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APPROXIMATION PROBLEMS

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November 1992

MAT 144/620-81



439

264/3

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Dedicated to L. Markus on the occasion of his 70th birthday

October 25, 1992

Abstract

The polynomial approach introduced in Fuhrmann [1991] is extended to cover the crucial area of AAK theory, namely the characterization of zero location of the Schmidt vectors of the Hankel operators. This is done using the duality theory developed in that paper but with a twist. First we get the standard, lower bound, estimates on the number of unstable zeroes of the minimal degree Schmidt vectors of the Hankel operator. In the case of the Schmidt vector corresponding to the smallest singular the lower bound is in fact achieved. This leads to a solution of a Bezout equation. We use this Bezout equation to introduce another Hankel operator which has singular values that are the inverse of the singular values of the original Hankel operator.

^{*}Earl Katz Family Chair in Algebraic System Theory

[†]Partially supported by the Israeli Academy of Sciences

Moreover the singular vectors are closely related to the original singular vectors. The lower bound estimates on the number of antistable zeroes of the new singular vectors lead to an upper bound estimate on the number of antistable zeroes of the original singular vectors. These two estimates turn out to be tight and give the correct number of antistable zeroes. From here the standard results on Hankel norm approximation and Nehari complementation follow easily.

1 INTRODUCTION

In Fuhrmann [1991] a polynomial approach to AAK theory, see Adamjan, Arov and Krein [1968a,1968b,1971,1978], was given. While many, one might dare to say even interesting, results were given in that paper, there was also a fundamental underlying weakness. At a crucial point the paper used the original AAK results concerning the number of antistable zeroes of the minimal degree singular vectors of the given Hankel operator. Thus that paper was not self contained. In point of fact, the determination of the zero location of Hankel singular values seems to be the bottleneck of AAK theory. No simple method was found to this problem, and even efforts at making a reasonably elementary exposition of AAK theory, e.g. Young [1988] and Partington [19??], have failed in this respect. Even the case of singular vectors corresponding to the largest singular vector, the only relatively easy case in AAK theory, which is disposed in a one line proof, is not really elementary inasmuch as it uses inner/outer factorizations.

The object of this paper is to address itself to this problem and to provide an elementary solution. It should be clarified at the outset that the context in which the problem is solved is that of rational functions. In fact the proof uses rationality in a crucial way. Thus the method, at least as presented in this paper, is not as general as others. However it has the advantage of simplicity. It can be truly said that this method brings AAK theory to a level that can be safely presented to the undergraduate student.

To achieve our goal we redevelop the theory with a little twist. The twist in our approach is that we focus first on the zeroes of the singular vector corresponding to the smallest singular vector. This turns out to be rather trivial to figure out. Once this is achieved a natural Bezout equation presents itself and leads to a related, one should really say a dual, Hankel operator. For both Hankel operators we have lower bound estimates on the number of antistable zeroes of the corresponding singular vectors. However the dual estimates translate into upper bound estimates of the original singular vectors. Moreover the estimates are tight, i.e. they determine the number of antistable zeroes of the minimal degree singular values. From this point the results on Hankel norm approximation follow as in Fuhrmann [1991].

A natural question presents itself. How was this approach been overlooked so long. It seems that the explanation lies in the tremendous authority of M.G. Krein. Once he put the limelight on the largest singular value, everything else remained in the dark.

In writing this paper a basic decision had to be made. Most of the development presented in the current approach is based on the results in Fuhrmann [1991]. It could be presented via a long list of pointers to the relevant parts of that paper. That would mean a short, but also nonreadable, presentation. The other alternative, the one eventually adopted, was to make this paper self contained. This means that there is substantial duplication of results, but the order of the development is different.

The paper is structured as follows. In section 2 we collect basic information on Hankel operators, invariant subspaces and their representation via Beurling's theorem. Next we introduce model intertwining operators. We do this using the frequency domain representation of the right translation semigroup. We study the basic properties of intertwining maps and in particular their invertibility properties. The important point here is the connection of invertibility to the solvability of an H_+^∞ Bezout equation. We follow this by defining Hankel operators. For the case of a rational, antistable function we give specific, Beurling type, representations for the cokernel and the image of the corresponding Hankel operator. Of importance is the connection between Hankel operators and intertwining maps. This connection, coupled with invertibility properties of intertwining maps are the key to duality theory.

In section 3 we do a detailed analysis of Schmidt pairs of a Hankel operator with scalar, rational symbol. Some important lemmas, due to AAK [1971], are rederived in this setting from an algebraic point of view. These lemmas lead to a polynomial formulation of the singular value singular vector equation of the Hankel operator. This equation, we refer to it as the Fundamental Polynomial Equation, is easily reduced, using the theory of polynomial models, to a standard eigenvalue problem.

Using nothing more than the polynomial division algorithm, the subspace of all singular vectors corresponding to a given singular value, is parametrized

via the minimal degree solution of the FPE. We obtain a connection between the minimal degree solution and the multiplicity of the singular value.

The FPE can be transformed, using a simple algebraic manipulation to a form that leads immediately to lower bound estimates on the the number of antistable zeroes of p_k , the minimal degree solution corresponding to the k -th Hankel singular value. This lower bound is shown to actually coincide with the degree of the minimal degree solution for the special case of the smallest singular value. Thus this polynomial turns out to be antistable. Another algebraic manipulation of the FPE leads to a Bezout equation over H_+^∞ . This provides the key to duality.

Section 4 has duality theory is its main theme. Using the previously obtained Bezout equation, we invert the intertwining map corresponding to the initial Hankel operator. The inverse intertwining map is related to a new Hankel operator which has inverse singular values to those of the original one. Moreover we can compute the Schmidt pairs corresponding to this Hankel operator in terms of the original Schmidt pairs.

Section 5 applies the previous information. The sme estimates on the number of antistable zeroes of the minimum degree solutions of the FPE that were obtained for the original Hankel operator Schmidt vectors are applied now to the new Hankel operator Schmidt vectors. Thus we obtain a second set of inequalities. The two sets of inequalities, taken together, lead to precise information on the number of antistable zeroes of the minimal degree solutions corresponding to all singular values. Utilizing this information leads to the solution of the Nehari problem as well as that of the general Hankel norm approximation problem.

It is fitting that the new insight into this problem came while preparing for a seminar at the Department of Mathematics of the University Kaiserslautern, where much of the research on the previous paper has been done. For providing this intellectually stimulating atmosphere I would like to thank D. Prätzel-Wolters. This particular piece of research was done while working on a large joint project with R. Ober. It is a pleasure to acknowledge his creative criticism and the endless conversations that no doubt helped in getting this work done.

2 PRELIMINARIES

Hankel operators are generally defined in the time domain and via the Fourier transform their frequency domain representation is obtained. We will skip this part and introduce Hankel operators directly as frequency domain objects. Our choice is to develop the theory of continuous time systems. This means that the relevant frequency domain spaces are the Hardy spaces of the left and right half planes. Thus we will study Hankel operators defined on half plane Hardy spaces rather than on those of the unit disc as was done by Adamjan, Arov and Krein [1971]. In this we follow the choice of Glover [1984]. This choice seems to be a very convenient one as all results on duality simplify significantly, due to the greater symmetry between the two half planes in comparison to the unit disc and its exterior.

