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STATE SPACE FORMULAS FOR COPRIME FACTORIZATION

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Dedicated to Professor T. Ando on his sixtieth birthday

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Abstract

In this paper we will give a uniform approach to the derivation of state space formulas of coprime factorizations, of different types, for rational matrix functions.

1 Introduction

The notion of coprimeness is as old as mathematics and goes back at least to the golden age of Greece, we refer to the Euclidean algorithm for the computation of the greatest common divisor of two integers.

Our interest in this paper lies in the representations of rational functions, i.e. quotients of coprime polynomials. By the Euclidean algorithm, or equivalently via ideal theory, coprimeness of two polynomials p, q is equivalent to the solvability of the Bezout equation

$$ap + bq = 1$$

over the ring of polynomials.

With changing our focus to the study of matrix rational functions, the use of left and right matrix fractions of the form

$$G = ND^{-1} = \overline{D}^{-1}\overline{N}$$

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with $N, \overline{N}, D, \overline{D}$ polynomial matrices. Such factorizations are called right and left coprime factorizations respectively if there exist polynomial matrix solution to the Bezout equations

$$XN + YD = I$$

and

$$\overline{NX} + \overline{DY} = I$$

respectively.

These polynomial coprime factorizations played an extremely important role in the development of algebraic system theory, and in particular in realization theory. In this connection we refer to Rosenbrock [1970], Fuhrmann [1976], Kailath [1980].

In a development parallel to system theory, operator theorists studied similar types of coprime factorizations, however over different rings (or rather algebras). The most prominent algebra in this connection is H^∞, the algebra of bounded analytic functions on the unit disc, or alternatively a half plane. In the wake of Beurling [1949] came the intensive study of shift operators. Cyclic vectors for the (right) shift operator were identified already by Beurling as outer functions. The next step was to determine the cyclic and noncyclic vectors of the backward shift. The noncyclic vectors of the backward shift are important inasmuch as they generalize the role of rational functions. The fundamental contribution in this connection is the work of Douglas, Shapiro and Shields [1971] and its generalization to the matrix case in Fuhrmann [1975]. The interesting point is that noncyclic vectors in H^2 are characterized in terms of special coprime factorizations over H^∞. We will refer to these factorizations as DSS (Douglas-Shapiro-Shields) factorizations.

As may be expected, the DSS factorization plays a central role in the development of infinite dimensional system theory. This is the theme of Fuhrmann [1981]. It is interesting to point out that the use of shift operators in infinite dimensional system theory predates their use in algebraic system theory, which was originated in Fuhrmann [1976].

Realization theory is but a tool in the development of control theory. Thus the real interest in the use of coprime factorizations is their application to the solution of control problems, in particular to the construction of stabilizing controllers. The cornerstone of this whole area is the Kucera-Youla parametrization of all stabilizing controllers which is based on coprime factorizations over H^∞. Pioneering works in this direction are Desoer et al. [1980], McFarlane and Glover [1989]. State space formulas for coprime factorizations were first developed by Khargonekar and Sontag [1982], Nett [1984]. The proof of coprimeness was done via explicit construction of doubly coprime factorizations. Specific choice was made for the solution of the Bezout equations, however no attempt was made to give an intrinsic characterization of the resulting doubly coprime factorization. We remedy this by showing that special choices lead to minimal McMillan degree doubly coprime factorizations. For the DSS factorization the state space formulas are due to Doyle [1984], and for the case of normalized coprime factorizations to Meyer and Franklin [1987], see also Vidyasagar [1985]. A polynomial approach to the derivation of normalized coprime factorizations was given in Fuhrmann and Ober [1992]. This method is powerful enough to lead to the unified derivation of state space formulas for various types of coprime factorizations, and this is the theme of this paper. For results concerning coprime factorization for nonlinear systems see e.g. Hammer [1985] and Verma [1988]

After some preliminary results on polynomial models we will present a unified approach to the derivation of state space formulas for coprime factorizations and normalized coprime factorizations

for various classes of functions. Thus we will study the classes of all rational functions of given McMillan degree, the class of unstable ones, bounded real and positive real functions. Except for the case of unstructured coprime factorizations, all other coprime factorizations are naturally given in terms of special indefinite metrics. Applications of these coprime factorizations, as well as the methods used to obtain them, to other control problems will be given in a subsequent paper.

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2 General factorizations

In this section we are going to analyze general factorizations of proper rational functions such that the factors are stable rational functions. The precise definition is as follows.

Definition 2.1 *Let G be a proper rational matrix-valued function. Then the factorization*

1. $G = NM^{-1}$ *is called a* right factorization (RF) *of G, if N, M are stable rational functions and M is invertible with proper inverse.*
   If N, M are right coprime, *i.e. if there exist stable rational functions $\tilde{U}$, $\tilde{V}$ such that*
   $$M\tilde{V} - N\tilde{U} = I,$$
   then the factorization is called a right coprime factorization (RCF).
2. $G = \tilde{M}^{-1}\tilde{N}$ *is called a* left factorization (LF) *of G, if $\tilde{N}, \tilde{M}$ are stable rational functions and $\tilde{M}$ is invertible with proper inverse.*
   If $\tilde{N}$, $\tilde{M}$ are left coprime, *i.e. if there exist stable rational functions U, V such that*
   $$V\tilde{M} - U\tilde{N} = I,$$
   then the factorization is called a left coprime factorization (LCF).

It is a standard result (see e.g. Vidyasagar [1985]) that right (left) coprime factorizations are unique up to right (left) multiplication by a stable rational function with proper stable inverse.

We are in particular interested in factorizations such that the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ equals the McMillan degree of G. The following Lemma shows that the McMillan degree of the function $\begin{pmatrix} M \\ N \end{pmatrix}$ is always larger than the McMillan degree of the $G = NM^{-1}$.

Lemma 2.1 *Let $G = NM^{-1}$ be a not necessarily coprime right factorization of the proper rational function G. Then,*

1. *if* $\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right)$ *is a realization of* $\begin{pmatrix} M \\ N \end{pmatrix}$ *then*
   $$G \equiv \left(\begin{array}{c|c} A_1 - B_1 D_1^{-1} C_1 & B_1 D_1^{-1} \\ \hline C_2 - D_2 D_1^{-1} C_1 & D_2 D_1^{-1} \end{array}\right)$$

is a realization of G.

2. *if we denote by $\delta(F)$ the McMillan degree of the proper rational function F then*

$$\delta(G) \leq \delta \begin{pmatrix} M \\ N \end{pmatrix}.$$

Proof: 1.) Note that, since $M = D_1 + C_1(sI - A_1)^{-1}B_1$, it follows that $M^{-1} = D_1^{-1} - D_1^{-1}C_1(sI - A_1 + B_1D_1^{-1}C_1)^{-1}B_1D_1^{-1}$. Therefore

$$G = NM^{-1} = [D_2 + C_2(sI - A_1)^{-1}B_1][D_1^{-1} - D_1^{-1}C_1(sI - A_1 + B_1D_1^{-1}C_1)^{-1}B_1D_1^{-1}]$$

$$= D_2D_1^{-1} + C_2(sI - A_1)^{-1}B_1D_1^{-1}$$
$$-D_2D_1^{-1}C_1(sI - A_1 + B_1D_1^{-1}C_1)^{-1}B_1D_1^{-1}$$
$$-C_2(sI - A_1)^{-1}B_1D_1^{-1}C_1(sI - A_1 + B_1D_1^{-1}C_1)^{-1}B_1D_1^{-1}$$

$$= D_2D_1^{-1} - D_2D_1^{-1}C_1(sI - A_1 + B_1D_1^{-1}C_1)^{-1}B_1D_1^{-1}$$
$$+C_2(sI - A_1)^{-1}[sI - A_1 + B_1D_1^{-1}C_1 - B_1D_1^{-1}C_1](sI - A_1 + B_1D_1^{-1}C_1)^{-1}B_1D_1^{-1}$$

$$= D_2D_1^{-1} + (C_2 - D_2D_1^{-1}C_1)(sI - A_1 + B_1D_1^{-1}C_1)^{-1}B_1D_1^{-1}.$$

2.) This follows immediately from part 1.) □

The following proposition gives a method to obtain factorizations using polynomial matrices. It establishes the existence of a factorization $G = NM^{-1}$ such that the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ equals the McMillan degree of G. A key step in the proof of this proposition is the following result that follows from the realization theory via polynomial models. For information on polynomial system and realization theory see Fuhrmann [1981].

Theorem 2.1 *Let $G = ND^{-1}$ be a coprime factorization and let (A, B, C) be a minimal realization of G. Let $G' = MD^{-1}$. Then G' has a realization (A, B, C_0) for some C_0.*

With the help of this theorem we can now prove the desired existence result of right factorizations with a given McMillan degree constraint.

Proposition 2.1 *Let G be a proper rational transfer function and let $G = ED^{-1}$ ($G = \overline{D}^{-1}\overline{E}$) be a polynomial right (left) coprime factorization. Let T ($\overline{T}$) be a square stable polynomial matrix of the same dimensions as D ($\overline{D}$), such that $N := ET^{-1}$ ($\tilde{N} := \overline{T}^{-1}\overline{E}$) and $M := DT^{-1}$ ($\tilde{M} := \overline{T}^{-1}\overline{D}$) are proper and M ($\tilde{M}$) has a proper inverse, then*

$$\begin{pmatrix} M \\ N \end{pmatrix} := \begin{pmatrix} DT^{-1} \\ ET^{-1} \end{pmatrix} \quad (\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix} := \begin{pmatrix} -\overline{T}^{-1}\overline{E} & \overline{T}^{-1}\overline{D} \end{pmatrix})$$

is a right (left) factorization of G ($-G$) and the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ ($\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix}$) equals the McMillan degree of G ($-G$).

Proof: The construction implies that NM^{-1} is a right factorization of G.

Let $G \equiv (A, B, C, D)$ be a minimal realization of G. Since $G = ED^{-1}$ and $M^{-1} = TD^{-1}$, Theorem 2.1 implies that M^{-1} has a realization given by

$$M^{-1} \equiv \left(\begin{array}{c|c} A & B \\ \hline C_0 & M^{-1}(\infty) \end{array} \right),$$

for some C_0. Hence M has a realization given by

$$M \equiv \left(\begin{array}{c|c} A - BM(\infty)C_0 & BM(\infty) \\ \hline -M(\infty)C_0 & M(\infty) \end{array} \right).$$

Since $N = ET^{-1}$ it follows again from Theorem 2.1 that

$$N \equiv \left(\begin{array}{c|c} A - BM(\infty)C_0 & BM(\infty) \\ \hline C_1 & D_1 \end{array} \right),$$

for some C_1 and D_1 and therefore

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BM(\infty)C_0 & BM(\infty) \\ \hline -M(\infty)C_0 & M(\infty) \\ C_1 & D_1 \end{array} \right).$$

This shows that there exists a right factorization whose state space realization has the same state-space as the realization of G. Therefore the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ is less than the McMillan degree of G and by Lemma 2.1 equal to the McMillan degree of G.

