Average required run times in seconds for simulating a single Monte Carlo path applying different simulation methods $\mathcal{P}^{b_1,b_2}$ for insurance portfolios of different sizes δ_0 in the case of an ongoing insurance business. The number of considered periods is $K = 200$.

If we apply coarser simulation methods, we may even increase the time saving as illustrated in Figure 4.1, where the corresponding run times are plotted against the sizes of the initial insurance portfolios. However, the additional amount of saved time gets smaller so that the run times of $\mathcal{P}^{2,10}$, $\mathcal{P}^{5,5}$, and $\mathcal{P}^{10,10}$ are nearly the same. Note that for $\mathcal{P}^{1,1}$, we obtain the transformed ages for grouping by the corresponding integer values, see Section 2.4.1. This is faster than following the procedure described in Section 2.4.2 for $b_1 = b_2 = 1$ and yields the same compressed insurance portfolio. As a consequence, the run time of $\mathcal{P}^{1,1}$ may be smaller than for coarser simulation methods, here $\mathcal{P}^{2,1}$. In most cases, less additional run time is required for larger δ_0. This is due to the fact that the number of generated cohorts increases slower for large insurance portfolios and is bounded from above due to logical constraints regarding the ages, as already observed in Figure 2.6 in Section 2.4.1. More importantly, new policies are merged into existing cohorts if possible.

Impacting factor: the number of considered periods

Next we investigate the dependence of the run time on the number K of considered periods. Here, we fix a time horizon of $\mathcal{T} = 50$ years and vary the period length Δt. The initial insurance portfolio consists of $\delta_0 = 250,000$ policies.

In the case of a run-off scenario, $\mathcal{P}^{0,0}$ requires 5.4 seconds per path, i.e. 900 minutes for a Monte Carlo simulation with $N = 10,000$ simulated paths if we choose a monthly discretization $\Delta t = 1/12$ yielding $K = 600$ considered periods. The method $\mathcal{P}^{1,1}$ only requires 0.03 seconds per path or 5 minutes for a Monte Carlo simulation, corresponding to a reduction of 99.6%. Again, simulating insurance portfolios obtained by coarser simulation methods, i.e. $\mathcal{P}^{b_1,b_2}$ with $\max\{b_1, b_2\} > 1$, are even faster.

In the case of an ongoing insurance business with new business, the required run times are much higher as displayed in Table 4.4.

K	Run time [s] $\mathcal{P}^{0,0}$	Run time [s] $\mathcal{P}^{1,1}$
50	4.8	1.0
100	24	2.1
200	157	5.0
600	12,296	19.5
1,200	650,000 (estimated)	46.5

Table 4.4.: Average required run times in seconds for simulating a single Monte Carlo path applying $\mathcal{P}^{0,0}$ and $\mathcal{P}^{1,1}$ for different numbers K of considered periods in the case of an ongoing insurance business. The initial size of the insurance portfolio is $\delta_0 = 250,000$.

Even more, the run time of the contract-specific simulation explodes indicating an enormous dependence on the amount of considered periods. More precisely, $\mathcal{P}^{0,0}$ requires now $12,296$ seconds per path, i.e. over $34,000$ hours for a Monte Carlo simulation if we have $K = 600$. The required run time of $\mathcal{P}^{1,1}$ is 19.5 seconds and thus more than 99.8% smaller.

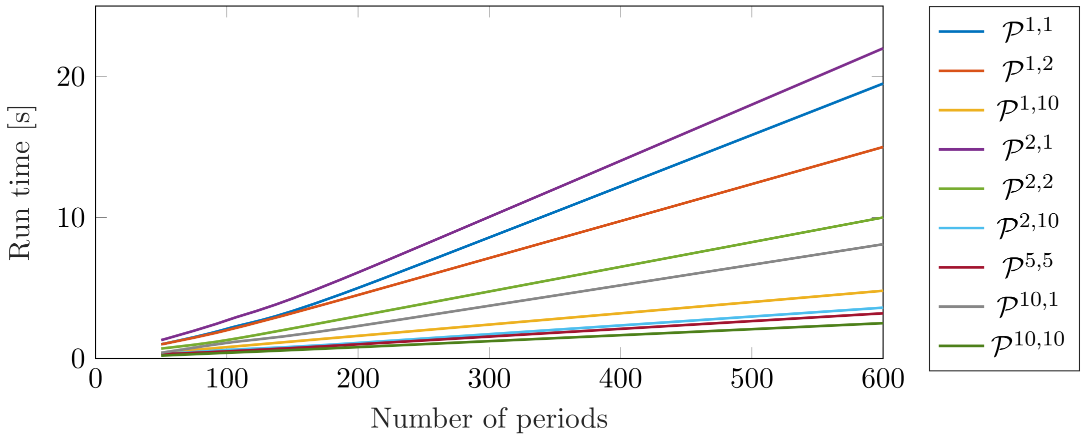

Figure 4.2.: Average required run times in seconds for simulating a single Monte Carlo path applying different simulation methods $\mathcal{P}^{b_1,b_2}$ for different numbers K of considered periods in the case of an ongoing insurance business. The initial size of the insurance portfolio is $\delta_0 = 250,000$.

As before, if we apply coarser simulation methods, the time saving may even be larger as illustrated in Figure 4.2, where the corresponding run times are plotted against the number K of considered periods. In contrast to Figure 4.1, where the required run time increases slower for large insurance portfolios, we now observe a proportional or even an over-proportional dependence. The additional amount of saved time gets smaller so that the run times of $\mathcal{P}^{2,10}$, $\mathcal{P}^{5,5}$, and $\mathcal{P}^{10,10}$ are nearly the same for the considered cases.

Development of the efficiency during the simulation

We close this section by elaborating on the development of the efficiency during the simulation. For this, we use the compression factor CF_k motivated in Section 2.4.3 and defined by equation (2.4.11). It reflects the proportion of the simulated (non-empty) cohorts M_k^{sim} to the actual size δ_k of the insurance portfolio according to $\mathcal{P}^{0,0}$, so that smaller values are associated with a more efficient compression and simulation method. In contrast to run times, the compression factor is an efficiency measure that is independent of the capability of computers. Note that it is not necessary to consider the run-off-case since here the number of non-empty cohorts decreases with the insurance portfolio size.

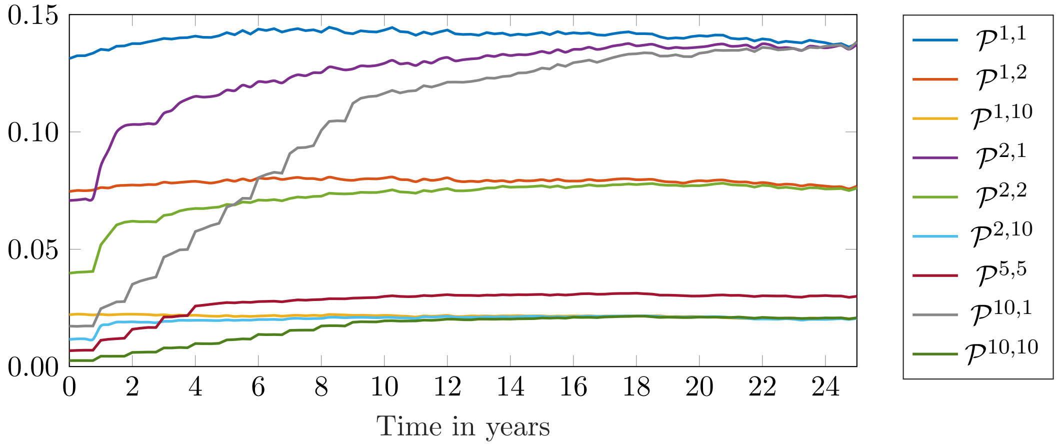

Figure 4.3.: Compression factors applying different simulation methods $\mathcal{P}^{b_1,b_2}$ in the case of an ongoing insurance business. The number of considered periods is $K = 100$ and the size of the insurance portfolio $\delta_0 = 10,000$.

Figure 4.3 displays the development of the compression factors CF_k applying different methods $\mathcal{P}^{b_1,b_2}$. We see that $\mathcal{P}^{1,1}$ has the worst efficiency which is due to the large number of generated cohorts. However, its efficiency remains stable throughout the simulation. This is not the case for e.g. $\mathcal{P}^{10,1}$ being very efficient in the short term but then the efficiency gets substantially worse. In the long term, it even has the same efficiency as $\mathcal{P}^{1,1}$. More generally, we observe that all methods $\mathcal{P}^{b_1,b_2}$ with $b_1 > 1$ suffer from a decline in efficiency throughout the simulation. This can be explained by the annually adjustment of the current ages of existing policyholders and the fact that the transformed ages of representative new customers are of the form $\underline{x}^{\min} + lb_1$ for a $l \geq 0$, cf. Section 2.4.2. Therefore, a merging of new cohorts and existing ones is not possible during the first $b_1 - 1$ years implying that M_k^{sim} increases (much) stronger than δ_k. After b_1 years, new cohorts can be merged again implying a slower increase in the cohorts and, eventually, a stationary development. Furthermore, we observe that the efficiency of a simulation method $\mathcal{P}^{b_1,b_2}$ with $b_1 > 1$ converges against the efficiency of $\mathcal{P}^{1,b_2}$. Since the exit ages remain constant, the methods $\mathcal{P}^{1,b_2}$ maintain their efficiencies throughout the simulation.

4.2.2. Approximation quality

In the following, we study the approximation quality of our developed simulation methods $\mathcal{P}^{b_1,b_2}$ for

$$(b_1, b_2) \in \{(1,1), (2,2), (5,5), (1,10), (10,1), (10,10)\} \tag{4.2.1}$$

in the case of a run-off and in the case of an ongoing insurance business with new business. Clearly, we need to restrict to smaller insurance portfolios and time horizons in order to be able to perform the required Monte Carlo simulations with an acceptable run time for $\mathcal{P}^{0,0}$, too. Therefore, we consider the given (uncompressed) insurance portfolio from Section 2.4.3 consisting of $\delta_0 = 10,000$ policies. The corresponding balance sheet is prepared according to the assumptions at the beginning of Section 4.2.

Each of the compression and simulation methods $\mathcal{P}^{b_1,b_2}$ with b_1, b_2 from (4.2.1) is applied $N = 10,000$ times. To investigate the direct impact of $\mathcal{P}^{b_1,b_2}$, in each run we use the same new business scenario and dynamics of the capital market, i.e. the n-th Monte Carlo path for $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ is based on the same generated random numbers, respectively. Thereby, we allow for a pathwise comparison between compressed and uncompressed simulation methods.

For selected quantities of interest X_k, we plot the average development according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ as a graphical visualization of the approximation quality, see equation (2.4.12) in Section 2.4.3. The robustness of the simulation methods is investigated by considering the worst-case approximation reflected by the Monte Carlo path $n^* = n^*\left(\mathcal{P}^{b_1,b_2}\right)$ defined by equation (2.4.13). The sample distributions at given time points t_k are compared using the total variation distance introduced in Section 2.2.3 by equation (2.2.4) with $J = 50$ intervals. We also look at the developments of the (estimated) cumulative default probabilities PD_k,

$$PD_k = P\left(\{Q_j < 0 \ \text{ for some } \ j \in \{0, \ldots, k\}\}\right), \tag{4.2.2}$$

according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$. For the investigations of the approximation quality, we also use the error measures AMSE AMAE, AMSPE, and AMAPE representing the average approximation error per period k, cf. Section 2.4.3.

For additional results, we refer to Appendix A.1.

Approximation quality in the case of a run-off

We first investigate the approximation quality in the case of a run-off scenario. Due to the absence of new customer arrivals, i.e. $\delta_k^{\text{new}} = 0$ for all k, we can consider larger time horizons. We choose $\mathcal{T} = 65$ with $\Delta t = 0.25$ yielding a total number of $K = 260$ considered periods.

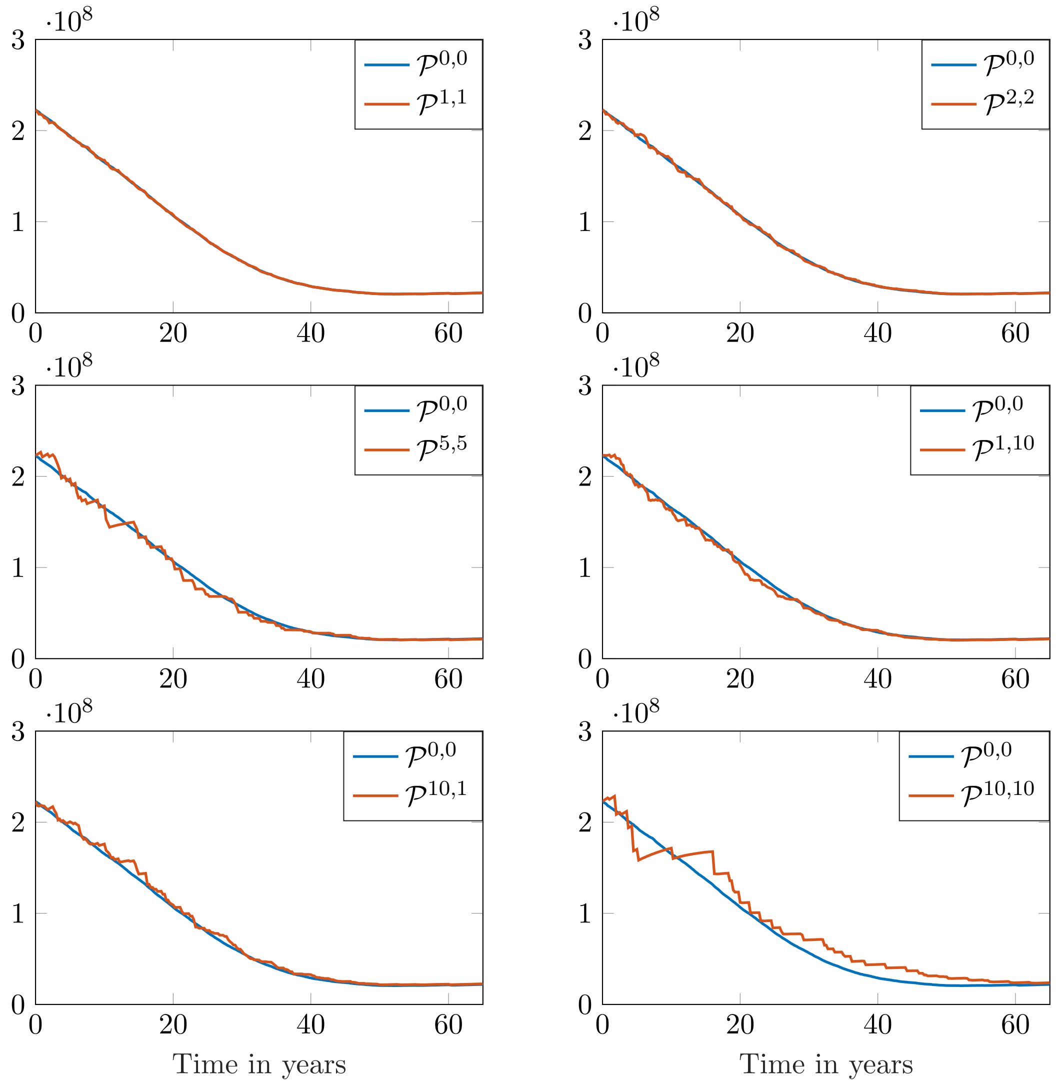

Figure 4.4.: Average development of the capital C_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of a run-off.

The average development of the capital C_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ is displayed in Figure 4.4. In the case of $\mathcal{P}^{1,1}$ and $\mathcal{P}^{2,2}$, we barely see any differences compared to $\mathcal{P}^{0,0}$. Also the coarser compression methods perform well but $\mathcal{P}^{10,10}$ does not provide very accurate approximations at most times. In all cases, the final average capital C_K approximately coincides with the one obtained by $\mathcal{P}^{0,0}$.

The respective approximation quality regarding the insurance portfolio size δ_k, the death and surrender benefit payments T_k and S_k, the liabilities to banks L_k, and the declared interest rate $\widehat{i}_k$ is similar, see the corresponding figures in Appendix A.1.

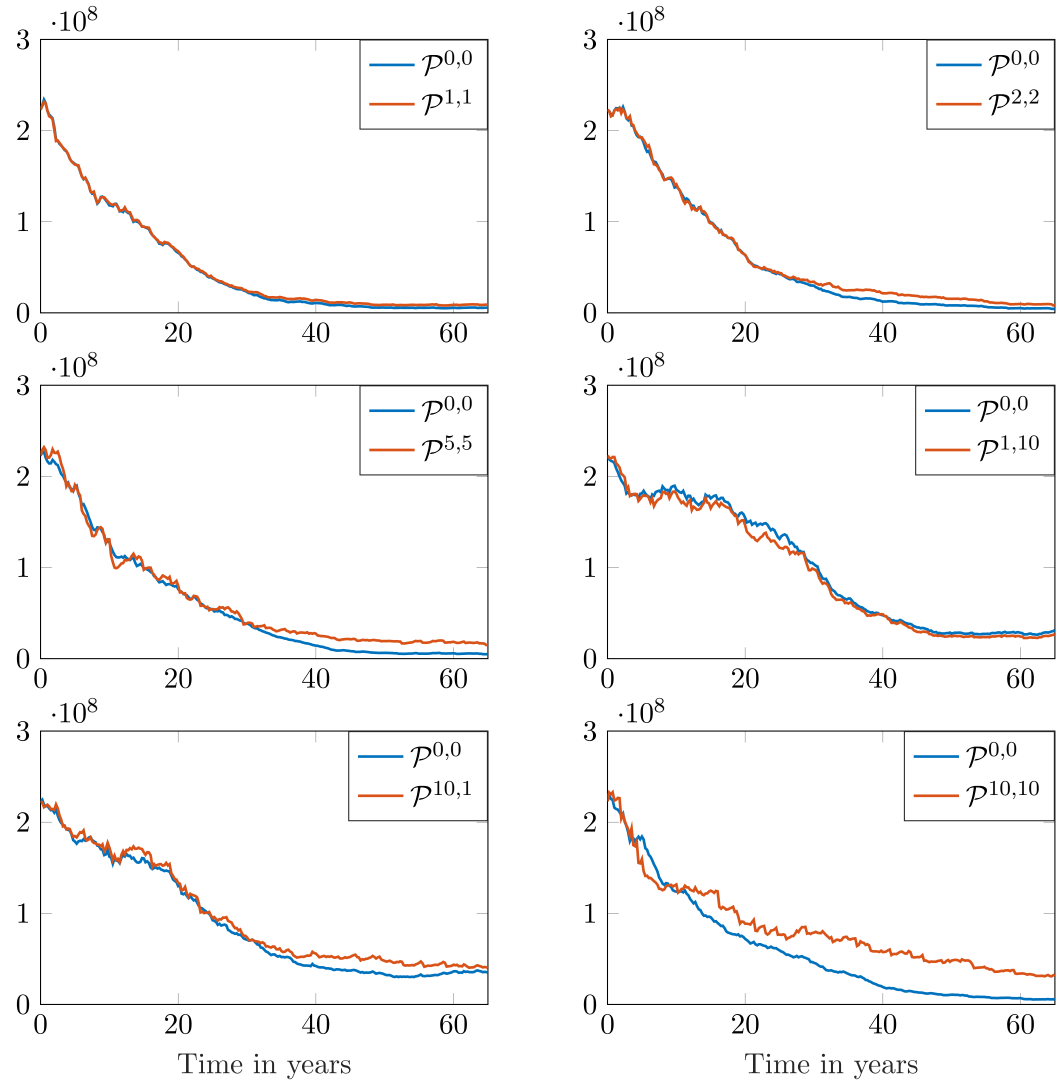

Figure 4.5.: Worst-case approximation paths n^* of the capital C_k in the case of a run-off.

The robustness of the simulation methods regarding the approximation of the capital C_k is investigated in Figure 4.5, where we show the paths $n^* = n^* \left(\mathcal{P}^{b_1,b_2}\right)$ corresponding to the worst-case approximations. The simulation method $\mathcal{P}^{1,1}$ still yields very good approximations at all times. $\mathcal{P}^{2,2}$ performs much worse in the long term contrasting the observations in the average-case in Figure 4.4, where the approximation in the long term was better than in the short term. Also $\mathcal{P}^{5,5}$, $\mathcal{P}^{10,1}$, and $\mathcal{P}^{10,10}$ perform much worse in the long term. Surprisingly, $\mathcal{P}^{1,10}$ yields very accurate approximations and is partly outperforming $\mathcal{P}^{2,2}$.

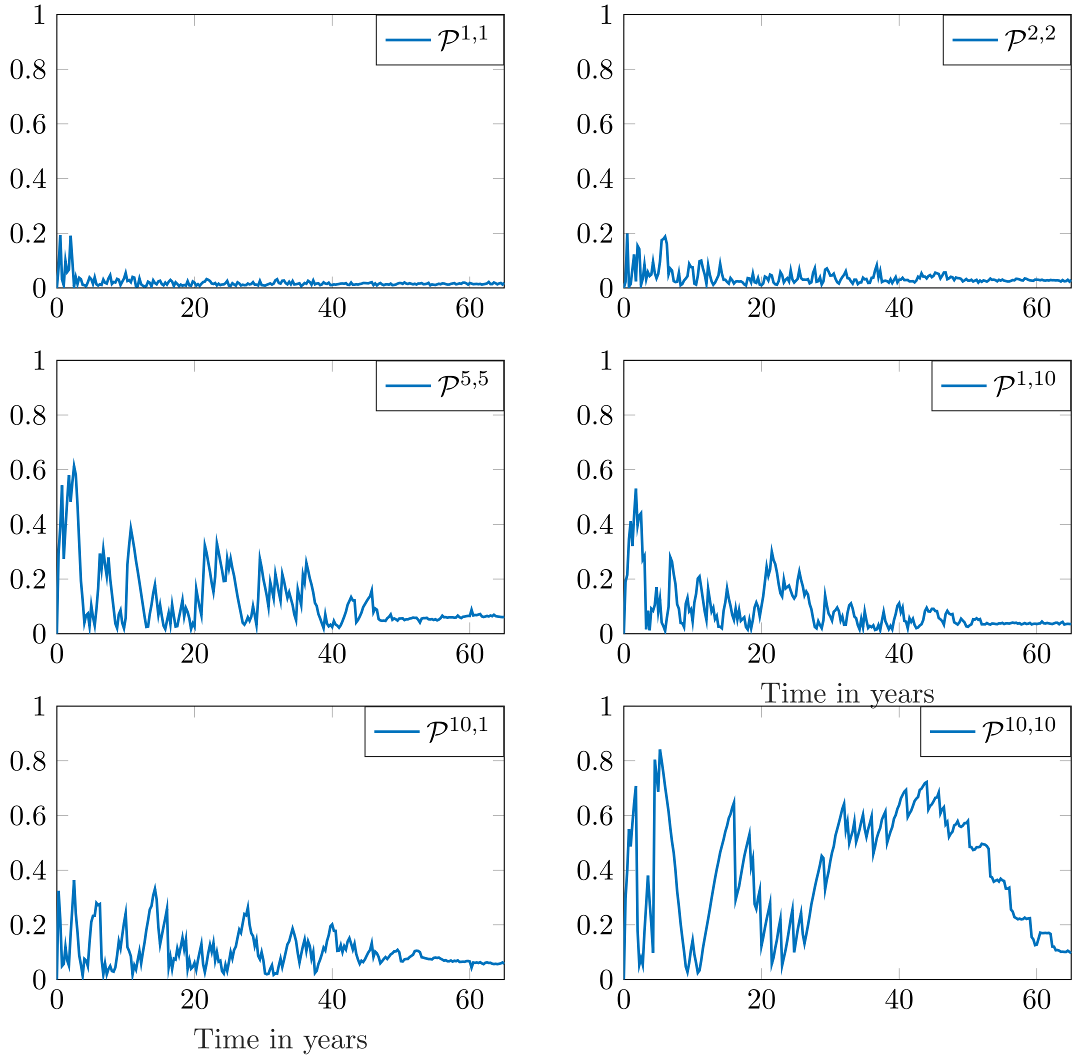

Figure 4.6.: Total variation distances of the sample distributions of the capital C_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of a run-off. The number of intervals in (2.2.4) is $J = 50$.

Figure 4.6 shows to which extent the sample distributions of the capital C_k according to $\mathcal{P}^{0,0}$ is affected by the compression and simulation methods $\mathcal{P}^{b_1,b_2}$. In the case of $\mathcal{P}^{1,1}$ and $\mathcal{P}^{2,2}$, we observe small values of the total variation distances indicating that the corresponding distributions did not change much. The other methods have a stronger influence, especially $\mathcal{P}^{10,10}$.

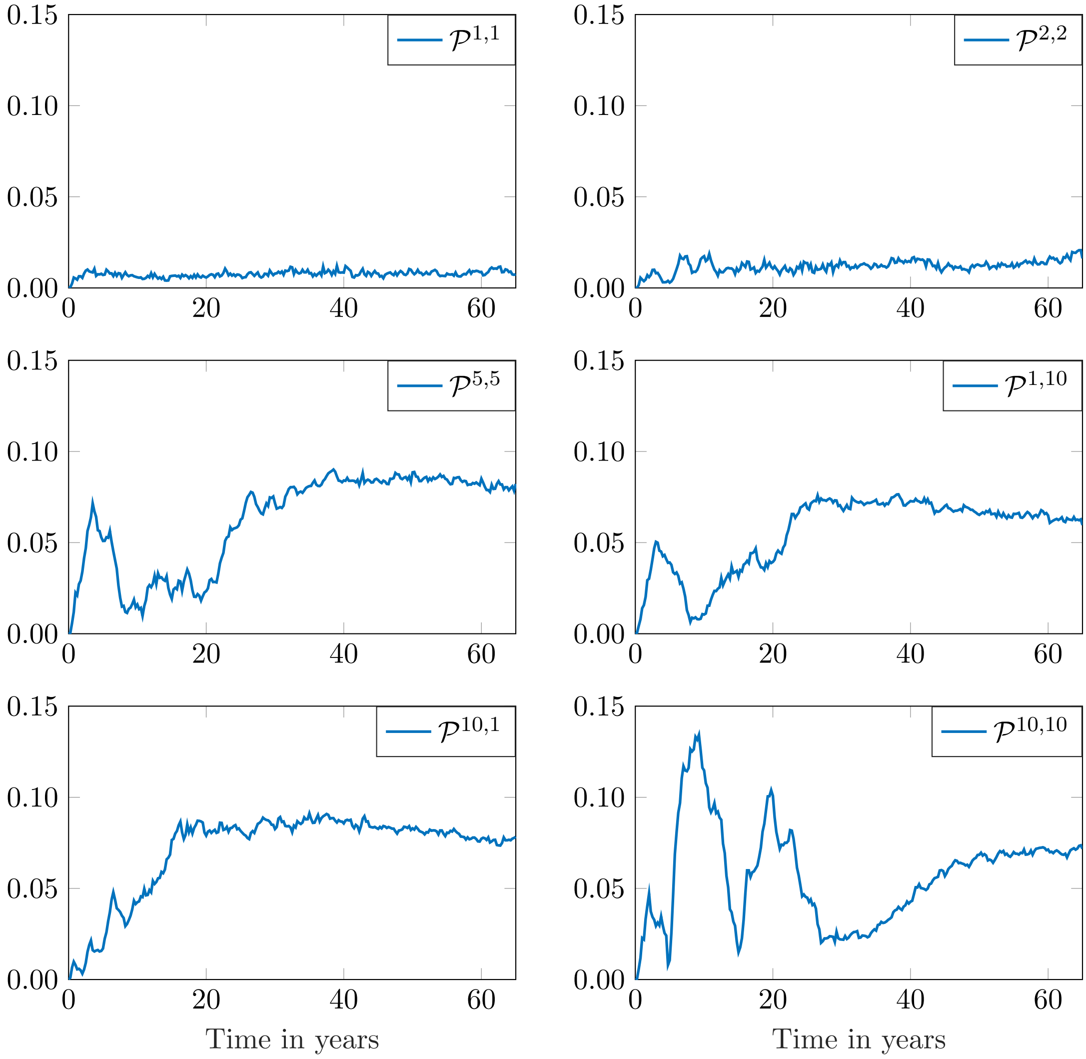

Figure 4.7.: Total variation distances of the sample distributions of the equity Q_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of a run-off. The number of intervals in (2.2.4) is $J = 50$.

The total variation distances of the sample distributions of the equity Q_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ is displayed in Figure 4.7. Again, in the case of $\mathcal{P}^{1,1}$ and $\mathcal{P}^{2,2}$, we observe the smallest values in general. It is interesting that in all cases, the values are much smaller than in Figure 4.6. This shows that the sample distributions of the equity according to $\mathcal{P}^{0,0}$ is much less affected by the simulation methods $\mathcal{P}^{b_1,b_2}$ than the distributions of the capital.

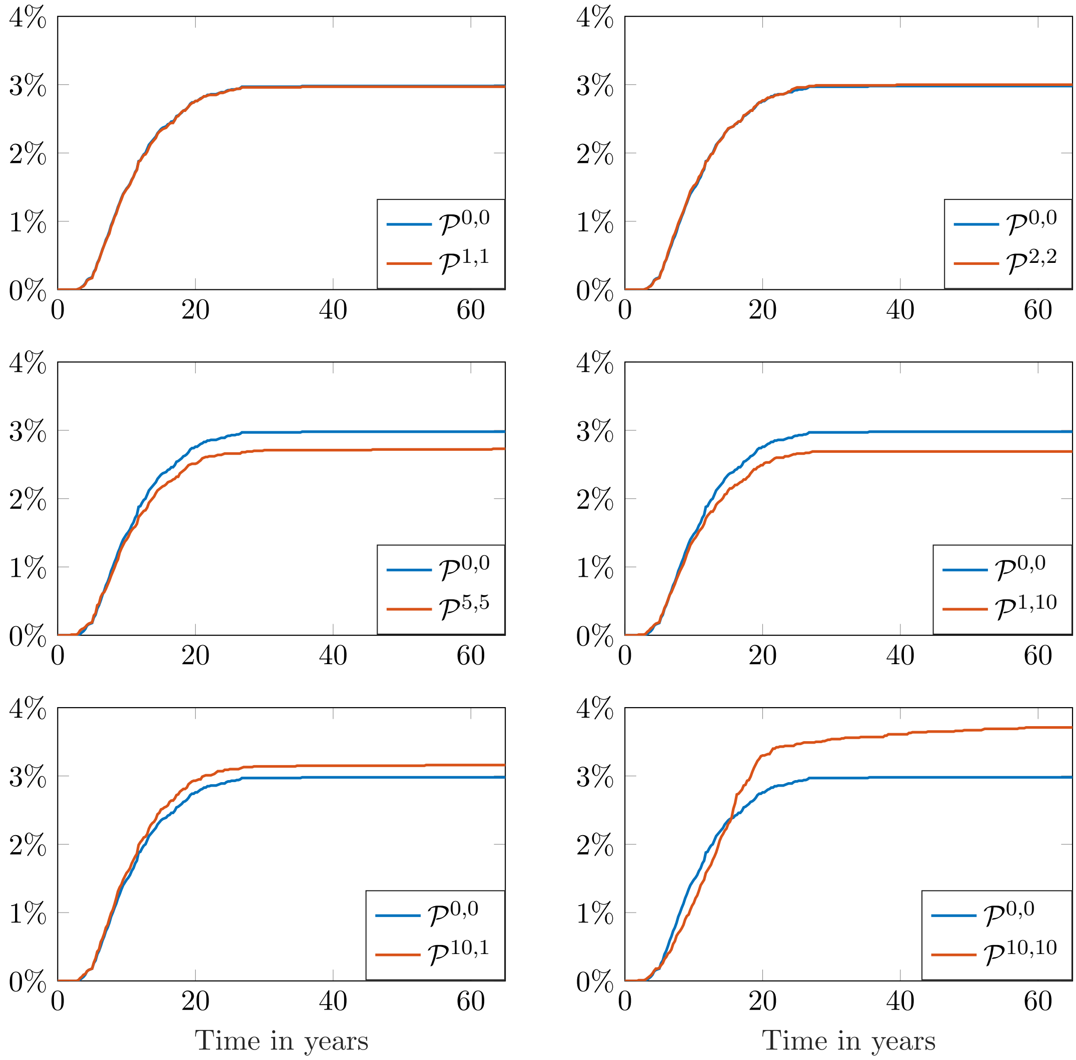

Figure 4.8.: Development of the default probability according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of a run-off.

Figure 4.8 shows the estimated default probabilities according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$. In the case of $\mathcal{P}^{1,1}$ and $\mathcal{P}^{2,2}$, we obtain nearly the same estimations, but also the other methods yield quite accurate approximations. Indeed, we could have presumed this from the observations in Figure 4.7 that the distribution of the equity under $\mathcal{P}^{0,0}$ is not much affected by the compression and simulation methods. It is interesting that both $\mathcal{P}^{5,5}$ and $\mathcal{P}^{1,10}$ underestimate the default probability at all times. This is not desirable and we would rather prefer an overestimation as we have for $\mathcal{P}^{10,1}$ and, in the long term, for $\mathcal{P}^{10,10}$.

To conclude the investigations in the run-off-case, we estimate the average approximation error per period of several quantities of interest X_k. Here, we consider additionally the actuarial reserve A_k, the bonus reserve B_k, the free reserve F_k, and the surplus G_k. The results are summarized in Table 4.5.

X_k	$\mathcal{P}^{1,1}$	$\mathcal{P}^{2,2}$	$\mathcal{P}^{5,5}$	$\mathcal{P}^{1,10}$	$\mathcal{P}^{10,1}$	$\mathcal{P}^{10,10}$
δ_k	192.1	$1.0{\cdot}10^3$	$1.4{\cdot}10^4$	$7.4{\cdot}10^3$	$5.7{\cdot}10^3$	$8.6{\cdot}10^4$
	7.2	16.6	69.4	49.8	42.5	177.5
C_k	$4.5{\cdot}10^{11}$	$2.7{\cdot}10^{12}$	$2.7{\cdot}10^{13}$	$1.4{\cdot}10^{13}$	$1.8{\cdot}10^{13}$	$2.0{\cdot}10^{14}$
	$4.7{\cdot}10^5$	$1.2{\cdot}10^6$	$3.7{\cdot}10^6$	$2.5{\cdot}10^6$	$3.0{\cdot}10^6$	$1.1{\cdot}10^7$
A_k	$2.9{\cdot}10^{11}$	$1.5{\cdot}10^{12}$	$2.0{\cdot}10^{13}$	$1.0{\cdot}10^{13}$	$1.1{\cdot}10^{13}$	$1.1{\cdot}10^{14}$
	$2.9{\cdot}10^5$	$6.7{\cdot}10^5$	$2.7{\cdot}10^6$	$1.9{\cdot}10^6$	$1.9{\cdot}10^6$	$6.5{\cdot}10^6$
B_k	$1.7{\cdot}10^{10}$	$8.9{\cdot}10^{10}$	$9.8{\cdot}10^{11}$	$3.0{\cdot}10^{11}$	$4.4{\cdot}10^{11}$	$4.4{\cdot}10^{12}$
	$8.1{\cdot}10^4$	$1.8{\cdot}10^5$	$6.0{\cdot}10^5$	$3.1{\cdot}10^5$	$3.8{\cdot}10^5$	$1.2{\cdot}10^6$
F_k	$3.7{\cdot}10^{10}$	$2.3{\cdot}10^{11}$	$9.8{\cdot}10^{11}$	$2.8{\cdot}10^{11}$	$6.0{\cdot}10^{11}$	$3.7{\cdot}10^{12}$
	$1.4{\cdot}10^5$	$3.3{\cdot}10^5$	$7.1{\cdot}10^5$	$3.8{\cdot}10^5$	$5.6{\cdot}10^5$	$1.4{\cdot}10^6$
Q_k	$1.4{\cdot}10^{10}$	$8.2{\cdot}10^{10}$	$4.8{\cdot}10^{11}$	$1.7{\cdot}10^{11}$	$2.8{\cdot}10^{11}$	$2.2{\cdot}10^{12}$
	$6.3{\cdot}10^4$	$1.5{\cdot}10^5$	$4.3{\cdot}10^5$	$2.9{\cdot}10^5$	$3.9{\cdot}10^5$	$8.0{\cdot}10^5$
L_k	$2.2{\cdot}10^{10}$	$4.3{\cdot}10^{11}$	$1.7{\cdot}10^{12}$	$7.7{\cdot}10^{10}$	$1.5{\cdot}10^{12}$	$2.1{\cdot}10^{14}$
	$2.4{\cdot}10^4$	$1.5{\cdot}10^5$	$4.5{\cdot}10^5$	$5.2{\cdot}10^4$	$3.7{\cdot}10^5$	$9.8{\cdot}10^6$
G_k	$1.8{\cdot}10^9$	$1.1{\cdot}10^{10}$	$5.0{\cdot}10^{10}$	$1.3{\cdot}10^{10}$	$2.3{\cdot}10^{10}$	$1.9{\cdot}10^{11}$
	$2.5{\cdot}10^4$	$6.6{\cdot}10^4$	$1.4{\cdot}10^5$	$6.9{\cdot}10^4$	$9.9{\cdot}10^4$	$2.8{\cdot}10^5$
E_k	$3.0{\cdot}10^{11}$	$9.1{\cdot}10^{11}$	$4.7{\cdot}10^{12}$	$2.0{\cdot}10^{12}$	$2.6{\cdot}10^{12}$	$1.5{\cdot}10^{13}$
	$3.2{\cdot}10^5$	$5.7{\cdot}10^5$	$1.4{\cdot}10^6$	$8.6{\cdot}10^5$	$9.6{\cdot}10^5$	$1.7{\cdot}10^6$
T_k	$1.0{\cdot}10^7$	$5.1{\cdot}10^7$	$5.7{\cdot}10^8$	$2.8{\cdot}10^8$	$2.0{\cdot}10^8$	$1.9{\cdot}10^9$
	$1.8{\cdot}10^3$	$4.4{\cdot}10^3$	$1.5{\cdot}10^4$	$1.1{\cdot}10^4$	$8.7{\cdot}10^3$	$2.7{\cdot}10^4$
S_k	$1.6{\cdot}10^7$	$8.5{\cdot}10^7$	$1.1{\cdot}10^9$	$5.5{\cdot}10^8$	$6.1{\cdot}10^8$	$6.0{\cdot}10^9$
	$2.4{\cdot}10^3$	$5.6{\cdot}10^3$	$2.2{\cdot}10^4$	$1.5{\cdot}10^4$	$1.5{\cdot}10^4$	$5.0{\cdot}10^4$
$\widehat{i}_k$	0.0043	0.0144	0.0300	0.0080	0.0213	0.1010
	0.0518	0.1505	0.3241	0.1297	0.2372	0.8814

Table 4.5.: Approximation quality for different quantities of interest X_k in terms of AMSE (upper rows) and AMAE (lower rows) applying different simulation methods $\mathcal{P}^{b_1,b_2}$ in the case of a run-off. For $X_k = \widehat{i}_k$, we use the corresponding relative error measures AMSPE (upper row) and AMAPE (lower row).