2.1 HARDY SPACES

Our setting will be that of Hardy spaces. Thus H_+^2 is the Hilbert space of all analytic functions in the open right half plane with

$$\|f\|^2 = \sup_{x>0} \frac{1}{\pi} \int_{-\infty}^{\infty} |f(x+iy)|^2 dy.$$

The space H_-^2 is similarly defined in the open left half plane. It is a theorem of Fatou that guarantees the existence of boundary values of H_{\pm}^2 -functions on the imaginary axis. Thus the spaces H_{\pm}^2 can be considered as closed subspaces of $L^2(i\mathbf{R})$, the space of Lebesgue square integrable functions on the imaginary axis. It follows from the Fourier-Plancherel and Paley-Wiener theorems that

$$L^2(i\mathbf{R}) = H_+^2 \oplus H_-^2,$$

with H_+^2 and H_-^2 the Fourier-Plancherel transforms of $L^2(0, \infty)$ and $L^2(-\infty, 0)$ respectively. Also H_+^∞ and H_-^∞ will denote the spaces of bounded analytic functions on the open right and left half planes respectively. These spaces can be considered as subspaces of $L^\infty(i\mathbf{R})$, the space of Lebesgue measurable and essentially bounded functions on the imaginary axis. An extensive discussion of these spaces can be found in Hoffman [1962], Duren [1970] and Garnett [1981].

We will define $f^*(s) = f(-\bar{s})^*$.

2.2 INVARIANT SUBSPACES

Before the introduction of Hankel operators we digress a bit on invariant subspaces of H_+^2 . Since we are using the half planes for our definition of the H^2 spaces, we do not have the shift operators conveniently at our disposal. This forces us to a slight departure from the usual convention.

The algebra H_+^∞ can be made an algebra of operators on H_+^2 by letting, for $\psi \in H^\infty$, induce a map $T_\psi : H_+^2 \longrightarrow H_+^2$ which is defined by

$$T_\psi f = \psi f, \quad f \in H_+^2. \quad (1)$$

The next proposition characterizes the adjoints of this class of operators.

Proposition 2.1 *Let $\psi \in H^\infty$ and T_ψ be defined by (1). The adjoint of T_ψ is given by*

$$T_\psi^* f = P_+ \psi^* f, \quad f \in H_+^2. \quad (2)$$

Both T_ψ and T_ψ^* are special cases of Toeplitz operators.

Definition 2.1 • *A subspace $M \subset$ of H_+^2 is called an invariant subspace if, for each $\psi \in H_+^\infty$ we have*

$$T_\psi M \subset M.$$

• *A subspace $M \subset$ of H_+^2 is called a backward invariant subspace if, for each $\psi \in H_+^\infty$ we have*

$$T_\psi^* M \subset M.$$

Clearly backward invariant subspaces are just orthogonal complements of invariant subspaces.

Invariant subspaces have been characterized by Beurling [1949]. For this we need the notion of an inner function.

Definition 2.2 A function $m \in H_+^\infty$ is called **inner** if $\|m\|_\infty \leq 1$ and its boundary values on the imaginary axis are unitary a.e.

Thus on the imaginary axis we have $m^*m = 1$. The next result, Beurling's theorem, is central. We quote it to put some results in the right perspective. We do not actually use it as in our setup we can directly calculate the relevant invariant subspaces and identify the corresponding inner functions. Thus we will not give a proof of this theorem.

Theorem 2.1 (Beurling) A nontrivial subspace $M \subset H_+^2$ is an invariant subspace if and only if

$$M = mH_+^2$$

for some inner function m .

2.3 MODEL OPERATORS AND INTERTWINING MAPS

Given an inner function $m \in H_+^\infty$ we consider the left invariant subspace $H(m) = \{mH_+^2\}^\perp = H_+^2 \ominus mH_+^2$. The algebra H_+^∞ , or equivalently the algebra of analytic Toeplitz operators, induce an algebra of bounded operators in $\{mH_+^2\}^\perp$. Thus for $\Theta \in H_+^\infty$ the maps $T_\Theta : H(m) \rightarrow H(m)$ are defined by

$$T_\Theta f = P_{H(m)} \Theta f, \quad \text{for } f \in H(m). \quad (3)$$

Clearly, if $\Theta \in H_+^\infty$, we have $\|T_\Theta\| \leq \|\Theta\|_\infty$.

We note that, for $t \leq 0$, the functions $\exp_\tau(s) = e^{-\tau s}$ are all in H_+^∞ . The operators T_{\exp_τ} form a strongly continuous semigroup of operators on $\{mH_+^2\}^\perp$. The following is a continuous time version of the Sarason [1968] commutant lifting theorem.

Theorem 2.2 A bounded operator X on $\{mH_+^2\}^\perp$ satisfies

$$XT_{\exp_\tau} = T_{\exp_\tau}X$$

for all $\tau \leq 0$ if and only if there exists a $\Theta \in H_+^\infty$ such that

$$X = T_\Theta.$$

The next theorem sums up duality properties of operators commuting with shifts.

Theorem 2.3 *Let $\Theta, m \in H_+^\infty$ with m an inner function, and let T_Θ be defined by*

$$T_\Theta f := P_{H(m)} \Theta f, \quad \text{for } f \in H(m).$$

Then

1. *Its adjoint T_Θ^* is given by*

$$T_\Theta^* f = P_+ \Theta^* f, \quad \text{for } f \in H(m).$$

2. *The operator $\tau_m : H(m) \longrightarrow H(m)$ defined by*

$$\tau_m f := m f^*$$

is unitary.

3. *The operators T_Θ and T_Θ^* are unitarily equivalent. More specifically we have*

$$T_\Theta \tau_m = \tau_m T_\Theta^*$$

Proof:

1. Let $f, g \in H(m)$. Then

$$\begin{aligned} (T_\Theta f, g) &= (P_{H(m)} \Theta f, g) = (m P_- m^* \Theta f, g) \\ &= (P_- m^* \Theta f, m^* g) = (m^* \Theta f, P_- m^* g) \\ &= (m^* \Theta f, m^* g) = (\Theta f, g) = (f, \Theta^* g) \\ &= (P_+ f, \Theta^* g) = (f, P_+ \Theta^* g) = (f, T_\Theta^* g). \end{aligned}$$

Here we used the fact that $g \in H(m)$ if and only if $m^* g \in H_-^2$.

2. Clearly the map τ_m , as a map in L^2 , is unitary. From the orthogonal direct sum decomposition

$$L^2 = H_-^2 \oplus H(m) \oplus m H_+^2$$

it follows, by conjugation, that

$$L^2 = m^* H_-^2 \oplus \{H_-^2 \ominus m^* H_-^2\} \oplus H_+^2.$$

Hence $m\{H_-^2 \ominus m^* H_-^2\} = H(m)$.

3. We compute

$$T_\Theta \tau_m f = T_\Theta m f^* = P_{H(m)} \Theta m f^* = m P_- m^* \Theta m f^* = m P_- \Theta f^*$$

Now

$$\tau_m T_\Theta^* = \tau_m (P_+ \Theta^* f) = m (P_+ \Theta^* f)^* = m P_- \Theta f^*.$$

■

The following spectral mapping theorem has been proved in Fuhrmann [1968a]. A vectorial generalization is given in Fuhrmann [1968b]. This will be instrumental in the analysis of Hankel operators restricted to their cokernels.