The statement concerning left factorizations is proved using the duality that $G = NM^{-1}$ is a right factorization if and only if $G^T = (M^Y)^{-1}N^T$ is a left factorization of G^T. □

In the following theorem all right factorizations $G = NM^{-1}$ of a proper rational function G are characterized such that the McMillan degree of G equals the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$. These factorizations are precisely those that can be obtained via the state feedback construction of Khargonekar and Sontag [1982] and later of Nett et. al. [1984]. Clearly this approach also provides a proof for the existence of right factorizations.

Theorem 2.2 *Let G be a proper rational function G and let $G \equiv \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$ be a minimal realization.*

Then $G = NM^{-1}$ is a, not necessarily coprime, right factorization of G such that the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ equals the McMillan degree of G if and only if there exists a state feedback F such that $A - BF$ is stable, and an invertible matrix D_1, s.t. $\begin{pmatrix} M \\ N \end{pmatrix}$ has a realization given by

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BF & BD_1 \\ \hline -F & D_1 \\ C - DF & DD_1 \end{array} \right).$$

Proof: Let $G = NM^{-1}$ be a right factorization. Let

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array} \right)$$

be a minimal realization and assume that G and $\begin{pmatrix} M \\ N \end{pmatrix}$ have the same McMillan degree. Since by assumption M has a proper inverse, D_1 is necessarily invertible. The stability of M and N implies that A_1 is stable. By Lemma 2.1 we have that

$$G \equiv \left(\begin{array}{c|c} A_1 - B_1 D_1^{-1} C_1 & B_1 D_1^{-1} \\ \hline C_2 - D_2 D_1^{-1} C_1 & D_2 D_1^{-1} \end{array} \right).$$

Since G and $\begin{pmatrix} M \\ N \end{pmatrix}$ have the same McMillan degree this implies that this realization of G is also minimal. Hence we can assume without loss of generality that

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \left(\begin{array}{c|c} A_1 - B_1 D_1^{-1} C_1 & B_1 D_1^{-1} \\ \hline C_2 - D_2 D_1^{-1} C_1 & D_2 D_1^{-1} \end{array} \right).$$

From here we can see that we have

$$D_2 = DD_1,$$

$$B_1 = BD_1,$$

$$A_1 = A + BC_1,$$

$$C_2 = C + DC_1.$$

Since A_1 is stable and $G \equiv \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$ and is minimal, this shows that $F := -C_1$ is a stabilizing state feedback such that,

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BF & BD_1 \\ \hline -F & D_1 \\ C - DF & DD_1 \end{array} \right).$$

Since M has a proper inverse by the assumption, this shows that $D_1 = M(\infty)$ is invertible.

Conversely, let D_1 be invertible and let F be such that $A - BF$ is stable. Define $\begin{pmatrix} M \\ N \end{pmatrix}$ by

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BF & BD_1 \\ \hline -F & D_1 \\ C - DF & DD_1 \end{array} \right).$$

Then clearly the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ is less than or equal to that of G since both have a realization on the same state-space. It can be verified easily that $G = NM^{-1}$. Hence by Lemma 2.1 $\begin{pmatrix} M \\ N \end{pmatrix}$ and G have the same McMillan degree.

□

The following corollary summarizes the analogous results concerning left factorizations.

Corollary 2.1 *Let G be a proper rational function and let* $G \equiv \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *be a minimal realization. Then $G = \tilde{M}^{-1}\tilde{N}$ is a, not necessarily coprime, left factorization of G such that the McMillan degree of $\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix}$ equals the McMillan degree of G if and only if there exists a output injection H, such that $A - HC$ is stable, and an invertible matrix $\overline{D}_1$, s.t. $\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix}$ has a realization given by*

$$\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - HC & HD - B & -H \\ \hline \overline{D}_1 C & -\overline{D}_1 D & \overline{D}_1 \end{array}\right).$$

Proof: The result can be obtained from the previous theorem, by using the duality that $G = NM^{-1}$ is a right factorization if and only if $G^T = (M^T)^{-1}N^T$ is a left factorization. □

In this theorem and corollary we examined right (left) factorizations. It was left open whether these factorizations are in fact coprime. In order to answer this question we will make use of so-called doubly coprime factorizations.

Definition 2.2 *The two proper stable rational block matrices*

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix}, \quad \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix},$$

with M ($\tilde{M}$) having a proper inverse, form a doubly coprime factorization *of the proper rational function G, if*

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}.$$

We are particularly interested in doubly coprime factorizations such that all three functions

$$G, \quad \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix}, \quad \begin{pmatrix} M & U \\ N & V \end{pmatrix}$$

have the same McMillan degree.

Before we can derive state-space realizations of the doubly coprime factorizations we need to state the following lemma, which is a key step in the proof of the subsequent corollary. It is a consequence of standard arguments in realization theory.

Lemma 2.2 *Let $G = [G_1, G_2]$ be a proper rational function. Assume that G_1 has McMillan degree n, then the McMillan degree of G is n if and only if for the Hankel operators H_{G_1} and H_{G_2} we have*

$$range(H_{G_2}) \subseteq range(H_{G_1}).$$

If this is the case and $G_1 \equiv \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right)$ *is a minimal state space realization, then G_2 has a state space realization of the form*

$$G_2 \equiv \left(\begin{array}{c|c} A_1 & L \\ \hline C_1 & D_2 \end{array}\right)$$

for some L and D_2.

The following corollary answers two questions. Given a rational function of McMillan degree n, all doubly coprime factors are characterized that have McMillan degree n. As a consequence of the construction, we see that a right factorization $G = NM^{-1}$ such that $\begin{pmatrix} M \\ N \end{pmatrix}$ has McMillan degree n is necessarily right coprime.

Corollary 2.2 *Let G be a proper rational function of McMillan degree n. All doubly coprime factorizations of G such that*

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix}, \quad \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix},$$

have McMillan degree n are given by

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - BF & BD_1 & BD_2 + H\overline{D}_1^{-1} \\ \hline -F & D_1 & D_2 \\ C - DF & DD_1 & \overline{D}_1^{-1} + DD_2 \end{array}\right)$$

and

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - HC & HD - B & -H \\ \hline -D_1^{-1}F - D_1^{-1}D_2\overline{D}_1C & D_1^{-1} + D_1^{-1}D_2\overline{D}_1D & -D_1^{-1}D_2\overline{D}_1 \\ \overline{D}_1C & -\overline{D}_1D & \overline{D}_1 \end{array}\right),$$

where $G \equiv \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is a minimal realization and F (H) is such that $A - BF$ ($A - HC$) is stable D_1, $\overline{D}_1$ are invertible and D_2 is arbitrary.

Proof: Let $\begin{pmatrix} M & U \\ N & V \end{pmatrix}, \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix}$ be doubly coprime factors of McMillan degree n. Then $G = NM^{-1}$ is a right factorization such that $\begin{pmatrix} M \\ N \end{pmatrix}$ has McMillan degree n. Let

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BF & BD_1 \\ \hline -F & D_1 \\ C - DF & DD_1 \end{array}\right)$$

be the minimal realization of $\begin{pmatrix} M \\ N \end{pmatrix}$ of Theorem 2.2, where F is a stabilizing state feedback and D_1 is invertible. By assumption U, V are such that the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ is equal to that of $\begin{pmatrix} M & U \\ N & V \end{pmatrix}$. By Lemma 2.2 thus has a $\begin{pmatrix} U \\ V \end{pmatrix}$ has a realization

$$\begin{pmatrix} U \\ V \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BF & L \\ \hline -F & D_2 \\ C - DF & D_3 \end{array}\right)$$

for some $L : \mathcal{K}^m \to X$, $D_2 : \mathcal{K}^m \to \mathcal{K}^p$ and $D_3 : \mathcal{K}^m \to \mathcal{K}^m$ and therefore,

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - BF & BD_1 & L \\ \hline -F & D_1 & D_2 \\ C - DF & DD_1 & D_3 \end{array}\right).$$

Similarly, the other factor has a state-space realization

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - HC & HD - B & -H \\ \hline \overline{L} & \overline{D}_3 & \overline{D}_2 \\ \overline{D}_1 C & -\overline{D}_1 D & \overline{D}_1 \end{array}\right),$$

for some $\overline{L} : X \to \mathcal{K}^m$, $\overline{D}_2 : \mathcal{K}^m \to \mathcal{K}^m$ and $\overline{D}_3 : \mathcal{K}^p \to \mathcal{K}^m$ and where H is a stabilizing output injection and $\overline{D}_1$ is invertible.

We first consider the feedthrough terms. Since we have a doubly coprime factorization we need to have that,

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \overline{D}_3 & \overline{D}_2 \\ -\overline{D}_1 D & \overline{D}_1 \end{pmatrix} \begin{pmatrix} D_1 & D_2 \\ DD_1 & D_3 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{D}_3 D_1 + \overline{D}_2 DD_1 & \overline{D}_3 D_2 + \overline{D}_2 D_3 \\ -\overline{D}_1 DD_1 + \overline{D}_1 DD_1 & -\overline{D}_1 DD_2 + \overline{D}_1 D_3 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{D}_3 D_1 + \overline{D}_2 DD_1 & \overline{D}_3 D_2 + \overline{D}_2 D_3 \\ 0 & -\overline{D}_1 DD_2 + \overline{D}_1 D_3 \end{pmatrix}.$$

Solving these equations we obtain after some calculations that

$$\overline{D}_3 = D_1^{-1} + D_1^{-1} D_2 \overline{D}_1 D,$$

$$D_3 = \overline{D}_1^{-1} + DD_2,$$

$$\overline{D}_2 = -D_1^{-1} D_2 \overline{D}_1.$$

Hence, we necessarily have that

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - BF & BD_1 & L \\ \hline -F & D_1 & D_2 \\ C - DF & DD_1 & \overline{D}_1^{-1} + DD_2 \end{array}\right)$$

and

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - HC & HD - B & -H \\ \hline \overline{L} & D_1^{-1} + D_1^{-1} D_2 \overline{D}_1 D & -D_1^{-1} D_2 \overline{D}_1 \\ \overline{D}_1 C & -\overline{D}_1 D & \overline{D}_1 \end{array}\right).$$

To determine L and $\overline{L}$, we calculate the state-space realizations of the cascaded system