Comparing all these numbers, we find the following priority order with respect to the average approximation quality:

$$\mathcal{P}^{1,1} \succ \mathcal{P}^{2,2} \succcurlyeq \mathcal{P}^{1,10} \succcurlyeq \mathcal{P}^{10,1} \succ \mathcal{P}^{5,5} \succ \mathcal{P}^{10,10}.$$

Indeed, for all considered quantities, $\mathcal{P}^{1,1}$ yields the smallest average approximation error per period, $\mathcal{P}^{5,5}$ the second largest, and $\mathcal{P}^{10,10}$ the largest. For the remaining methods, there is no strict priority order possible but we see corresponding tendencies. For example, $\mathcal{P}^{2,2}$ is mostly in the second place but regarding the liabilities to banks L_k and the declared interest rate $\widehat{i}_k$, $\mathcal{P}^{1,10}$ performs better. Regarding the insurance portfolio size δ_k and the death benefit payments T_k, $\mathcal{P}^{1,10}$ performs worse than $\mathcal{P}^{10,1}$. The priority orders of the preferences according to the error measures AMSE and AMAE coincide in all cases.

Approximation quality in the case of an ongoing insurance business

Now we investigate the approximation quality in the case of an ongoing insurance business with new customer arrivals. The time horizon is $\mathcal{T} = 25$ with $\Delta t = 0.25$ yielding a total number of $K = 100$ considered periods.

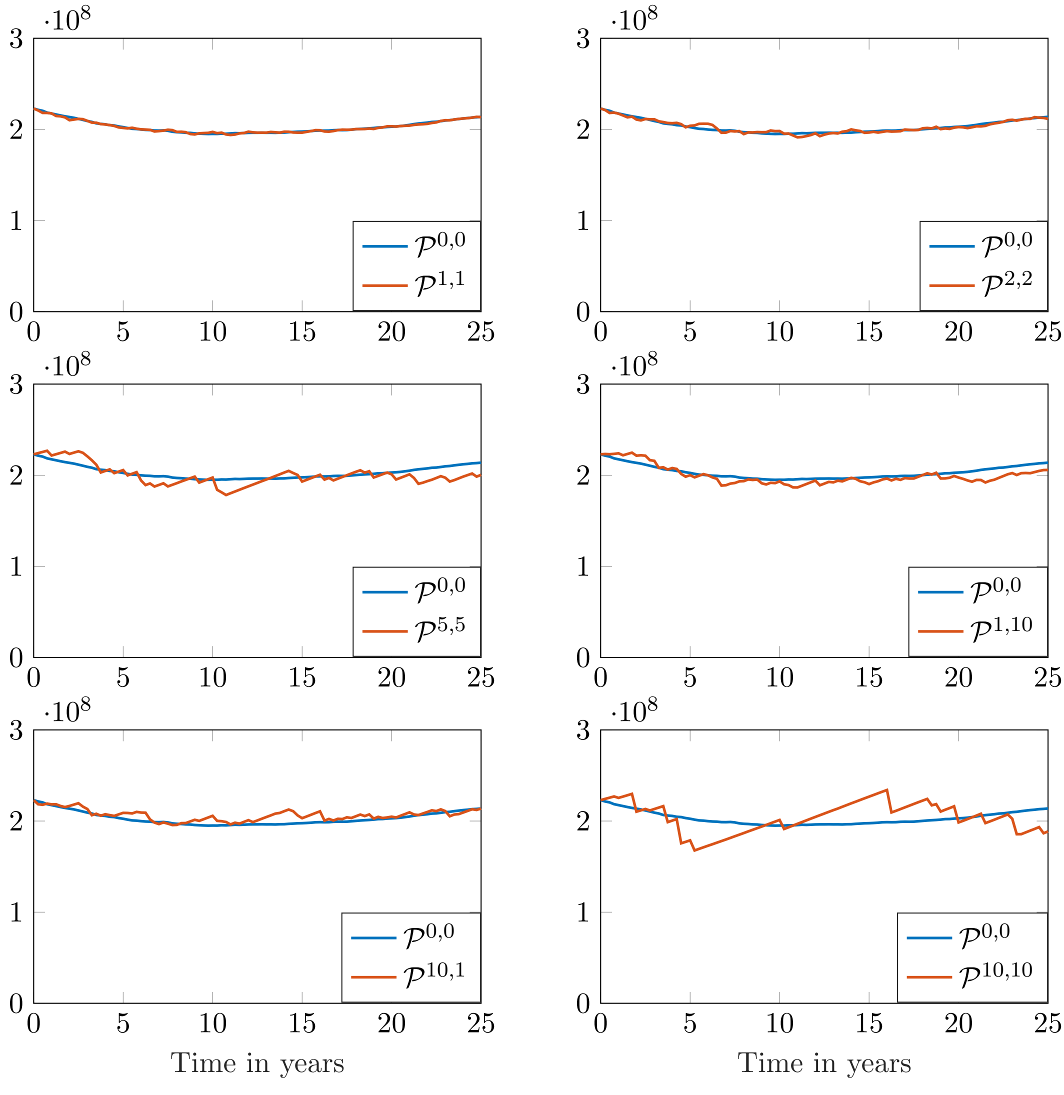

Figure 4.9.: Average development of the capital C_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of an ongoing insurance business.

The average development of the capital C_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ is shown in Figure 4.9. The respective approximation quality is comparable with the one in the run-off-case. $\mathcal{P}^{1,1}$ and $\mathcal{P}^{2,2}$ perform very well and also the coarser simulation methods mostly yield satisfying results. However, the approximation provided by $\mathcal{P}^{10,10}$ is quite inaccurate at most times.

As we can see in the corresponding figures in Appendix A.1, the last observations also hold for the insurance portfolio size δ_k, the death and surrender benefit payments T_k and S_k, and the declared interest rate $\widehat{i}_k$.

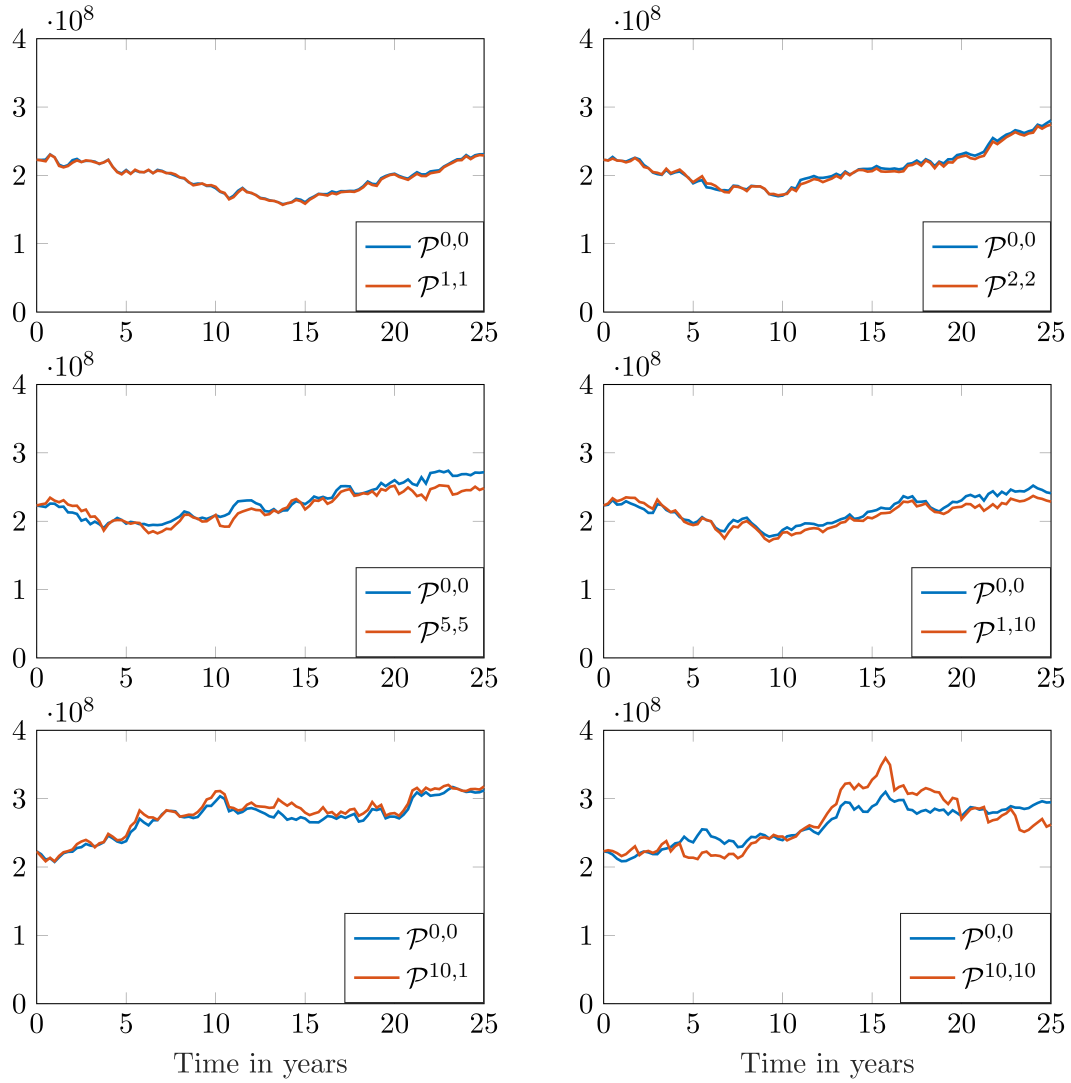

Figure 4.10.: Worst-case approximation paths n^* of the capital C_k in the case of an ongoing insurance business.

In Figure 4.10, we illustrate the robustness of the simulation methods regarding the approximation of the capital C_k. In contrast to the run-off-case, where the worst-case approximation applying $\mathcal{P}^{2,2}$, $\mathcal{P}^{5,5}$, and $\mathcal{P}^{10,1}$ is much worse in the long term, now all methods maintain their good performance. Again, especially $\mathcal{P}^{1,1}$ yields very good approximations at all times.

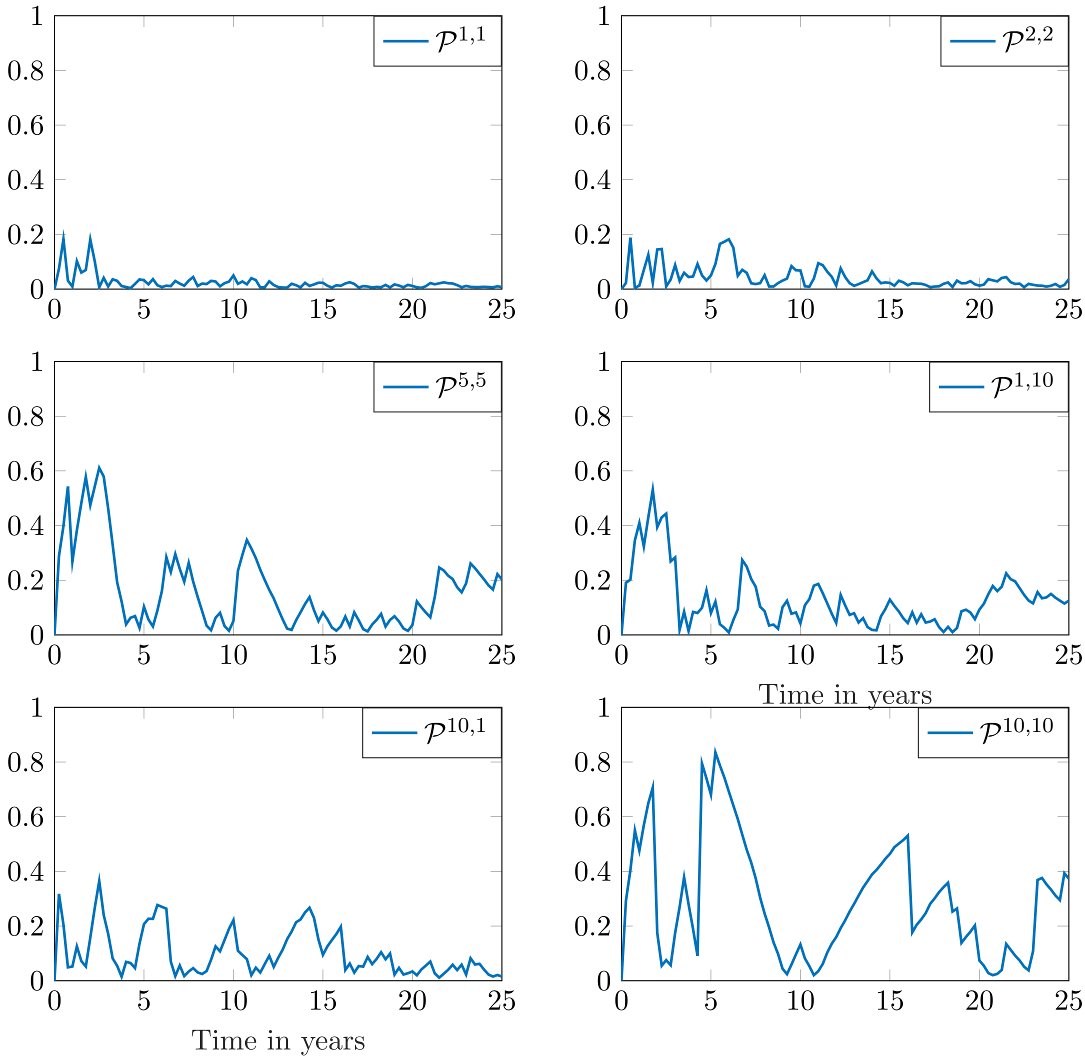

Figure 4.11.: Total variation distances of the sample distributions of the capital C_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of an ongoing insurance business. The number of intervals in (2.2.4) is $J = 50$.

Figure 4.11 displays the total variation distances regarding the capital C_k according to the simulation methods. They are each of a similar size as in the run-off-case or even smaller and tend to decrease in the long term. This shows that despite the uncertainty induced by the new business, the sample distributions of C_k according to $\mathcal{P}^{0,0}$ are well approximated by the simulation methods.

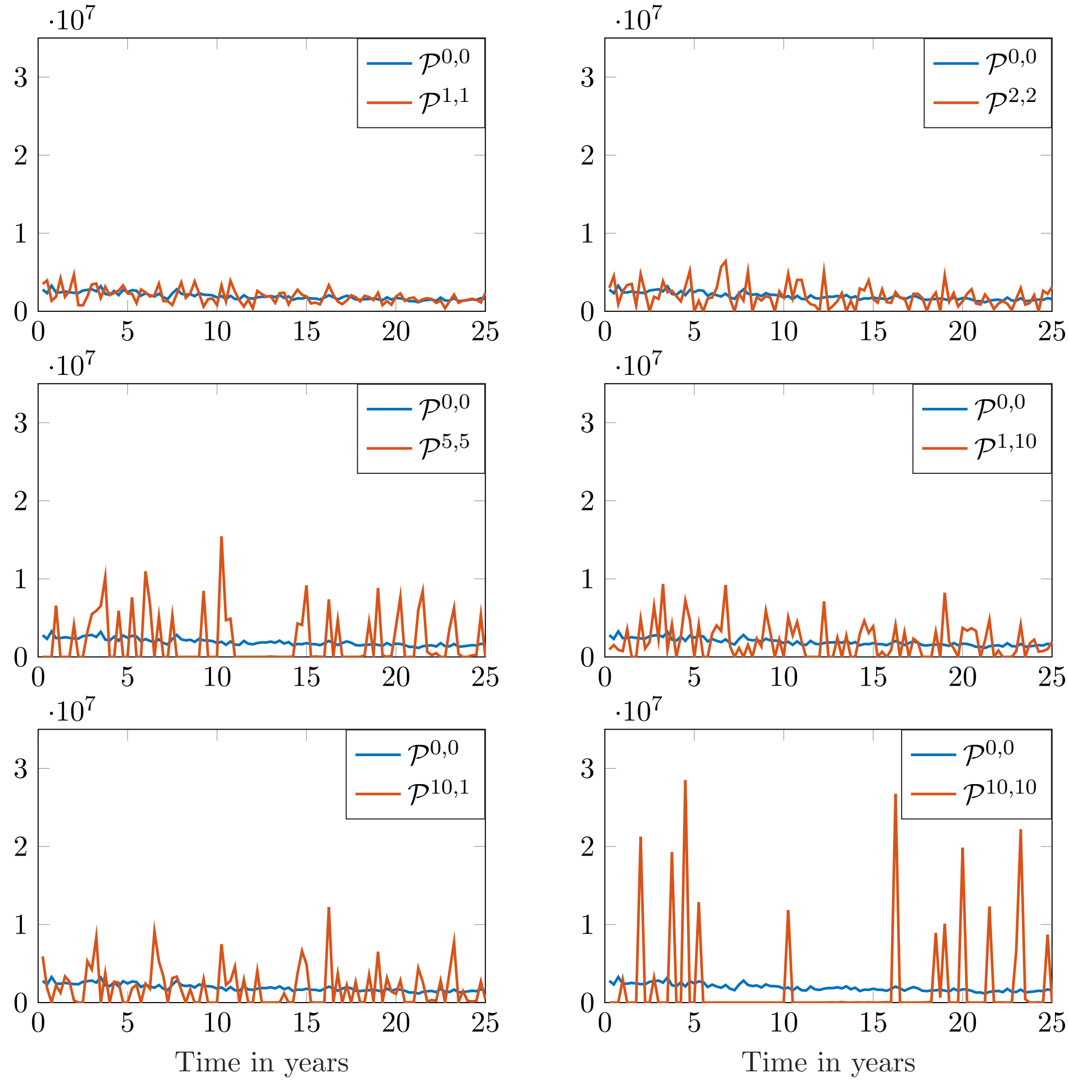

Figure 4.12.: Average development of the survival benefit payments E_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of an ongoing insurance business.

The performance of approximating the survival payments E_k is not satisfying as we can conclude from Figure 4.12. Only $\mathcal{P}^{1,1}$ yields quite accurate approximations. Especially in the case of $\mathcal{P}^{10,10}$, we obtain a very poor performance. There are many times where no survival payments are paid at all while at others we observe large peaks. This can be explained by the small number of generated cohorts that contain accordingly many policies expiring all at the same time. Indeed, according to $\mathcal{P}^{10,10}$, the insured collective consisting of $\delta_0 = 10,000$ policies is grouped into 26 cohorts, cf. Figure 2.11 in Section 2.4.3, and from Figure 4.3 we can conclude that this number does not increase a lot.

X_k	$\mathcal{P}^{1,1}$	$\mathcal{P}^{2,2}$	$\mathcal{P}^{5,5}$	$\mathcal{P}^{1,10}$	$\mathcal{P}^{10,1}$	$\mathcal{P}^{10,10}$
δ_k	521.6	$2.8{\cdot}10^3$	$5.4{\cdot}10^4$	$3.0{\cdot}10^4$	$1.3{\cdot}10^4$	$2.2{\cdot}10^5$
	17.2	39.2	184.5	145.0	87.7	382.4
C_k	$8.6{\cdot}10^{11}$	$4.3{\cdot}10^{12}$	$7.0{\cdot}10^{13}$	$3.9{\cdot}10^{13}$	$3.1{\cdot}10^{13}$	$2.9{\cdot}10^{14}$
	$7.1{\cdot}10^5$	$1.6{\cdot}10^6$	$6.7{\cdot}10^6$	$5.2{\cdot}10^6$	$4.3{\cdot}10^6$	$1.4{\cdot}10^7$
A_k	$7.3{\cdot}10^{11}$	$3.7{\cdot}10^{12}$	$5.8{\cdot}10^{13}$	$3.2{\cdot}10^{13}$	$2.5{\cdot}10^{13}$	$2.5{\cdot}10^{14}$
	$6.4{\cdot}10^5$	$1.4{\cdot}10^6$	$6.2{\cdot}10^6$	$4.8{\cdot}10^6$	$3.8{\cdot}10^6$	$1.3{\cdot}10^7$
B_k	$1.6{\cdot}10^{10}$	$6.7{\cdot}10^{10}$	$7.6{\cdot}10^{11}$	$2.9{\cdot}10^{11}$	$2.4{\cdot}10^{11}$	$3.0{\cdot}10^{12}$
	$8.2{\cdot}10^4$	$1.7{\cdot}10^5$	$5.3{\cdot}10^5$	$3.2{\cdot}10^5$	$3.0{\cdot}10^5$	$1.1{\cdot}10^6$
F_k	$2.2{\cdot}10^{10}$	$9.4{\cdot}10^{10}$	$6.4{\cdot}10^{11}$	$3.8{\cdot}10^{11}$	$5.4{\cdot}10^{11}$	$3.7{\cdot}10^{12}$
	$1.1{\cdot}10^5$	$2.2{\cdot}10^5$	$5.9{\cdot}10^5$	$4.6{\cdot}10^5$	$5.4{\cdot}10^5$	$1.4{\cdot}10^6$
Q_k	$6.0{\cdot}10^9$	$2.8{\cdot}10^{10}$	$1.6{\cdot}10^{11}$	$7.5{\cdot}10^{10}$	$1.2{\cdot}10^{11}$	$8.3{\cdot}10^{11}$
	$3.4{\cdot}10^4$	$7.0{\cdot}10^4$	$1.9{\cdot}10^5$	$1.5{\cdot}10^5$	$2.2{\cdot}10^5$	$4.3{\cdot}10^5$
L_k	0	0	$1.8{\cdot}10^5$	0	0	$9.9{\cdot}10^{11}$
	0	0	0.4	0	0	$1.6{\cdot}10^5$
G_k	$1.6{\cdot}10^9$	$7.5{\cdot}10^9$	$5.3{\cdot}10^{10}$	$2.2{\cdot}10^{10}$	$2.6{\cdot}10^{10}$	$2.7{\cdot}10^{11}$
	$2.6{\cdot}10^4$	$5.5{\cdot}10^4$	$1.6{\cdot}10^5$	$1.0{\cdot}10^5$	$1.1{\cdot}10^5$	$3.3{\cdot}10^5$
E_k	$7.1{\cdot}10^{11}$	$2.0{\cdot}10^{12}$	$1.1{\cdot}10^{13}$	$4.8{\cdot}10^{12}$	$5.8{\cdot}10^{12}$	$3.6{\cdot}10^{13}$
	$6.4{\cdot}10^5$	$1.1{\cdot}10^6$	$2.7{\cdot}10^6$	$1.8{\cdot}10^6$	$1.9{\cdot}10^6$	$3.5{\cdot}10^6$
T_k	$2.4{\cdot}10^7$	$1.3{\cdot}10^8$	$1.1{\cdot}10^9$	$5.4{\cdot}10^8$	$3.9{\cdot}10^8$	$4.2{\cdot}10^9$
	$3.7{\cdot}10^3$	$9.2{\cdot}10^3$	$2.6{\cdot}10^4$	$1.9{\cdot}10^4$	$1.5{\cdot}10^4$	$5.0{\cdot}10^4$
S_k	$3.8{\cdot}10^7$	$1.9{\cdot}10^8$	$3.0{\cdot}10^9$	$1.6{\cdot}10^9$	$1.2{\cdot}10^9$	$1.3{\cdot}10^{10}$
	$4.7{\cdot}10^3$	$1.0{\cdot}10^4$	$4.5{\cdot}10^4$	$3.4{\cdot}10^4$	$2.6{\cdot}10^4$	$9.1{\cdot}10^4$
$\widehat{i}_k$	$5.5{\cdot}10^{-6}$	$2.2{\cdot}10^{-5}$	$1.7{\cdot}10^{-4}$	$6.9{\cdot}10^{-5}$	$6.7{\cdot}10^{-5}$	$7.7{\cdot}10^{-4}$
	0.0137	0.0273	0.0822	0.0512	0.0498	0.1761

Table 4.6.: Approximation quality for different quantities of interest X_k in terms of AMSE (upper rows) and AMAE (lower rows) applying different simulation methods $\mathcal{P}^{b_1,b_2}$ in the case of an ongoing insurance business. For $X_k = \widehat{i}_k$, we use the corresponding relative error measures AMSPE (upper row) and AMAPE (lower row).

Table 4.6 summarizes the estimated average approximation error per period of several quantities of interest X_k in the case of an ongoing insurance business with new business.[2] The size orders are comparable with the corresponding ones in the run-off-scenario (Table 4.5). Also the priority order with respect to the average approximation quality is nearly the same:

$$\mathcal{P}^{1,1} \succ \mathcal{P}^{2,2} \succ \mathcal{P}^{10,1} \succcurlyeq \mathcal{P}^{1,10} \succcurlyeq \mathcal{P}^{5,5} \succ \mathcal{P}^{10,10}.$$

Remarkable is that now $\mathcal{P}^{10,1}$ mostly performs better than $\mathcal{P}^{1,10}$ with exceptions for the free reserve F_k, the equity Q_k, the surplus G_k, and the survival benefit payments E_k. For

[2]According to $\mathcal{P}^{0,0}$, there is no demand for credits in any considered scenario.

all considered quantities, $\mathcal{P}^{2,2}$ yields the second smallest average approximation error per period.

Except for the equity Q_k, the priority orders of the preferences according to the error measures AMSE and AMAE coincide. Indeed, according to the latter, $\mathcal{P}^{5,5}$ performs better than $\mathcal{P}^{10,1}$ in approximating Q_k.

4.2.3. Summary of the observations and recommendations

We complete this section by summarizing some of the observations from the last two sections with the purpose of developing corresponding guidelines.

Section 4.2.1 showed that simulating all policies is not a practicable method. In particular in the case of an ongoing insurance business with new customer arrivals in every period, the required run time for simulating a large, uncompressed insurance portfolio explodes if we consider large time horizons with many periods. Furthermore, depending on the available computer equipment, memory problems may occur. This indicates that approximating simulation methods are not optional but necessary. In the following, we propose priority orders for the considered simulation methods regarding different performance criteria.

Efficiency criterion 1: run time

Regarding the run time, we observed a dependence on the size of the insurance portfolio and a particularly strong one on the number of considered periods. Keeping in mind that we need to generate at least 10,000 paths per Monte Carlo simulation for each considered scenario and parameter specification, even small differences in the pathwise run time may significantly impact the total time exposure. Regarding the simulation methods $\mathcal{P}^{b_1,b_2}$ with b_1, b_2 from equation (4.2.1), we thus find the following strict priority order:

$$\mathcal{P}^{10,10} \succ \mathcal{P}^{5,5} \succ \mathcal{P}^{1,10} \succ \mathcal{P}^{10,1} \succ \mathcal{P}^{2,2} \succ \mathcal{P}^{1,1}.$$

Efficiency criterion 2a: compression factor (for smaller $\mathcal{T}$)

Looking at the compression factor in Figure 4.3, we saw that there are substantial differences regarding the development of the methods' efficiencies throughout the simulation. For those methods $\mathcal{P}^{b_1,b_2}$ with $b_1 > 1$, we observed a (partly very strong) decline in efficiency, while the ones of $\mathcal{P}^{1,b_2}$ remained stable from the beginning. Yet for smaller time horizons $\mathcal{T}$, we would prefer $\mathcal{P}^{b_1,b_2}$ with $b_1 > 1$ over $\mathcal{P}^{1,b_2}$:

$$\mathcal{P}^{10,10} \succ \mathcal{P}^{5,5} \succapprox \mathcal{P}^{1,10} \succ \mathcal{P}^{10,1} \succ \mathcal{P}^{2,2} \succ \mathcal{P}^{1,1}.$$

Efficiency criterion 2b: compression factor (for larger $\mathcal{T}$)

Due to the convergence towards the efficiencies of $\mathcal{P}^{1,b_2}$, we are indifferent between $\mathcal{P}^{b_1,b_2}$ and $\mathcal{P}^{1,b_2}$ after a certain amount of time (e.g. after 10 and 25 years for $b_2 = 10$ and $b_2 = 1$, respectively) yielding the following priority order for larger values of $\mathcal{T}$:

$$\mathcal{P}^{10,10} \sim \mathcal{P}^{1,10} \succ \mathcal{P}^{5,5} \succ \mathcal{P}^{2,2} \succ \mathcal{P}^{10,1} \sim \mathcal{P}^{1,1}.$$

Approximation quality

In Section 4.2.2, we observed that the less efficient strategies tend to perform better. In all cases, $\mathcal{P}^{1,1}$ provided the best approximation and $\mathcal{P}^{10,10}$ mostly the worst.

Approximation quality in case of run-off

Regarding the group of quantities of interest considered in Table 4.5, we found the following priority order in the case of a run-off:

$$\mathcal{P}^{1,1} \succ \mathcal{P}^{2,2} \succneqq \mathcal{P}^{1,10} \succneqq \mathcal{P}^{10,1} \succ \mathcal{P}^{5,5} \succ \mathcal{P}^{10,10}.$$

However, we saw in the previous figures that the approximation quality may differ significantly among single quantities of interests. In particular, $\mathcal{P}^{1,10}$ outperformed $\mathcal{P}^{2,2}$ regarding the worst-case approximation of the capital C_k in the long term, cf. Figure 4.5.

Approximation quality in case of ongoing insurance business

In the case of an ongoing insurance business, we saw that the approximation quality remains stable throughout the simulations. Regarding the group of quantities of interest considered in Table 4.6, we found the following priority order:

$$\mathcal{P}^{1,1} \succ \mathcal{P}^{2,2} \succ \mathcal{P}^{10,1} \succneqq \mathcal{P}^{1,10} \succneqq \mathcal{P}^{5,5} \succ \mathcal{P}^{10,10}.$$

Note that the primary purpose of Table 4.6 is the comparison of the corresponding simulation methods. In order to decide if a single value reflects an accurate approximation or not, we can relate it to the average value of X_k per period. For example, the obtained AMAE for C_k corresponds to 0.3% and 6.9% of the average capital per period according to $\mathcal{P}^{1,1}$ and $\mathcal{P}^{10,10}$, respectively.

Approximation quality regarding default probability

All simulation methods performed very well in approximating the estimated default probability in both considered business forms. Based on the assumption that we would rather like to overestimate the default probability instead of underestimating it, we propose the following priority order:

$$\mathcal{P}^{1,1} \sim \mathcal{P}^{2,2} \succ \mathcal{P}^{10,1} \succ \mathcal{P}^{10,10} \succneqq \mathcal{P}^{1,10} \sim \mathcal{P}^{5,5}.$$

Conclusion and method selection

In this section, we sketched how to proceed in choosing the appropriate compression and simulation method. As in many cases, the choice breaks down in selecting an efficient method associated with a tolerable error level that depends on the specific application. From a practitioner's point of view, the capability of computers may constrain the class of applicable simulation methods. If not stated otherwise, we apply the compression and simulation method $\mathcal{P}^{1,1}$ for the remainder of this thesis.

4.3. Run-off and ongoing insurance business

In this section, we compare a run-off scenario with an ongoing business where in addition new customers arrive in course of time. Among others, we analyze the effects of incorporating stationary new business on the development and structure of the future balance sheets. For the nearly stationary setting, we choose $(\alpha_k, \beta_k) = (1, 1)$ yielding $\Lambda_k \sim \mathcal{U}(0.5\% \cdot \delta_0, 2.2\% \cdot \delta_0)$. We perform a Monte Carlo simulation consisting of $N = 10,000$ paths. With the objective of a good comparability of both settings, we start with the same balance sheet and insurance portfolio. Moreover, surrender probabilities are modeled homogeneously by equation (2.3.5). If not stated otherwise, the input parameters and the distributional assumptions are taken from Table 4.1.

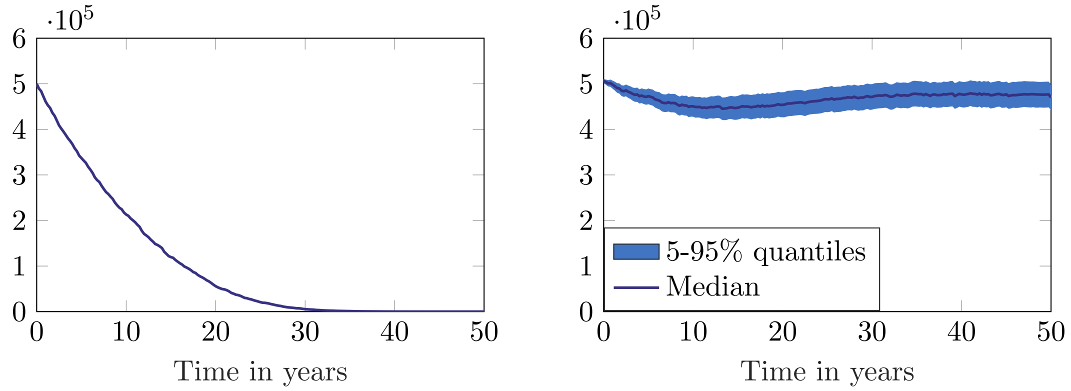

Figure 4.13.: Development of the size of the insurance portfolio. Left: run-off, right: ongoing insurance business.

Figure 4.13 displays the size of the insurance portfolio in course of time. In the run-off-case, it is falling monotonously and after 30 years, there only remains about 1% of the contracts. In the ongoing-business-case, it first decreases and then becomes stable. The development in both cases heavily depends on the distribution of the biometric parameters of the insured collective. The deterministic decrement results from our approach of modeling mortality and cancellation, whereas the random numbers of new customers induce uncertainty in the ongoing insurance business-case. This, and the independence of the random capital market's variations also explain the development of the actuarial reserve in Figure 4.14.

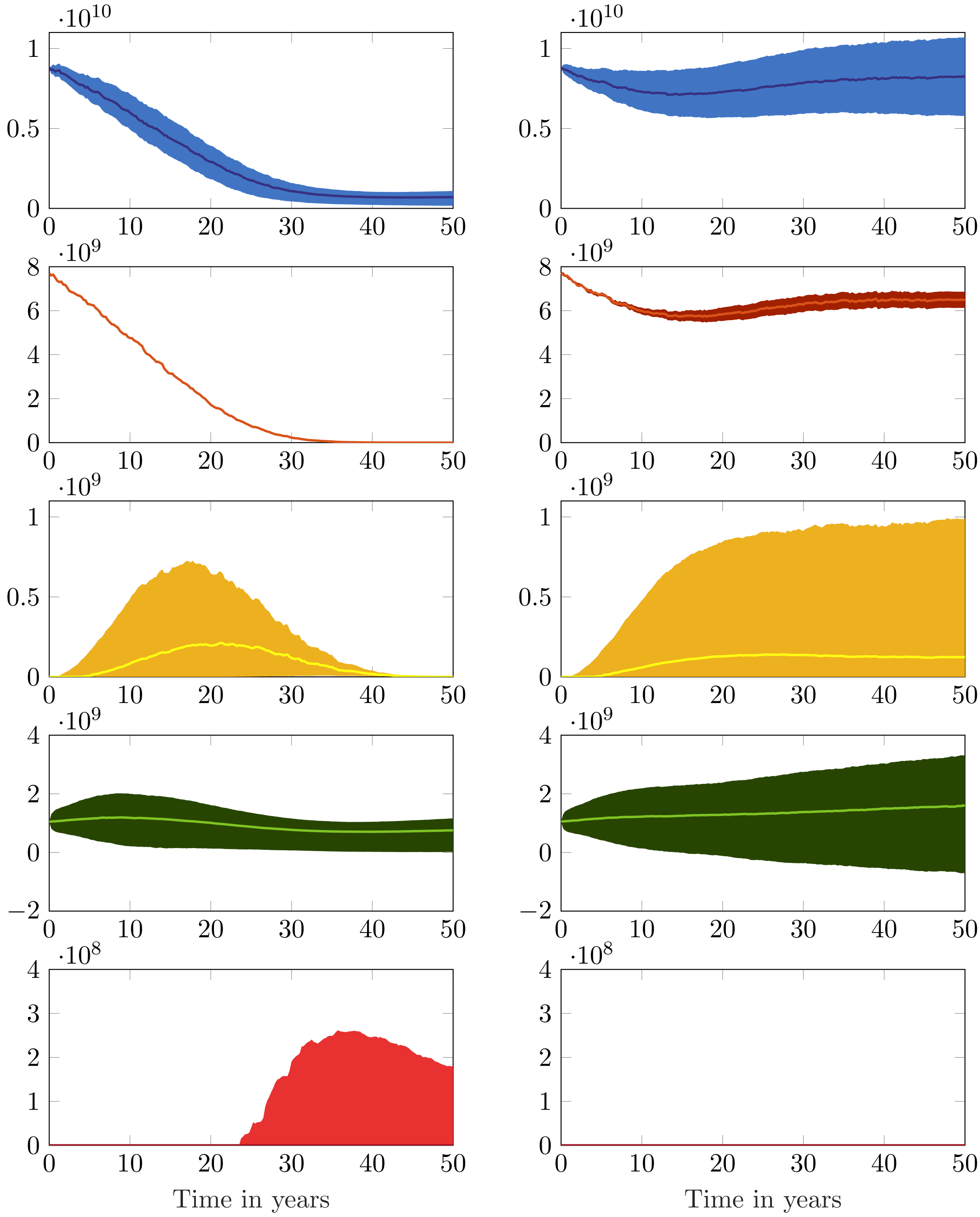

Figure 4.14.: Median of (aggregated) balance sheet positions with corresponding 5-95% quantiles. Top to bottom: capital, actuarial reserve, bonus reserve, own funds, liabilities to banks. Left: run-off, right: ongoing insurance business.

In Figure 4.14, we show the development of the (aggregated) balance sheet positions. The corresponding 5-95% quantiles are illustrated by colored areas. Only during the first years, the developments in both cases look similar. Then, the effect of including new business becomes clearly visible. While capital C_k and actuarial reserve A_k continue to decrease in the run-off scenario, they become more and more stable in the case of an ongoing business. The bonus reserve B_k is built up in the first years and then reduces (on the left) or becomes stable (on the right). Looking at the size of the quantile distance illustrated by the width of the colored areas, the uncertainty regarding own funds $F_k + Q_k$ increases in the case of an ongoing business, while it reduces from year 10 onwards in the other case. Demand for credits can only be observed in the long term in the case of a run-off. But even there, the median equals zero at all times.