Theorem 2.4 (Fuhrmann) *Let $\Theta, m \in H_+^\infty$ with m an inner function. The following statements are equivalent.*

1. *The operator T_Θ defined in (3) is invertible.*
2. *There exists a $\delta > 0$ such that*

$$|\Theta(s)| + |m(s)| \geq \delta, \quad \text{for all } s \text{ with } \operatorname{Re} s > 0. \quad (4)$$

3. *There exist $\xi, \eta \in H_+^\infty$ that solve the Bezout equation*

$$\xi \Theta + \eta m = 1. \quad (5)$$

In this case we have

$$T_\Theta^{-1} = T_\xi.$$

Proof: We will not give a proof which can be found in Fuhrmann [1968, 1981].

We remark only that by the Carleson corona theorem, Carleson [1962], the strong coprimeness condition of (4) is equivalent to the solvability of the Bezout equation (5) over H^∞ . ■

2.4 HANKEL OPERATORS

We proceed to define Hankel operators and we do this directly in the frequency domain. Readers interested in the time domain definition and the details of the transformation into frequency domain are referred to Fuhrmann [1981], Glover [1984].

Definition 2.3 Given a function $\phi \in L^\infty(i\mathbf{R})$ the Hankel operator $H_\phi : H_+^2 \longrightarrow H_-^2$ is defined by

$$H_\phi f = P_-(\phi f), \text{ for } f \in H_+^2. \quad (6)$$

The adjoint operator $(H_\phi)^* : H_-^2 \longrightarrow H_+^2$ is given by

$$(H_\phi)^* f = P_+(\phi^* f), \text{ for } f \in H_-^2. \quad (7)$$

Here $\phi^*(z) = \overline{\phi(-\bar{z})}$.

In the algebraic theory of Hankel operators the kernel and image of a Hankel operator are directly related to the coprime factorization of the symbol over the ring of polynomials. The details can be found for example in Fuhrmann [1983]. In the same way the kernel and image of a large class of Hankel operators are related to a coprime factorization over H^∞ . This theme, originating in the work of Douglas, Shapiro and Shields [1971] and that of D. N. Clark, see Helton [1974] and Fuhrmann [1975] is developed extensively in Fuhrmann [1981]. Of course if the symbol of the Hankel operator is rational and in H_-^∞ these two coprime factorizations are easily related.

Thus assume $\phi = \frac{n}{d} \in H_-^\infty$ and $n \wedge d = 1$. So our assumption is that d is antistable. In spite of the slight ambiguity we will write $n = \deg d$. It will always be clear from the context what n means. This leads to

$$\phi = \frac{n}{d} = \frac{n}{d^*} \frac{d^*}{d}.$$

Thus

$$\phi = m^* \eta$$

with

$$\eta = \frac{n}{d^*}, \quad m = \frac{d}{d^*}$$

is a coprime factorization in H_+^∞ .

The next theorem discusses the functional equation of Hankel operators. It can be shown, quite easily using the commutant lifting theorem of Sz.-Nagy and Foias [1970], that the Hankel operators are the only solutions of this functional equation. For more information one can consult Nikolskii [1985].

Theorem 2.5 1. For every $\psi \in H_+^\infty$ the Hankel operator H_ϕ satisfies the functional equation

$$P_- \psi H_\phi f = H_\phi \psi f, \quad f \in H_+^2. \quad (8)$$

2. $\text{Ker} H_\phi$ is an invariant subspace, i.e. for $f \in \text{Ker} H_\phi$ and $\psi \in H_+^\infty$ we have $\psi f \in \text{Ker} H_\phi$.

It follows from a theorem of Beurling [1949] that $\text{Ker} H_\phi = m H_+^2$ for some inner function $m \in H_+^\infty$. Since we are dealing with the rational case the next theorem can make this more specific and characterizes the kernel and image of a Hankel operator and also clarifies the connection between them and polynomial and rational models. A closely related derivation can be found in Young [1983] and Lindquist and Picci [1985].

Theorem 2.6 Let $\phi = \frac{n}{d} \in H_-^\infty$ and $n \wedge d = 1$ Then

1. $\text{Ker} H_\phi = \frac{d}{d^*} H_+^2$
2. $\{\text{Ker} H_\phi\}^\perp = \{\frac{d}{d^*} H_+^2\}^\perp = X^{d^*}$
3. $\text{Im} H_\phi = H_-^2 \ominus \frac{d^*}{d} H_-^2 = X^d$

Proof: $\{Ker H_\phi\}^\perp$ contains only rational functions. Let $f = \frac{p}{q} \in \{\frac{d}{d^*} H_+^2\}^\perp$, then $\frac{d^* p}{dq} \in H_-^2$. So $q \mid d^* p$. But, as $p \wedge q = 1$ it follows that $q \mid d^*$, i.e. $d^* = qr$. Hence $f = \frac{rp}{d^*} \in X^{d^*}$.

Conversely, let $\frac{p}{d^*} \in X^{d^*}$ then, $\frac{p}{d^*} = \frac{p}{d} \frac{d}{d^*}$ or $\frac{d^*}{d} \frac{p}{d^*} \in H_-^2$. So we have $\frac{p}{d^*} \in \{\frac{d}{d^*} H_+^2\}^\perp$. ■

The previous theorem, though of an elementary nature, is central to all further development as it provides the direct link between the infinite dimensional object, namely the Hankel operator, and the well developed theory of polynomial and rational models. This link will be continuously exploited.

There is a very close connection between a wide class of Hankel operators and intertwining maps. This is summarized in the following.

Theorem 2.7 *A map $H : H_+^2 \longrightarrow H_-^2$ is a Hankel operator with a nontrivial kernel mH_+^2 , with m inner, if and only if we have*

$$H = m^* X P_{H(m)}$$

where $X : H(m) \longrightarrow H(m)$ is an intertwining map, i.e. of the form

$$X = T_\Theta$$

for some $\Theta \in H_+^\infty$.

Proof: Assume $X = T_\Theta$ with $\Theta \in H_+^\infty$. We define $H : H_+^2 \longrightarrow H_-^2$ by

$$Hf = m^* X P_{H(m)} f.$$

Then, since m is inner and $P_{H(m)}$ an orthogonal projection, the operator H

is bounded. Moreover

$$\begin{aligned}
H \exp_\tau f &= m^* X P_{H(m)} \exp_\tau f \\
&= m^* P_{H(m)} \Theta P_{H(m)} \exp_\tau P_{H(m)} f \\
&= m^* P_{H(m)} \Theta \exp_\tau P_{H(m)} f \\
&= m^* P_{H(m)} \exp_\tau \Theta P_{H(m)} f \\
&= m^* P_{H(m)} \exp_\tau P_{H(m)} \Theta P_{H(m)} f \\
&= m^* m P_- m^* \exp_\tau P_{H(m)} \Theta P_{H(m)} f \\
&= P_- m^* \exp_\tau P_{H(m)} \Theta P_{H(m)} f \\
&= P_- \exp_\tau m^* m P_- m^* \Theta P_{H(m)} f \\
&= P_- \exp_\tau m^* X P_{H(m)} f \\
&= P_- \exp_\tau H f
\end{aligned}$$

Thus H satisfies the functional equation of a Hankel operator.