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} \equiv$$

$$\left(\begin{array}{cc|cc} A - HC & \begin{bmatrix} HD - B & -H \end{bmatrix} \begin{bmatrix} -F \\ C - DF \end{bmatrix} & \multicolumn{2}{c}{\begin{bmatrix} HD - B & -H \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ DD_1 & \overline{D}_1^{-1} + DD_2 \end{bmatrix}} \\ 0 & A - BF & \multicolumn{2}{c}{BD_1 L} \\ \hline \overline{L} & \begin{bmatrix} D_1^{-1} + D_1^{-1} D_2 \overline{D}_1 D & -D_1^{-1} D_2 \overline{D}_1 \end{bmatrix} \begin{bmatrix} -F \\ C - DF \end{bmatrix} & I & 0 \\ \overline{D}_1 C & \begin{bmatrix} -\overline{D}_1 D & \overline{D}_1 \end{bmatrix} \begin{bmatrix} -F \\ C - DF \end{bmatrix} & 0 & I \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} A-HC & BF-HC & -BD_1 & -BD_2 - H\overline{D}_1^{-1} \\ 0 & A-BF & BD_1 & L \\ \hline \overline{L} & -D_1^{-1}F - D_1^{-1}D_2\overline{D}_1 C & I & 0 \\ \overline{D}_1 C & \overline{D}_1 C & 0 & I \end{array} \right).$$

A state-space transformation by $\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}$ gives

$$\left(\begin{array}{cc|cc} A-HC & 0 & 0 & -BD_2 - H\overline{D}_1^{-1} + L \\ 0 & A-BF & BD_1 & L \\ \hline \overline{L} & -D_1^{-1}F - D_1^{-1}D_2\overline{D}_1 C - \overline{L} & I & 0 \\ \overline{D}_1 C & 0 & 0 & I \end{array} \right).$$

Consider the $(2,2)$ subsystem

$$I \equiv \left(\begin{array}{c|c} A-HC & -BD_2 - H\overline{D}_1^{-1} + L \\ \hline \overline{D}_1 C & I \end{array} \right)$$

and the $(1,1)$ subsystem

$$I \equiv \left(\begin{array}{c|c} A-BF & BD_1 \\ \hline -D_1^{-1}F - D_1^{-1}D_2\overline{D}_1 C - \overline{L} & I \end{array} \right).$$

Since $\overline{D}_1$ and D_1 are invertible, the first system is observable and the second system is reachable. Hence these two systems are I if and only if

$$-BD_2 - H\overline{D}_1^{-1} + L = 0$$

$$-D_1^{-1}F - D_1^{-1}D_2\overline{D}_1 C - \overline{L} = 0,$$

or if and only if

$$L = BD_2 + H\overline{D}_1^{-1}$$

and

$$\overline{L} = -D_1^{-1}F - D_1^{-1}D_2\overline{D}_1 C.$$

Conversely, let D_2 be arbitrary and let $\begin{pmatrix} M & U \\ N & V \end{pmatrix}$ and $\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix}$ be defined through the state-space realizations in the statement of the corollary. Then it can be checked in a straightforward way that

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

By construction, the McMillan degrees of these two functions are less than or equal to that of G. Lemma 2.1 then implies that the McMillan degrees of all three functions are the same.

□

The following corollary states that McMillan degree n factors of rational functions of McMillan degree n are necessarily coprime.

Corollary 2.3 *Let $G = NM^{-1}$ be a, not necessarily coprime, right factorization of G. If the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ equals the McMillan degree of G then N an M are right coprime.*

Similarly, let $G = \tilde{M}^{-1}\tilde{N}$ be a, not necessarily coprime, left factorization of G. If the McMillan degree of $\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix}$ equals the McMillan degree of G then $\tilde{N}$ an $\tilde{M}$ are left coprime.

Proof: This follows immediately from the previous corollary where solutions to the Bezout equations were constructed. □

3 Antistable functions

One of the main purposes of this paper is to derive state-space realizations of factorizations that are normalized in certain ways. The first class of systems for which we are going to consider normalized factorizations is the class of antistable functions. By an antistable function we mean a function whose poles are in the open right half plane. Here the factorization is normalized so that the denominator M is inner, i.e. $M^*M = I$. This type of factorization has been introduced by Douglas, Shapiro and Shields [1971] for scalar functions and by Fuhrmann [1981] for matrix-valued functions. It is therefore referred to as the Douglas-Shapiro-Shields factorization (DDS). This factorization is amongst other applications particularly important in the theory of Hankel operators. State space formulae for DSS-factorizations appear in the control literature, see e.g. Doyle [1984].

Let G be a proper antistable function, i.e. all poles of G are in the open right half plane. A right (left) coprime factorization $G = NM^{-1}$ ($G = \tilde{M}^{-1}\tilde{N}$) is called a right (left) *Douglas-Shapiro-Shields (DSS)* factorization if $M^*M = I$ ($\tilde{M}\tilde{M}^* = I$.)

The existence of a DSS factorization is guaranteed by the following proposition.

Proposition 3.1 *Let G be an antistable proper rational function of McMillan degree n. Then there exists a right (left) factorization*

$$G = NM^{-1} \quad (G = \tilde{M}^{-1}\tilde{N})$$

with $M^*M = I$ ($\tilde{M}^*\tilde{M} = I$). Moreover, M and $\tilde{M}$ have McMillan degree n. The right (left) factorization with this property is unique up to right (left) multiplication by a unitary constant matrix. All such factorizations are coprime and such that $\begin{pmatrix} M \\ N \end{pmatrix}$ ($\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix}$) has McMillan degree n. Moreover, M and $\tilde{M}$ have McMillan degree n.

Proof: Let $G = ED^{-1}$ be a right polynomial coprime factorization. By assumption D is antistable. Let T be a square stable spectral factor of D^*D, i.e. $D^*D = T^*T$. Then

$$\begin{pmatrix} M \\ N \end{pmatrix} := \begin{pmatrix} DT^{-1} \\ ET^{-1} \end{pmatrix}$$

defines by Proposition 2.1 right factors of G, i.e. $G = NM^{-1}$ with N, M and M^{-1} proper and $\begin{pmatrix} M \\ N \end{pmatrix}$ has the same McMillan degree as G. Hence the factorization is coprime by Corollary 2.3. Clearly, $M^*M = I$. Since D is antistable and T is stable, there are no pole-zero cancellations and therefore M is of McMillan degree n. Let $G = NM^{-1} = N_1M_1^{-1}$ be two DSS factorizations of G.

Since both are coprime factorizations, there exists a stable function Q with proper stable inverse that relates the two factorizations (see e.g. Vidyasagar [1985]). In particular, $M = M_1 Q$. Since $I = M^* M = Q^* M_1^* M_1 Q = Q^* Q$, this shows that $Q = Q^{-*}$. Since Q is stable with proper stable inverse, this implies that Q must be a constant unitary matrix.

The statement concerning left factorizations follows analogously. □

The following Lemma will be important in the proofs of the main theorems of this and subsequent sections.

Lemma 3.1 *Let $U(s)$ be a proper rational function such that*

$$U(s) = J_1 U(s)^{-*} J_2, \quad s \in C \setminus \sigma(U),$$

where J_i, $i = 1, 2$, are constant matrices such that $J_i = J_i^{-1} = J_i^*$, $i = 1, 2$. Let $U \equiv \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ be a minimal realization of $U(s)$. Then,

1. *$D = J_1 D^{-*} J_2$.*
2. *$U(s)$ has another minimal realization of the form*

$$U \equiv \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \equiv \left(\begin{array}{c|c} -A^* + C^* J_1 D J_2 B^* & C^* J_1 D \\ \hline D J_2 B^* & D \end{array}\right),$$

and there exists a unique non-singular state-space transformation Y between these two realizations such that $Y = Y^*$, and

$$YA = (-A^* + C^* J_1 D J_2 B^*) Y,$$

$$YB = C^* J_1 D,$$

$$C = D J_2 B^* Y.$$

Proof: Note that

$$U^{-1} \equiv \left(\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right)$$

and

$$U^{-*} \equiv \left(\begin{array}{c|c} -A^* + C^* D^{-*} B^* & C^* D^{-*} \\ \hline D^{-*} B^* & D^{-*} \end{array}\right)$$

and therefore

$$U = J_1 U^{-*} J_2 \equiv \left(\begin{array}{c|c} -A^* + C^* D^{-*} B^* & C^* D^{-*} J_2 \\ \hline J_1 D^{-*} B^* & J_1 D^{-*} J_2 \end{array}\right).$$

Since $U = J_1 U^{-*} J_2$ we clearly have that $D = J_1 D^{-*} J_2$. Since $U \equiv \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is a minimal realization there exists a unique non-singular state-space transformation T, between this and the second realization of U, i.e.

$$C = J_1 D^{-*} B^* T,$$

$$B = T^{-1}C^*D^{-*}J_2,$$

$$A = T^{-1}(-A^* + C^*D^{-*}B^*)T.$$

Dualizing these equations, we obtain,

$$C^* = T^*BD^{-1}J_1,$$

$$B^* = J_2D^{-1}CT^{-*},$$

$$A^* = T^*(-A + BD^{-1}C)T^{-*}.$$

Solving for A, B and C we have,

$$C = DJ_2B^*T^* = J_1D^{-*}B^*T^*$$

$$B = T^{-*}C^*J_1D = T^{-*}C^*D^{-*}J_2,$$

$$A = -T^{-*}A^*T^* + BD^{-1}C = -T^{-*}AT^* + T^{-*}C^*D^{-*}J_2D^{-1}J_1D^{-*}B^*T^*$$

$$= T^{-*}(-A^* + C^*D^{-*}B^*)T^*.$$

But this shows that T^* is also a state-space transformation between the two systems. The uniqueness of the transformation therefore implies that $T^* = T$. The form of the realization of U given in the statement, follows immediately from the above identities. □

Proposition 3.1 showed that a right (left) DSS-factorization is unique up to right multiplication by a constant unitary matrix. The factorization is such that the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ equals the McMillan degree of G. In the following theorem all doubly-coprime factorizations are characterized which are such that $\begin{pmatrix} M \\ N \end{pmatrix}$ forms a DSS-factorization of G.