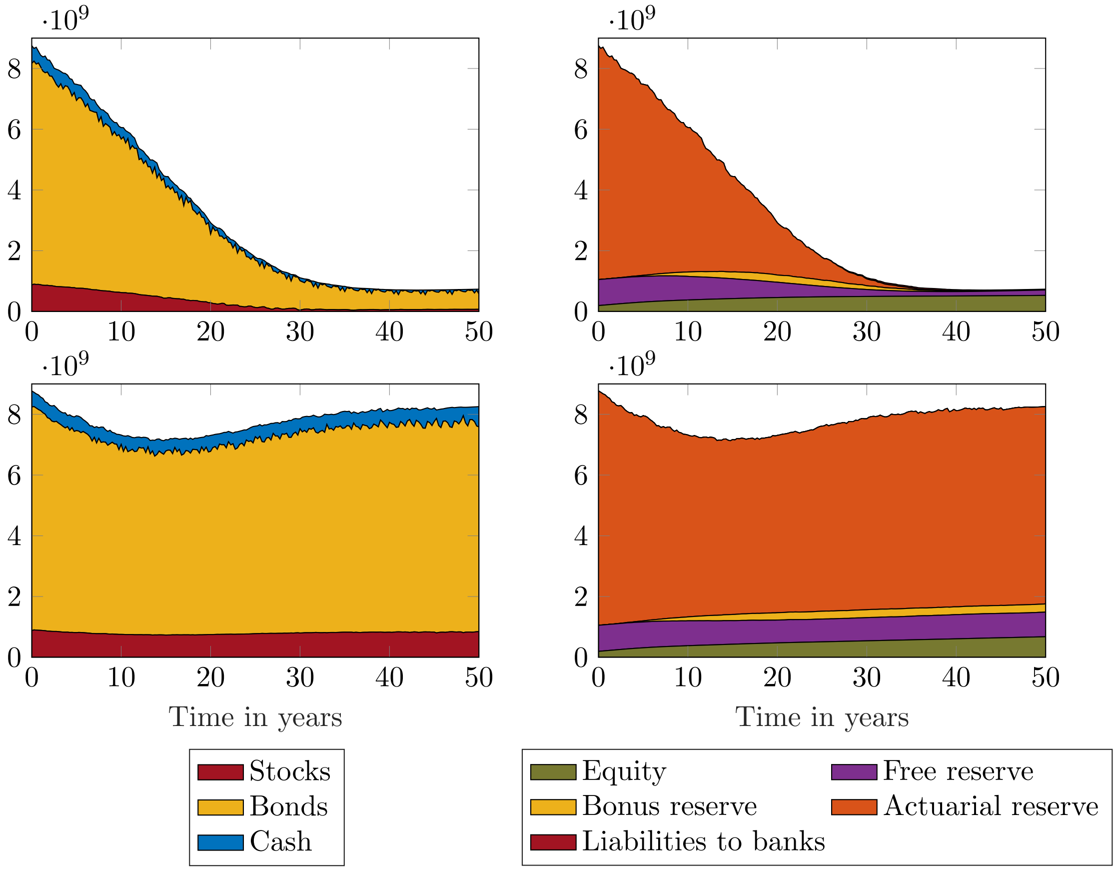

Figure 4.15.: Expected balance sheet structure. Top: run-off, bottom: ongoing insurance business. Left: assets, right: liabilities.

In Figure 4.15, we illustrate the expected structure of the balance sheet in the case of a run-off and an ongoing business with stationary new business. In particular, the graphs visualize the fulfillment of the fundamental balance sheet equation.

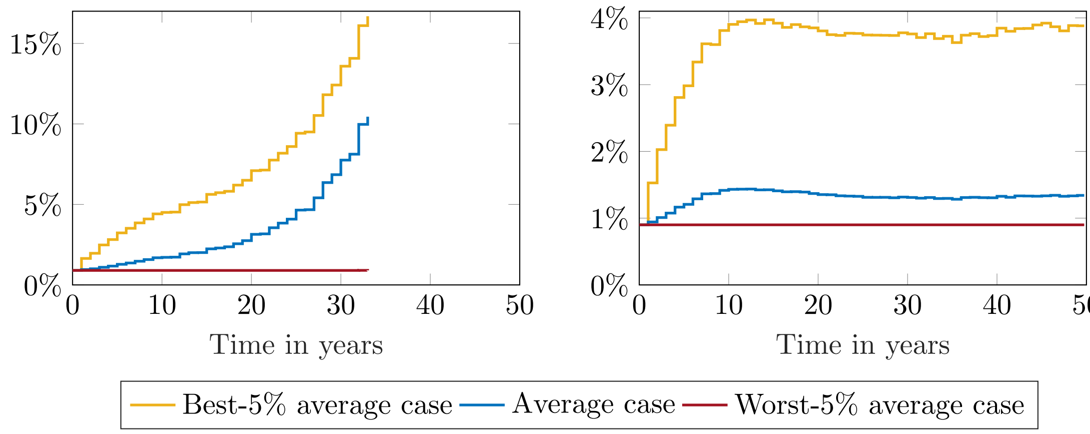

Figure 4.16.: Declared interest rate $\widehat{i}_k$. Left: run-off, right: ongoing insurance business.

In Figure 4.16, we see the annually declared interest rate $\widehat{i}_k$. On both sides, only the guaranteed rate $\widehat{i}_G = 0.9\%$ is paid in the worst-5% average case. Let us now focus on the run-off scenario for a moment. On average and in the best-5% average case, the interest rate increases in course of time, and especially fast after 22 years. This is caused by the stronger decrement of the technical reserve V_k compared to the free reserve F_k yielding constantly increasing reserve rates γ_k and thus higher interest rates, cf. equation (3.5.5). After 33 years, when there are only less than 0.5% of the initial policyholders left, the interest rates get unrealistically large.[3] Note that in other studies sometimes an upper bound is put on the declared interest rate, e.g. 10% in Gerstner et al. [27], which we do not do here. Looking at the right-hand side, we see that here the declared interest rate becomes stable. Already after six years, there are no adjustments larger than 0.3 percentage points. In the medium and long term, the policyholders can expect 1.3% on average and 3.8% in the best-5% average case.

4.4. Ongoing business with alternative new business scenarios

Now we investigate for ongoing insurance business the effect of non-stationary new business. For this, we specify four alternative patterns of new contract arrivals and study the effects on the expected balance sheet structure. The case of a stationary new business is set as a benchmark (Scenario 0). Scenarios 1 and 2 correspond to a gradually expanding and a decreasing new business. Positive and negative shocks on the expected future number of new customers are considered in the last two scenarios. The chosen shape parameters of the beta distribution for modeling the new contract arrivals are displayed in Table 4.7 at the end of this section. If not stated otherwise, the input parameters and the distributional assumptions are taken from Table 4.1.

[3] After 48 years, when there are no longer any contracts in the insurance portfolio, we would observe interest rates of 27%, 26%, and 16% in the considered average cases.

The corresponding new business scenarios and the resulting size of the insurance portfolios are illustrated in Figure 4.17 where, e.g., the first row reflects the scenario of a stationary new business (NBS 0) and the last row reflects the scenario with a negative shock on the expected future number of new customers after 25 years (NBS 4).

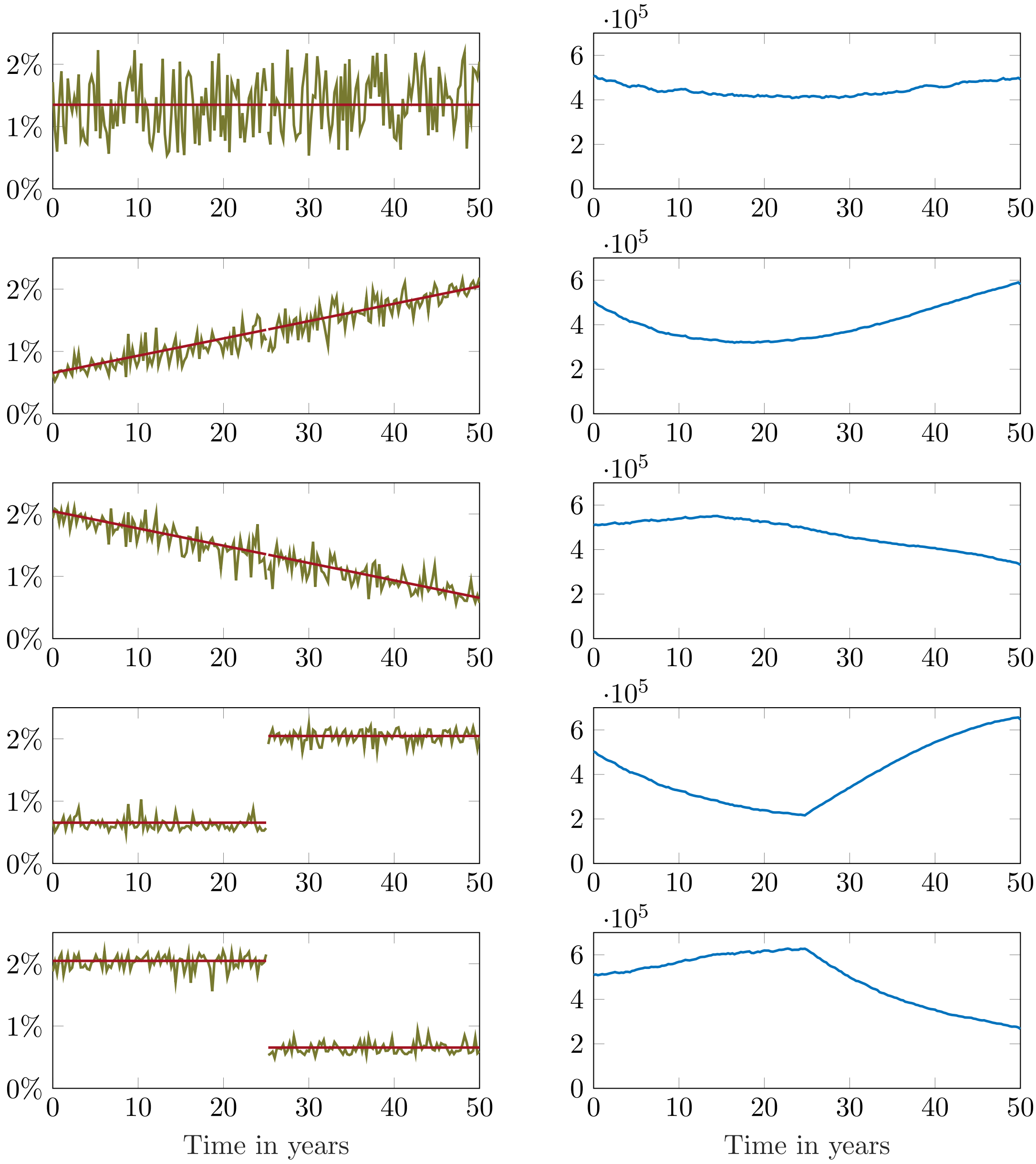

Figure 4.17.: New business scenarios NBS 0 (top) to NBS 4 (bottom). Left: number of new customers per period (expected number in red, one path in green), right: corresponding path of the insurance portfolio size.

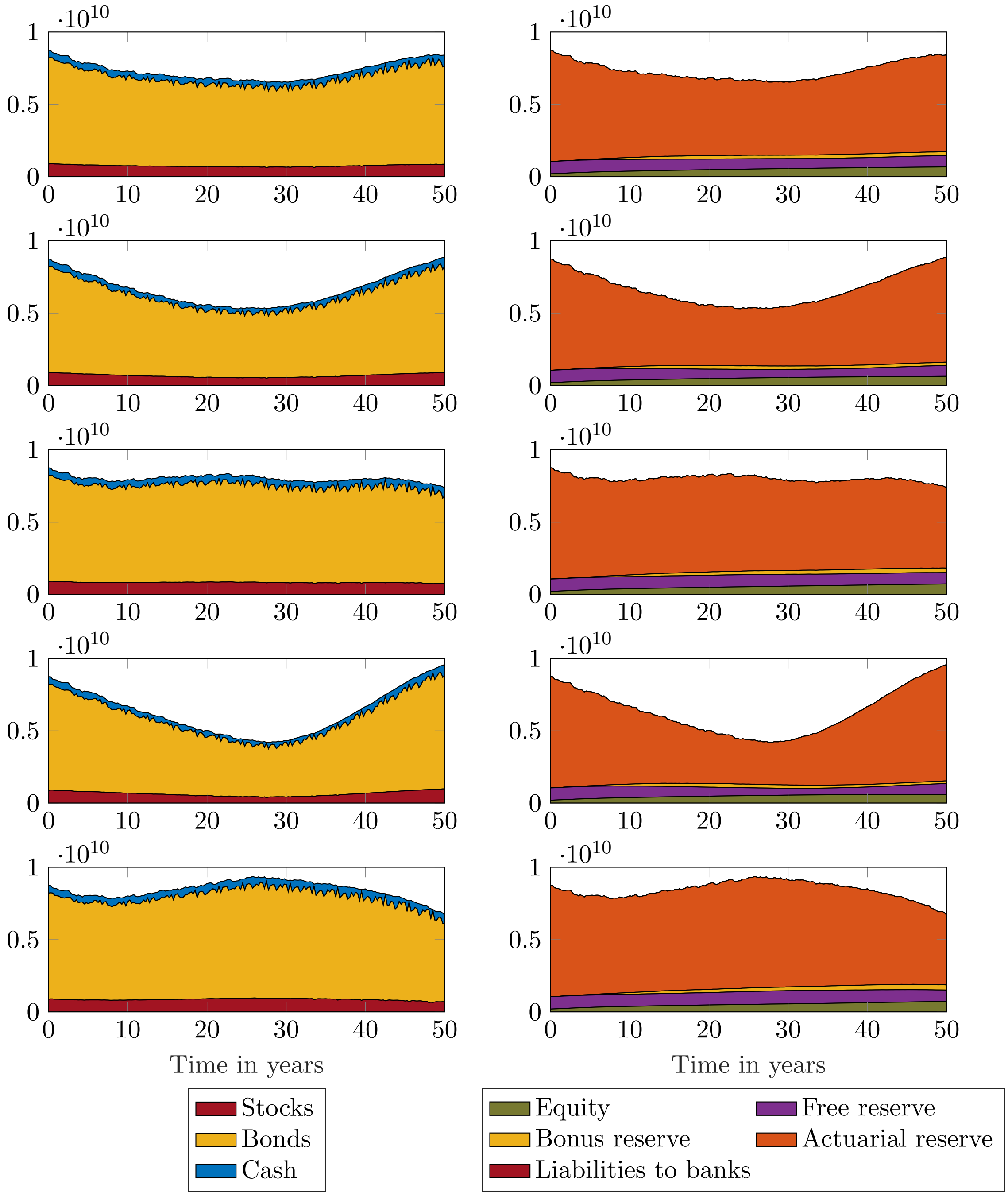

Figure 4.18.: Expected balance sheet structure for NBS 0 (top) to NBS 4 (bottom). Left: assets, right: liabilities.

The expected balance sheet structures within the considered scenarios are shown in Figure 4.18. We can clearly see the dependence on the development of the future new business. However, one can also observe stability on average regarding the target of a constant stock portion according to the CM strategy and the amount of equity or the reserves. Even in the extreme scenarios NBS 3 and NBS 4, the bonus reserve, the free reserve, and the equity remain stable which is in line with the life insurer's objectives of a smooth surplus participation and the preservation of enough own funds for future uncertainties.

NBS	Shape parameters α_k and β_k
0	$(\alpha_k, \beta_k) = (1, 1)$ for $k = 1, \ldots, K$
1	$(\alpha_1, \beta_1) = (2, 20)$, $(\alpha_K, \beta_K) = (20, 2)$, and for $k = 0, \ldots, K-2$:
	$\alpha_{k+1} = \alpha_1 + (\alpha_K - \alpha_1)\frac{k}{K-1}$, $\beta_{k+1} = \beta_1 - (\beta_1 - \beta_K)\frac{k}{K-1}$
2	$(\alpha_1, \beta_1) = (20, 2)$, $(\alpha_K, \beta_K) = (2, 20)$, and for $k = 0, \ldots, K-2$:
	$\alpha_{k+1} = \alpha_1 - (\alpha_1 - \alpha_K)\frac{k}{K-1}$, $\beta_{k+1} = \beta_1 + (\beta_K - \beta_1)\frac{k}{K-1}$
3	$(\alpha_k, \beta_k) = (2, 20)$ for $k = 1, \ldots, \frac{K}{2}$ and
	$(\alpha_k, \beta_k) = (20, 2)$ for $k = \frac{K}{2} + 1, \ldots, K$
4	$(\alpha_k, \beta_k) = (20, 2)$ for $k = 1, \ldots, \frac{K}{2}$ and
	$(\alpha_k, \beta_k) = (2, 20)$ for $k = \frac{K}{2} + 1, \ldots, K$

Table 4.7.: Parameterization of the considered new business scenarios (NBS).

4.5. CM and CPPI strategies

In the following, we compare the two investment strategies from Section 3.5.1 in the case of an ongoing insurance business with stationary new business. They essentially differ in terms of the capital invested in stocks. In order to analyze the specific characteristics and behavior of these strategies, we first consider a single simulation path for each strategy based on the same sequence of generated random numbers and perform Monte Carlo simulations later. If not stated otherwise, the input parameters and the distributional assumptions are taken from Table 4.1.

4.5.1. Performance within a crash-free capital market

Figure 4.19 shows the path of the stock prices in this exemplary scenario and the resulting reached stock ratios over time. Before the investment, i.e. at the end of the previous period, they can be larger than the maximum or target value, since funds needed for financing the disbursements are firstly taken from expired bonds, see Section 3.5.3. After the investment, however, this is not possible. Following the CPPI strategy, the amount of funds invested in stocks is implicitly linked to the actual stock prices via the free reserve and equity. They contain the generated surpluses depending partly on the past stock prices. This characteristic can be observed in Figure 4.19: during the first years, the stock initially performed well and then decreased substantially. As a result, the stock ratio increased to over 30% and then dropped to zero around 3 years. At all times, increasing or decreasing stock ratios are traceable to corresponding variations of the stock prices. Note that the described dependence would be even more visible if we had no or a higher maximum stock ratio (here 35%). In the CM-case, we observe a completely different development. After the reallocation of assets, the stock ratio equals almost always the target value of 10%.

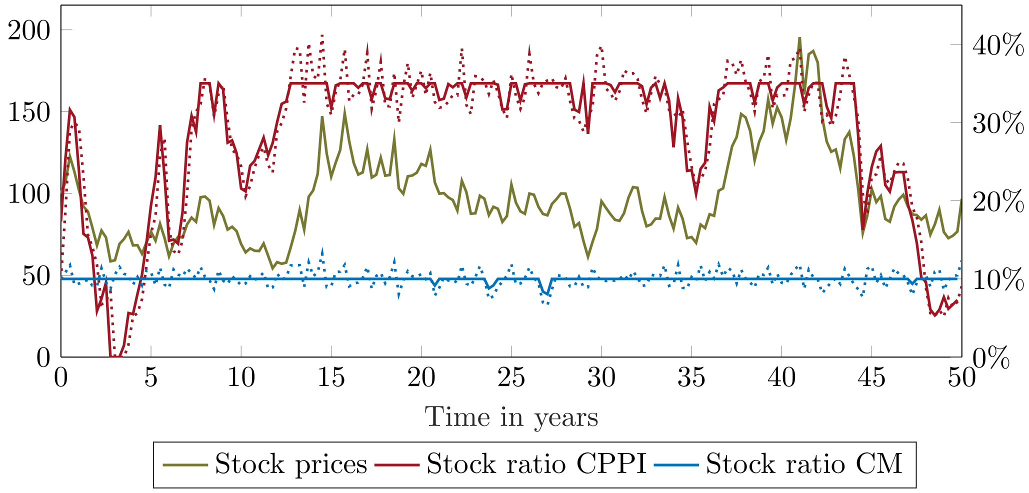

Figure 4.19.: Reached stock ratio before (dotted) and after the reallocation of assets. The left scale is only for the stock prices.

As we can see in Figure 4.20, the differences between the two strategies are also reflected by the amount of bought bonds due to investment and the value of sold stocks due to payment of benefits and repayment of credits. The CPPI strategy yields a volatile development for both positions. There are many points in time where no bonds are bought at all, while at others we see large purchases. As a result, the life insurer often needs to sell stocks, since the amount of expired bonds does not suffice. For example, after six years we observe the first large peak which is due to the fact that no bonds were bought at time $t_k = 3$. Instead, at this time the life insurer takes loans by short-selling or emitting bonds, here illustrated by a negative amount of bought bonds. The reason why no stocks were sold after three years (although no bonds were bought at time $t_0 = 0$) is that the stock ratio was 0, see Figure 4.19. In the CM-case, the amount of bought bonds is always non-negative and the development seems to be quite balanced over time. As a consequence, selling of stocks due to financing of disbursements is less frequent. The periodic pattern in the value of bought bonds in both cases can be explained by the assumption that liquid funds are first invested in stocks for each strategy, while the remaining part is used to buy bonds having a fixed duration of τ years (here $\tau = 3$) and which are held until maturity. More generally, the investments are determined by a variety of factors, including the chosen strategy, the development of prices, but also the disbursements depending partly on the distribution of biometric parameters within the insurance portfolio.

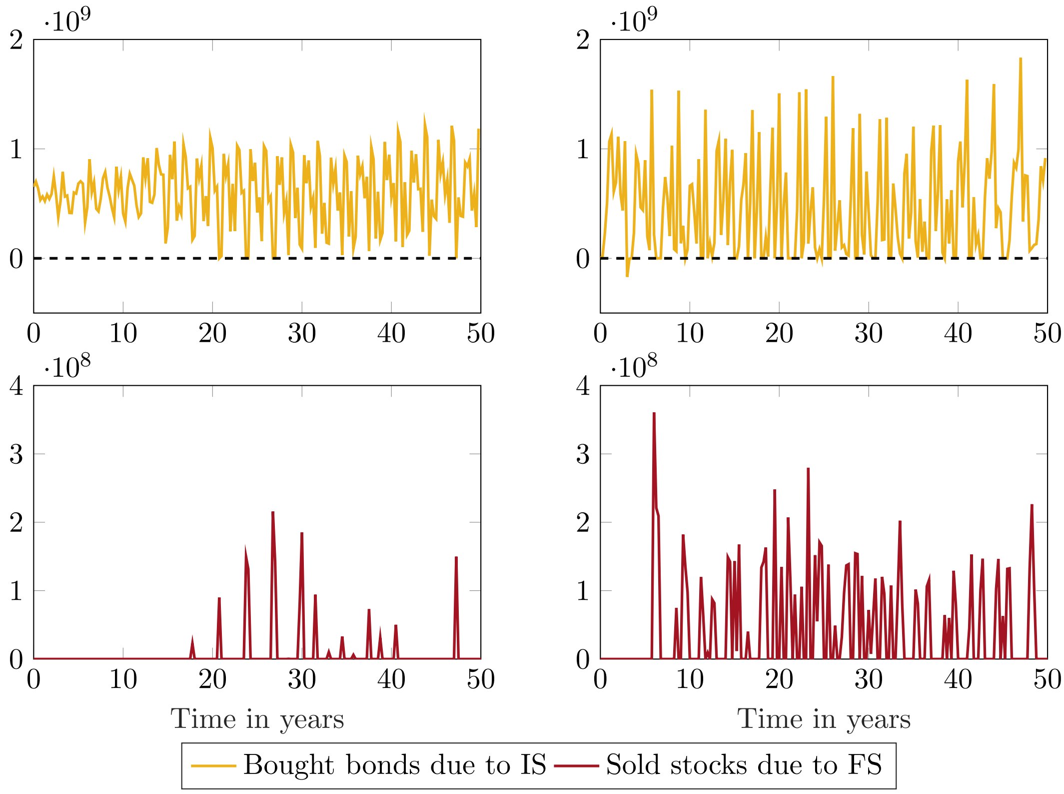

Figure 4.20.: Amount of bought bonds due to investment strategy and value of sold stocks due to financing strategy. Left: CM, right: CPPI.

We complete this section by measuring the performance of the CM and CPPI strategy in terms of different criteria reflecting both the life insurer's and the policyholders' point of view.

Criterion	CM	CPPI
$C_K \cdot 10^9$	8.41 (1.39)	9.12 (3.39)
$(F_K + Q_K) \cdot 10^9$	1.46 (1.15)	1.71 (2.54)
$L_K \cdot 10^7$	0 (0)	3.59 (10.19)
PD_K in %	29.75	53.30
$B_K \cdot 10^8$	2.59 (3.32)	6.88 (10.04)
$\widehat{i}_K$ in %	1.27 (0.73)	1.76 (1.68)
$\mathcal{B}_K \cdot 10^7$	13.84 (0.83)	14.99 (2.62)

Table 4.8.: Estimated default probability and arithmetic means together with sample standard deviations (in brackets) of different criteria applying the CM and CPPI strategy. The estimations are based on $N = 10,000$ simulated paths.

In more detail, these are the total value of assets C_K, the amount of own funds $F_K + Q_K$, the liabilities to banks L_K, the default probability PD_K, the bonus reserve B_K, the declared interest rate $\widehat{i}_K$, and the aggregated benefit payments $\mathcal{B}_K$ at the end of the considered time horizon, i.e. at time t_K. Based on $N = 10,000$ simulated paths, we calculate the corresponding arithmetic means together with the sample standard deviations and estimate the cumulative default probability PD_K. The results are shown in Table 4.8. We observe that for the considered parameter set, the CPPI strategy yields on average larger amounts of capital C_K, own funds $F_K + Q_K$, bonus reserves B_K, and a higher declared interest rate $\widehat{i}_K$. At the same time, the default probability PD_K is much larger and there is an increased demand for credit liabilities L_K. From the corresponding sample standard deviations we conclude that the insurance business tends to be more stable if we apply the CM strategy.

4.5.2. Performance within a capital market with crashes

In this section, we investigate the robustness of the considered investment strategies. We consider again an ongoing business with stationary new business. In contrast to the section before, we now allow for crashes in the stock and bond markets.[4] First, we assume a setting with a single crash either in the stock markets or in the bond markets at a fixed time and of a fixed crash size. More specifically, the time of occurrence and the intensity of the crash are chosen in advance and then $N = 10,000$ realizations of the capital market are generated. This can be seen as a worst-case approach. Indeed, fixed crash sizes can be interpreted as stochastic crash sizes attaining a predefined upper bound. The life insurer has no prior knowledge about the time and the size of the fixed crash. Later, we expand our considerations to more general crash scenarios.

In the case of a stock market crash $\left(t^{\mathrm{C}}, z^{\mathrm{C}}\right)^s = \left(t^{\mathrm{C},s}, z^{\mathrm{C},s}\right)$ the stock prices decrease instantly by a factor $z^{\mathrm{C},s}$ at time $t^{\mathrm{C},s}$. Regarding bond market crashes one needs to take correlations between different bonds into account, in addition to the time and the size of crashes. Since we want to investigate robustness of the investment strategies and stability of the balance sheets even in extreme scenarios, we assume a perfect correlation implying that a bond market crash affects all held bonds to the same extent. At crash time $t^{\mathrm{C},b}$, a fraction $z^{\mathrm{C},b}$ of all held bonds defaults completely.[5] Correspondingly, the number of held bonds with different times to maturity is decreased instantly. Therefore, the bond market crash $\left(t^{\mathrm{C}}, z^{\mathrm{C}}\right)^b = \left(t^{\mathrm{C},b}, z^{\mathrm{C},b}\right)$ also leads to liquidity shocks in the following periods.

Figure 4.21 displays five realizations of the stock price process and its expected development illustrating a crash of size $z^{\mathrm{C},s} = 0.4$ after 25 years.

[4]Here, we assume that crashes are caused by exogenous factors. Alternatively, bond market crashes could be caused by (instantly) rising interest rates.

[5]At this time point, this is equivalent to assume that the prices of all held bonds decrease instantly by the same factor $z^{\mathrm{C},b}$.

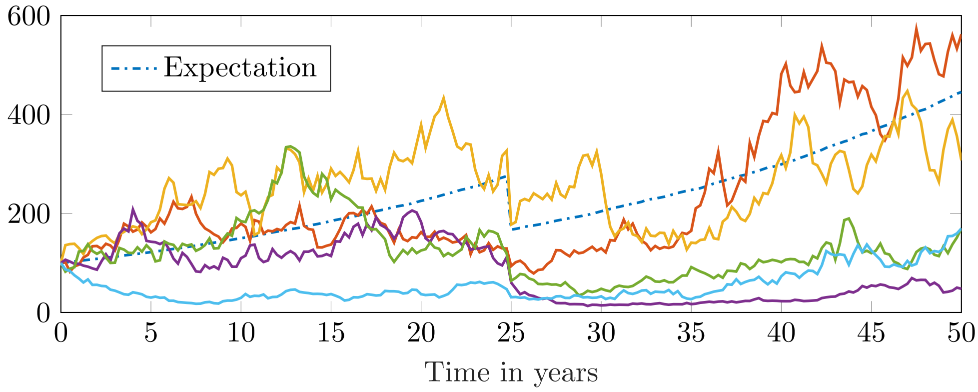

Figure 4.21.: Five realizations of the stock price process and its expected development in the case of a stock market crash $\left(t^{\mathrm{C}}, z^{\mathrm{C}}\right)^{s} = (25, 0.4)^{s}$.

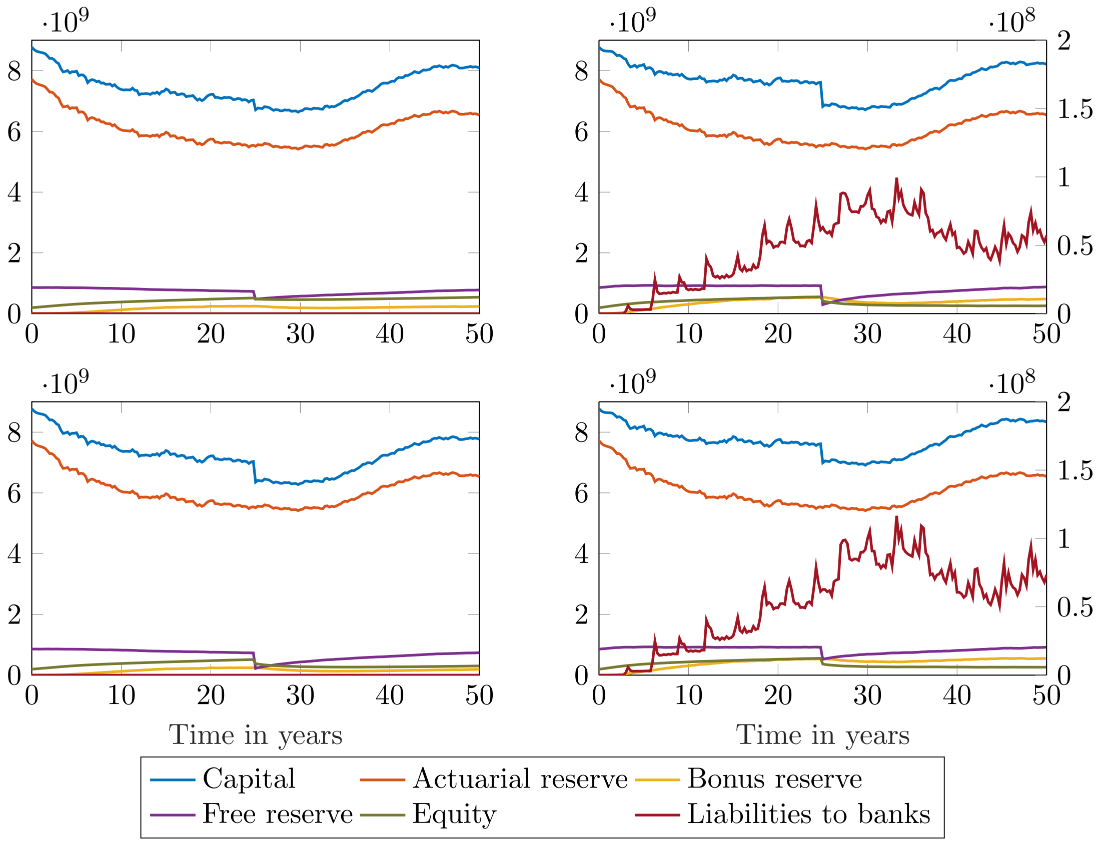

Figure 4.22.: Expected balance sheet positions in the case of a stock market crash $\left(t^{\mathrm{C}}, z^{\mathrm{C}}\right)^{s} = (25, 0.4)^{s}$ (top) and in the case of a bond market crash $\left(t^{\mathrm{C}}, z^{\mathrm{C}}\right)^{b} = (25, 0.1)^{b}$ (bottom). The right scale on the right-hand side is only for the liabilities to banks. Left: CM, right: CPPI.

The averages of the development of the (aggregated) balance sheet positions for the two investment strategies in this scenario are shown in the first row of Figure 4.22. The liabilities to banks equal zero in the CM-case at all times, in contrast to the CPPI-case where it increases. Especially after the stock market crash we observe larger values about 10^8. The actuarial reserve is not effected due to its independence from the capital market's variations while free reserve and equity suffer a lot. It is striking that the overall exposure seems to be higher using the CPPI strategy. Indeed, the equity was about 10% higher than in the CM-case shortly before the crash but then decreased tremendously. In the end, we observe $0.5 \cdot 10^9$ and $0.3 \cdot 10^9$ in the CM- and in the CPPI-case, respectively. The second row of Figure 4.22 shows the influence of a bond market crash of size $z^{C,b} = 0.1$ after 25 years. Now the crash is also clearly visible in the CM-case and the overall exposure seems to be of comparable size in both cases. At the end, the bonus reserve in the CPPI-case is even three times larger than in the CM-case.

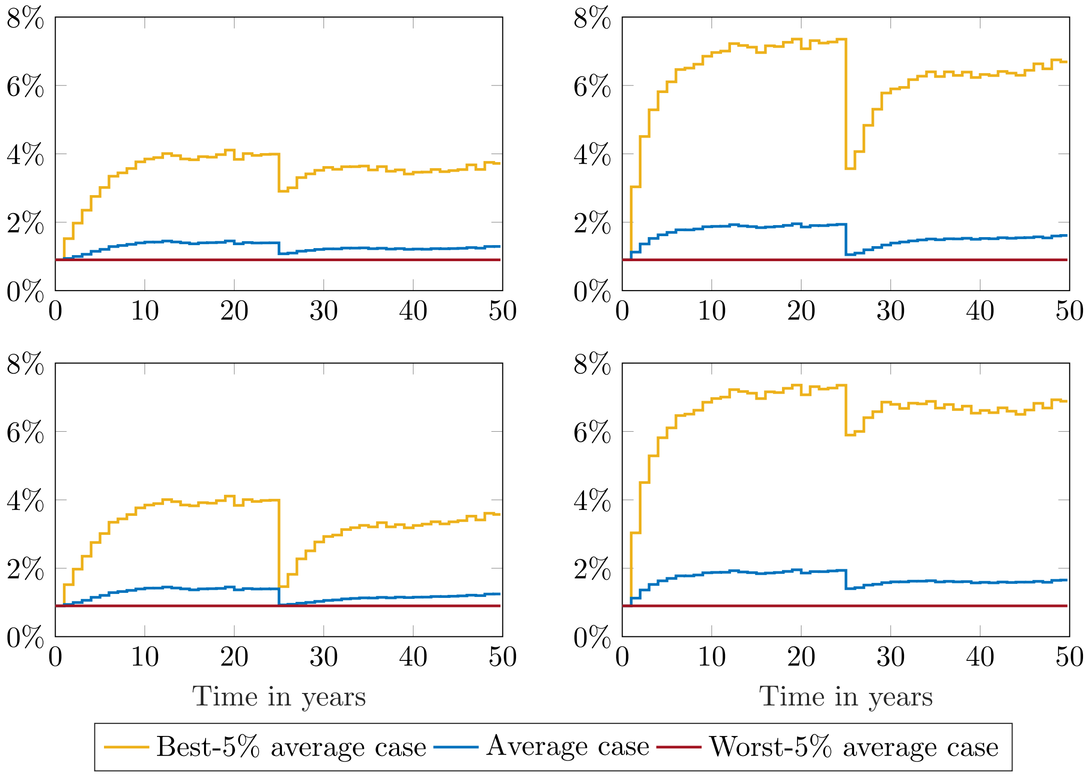

Figure 4.23.: Declared interest rate $\widehat{i}_k$ in the case of a stock market crash $\left(t^{\mathrm{C}}, z^{\mathrm{C}}\right)^s = (25, 0.4)^s$ (top) and in the case of a bond market crash $\left(t^{\mathrm{C}}, z^{\mathrm{C}}\right)^b = (25, 0.1)^b$ (bottom). Left: CM, right: CPPI.

The latter observations are also reflected in Figure 4.23 visualizing the impact of the stock market crash (first row) and the impact of the bond market crash (second row) on the annual declared interest rate. In the first case and applying the CPPI strategy, it is approximately reduced by half while applying the CM strategy, it is adjusted by 1.2 percentage points (pp) in the best-5% average case and 0.4pp in the average case corresponding to a reduction of less than 30%. In the case of a bond market crash, the declared interest rate suffers more applying the CM strategy: in the best-5% average case, we observe a reduction of 2.5pp (over 60%) compared to 1.5pp (20%) if applying the CPPI strategy. However, in the average case the relative change is similar for both strategies.

Now we investigate the robustness of the strategies in more detail by considering varying crash scenarios $\left(t^{\mathrm{C}}, z^{\mathrm{C}}\right)^{\cdot}$ regarding the stock and bond markets and measuring their impact on the performance within the corresponding crash-free capital market. More specifically, on the basis of $N = 10,000$ simulated paths, we average over all simulations and all periods to get the average change per period as a number for illustration. As before, we select different criteria reflecting both the life insurer's and the policyholders' point of view. In more detail, these are the own funds $F_k + Q_k$, the liabilities to banks L_k, the declared interest rate $\widehat{i}_k$, and the benefit payments $\mathcal{B}_k$. The influence is measured in terms of absolute changes if non-positive values are possible, otherwise in terms of relative changes (in %) or in percentage points (pp). The results are shown in Table 4.9, where we also display the impact on the 50-year default probability PD_K.

$\left(t^{\mathrm{C}}, z^{\mathrm{C}}\right)^{\cdot}$	$\Delta\left(F_k + Q_k\right) \cdot 10^8$	ΔL_k	ΔPD_K in pp	$\Delta\widehat{i}_k$ in pp	$\Delta\mathcal{B}_k$ in %
$(1, 0.1)^s$	$-0.5/-1.7$	$0/1.1 \cdot 10^7$	2.25/5.09	$-0.02/-0.10$	$-0.25/-1.02$
$(1, 0.1)^b$	$-6.2/-6.8$	$0/9.4 \cdot 10^7$	44.26/23.74	$-0.15/-0.32$	$-1.63/-3.22$
$(1, 0.4)^s$	$-2.2/-9.1$	$0/1.3 \cdot 10^8$	11.83/31.97	$-0.08/-0.43$	$-0.91/-4.22$
$(1, 0.4)^b$	$-32.9/-32.0$	$6.6 \cdot 10^3/8.0 \cdot 10^8$	68.67/46.45	$-0.37/-0.85$	$-3.49/-7.48$
$(25, 0.1)^s$	$-0.2/-0.6$	$0/4.9 \cdot 10^5$	1.30/1.99	$-0.01/-0.05$	$-0.14/-0.47$
$(25, 0.1)^b$	$-2.5/-2.6$	$0/1.1 \cdot 10^7$	16.69/7.83	$-0.10/-0.12$	$-0.95/-1.08$
$(25, 0.4)^s$	$-1.0/-3.1$	$0/4.0 \cdot 10^6$	5.71/12.22	$-0.05/-0.19$	$-0.52/-1.71$
$(25, 0.4)^b$	$-12.9/-12.1$	$2.2 \cdot 10^1/4.1 \cdot 10^7$	68.67/33.07	$-0.19/-0.36$	$-1.52/-2.86$
$(45, 0.1)^s$	$-0.1/-0.2$	$0/8.4 \cdot 10^4$	0.70/0.75	$-0.01/-0.02$	$-0.02/-0.05$
$(45, 0.1)^b$	$-0.8/-0.7$	$0/1.7 \cdot 10^6$	10.59/5.45	$-0.03/-0.04$	$-0.08/-0.09$
$(45, 0.4)^s$	$-0.3/-0.9$	$0/9.9 \cdot 10^5$	3.20/7.29	$-0.02/-0.06$	$-0.05/-0.16$
$(45, 0.4)^b$	$-3.2/-2.9$	$0/6.7 \cdot 10^6$	68.66/25.89	$-0.04/-0.08$	$-0.09/-0.19$

Table 4.9.: Absolute or relative changes due to a predefined crash regarding the stock markets $\left(t^{\mathrm{C}}, z^{\mathrm{C}}\right)^s$ and the bond markets $\left(t^{\mathrm{C}}, z^{\mathrm{C}}\right)^b$ applying the CM/CPPI strategy.