Conversely, assume $H = H_\phi$ is a Hankel operator with a nontrivial kernel given by mH_+^2 , for some inner function $m \in H^\infty$. Then we define a map $T : H(m) \longrightarrow H(m)$ by

$$Tf = mHf, \quad f \in H(m).$$

Then we compute

$$\begin{aligned}
TP_{H(m)} \exp_\tau f &= mHP_{H(m)} \exp_\tau f \\
&= mH \exp_\tau f \\
&= mP_- \exp_\tau Hf \\
&= mP_- m^* m \exp_\tau Hf \\
&= P_{H(m)} \exp_\tau mHf \\
&= P_{H(m)} \exp_\tau Tf
\end{aligned}$$

So T is an intertwining map. ■

The previous theorem opens the way to prove Nehari's theorem from Sarason's lifting theorem as well as prove Sarason's theorem from Nehari's. This equivalence is known for a long time and can be found in Page [1970], Nikolskii [1985] and AAK [1968] to cite a few references.

Hankel operators in general and those with rational symbol in particular are never invertible. Still we may want to invert the Hankel operator as a

map from its cokernel, i.e. the orthogonal complement of its kernel, to its image. We saw that such a restriction of a Hankel operator is of considerable interest because of its connection to intertwining maps of model operators. Now theorem 2.4 gave a full characterization of invertibility properties of intertwining maps. These can be applied now to the inversion of the restricted Hankel operators. This will turn out to be of great importance in the development of duality theory.

3 SCHMIDT PAIRS OF RATIONAL HANKEL OPERATORS

It is quite well known, see Gohberg and Krein [1969], that singular values of operators are closely related to the problem of best approximation by operators of finite rank. That this basic method could be applied to the approximation of Hankel operators by Hankel operators of lower ranks through the detailed analysis of singular values and the corresponding Schmidt pairs is a fundamental contribution of Adamjan, Arov and Krein.

We recall that, given a bounded operator A on a Hilbert space, μ is a *singular value* of A if there exists a nonzero vector f such that

$$A^*Af = \mu^2 f.$$

Rather than solve the previous equation we let $g = \frac{1}{\mu}Af$ and go over to the equivalent system

$$\begin{cases} Af = \mu g \\ A^*g = \mu f \end{cases},$$

i.e. μ is a singular value of both A and A^* .

The analysis of Schmidt pairs of Hankel operators goes back to Adamjan, Arov and Krein [1971]. Here, for the rational case we present an algebraic derivation of some of their results.

We proceed to compute the singular vectors of the Hankel operator H_ϕ . In view of the preceding remarks we have to solve

$$H_\phi f = \mu g$$

$$H_\phi^* g = \mu f$$

or

$$P_- \frac{n}{d} \frac{p}{d^*} = \mu \frac{\hat{p}}{d}$$

$$P_+ \frac{n^*}{d^*} \frac{\hat{p}}{d} = \mu \frac{p}{d^*}$$

This means there exist polynomials π and ξ such that

$$\frac{n}{d} \frac{p}{d^*} = \mu \frac{\hat{p}}{d} + \frac{\pi}{d^*}$$

$$\frac{n^*}{d^*} \frac{\hat{p}}{d} = \mu \frac{p}{d^*} + \frac{\xi}{d}$$

These equations can be rewritten as polynomial equations

$$np = \mu d^* \hat{p} + d\pi \tag{9}$$

$$n^* \hat{p} = \mu dp + d^* \xi. \tag{10}$$

Equation (9), considered as an equation modulo the polynomial d , is not an eigenvalue equation as there are too many unknowns. More specifically, we have to find the coefficients of both p and \hat{p} . To overcome this difficulty we study in more detail the structure of Schmidt pairs of Hankel operators.

Lemma 3.1 *Let $\{\frac{p}{d^*}, \frac{\hat{p}}{d}\}$ and $\{\frac{q}{d^*}, \frac{\hat{q}}{d}\}$ be two Schmidt pairs of the Hankel operator $H_{\frac{n}{d}}$, corresponding to the same singular value μ . Then*

$$\frac{p}{\hat{p}} = \frac{q}{\hat{q}},$$

i.e. this ratio is independent of the Schmidt pair.

Proof: The polynomials p, \hat{p} correspond to one Schmidt pair and let the polynomials q, \hat{q} correspond to another Schmidt pair, i.e.

$$nq = \mu d^* \hat{q} + d\rho \quad (11)$$

$$n^* \hat{q} = \mu d q + d^* \eta. \quad (12)$$

Now, from equations (9) and (12) we get

$$0 = \mu d(p\hat{q} - q\hat{p}) + d^*(\xi\hat{q} - \eta\hat{p}).$$

Since d and d^* are coprime it follows that $d^* \mid p\hat{q} - q\hat{p}$. On the other hand, from equations (9) and (11), we get

$$0 = \mu d^*(\hat{p}q - \hat{q}p) + d(\pi q - \rho p),$$

and hence that $d \mid \hat{p}q - \hat{q}p$. Now both d and d^* divide $\hat{p}q - \hat{q}p$, and, as $\deg(\hat{p}q - \hat{q}p) < \deg d + \deg d^*$ it follows that

$$\hat{p}q - \hat{q}p = 0.$$

Equivalently

$$\frac{p}{\hat{p}} = \frac{q}{\hat{q}}$$

i.e. $\frac{p}{\hat{p}}$ is independent of the particular Schmidt pair associated to the singular value μ . ■

Lemma 3.2 *Let $\{\frac{p}{d^*}, \frac{\hat{p}}{d}\}$ be a Schmidt pair associated with the singular value μ . Then $\frac{p}{\hat{p}}$ is unimodular or all pass.*

Proof: Going back to equation (10) and the dual of (9) we have

$$n^* \hat{p} = \mu d p + d^* \xi$$

$$n^* p^* = \mu d(\hat{p})^* + d^* \pi^*$$

It follows that

$$0 = \mu d(pp^* - \hat{p}(\hat{p})^*) + d^*(\xi p^* - \pi^* \hat{p})$$

and hence $d^* \mid (pp^* - \hat{p}(\hat{p})^*)$. By symmetry also $d \mid (pp^* - \hat{p}(\hat{p})^*)$, and so necessarily

$$pp^* - \hat{p}(\hat{p})^* = 0.$$

This can be rewritten as

$$\frac{p}{\hat{p}} \frac{p^*}{(\hat{p})^*} = 1,$$

i.e. $\frac{p}{\hat{p}}$ is all pass. ■

We will say that a pair of polynomials (p, \hat{p}) , with $\deg p, \deg \hat{p} < \deg d$, is a *solution pair* if there exist polynomials π and ξ such that equations (9) and (10) are satisfied.

The next lemma characterizes all solution pairs.