Theorem 3.1 *Let G be an antistable proper rational transfer function of McMillan degree n with minimal state-space realization (A, B, C, D). Then all doubly coprime factorizations, such that $G = NM^{-1}$, with $MM^* = I$, and $G = \tilde{M}^{-1}\tilde{N}$ with $\tilde{M}\tilde{M}^* = I$, and the McMillan degrees of $\begin{pmatrix} M & U \\ N & V \end{pmatrix}$ and $\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix}$ are n, are given by*

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - BB^*Y & BD_1 & BD_2 + ZC^*\overline{D}_1^* \\ \hline -B^*Y & D_1 & D_2 \\ C - DB^*Y & DD_1 & \overline{D}_1^* + DD_2 \end{array}\right),$$

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - ZC^*C & ZC^*D - B & -ZC^* \\ \hline -D_1^*B^*Y - D_1^*D_2\overline{D}_1C & D_1^* + D_1^*D_2\overline{D}_1D & -D_1^*D_2\overline{D}_1 \\ \overline{D}_1C & -\overline{D}_1D & \overline{D}_1 \end{array}\right),$$

where D_1 is arbitrary such that $D_1 = D_1^{-*}$, $\overline{D}_1$ is arbitrary such that $\overline{D}_1 = \overline{D}_1^{-*}$, D_2 arbitrary, and Y and Z are the unique positive definite solutions to the Riccati equations

$$AY + YA^* - YBB^*Y = 0,$$

$$A^*Z + ZA - ZC^*CZ = 0.$$

Proof: Let $G = NM^{-1}$ be a DSS factorization such that $\begin{pmatrix} M \\ N \end{pmatrix}$ has McMillan degree n. This exists by Proposition 3.1. By Theorem 2.2 any right factorization $G = NM^{-1}$ such $\begin{pmatrix} M \\ N \end{pmatrix}$ has McMillan degree n is of the form

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BF & BD_1 \\ \hline -F & D_1 \\ C - DF & DD_1 \end{array}\right),$$

where F is a stabilizing feedback and $D_1 = M(\infty)$ is invertible. Since M is such that $M^*M = I$, we have that

$$M = M^{-*}.$$

Since M has McMillan degree n, the realization

$$M \equiv \left(\begin{array}{c|c} A - BF & BD_1 \\ \hline -F & D_1 \end{array}\right)$$

is minimal. Lemma 3.1 now implies that there exists a unique non-singular state-space transformation $\hat{Y} = \hat{Y}^*$ such that

$$D_1 = D_1^{-*},$$

$$\hat{Y}BD_1 = -F^*D_1,$$

$$-F = D_1D_1^*B^*\hat{Y} = B^*\hat{Y},$$

$$\hat{Y}(A - BF) = (-A^* + F^*B^* - F^*D_1D_1^*B^*)\hat{Y} = -A^*\hat{Y}.$$

Using that $F = -B^*\hat{Y}$ we can rewrite the equation

$$\hat{Y}(A - BF) = -A^*\hat{Y}$$

as

$$A^*\hat{Y} + \hat{Y}A + \hat{Y}BB^*\hat{Y} = 0.$$

Setting $Y := -\hat{Y}$, this equation is equivalent to the more conventional equation,

$$A^*Y + YA - YBB^*Y = 0.$$

Since Y is invertible, this Riccati equation is equivalent to the Lyapunov equation,

$$Y^{-1}A^* + AY^{-1} - BB^* = 0,$$

which shows that Y^{-1} and therefore Y is positive definite. Since A^* is antistable, Y^{-1} is the unique positive definite solution of this equation. Hence Y is the unique positive definite solution to the Riccati equation. A state space realization of $\begin{pmatrix} M \\ N \end{pmatrix}$ is given by

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BB^*Y & BD_1 \\ \hline -B^*Y & D_1 \\ C - DB^*Y & DD_1 \end{array} \right),$$

where D_1 is arbitrary such that $D_1 = D_1^{-*}$.

The expressions for the doubly coprime factors now follow from Corollary 2.2.

An analogous argument or the duality consideration that $G = NM^{-1}$ is a right factorization if and only if $G^T = (M^T)^{-1}N^T$, shows that a state space realization of $\left[-\tilde{N} \;\; \tilde{M} \right]$ is given by

$$\left[-\tilde{N} \;\; \tilde{M} \right] \equiv \left(\begin{array}{c|cc} A - ZC^*C & ZC^*D - B & -ZC^* \\ \hline \overline{D}_1 C & -\overline{D}_1 D & \overline{D}_1 \end{array} \right),$$

where Z is the unique positive solution of the Riccati equation

$$AZ + ZA^* - ZC^*CZ = 0,$$

and $\overline{D}_1$ is arbitrary such that $\overline{D}_1 = \overline{D}_1^{-*}$. The remaining part of the argument is analogous to the above derivation.

Conversely, let Y be the unique positive definite solution to the Riccati equation

$$A^*Y + YA - YBB^*Y = 0.$$

Constructing the DSS factorization of G as in Proposition 3.1, proceeding as above and using the uniqueness of the solution Y, shows that

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BB^*Y & BD_1 \\ \hline -B^*Y & D_1 \\ C - DB^*Y & DD_1 \end{array} \right),$$

gives a realization of the DSS factors of G. Hence $F = B^*Y$ is a stabilizing feedback and the state-space construction gives indeed the required factorizations.

□

The expressions for the doubly-coprime factorizations can be simplified if we are only interested in a particular factorization and not in all of them. The choice $D_1 = I$, $\overline{D}_1^* = I$ and $D_2 = 0$ would lead to such a simplification.

As part of the proof of the theorem we have also shown the well-known result that a certain degenerate Riccati equation has a stabilizing solution.

Corollary 3.1 *Let (A, B, C, D) be an antistable continuous-time minimal system. Then there exists a unique positive definite solution Y (Z) of the Riccati equation*

$$AY + YA^* - YBB^*Y = 0 \quad (A^*Z + ZA - ZC^*CZ = 0).$$

This solution is such that $A - BB^*Y$ $(A - ZC^*C)$ is stable, i.e. all eigenvalues of $A - BB^*Y$ $(A - ZC^*C)$ are in the open left half plane.

Proof: This statement was proved as part of the proof of the theorem. □

4 Minimal systems

Normalized coprime factorizations proved to be powerful methods in control theoretic problems, especially in the area of robust control (see e.g. Vidyasagar [1985], McFarlane and Glover [1989], Fuhrmann and Ober [1992]). For their relevance in parametrization problems, see Ober and McFarlane [1989].

Let

$$J_L = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

A right (coprime) factorization $G = NM^{-1}$ of G is called a *J_L-RF (J_L-RCF)* of G if

$$\begin{pmatrix} M^* & N^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = M^*M + N^*N = I.$$

Similarly a left coprime factorization $G = \overline{M}^{-1}\overline{N}$ the transfer function G is called *J_L-LF* (a *J_L-LCF*) of G if

$$\overline{N}\overline{N}^* + \overline{M}\overline{M}^* = I.$$

The following proposition guarantees the existence of such factorizations. The existence and uniqueness result is due to Vidyasagar (see e.g. Vidyasagar [1985]).

Proposition 4.1 *Let G be a proper rational function of McMillan degree n. Then there exists a J_L-right (left) factorization,*

$$G = NM^{-1} \quad (G = \tilde{M}^{-1}\tilde{N}).$$

This factorization is right (left) coprime and is unique up to right (left) multiplication by a constant unitary matrix. All such factorizations are such that $\begin{pmatrix} M \\ N \end{pmatrix}$ ($\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix}$) is of McMillan degree n.

Proof: Let $G = ED^{-1}$ be a right polynomial coprime factorization. Let T be a square stable spectral factor of $E^*E + D^*D$, i.e. $T^*T = E^*E + D^*D$. Then

$$\begin{pmatrix} M \\ N \end{pmatrix} := \begin{pmatrix} DT^{-1} \\ ET^{-1} \end{pmatrix}$$

defines by Proposition 2.1 right factors of G, i.e. $G = NM^{-1}$ with N, M and M^{-1} proper and $\begin{pmatrix} M \\ N \end{pmatrix}$ of McMillan degree n. This also implies by Corollary 2.3 that the factorization is coprime. Let now $G = N_1M_1^{-1}$ be another J_L-right coprime factorization. Then there exists a stable Q with proper stable inverse (Vidyasagar [1985]) such that $M = M_1Q$ and $N = N_1Q$. But $I = M^*M + N^*N = Q^*M_1^*M_1Q + Q^*N_1^*N_1Q = Q^*Q$, which shows that $Q = Q^{-*}$, which implies that Q is a constant matrix.

The statement concerning left factorizations follows analogously. □

Note that for J_L-factorizations $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ we have that

$$\begin{pmatrix} M & -\tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix}^* \begin{pmatrix} M & -\tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

The following Lemma will be useful in the proof of the subsequent theorem.

Lemma 4.1 *Consider*

$$\begin{pmatrix} D_1 & -D^*\overline{D}_1^* \\ DD_1 & \overline{D}_1^* \end{pmatrix} = \begin{pmatrix} I & -D^* \\ D & I \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & \overline{D}_1^* \end{pmatrix},$$

with D_1 and $\overline{D}_1$ invertible. Then

1.
$$\begin{pmatrix} D_1 & -D^*\overline{D}_1^* \\ DD_1 & \overline{D}_1^* \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} D_1^{-1} & 0 \\ 0 & \overline{D}_1^{-*} \end{pmatrix} \begin{pmatrix} I & D^* \\ -D & I \end{pmatrix} \begin{pmatrix} (I+D^*D)^{-1} & 0 \\ 0 & (I+DD^*)^{-1} \end{pmatrix}.$$

2. *If*
$$\begin{pmatrix} D_1 & -D^*\overline{D}_1^* \\ DD_1 & \overline{D}_1^* \end{pmatrix} = \begin{pmatrix} D_1 & -D^*\overline{D}_1^* \\ DD_1 & \overline{D}_1^* \end{pmatrix}^{-*},$$
then
$$D_1D_1^* = (I+D^*D)^{-1},$$
$$\overline{D}_1^*\overline{D}_1 = (I+DD^*)^{-1}.$$

Proof: The statements are checked in a straightforward way. □

The following theorem characterizes all doubly-coprime factorizations such that $G = NM^{-1}$ is a J_L right coprime factorization. The state-space formulae for J_L- factorizations are not new. They are due to Meyer and Franklin [1987] for the strictly proper case and due to Vidyasagar [1988] for the general case. Our proof is however new in that the formulae are derived in a systematic way starting from the known existence of such factorizations.

Theorem 4.1 *Let (A, B, C, D) be a minimal realization of the proper rational function G of McMillan degree n.*

Then all doubly coprime factorizations of G, such that

1. *$G = NM^{-1}$ is a J_L-RF and $G = \tilde{M}^{-1}\tilde{N}$ is a J_L-LF,*
2. $$\begin{pmatrix} M & U \\ N & V \end{pmatrix}, \quad \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix},$$
are of McMillan degree n,

are given by

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \equiv \left(\begin{array}{c|cc} A-BF & BD_1 & BD_2 + H\overline{D}_1^{-1} \\ \hline -F & D_1 & D_2 \\ C-DF & DD_1 & \overline{D}_1^{-1} + DD_2 \end{array} \right),$$

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \equiv \left(\begin{array}{c|cc} A-HC & HD-B & -H \\ \hline -D_1^{-1}F - D_1^{-1}D_2\overline{D}_1C & D_1^{-1} + D_1^{-1}D_2\overline{D}_1D & -D_1^{-1}D_2\overline{D}_1 \\ \overline{D}_1C & -\overline{D}_1D & \overline{D}_1 \end{array} \right),$$

where

- D_1 *is such that* $D_1 D_1^* = (I + D^*D)^{-1}$.
- $\overline{D}_1$ *is such that* $\overline{D}_1^* \overline{D}_1 = (I + DD^*)^{-1}$.
- D_2 *is arbitrary* .
- Y *and* Z *are solutions of the Riccati equations*

$$0 = (A - B(I + D^*D)^{-1}D^*C)^*Y + Y(A - B(I + D^*D)^{-1}D^*C)$$

$$-YB(I + D^*D)^{-1}B^*Y + C^*(I + DD^*)^{-1}C,$$

$$0 = (A - B(I + D^*D)^{-1}D^*C)Z + Z(A - B(I + D^*D)^{-1}D^*C)^*$$

$$-ZC^*(I + DD^*)^{-1}CZ + B(I + D^*D)^{-1}B^*,$$

such that $A - BF$ *and* $A - HC$ *are stable, where*

$$F = (I + D^*D)^{-1}D^*C + (I + D^*D)^{-1}B^*Y,$$

$$H = BD^*(I + DD^*)^{-1} + ZC^*(I + DD^*)^{-1}.$$

Proof: Let G have a minimal realization (A, B, C, D). Let $G = NM^{-1}$ be a right factorization G such that

$$\begin{pmatrix} M^* & N^* \end{pmatrix} J_L \begin{pmatrix} M \\ N \end{pmatrix} = I.$$