Clearly, the changes are larger for greater crash-intensities and since we averaged over all periods, they are smaller if the crash occurs at later time points. In the case of stock market crashes, applying the CM strategy there is no demand for credits in any considered scenario and the performance is always much less affected than that in the CPPI-case. Regarding bond market crashes, the differences between the strategies are much smaller. Even more, the default probability and partly the own funds are now more affected when applying the CM strategy. It is striking that in the considered scenarios bond market crashes have a larger impact on the performance than stock market crashes. This is due to the fact that most funds are invested in bonds, see Figure 4.19.

We complete this section by expanding our investigations to more general settings. In contrast to before, we consider random crash times and sizes regarding both the stock and bond markets. In particular, we now allow for several crashes within the considered time horizon but we do not assume that there always has to be a crash.

The independent waiting times $W_1, W_2, \dots$ for the crashes are modeled by an exponential distribution with parameter $\frac{1}{\mathcal{T}}$ such that the l-th crash time is given by

$$t_l^{\mathrm{C}} = \sum_{j=1}^{l} W_j, \quad W_1, W_2, \dots \text{ iid}, \quad W_1 \sim \mathrm{Exp}\left(\frac{1}{\mathcal{T}}\right).$$

This approach leads to an average amount of one crash within the considered time horizon $\mathcal{T}$, but also allows for crash-free scenarios and several crashes. The independent crash sizes z_l^{C} are modeled by a beta distribution with parameters 2 and 6, i.e.

$$z_l^{\mathrm{C}} \sim \mathrm{Beta}\,(2, 6)\,.$$

Hence, the expected value of the crash size is 0.25 as in the deterministic crash scenarios. We perform several Monte Carlo simulations consisting each of $N = 10,000$ simulated paths and calculate the established criteria. We consider crashes only in stock markets, only in bonds markets, independent crashes in both markets, and coupled crashes in both markets where the crashes of independent sizes occur in both markets at the same (random) time. Table 4.10 summarizes the results.

Crashes in	$\Delta\,(F_k + Q_k) \cdot 10^8$	ΔL_k	ΔPD_K in pp	$\Delta\widehat{i_k}$ in pp	$\Delta\mathcal{B}_k$ in %
Stocks	$-0.7/-2.0$	$0/10.0 \cdot 10^6$	4.08/7.47	$-0.03/-0.11$	$-0.29/-1.01$
Bonds	$-7.9/-8.0$	$6.8 \cdot 10^5/8.5 \cdot 10^7$	34.49/17.76	$-0.12/-0.22$	$-1.02/-1.85$
Both	$-8.4/-9.6$	$4.1 \cdot 10^5/9.5 \cdot 10^7$	36.16/21.97	$-0.13/-0.30$	$-1.20/-2.57$
Both at once	$-8.7/-10.0$	$2.2 \cdot 10^5/1.1 \cdot 10^8$	37.50/23.12	$-0.12/-0.28$	$-1.09/-2.26$

Table 4.10.: Absolute or relative changes due to random crashes in markets of stocks, bonds, and both applying the CM/CPPI strategy. The last row means that both, stock and bond markets, crash at the same (random) time.

We see that the observations made so far manifest themselves again. The CM strategy (with a target stock ratio of $\pi^{s,\mathrm{tar}} = 10\%$) is more robust against possible stock market crashes and, to a smaller extent, for most criteria also against possible bond market crashes. In the CM-case, the latter leads to much higher default probabilities compared to a crash-free scenario but they are still smaller than in the CPPI-case. Furthermore, significant amounts of liabilities to banks are now also observed in the CM-case.

From the last two sections we can conclude that for the considered parameter set and scenarios, the CPPI strategy yields on average larger bonus reserves, free reserves, and declared interest rates. At the same time, it leads to a higher default probability and in most cases to a greater exposure against possible capital market crashes.

4.6. Sensitivity analysis

The capital market's true parameters are, in general, not known but have to be estimated on the basis of historical data. However, estimations are always associated with a certain degree of uncertainty. In the following, we study this issue using the long-term mean $\frac{b}{a}$ of the short rate process as a key parameter. We specify three different scenarios. In one case, it equals the guaranteed interest rate $\widehat{i}_G = 0.9\%$. In the other cases, it is 0.4 percentage points above or below the guaranteed rate. If not stated otherwise, all other initial values remain the same, cf. Table 4.1. Here, we consider again a life insurer with stationary new business applying the CM strategy for investments. The corresponding probabilities and expectations are each estimated on the basis of $N = 10,000$ simulations.

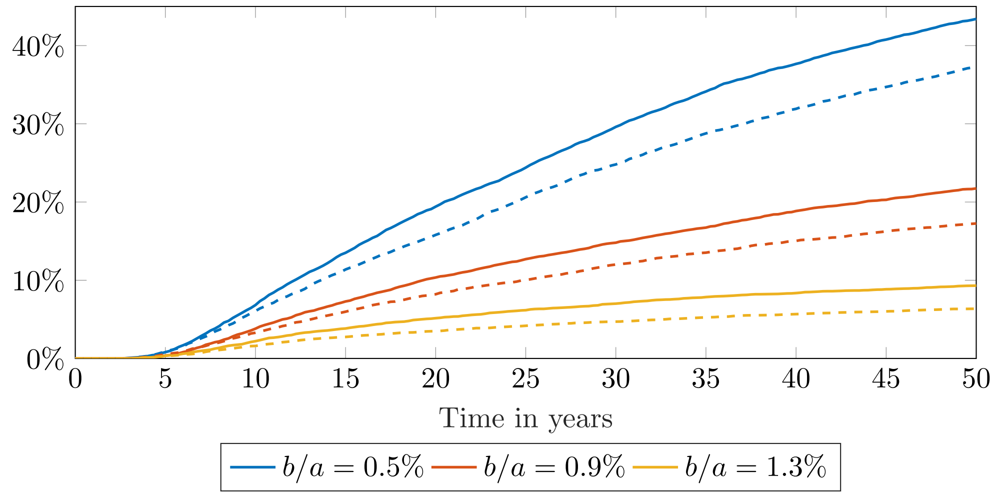

Figure 4.24.: Development of the default probability within the three scenarios. The dashed lines correspond to $\lambda_0 = 0$.

In Figure 4.24, we show the development of the default probabilities PD_k, i.e. the probabilities of the events $\{Q_j < 0 \text{ for some } j \in \{0, \dots, k\}\}$. Within the first years, they are (almost) zero for all scenarios due to the initial amount of own funds $F_0 + Q_0$. From year 5 onwards, we observe significant differences getting larger as time goes on. The default probability increases as the long-term mean decreases (slightly). This indicates a high sensitivity with respect to the long-term mean of the short rate, which is characteristic for life insurers writing long-term insurance business. It is also noticeable that the influence is non-symmetric. The probability of default is more affected by negative deviations than by positive ones. The dashed lines correspond to the results if the market price of interest rate risk parameter λ_0 is zero, i.e. if the real world coincides with the risk-neutral world ($P = Q$). In that case, we obtain larger bond prices and thus increased asset values which reduce the risk and thus lead to smaller default probabilities compared to $\lambda_0 = 2\%$.

An important objective could be to keep the default probability within a certain time horizon, say during the next 10 years, under a predefined threshold, say 5%. As illustrated in Figure 4.25, this can be achieved by increasing the initial fraction ψ_0 of own funds.[6] Note that we ensured a non-negative value of initial equity by reducing the reserve rate γ_0 correspondingly. As expected, the probabilities decrease if the own funds increase, especially fast for smaller values of ψ_0. Here, a default probability equal to the threshold of 5% requires initial fractions of approximately 9.9%, 11.3%, and 13.2% if the long-term mean is 1.3%, 0.9%, or 0.5%, respectively. In the case $\lambda_0 = 0$, we need less own funds to obtain the same default probability.

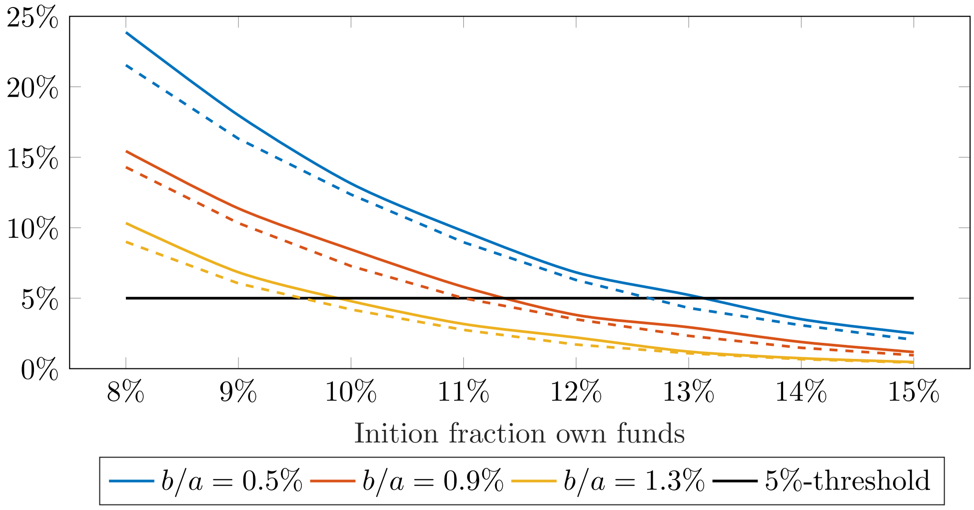

Figure 4.25.: Development of the 10-year default probability depending on own funds within the three scenarios. The dashed lines correspond to $\lambda_0 = 0$.

The declared interest rates $\widehat{i}_k$ on average are displayed in the upper part of Figure 4.26. During the first year, only the guaranteed rate $\widehat{i}_G = 0.9\%$ is paid. Then, interest rates increase before they become stable. Significant differences between the three scenarios are observable from year 5 on. In contrast to the default probabilities, these do not increase steadily with time but remain approximately constant. In addition, now the positive deviations in the long-term mean have a (slightly) stronger influence. In the medium and long term, the average values are 1.2%, 1.5%, and 1.9% if $\frac{b}{a}$ equals 0.5%, 0.9%, and 1.3%. A complementary benchmark, additionally to the declared interest rate, could be the probability that only the guaranteed interest rate $\widehat{i}_G$ will be paid as illustrated in the lower part of Figure 4.26. The probabilities decrease during the first years and then become quite stable. As expected, larger values for the long-term mean result in lower probabilities that the declared interest rate equals the guaranteed rate. As before, the differences between the scenarios remain approximately constant over time. In the case $\lambda_0 = 0$, we obtain similar results.

[6]In practice, a typical drawback would be higher costs of equity.

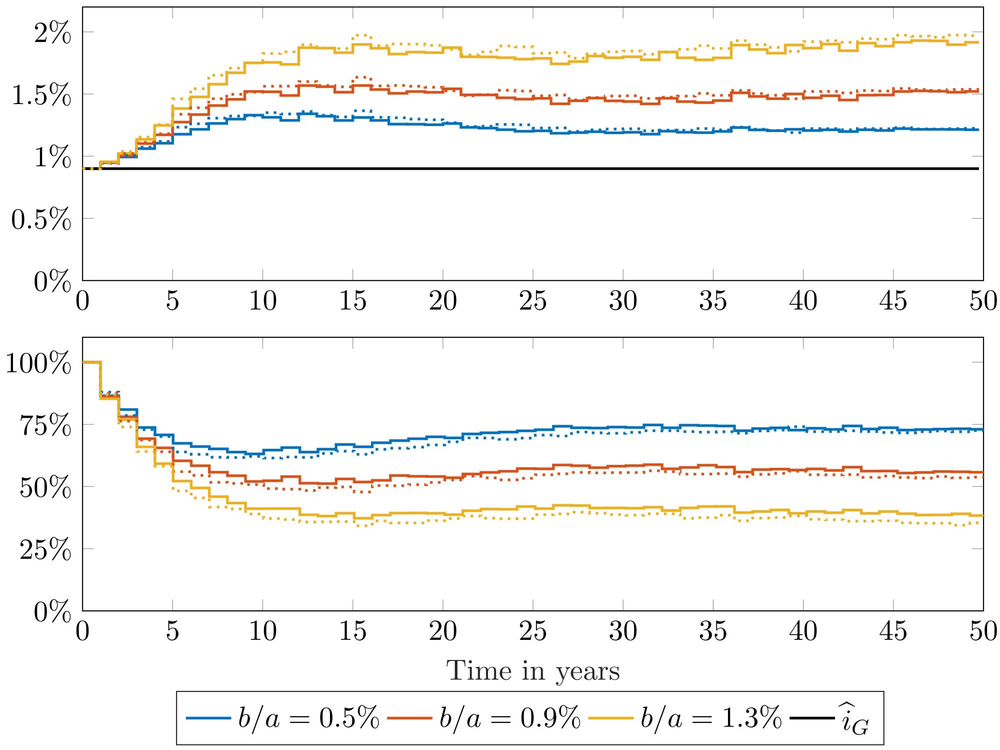

Figure 4.26.: Average declared interest rate (top) and probability that only the guaranteed interest rate is paid (bottom) within the three scenarios. The dotted lines correspond to $\lambda_0 = 0$.

We complete this chapter by investigating more the obtained high sensitivity, here exemplary for the market price of interest rate risk parameter λ_0. The long-term mean $\frac{b}{a}$ of the short rate process is assumed to coincide with the guaranteed interest rate $\widehat{i}_G$, i.e. $\frac{b}{a} = 0.9\%$. The long-term mean under the risk-neutral measure Q is $\frac{\tilde{b}}{\tilde{a}}$ with $\tilde{b} = b - \lambda_0 \sigma_r$ and $\tilde{a} = a$, cf. Section 3.3.1.

Figure 4.27 displays the arithmetic means of several quantities of interest at the end of the considered time horizon for different values of λ_0. We see that both profitability and security of the life insurance business decrease as the market price of interest rate risk parameter increases. We observe that the dependence is particularly strong for the default probability PD_K, while the aggregated benefit payments $\mathcal{B}_K$ are less affected.

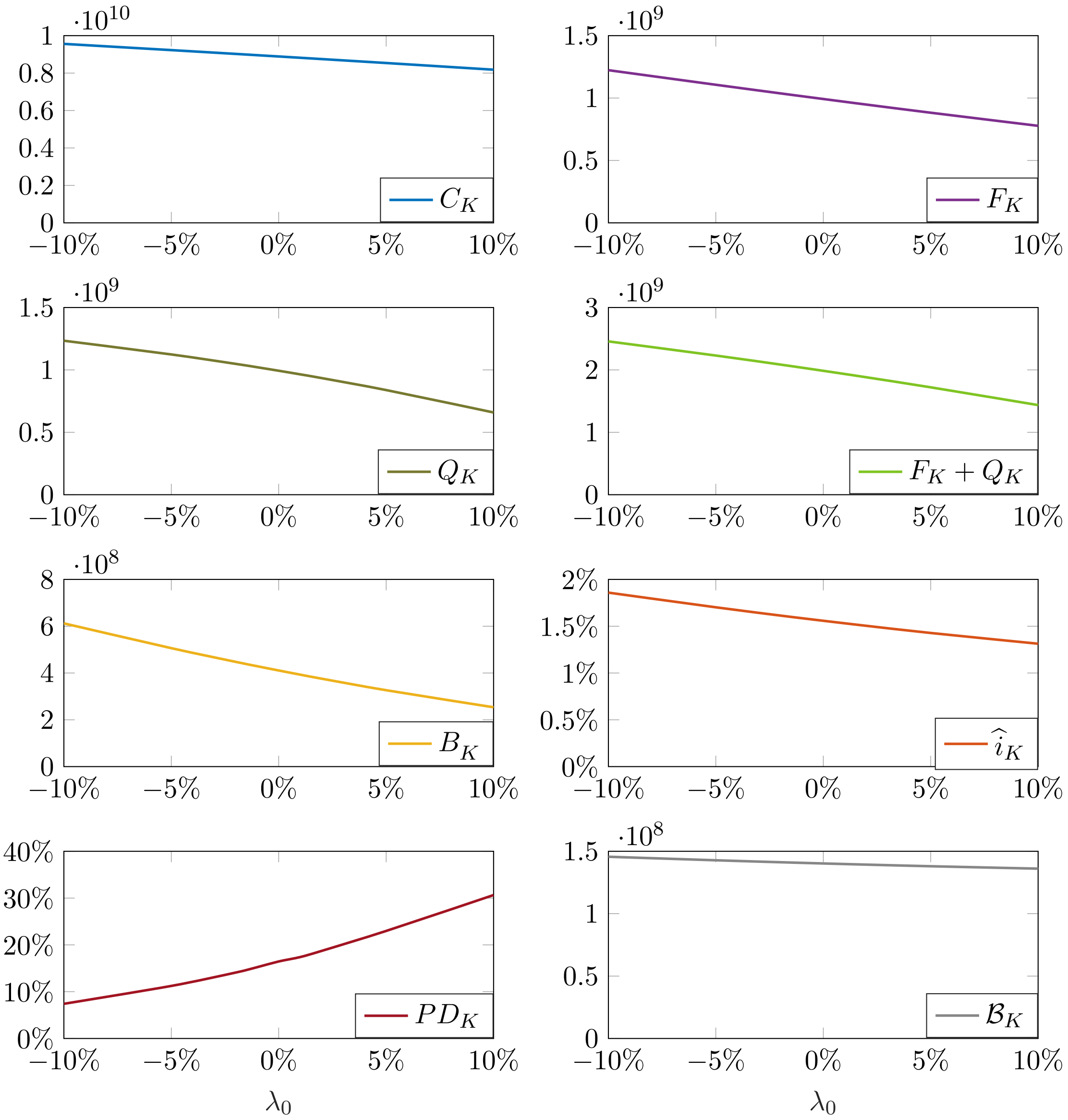

Figure 4.27.: Quantities of interest depending on the market price of interest rate risk parameter λ_0.

5. Further Applications of the General ALM Model: Refinements and New Strategies

This chapter builds up on Chapter 3. We present several refinements, alternative modeling approaches, and further applications of the general ALM model emphasizing its flexibility and universal applicability. In Section 5.1, we derive new investment strategies that aim at meeting requirements on the asset allocation induced by the insurer's liabilities. In Section 5.2, we develop new strategies for financing the periodic disbursements. In particular, we allow for selling bonds before maturity, thereby relaxing the assumptions made in Section 3.5.1. For this, we develop an algorithm for the successive sale of bonds to reduce a defined gap in funds while taking into account a safety amount that should be kept for future obligations, see Section 5.3. In Section 5.4, we prove that the refined ALM model respects the fundamental balance sheet equation, and provide an alternative strategy for the use of surpluses in Section 5.5.

5.1. Development of new investment strategies

In this section, we motivate and develop new investment strategies that take into account dependencies between the life insurer's assets and liabilities. For this, we introduce a quantity that represents the required capital needed for new bond investments, the *coverage fund*.

5.1.1. Motivation and coverage funds

In Section 3.5.1 we specified the asset allocation by introducing two prominent investment strategies: the CM and the CPPI strategy. Both are characterized by the management's target for the stock position, while the remaining liquid capital is invested in bonds with a fixed duration.

In simulation studies, we observed a periodic pattern and pronounced peaks in the amount of bought bonds for both strategies within a pathwise consideration.[1] That characteristic behavior results from the assumption that bonds are held until maturity and from the strong dependence of the asset allocation on the periodic obligations. A substantial part of the disbursements consists of survival benefit payments, whose guaranteed part is independent of the random capital market's variations. Therefore, the dependencies are still visible on average as we can see in Figure 5.1. For the corresponding simulations, we

[1] See Figure 4.20 in Section 4.5.1.

used the parameters from Table 4.1 with a minimum stock ratio $\pi_k^{s,\min} = 10\%$. Furthermore, we considered a life insurer with stationary new business applying the CM strategy for investments.[2]

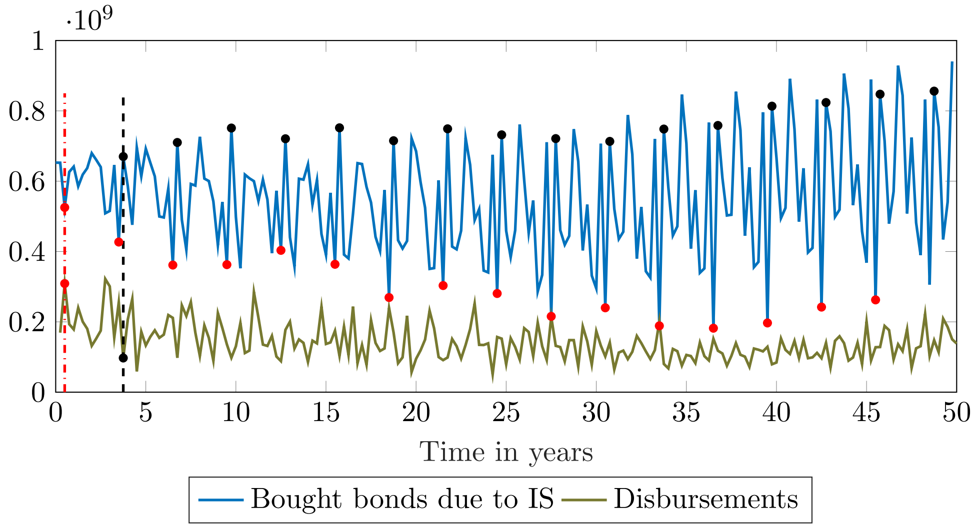

Figure 5.1.: Illustration of the dependence of the bond strategy on the disbursements. The red (black) dashed line represents the first time where a high (low) amount of disbursements led to a low (high) value of bought bonds. It also marks the beginning of the shown series of local minimums (maximums) of bought bonds.

The large amount of approximately $0.3 \cdot 10^9$ of due disbursements after 0.5 years sets the beginning of the series of local minimums in the amount of bought bonds illustrated by the red points in Figure 5.1. Likewise, the comparatively large purchase of bonds at time $t_k = 3.75$, being the start of the series of local maximums in the amount of bought bonds (the black points), is a direct consequence of the small amount of disbursements due at this time.

The last figure further illustrates that the structure of the insurance portfolio heavily affects the asset allocation. The assumption that bonds are held until maturity reinforces the observed behavior of the amount of bought bonds. More evolved investment strategies should take into account the future disbursements by reconciling the bond investments accordingly. Ideally, payouts of the expired bonds should be sufficient to meet the due obligations. In the following, we thus introduce the concept of coverage funds that forms the basis of the investment strategies which we develop later.

As newly purchased bonds are assumed to have duration τ, the coverage fund CF_{k-1} at time t_{k-1} admits the following, general representation:

$$CF_{k-1} = \mathbb{E}\left[D_{k-1+\frac{\tau}{\Delta t}} \,|\, \mathcal{F}_{k-1}\right] p\left(t_{k-1}, t_{k-1}+\tau\right). \tag{5.1.1}$$

[2]Applying the CPPI strategy, the observed periodic pattern would be even more pronounced.

The conditional expectation describes the estimated disbursements $D_{k-1+\frac{\tau}{\Delta t}}$ at time $t_{k-1+\frac{\tau}{\Delta t}}$, given the available information $\mathcal{F}_{k-1}$ up to time t_{k-1}.

Remark 5.1.1 (Estimation of the future disbursements)**.** As we can see in equation (5.1.1), the main component of the coverage fund is the conditional expectation of the future disbursements. Its exact calculation is not possible since that would require knowledge about the true distributions of the involved quantities and about the future new business. Nevertheless, in some cases, quite accurate approximations are possible, e.g. if there are no mass cancellations nor other extreme events. More details are provided in Section 6.2.4.

5.1.2. Coverage fund strategies

As before, the asset reallocation at the beginning of period k requires the calculation of the tied up capital C^B_{k-1} and the position of liquid funds C^L_{k-1}. For this, we proceed analogously to Section 3.5.1. However, we introduce additional quantities and adjust the notation at some places. This is due to the new setting, where we e.g. now explicitly allow for prior selling of held bonds.

Starting with the previous balance sheet with the life insurer's total assets C_{k-1} at time t_{k-1}, the updated capital $C_{(k-1)^+}$ is given by

$$C_{(k-1)^+} = C_{k-1} - \xi^{b,l}_{(k-1)^+} - \xi^{s,l}_{(k-1)^+} + \left(P_{k-1} - L^+_{k-1}\right)^+ . \tag{5.1.2}$$

The new quantity $\xi^{b,l}_{(k-1)^+}$ denotes the aggregated amount of sold bonds at the beginning of period k according to the applied financing strategy and is specified in Section 5.3.2.

The tied up capital C^B_{k-1} and the position of liquid funds C^L_{k-1} are given by

$$C^B_{k-1} = C^b_{k-1} - \xi^{b,l}_{(k-1)^+} + \min\left\{\pi^{s,\min}_{k-1} C^s_{k-1}, \pi^{s,\max}_{k} C_{(k-1)^+}\right\}$$

and

$$\begin{aligned} C^L_{k-1} = &\left(1 - \pi^{s,\min}_{k-1}\right) C^s_{k-1} - \xi^{s,l}_{(k-1)^+} + C^c_{k-1} + \left(P_{k-1} - L^+_{k-1}\right)^+ \\ &+ \left(\pi^{s,\min}_{k-1} C^s_{k-1} - \pi^{s,\max}_{k} C_{(k-1)^+}\right)^+ , \end{aligned}$$

so that the updated capital $C_{(k-1)^+}$ can still be written as

$$C_{(k-1)^+} = C^L_{k-1} + C^B_{k-1}.$$

After the reallocation, $C_{(k-1)^+}$ comprises only stocks and bonds with different times to maturity, i.e. we then have

$$C_{(k-1)^+} = C^s_{(k-1)^+} + C^b_{(k-1)^+},$$

as in Section 3.5.1. However, the updated value $C^b_{(k-1)^+}$ of the bond portfolio now is

$$C^b_{(k-1)^+} = C^b_{k-1} - \xi^{b,l}_{(k-1)^+} - \xi^{b-}_{k-1} + \varphi^b\left(t_{(k-1)^+}; t_{k-1}+\tau\right) p\left(t_{k-1}, t_{k-1}+\tau\right),$$

where ξ^{b-}_{k-1} corresponds to the aggregated amount of sold bonds according to the applied investment strategy and is specified in Section 5.3.1. The number of held stocks and the stock ratio after the reallocation of assets is given as before by equations (3.5.3) and (3.5.4).

In the following, we introduce three coverage fund-based investment strategies, particularly specifying the number $\varphi^b\left(t_{(k-1)^+}; t_{k-1}+\tau\right)$ of newly purchased bonds with duration τ and the updated stock position $C^s_{(k-1)^+}$.

SCF strategy (strict coverage fund)

The fulfillment of the coverage fund has a high priority in the sense that the management even sells stocks to invest more in bonds if needed. Taking into account the minimum and maximum stock ratio, the money invested in stocks is given by

$$C^s_{(k-1)^+} = \min\left\{\left(C^L_{k-1} - CF_{k-1}\right)^+ + \pi^{s,\min}_{k-1} C^s_{k-1}, \pi^{s,\max}_k C_{(k-1)^+}\right\}.$$

RCF strategy (relaxed coverage fund)

Here, the goal of a sufficient large coverage fund has a smaller priority compared to the SCF strategy. More specifically, bond investments no longer justify selling of stocks.[3] Instead, only cash is used to buy new bonds. The adjusted stock position can be written as

$$C^s_{(k-1)^+} = \min\left\{\max\left\{C^L_{k-1} - CF_{k-1}, C^s_{k-1} - \xi^{s,l}_{(k-1)^+}\right\}, \pi^{s,\max}_k C_{(k-1)^+}\right\}.$$

Applying the SCF or the RCF strategy, the number of newly purchased bonds with duration τ is given by

$$\varphi^b\left(t_{(k-1)^+}; t_{k-1}+\tau\right) = \frac{C^L_{k-1} - \left(C^s_{(k-1)^+} - \min\left\{\pi^{s,\min}_{k-1} C^s_{k-1}, \pi^{s,\max}_k C_{(k-1)^+}\right\}\right)}{p\left(t_{k-1}, t_{k-1}+\tau\right)}.$$

Furthermore, no bonds are sold, i.e. we have $\xi^{b-}_{k-1} = 0$.

DCF strategy (dynamic coverage fund)

The motivation of the DCF strategy is the assumption that it might be more desirable to sell excessive bonds instead of stocks to increase the liquid funds for new bond investments. In order to have excessive bonds, we need to build up some kind of buffer at times where there is a lot of available cash. For this, we introduce the coverage fund multiplier $\lambda^{\mathrm{CF}}_{k-1} \geq 1$.

[3] Nevertheless, stocks may be sold according to the financing strategy or due to the maximum stock ratio.

According to the DCF strategy, the money invested in stocks can be represented as

$$C^s_{(k-1)^+} = \min\left\{\max\left\{C^L_{k-1} - \lambda^{\mathrm{CF}}_{k-1} CF_{k-1}, C^s_{k-1} - \xi^{s,l}_{(k-1)^+}\right\}, \pi^{s,\max}_k C_{(k-1)^+}\right\},$$

which is the same structure as in the RCF strategy. If the available cash does not suffice for a bond investment of volume CF_{k-1}, there exists a coverage fund gap CF^{gap}_{k-1}:

$$CF^{\mathrm{gap}}_{k-1} = \left(CF_{k-1} - \left(C^L_{k-1} - \left(C^s_{(k-1)^+} - \min\left\{\pi^{s,\min}_{k-1} C^s_{k-1}, \pi^{s,\max}_k C_{(k-1)^+}\right\}\right)\right)\right)^+.$$

By selling bonds, the insurer now tries to reduce that gap. The exact procedure is described in Section 5.3.1, where we derive an algorithm for the iterative sale of excessive bonds. Here, we also specify the excess and the aggregated amount ξ^{b-}_{k-1} of sold bonds. The latter corresponds to additional cash that is fully used to increase the number of newly purchased bonds with duration τ:

$$\varphi^b\left(t_{(k-1)^+}; t_{k-1} + \tau\right) = \frac{C^L_{k-1} - \left(C^s_{(k-1)^+} - \min\left\{\pi^{s,\min}_{k-1} C^s_{k-1}, \pi^{s,\max}_k C_{(k-1)^+}\right\}\right) + \xi^{b-}_{k-1}}{p\left(t_{k-1}, t_{k-1} + \tau\right)}.$$

5.2. Development of new financing strategies

In this section, we develop two more strategies for financing the periodic obligations. Compared to Section 3.5.3, we now additionally allow for selling held bonds. Therefore, the new strategies can be seen as extensions of FS 1 and FS 2. All other measures and the associated assumptions from Section 3.5.3 remain unchanged. For completeness, we list the available measures:

- take funds from expired bonds,
- sell bonds,
- sell stocks,
- raise short-term credits,
- take funds from premium income, and
- raise long-term credits.

According to the new notation, the amount of expired bonds at the end of period k is

$$C^{b-}_k = \varphi^b\left(t_k; t_{k-\frac{\tau}{\Delta t}} + \tau\right) p\left(t_k, t_k\right) = \varphi^b\left(t_k; t_k\right).$$

As before, assets can be sold at the end of period k or at the beginning of the next one. The corresponding aggregated market values of sold bonds are denoted by $\xi^{b,l}_k$ and $\xi^{b,l}_{k^+}$, respectively, and are specified in Section 5.3.2. As selling bonds before maturity was excluded in Section 3.5.3, we have $\xi^{b,l}_k = \xi^{b,l}_{k^+} = 0$ for financing strategies FS 1 and FS 2.

Financing strategy 3 (FS 3)

We again assume that the life insurer prefers selling excessive bonds instead of stocks yielding the following priority order:

1. take funds from expired bonds C_k^{b-},
2. sell bonds $\xi_k^{b,l}$,
3. sell stocks $\xi_k^{s,l}$,
4. raise short-term credits L_k^+,
5. take funds from premium income P_k, and
6. raise long-term credits L_k^{new}.

At first, funds are taken from expired bonds C_k^{b-}. If these payouts are not sufficient, there exists a demand for cash C_k^{c+} given by

$$C_k^{c+} = \left(D_k - C_k^{b-}\right)^+ .$$

To reduce that demand for cash, the life insurer first sells excessive bonds and then stocks in the amounts of $\xi_k^{b,l}$ and $\xi_k^{s,l}$, respectively. The bridging loan can thus be written as

$$L_k^+ = C_k^{c+} - \xi_k^{b,l} - \xi_k^{s,l},$$

where

$$\xi_k^{s,l} = \min\left\{C_k^{c+} - \xi_k^{b,l}, \left(1 - \pi_k^{s,\min}\right) \varphi^s_{(k-1)^+} s_k\right\}.$$

At the beginning of period $k+1$ premium income P_k is used to repay the bridging loan but no assets are sold, i.e. we have $\xi_{k^+}^{b,l} = \xi_{k^+}^{s,l} = 0$. The long-term credits to enter newly thus amount to

$$L_k^{\text{new}} = \left(L_k^+ - P_k\right)^+ .$$

Financing strategy 4 (FS 4)

The motivation for the following strategy is the same as for FS 2, i.e. we aim to reduce the amount of traded assets. The order of the individual measures are:

1. take funds from expired bonds C_k^{b-},
2. raise short-term credits L_k^+,
3. take funds from premium income P_k,

4. sell bonds $\xi_{k+}^{b,l}$,
5. sell stocks $\xi_{k+}^{s,l}$, and
6. raise long-term credits L_k^{new}.

If funds from expired bonds C_k^{b-} do not suffice, we directly raise a short-term credit instead of selling assets, i.e. we have $\xi_k^{b,l} = \xi_k^{s,l} = 0$. The bridging loan is given by

$$L_k^+ = \left(D_k - C_k^{b-}\right)^+,$$

and premium income P_k is used for its repayment at the beginning of the following period. If this is not sufficient, there remains a demand for cash:

$$C_k^{c+} = \left(L_k^+ - P_k\right)^+.$$

As before, the life insurer first sells excessive bonds and then stocks. Accordingly, the long-term credits to enter newly can be represented as

$$L_k^{\text{new}} = C_k^{c+} - \xi_{k+}^{b,l} - \xi_{k+}^{s,l},$$

where

$$\xi_{k+}^{s,l} = \min\left\{C_k^{c+} - \xi_{k+}^{b,l}, \left(1 - \pi_k^{s,\min}\right) C_k^s\right\}$$

is the market value of sold stocks.

5.3. Selling of excessive bonds and transaction costs

In this section, we describe the procedure of selling bonds in more detail. Naturally, for this there are many strategies possible since the life insurer holds bonds with different times to maturity. The algorithms developed in this thesis are motivated by the concept of coverage funds, too. More specifically, we only sell excessive bonds whose later payouts are probably not needed to finance the due obligations. This is in line with the permanent assumption that funds are first taken from expired bonds. The priority order for selling bonds is then determined by the corresponding numbers of excessive bonds with different times to maturity.

According to the strategies introduced in this thesis, selling of excessive bonds can be triggered by a positive coverage fund gap and by a positive demand for cash. While the former is associated with the DCF investment strategy at the beginning of a period, the latter is associated with the financing strategy either at the end of a period (FS 3) or at the beginning of a period before the asset reallocation (FS 4). In all cases, the liquidations intend to raise cash in order to reduce a certain gap in funds, and the respective excess is obtained by comparing the actual held number of bonds with a certain time to maturity with the expected disbursements at that time.[4] Nevertheless, due to different notations, we distinguish in the following between bond sales due to investment strategies and due to financing strategies. Let us now fix a period k, i.e. we consider the time interval $[t_{k-1}, t_k]$.

5.3.1. Selling of excessive bonds due to investment strategies

According to the DCF investment strategy, bonds are sold to reduce the coverage fund gap CF_{k-1}^{gap}. The number of bonds with maturity at time $t_k, t_{k+1}, \ldots, t_{k-2+\frac{\tau}{\Delta t}}$ held at the end of period $[t_{k-2}, t_{k-1}]$ is $\varphi^b\left(t_{k-1}; t_i + \tau\right)$, for $i = k - \frac{\tau}{\Delta t}, k - \frac{\tau}{\Delta t} + 1, \ldots, k-2$. The bonds were purchased at times t_i. The corresponding excessive number can be calculated by

$$\varphi^{b,\text{excess}}\left(t_{(k-1)^+}; t_i + \tau\right) = \left(\varphi^b\left(t_{k-1}; t_i + \tau\right) - \varphi^{b,l}\left(t_{(k-1)^+}; t_i + \tau\right) - \mathbb{E}\left[D_{i+\frac{\tau}{\Delta t}} \mid \mathcal{F}_{k-1}\right]\right)^+,$$

where $\varphi^{b,l}\left(t_{(k-1)^+}; t_i + \tau\right)$ corresponds to the number of sold bonds according to the chosen financing strategy. The life insurer now successively sells excessive bonds until there is no coverage fund gap or no excess anymore. The corresponding order is given by sorting the numbers $\varphi^{b,\text{excess}}\left(t_{(k-1)^+}; t_i + \tau\right)$ in descending order. The exact procedure is described in Algorithm 1.

The actual number of held bonds with maturity at time $t_i + \tau$, $i = k - \frac{\tau}{\Delta t}, \ldots, k-2$, after potential sales due to the applied financing and investment strategy is

$$\varphi^b\left(t_{(k-1)^+}; t_i + \tau\right) = \varphi^b\left(t_{k-1}; t_i + \tau\right) - \varphi^{b,l}\left(t_{(k-1)^+}; t_i + \tau\right) - \varphi^{b-}\left(t_{(k-1)^+}; t_i + \tau\right),$$

where $\varphi^{b,l}\left(t_{(k-1)^+}; t_i + \tau\right)$ and $\varphi^{b-}\left(t_{(k-1)^+}; t_i + \tau\right)$ are determined in Section 5.3.2 and by Algorithm 1, respectively.

[4] Note that the actual market value of the bonds is irrelevant here. Important is only the number of held bonds since they all pay one unit of money at maturity, respectively.