Lemma 3.3 *Let μ be a singular value of the Hankel operator $H_{\frac{n}{d}}$. Then there exists a unique, up to a constant factor, solution pair (p, \hat{p}) , of minimal degree. The set of all solutions pairs is given by*

$$\{(q, \hat{q}) \mid q = pa, \hat{q} = \hat{p}a, \deg a < \deg q - \deg p\}.$$

Proof: Clearly, if μ is a singular value of the Hankel operator, then a nonzero solution pair (p, \hat{p}) , of minimal degree exists. Let (q, \hat{q}) be any other solution, pair with $\deg q, \deg \hat{q} < \deg d$. By the division rule for polynomials $q = ap + r$ with $\deg r < \deg p$. Similarly, $\hat{q} = \hat{a}\hat{p} + \hat{r}$ with $\deg \hat{r} < \deg \hat{p}$. From Equation (9) we get

$$n(ap) = \mu d^*(a\hat{p}) + d(a\pi) \tag{13}$$

whereas Equation (11) yields

$$n(ap + r) = \mu d^*(\hat{a}\hat{p} + \hat{r}) + d(\tau) \tag{14}$$

By subtraction we obtain

$$nr = \mu d^*((\hat{a} - a)\hat{p} + \hat{r}) + d(\tau - a\pi) \tag{15}$$

Similarly from Equation (10) we get

$$n^*(\hat{a}\hat{p} + \hat{r}) = \mu d(ap + r) + d^*\xi. \quad (16)$$

whereas Equation (10) yields

$$n^*(a\hat{p}) = \mu d(ap) + d^*(a\xi). \quad (17)$$

Subtracting the two gives

$$n^*((\hat{a} - a)\hat{p} + \hat{r}) = \mu dr + d^*(\eta - a\xi). \quad (18)$$

Equations (15) and (18) imply that $\{\frac{r}{d^*}, \frac{(\hat{a} - a)\hat{p} + \hat{r}}{d}\}$ is a μ Schmidt pair. Since necessarily $\deg r = \deg(\hat{a} - a)\hat{p} + \hat{r}$ we get $\hat{a} = a$. Finally, since we assumed (p, \hat{p}) to be of minimal degree we must have $r = \hat{r} = 0$.

Conversely, if a is any polynomial with $\deg a < \deg d - \deg p$ then from Equations (9) and (10) it follows by multiplication that $(pa, \hat{p}a)$ is also a solution pair. ■

Lemma 3.4 *Let p, q be coprime polynomials with real coefficients such that $\frac{p}{q}$ is all pass. Then $q = \pm p^*$.*

Proof: Since $\frac{p}{q}$ is all pass, it follows that

$$\frac{p p^*}{q q^*} = 1$$

or $pp^* = qq^*$. As the polynomials p and q are coprime it follows that $p \mid q^*$ and hence $q^* = \pm p$. ■

In the general case we have the following.

Lemma 3.5 *Let p, q be polynomials with real coefficients such that $p \wedge q = 1$ and $\frac{p}{q}$ is all pass. Then, with $r = p \wedge \hat{p}$, we have*

$$p = rs$$

$$\hat{p} = \pm rs^*$$

Proof: Write $p = rs$, $\hat{p} = r\hat{s}$. Then $s \wedge \hat{s} = 1$ and $\frac{s}{\hat{s}}$ is all pass. The result follows by applying the previous lemma. ■

The next theorem is of central importance due to the fact that it reduces the analysis to one polynomial. Thus we get an equation which is easily reduced to an eigenvalue problem.

Theorem 3.1 *Let μ be a singular value of H_ϕ and let (p, \hat{p}) be a nonzero, minimal degree solution pair of Equations (9) and (10). Then p is a solution of*

$$np = \lambda d^* p^* + d\pi, \quad (19)$$

with λ real and $|\lambda| = \mu$.

Proof: Let (p, \hat{p}) be a nonzero, minimal degree solution pair of Equations (9) and (10). By taking their adjoints we can easily see that (\hat{p}^*, p^*) is also a nonzero, minimal degree solution pair. By uniqueness of such a solution, i.e. by Lemma 3.3, we have

$$\hat{p}^* = \epsilon p. \quad (20)$$

Since $\frac{\hat{p}}{p}$ is all pass and both polynomials are real we have $\epsilon = \pm 1$. Let us put $\lambda = \epsilon\mu$, then (19) can be rewritten as

$$\hat{p} = \epsilon p^*$$

and so (19) follows from (9). ■

We will refer to equation (19) as the **fundamental polynomial equation**. It will be the source of all future derivations.

Corollary 3.1 *Let μ_i be a singular value of H_ϕ and let p_i be the minimal degree solution of the fundamental polynomial equation, i.e.*

$$np_i = \lambda_i d^* p_i^* + d\pi_i. \quad (21)$$

Then

1.

$$\deg p_i = \deg p_i^* = \deg \pi_i.$$

2. Putting $p_i(z) = \sum_{j=0}^{n-1} p_{i,j} z^j$ and $\pi_i(z) = \sum_{j=0}^{n-1} \pi_{i,j} z^j$ we have the equality

$$\pi_{i,n-1} = \lambda_i p_{i,n-1}. \quad (22)$$

Corollary 3.2 *Let p be a minimal degree solution of equation (19). Then*

1. *The set of all singular vectors of the Hankel operator $H_{\frac{n}{2}}$, corresponding to the singular value μ , is given by*

$$\text{Ker}(H_{\frac{n}{2}}^* H_{\frac{n}{2}} - \mu^2 I) = \left\{ \frac{pa}{d^*} \mid a \in R[z], \deg a < \deg d - \deg p \right\}$$

2. *The multiplicity of $\mu = \|H_\phi\|$ as a singular value of H_ϕ is equal to $m = \deg d - \deg p$ where p is the minimum degree solution of (19).*

3. *There exists a constant c such that $c + \frac{n}{d}$ is a constant multiple of an antistable all-pass function if and only if $\mu_1 = \dots = \mu_n$.*

Proof: We will prove (3) only. Assume all singular values are equal to μ . Thus the multiplicity of μ is $\deg d$. Hence the minimal degree solution p of (19) is a constant and so is π . Putting $c = -\frac{\pi}{p}$ then (19) can be rewritten as

$$\frac{n}{d} + c = \lambda \frac{d^* p^*}{dp},$$

and this is a multiple of an antistable all-pass function.

Conversely assume, without loss of generality, that $\frac{n}{d} + c$ is antistable all-pass. Then the induced Hankel operator is isometric and all its singular values are equal to 1. ■

Part 3 of the corollary is due to Glover [1984].

The following simple proposition is important in the study of zeroes of singular vectors.

Proposition 3.1 *Let μ_k be asingular value of H_ϕ and let p_k be the minimal degree solution of*

$$np_k = \lambda_k d^* p_k^* + d\pi_k \quad (23)$$

Then

- *The polynomials p_k and p_k^* are coprime.*
- *The polynomial p_k has no imaginary axis zeroes.*

Proof:

- Let $e = p_k \wedge p_k^*$. Without loss of generality we may assume that $e = e^*$. The polynomial e has no imaginary axis zeroes, for that would imply that e and π_k have a nontrivial common divisor. Thus the fundamental polynomial equation could be divided by a suitable polynomial factor. This in contradiction to the assumption that p_k is a minimal degree solution.
- This clearly follows from the first part.

■

The fundamental polynomial equation is easily reduced to either a generalized eigenvalue equation or to a regular eigenvalue equation. There are several reductions of this kind in the literature, e.g. Kung [1980], Harshavardhana, Jonckheere and Silverman [1984], to cite a few. Another simple approach to this, using polynomial models is presented in Fuhrmann [1991].