Such a factorization exists by Proposition 4.1 and has the same McMillan degree as G. By Theorem 2.2 any right factorization $\begin{pmatrix} M \\ N \end{pmatrix}$ such that the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ is the same as that of G, has a state space realization of the form,

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BF & BD_1 \\ \hline -F & D_1 \\ C - DF & DD_1 \end{array} \right),$$

where F is a stabilizing state feedback and D_1 is invertible. Similarly, a left factorization $G = \tilde{M}^{-1}\tilde{N}$ such that

$$\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix} J_L \begin{pmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{pmatrix} = I,$$

exists and is of McMillan degree n and has a state-space representation of the form

$$\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - HC & HD - B & -H \\ \hline \overline{D}_1 C & -\overline{D}_1 D & \overline{D}_1 \end{array} \right),$$

where H is a stabilizing output injection. Since $\begin{pmatrix} M \\ N \end{pmatrix}$ is stable and of McMillan degree n and $\begin{pmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{pmatrix}$ is antistable and also of McMillan degree n, the function

$$\begin{pmatrix} M & -\tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix} \equiv \left(\begin{array}{cc|cc} A-BF & 0 & BD_1 & 0 \\ 0 & -A^*+C^*H^* & 0 & C^*\overline{D}_1^* \\ \hline -F & B^*-D^*H^* & D_1 & -D^*\overline{D}_1^* \\ C-DF & H^* & DD_1 & \overline{D}_1^* \end{array}\right)$$

$$=: \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)$$

has McMillan degree $2n$. Note that since

$$\begin{pmatrix} M & -\tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix}^{-*} = \begin{pmatrix} M & -\tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix},$$

we have by Lemma 3.1 that $\begin{pmatrix} M & -\tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix}$ has two equivalent realizations

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)$$

and

$$\left(\begin{array}{c|c} -\mathcal{A}^* + \mathcal{C}^*\mathcal{D}\mathcal{B}^* & \mathcal{C}^*\mathcal{D} \\ \hline \mathcal{D}\mathcal{B}^* & \mathcal{D} \end{array}\right).$$

Since $\mathcal{D} = \mathcal{D}^{-1}$, Lemma 4.1 implies that $\mathcal{D}$ and $\overline{D}_1$ are such that

$$D_1 D_1^* = (I + D^*D)^{-1},$$

$$\overline{D}_1^*\overline{D}_1 = (I + DD^*)^{-1}.$$

We need to compute $-\mathcal{A}^* + \mathcal{C}^*\mathcal{D}\mathcal{B}^*$,

$$\begin{aligned}
&-\mathcal{A}^* + \mathcal{C}^*\mathcal{D}\mathcal{B}^* \\
&= \begin{pmatrix} -A^* + F^*B^* & 0 \\ 0 & A - HC \end{pmatrix} \\
&\quad + \begin{pmatrix} -F^* & C^* - F^*D^* \\ B - HD & H \end{pmatrix} \begin{pmatrix} I & -D^* \\ D & I \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & \overline{D}_1^* \end{pmatrix} \begin{pmatrix} D_1^*B^* & 0 \\ 0 & \overline{D}_1 C \end{pmatrix} \\
&= \begin{pmatrix} -A^* + F^*B^* & 0 \\ 0 & A - HC \end{pmatrix} \\
&\quad + \begin{pmatrix} -F^*(I + D^*D) + C^*D & C^* \\ B & -BD^* + H(I + DD^*) \end{pmatrix} \begin{pmatrix} D_1D_1^* & 0 \\ 0 & \overline{D}_1^*\overline{D}_1 \end{pmatrix} \begin{pmatrix} B^* & 0 \\ 0 & C \end{pmatrix} \\
&= \begin{pmatrix} -A^* + C^*DD_1D_1^*B^* & C^*\overline{D}_1^*\overline{D}_1 C \\ BD_1D_1^*B^* & A - BD^*\overline{D}_1^*\overline{D}_1 C \end{pmatrix}.
\end{aligned}$$

Since $\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)$ and $\left(\begin{array}{c|c} -\mathcal{A}^* + \mathcal{C}^*\mathcal{D}\mathcal{B}^* & \mathcal{C}^*\mathcal{D} \\ \hline \mathcal{D}\mathcal{B}^* & \mathcal{D} \end{array}\right)$ are both minimal realizations of $\begin{pmatrix} M & -\tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix}$, there exists a unique non-singular state-space transformation $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$, which by Lemma 3.1 is such that $\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} Y_{11}^* & Y_{21}^* \\ Y_{12}^* & Y_{22}^* \end{pmatrix}$ and such that

$$\mathcal{C} = \mathcal{D}\mathcal{B}^*Y,$$

$$Y\mathcal{B} = \mathcal{C}^*\mathcal{D},$$

$$Y\mathcal{A} = (-\mathcal{A}^* + \mathcal{C}^*\mathcal{D}\mathcal{B}^*)Y.$$

More explicitly, we have writing the equation $Y\mathcal{B} = \mathcal{C}^*\mathcal{D}$ componentwise,

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix}\begin{pmatrix} BD_1 & 0 \\ 0 & C^*\overline{D}_1^* \end{pmatrix}$$

$$= \begin{pmatrix} -F^* & C^* - F^*D^* \\ B - HD & H \end{pmatrix}\begin{pmatrix} I & -D^* \\ D & I \end{pmatrix}\begin{pmatrix} D_1 & 0 \\ 0 & \overline{D}_1^* \end{pmatrix}$$

and therefore

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & C^* \end{pmatrix} = \begin{pmatrix} -F^*(I + D^*D) + C^*D & C^* \\ B & -BD^* + H(I + DD^*) \end{pmatrix}.$$

Hence, we have that

$$-F^*(I + D^*D) + C^*D = Y_{11}B,$$

$$C^* = Y_{12}C^*,$$

$$H(I + DD^*) - BD^* = Y_{22}C^*,$$

$$B = Y_{12}^*B,$$

which shows that

$$F = (I + D^*D)^{-1}D^*C - (I + D^*D)^{-1}B^*Y_{11},$$

$$H = BD^*(I + DD^*)^{-1} + Y_{22}C^*(I + DD^*)^{-1}.$$

Writing $Y\mathcal{A} = (-\mathcal{A}^* + \mathcal{C}^*\mathcal{D}\mathcal{B}^*)Y$, componentwise, we have for the $(1,1)$ entry,

$$Y_{11}(A - BF) = (-A^* + C^*DD_1D_1^*B^*)Y_{11} + C^*\overline{D}_1^*\overline{D}_1CY_{12}^*,$$

and using the above identities, this gives,

$$0 = (A^* - C^*DD_1D_1^*B^*)Y_{11} + Y_{11}(A - B[(I + D^*D)^{-1}D^*C - (I + D^*D)^{-1}B^*Y_{11}])$$

$$-C^*\overline{D}_1^*\overline{D}_1C$$

$$= (A^* - C^*D(I + D^*D)^{-1}B^*)Y_{11} + Y_{11}(A - B(I + D^*D)^{-1}D^*C) +$$

$$Y_{11}B(I+D^*D)^{-1}B^*Y_{11} - C^*(I+DD^*)^{-1}C.$$

Setting $Y := -Y_{11}$ we obtain the Riccati equation

$$0 = (A - B(I+D^*D)^{-1}D^*C)^*Y + Y(A - B(I+D^*D)^{-1}D^*C)$$
$$-YB(I+D^*D)^{-1}B^*Y + C^*(I+DD^*)^{-1}C.$$

Moreover, with $F = (I+D^*D)^{-1}D^*C + (I+D^*D)^{-1}B^*Y$ we have that $A - BF$ is stable.

Evaluating the $(2,2)$ entry we obtain,

$$Y_{22}(-A^* + C^*H^*) = BD_1D_1^*B^*Y_{12} + (A - BD^*\overline{D_1^*D_1}C)Y_{22},$$

or,

$$0 = (A - BD^*(I+DD^*)^{-1}C)Z + Z(A^* - C^*(I+DD^*)^{-1}DB^*)$$
$$-ZC^*(I+DD^*)^{-1}CZ + B(I+D^*D)^{-1}B^*,$$

where we have set $Z := Y_{22}$. Note that $A - HC$ is stable with $H = BD^*(I+DD^*)^{-1} + ZC^*(I+DD^*)^{-1}$.

It can be verified in a straightforward but tedious way that if state space representations are given as in the statement of the theorem that the transfer functions of these representations give doubly coprime factorizations with the required properties.

□

In the proof of the theorem we also established the well-known result that the algebraic Riccati equation has a stabilizing solution.

Corollary 4.1 *Let (A,B,C,D) be a minimal continuous-time system. Then there exist hermitian solutions Y and Z of the Riccati equations*

$$0 = (A - B(I+D^*D)^{-1}D^*)^*Y + Y(A - B(I+D^*D)^{-1}D^*C)$$
$$-YB(I+D^*D)^{-1}B^*Y + C^*(I+DD^*)^{-1}C,$$

respectively,

$$0 = (A - B(I+D^*D)^{-1}D^*C)Z + Z(A - B(I+D^*D)^{-1}D^*C)^*$$
$$-ZC^*(I+DD^*)^{-1}CZ + B(I+D^*D)^{-1}B^*,$$

such that $A - BF$ and $A - HC$ are stable, where

$$F = (I+D^*D)^{-1}D^*C + (I+D^*D)^{-1}B^*Y$$
$$H = BD^*(I+DD^*)^{-1} + ZC^*(I+DD^*)^{-1}.$$

Proof: This statement was prcved as part of the proof of the theorem. □

5 Bounded-real functions

In this section we are going to consider stable rational functions that are bounded in magnitude by 1 in the right half plane. We are going to define factorizations for this class of functions and are going to give the corresponding doubly coprime factorizations.