Algorithm 1 Sale of excessive bonds due to investment strategies (here: DCF)

Set $\eta \leftarrow 0, i_\eta \leftarrow \emptyset, CF_{k-1}^{\text{gap},(\eta)} \leftarrow CF_{k-1}^{\text{gap}}$.

while $\max\left\{\varphi^{b,\text{excess}}\left(t_{(k-1)^+}; t_i + \tau\right) : i \in \left\{k - \frac{\tau}{\Delta t}, \ldots, k-2\right\} \setminus \{i_0, \ldots, i_\eta\}\right\} > 0$ and $CF_{k-1}^{\text{gap},(\eta)} > 0$ **do**

  Update number of iteration: $\eta \leftarrow \eta + 1$

  Find index i_η associated with the largest excess:

$$i_\eta \leftarrow \operatorname{argmax}\left\{\varphi^{b,\text{excess}}\left(t_{(k-1)^+}; t_i + \tau\right) : i \in \left\{k - \frac{\tau}{\Delta t}, \ldots, k-2\right\} \setminus \{i_0, \ldots, i_{\eta-1}\}\right\}$$

  Sell $\varphi^{b-}\left(t_{(k-1)^+}; t_{i_\eta} + \tau\right)$ excessive bonds with maturity date $t_{i_\eta} + \tau$:

$$\varphi^{b-}\left(t_{(k-1)^+}; t_{i_\eta} + \tau\right) = \min\left\{\frac{CF_{k-1}^{\text{gap},(\eta-1)}}{p\left(t_{k-1}, t_{i_\eta} + \tau\right)}, \varphi^{b,\text{excess}}\left(t_{(k-1)^+}; t_{i_\eta} + \tau\right)\right\}$$

  Update remaining coverage fund gap:

$$CF_{k-1}^{\text{gap},(\eta)} \leftarrow CF_{k-1}^{\text{gap},(\eta-1)} - \varphi^{b-}\left(t_{(k-1)^+}; t_{i_\eta} + \tau\right) p\left(t_{k-1}, t_{i_\eta} + \tau\right)$$

end while

Set $\eta^{\max} \leftarrow \eta$.

for $i \in \left\{k - \frac{\tau}{\Delta t}, \ldots, k-2\right\} \setminus \{i_0, \ldots, i_{\eta^{\max}}\}$ **do**

  $\varphi^{b-}\left(t_{(k-1)^+}; t_i + \tau\right) = 0$

end for

Calculate aggregated amount of sold bonds:

$$\xi_{k-1}^{b-} = \sum_{i=k-\frac{\tau}{\Delta t}}^{k-2} \varphi^{b-}\left(t_{(k-1)^+}; t_i + \tau\right) p\left(t_{k-1}, t_i + \tau\right)$$

5.3.2. Selling of excessive bonds due to financing strategies

Analogously to the previous section, we now describe the procedure of selling excessive bonds to reduce the cash demand C_k^{c+} at time t_k. The number of bonds with maturity at time $t_{k+1}, t_{k+2}, \ldots, t_{k-1+\frac{\tau}{\Delta t}}$ held after the reallocation of assets at the beginning of period $[t_{k-1}, t_k]$ is $\varphi^b\left(t_{(k-1)^+}; t_i + \tau\right)$, for $i = k - \frac{\tau}{\Delta t} + 1, k - \frac{\tau}{\Delta t}, \ldots, k-1$. Throughout the interval (t_{k-1}, t_k), there are no changes nor adjustments possible implying that the corresponding excessive number at time t_k is given by

$$\varphi^{b,\text{excess}}\left(t_k; t_i+\tau\right) = \left(\varphi^{b}\left(t_{(k-1)^+}; t_i+\tau\right) - \mathbb{E}\left[D_{i+\frac{\tau}{\Delta t}} \mid \mathcal{F}_k\right]\right)^{+}.$$

As before, the life insurer now successively sells excessive bonds until there is no demand for cash or no excess anymore. The corresponding order is again given by sorting the numbers $\varphi^{b,\text{excess}}\left(t_k; t_i+\tau\right)$ in descending order. The exact procedure is described in Algorithm 2.

Algorithm 2 Sale of excessive bonds due to financing strategies (here: FS 3)

Set $\eta \leftarrow 0, i_\eta \leftarrow \emptyset, C_k^{c+,(\eta)} \leftarrow C_k^{c+}$.

while $\max\left\{\varphi^{b,\text{excess}}\left(t_k; t_i+\tau\right) : i \in \left\{k-\frac{\tau}{\Delta t}+1, \ldots, k-1\right\} \setminus \{i_0, \ldots, i_\eta\}\right\} > 0$ and $C_k^{c+,(\eta)} > 0$ **do**

  Update number of iteration: $\eta \leftarrow \eta + 1$

  Find index i_η associated with the largest excess:

$$i_\eta \leftarrow \operatorname{argmax}\left\{\varphi^{b,\text{excess}}\left(t_k; t_i+\tau\right) : i \in \left\{k-\frac{\tau}{\Delta t}+1, \ldots, k-1\right\} \setminus \{i_0, \ldots, i_{\eta-1}\}\right\}$$

  Sell $\varphi^{b,l}\left(t_k; t_{i_\eta}+\tau\right)$ excessive bonds with maturity date $t_{i_\eta}+\tau$:

$$\varphi^{b,l}\left(t_k; t_{i_\eta}+\tau\right) = \min\left\{\frac{C_k^{c+,(\eta-1)}}{p\left(t_k, t_{i_\eta}+\tau\right)}, \varphi^{b,\text{excess}}\left(t_k; t_{i_\eta}+\tau\right)\right\}$$

  Update remaining demand for cash:

$$C_k^{c+,(\eta)} \leftarrow C_k^{c+,(\eta-1)} - \varphi^{b,l}\left(t_k; t_{i_\eta}+\tau\right) p\left(t_k, t_{i_\eta}+\tau\right)$$

end while

Set $\eta^{\max} \leftarrow \eta$

for $i \in \left\{k-\frac{\tau}{\Delta t}+1, \ldots, k-1\right\} \setminus \{i_0, \ldots, i_{\eta^{\max}}\}$ **do**

  $\varphi^{b,l}\left(t_k; t_i+\tau\right) = 0$

end for

Calculate aggregated amount of sold bonds:

$$\xi_k^{b,l} = \sum_{i=k-\frac{\tau}{\Delta t}+1}^{k-1} \varphi^{b,l}\left(t_k; t_i+\tau\right) p\left(t_k, t_i+\tau\right)$$

The actual number of held bonds with maturity at time $t_i + \tau$, $i = k - \frac{\tau}{\Delta t} + 1, \ldots, k-1$, after potential sales due to the applied financing strategy is

$$\varphi^b\left(t_k; t_i + \tau\right) = \varphi^b\left(t_{(k-1)^+}; t_i + \tau\right) - \varphi^{b,l}\left(t_k; t_i + \tau\right), \tag{5.3.1}$$

where $\varphi^{b,l}\left(t_k; t_i + \tau\right)$ is determined by Algorithm 2. For financing strategy FS 4, where bonds are sold at the beginning of a period, we substitute $\varphi^{b,l}\left(t_k; t_i + \tau\right)$ and $\xi_k^{b,l}$ by $\varphi^{b,l}\left(t_{k+}; t_i + \tau\right)$ and $\xi_{k+}^{b,l}$, respectively, to obtain the corresponding number and aggregated amount of sold bonds at the beginning of period $[t_k, t_{k+1}]$.

5.3.3. Transaction costs

The performance of the new developed investment and financing strategies will later be investigated in the presence of transaction costs. More specifically, we consider proportional transaction costs that depend on the volume of traded assets. As only single prices for bonds and stocks are quoted at each time point t_k, we aggregate $\xi_k^{b,l}$ and $\xi_{k+}^{b,l}$ as well as $\xi_k^{s,l}$ and $\xi_{k+}^{s,l}$, respectively.

In the following, we list the amount of traded assets at time t_k, where we also differentiate between assets that are bought or sold due to the investment strategy (IS) or due to the financing strategy (FS):

- emitted bonds with duration τ due to FS: $\xi_k^l = \varphi_k^l p\left(t_k, t_k + \tau\right)$,
- sold stocks due to FS: $\xi_k^{s,l} + \xi_{k+}^{s,l}$,
- aggregated amount of sold bonds due to FS: $\xi_k^{b,l} + \xi_{k+}^{b,l}$,
- bought bonds with duration τ due to IS: $\xi_k^{b+} = \varphi^b\left(t_{k+}; t_k + \tau\right) p\left(t_k, t_k + \tau\right)$,
- aggregated amount of sold bonds due to IS: ξ_k^{b-},
- bought stocks due to IS: $\xi_k^{s+} = \left(C_{k+}^s - \left(C_k^s - \xi_{k+}^{s,l}\right)\right)^+$, and
- sold stocks due to IS: $\xi_k^{s-} = \left(\left(C_k^s - \xi_{k+}^{s,l}\right) - C_{k+}^s\right)^+$.

The proportional transaction costs are characterized by corresponding cost factors c^b and c^s for bonds and stocks. Furthermore, we assume equal costs for buying and selling of assets. At time t_k, the total amount C_k^{cost} of transaction costs can be calculated by

$$C_k^{\text{cost}} = c^s C_k^{s,\text{traded}} + c^b C_k^{b,\text{traded}}, \tag{5.3.2}$$

where the traded volumes $C_k^{s,\text{traded}}$ and $C_k^{b,\text{traded}}$ of stocks and bonds are given by

$$C_k^{s,\text{traded}} = \xi_k^{s,l} + \xi_{k+}^{s,l} + \xi_k^{s+} + \xi_k^{s-}$$

and

$$C_k^{b,\text{traded}} = \xi_k^l + \xi_k^{b,l} + \xi_{k^+}^{b,l} + \xi_k^{b+} + \xi_k^{b-}.$$

5.4. Balance sheet equation in the refined ALM model

In this section, we prove that also in the refined ALM model, where we allow for selling bonds before maturity, the fundamental balance sheet equation holds at all times. Most of the components of the balance sheet are defined as before in Section 3.7. However, due to the new setting and the adjusted notation, we obtain slightly different representations at some places. If not stated otherwise, the assumptions from Section 3.7, e.g. the order of the business steps, remain unchanged.

According to the new financing strategies, funds are still first taken from expired bonds C_k^{b-} implying that the position of cash C_k^c is defined as before by equation (3.7.1). The stock position C_k^s can now be generally represented by

$$\begin{aligned} C_k^s &= \varphi_{(k-1)^+}^s s_k - \left(D_k - C_k^{b-}\right)^+ + \xi_k^{b,l} + L_k^+ \\ &= \varphi_{(k-1)^+}^s s_k - \xi_k^{s,l}, \end{aligned}$$

where the precise aggregated amount $\xi_k^{b,l}$ of sold bonds, the value $\xi_k^{s,l}$ of sold stocks, and the demand for credits L_k^+ are determined by the chosen financing strategy. Note that for strategies FS 2 and FS 4, we have $\xi_k^{s,l} = 0$ since assets are only sold at the beginning of a period. Among the strategies introduced in this thesis, $\xi_k^{b,l} > 0$ is only possible if we apply financing strategy FS 3. The bond part C_k^b is given by

$$C_k^b = \sum_{i=k-\frac{\tau}{\Delta t}+1}^{k-1} \varphi^b\left(t_k; t_i + \tau\right) p\left(t_k, t_i + \tau\right)$$

with $\varphi^b\left(t_k; t_i + \tau\right)$ from equation (5.3.1).

The total value C_k of the life insurer's assets and all liability positions from the right side of the balance sheet are defined as before, see equation (3.7.2) and the corresponding equations in Section 3.7.2, respectively.

According to the adjusted notation of the number of held bonds, the total surplus G_k for the classic endowment insurance can be written as

$$\begin{aligned} G_k = \varphi_{(k-1)^+}^s \Delta s_k &+ \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi^b\left(t_k; t_i + \tau\right) \Delta p_{k,i} - i_k\left(V_{k-1} + P_{k-1}\right) \\ &- \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi_i^l \Delta p_{k,i} + \left(\frac{1}{\vartheta} - 1\right) S_k. \end{aligned} \tag{5.4.1}$$

The following theorem shows that also in our new setting, the life insurer's business activities lead to equal sums of assets and the liabilities as displayed in Table 3.2.

Theorem 5.4.1 (Verification of the refined model). *Consider the endowment insurance with surrender factor $\vartheta > 0$ and suppose that the sum of all assets equals the sum of all liabilities at the start of the simulation, i.e., $C_0 = A_0 + B_0 + F_0 + Q_0 + L_0$. Then, the fundamental balance sheet equation is fulfilled at any time, i.e. it holds*

$$C_k = A_k + B_k + F_k + Q_k + L_k$$

for all $k = 0, \dots, K$.

Proof. We prove the statement by induction over k. By assumption, the equality holds at time t_0. As induction hypothesis, we assume that the equality

$$C_{k-1} = A_{k-1} + B_{k-1} + F_{k-1} + Q_{k-1} + L_{k-1}$$

holds for all times t_i with $i \le k-1 < K$. For the induction step, we first decompose the total capital C_k into

$$C_k = C_k^b + C_k^s + C_k^c.$$

Taking price changes from period k, the demand for credits L_k^+, and the disbursements D_k at time t_k into account, the latter equation is linked to the stock and bond part after the reallocation of assets at time t_{k-1} via

$$C_k = C^b_{(k-1)^+} + C^s_{(k-1)^+} + \varphi^s_{(k-1)^+} \Delta s_k + \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi^b\left(t_k; t_i + \tau\right) \Delta p_{k,i} + L_k^+ - D_k.$$

Using

$$\begin{aligned} D_k &= \mathcal{B}_k + L_k^-, \\ \mathcal{B}_k &= E_k + T_k + S_k, \\ C_{(k-1)^+} &= C^b_{(k-1)^+} + C^s_{(k-1)^+}, \end{aligned}$$

and plugging in the representation of $C_{(k-1)^+}$ from equation (5.1.2), we get

$$\begin{aligned} C_k = C_{k-1} &- \xi^{b,l}_{(k-1)^+} - \xi^{s,l}_{(k-1)^+} + \left(P_{k-1} - L^+_{k-1}\right)^+ + \varphi^s_{(k-1)^+} \Delta s_k \\ &+ \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi^b\left(t_k; t_i + \tau\right) \Delta p_{k,i} + L_k^+ - E_k - T_k - S_k - L_k^-. \end{aligned}$$

According to the specific representation of the surplus G_k from equation (5.4.1), we can write

$$C_k = C_{k-1} - \xi^{b,l}_{(k-1)^+} - \xi^{s,l}_{(k-1)^+} + \left(P_{k-1} - L^+_{k-1}\right)^+ + G_k + i_k\left(V_{k-1} + P_{k-1}\right) - \frac{1}{\vartheta}S_k$$
$$+ \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi^l_i \Delta p_{k,i} + L^+_k - E_k - T_k - L^-_k .$$

Using the recursive scheme of the technical reserve V_k from Proposition 3.7.1 (iii) and the induction hypothesis, we get

$$C_k = A_{k-1} + B_{k-1} + F_{k-1} + Q_{k-1} + L_{k-1} - \xi^{b,l}_{(k-1)^+} - \xi^{s,l}_{(k-1)^+} + \left(P_{k-1} - L^+_{k-1}\right)^+ + G_k$$
$$+ V_k - \left(V_{k-1} + P_{k-1}\right) + \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi^l_i \Delta p_{k,i} + L^+_k - L^-_k .$$

The relation between own funds and surplus in Remark 3.7.2 and $V_{k-1} = A_{k-1} + B_{k-1}$ imply

$$C_k = F_k + Q_k + L_{k-1} - \xi^{b,l}_{(k-1)^+} - \xi^{s,l}_{(k-1)^+} + \left(P_{k-1} - L^+_{k-1}\right)^+ + V_k - P_{k-1}$$
$$+ \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi^l_i \Delta p_{k,i} + L^+_k - L^-_k .$$

Using equation (3.7.6) for L_{k-1} and observing the equality

$$L^{\text{new}}_{k-1} = \left(P_{k-1} - L^+_{k-1}\right)^+ - \xi^{b,l}_{(k-1)^+} - \xi^{s,l}_{(k-1)^+} - P_{k-1} + L^+_{k-1}$$

for both financing strategies, see also equation (3.5.13), we can write

$$C_k = F_k + Q_k + V_k + L^{\text{new}}_{k-1} + \sum_{i=k-\frac{\tau}{\Delta t}}^{k-2} \varphi^l_i p\left(t_{k-1}, t_i + \tau\right) + \sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi^l_i \Delta p_{k,i} + L^+_k - L^-_k .$$

Since

$$\begin{aligned}
L^{\text{new}}_{k-1} &= \varphi^l_{k-1} p\left(t_{k-1}, t_{k-1} + \tau\right), \\
L^-_k &= \varphi^l_{k-\frac{\tau}{\Delta t}} p\left(t_k, t_{k-\frac{\tau}{\Delta t}} + \tau\right), \\
\Delta p_{k,i} &= p\left(t_k, t_i + \tau\right) - p\left(t_{k-1}, t_i + \tau\right),
\end{aligned}$$

the latter equation becomes

$$\begin{aligned} C_k &= F_k + Q_k + V_k + \sum_{i=k-\frac{\tau}{\Delta t}+1}^{k-1} \varphi_i^l p\left(t_k, t_i + \tau\right) + L_k^+ \\ &= A_k + B_k + F_k + Q_k + L_k \end{aligned}$$

which completes the proof. □

5.5. Alternative strategy for the use of surpluses

We complete this chapter by proposing an alternative modeling approach of the management decisions regarding the use of surpluses. Thereby, we further demonstrate the flexibility of our general ALM model.

In Section 3.4, we introduced the balance sheet position "free reserve" that represents the aggregated amount of unappropriated and unallocated surpluses. As these funds are not assigned to individual insured, they can be used to cover future losses under strict conditions. In Section 3.7.2, we thus assumed that the free reserve F_{k-1} fully absorbs losses $G_k < 0$ while the withdrawals are limited to the available funds. If these are not sufficient, the shareholders absorb the remaining loss. In any case, a fixed portion αG_k of a positive surplus $G_k > 0$ is deposited in the free reserve F_{k-1} and the remaining amount is credited to the equity Q_{k-1}. According to that procedure, we obtained the representations (3.7.10) and (3.7.11) of the free reserve F_{k-1} and the equity Q_{k-1}, respectively.

In the following, we introduce an alternative strategy for the use of surpluses and the resulting risk sharing between policyholders and shareholders. In contrast to the method from Section 3.7.2, the allocation of surpluses now depends on whether the equity is negative or positive.[5] Furthermore, we aim at deriving a procedure for the splitting of losses that adequately reflects the past surplus allocation. This is motivated by our assumption that even if approval for withdrawals from the free reserve to cover losses is taken for granted, it is questionable if regulators would allow the life insurer to only use funds from F_{k-1}.[6] However, as pointed out by Burkhart et al. [14], there are hardly any generally applicable legal regulations in this regard since each case has to be considered individually.

We denote by G_k^F and G_k^Q the portions of the surplus that are credited to ($G_k \geq 0$) or absorbed by ($G_k < 0$) the free reserve F_{k-1} and the equity Q_{k-1}, respectively, such that

$$G_k^F + G_k^Q = G_k.$$

[5]The event of a negative equity is considered as technical ruin and formed also the basis for defining the default probability, see equation (4.2.2). In such cases, we do not stop our simulations implying that the equity can get positive again. For comparison, the free reserve is always non-negative. In practice, a possible restriction in the case of a technical ruin could be that the life insurer is not allowed to sell new contracts, which we do not consider in this thesis.

[6]In practice, it is likely that other options like the realization of unrealized gains need to be checked before using funds from the free reserve.

In particular, we always ensure that surpluses are still completely allocated every period, i.e. distributed between F_{k-1} and Q_{k-1}. Therefore, new proofs regarding the fulfillment of the balance sheet equation are not required.

The alternative strategy for the use of surpluses is specified in the following, where we distinguish different cases.

Case 1: $G_k \geq 0$ and $Q_{k-1} \geq 0$

In this case, we proceed as in Section 3.7.2, i.e. a fixed portion

$$G_k^F = \alpha G_k$$

is deposited in the free reserve F_{k-1} and the remaining amount

$$G_k^Q = (1-\alpha)\, G_k$$

is credited to the equity Q_{k-1}.

Case 2: $G_k \geq 0$ and $Q_{k-1} < 0$

If the equity is negative, the positive surplus is completely used to repay the liabilities to the shareholders. Only the surpassing part $(G_k - |Q_{k-1}|)^+$ is allocated between the free reserve and the equity as in Case 1. Accordingly, we have

$$G_k^F = \alpha\, (G_k - |Q_{k-1}|)^+$$

and

$$G_k^Q = G_k - G_k^F.$$

Case 3: $G_k < 0$ and $Q_{k-1} \geq 0$

If the equity is non-negative, losses $G_k < 0$ are shared between policyholders and shareholders in the same proportion as positive surpluses were allocated in the last τ^G years. This yields

$$G_k^F = \frac{\sum_{i=k-\frac{\tau^G}{\Delta t}}^{k-1} \left(G_i^F\right)^+}{\sum_{i=k-\frac{\tau^G}{\Delta t}}^{k-1} (G_i)^+} G_k$$

and

$$G_k^Q = G_k - G_k^F.$$

Case 4: $G_k < 0$ and $Q_{k-1} < 0$

Here, we assume that a larger amount of equity has a higher priority than a potential fair risk sharing as in Case 3. Therefore, the free reserve fully absorbs losses $G_k < 0$ while the withdrawals are limited to the available funds as in Section 3.7.2. Correspondingly, we obtain

$$G_k^F = \max\{G_k, -F_{k-1}\}$$

and

$$G_k^Q = G_k - G_k^F.$$

In any case, the free reserve F_k and the equity Q_k can be represented as

$$F_k = F_{k-1} + G_k^F \qquad \text{and} \qquad Q_k = Q_{k-1} + G_k^Q,$$

respectively.

6. Simulation Studies: Part II

In this chapter, we perform several simulation studies to illustrate the refinements and the alternative modeling approaches from Chapter 5. We consider again a classic endowment insurance equipped with the specifications from Section 3.6.2. If not stated otherwise, we use the basic capital market model from Section 3.3.1.

In Section 6.1, we provide an extensive comparison study of all introduced investment and financing strategies for different new business scenarios. The performance is investigated taking into account both the life insurer's and the policyholders' point of view in the presence of transaction costs. Further applications of the general and the refined ALM model are illustrated in Section 6.2. This includes the modeling of the capital market, the used method for the annual interest rate declaration, and the strategy for the use of surpluses.

6.1. Performance of the investment and financing strategies

In this section, we investigate the performance of all investment and financing strategies developed in this thesis. For the sake of clarity, they are summarized in Table 6.1 together with their corresponding main characteristics.

For each pair $(\mathrm{IS}, \mathrm{FS})$ of investment and financing strategies, we simulate the resulting ALM model $N = 10,000$ times. To investigate the direct impact of the strategies, we each use the same dynamics of the capital market, i.e. the n-th Monte Carlo path for the pairs $(\mathrm{IS}, \mathrm{FS})$ with corresponding IS and FS is based on the same generated random numbers, respectively. Thereby, we allow for a pathwise comparison. The input parameters and the distributional assumptions are taken from Table 4.1, except for the minimum stock ratio which is set to $\pi_k^{s,\min} = 10\%$. Regarding the new business, we consider four different scenarios.

IS	Section	Main characteristics
CM	3.5.1	The target is a constant stock ratio.
CPPI	3.5.1	The stock investments are linked to the amount of own funds.
SCF	5.1.2	The bond investments are linked to the expected amount of disbursements. The corresponding coverage fund has a high priority, i.e. stocks are sold if necessary.
RCF	5.1.2	Same as for SCF but the corresponding coverage fund has a lower priority, i.e. no assets are sold.
DCF	5.1.2	Same as for RCF but the bond investments are increased such that excessive bonds can be sold if necessary.

FS	Section	Main characteristics
FS 1	3.5.3	If payouts from expired bonds are not sufficient, stocks are sold before raising a credit and preparing the balance sheet.
FS 2	3.5.3	If payouts from expired bonds are not sufficient, a credit is directly raised before preparing the balance sheet. Stocks may be sold at the beginning of the following period.
FS 3	5.2	Same as for FS 1 but, in addition, excessive bonds are sold if necessary before selling stocks.
FS 4	5.2	Same as for FS 2 but, in addition, excessive bonds are sold if necessary before selling stocks.

Table 6.1.: Overview of the investment and financing strategies developed in this thesis. The last column only describes the respective main characteristics, while details can be found in the corresponding sections.

Table 6.2 displays the chosen shape parameters of the beta distribution for modeling the new contract arrivals. The corresponding new business scenarios and the resulting size of the insurance portfolios are illustrated in Figure 6.1 where, e.g., the first row reflects the scenario with a negative shock on the expected future number of new customers after 25 years (NBS 1) and the last row reflects the scenario of a stationary new business (NBS 4).

NBS	Shape parameters α_k and β_k
1	$(\alpha_k, \beta_k) = (20, 2)$ for $k = 1, \dots, \frac{K}{2}$ and $(\alpha_k, \beta_k) = (2, 5)$ for $k = \frac{K}{2} + 1, \dots, K$
2	$(\alpha_1, \beta_1) = (2, 5)$, $(\alpha_K, \beta_K) = (5, 2)$, and for $k = 0, \dots, K-2$: $\alpha_{k+1} = \alpha_1 + (\alpha_K - \alpha_1)\frac{k}{K-1}$, $\beta_{k+1} = \beta_1 - (\beta_1 - \beta_K)\frac{k}{K-1}$
3	$(\alpha_1, \beta_1) = (20, 2)$, $(\alpha_K, \beta_K) = (2, 5)$, and for $k = 0, \dots, K-2$: $\alpha_{k+1} = \alpha_1 - (\alpha_1 - \alpha_K)\frac{k}{K-1}$, $\beta_{k+1} = \beta_1 + (\beta_K - \beta_1)\frac{k}{K-1}$
4	$(\alpha_k, \beta_k) = (1, 1)$ for $k = 1, \dots, K$

Table 6.2.: Parameterization of the considered new business scenarios (NBS).

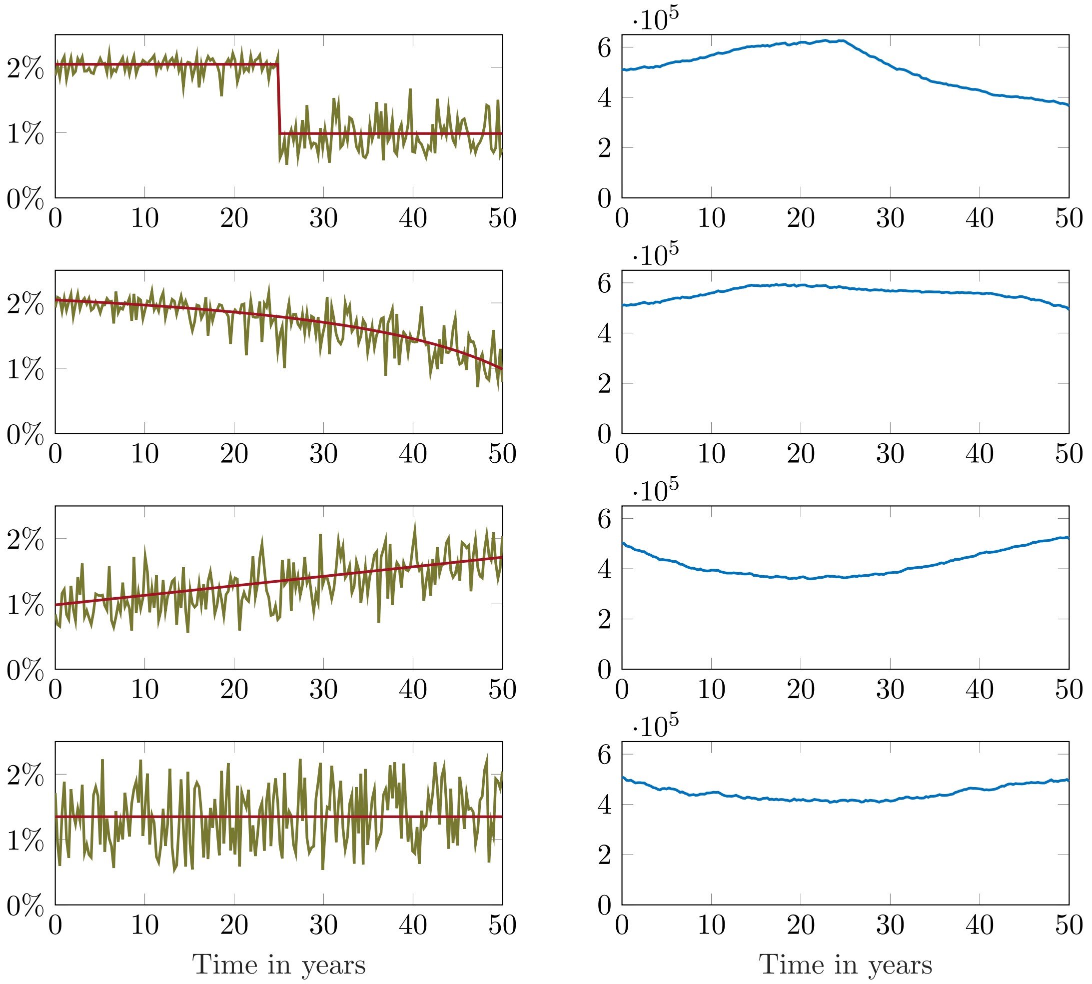

Figure 6.1.: NBS 1 (top) to NBS 4 (bottom). Left: number of new customers per period (expected number in red, one path in green), right: corresponding path of the insurance portfolio size.

We assume that the life insurer has no prior knowledge about the future new business and thus assumes a stationary development. This motivates the choice of a constant value for $\mathbb{E}\left[D_{k-1+\frac{\tau}{\Delta t}} \,|\, \mathcal{F}_{k-1}\right]$, $k = 1, \ldots, K$, in the general representation (5.1.1). In this section, we thus choose the following form of the coverage fund CF_{k-1}:

$$CF_{k-1} = 4 \cdot 10^8 \cdot p\left(t_{k-1}, t_{k-1} + \tau\right). \tag{6.1.1}$$

The specific value of $\mathbb{E}\left[D_{k-1+\frac{\tau}{\Delta t}} \,|\, \mathcal{F}_{k-1}\right]$ is derived from prior simulations[1] and taking into account a certain safety loading. As coverage fund multiplier we use $\lambda_{k-1}^{\mathrm{CF}} = 1.5$.

[1] See Figure 5.1 in Section 5.1.1.

The development of new investment strategies in Section 5.1 was motivated by the observed periodic pattern with pronounced peaks in the amount of bought bonds if we apply the CM or the CPPI strategy. This characteristic can be seen as a weakness since it reflects the fact that both investment strategies are not well oriented to the obligations induced by the insured collective. In the following, we thus examine how the bond and stock investments differ if we apply the other investment strategies. Here, we consider the scenario of a stationary new business (NBS 4) and use financing strategy FS 1 as we did in the CM-case in Figure 5.1. The results for the remaining new business scenarios are shown in Appendix A.2.

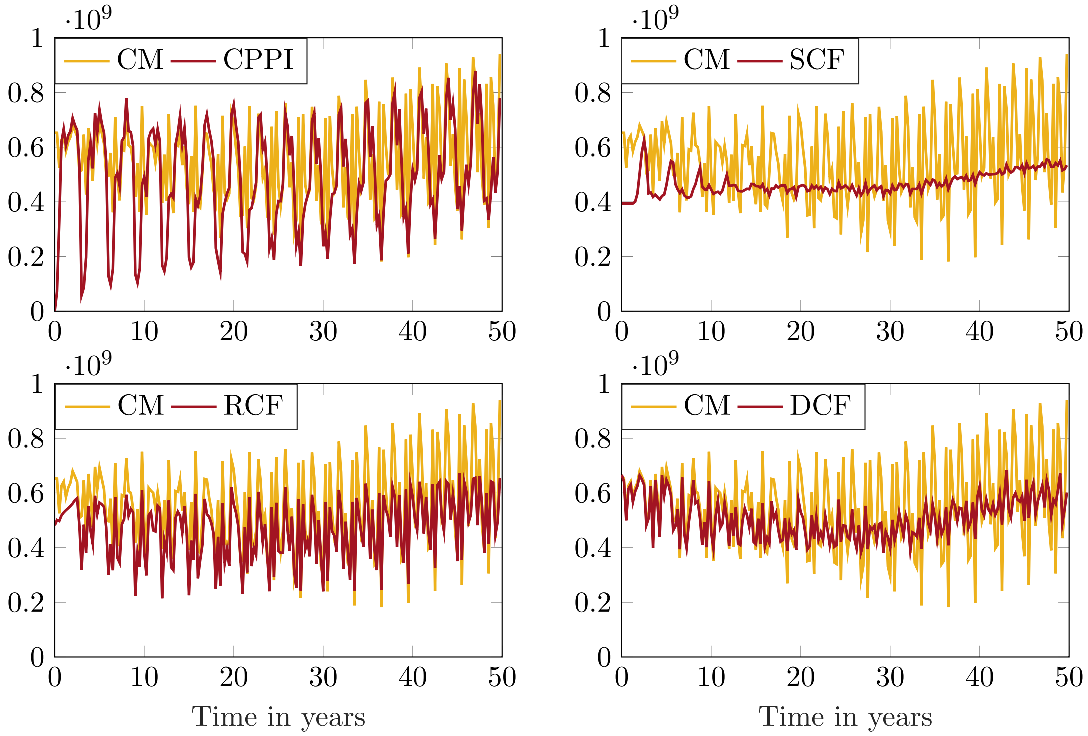

Figure 6.2.: Amount of bought bonds applying different investment strategies and financing strategy FS 1 in the case of a stationary new business (NBS 4).

Figure 6.2 shows the average amount of bought bonds if we apply different investment strategies. Compared to the CM-case, the CPPI strategy yields a development that is more volatile in the first 20 years but less volatile in the long term.[2] The coverage fund-based strategies yield even less volatile developments and, except for the RCF strategy, there is no longer a periodic pattern with pronounced peaks. Especially if we apply the SCF strategy, the average bond investments are quite smooth from year 12 onwards.

[2]Note that for the path considered in the upper part of Figure 4.20, the development in the CPPI-case was more volatile in the whole time horizon.

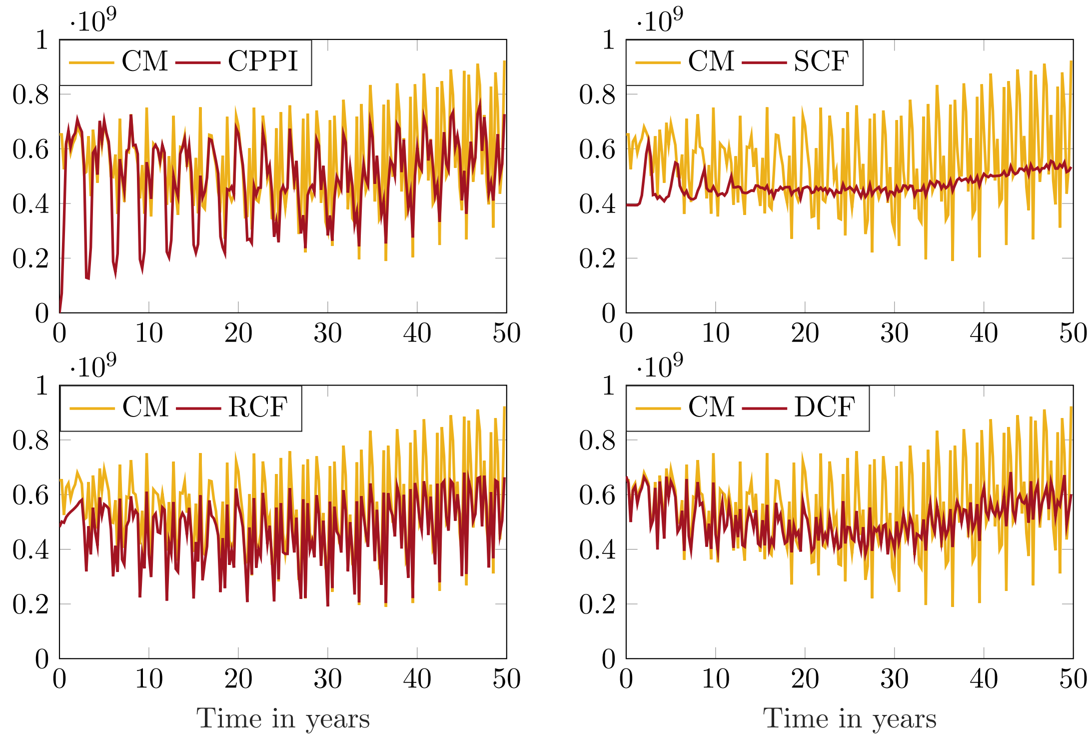

Figure 6.3.: Amount of bought bonds applying different investment strategies and financing strategy FS 4 in the case of a stationary new business (NBS 4).

Figure 6.3 illustrates the respective average amounts of bought bonds if we apply financing strategy FS 4, where we additionally allow for selling excessive bonds. Compared to Figure 6.2, we barely see any differences. This indicates that, in contrast to the investment strategies, the considered financing strategies do not significantly impact the bond investments.

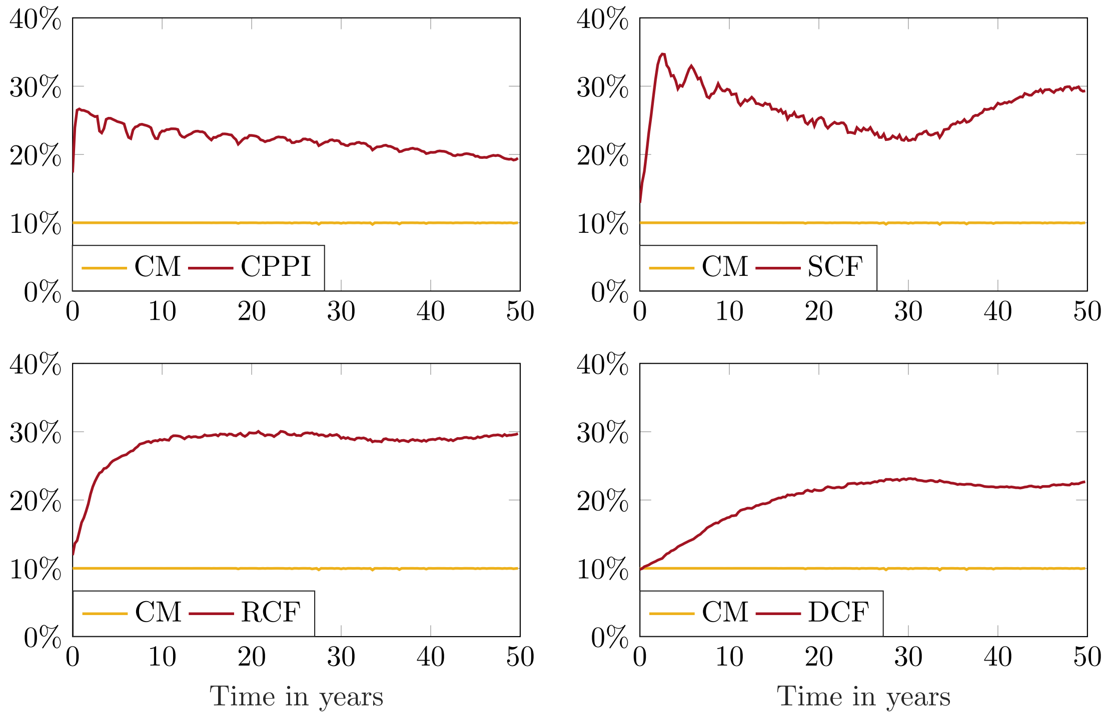

Figure 6.4.: Stock ratio after reallocation of assets applying different investment strategies and financing strategy FS 1 in the case of a stationary new business (NBS 4).