3.1 Zeroes of singular vectors

We begin now the study of the zero location of the numerator polynomials of singular vectors. This is of course the same as the study of the zeroes of minimal degree solutions of equation (19). The following proposition provides a lower bound on the number of zeroes the minimal degree solutions of (19) can have in the open left half plane. However the lower bound is sharp in one special case. This is enough to lead us eventually to a full characterization, given originally by Adamjan, Arov and Krein [1968], and this will be given in Theorem 5.1.

Proposition 3.2 *Let $\phi = \frac{n}{d} \in H_-^\infty$. Let μ_k be a singular value of H_ϕ satisfying $\mu_1 \geq \dots \geq \mu_{k-1} > \mu_k = \dots = \mu_{k+\nu-1} > \mu_{k+\nu} \geq \dots \geq \mu_n$ i.e. μ_k is a singular value of multiplicity ν . Let p_k be the minimum degree solution of (19) corresponding to μ_k . Then the number of antistable zeroes of p_k are $\geq k - 1$.*

If μ_n is the smallest singular value of H_ϕ and is of multiplicity ν , i.e. $\mu_1 \geq \dots \geq \mu_{n-\nu} > \mu_{n-\nu+1} = \dots = \mu_n$, and $p_{n-\nu+1}$ is the corresponding minimum degree solution of (19), then all the zeroes of $p_{n-\nu+1}$ are antistable.

Proof: From Equation 19, i.e.

$$np_k = \lambda_k d^* p_k^* + d\pi_k, \quad (24)$$

we get, dividing by dp_k ,

$$\frac{n}{d} - \frac{\pi_k}{p_k} = \lambda_k \frac{d^* p_k^*}{dp_k}$$

which implies of course that

$$\|H_{\frac{n}{d}} - H_{\frac{\pi_k}{p_k}}\| \leq \left\| \frac{n}{d} - \frac{\pi_k}{p_k} \right\|_\infty = \mu_k \left\| \frac{d^* p_k^*}{dp_k} \right\|_\infty = \mu_k.$$

This means, by the definition of singular values, that $\text{rank} H_{\frac{\pi_k}{p_k}} \geq k - 1$. But this implies, by Kronecker's theorem, that the number of antistable poles of $\frac{\pi_k}{p_k}$ which is the same as the number of antistable zeroes of p_k is $\geq k - 1$.

If μ_n is the smallest singular value and has multiplicity ν , and $p_{n-\nu+1}$ is the minimal degree solution of Equation (19), then it has degree $n - \nu$. But

by the previous part it must have at least $n - \nu$ antistable zeroes. So this implies that all the zeroes of $p_{n-\nu+1}$ are antistable. ■

The previous result is extremely important from our point of view. It shifts the focus from the largest singular value, the starting point in all derivations so far, to the smallest singular value. Certainly the derivation is elementary, inasmuch as we use only the definition of singular values and Kronecker's theorem. The great advantage is that at this stage we can solve an important Bezout equation which is the key to duality theory.

We have now at hand all that is needed to obtain the optimal Hankel norm approximant corresponding to the smallest singular value. We shall delay this analysis to a later stage and develop duality theory first.

From Equation (19) we obtain, dividing by $\lambda_n d^* p_n^*$, the Bezout equation

$$\frac{n}{d^*} \left(\frac{1}{\lambda_n} \frac{p_n}{p_n^*} \right) - \frac{d}{d^*} \left(\frac{1}{\lambda_n} \frac{\pi_n}{p_n^*} \right) = 1. \quad (25)$$

Since the polynomials p_n and d are antistable all four functions appearing in the Bezout equation are in H_+^∞ . We shall discuss next the implications of this Bezout equation.

4 DUALITY

In this section we develop a duality theory in the context of Hankel norm approximation problems. There are three operations applied to a given, antistable, transfer function. Namely, inversion of the restricted Hankel operator, taking the adjoint map and finally one sided multiplication by unitary operators. The last two operations do not change the singular values, whereas the first operation inverts them.

We will say that two Hilbert space operators $T : H_1 \rightarrow H_2$ and $T' : H_3 \rightarrow H_4$ are *equivalent* if there exist unitary operators $U : H_1 \rightarrow H_3$ and $V : H_2 \rightarrow H_4$ such that

$$VT = T'U.$$

Lemma 4.1 *Let $T : H_1 \rightarrow H_2$ and $T' : H_3 \rightarrow H_4$ be equivalent. Then T and T' have the same singular values.*

Proof: Let $T^*Tx = \mu^2x$. Since $VT = T'U$ it follows that

$$U^*T'^*T'Ux = T^*V^*VTx = T^*Tx = \mu^2x,$$

or

$$T'^*T'(Ux) = \mu^2(Ux).$$

■

The following proposition is bordering on the trivial and no proof need be given. However, when applied to Hankel operators it has far reaching implications. In fact it provides a key to duality theory and leads eventually to the proof of the AAK results.

Proposition 4.1 *Let T be an invertible linear transformation. Then, if x is a singular vector of the operator T corresponding to the singular value μ , i.e. $T^*Tx = \mu^2x$ then*

$$T^{-1}(T^{-1})^*x = \mu^{-2}x$$

i.e. x is also a singular vector for $(T^{-1})^$ corresponding to the singular value μ^{-1} .*

In view of this proposition, it is of interest to compute $[(H_\phi|H(m))^{-1}]^*$. Before proceeding with this we compute the inverse of a related operator. This is a special case of Theorem 2.4 for the rational case. Note that, since $\|T_\Theta^{-1}\| = \mu_n^{-1}$, there exists, by Sarason's theorem, a $\xi \in H_+^\infty$ such that $T_\Theta^{-1} = T_\xi$ and $\|\xi\|_\infty = \mu_n^{-1}$. The next theorem provides this ξ . For an algebraic analogue of the next two theorems we refer to Helmke and Fuhrmann [1989].

Theorem 4.1 *Let $\phi = \frac{n}{d} \in H_-^\infty$. Then $\theta = \frac{n}{d^*} \in H_+^\infty$. The operator T_θ defined by equation (3) is invertible and its inverse given by $T_{\frac{1}{\lambda_n} \frac{p_n}{p_n^*}}$ where λ_n is the last signed singular value of H_ϕ and p_n is the minimal degree solution of*

$$np_n = \lambda_n d^* p_n^* + d\pi_n.$$

Proof: From the previous equation we obtain the Bezout equation

$$\frac{n}{d^*} \left(\frac{1}{\lambda_n} \frac{p_n}{p_n^*} \right) - \frac{d}{d^*} \left(\frac{\pi_n}{\lambda_n p_n^*} \right) = 1. \quad (26)$$

By Theorem 3.2 the polynomial p_n is antistable so $\frac{p_n}{p_n^*} \in H_+^\infty$. This, by Theorem 2.4 implies the result. ■

It is well known that stabilizing controllers are related to solutions of Bezout equations over H^∞ . Thus we expect Equation (25) to lead to a stabilizing controller. The next corollary is a result of this type.