Definition 5.1 *A proper stable rational function is called* bounded real, *if*

$$I - G^*(i\omega)G(i\omega) > 0,$$

for all $\omega \in \Re \cup \{\pm\infty\}$. *Let*

$$J_B = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

A right (coprime) factorization of $G = NM^{-1}$ *is called a* J_B-normalized right (coprime) factorization *(*$J_B - RF$ *respectively* $J_B - RCF$*) of* G *if*

$$\begin{pmatrix} M^* & N^* \end{pmatrix} J_B \begin{pmatrix} M \\ N \end{pmatrix} = I.$$

Similarly, a left (coprime) factorization is called a J_B-normalized left (coprime) factorization *(*$J_B - LF$ *respectively* $J_B - LCF$*) of* G *if*

$$-\begin{pmatrix} \tilde{N} & \tilde{M} \end{pmatrix} J_B \begin{pmatrix} \tilde{N}^* \\ \tilde{M}^* \end{pmatrix} = I.$$

The existence and uniqueness of such factorizations is established in the following proposition.

Proposition 5.1 *Let* G *be a proper rational bounded-real function of McMillan degree* n. *Then there exists a* J_B*-right (left) factorization,*

$$G = NM^{-1} \quad (G = \tilde{M}^{-1}\tilde{N}).$$

This factorization is right (left) coprime and is unique up to right (left) multiplication by a constant unitary matrix. All such factorizations are such that $\begin{pmatrix} M \\ N \end{pmatrix}$ *(*$\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix}$*) is of McMillan degree* n.

Proof: Let $G = ED^{-1}$ be a right polynomial coprime factorization. Let T be a square stable spectral factor of $D^*D - E^*E$, i.e. $T^*T = D^*D - E^*E$. Then

$$\begin{pmatrix} M \\ N \end{pmatrix} := \begin{pmatrix} DT^{-1} \\ ET^{-1} \end{pmatrix}$$

defines by Proposition 2.1 right factors of G, i.e. $G = NM^{-1}$ with N, M and M^{-1} proper and $\begin{pmatrix} M \\ N \end{pmatrix}$ of McMillan degree n. This also implies by Corollary 2.3 that the factorization is coprime. The remaining parts of the proof follow in the standard way. □

Some properties of J_B-factorizations are summarized in the following lemma.

Lemma 5.1 *Let $G = NM^{-1}$ be a J_B-RF and $G = \tilde{M}^{-1}\tilde{N}$ a J_B-LF of the stable bounded-real function G. Then*

1. $$\begin{pmatrix} M^* & N^* \\ \tilde{N} & \tilde{M} \end{pmatrix} J_B \begin{pmatrix} M & \tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

2. $$\begin{pmatrix} M & \tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix} = J_B \begin{pmatrix} M & \tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix}^{-*} J_B.$$

Proof: The statements are easily verified. □

The following Lemma will be useful in the proof the subsequent theorem.

Lemma 5.2 *Consider*

$$\begin{pmatrix} D_1 & D^*\overline{D}_1^* \\ DD_1 & \overline{D}_1^* \end{pmatrix} = \begin{pmatrix} I & D^* \\ D & I \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & \overline{D}_1^* \end{pmatrix},$$

with D_1 and $\overline{D}_1$ invertible and D such that $I - DD^* > 0$. Then

1. $$\begin{pmatrix} D_1 & D^*\overline{D}_1^* \\ DD_1 & \overline{D}_1^* \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} D_1^{-1} & 0 \\ 0 & \overline{D}_1^{-*} \end{pmatrix} \begin{pmatrix} I & -D^* \\ -D & I \end{pmatrix} \begin{pmatrix} (I - D^*D)^{-1} & 0 \\ 0 & (I - DD^*)^{-1} \end{pmatrix}.$$

2. *If*

$$\begin{pmatrix} D_1 & D^*\overline{D}_1^* \\ DD_1 & \overline{D}_1^* \end{pmatrix} = J_B \begin{pmatrix} D_1 & D^*\overline{D}_1^* \\ DD_1 & \overline{D}_1^* \end{pmatrix}^{-*} J_B,$$

then

$$D_1 D_1^* = (I - D^*D)^{-1},$$

$$\overline{D}_1^*\overline{D}_1 = (I - DD^*)^{-1}.$$

Proof: The statements are checked in a straightforward way. □

We are now in a position to characterize all doubly coprime factorizations so that $G = NM^{-1}(= \tilde{M}^{-1}\tilde{N})$ is a J_B-RF (J_B-LF).

Theorem 5.1 *Let (A, B, C, D) be a minimal realization of the proper bounded-real rational function G of McMillan degree n.*

Then all doubly coprime factorizations of G, such that

1. *$G = NM^{-1}$ is a J_B-RF and $G = \tilde{M}^{-1}\tilde{N}$ is a J_B-LF,*

2. $$\begin{pmatrix} M & U \\ N & V \end{pmatrix}, \quad \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix},$$

are of McMillan degree n,

are given by

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - BF & BD_1 & BD_2 + H\overline{D}_1^{-1} \\ \hline -F & D_1 & D_2 \\ C - DF & DD_1 & \overline{D}_1^{-1} + DD_2 \end{array}\right),$$

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - HC & HD - B & -H \\ \hline -D_1^{-1}F - D_1^{-1}D_2\overline{D}_1 C & D_1^{-1} + D_1^{-1}D_2\overline{D}_1 D & -D_1^{-1}D_2\overline{D}_1 \\ \overline{D}_1 C & -\overline{D}_1 D & \overline{D}_1 \end{array}\right),$$

where

- D_1 *is such that* $D_1 D_1^* = (I - D^*D)^{-1}$.
- $\overline{D}_1$ *is such that* $\overline{D}_1^*\overline{D}_1 = (I - DD^*)^{-1}$.
- D_2 *is arbitrary* .
- Y *and* Z *are solutions of the Riccati equations*

$$0 = (A + B(I - D^*D)^{-1}D^*C)^*Y + Y(A + B(I - D^*D)^{-1}D^*C)$$
$$+YB(I - D^*D)^{-1}B^*Y + C^*(I - DD^*)^{-1}C,$$
$$0 = (A + B(I - D^*D)^{-1}D^*C)Z + Z(A + B(I - D^*D)^{-1}D^*C)^*$$
$$+ZC^*(I - DD^*)^{-1}CZ + B(I - D^*D)^{-1}B^*,$$

such that $A - BF$ *and* $A - HC$ *are stable, where*

$$F = -(I - D^*D)^{-1}D^*C - (I - D^*D)^{-1}B^*Y,$$
$$H = -BD^*(I - DD^*)^{-1} - ZC^*(I - DD^*)^{-1}.$$

Proof: Let G have a minimal realization (A, B, C, D). Let $G = NM^{-1}$ be a right factorization G such that

$$\begin{pmatrix} M^* & N^* \end{pmatrix} J_B \begin{pmatrix} M \\ N \end{pmatrix} = I.$$

Such a factorization exists by Proposition 4.1 and has the same McMillan degree as G. By Theorem 2.2 any right factorization $\begin{pmatrix} M \\ N \end{pmatrix}$ such that the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ is the same as that of G has a state space realization of the form,

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BF & BD_1 \\ \hline -F & D_1 \\ C - DF & DD_1 \end{array}\right),$$

where F is a stabilizing state feedback and D_1 is invertible. Similarly, a left factorization $G = \tilde{M}^{-1}\tilde{N}$ such that

$$\begin{pmatrix} \tilde{N} & \tilde{M} \end{pmatrix} J_B \begin{pmatrix} \tilde{N}^* \\ \tilde{M}^* \end{pmatrix} = -I,$$

exists and is of McMillan degree n and has a state-space representation of the form

$$\begin{pmatrix} \tilde{N} & \tilde{M} \end{pmatrix} \equiv \left(\begin{array}{c|cc} A-HC & B-HD & -H \\ \hline \overline{D}_1 C & \overline{D}_1 D & \overline{D}_1 \end{array}\right),$$

where H is a stabilizing output injection. Since $\begin{pmatrix} M \\ N \end{pmatrix}$ is stable and of McMillan degree n and $\begin{pmatrix} \tilde{N}^* \\ \tilde{M}^* \end{pmatrix}$ is antistable and also of McMillan degree n, the function

$$\begin{pmatrix} M & \tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix} \equiv \left(\begin{array}{cc|cc} A-BF & 0 & BD_1 & 0 \\ 0 & -A^*+C^*H^* & 0 & C^*\overline{D}_1^* \\ \hline -F & D^*H^*-B^* & D_1 & D^*\overline{D}_1^* \\ C-DF & H^* & DD_1 & \overline{D}_1^* \end{array}\right)$$

$$=: \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)$$

has McMillan degree $2n$. Note that since

$$\begin{pmatrix} M & \tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix} = J_B \begin{pmatrix} M & \tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix}^{-*} J_B,$$

we have by Lemma 3.1 that $\begin{pmatrix} M & \tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix}$ has the two equivalent realizations

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)$$

and

$$\left(\begin{array}{c|c} -\mathcal{A}^* + \mathcal{C}^* J_B \mathcal{D} J_B \mathcal{B}^* & \mathcal{C}^* J_B \mathcal{D} \\ \hline \mathcal{D} J_B \mathcal{B}^* & \mathcal{D} \end{array}\right).$$

Since $\mathcal{D} = J_B \mathcal{D}^{-1} J_B$, Lemma 4.1 implies that $\mathcal{D}$ and $\overline{D}_1$ are such that

$$D_1 D_1^* = (I - D^* D)^{-1},$$

$$\overline{D}_1^* \overline{D}_1 = (I - DD^*)^{-1}.$$

We need to compute $-\mathcal{A}^* + \mathcal{C}^* J_B \mathcal{D} J_B \mathcal{B}^*$,

$$-\mathcal{A}^* + \mathcal{C}^* J_B \mathcal{D} J_B \mathcal{B}^*$$

$$= \begin{pmatrix} -A^* + F^* B^* & 0 \\ 0 & A - HC \end{pmatrix}$$

$$+ \begin{pmatrix} -F^* & C^* - F^* D^* \\ HD - B & H \end{pmatrix} \begin{pmatrix} I & -D^* \\ -D & I \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & \overline{D}_1^* \end{pmatrix} \begin{pmatrix} D_1^* B^* & 0 \\ 0 & \overline{D}_1 C \end{pmatrix}$$

$$= \begin{pmatrix} -A^* + F^* B^* & 0 \\ 0 & A - HC \end{pmatrix}$$

$$+\begin{pmatrix} -F^*(I-D^*D)-C^*D & C^* \\ -B & BD^*+H(I-DD^*) \end{pmatrix}\begin{pmatrix} D_1D_1^* & 0 \\ 0 & \overline{D}_1^*\overline{D}_1 \end{pmatrix}\begin{pmatrix} B^* & 0 \\ 0 & C \end{pmatrix}$$

$$=\begin{pmatrix} -A^*-C^*DD_1D_1^*B^* & C^*\overline{D}_1^*\overline{D}_1C \\ -BD_1D_1^*B^* & A+BD^*\overline{D}_1^*\overline{D}_1C \end{pmatrix}.$$