In Figure 6.4, we show the average reached stock ratios after the reallocation of assets if we apply different investment strategies. In the CPPI-case, we clearly see a periodic pattern since bonds are held until maturity according to financing strategy FS 1. If we apply the DCF strategy, the stock ratio increases gradually (after one year it is 10.5%) and then becomes stable around 22% in the long term. In contrast, the average stock ratios increase much stronger in the first year to over 26%, 23%, and 16% if we apply the CPPI, the SCF, and the RCF strategy, respectively. In the CPPI-case the stock ratio then gradually decreases while in the RCF-case it further increases and becomes stable around 29% from year 10 onwards. If we apply the SCF strategy, the stock ratio even increases to over 34% in the second year. Eventually, it gradually decreases to 22% in year 28 before it increases again to 29%. In the CM-case, the stock ratio equals almost always the target value of 10%.

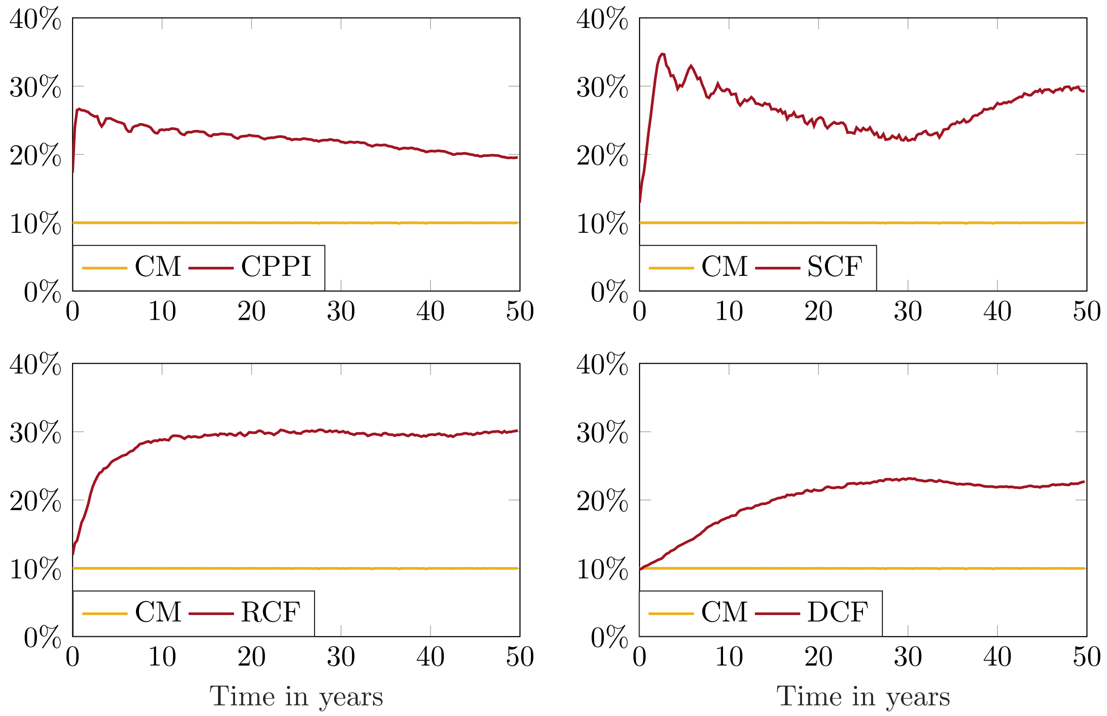

Figure 6.5.: Stock ratio after reallocation of assets applying different investment strategies and financing strategy FS 4 in the case of a stationary new business (NBS 4).

Figure 6.5 displays the respective average stock ratio after the reallocation of assets if we apply financing strategy FS 4. Compared to Figure 6.4, we now obtain a much smoother development in the CPPI-case. This results from selling excessive bonds at the beginning of a period. In the other cases, the effect is less visible.

As in Section 4.5, we now measure the strategies' performances in terms of different criteria reflecting both the life insurer's and the policyholders' point of view. However, in this section we use modified criteria and include additional ones like, for example, the average ratio of transaction costs per period. In the following, we describe the corresponding criteria in more detail.

Criterion 1: default probability

The cumulative default probability is defined in equation (4.2.2) in Section 4.2.2 and represents an established measure for the risk of the life insurance business. Here, we look at the 5-year and 50-year default probability $PD_{\frac{5}{\Delta t}}$ and PD_K, respectively.

Criterion 2: declared interest rate

Also, the annual declared interest rate $\widehat{i}_k$ from equation (3.5.5) in Section 3.5.2 was already used in various simulation studies. In contrast to before, we now consider the average interest rate within the intervals $(0, 5]$ and $(5, 50]$. Thereby, we differentiate between short- (or medium) and long-term developments.

Criterion 3: reserve rate

Recall that the reserve rate γ_k describes the ratio of the free reserve F_k and the total reserves $F_k + V_k$ and forms the basis of the annual interest rate declaration, see equations (3.5.5) and (3.5.6) in Section 3.5.2. As buffer against future losses, we introduced a target value γ. For the first sub-criterion, we estimate for each period k the probability that the objective $\gamma_k \geq \gamma$ is achieved. We then average over all periods to get the corresponding average probability as a number for illustration. As a second sub-criterion, we take the average reserve rate per period.

Criterion 4: own funds

Analogously to the previous criterion, we now consider the average probability that the fraction of own funds ψ_k with

$$\psi_k = \frac{F_k + Q_k}{C_k}$$

is greater than or equal a target value ψ that might be prescribed by regulating authorities.[3] As a second sub-criterion, we take the average fraction of own funds per period.

Criterion 5: transaction costs

The performance of the strategies is also measured in terms of transaction costs. In the following simulation studies, we calculate the proportional transaction costs C_k^{cost} at time k by equation (5.3.2) with

$$c^b = \frac{0.1}{1000} \qquad \text{and} \qquad c^s = \frac{3}{1000},$$

and display the relative transaction costs $\frac{C_k^{\text{cost}}}{C_k}$. More specifically, we average over all simulations and all periods to get the average ratio of transaction costs as a number for illustration.

Criterion 6: liabilities to banks

The last criterion is the average ratio $\frac{L_k}{C_k}$ of credit liabilities L_k and total assets C_k. As before, we average over all simulations and all periods to get the average ratio of liabilities to banks as a number for illustration.

[3]Here, we set $\psi = 10\%$.

The results of the comparison study are shown in the following tables, one for each new business scenario. For every (sub-) criterion, we display in bold the best value and those values that deviate by less than 5% from the best value.

IS	$PD_{\frac{5}{\Delta t}}, PD_K$	$\widehat{i}_k$ $[0,5),[5,50)$	$\gamma_k \geq \gamma, \gamma_k$	$\psi_k \geq \psi, \psi_k$	$\frac{C_k^{\text{cost}}}{C_k}$	$\frac{L_k}{C_k}$
CM	0.46, **35.74**	1.01, 1.32	51.80, 10.20	**74.94**, **14.54**	0.03	**0.00**
	0.46, **35.74**	1.01, 1.32	51.80, 10.20	**74.92**, **14.53**	0.03	0.07
	0.46, **35.66**	1.01, 1.32	51.80, 10.20	**74.94**, **14.54**	0.03	**0.00**
	0.46, **35.72**	1.01, 1.32	51.81, 10.20	**74.93**, **14.53**	0.03	0.06
CPPI	5.96, 60.06	**1.34**, 1.81	43.19, 10.48	56.54, 12.50	0.07	0.79
	5.94, 60.05	**1.34**, 1.81	43.21, 10.48	56.41, 12.44	0.06	1.17
	6.19, 60.47	**1.35**, 1.82	43.18, 10.50	56.35, 12.36	0.06	**0.00**
	6.02, 60.15	**1.35**, 1.81	43.16, 10.48	56.31, 12.30	0.06	0.40
SCF	16.71, 65.21	**1.32**, **2.08**	**55.93**, **12.63**	65.65, 12.83	0.05	**0.00**
	16.71, 65.21	**1.32**, **2.08**	**55.93**, **12.63**	65.65, 12.83	0.05	**0.00**
	16.71, 65.21	**1.32**, **2.08**	**55.93**, **12.63**	65.65, 12.83	0.05	**0.00**
	16.71, 65.21	**1.32**, **2.08**	**55.93**, **12.63**	65.65, 12.83	0.05	**0.00**
RCF	5.44, 60.14	1.17, **2.00**	**54.65**,**12.29**	67.88, **14.50**	0.02	**0.00**
	5.44, 60.27	1.17, **2.00**	**54.77**, **12.32**	67.89, **14.50**	0.02	0.04
	5.44, 60.21	1.17, **2.00**	**54.74**, **12.31**	67.90, **14.50**	0.02	**0.00**
	5.44, 60.27	1.17, **2.00**	**54.80**, **12.32**	67.90, **14.50**	0.02	0.04
DCF	**0.22**, 48.58	1.04, 1.72	50.64, 11.18	70.77, **15.06**	**0.01**	**0.00**
	0.22, 48.58	1.04, 1.72	50.64, 11.18	70.77, **15.06**	**0.01**	**0.00**
	0.22, 48.58	1.04, 1.72	50.64, 11.18	70.77, **15.06**	**0.01**	**0.00**
	0.22, 48.58	1.04, 1.72	50.64, 11.18	70.77, **15.06**	**0.01**	**0.00**

Table 6.3.: Performance of the strategies in NBS 1. For each investment strategy IS, the four rows correspond to the four financing strategies FS. The transaction costs ratios $\frac{C_k^{\text{cost}}}{C_k}$ are denoted in per mil and all other criteria in percentage terms.

The performance of the strategies in the new business scenario with a negative shock on the expected future number of new customers after 25 years (NBS 1) are shown in Table 6.3. We see that there are substantial differences among the investment strategies but, except for the liabilities to banks, no or marginal differences among the financing strategies. Furthermore, for each investment strategy we can find a (sub-) criterion where it performs the best. Regarding CM and CPPI strategies, the results are in line with the observations from Section 4.5.[4]

[4]Note that here we consider the fraction instead of the total amount of own funds.

IS	$PD_{\frac{5}{\Delta t}}, PD_K$	$\widehat{i}_k$ $[0,5),[5,50)$	$\gamma_k \geq \gamma, \gamma_k$	$\psi_k \geq \psi, \psi_k$	$\frac{C_k^{\text{cost}}}{C_k}$	$\frac{L_k}{C_k}$
CM	0.46, **37.03**	1.01, 1.29	50.56, 9.98	**74.35**, **14.22**	0.03	**0.00**
	0.46, **37.03**	1.01, 1.29	50.56, 9.98	**74.34**, **14.22**	0.03	0.04
	0.46, **37.00**	1.01, 1.29	50.57, 9.98	**74.36**, **14.23**	0.03	**0.00**
	0.46, **37.04**	1.01, 1.29	50.57, 9.98	**74.34**, **14.22**	0.03	0.04
CPPI	5.96, 60.89	**1.34**, 1.77	42.45, 10.29	56.14, 12.40	0.07	0.43
	5.94, 60.93	**1.34,** 1.77	42.46, 10.29	56.02, 12.34	0.06	0.78
	6.19, 61.30	**1.35**, 1.78	42.45, 10.31	55.95, 12.35	0.06	**0.00**
	6.02, 60.93	**1.34**, 1.78	42.43, 10.29	55.91, 12.29	0.06	0.37
SCF	16.71, 66.62	**1.32**, **2.05**	**55.78**, **12.52**	65.44, 13.04	0.05	**0.00**
	16.71, 66.62	**1.32**, **2.05**	**55.78**, **12.52**	65.44, 13.04	0.05	**0.00**
	16.71, 66.62	**1.32**, **2.05**	**55.78**, **12.52**	65.44, 13.04	0.05	**0.00**
	16.71, 66.62	**1.32**, **2.05**	**55.78**, **12.52**	65.44, 13.04	0.05	**0.00**
RCF	5.45, 61.25	1.17, **1.95**	**53.96**, **12.08**	67.55, **14.42**	0.02	**0.00**
	5.45, 61.32	1.17, **1.95**	**54.05**, **12.10**	67.57, **14.42**	0.02	0.02
	5.45, 61.29	1.17, **1.95**	**53.98**, **12.09**	67.56, **14.42**	0.02	**0.00**
	5.45, 61.32	1.17, **1.95**	**54.05**, **12.10**	67.57, **14.42**	0.02	0.02
DCF	**0.22**, 49.91	1.04, 1.67	49.67, 10.93	70.27, **14.81**	**0.01**	**0.00**
	0.22, 49.91	1.04, 1.67	49.67, 10.93	70.26, **14.81**	**0.01**	**0.00**
	0.22, 49.91	1.04, 1.67	49.67, 10.93	70.26, **14.81**	**0.01**	**0.00**
	0.22, 49.91	1.04, 1.67	49.67, 10.93	70.26, **14.81**	**0.01**	**0.00**

Table 6.4.: Performance of the strategies in NBS 2. For each investment strategy IS, the four rows correspond to the four financing strategies FS. The transaction costs ratios $\frac{C_k^{\text{cost}}}{C_k}$ are denoted in per mil and all other criteria in percentage terms.

The performance of the strategies in the case of a gradually decreasing new business (NBS 2) is displayed in Table 6.4. Compared to NBS 1, they perform worse such that, for example, the average reserve rate γ_k per period is now smaller than the target value γ if we apply the CM strategy. Nevertheless, the best values for each (sub-) criterion correspond to the same strategies as in NBS 1.

IS	$PD_{\frac{5}{\Delta t}}, PD_K$	$\widehat{i}_k$ $[0,5),[5,50)$	$\gamma_k \geq \gamma, \gamma_k$	$\psi_k \geq \psi, \psi_k$	$\frac{C_k^{\text{cost}}}{C_k}$	$\frac{L_k}{C_k}$
CM	0.39, **27.76**	1.02, 1.38	**54.52**, 10.61	**80.84**, **16.48**	0.03	**0.00**
	0.39, **27.76**	1.02, 1.38	**54.52**, 10.61	**80.82**, **16.47**	0.03	0.07
	0.39, **27.79**	1.02, 1.38	**54.53**, 10.61	**80.84**, **16.48**	0.03	**0.00**
	0.39, **27.75**	1.02, 1.39	**54.53**, 10.61	**80.83**, **16.47**	0.03	0.07
CPPI	5.63, 53.64	**1.36**, 1.93	46.10, 11.13	61.45, 14.77	0.07	0.85
	5.63, 53.75	**1.36**, 1.93	46.11, 11.13	61.32, 14.69	0.06	1.27
	5.93, 54.46	**1.36**, **1.95**	46.07, 11.15	61.18, 14.64	0.06	**0.00**
	5.75, 54.04	**1.36**, 1.94	46.06, 11.14	61.19, 14.58	0.06	0.40
SCF	15.92, 56.74	**1.33**, 1.92	49.44, 11.50	65.69, 14.42	0.04	**0.00**
	15.92, 56.74	**1.33**, 1.92	49.44, 11.50	65.69, 14.42	0.04	**0.00**
	15.92, 56.74	**1.33**, 1.92	49.44, 11.50	65.69, 14.42	0.04	**0.00**
	15.92, 56.74	**1.33**, 1.92	49.44, 11.50	65.69, 14.42	0.04	**0.00**
RCF	4.75, 51.44	1.18, **2.04**	**54.93**, **12.43**	72.00, **16.65**	0.02	**0.00**
	4.75, 52.14	1.18, **2.05**	**55.36**, **12.51**	72.04, **16.64**	0.02	0.07
	4.75, 51.53	1.18, **2.04**	**55.06**, **12.45**	72.05, **16.65**	0.02	**0.00**
	4.75, 52.17	1.18, **2.05**	**55.37**, **12.51**	72.05, **16.64**	0.02	0.07
DCF	**0.21**, 37.55	1.05, 1.75	**52.64**, 11.44	**76.84**, **17.13**	**0.01**	**0.00**
	0.21, 37.54	1.05, 1.75	**52.67**, 11.44	**76.85**, **17.13**	**0.01**	**0.00**
	0.21, 37.54	1.05, 1.75	**52.65**, 11.44	**76.84**, **17.13**	**0.01**	**0.00**
	0.21, 37.54	1.05, 1.75	**52.67**, 11.44	**76.85**, **17.13**	**0.01**	**0.00**

Table 6.5.: Performance of the strategies in NBS 3. For each investment strategy IS, the four rows correspond to the four financing strategies FS. The transaction costs ratios $\frac{C_k^{\text{cost}}}{C_k}$ are denoted in per mil and all other criteria in percentage terms.

The performance of the strategies in the scenario of a gradually expanding new business (NBS 3) is displayed in Table 6.5. Nearly all strategies perform now better than in the scenarios above, especially regarding the long-term default probability. An exception is here the SCF strategy performing much worse according to the reserve rate criteria and the interest rate declaration in the long term.

IS	$PD_{\frac{5}{\Delta t}}, PD_K$	$\widehat{i}_k$ $[0,5), [5,50)$	$\gamma_k \geq \gamma, \gamma_k$	$\psi_k \geq \psi, \psi_k$	$\frac{C_k^{\text{cost}}}{C_k}$	$\frac{L_k}{C_k}$
CM	0.40, **29.59**	1.02, 1.36	**53.73**, 10.48	**79.37**, **15.94**	0.03	**0.00**
	0.40, **29.59**	1.02, 1.36	**53.73**, 10.48	**79.35**, **15.93**	0.03	0.07
	0.40, **29.61**	1.02, 1.36	**53.73**, 10.48	**79.37**, **15.94**	0.03	**0.00**
	0.40, **29.59**	1.02, 1.36	**53.73**, 10.48	**79.36**, **15.93**	0.03	0.06
CPPI	5.73, 55.34	**1.35**, 1.89	45.24, 10.93	60.12, 14.14	0.07	0.74
	5.73, 55.38	**1.35**, 1.89	45.26, 10.94	60.00, 14.07	0.06	1.15
	6.02, 56.00	**1.36**, 1.91	45.24, 10.96	59.91, 14.03	0.06	**0.00**
	5.87, 55.73	**1.36**, 1.90	45.24, 10.94	59.88, 13.98	0.06	0.40
SCF	16.17, 60.07	**1.33**, **1.98**	51.16, **11.83**	64.88, 14.05	0.04	**0.00**
	16.17, 60.07	**1.33**, **1.98**	51.16, **11.83**	64.88, 14.05	0.04	**0.00**
	16.17, 60.07	**1.33**, **1.98**	51.16, **11.83**	64.88, 14.05	0.04	**0.00**
	16.17, 60.07	**1.33**, **1.98**	51.16, **11.83**	64.88, 14.05	0.04	**0.00**
RCF	4.96, 53.81	1.18, **2.02**	**54.73**, **12.35**	70.84, **16.07**	0.02	**0.00**
	4.96, 54.15	1.18, **2.03**	**55.07**, **12.42**	70.89, **16.07**	0.02	0.06
	4.96, 53.83	1.18, **2.02**	**54.83**, **12.37**	70.88, **16.08**	0.02	**0.00**
	4.96, 54.16	1.18, **2.03**	**55.08**, **12.42**	70.89, **16.07**	0.02	0.06
DCF	**0.22**, 39.74	1.04, 1.72	51.86, 11.29	75.36, **16.54**	**0.01**	**0.00**
	0.22, 39.74	1.04, 1.72	51.88, 11.30	75.37, **16.55**	**0.01**	**0.00**
	0.22, 39.74	1.04, 1.72	51.86, 11.29	75.36, **16.54**	**0.01**	**0.00**
	0.22, 39.74	1.04, 1.72	51.88, 11.30	75.37, **16.55**	**0.01**	**0.00**

Table 6.6.: Performance of the strategies in NBS 4. For each investment strategy IS, the four rows correspond to the four financing strategies FS. The transaction costs ratios $\frac{C_k^{\text{cost}}}{C_k}$ are denoted in per mil and all other criteria in percentage terms.

Finally, the performance of the strategies in the scenario of a stationary new business (NBS 4) is displayed in Table 6.6 and is similar to NBS 3 for the most part. However, we observe now a much larger default probability if we apply the SCF strategy.

6.1.1. Summary of the observations

In the following, we summarize our observations and derive recommendations regarding the applied investment and financing strategy.

The comparison study shows that there is no strategy (pair) that performs best regarding all criteria in any of the considered new business scenarios. Indeed, if we are indifferent regarding the relevance of the considered (sub-) criteria, the respective performances are quite balanced. This is further illustrated in Table 6.7, where we added up the criteria (sub-criteria in brackets) associated with the best performance for each investment strategy and each new business scenario.[5]

[5]For each investment strategy we chose the best-performing financing strategy, respectively.

IS	NBS 1	NBS 2	NBS 3	NBS 4	Sum
CM	3 (4)	3 (4)	4 (5)	4 (5)	14 (18)
CPPI	2 (2)	2 (2)	2 (3)	2 (2)	8 (9)
SCF	3 (5)	3 (5)	2 (2)	3 (4)	11 (16)
RCF	4 (5)	4 (5)	4 (5)	4 (5)	16 (20)
DCF	4 (4)	4 (4)	5 (6)	4 (4)	17 (18)

Table 6.7.: Sum of the criteria (sub-criteria in brackets) being associated with the best performance for each investment strategy and each NBS. In total, there are 6 criteria (10 sub-criteria). The last column aggregates the performance over all new business scenarios.

The smallest long-term default probability is obtained if we apply the CM strategy. However, in the short-term, the DCF strategy performs better. In the first five years, the highest declared interest rates on average are obtained if we apply the CPPI strategy. In the long term, SCF and RCF often perform better, the former especially in the scenarios with decreasing new business (NBS 1 and NBS 2). The average reserve ratio per period is significantly smaller if we apply the CM strategy, especially compared to the application of CF strategies. Regarding the average ratios of own funds and transaction costs per period, the DCF strategy outperforms all other strategies.

In all considered new business scenarios, the financing strategy has no significant influence if we apply the SCF or the DCF strategy. If we apply the CM or the RCF strategy, financing strategy FS 1 and FS 3 perform better than the other two, and if we apply the CPPI strategy, financing strategy FS 3 performs best. Therefore, if one is not sure which investment strategy to use or if a switching between investment strategies throughout the simulation is planned, financing strategy FS 3 can be recommended.

6.2. Further applications of the refined ALM model

The following simulation studies illustrate the effects of further refinements and alternative modeling approaches of components of the general ALM model from Chapter 3. This includes the modeling of the capital market, the used method for the annual interest rate declaration, and the strategy for the use of surpluses. In each case, we select a basic setting and measure the impact on several quantities of interest. We use some of the established criteria from Section 6.1 and, additionally, the periodic surplus G_k. Similar to Section 4.5.2, we average over all simulations and all periods to get the average change per period as a number for illustration.

For each Monte Carlo simulation we simulate $N = 10,000$ paths. With the objective of a good comparability in the respective settings, we start with the same balance sheet and insurance portfolio. If not stated otherwise, the input parameters and the distributional assumptions are taken from Table 4.2. According to the obtained performances and the derived recommendations in Section 6.1.1, we use strategies DCF and FS 3 for investing and financing, respectively. The form of the coverage fund CF_{k-1} is given by equation (6.1.1) and the corresponding multiplier is $\lambda_{k-1}^{\mathrm{CF}} = 1.5$. In addition, we apply the CM strategy that was mainly used in the simulation studies in Section 4.

We consider a life insurer with a gradually decreasing new business which is motivated by the results from Section 6.1 indicating that the performances of CM and DCF strategies differ more in such cases. The shape parameters of the beta distribution for modeling the new contract arrivals are

$$(\alpha_1, \beta_1) = (16, 3), (\alpha_K, \beta_K) = (4, 12),$$

and for $k = 0, \ldots, K-2$

$$\alpha_{k+1} = \alpha_1 - (\alpha_1 - \alpha_K)\frac{k}{K-1}, \beta_{k+1} = \beta_1 + (\beta_K - \beta_1)\frac{k}{K-1}.$$

The corresponding new business scenario and the resulting size of the insurance portfolio are illustrated in Figure 6.6.

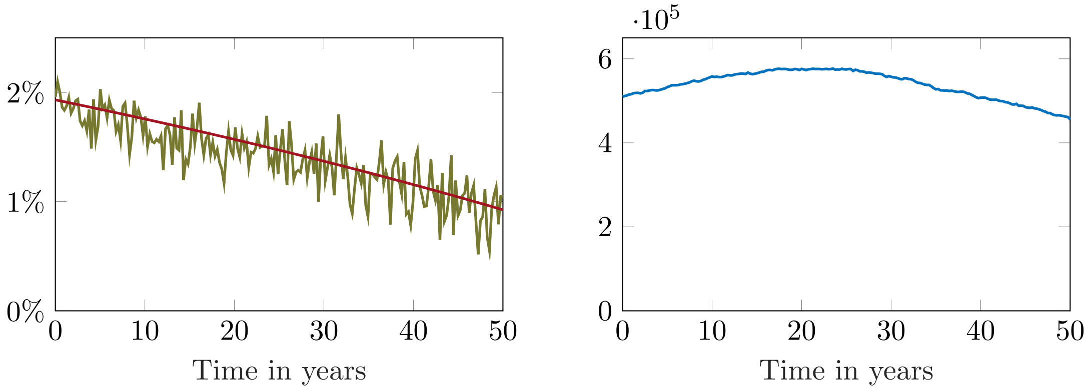

Figure 6.6.: Considered new business scenario. Left: number of new customers per period (expected number in red, one path in green), right: corresponding path of the insurance portfolio size.

6.2.1. Simulation of the Heston model

In most of the simulation studies we used the basic capital market model from Section 3.3.1, where stock prices follow a geometric Brownian motion with dynamics (3.3.4) as in the Black-Scholes model. A more general model was applied in Section 4.5.2, where we allowed for crashes in the stock and bond markets.

In this section, we investigate the impact of applying the Heston instead of the Black-Scholes model for the simulation of stock prices. By allowing for a non-constant volatility of the stock price process, the resulting capital market model can be seen as an extension of the basic model. In the Heston model, the stock price and variance processes are given by the coupled system of stochastic differential equations (3.3.6) and (3.3.7) in Section 3.3.2. Regarding the latter, we choose $\nu_0 = 0.04, \kappa = 2, \theta_\nu = 0.04, \sigma_\nu = 0.2$, and $\rho_{s\nu} = -0.9$.

In particular, the long-term mean θ_ν of the variance process coincides with the constant squared volatility $\sigma_s^2 = 0.2^2$ of the stock price process in the Black-Scholes model. The remaining parameters of the capital market model are taken from Table 4.2.

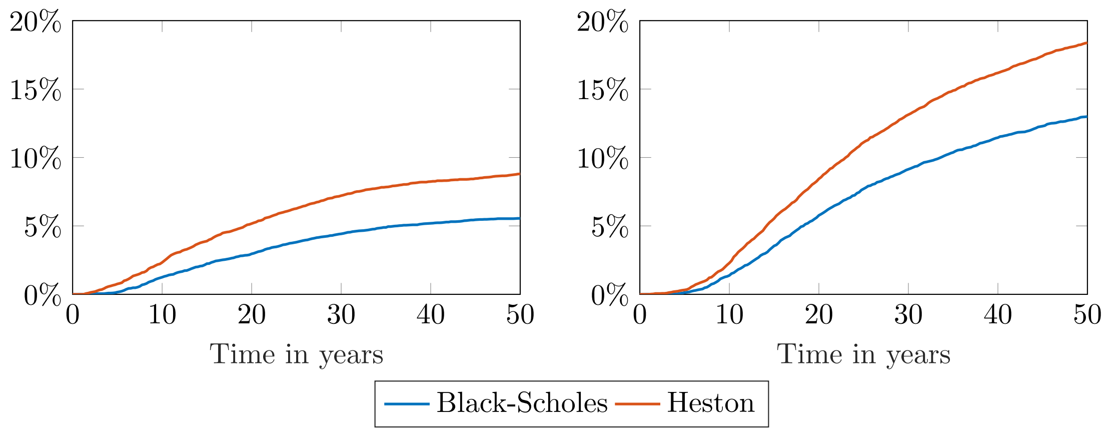

Figure 6.7.: Development of the default probability if stock prices are modeled by the Black-Scholes and by the Heston model, respectively. Left: CM, right: DCF.

Figure 6.7 displays the development of the default probability in the considered settings. The stochastic volatility in the Heston model incorporates additional uncertainty in the ALM model which results in larger default probabilities PD_k. In the CM-case, PD_K is increased by 3.3pp (58.6%) and in the DCF-case it is increased by 5.4pp (41.5%).

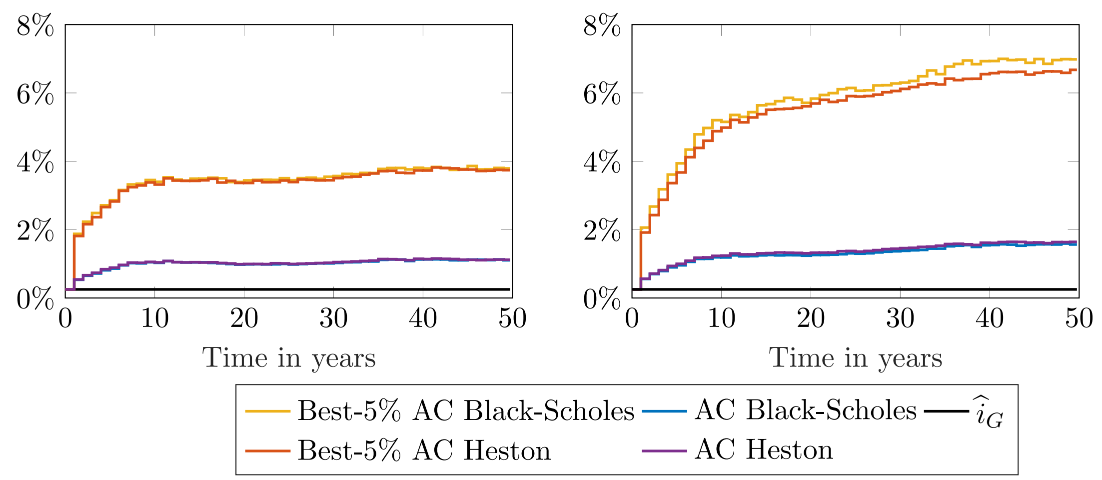

Figure 6.8.: Development of the declared interest rate if stock prices are modeled by the Black-Scholes and by the Heston model, respectively. Left: CM, right: DCF.

In comparison, the annual declared interest rate shown in Figure 6.8 is much less affected by the alternative approach for modeling stock prices. Especially in the CM-case, the difference is negligible. Applying the DCF strategy, the declared interest rates in the best-5% average case (AC) are 0.3pp smaller in the long term if we apply the Heston model, while on average they are slightly increased (by 0.08pp). In the worst-5% average case, only the guaranteed rate $\widehat{i}_G = 0.25\%$ is paid in any case.

IS	$\Delta(\gamma_k \geq \gamma)$	$\Delta\gamma_k$	$\Delta(\psi_k \geq \psi)$	$\Delta\psi_k$	$\Delta G_k \cdot 10^5$	$\Delta\mathcal{B}_k$ in %
CM	1.02	0.07	−1.03	−0.23	−3.20	0.46
DCF	2.17	0.30	−1.84	−0.40	−7.66	1.93

Table 6.8.: Absolute or relative changes due to applying the Heston model for the simulation of stock prices. If not stated otherwise, the change is measured in percentage points.

Table 6.8 displays further quantities that are affected by the alternative modeling of stock prices. As we can see, the effects are stronger if we apply the DCF investment strategy. This results from the fact that here the average stock ratio is significantly larger than in the CM-case, see e.g. Figure 6.4. However, they are still small compared to the impact on the default probability.

6.2.2. Alternative interest rate declaration

In the following, we compare the two methods for declaring the annual interest rate introduced in Section 3.5.2. In more detail, these are the basic method from equation (3.5.5) which was always used so far and the alternative method from equation (3.5.9) in Remark 3.5.1. For the stock price modeling, we use the Heston model.

According to the alternative method, the declared interest rate is directly linked to the obtained investment returns. For period k, the portfolio return R_k is defined as weighted average of the aggregated bond and stock returns, i.e. by

$$R_k = \left(1 - \pi^s_{(k-1)^+}\right) \frac{\sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi^b\left(t_k; t_i + \tau\right) \Delta p_{k,i}}{\sum_{i=k-\frac{\tau}{\Delta t}}^{k-1} \varphi^b\left(t_k; t_i + \tau\right) p\left(t_{k-1}, t_i + \tau\right)} + \pi^s_{(k-1)^+} \frac{\Delta s_k}{s_{k-1}}$$

with $\Delta s_k = s_k - s_{k-1}$ and $\Delta p_{k,i} = p\left(t_k, t_i + \tau\right) - p\left(t_{k-1}, t_i + \tau\right)$. At the beginning of the simulation, we choose $R_0 = \widehat{i}_G$. For the reserve rate-depending interest rate adjustment term i_k^{adjust}, we choose

$$\gamma_{k-1}^{\min} = 10\% \qquad \text{and} \qquad \gamma_{k-1}^{\max} = 20\%.$$

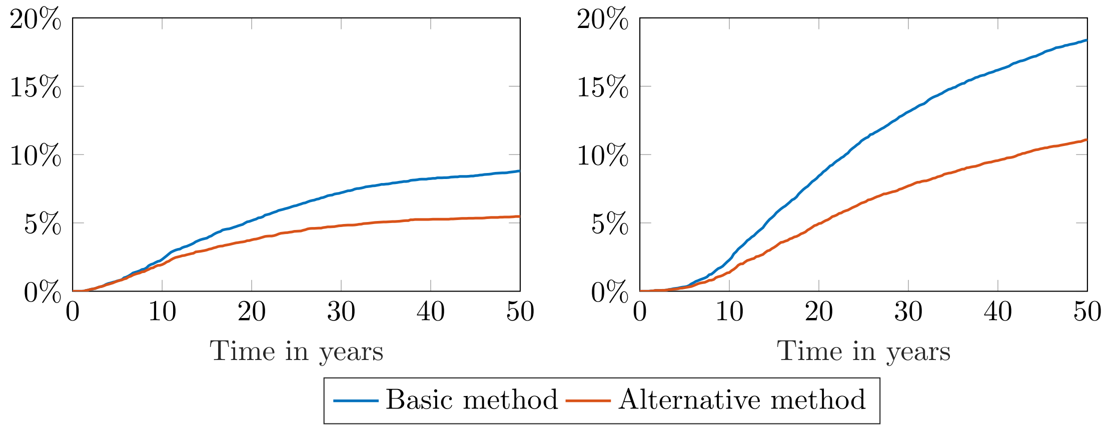

Figure 6.9.: Development of the default probability for both methods of declaring the annual interest rate. Left: CM, right: DCF.

Figure 6.9 shows the development of the default probability if we use the basic and the alternative interest rate declaration procedure, respectively. Only during the first years, we obtain the same developments in both cases. Then, the alternative method leads to smaller default probabilities and the differences increase steadily with time. After 50 years, PD_K is decreased by 3.3pp (37.8%) and by 7.3pp (39.7%) if we apply the CM and the DCF strategy, respectively.

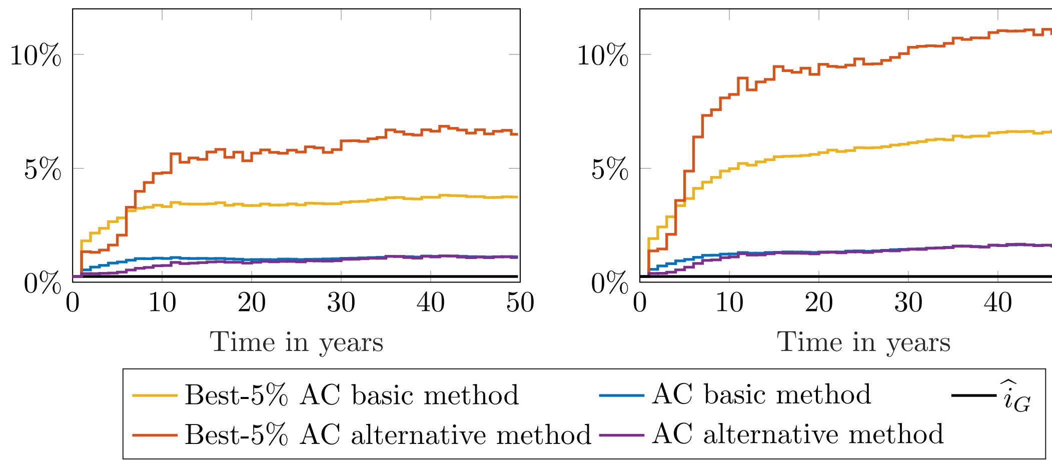

Figure 6.10.: Development of the declared interest rate for both methods of declaring the annual interest rate. Left: CM, right: DCF.

As we can see in Figure 6.10, the average interest rate in the medium and long term is not affected by the alternative method. In the first years, we even obtain smaller values. However, in the best-5% average case, the alternative method yields much larger declared interest rates already after few years. In the worst-5% average case, only the guaranteed rate $\widehat{i}_G = 0.25\%$ is paid in any case.

Note that smaller values for $\gamma_k^{\min}$ or $\gamma_k^{\max}$ may increase the declared interest rate, yet also the default probability substantially.[6] As illustrated in Figure 6.11 for the DCF-case, a decrease in the default probabilities to a certain degree can be achieved by increasing $\gamma_k^{\max}$.

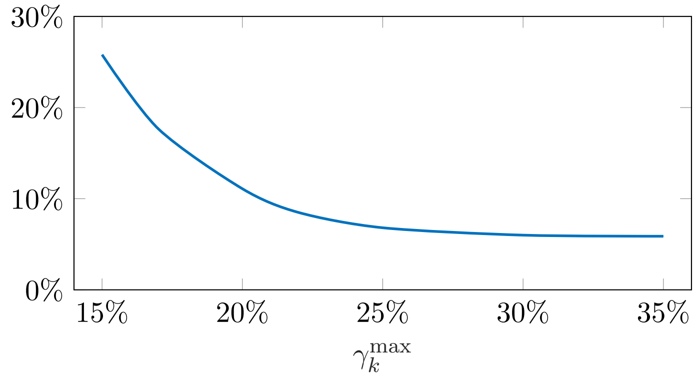

Figure 6.11.: Development of the 50-year default probability depending on $\gamma_{k-1}^{\max}$ if we apply the DCF strategy.

For the basic interest rate declaration procedure, the control parameter would be the distribution ratio ω and the target reserve rate γ.