Corollary 4.1 *Let $\phi = \frac{n}{d} \in H_-^\infty$. The controller $k = \frac{p_n}{\pi_n}$ stabilizes ϕ . If the multiplicity of μ_n is m there exists a stabilizing controller of degree $n - m$.*

Proof: Since p_n is antistable, we get from (19) that $np_n - d\pi_n = \lambda_n d^* p_n^*$ is stable. We compute

$$\frac{\phi}{1 - k\phi} = \frac{\frac{n}{d}}{1 - \frac{p_n}{\pi_n} \frac{n}{d}} = \frac{-n\pi_n}{np_n - d\pi_n} = \frac{-n\pi_n}{\lambda_n d^* p_n^*} \in H_+^\infty.$$

■

This corollary is related to questions of robust control. For more on this see Glover [1986].

Theorem 4.2 *Let $\phi = \frac{n}{d} \in H_-^\infty$. Let $H : X^{d^*} \longrightarrow X^d$ be defined by $H = H_\phi|X^{d^*}$. Then*

1. $H_\phi^{-1} : X^d \longrightarrow X^{d^*}$ is given by

$$H_\phi^{-1}h = \frac{1}{\lambda_n} \frac{d}{d^*} P_- \frac{p_n}{p_n^*} h \quad (27)$$

2. $(H_\phi^{-1})^* : X^{d^*} \longrightarrow X^d$ is given by

$$(H_\phi^{-1})^* f = \frac{1}{\lambda_n} \frac{d^*}{d} P_+ \frac{p_n^*}{p_n} f \quad (28)$$

Proof:

1. Let $m = \frac{d}{d^*}$ and let T be the map given by $T = mH_{\frac{n}{d}}$. Thus we have the following commutative diagram

$$\begin{array}{ccc} X^{d^*} & \xrightarrow{H_\phi} & X^d \\ T & \searrow & \downarrow m \\ & & X^{d^*} \end{array}$$

Now

$$\begin{aligned} Tf &= \frac{d}{d^*} P_- \frac{n}{d} f = \frac{d}{d^*} P_- \frac{d^*}{d} \frac{n}{d^*} f \\ &= P_{H(\frac{d}{d^*})} \frac{n}{d^*} f = P_{X^{d^*}} \frac{n}{d^*} f \end{aligned}$$

i.e. $T = T_\theta$ where $\theta = \frac{n}{d^*}$. Now, from $T_\theta = mH_{\frac{n}{d}}$ we have, by Theorem 4.1,

$$T_\theta^{-1} = T \frac{1}{\lambda_n} \frac{p_n}{p_n^*}.$$

So, for $h \in X^d$,

$$\begin{aligned} H_{\frac{n}{d}}^{-1}h &= \frac{1}{\lambda_n} P_{H(\frac{d}{d^*})} \frac{p_n}{p_n^*} \frac{d}{d^*} h \\ &= \frac{1}{\lambda_n} \frac{d}{d^*} P_- \frac{d^*}{d} \frac{p_n}{p_n^*} \frac{d}{d^*} h \\ &= \frac{1}{\lambda_n} \frac{d}{d^*} P_- \frac{p_n}{p_n^*} h. \end{aligned} \quad (29)$$

2. Equation (29) can be written also as

$$H_{\frac{n}{2}}^{-1}h = T \frac{1}{\lambda_n} \frac{p_n}{p_n^*} mh.$$

Therefore, using Theorem 2.3, we have, for $f \in X^{d^*}$,

$$(H_{\phi}^{-1})^* f = m^* (T \frac{1}{\lambda_n} \frac{p_n}{p_n^*})^* = \frac{d^*}{d} P_+ \frac{1}{\lambda_n} \frac{p_n}{p_n^*} f = \frac{1}{\lambda_n} \frac{d^*}{d} P_+ \frac{p_n^*}{p_n} f.$$

■

Corollary 4.2 *There exist polynomials α_i , of degree $\leq n-2$, such that*

$$\lambda_i p_n^* p_i - \lambda_n p_n p_i^* = \lambda_i d^* \alpha_i, \quad i = 1, \dots, n-1.$$

This holds also formally for $i = n$ with $\alpha_n = 0$.

Proof: Since

$$H_{\frac{n}{2}} \frac{p_i}{d^*} = \lambda_i \frac{p_i^*}{d}$$

it follows that

$$(H_{\frac{n}{2}}^{-1})^* \frac{p_i}{d^*} = \lambda_i^{-1} \frac{p_i^*}{d}.$$

So, using equation (28), we have

$$\frac{1}{\lambda_n} \frac{d^*}{d} P_+ \frac{p_n^*}{p_n} \frac{p_i}{d^*} = \frac{1}{\lambda_i} \frac{p_i^*}{d},$$

i.e.

$$\frac{\lambda_n}{\lambda_i} \frac{p_i^*}{d^*} = P_+ \frac{p_n^*}{p_n} \frac{p_i}{d^*}.$$

This implies, by partial fraction decomposition, the existence of polynomials α_i , $i = 1, \dots, n$ such that $\deg \alpha_i < \deg p_n = n-1$, and

$$\frac{p_n^*}{p_n} \frac{p_i}{d^*} = \frac{\lambda_n}{\lambda_i} \frac{p_i^*}{d^*} + \frac{\alpha_i}{p_n},$$

i.e.

$$\lambda_i p_n^* p_i - \lambda_n p_n p_i^* = \lambda_i d^* \alpha_i. \quad (30)$$

■

We saw, in Theorem 4.2, that for the Hankel operator H_ϕ the map $(H_\phi^{-1})^*$ is not a Hankel map. However there is an equivalent Hankel map. We sum this up in the following.

Theorem 4.3 Let $\phi = \frac{n}{d} \in H_-^\infty$. Let $H : X^{d^*} \longrightarrow X^d$ be defined by $H = H_\phi|X^{d^*}$. Then

1. The operator $(H_\phi^{-1})^*$ is equivalent to the Hankel operator $H \frac{1}{\lambda_n} \frac{d^* p_n}{dp_n^*}$.
2. The Hankel operator $H \frac{1}{\lambda_n} \frac{d^* p_n}{dp_n^*}$ has singular values $\mu_1^{-1} < \dots < \mu_n^{-1}$.
3. The Schmidt pairs of $H \frac{1}{\lambda_n} \frac{d^* p_n}{dp_n^*}$ are $\{\frac{p_i^*}{d^*}, \frac{p_i}{d}\}$.