Since $\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)$ and $\left(\begin{array}{c|c} -\mathcal{A}^*+\mathcal{C}^*J_B\mathcal{D}J_B\mathcal{B}^* & \mathcal{C}^*J_B\mathcal{D} \\ \hline \mathcal{D}J_B\mathcal{B}^* & \mathcal{D} \end{array}\right)$ are both minimal realizations of $\begin{pmatrix} M & \tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix}$, there exists a unique non-singular state-space transformation $Y=\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$, which by Lemma 3.1 is such that $\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}=\begin{pmatrix} Y_{11}^* & Y_{21}^* \\ Y_{12}^* & Y_{22}^* \end{pmatrix}$, and such that

$$\mathcal{C}=\mathcal{D}J_B\mathcal{B}^*Y,$$

$$Y\mathcal{B}=\mathcal{C}^*J_B\mathcal{D},$$

$$Y\mathcal{A}=(-\mathcal{A}^*+\mathcal{C}^*J_B\mathcal{D}J_B\mathcal{B}^*)Y.$$

More explicitly, we have writing the equation $Y\mathcal{B}=\mathcal{C}^*J_B\mathcal{D}$ componentwise,

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix}\begin{pmatrix} BD_1 & 0 \\ 0 & C^*\overline{D}_1^* \end{pmatrix}$$

$$=\begin{pmatrix} -F^* & C^*-F^*D^* \\ HD-B & H \end{pmatrix}\begin{pmatrix} I & D^* \\ -D & -I \end{pmatrix}\begin{pmatrix} D_1 & 0 \\ 0 & \overline{D}_1^* \end{pmatrix}$$

and therefore

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & C^* \end{pmatrix}=\begin{pmatrix} -F^*(I-D^*D)-C^*D & -C^* \\ -B & -BD^*-H(I-DD^*) \end{pmatrix}.$$

Hence, we have that

$$-F^*(I-D^*D)-C^*D=Y_{11}B,$$

$$-C^*=Y_{12}C^*,$$

$$-H(I-DD^*)-BD^*=Y_{22}C^*,$$

$$-B=Y_{12}^*B,$$

which shows that

$$F=-(I-D^*D)^{-1}D^*C-(I-D^*D)^{-1}B^*Y_{11},$$

$$H=-BD^*(I-DD^*)^{-1}-Y_{22}C^*(I-DD^*)^{-1}.$$

Writing $Y\mathcal{A}=(-\mathcal{A}^*+\mathcal{C}^*J_B\mathcal{D}J_B\mathcal{B}^*)Y$, componentwise, we have for the $(1,1)$ entry,

$$Y_{11}(A-BF)=(-A^*-C^*DD_1D_1^*B^*)Y_{11}+C^*\overline{D}_1^*\overline{D}_1CY_{12}^*,$$

and using the above identities, this gives,

$$0 = (A^* + C^*DD_1D_1^*B^*)Y_{11} + Y_{11}(A + B[(I - D^*D)^{-1}D^*C + (I - D^*D)^{-1}B^*Y_{11}])$$

$$+C^*\overline{D_1^*}\overline{D}_1C$$

$$= (A^* + C^*D(I - D^*D)^{-1}B^*)Y + Y(A + B(I - D^*D)^{-1}D^*C)+$$

$$YB(I - D^*D)^{-1}B^*Y + C^*(I - DD^*)^{-1}C,$$

where we have set $Y := Y_{11}$. Evaluating the $(2,2)$ entry we obtain,

$$Y_{22}(-A^* + C^*H^*) = -BD_1D_1^*B^*Y_{12} + (A + BD^*\overline{D_1^*}\overline{D}_1C)Y_{22},$$

or,

$$0 = (A + BD^*(I - DD^*)^{-1}C)Z + Z(A^* + C^*(I - DD^*)^{-1}DB^*)$$

$$+ZC^*(I - DD^*)^{-1}CZ + B(I - D^*D)^{-1}B^*,$$

where we have set $Z := Y_{22}$. Note that $A - BF$ and $A - HC$ are stable with F and H as above.

It can be verified in a straightforward but tedious way that if state space representations are given as in the statement of the theorem that the transfer functions of these representations give doubly coprime factorizations with the required properties.

□

In the proof of the theorem we also gave a proof of the well-known result that the so called bounded-real Riccati equation has a stabilizing solution.

Corollary 5.1 *Let (A, B, C, D) be a minimal realization of a bounded-real rational function. Then there exist hermitian solutions Y respectively Z to the Riccati equations*

$$0 = (A + B(I - D^*D)^{-1}D^*B^*C)^* + Y(A + B(I - D^*D)^{-1}D^*C)+$$

$$YB(I - D^*D)^{-1}B^*Y + C^*(I - DD^*)^{-1}C,$$

respectively,

$$0 = (A + B(I - DD^*)^{-1}D^*C)Z + Z(A + B(I - DD^*)^{-1}D^*C)^*$$

$$+ZC^*(I - DD^*)^{-1}CZ + B(I - D^*D)^{-1}B^*,$$

such that $A - BF$ and $A - HC$ respectively are stable, where

$$F = -(I - D^*D)^{-1}D^*C - (I - D^*D)^{-1}B^*Y,$$

$$H = -BD^*(I - DD^*)^{-1} - ZC^*(I - DD^*)^{-1}.$$

Proof: This statement was proved as part of the proof of the theorem. □

6 Positive-real functions

Positive real functions are of importance in many areas of system and control theory, e.g. in stochastic system theory or in adaptive control. We are now going to define what we mean by positive-real functions and by the J_P-factorization of such functions.

Definition 6.1 *A proper square stable rational function is called* positive-real, *if*

$$G(i\omega) + G^*(i\omega) > 0,$$

for all $\omega \in \Re \cup \{\pm\infty\}$. *Let*

$$J_P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

A right (coprime) factorization of $G = NM^{-1}$ *is called a* J_P-normalized right (coprime) factorization *(*$J_P - RF$ *respectively* $J_P - RCF$*) of* G *if*

$$\begin{pmatrix} M^* & N^* \end{pmatrix} J_P \begin{pmatrix} M \\ N \end{pmatrix} = I.$$

Similarly, a left (coprime) factorization is called a J_P-normalized left (coprime) factorization *(*$J_P - LF$ *respectively* $J_P - LCF$*) of* G *if*

$$\begin{pmatrix} \tilde{N} & \tilde{M} \end{pmatrix} J_P \begin{pmatrix} \tilde{N}^* \\ \tilde{M}^* \end{pmatrix} = I.$$

The following proposition establishes the existence of J_P-factorizations.

Proposition 6.1 *Let* G *be a proper rational positive-real function of McMillan degree* n. *Then there exists a* J_P*-right (left) factorization,*

$$G = NM^{-1} \quad (G = \tilde{M}^{-1}\tilde{N}).$$

This factorization is right (left) coprime and is unique up to right (left) multiplication by a constant unitary matrix. All such factorizations are such that $\begin{pmatrix} M \\ N \end{pmatrix}$ $(\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix})$ *is of McMillan degree* n.

Proof: Let $G = ED^{-1}$ be a right polynomial coprime factorization. Let T be a square stable spectral factor of $E^*D + D^*E$, i.e. $T^*T = E^*D + D^*E$ and proceed as previously. □

Some properties of J_P-factorizations are summarized in the following lemma.

Lemma 6.1 *Let* $G = NM^{-1}$ *be a* J_P*-RF and* $G = \tilde{M}^{-1}\tilde{N}$ *a* J_P*-LF of the positive-real function* G. *Then*

1. $$\begin{pmatrix} M & -\tilde{M}^* \\ N & \tilde{N}^* \end{pmatrix}^* J_P \begin{pmatrix} M & -\tilde{M}^* \\ N & \tilde{N}^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

2. $$\begin{pmatrix} M & -\tilde{M}^* \\ N & \tilde{N}^* \end{pmatrix} = J_P \begin{pmatrix} M & -\tilde{M}^* \\ N & \tilde{N}^* \end{pmatrix}^{-*} J_B.$$

Proof: The statements are easily verified. □

A few useful identities are given in the following Lemma.

Lemma 6.2 *Consider*

$$\begin{pmatrix} D_1 & -\overline{D}_1^* \\ DD_1 & D^*\overline{D}_1^* \end{pmatrix} = \begin{pmatrix} I & -I \\ D & D^* \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & \overline{D}_1^* \end{pmatrix},$$

with D_1 and $\overline{D}_1$ invertible and D is square such that $D + D^* > 0$. Then

1. $$\begin{pmatrix} D_1 & -\overline{D}_1^* \\ DD_1 & D^*\overline{D}_1^* \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} D_1^{-1} & 0 \\ 0 & \overline{D}_1^{-*} \end{pmatrix} \begin{pmatrix} D^* & I \\ -D & I \end{pmatrix} \begin{pmatrix} (D+D^*)^{-1} & 0 \\ 0 & (D+D^*)^{-1} \end{pmatrix}.$$

2. *If*

$$\begin{pmatrix} D_1 & -\overline{D}_1^* \\ DD_1 & D^*\overline{D}_1^* \end{pmatrix} = J_P \begin{pmatrix} D_1 & -\overline{D}_1^* \\ DD_1 & D^*\overline{D}_1^* \end{pmatrix}^{-*} J_B,$$

then

$$D_1 D_1^* = (D + D^*)^{-1},$$

$$\overline{D}_1^*\overline{D}_1 = (D + D^*)^{-1}.$$

Proof: The statements are checked in a straightforward way. □

We can now characterize doubly coprime factorizations for positive-real functions.