IS	$\Delta\left(\gamma_k \geq \gamma\right)$	$\Delta\gamma_k$	$\Delta\left(\psi_k \geq \psi\right)$	$\Delta\psi_k$	$\Delta G_k \cdot 10^6$	$\Delta\mathcal{B}_k$ in %
CM	14.51	3.26	2.54	3.48	3.93	−1.73
DCF	7.86	1.50	4.09	2.26	2.13	−1.01

Table 6.9.: Absolute or relative changes due to applying the alternative interest rate declaration. If not stated otherwise, the change is measured in percentage points.

Table 6.9 summarizes the impact of the alternative method on further quantities. This time, the effects are mostly larger if we apply the CM investment strategy. Together with the above observations, we conclude that for the considered settings the alternative method for the interest rate declaration performs better than the basic method, even though the total benefit payments are (slightly) decreased.

6.2.3. Alternative strategy for the use of surpluses

In the following, we compare the two introduced strategies for the use of surpluses, i.e. for the specified allocation of the periodic surplus G_k between free reserve F_{k-1} and equity Q_{k-1}. The main differences are that according to the basic method from Section 3.7.2, a fixed portion of a positive surplus is deposited in the free reserve which also primarily

[6]For example, choosing $\gamma_k^{\min} = 5\%$ and $\gamma_k^{\max} = 10\%$ in the DCF-case increases the average interest rate in the long term about 0.2pp while the default probability is increased to $PD_K = 66.3\%$, i.e. by over 55pp.

covers losses. In contrast, the alternative method from Section 5.5 aims at providing a potential fair splitting of losses. At the same time, repaying debts to shareholders yet has the highest priority. Regarding Case 3, we choose $\tau^G = 5$ implying that if the equity is non-negative, losses are shared in the same proportion as positive surpluses were allocated in the last 5 years. As before, we use the Heston model for the simulation of stock prices.

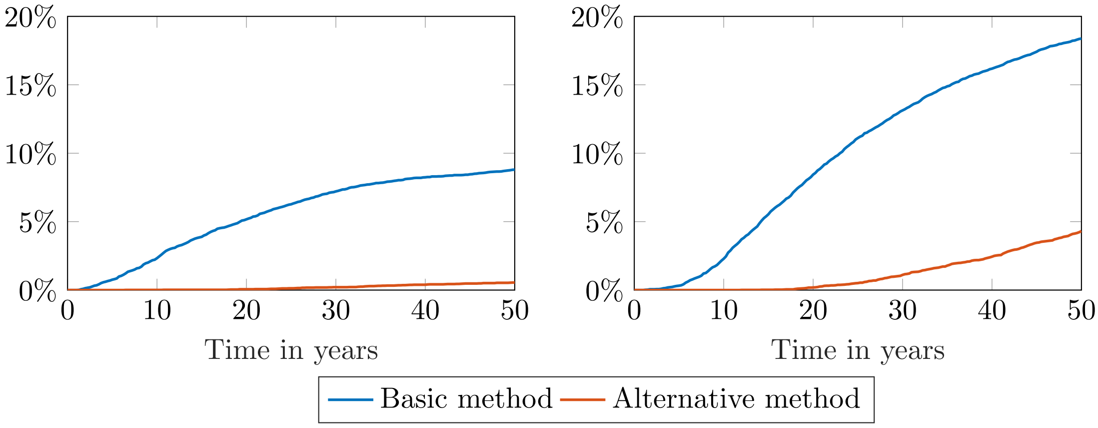

Figure 6.12.: Development of the default probability for both methods for the use of surpluses. Left: CM, right: DCF.

Figure 6.12 illustrates the development of the default probability if we use the basic and the alternative method for the use of surpluses, respectively. Clearly, the alternative strategy yields substantially smaller values, especially in the CM-case. Indeed, the default probability is smaller than 0.1% and 0.4% in the first 23 years and PD_K is decreased by 8.3pp (93.8%) and by 14.1pp (76.6%) if we apply the CM and DCF strategy, respectively.

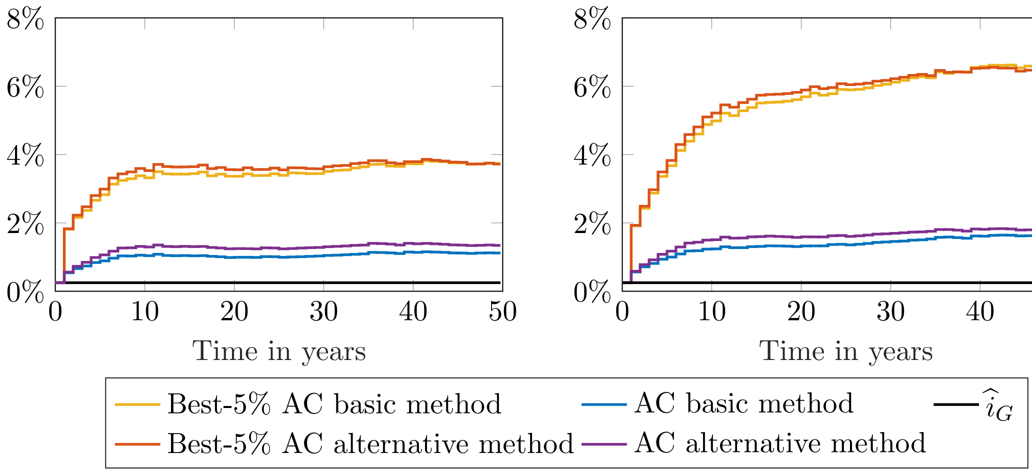

Figure 6.13.: Development of the declared interest rate for both methods for the use of surpluses. Left: CM, right: DCF.

In Figure 6.13, we show the declared interest rate if we use the basic and the alternative method for the use of surpluses. Especially in the average case, the corresponding differences remain approximately stable from year 10 onwards. In contrast to the section before, now the best-5% average case is less affected. In addition, the benefit payments are now increased and the surplus decreased if we apply the alternative method as we can see Table 6.10. Here, we also see that the differences between the CM and the DCF strategy are smaller compared to Table 6.9.

IS	$\Delta\left(\gamma_k \geq \gamma\right)$	$\Delta\gamma_k$	$\Delta\left(\psi_k \geq \psi\right)$	$\Delta\psi_k$	$\Delta G_k \cdot 10^6$	$\Delta\mathcal{B}_k$ in %
CM	11.61	1.34	−6.51	−4.25	−6.50	2.60
DCF	8.74	0.98	−9.10	−4.24	−7.25	2.60

Table 6.10.: Absolute or relative changes due to applying the alternative strategy for the use of surpluses. If not stated otherwise, the change is measured in percentage points.

The effects of combining both the alternative interest rate declaration and the alternative strategy for the use of surpluses are illustrated in Appendix A.3 and complete the investigations from the last two sections.

6.2.4. Outlook: refinement of the CF strategies

The following outlook illustrates that there is still potential to further improve the strategies developed in this thesis. Here, we focus on the CF strategies that are based on the coverage fund CF_{k-1} given by the general representation (5.1.1) in Section 5.1.2, i.e. by

$$CF_{k-1} = \mathbb{E}\left[D_{k-1+\frac{\tau}{\Delta t}} \,|\, \mathcal{F}_{k-1}\right] p\left(t_{k-1}, t_{k-1}+\tau\right).$$

So far, we used a constant value for the expected amount $\mathbb{E}\left[D_{k-1+\frac{\tau}{\Delta t}} \,|\, \mathcal{F}_{k-1}\right]$ of disbursements. We saw in Section 6.1 that this already led to good performances. Especially the DCF strategy performed well regarding many different criteria in all considered new business scenarios, and the last sections showed that the overall performance can be even increased if we apply alternative strategies for the interest rate declaration and the use of surpluses.

The refined versions of the CF strategies proposed in this section aim at estimating the dynamics of the future disbursements, thereby adjusting correspondingly the required coverage funds and thus the new bond investments. Recall that the disbursements D_{k-1+i} at time t_{k-1+i}, for $i = 1, \ldots, \frac{\tau}{\Delta t}$, consist of benefit payments $\mathcal{B}_{k-1+i}$ and expired long-term credits L^{-}_{k-1+i}, i.e.

$$D_{k-1+i} = \mathcal{B}_{k-1+i} + L^{-}_{k-1+i}.$$

The last term is already known since these correspond to the number $\varphi^{l}_{k-1+i-\frac{\tau}{\Delta t}}$ of emitted bonds at time $t_{k-1+i-\frac{\tau}{\Delta t}}$, see equation (3.5.12) in Section 3.5.3. The benefit payments can be decomposed into survival, death, and surrender benefit payments according to equation (3.6.1) in Section 3.6.1. Taking into account a safety loading λ^D, this yields

$$\begin{aligned}\mathbb{E}\left[D_{k-1+i} \,|\, \mathcal{F}_{k-1}\right] &= (1+\lambda^D) \\ &\cdot \left(\mathbb{E}\left[E_{k-1+i} \,|\, \mathcal{F}_{k-1}\right] + \mathbb{E}\left[T_{k-1+i} \,|\, \mathcal{F}_{k-1}\right] + \mathbb{E}\left[S_{k-1+i} \,|\, \mathcal{F}_{k-1}\right] + \varphi^{l}_{k-1+i-\frac{\tau}{\Delta t}}\right).\end{aligned} \tag{6.2.1}$$

The safety loading is incorporated to cope with upper deviations from the expected disbursements since we ignore the future number of new customers, for example. The single conditional expectations of (6.2.1) are calculated approximately according to

$$\begin{aligned}\mathbb{E}\left[E_{k-1+i} \,|\, \mathcal{F}_{k-1}\right] &= \sum_{m=1}^{M_k} \mathbb{1}_{\left\{d^m_{k-1+i}=0\right\}} \cdot \left(\delta^m_{k-1} + \delta^{\text{new}}_{k-1}\right) \\ &\quad \cdot \left(A^m_{(k-1)^+}\left(1+i_G\right)^{d^m_{k-1}} + B^m_{(k-1)^+}\left(1+\widehat{i}_k\right)^{d^m_{k-1}} + \sum_{j=1}^{d^m_{k-1}} \left(1+i_G\right)^j P^m_{(k-1)^+}\right),\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[T_{k-1+i} \,|\, \mathcal{F}_{k-1}\right] &= \sum_{m=1}^{M_k} \mathbb{1}_{\left\{d^m_{k-1+i}\geq 0\right\}} \cdot \left(\delta^m_{k-1} + \delta^{\text{new}}_{k-1}\right) q^m_{k-1+i} \\ &\quad \cdot \left(A^m_{(k-1)^+}\left(1+i_G\right)^{d^m_{k-1}} + B^m_{(k-1)^+}\left(1+\widehat{i}_k\right)^{d^m_{k-1}} + \sum_{j=1}^{d^m_{k-1}} \left(1+i_G\right)^j P^m_{(k-1)^+}\right),\end{aligned}$$

and

$$\mathbb{E}\left[S_{k-1+i}\,|\,\mathcal{F}_{k-1}\right] = \sum_{m=1}^{M_k} \mathbb{1}_{\{d^m_{k-1+i}>0\}} \cdot \left(\delta^m_{k-1} + \delta^{\text{new}}_{k-1}\right) u^m_{k-1+i} \left(1 - q^m_{k-1+i}\right) \vartheta$$
$$\cdot \left(A^m_{(k-1)^+} \left(1 + i_G\right)^{d^m_{k-1}} + B^m_{(k-1)^+} \left(1 + \widehat{i}_k\right)^{d^m_{k-1}} + \sum_{j=1}^{d^m_{k-1}} \left(1 + i_G\right)^j P^m_{(k-1)^+} \right).$$

Following this approach, we can precisely estimate the dynamics of the future disbursements as displayed in Figure 6.14, where we chose $\lambda^D = 30\%$ as safety loading. Note that we did not make any assumptions regarding the new business scenario.

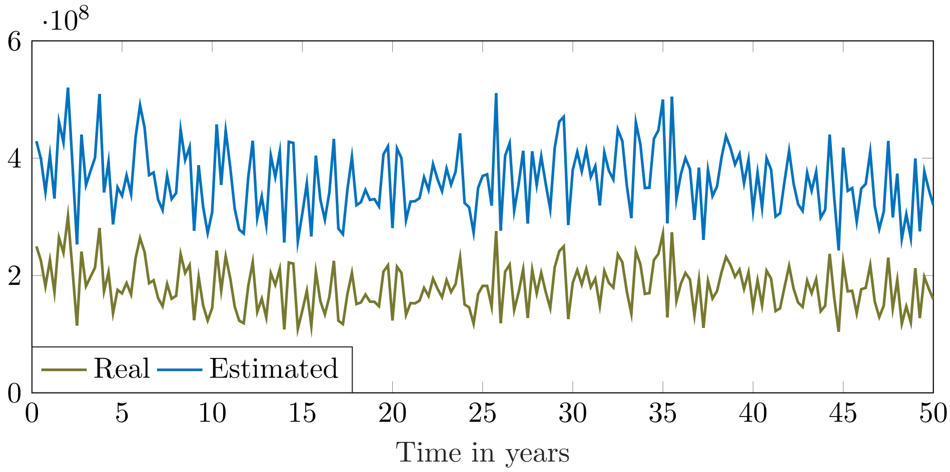

Figure 6.14.: Development of the average disbursements. The safety loading for the estimation is $\lambda^D = 30\%$.

We close this section by investigating the effect of the above refinement on the performance of the DCF strategy. The general distributional assumptions and the parameter specification are as before. Furthermore, we use the alternative strategy for the use of surpluses, i.e. we are in the setting of Section 6.2.3. The performance of the (basic) DCF strategy in that section serves as a benchmark and the changes due to applying the refined version are summarized in Table 6.11.

λ^D	ΔPD_K	$\Delta\widehat{i}_k$	$\Delta\left(\gamma_k \geq \gamma\right)$	$\Delta\gamma_k$	$\Delta\left(\psi_k \geq \psi\right)$	$\Delta\psi_k$	$\Delta G_k \cdot 10^5$	$\Delta\mathcal{B}_k$ in %
30%	0.90	0.07	−0.98	0.07	−1.37	0.04	3.07	0.72
50%	0.00	0.01	−0.08	0.01	−0.12	0.01	0.57	0.07

Table 6.11.: Absolute or relative changes due to applying the refined DCF strategy. If not stated otherwise, the change is measured in percentage points.

Choosing $\lambda^D = 30\%$, the arithmetic mean of the estimated disbursements per period is now $3.63 \cdot 10^8$ and thus smaller than in Section 6.2.3, where we assumed

$$\mathbb{E}\left[D_{k-1+\frac{\tau}{\Delta t}} \mid \mathcal{F}_{k-1}\right] = 4 \cdot 10^8.$$

Correspondingly, the average bond investments are also smaller. Indeed, the average stock ratio after the asset reallocation is 1.99pp larger which leads to increased interest rates, surpluses, and benefit payments but also to increased default probabilities. If we increase the safety loading to 50%, the average stock ratio after the asset reallocation is only 0.23pp larger. We obtain the same 50-year default probability and also the other criteria are much less affected on average.

7. Conclusion

In this thesis, we developed a stochastic asset-liability management model for life insurers allowing for different strategies for investing in the capital market and for financing the due obligations. The model is consistent in the sense that it respects the fundamental balance sheet equation at the end of every period according to the principle of double-entry bookkeeping as required in accounting. The framework is kept universal, such that the realization of alternative modeling approaches or the adaption to different insurance products are straightforward. We elaborated on this by proposing varying strategies e.g. for the annual interest rate declaration and the use of surpluses but also by applying different capital market models. In addition, we illustrated that the model is capable of simulating different business forms of a life insurer. As examples we considered the two important cases of a run-off scenario and an ongoing insurance business with stationary new business but also more dynamic scenarios with shocks on the expected future number of new customers.

Motivated by the observation that a naive contract-specific simulation is not applicable if we consider large, heterogeneous insurance portfolios with new business, we proposed different compression and simulation methods, where we explicitly generate cohorts and integrate new contracts to maintain efficiency throughout the simulation. Extensive Monte Carlo studies allowed us to derive respective recommendations as to which simulation method can be used for which purposes and under which conditions. We saw that those methods $\mathcal{P}^{b_1,b_2}$ with $b_1 > 1$ suffer from a decline in efficiency throughout the simulation while the ones of $\mathcal{P}^{1,b_2}$ remained stable from the beginning. For smaller time horizons, we would prefer $\mathcal{P}^{b_1,b_2}$ with $b_1 > 1$, but due to the convergence towards the efficiencies of $\mathcal{P}^{1,b_2}$, we are indifferent after a certain amount of time. In general, the approximation quality of the simulation methods remained stable, even in the case of an ongoing insurance business, where new contract arrivals induce additional uncertainty. For estimating the default probability, which represents an important measure for the overall risk, we observed that it suffices to apply quite coarse simulation methods that are very efficient. Yet we found that, at some point, it is not reasonable to further increase the degree of compression being generally associated with a worse approximation quality since the additional amount of saved run time gets smaller. While most of the developed simulation methods provide a good approximation quality, especially $\mathcal{P}^{1,1}$, we saw that there is still potential for improvement, e.g. regarding the approximation of the survival benefit payments. Furthermore, a topic of further research could be to establish a link between the selected simulation methods and the resulting performances.

The general goal of this thesis was always the simulation of large insurance portfolios and the investigation of factors potentially disturbing the stability of the corresponding balance sheets. We elaborated on the intrinsic risk of selecting a model and fixing a certain parameter configuration. Using a sensitivity analysis, we discussed the strong influence of single parameters on the performance. Keeping in mind that the simulation requires to model or estimate a lot of input parameters, this shows that one has to be careful with the interpretation of single observations. Nevertheless, the application of such stochastic simulations can support management decisions by illustrating the long-term effects of potential measures, as for example the impact of specific investment and financing strategies, interest rate fixings, or the applied method for the use of surpluses. Indeed, from this thesis, one can derive several fundamental observations regarding the profitability and the security of the life insurance business. We could identify several risk drivers that may threaten the life insurer's liquidity and long-term solvency. For example, we saw that especially bond market crashes can lead to extreme liquidity shocks in the short term and thus constitute a substantial risk. This setting covers the extension to corporate bond investments and is related to times of high inflation, where instantly or sharp rising interest rates imply strongly decreasing bond prices. We saw that also run-off scenarios are associated with risks since missing new premiums imply a potential increased demand for credit liabilities. It is particularly remarkable that even in different new business scenarios and in the presence of stock and bond market crashes, we observed a certain stationarity in the long term.

Furthermore, several extensive comparison studies gave insight on the performance and robustness of the introduced strategies for investing in the capital market and for financing the periodic obligations, also in the presence of transaction costs. Here, we always took into account different criteria reflecting both the policyholders' and the insurer's point of view. We observed that none of the strategies performed best regarding all of them but we were able to identify criteria-depending superior investment and financing strategies. For example, we saw that the CPPI strategy is much more risky than the simple-structured CM strategy but yields on average higher declared interest rates. Even though we assumed that there is no liquidity gap at the beginning of the first period and that the initial amount of cash equals the average excessive value of expiring bonds, we observed a periodic pattern in the value of bought bonds in both cases. This showed that neither CM nor CPPI strategy are well oriented to the obligations induced by the (compressed) insured collective, and motivated to develop more evolved strategies that take into account the future disbursements by reconciling the bond investments accordingly. We showed that the resulting coverage fund-based strategies perform well regarding many criteria and that performances can be further improved for some of them by additional refinements. In practice, choosing the applied strategies might additionally depend on the management's subjective assessment of which criterion has the highest priority.

Appendix A. Simulation Studies: Additional Results

A.1. Compression and simulation methods: approximation quality

In the following, we provide additional results regarding the approximation quality of our developed simulation methods $\mathcal{P}^{b_1,b_2}$ with b_1, b_2 from (4.2.1) in the case of a run-off and in the case of an ongoing insurance business. They complete the investigations in Section 4.2.2 and approve the recommendations in Section 4.2.3.

Approximation quality in the case of a run-off

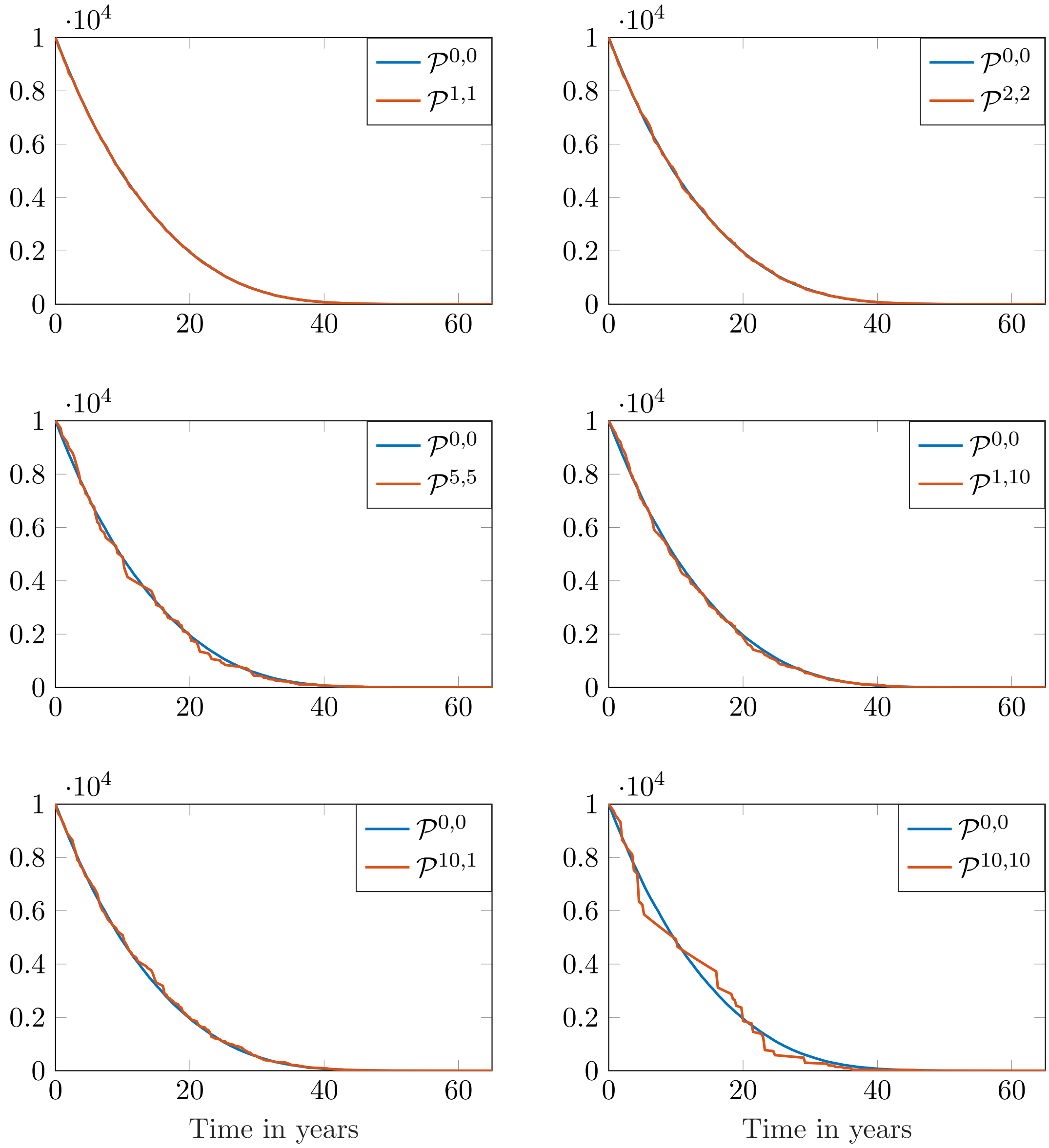

Figure A.1.: Development of the insurance portfolio size δ_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of a run-off.

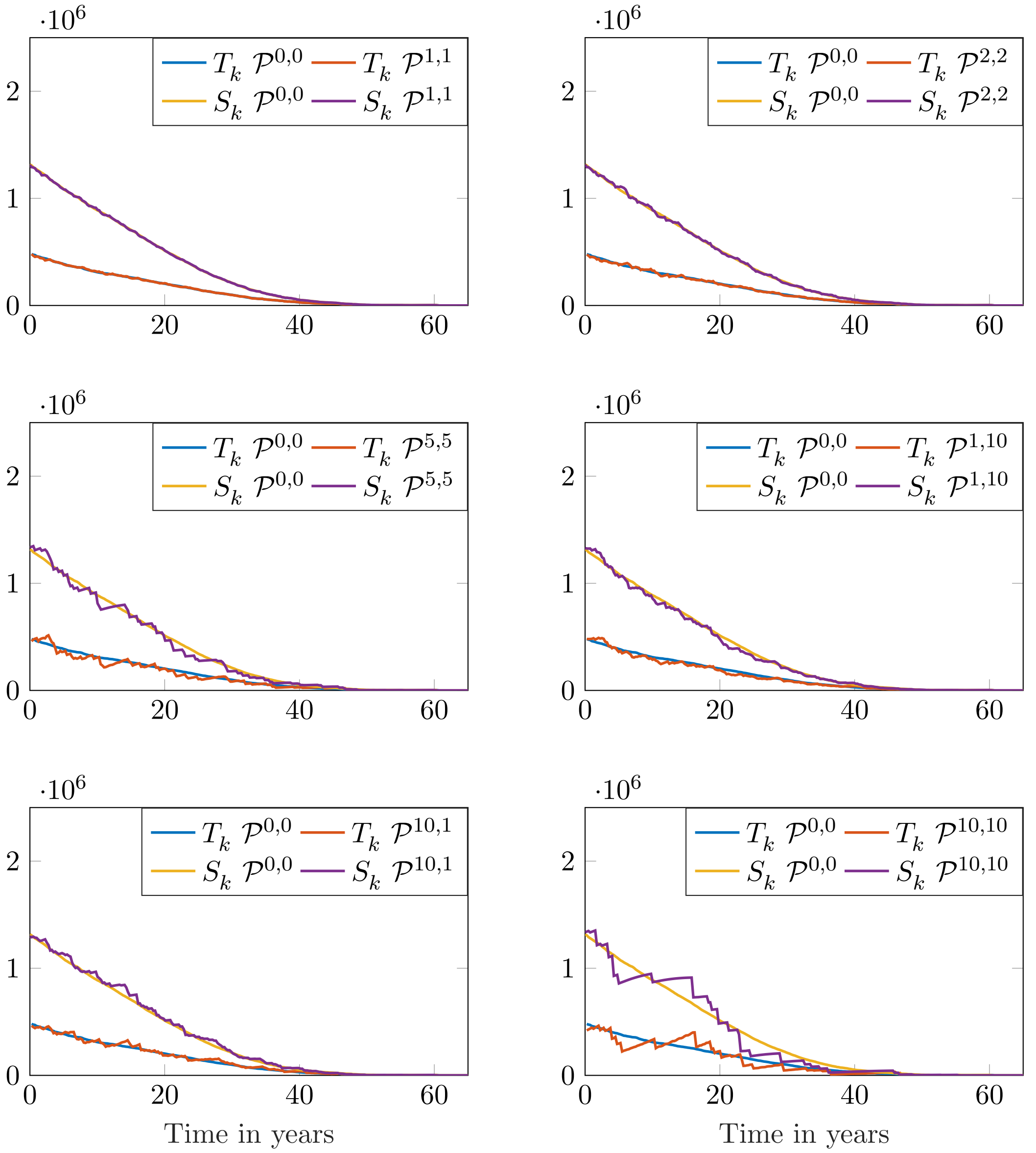

Figure A.2.: Average development of the death and surrender benefit payments T_k and S_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of a run-off.

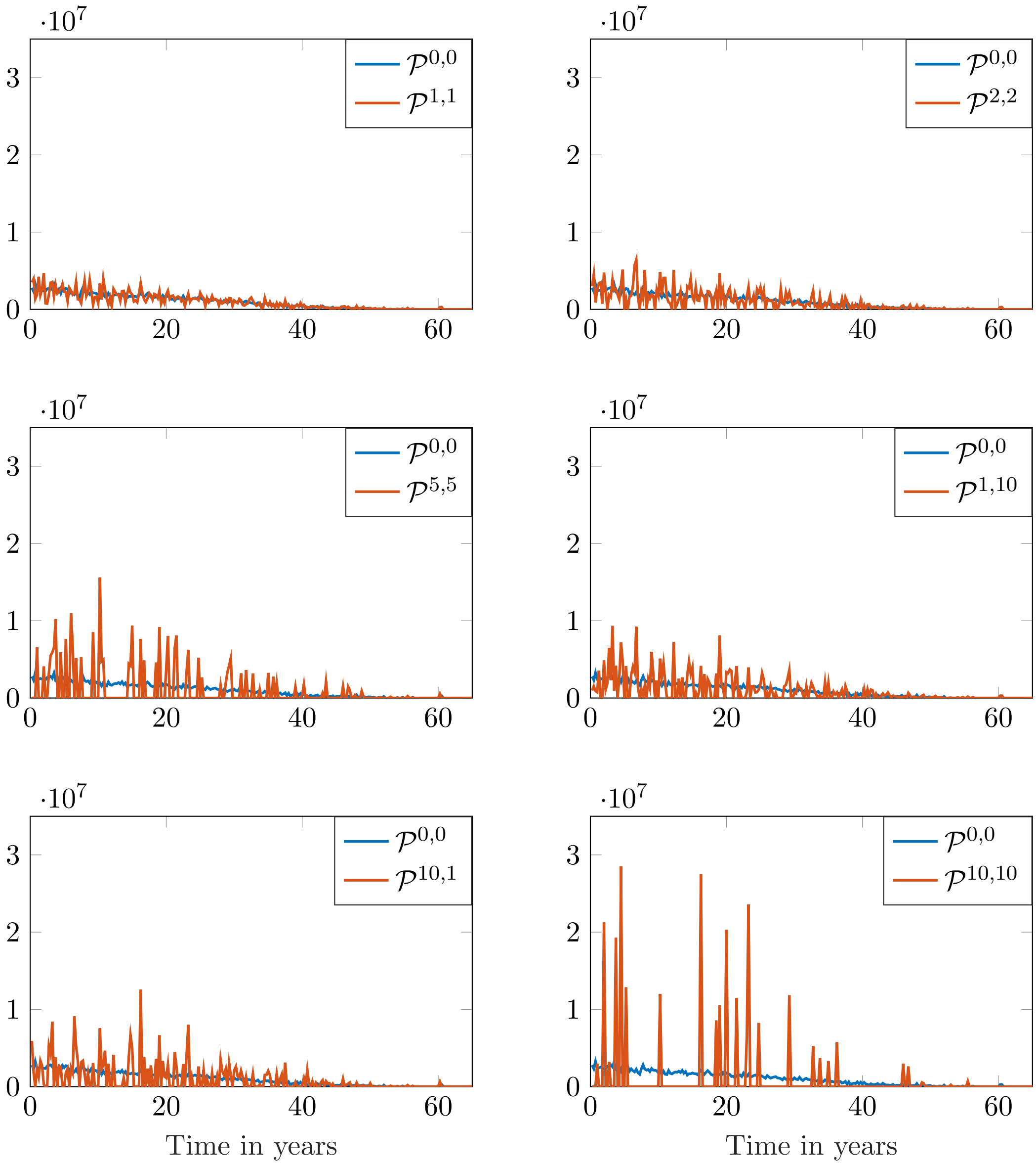

Figure A.3.: Average development of the survival benefit payments E_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of a run-off.

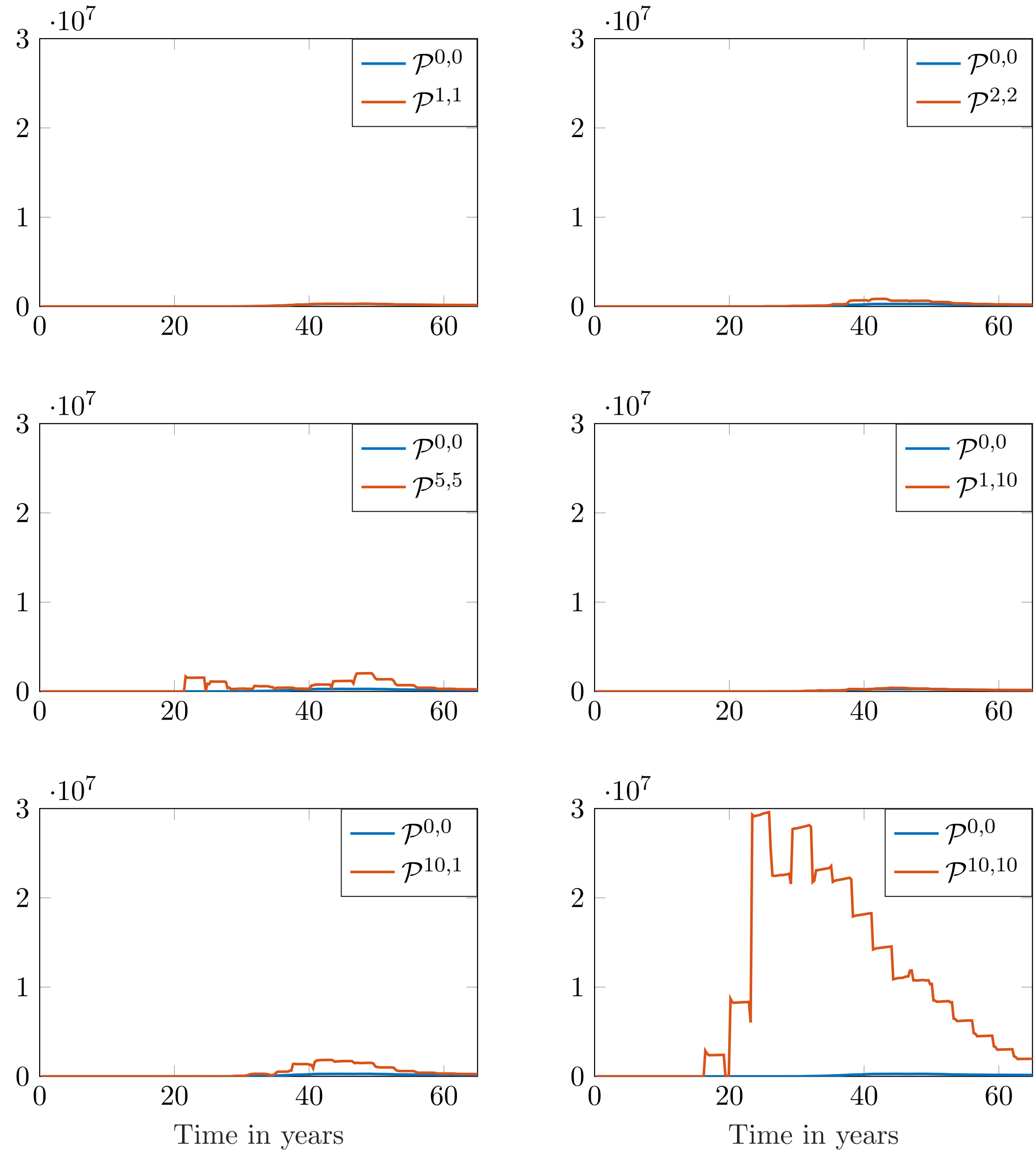

Figure A.4.: Average development of the liabilities to banks L_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of a run-off.

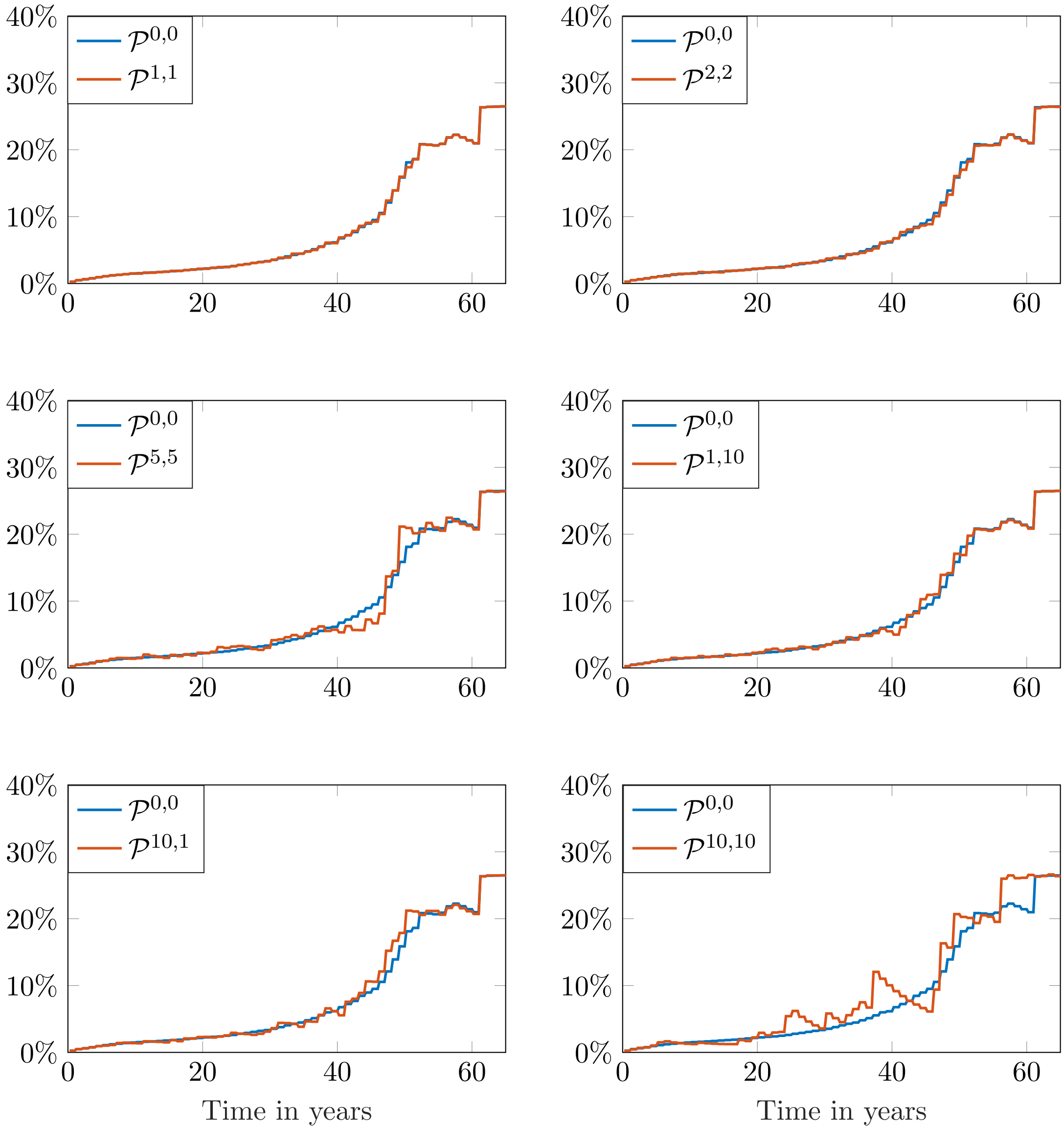

Figure A.5.: Average development of the declared interest rate $\widehat{i}_k$ according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of a run-off.

Approximation quality in the case of an ongoing insurance business

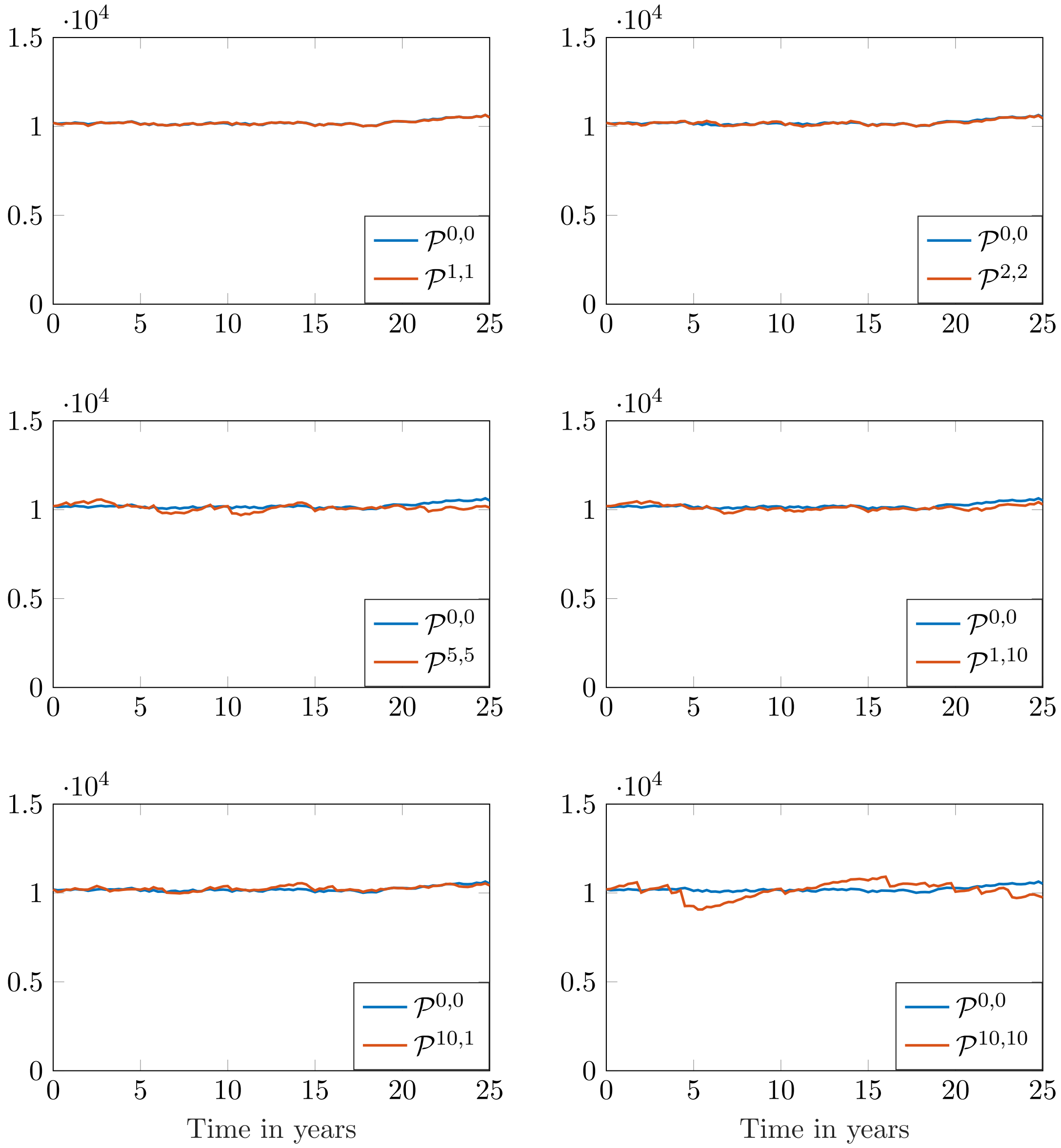

Figure A.6.: Development of the insurance portfolio size δ_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of an ongoing insurance business.

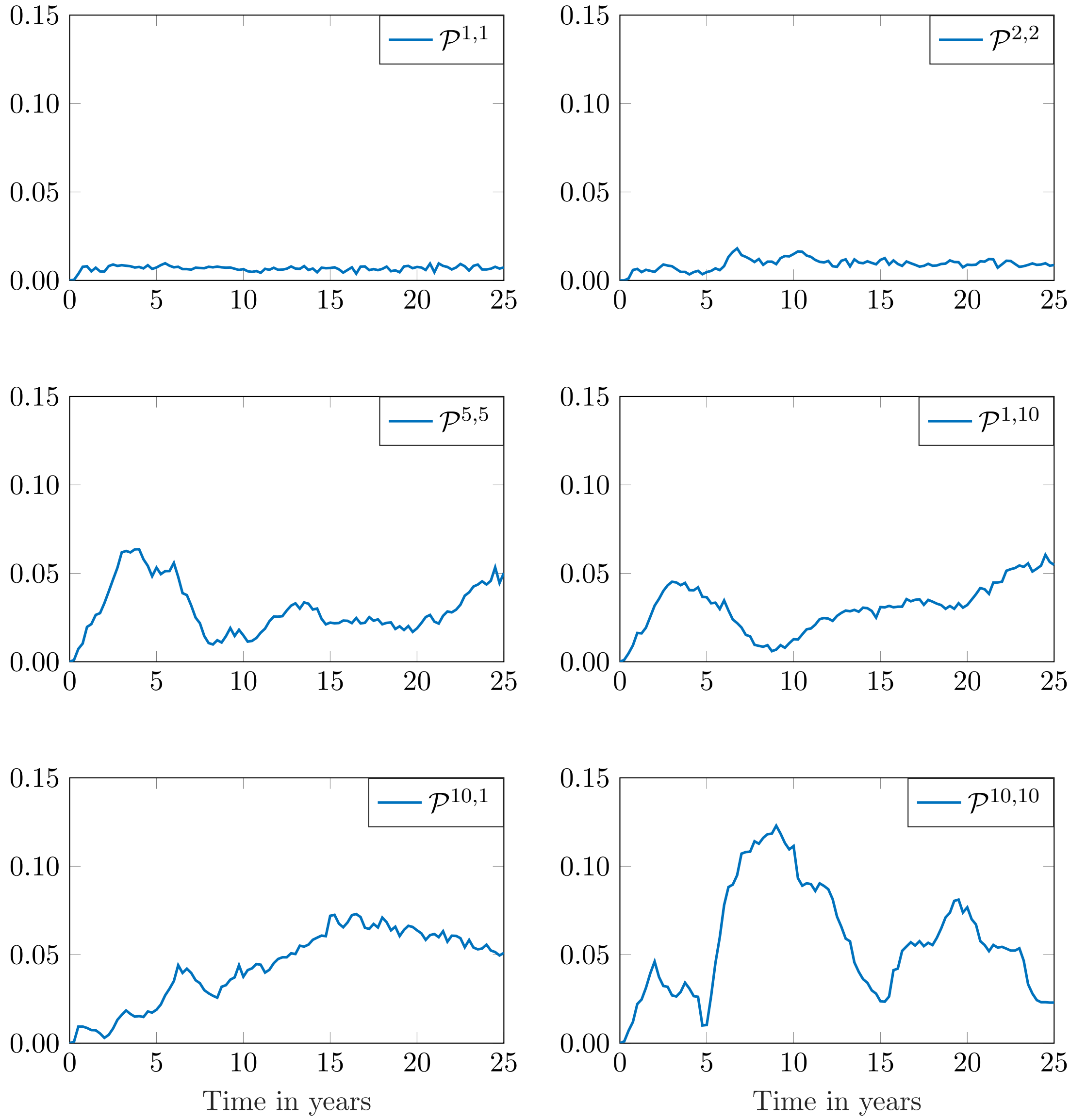

Figure A.7.: Total variation distances of the sample distributions of the equity Q_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of an ongoing insurance business. The number of intervals in (2.2.4) is $J = 50$.

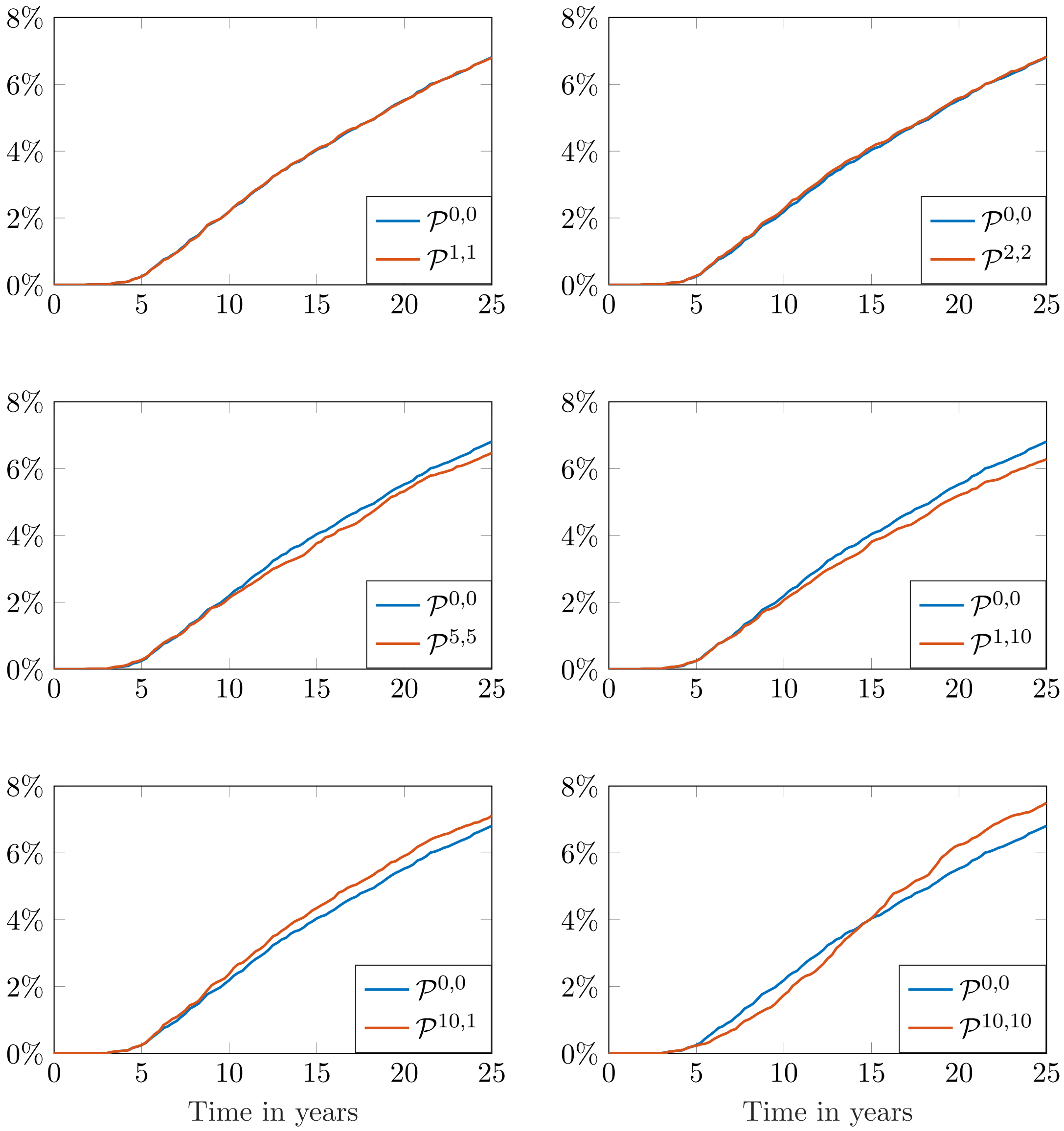

Figure A.8.: Development of the default probability according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of an ongoing insurance business.

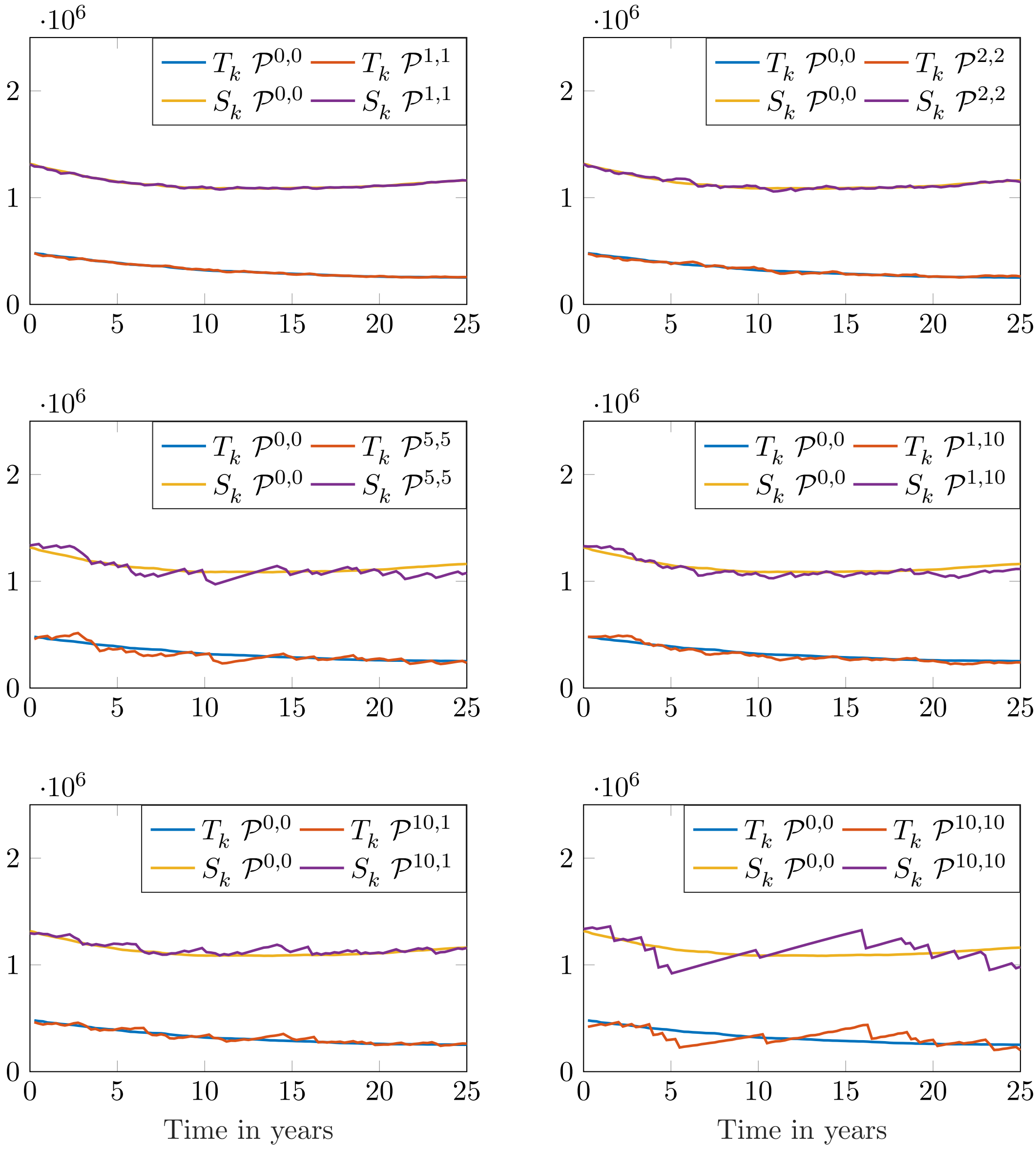

Figure A.9.: Average development of the death and surrender benefit payments T_k and S_k according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$.

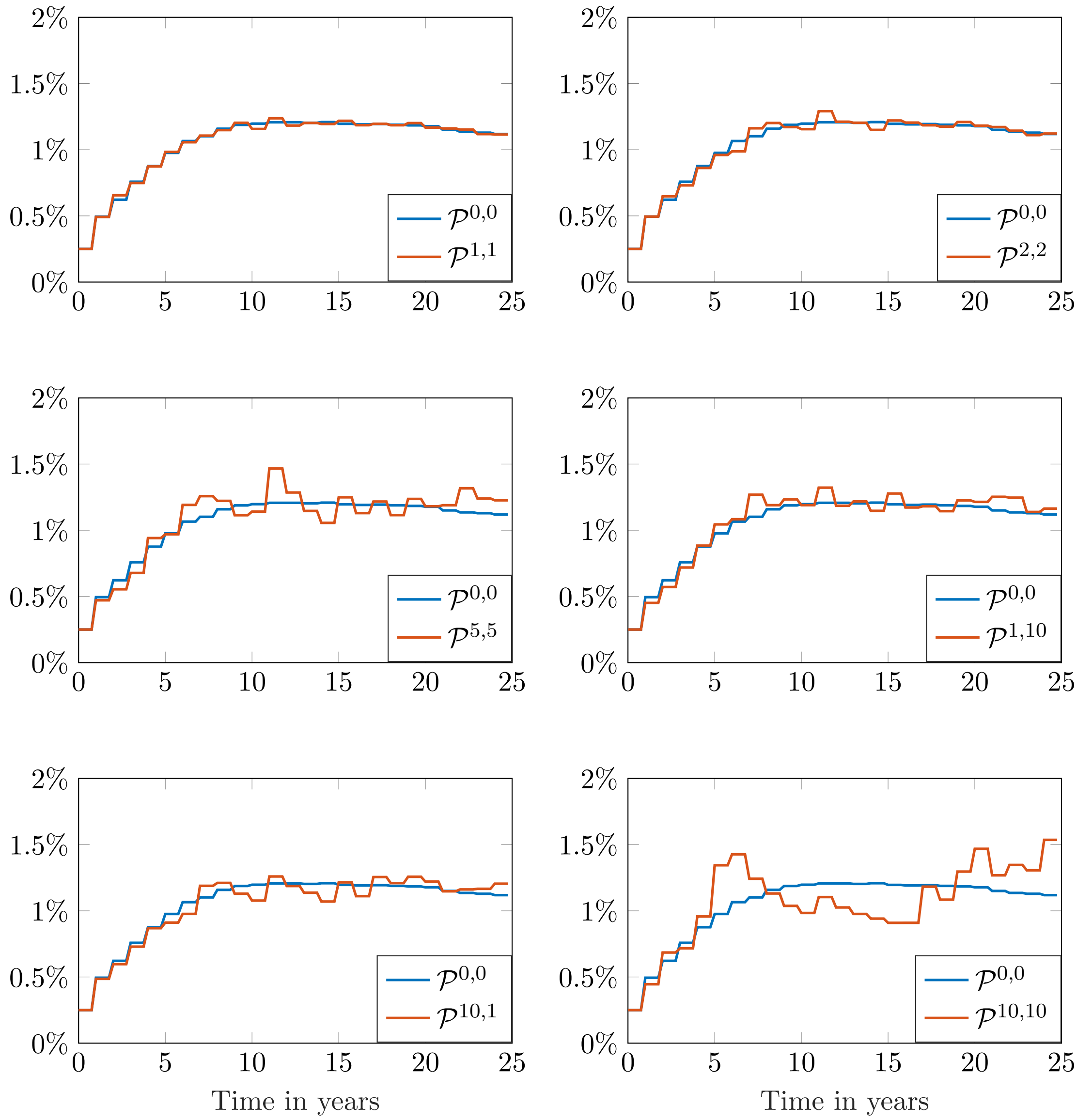

Figure A.10.: Average development of the declared interest rate $\widehat{i}_k$ according to $\mathcal{P}^{0,0}$ and $\mathcal{P}^{b_1,b_2}$ in the case of an ongoing insurance business.

A.2. Performance of the investment and financing strategies

In the following, we provide additional results regarding the investment strategies developed in this thesis. More specifically, we show the average amount of bought bonds and the stock ratio after the asset reallocation in the case of NBS 1, NBS 2, and NBS 3. Thereby, we complete the investigations in Section 6.1. The corresponding new business scenarios were parameterized in Table 6.2 and illustrated in Figure 4.17. As financing strategy, we use FS 4.

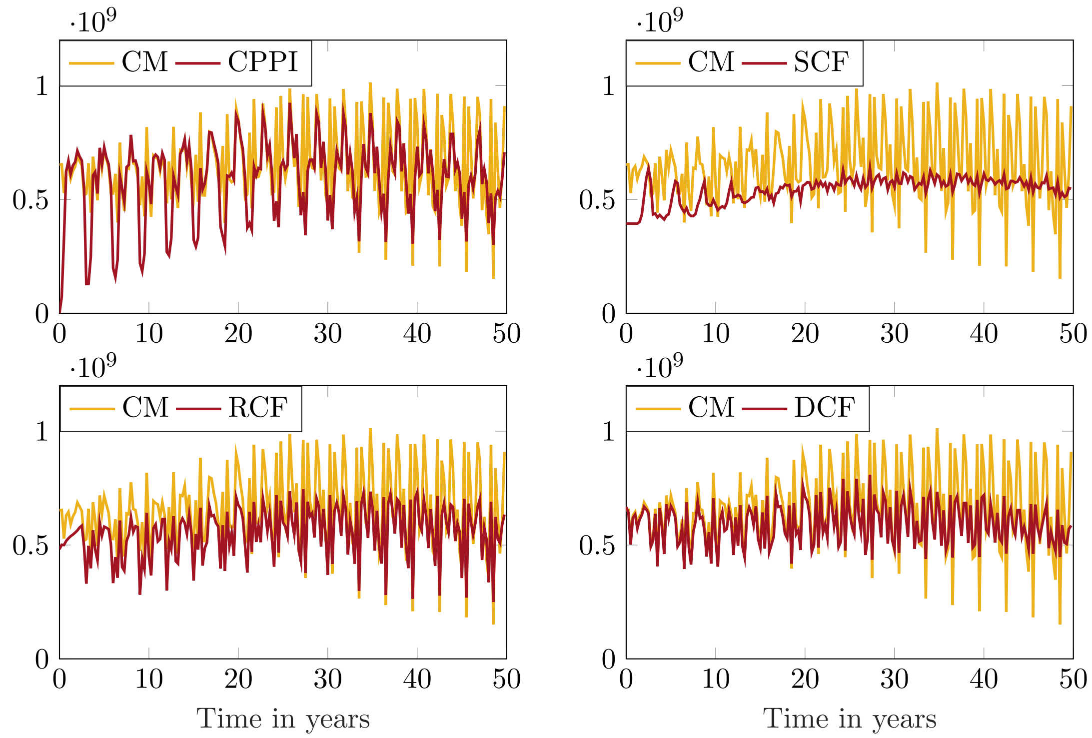

Figure A.11.: Amount of bought bonds applying different investment strategies and financing strategy FS 4 in the case of NBS 1.

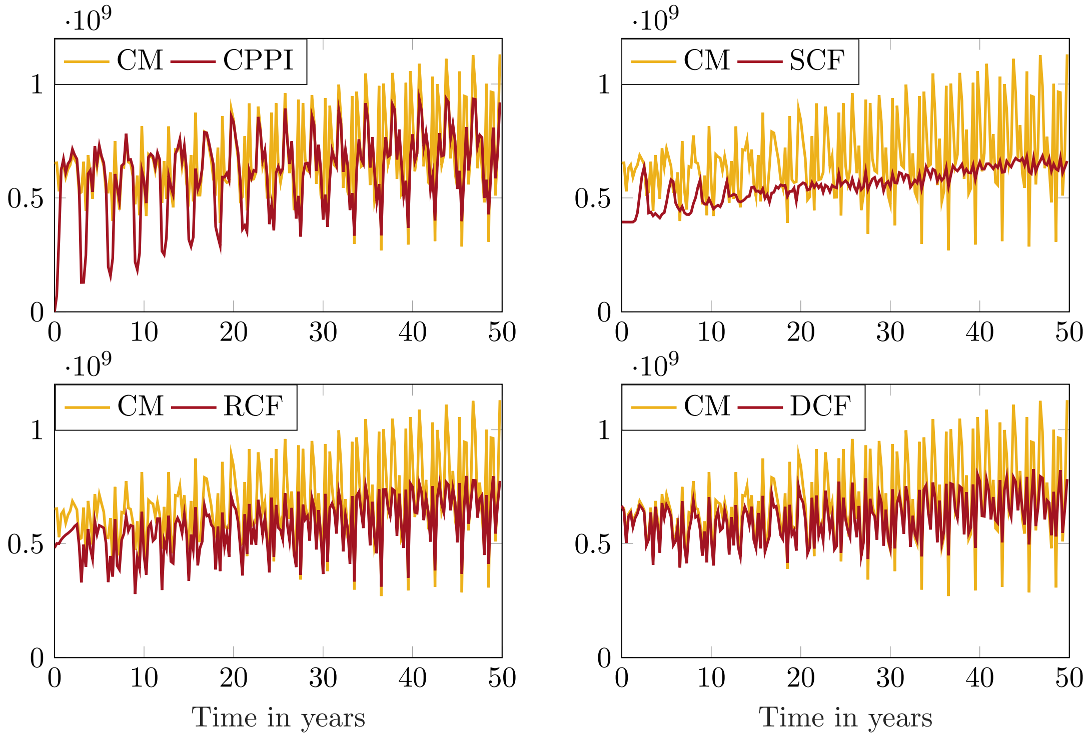

Figure A.12.: Amount of bought bonds applying different investment strategies and financing strategy FS 4 in the case of NBS 2.

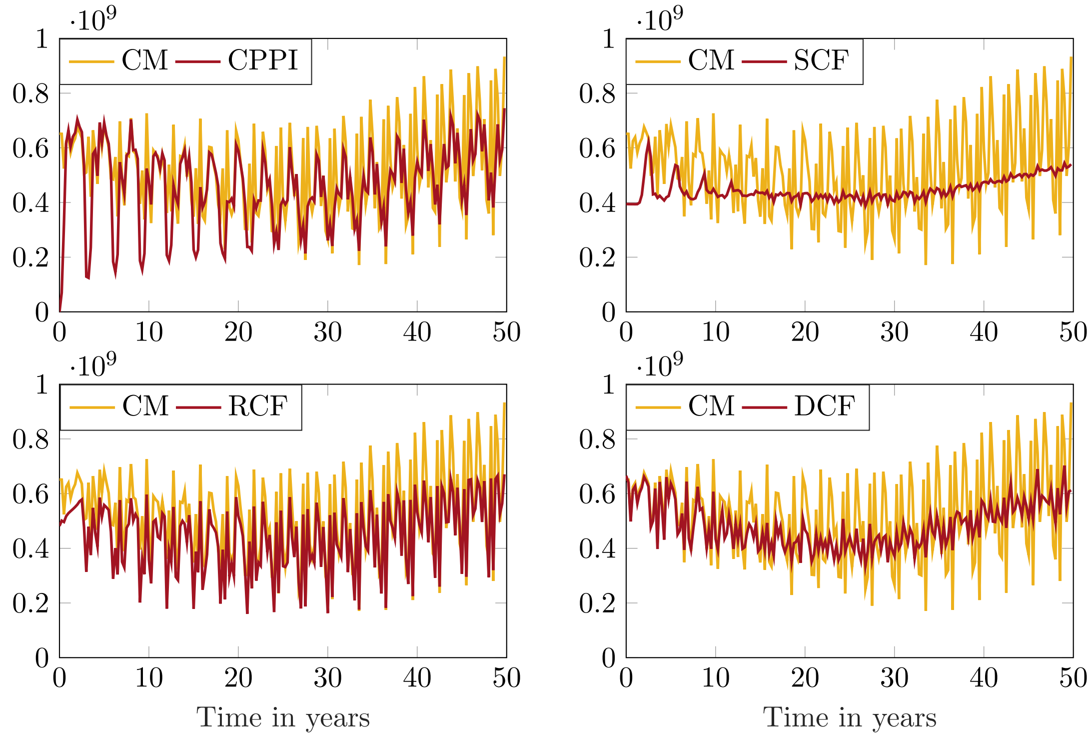

Figure A.13.: Amount of bought bonds applying different investment strategies and financing strategy FS 4 in the case of NBS 3.

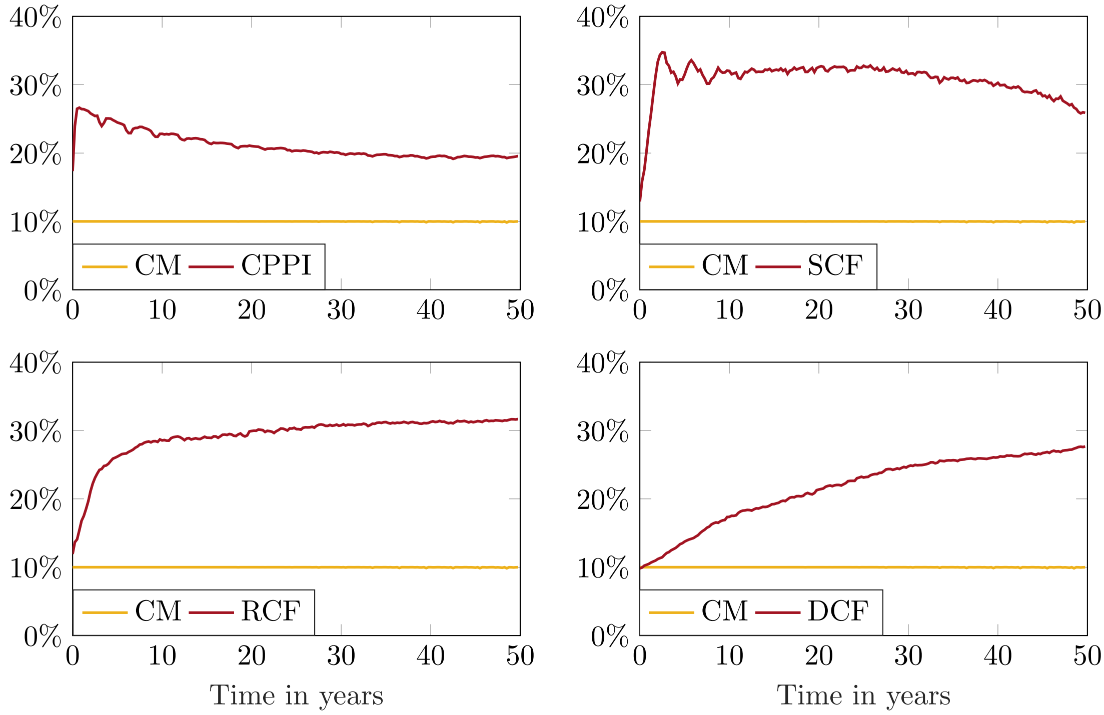

Figure A.14.: Stock ratio after reallocation of assets applying different investment strategies and financing strategy FS 4 in the case of NBS 1.

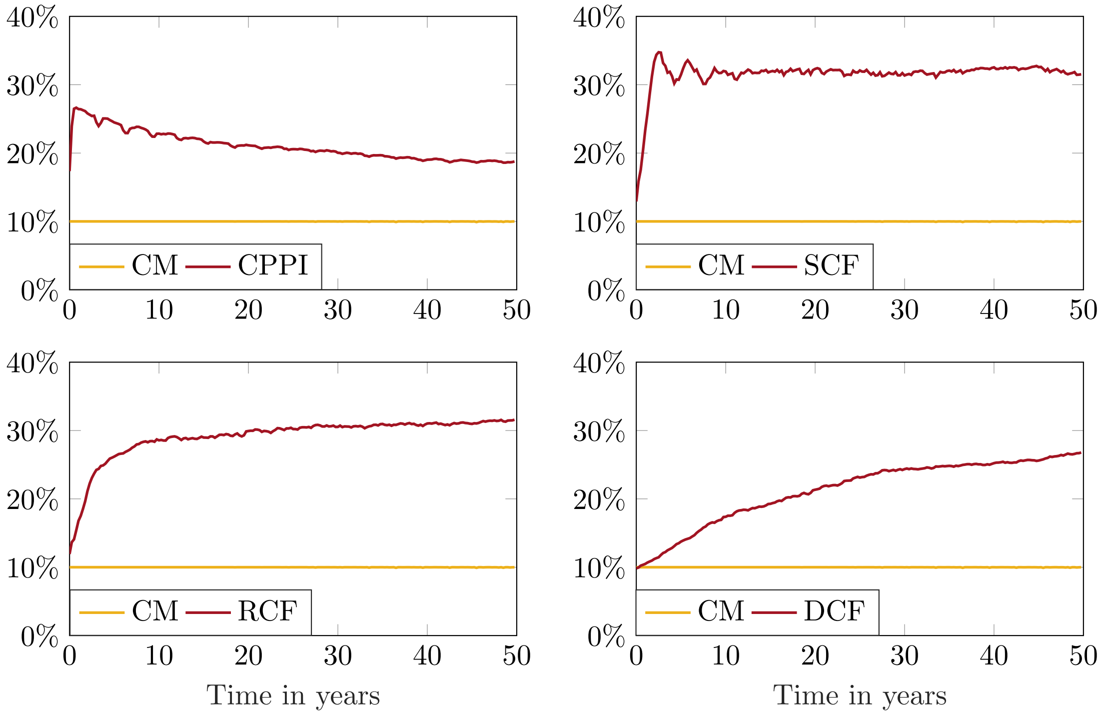

Figure A.15.: Stock ratio after reallocation of assets applying different investment strategies and financing strategy FS 4 in the case of NBS 2.

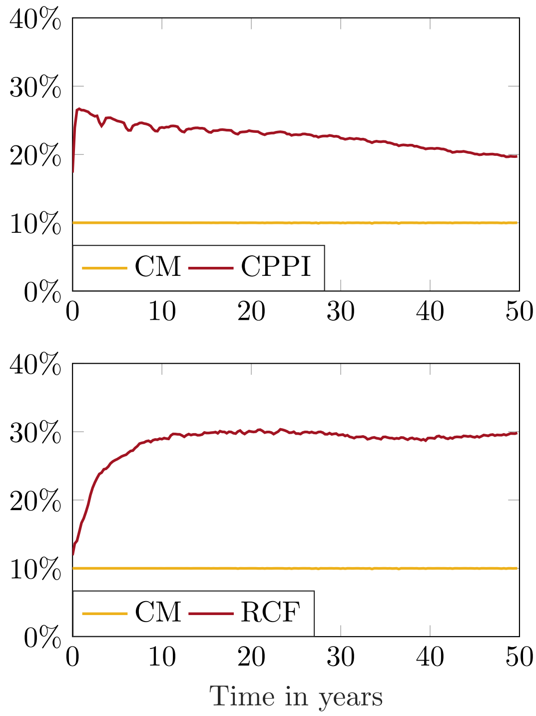

Figure A.16.: Stock ratio after reallocation of assets applying different investment strategies and financing strategy FS 4 in the case of NBS 3.

A.3. Alternative interest rate declaration and strategy for the use of surpluses

Here, we provide additional results regarding further applications of the introduced ALM model. More specifically, we show the effects from combining the alternative interest rate declaration from equation (3.5.9) in Remark 3.5.1 with the alternative strategy for the use of surpluses from Section 5.5. For the former, we choose $R_0 = \widehat{i}_G$, $\gamma_k^{\min} = 10\%$, and $\gamma_k^{\max} = 20\%$. We use the Heston model for the simulation of stock prices. The presented results complete the corresponding investigations from Sections 6.2.3 and 6.2.2.

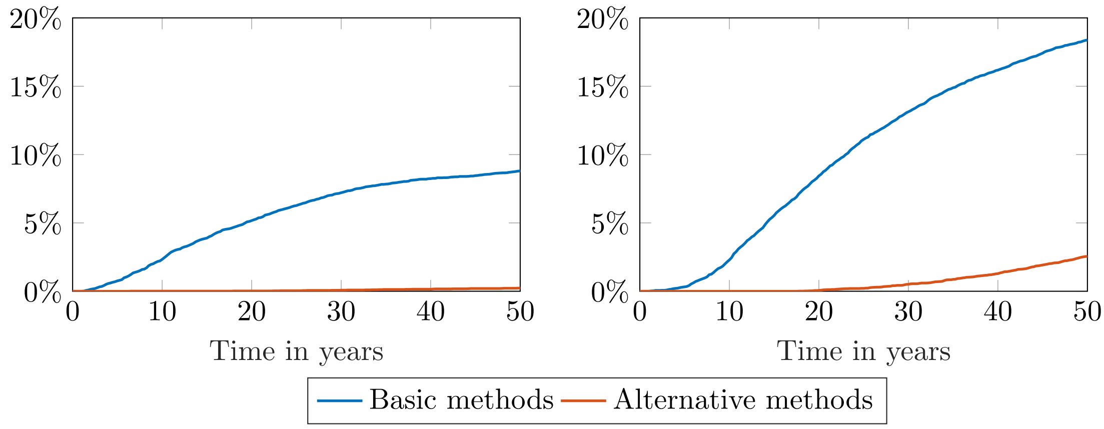

Figure A.17.: Development of the default probability for both methods of declaring the annual interest rate and for the use of surpluses. Left: CM, right: DCF.

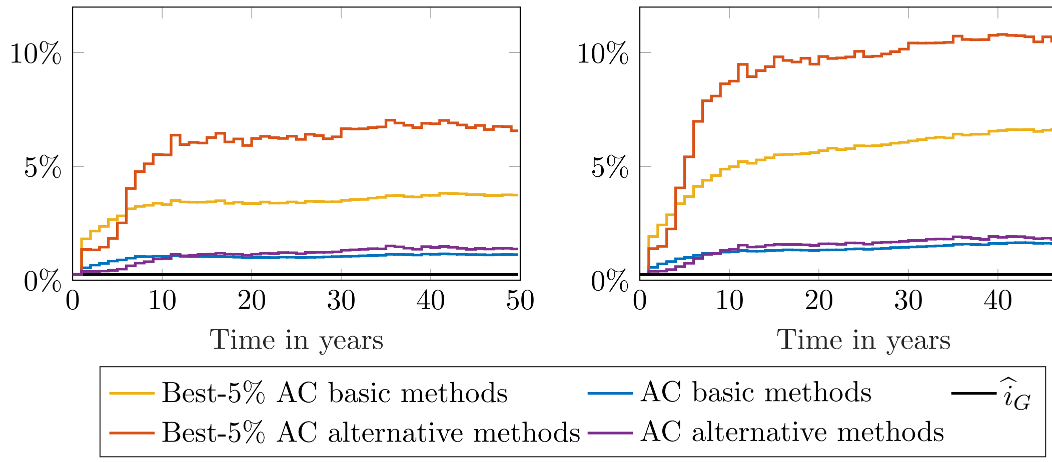

Figure A.18.: Development of the declared interest rate for both methods of declaring the annual interest rate and for the use of surpluses. Left: CM, right: DCF.

IS	$\Delta(\gamma_k \geq \gamma)$	$\Delta\gamma_k$	$\Delta(\psi_k \geq \psi)$	$\Delta\psi_k$	$\Delta G_k \cdot 10^6$	$\Delta\mathcal{B}_k$ in %
CM	24.48	5.31	1.36	−0.07	−2.75	0.85
DCF	19.33	3.38	−1.05	−1.67	−5.59	1.70

Table A.1.: Absolute or relative changes due to applying the alternative methods of declaring the annual interest rate and for the use of surpluses. If not stated otherwise, the change is measured in percentage points.

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