Theorem 6.1 *Let (A, B, C, D) be a minimal realization of the proper positive-real rational function G of McMillan degree n.*

Then all doubly coprime factorizations of G, such that

1. *$G = NM^{-1}$ is a J_P-RF and $G = \tilde{M}^{-1}\tilde{N}$ is a J_B-LF,*
2. $$\begin{pmatrix} M & U \\ N & V \end{pmatrix}, \quad \begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix},$$
   are of McMillan degree n.

are given by

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - BF & BD_1 & BD_2 + H\overline{D}_1^{-1} \\ \hline -F & D_1 & D_2 \\ C - DF & DD_1 & \overline{D}_1^{-1} + DD_2 \end{array}\right),$$

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - HC & HD - B & -H \\ \hline -D_1^{-1}F - D_1^{-1}D_2\overline{D}_1C & D_1^{-1} + D_1^{-1}D_2\overline{D}_1D & -D_1^{-1}D_2\overline{D}_1 \\ \overline{D}_1C & -\overline{D}_1D & \overline{D}_1 \end{array}\right),$$

where

- D_1 *is such that* $D_1 D_1^* = (D + D^*)^{-1}$.
- $\overline{D}_1$ *is such that* $\overline{D}_1^* \overline{D}_1 = (D + D^*)^{-1}$.
- D_2 *is arbitrary* .
- Y *and* Z *are solutions of the Riccati equations*

$$0 = (A - B(D + D^*)^{-1}C)^*Y + Y(A - B(D + D^*)^{-1}C) + YB(D + D^*)^{-1}B^*Y + C^*(D + D^*)^{-1}C,$$

$$0 = (A - B(D + D^*)^{-1}C)Z + Z(A - B(D + D^*)^{-1}C) + ZC^*(D + D^*)^{-1}CZ + B(D + D^*)^{-1}B^*,$$

such that $A - BF$ *and* $A - HC$ *are stable, with*

$$F = (D + D^*)^{-1}C - (D + D^*)^{-1}B^*Y,$$

$$H = B(D + D^*)^{-1} - ZC^*(D + D^*)^{-1}.$$

Proof: Let G have a minimal realization (A, B, C, D). Let $G = NM^{-1}$ be a right factorization G such that

$$\begin{pmatrix} M^* & N^* \end{pmatrix} J_P \begin{pmatrix} M \\ N \end{pmatrix} = I.$$

Such a factorization exists by Proposition 4.1 and has the same McMillan degree as G. By Theorem 2.2 any right factorization $\begin{pmatrix} M \\ N \end{pmatrix}$ such that the McMillan degree of $\begin{pmatrix} M \\ N \end{pmatrix}$ is the same as that of G has a state space realization of the form,

$$\begin{pmatrix} M \\ N \end{pmatrix} \equiv \left(\begin{array}{c|c} A - BF & BD_1 \\ \hline -F & D_1 \\ C - DF & DD_1 \end{array}\right),$$

where F is a stabilizing state feedback and D_1 is invertible. Similarly, a left factorization $G = \tilde{M}^{-1}\tilde{N}$ such that

$$\begin{pmatrix} -\tilde{M} & \tilde{N} \end{pmatrix} J_P \begin{pmatrix} -\tilde{M}^* \\ \tilde{N}^* \end{pmatrix} = -I,$$

exists and is of McMillan degree n and has a state-space representation of the form

$$\begin{pmatrix} -\tilde{M} & \tilde{N} \end{pmatrix} \equiv \left(\begin{array}{c|cc} A - HC & H & B - HD \\ \hline D_1C & -D_1 & D_1D \end{array}\right),$$

where H is a stabilizing output injection. Since $\begin{pmatrix} M \\ N \end{pmatrix}$ is stable and of McMillan degree n and $\begin{pmatrix} -\tilde{M}^* \\ \tilde{N}^* \end{pmatrix}$ is antistable and also of McMillan degree n, the function

$$\begin{pmatrix} M & -\tilde{M}^* \\ N & \tilde{N}^* \end{pmatrix} \equiv \left(\begin{array}{cc|cc} A-BF & 0 & BD_1 & 0 \\ 0 & -A^*+C^*H^* & 0 & C^*\overline{D}_1^* \\ \hline -F & -H^* & D_1 & -\overline{D}_1^* \\ C-DF & D^*H^*-B^* & DD_1 & D^*\overline{D}_1^* \end{array}\right)$$

$$=: \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)$$

has McMillan degree $2n$. Note that since

$$\begin{pmatrix} M & -\tilde{M}^* \\ N & \tilde{N}^* \end{pmatrix} = J_P \begin{pmatrix} M & -\tilde{M}^* \\ N & \tilde{N}^* \end{pmatrix}^{-*} J_B,$$

we have by Lemma 3.1 that $\begin{pmatrix} M & -\tilde{M}^* \\ N & \tilde{N}^* \end{pmatrix}$ has the two equivalent realizations

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)$$

and

$$\left(\begin{array}{c|c} -\mathcal{A}^* + \mathcal{C}^* J_P \mathcal{D} J_B \mathcal{B}^* & \mathcal{C}^* J_P \mathcal{D} \\ \hline \mathcal{D} J_B \mathcal{B}^* & \mathcal{D} \end{array}\right).$$

Since $\mathcal{D} = J_p \mathcal{D}^{-*} J_B$, Lemma 4.1 implies that $\mathcal{D}$ and $\overline{D}_1$ are such that

$$D_1 D_1^* = (D + D^*)^{-1},$$

$$\overline{D}_1^* \overline{D}_1 = (D + D^*)^{-1}.$$

We need to compute $-\mathcal{A}^* + \mathcal{C}^* J_P \mathcal{D} J_B \mathcal{B}^*$,

$$-\mathcal{A}^* + \mathcal{C}^* J_P \mathcal{D} J_B \mathcal{B}^*$$

$$= \begin{pmatrix} -A^* + F^*B^* & 0 \\ 0 & A - HC \end{pmatrix}$$

$$+ \begin{pmatrix} -F^* & C^* - F^*D^* \\ -H & HD - B \end{pmatrix} J_P \begin{pmatrix} I & -I \\ D & D^* \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & \overline{D}_1^* \end{pmatrix} J_B \begin{pmatrix} D_1^* B^* & 0 \\ 0 & \overline{D}_1 C \end{pmatrix}.$$

$$= \begin{pmatrix} -A^* + F^*B^* & 0 \\ 0 & A - HC \end{pmatrix}$$

$$+ \begin{pmatrix} -F^* & C^* - F^*D^* \\ -H & HD - B \end{pmatrix} \begin{pmatrix} D & -D^* \\ I & I \end{pmatrix} \begin{pmatrix} D_1 D_1^* & 0 \\ 0 & \overline{D}_1^* \overline{D}_1 \end{pmatrix} \begin{pmatrix} B^* & 0 \\ 0 & C \end{pmatrix}$$

$$= \begin{pmatrix} -A^* + F^*B^* & 0 \\ 0 & A - HC \end{pmatrix}$$

$$+ \begin{pmatrix} -F^*(D + D^*) + C^* & C^* \\ -B & H(D + D^*) - B \end{pmatrix} \begin{pmatrix} D_1 D_1^* & 0 \\ 0 & \overline{D}_1^* \overline{D}_1 \end{pmatrix} \begin{pmatrix} B^* & 0 \\ 0 & C \end{pmatrix}$$

$$= \begin{pmatrix} -A^* + C^*D_1D_1^*B^* & C^*\overline{D}_1^*\overline{D}_1C \\ -BD_1D_1^*B^* & A - B\overline{D}_1^*\overline{D}_1C \end{pmatrix}.$$

Since $\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)$ and $\left(\begin{array}{c|c} -\mathcal{A}^* + \mathcal{C}^*J_P\mathcal{D}J_B\mathcal{B}^* & \mathcal{C}^*J_P\mathcal{D} \\ \hline \mathcal{D}J_B\mathcal{B}^* & \mathcal{D} \end{array}\right)$ are both minimal realizations of $\begin{pmatrix} M & -\tilde{M}^* \\ N & \tilde{N}^* \end{pmatrix}$, there exists a unique non-singular state-space transformation $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$, which by Lemma 3.1 is such that $\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} Y_{11}^* & Y_{21}^* \\ Y_{12}^* & Y_{22}^* \end{pmatrix}$, and such that

$$\mathcal{C} = \mathcal{D}J_B\mathcal{B}^*Y,$$

$$Y\mathcal{B} = \mathcal{C}^*J_P\mathcal{D},$$

$$Y\mathcal{A} = (-\mathcal{A}^* + \mathcal{C}^*J_P\mathcal{D}J_B\mathcal{B}^*)Y.$$

More explicitly, we have writing the equation $Y\mathcal{B} = \mathcal{C}^*J_P\mathcal{D}$ componentwise,

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix}\begin{pmatrix} BD_1 & 0 \\ 0 & C^*\overline{D}_1^* \end{pmatrix}$$

$$= \begin{pmatrix} -F^* & C^* - F^*D^* \\ -H & HD - B \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} I & -I \\ D & D^* \end{pmatrix}\begin{pmatrix} D_1 & 0 \\ 0 & \overline{D}_1^* \end{pmatrix}$$

and therefore

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & C^* \end{pmatrix} = \begin{pmatrix} -F^*(D+D^*) + C^* & -C^* \\ -B & -H(D+D^*) + B \end{pmatrix}.$$

Hence, we have that

$$-F^*(D+D^*) + C^* = Y_{11}B,$$

$$-C^* = Y_{12}C^*,$$

$$-H(D+D^*) + B = Y_{22}C^*,$$

$$-B = Y_{12}^*B,$$

which shows that

$$F = (D+D^*)^{-1}C - (D+D^*)^{-1}B^*Y_{11},$$

$$H = B(D+D^*)^{-1} - Y_{22}C^*(D+D^*)^{-1}.$$

Writing $Y\mathcal{A} = (-\mathcal{A}^* + \mathcal{C}^*J_P\mathcal{D}J_B\mathcal{B}^*)Y$, componentwise, we have for the $(1,1)$ entry,

$$Y_{11}(A - BF) = (-A^* + C^*D_1D_1^*B^*)Y_{11} + C^*\overline{D}_1^*\overline{D}_1CY_{12}^*,$$

and using the above identities, this gives,

$$0 = (A^* - C^*D_1D_1^*B^*)Y_{11} + Y_{11}(A - B[(D+D^*)^{-1}C + (D+D^*)^{-1}B^*Y_{11}])$$

$$+C^*\overline{D}_1^*\overline{D}_1C$$

$$= (A^* - C^*(D+D^*)^{-1}B^*)Y + Y(A - B(D+D^*)^{-1}C) +$$
$$YB(D+D^*)^{-1}B^*Y + C^*(D+D^*)^{-1}C,$$

where we have set $Y := Y_{11}$. Evaluating the $(2,2)$ entry we obtain,

$$Y_{22}(-A^* + C^*H^*) = -BD_1D_1^*B^*Y_{12} + (A - B\overline{D}_1^*\overline{D}_1C)Y_{22},$$

or,

$$0 = (A - B(D+D^*)^{-1}C)Z + Z(A^* - C^*(D+D^*)^{-1}B^*)$$
$$+ZC^*(D+D^*)^{-1}CZ + B(D+D^*)^{-1}B^*.$$

The remaining part of the proof is analogous to the equivalent steps in the previous theorems. □

In the proof of the theorem we also established the well-known result that positive-real Riccati equations have a stabilizing solution.

Corollary 6.1 *Let (A, B, C, D) be a minimal realization of a positive-real rational function. Then there exist hermitian solutions Y respectively Z to the Riccati equations*

$$0 = (A^* - C^*(D+D^*)^{-1}B^*)Y + Y(A - B(D+D^*)^{-1}C) +$$
$$YB(D+D^*)^{-1}B^*Y + C^*(D+D^*)^{-1}C.$$

respectively,

$$0 = (A - B(D+D^*)^{-1}C)Z + Z(A^* - C^*(D+D^*)^{-1}B^*)$$
$$+ZC^*(D+D^*)^{-1}Z + B(D+D^*)^{-1}B^*.$$

such that $A - BF$ and $A - HC$ are stable, where

$$F = (D+D^*)^{-1}C - (D+D^*)^{-1},$$
$$H = B(D+D^*)^{-1} - ZC^*(D+D^*)^{-1}.$$

Proof: This statement was proved as part of the proof of the theorem. □

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