Proof: We saw that

$$(H_{\frac{n}{d}}^{-1})^* = \frac{d^*}{d} T^* \frac{1}{\lambda_n} \frac{p_n}{p_n^*}.$$

Since multiplication by $\frac{d^*}{d}$ is a unitary map of X^{d^*} onto X^d , the operator $(H_{\frac{n}{d}}^{-1})^*$ has, by Lemma 4.1, the same singular values as $T^* \frac{1}{\lambda_n} \frac{p_n}{p_n^*}$. These are

the same as those of the adjoint operator $T \frac{1}{\lambda_n} \frac{p_n}{p_n^*}$. However the last operator is equivalent to the Hankel operator $H \frac{1}{\lambda_n} \frac{d^* p_n}{dp_n^*}$. Indeed,

$$\frac{d^*}{d} T \frac{1}{\lambda_n} \frac{p_n}{p_n^*} f = \frac{d^*}{d} P_{H(\frac{d^*}{d})} \frac{1}{\lambda_n} \frac{p_n}{p_n^*} f = \frac{d^*}{d} \frac{d}{d^*} P_- \frac{d^*}{d} \frac{1}{\lambda_n} \frac{p_n}{p_n^*} f = H \frac{1}{\lambda_n} \frac{d^* p_n}{dp_n^*} f.$$

This Hankel operator has singular values $\mu_1^{-1} < \dots < \mu_n^{-1}$. and its Schmidt pairs are $\{\frac{p_i^*}{d^*}, \frac{p_i}{d}\}$. Indeed

$$H \frac{d^* p_n}{dp_n^*} \frac{p_i^*}{d^*} = P_- \frac{d^* p_n}{dp_n^*} \frac{p_i^*}{d^*} = P_- \frac{p_n p_i^*}{dp_n^*}.$$

Now, from equation (30) we get

$$p_n p_i^* = \frac{\lambda_i}{\lambda_n} p_n^* p_i - \frac{\lambda_i}{\lambda_n} d^* \alpha_i,$$

or taking the dual of that equation

$$p_n p_i^* = \frac{\lambda_n}{\lambda_i} p_n^* p_i + d \alpha_i^*,$$

So

$$\frac{p_n p_i^*}{dp_n^*} = \frac{\lambda_n}{\lambda_i} \frac{p_n^* p_i}{dp_n^*} + \frac{d \alpha_i^*}{dp_n^*} = \frac{\lambda_n}{\lambda_i} \frac{p_i}{d} + \frac{\alpha_i^*}{p_n^*}.$$

Hence

$$P_- \frac{p_n p_i^*}{dp_n^*} = \frac{\lambda_n}{\lambda_i} \frac{p_i}{d}.$$

Therefore

$$\frac{1}{\lambda_n} H \frac{d^* p_n}{dp_n^*} \frac{p_i^*}{d^*} = \frac{1}{\lambda_i} \frac{p_i}{d}.$$

■

5 HANKEL NORM APPROXIMATION

The duality results obtained before allow us now to complete our study of the zero structure of minimal degree solutions of the fundamental polynomial equation (19). This in turn leads to an elementary proof of the central theorem in the AAK theory.

Theorem 5.1 (Adamjan, Arov and Krein) *Let $\phi = \frac{n}{d} \in H_-^\infty$. Let μ_k be a singular value of H_ϕ satisfying $\mu_1 \geq \dots \geq \mu_{k-1} > \mu_k = \dots = \mu_{k+\nu-1} > \mu_{k+\nu} \geq \dots \geq \mu_n$ i.e. μ_k is a singular value of multiplicity ν . Let p_k be the minimum degree solution of (19) corresponding to μ_k . Then the number of antistable zeroes of p_k is exactly $k - 1$.*

If μ_1 is the largest singular value of H_ϕ and is of multiplicity ν , i.e. $\mu_1 = \dots = \mu_\nu > \mu_{\nu+1} \geq \dots \geq \mu_n$, and p_1 is the corresponding minimum degree solution of (19), then all the zeroes of p_1 are stable, this is equivalent to saying that p_1 is outer.

Proof: We saw, in the proof of Proposition 3.2, that the number of antistable zeroes of p_k is $\geq k - 1$. Now, by Theorem 4.3, p_k^* is the minimum degree solution of the fundamental equation corresponding to the transfer function $\frac{1}{\lambda_n} \frac{d^* p_n}{dp_n^*}$ and the singular value $\mu_{k+\nu-1}^{-1} = \dots = \mu_k^{-1}$. Clearly we have $\mu_n^{-1} \geq \dots \geq \mu_{k+\nu}^{-1} > \mu_{k+\nu-1}^{-1} = \dots = \mu_k^{-1} > \mu_{k-1}^{-1} \geq \dots \geq \mu_1^{-1}$. In particular, applying Proposition 3.2, the number of antistable zeroes of p_k^* is $\geq n - k - \nu + 1$. Since the degree of p_k^* is $n - \nu$ it follows that the number of stable zeroes of p_k^* is $\leq k - 1$. However this is the same as saying the number of antistable zeroes of p_k is $\leq k - 1$. Combining the two inequalities, it follows that the number of antistable zeroes of p_k is exactly $k - 1$.

The first part implies that the minimum degree solution of (19) has only stable zeroes, i.e. it is an outer function. ■

We now come to apply some results of the previous section to the case of Hankel norm approximation. We use here the characterization of singular values as approximation numbers, see Gohberg and Krein [1969] for an extensive treatment of this topic.

Theorem 5.2 (Adamjan, Arov and Krein) *Let $\phi = \frac{n}{d} \in H_-^\infty$ be a scalar, strictly proper, transfer function, with n and d coprime polynomials and d is*

monic of degree n . Assume that $\mu_1 \geq \dots \geq \mu_{k-1} > \mu_k = \dots = \mu_{k+\nu-1} > \mu_{k+\nu} \geq \dots \geq \mu_n > 0$ are the singular values of H_ϕ . Then

$$\begin{aligned} \mu_k &= \inf \{ \|H_\phi - A\| \mid \text{rank } A \leq k-1 \} \\ &= \inf \{ \|H_\phi - H_\psi\| \mid \text{rank } H_\psi \leq k-1 \} \\ &= \inf \{ \|\phi - \psi\|_\infty \mid \psi \in H_{[k-1]}^\infty \} \end{aligned} \quad (31)$$

Moreover, the infimum is attained on a unique function $\psi_k = \phi - \frac{H_\phi f_k}{f_k} = \phi - \mu_k \frac{g}{f}$, where (f_k, g_k) is an arbitrary Schmidt pair of H_ϕ that corresponds to μ_k .

Proof: Given $\psi \in H_{[k-1]}^\infty$, we clearly have

$$\begin{aligned} \mu_k &= \inf \{ \|H_\phi - A\| \mid \text{rank } A \leq k-1 \} \\ &\leq \inf \{ \|H_\phi - H_\psi\| \mid \text{rank } H_\psi \leq k-1 \} \\ &\leq \inf \{ \|\phi - \psi\|_\infty \mid \psi \in H_{[k-1]}^\infty \} \end{aligned} \quad (32)$$

so the proof will be complete if we can exhibit a function $\psi_k \in H_{[k-1]}^\infty$ for which the equality $\mu_k = \|\phi - \psi\|_\infty$ holds. To this end let p_k be the minimal degree solution of (19), and define $\psi_k = \frac{\pi_k}{p_k}$. From the equation

$$np_k = \lambda_k d^* p_k^* + d\pi_k$$

we get, dividing by dp_k , that

$$\frac{n}{d} - \frac{\pi_k}{p_k} = \lambda_k \frac{d^* p_k^*}{dp_k}.$$

This is of course equivalent to

$$\psi_k = \frac{\pi_k}{p_k} = \frac{n}{d} - \lambda_k \frac{d^* p_k^*}{dp_k} = \phi - \frac{H_\phi f_k}{f_k}.$$

So

$$\|\phi - \psi\|_\infty = \left\| \frac{n}{d} - \frac{\pi_k}{p_k} \right\|_\infty = \left\| \lambda_k \frac{d^* p_k^*}{dp_k} \right\|_\infty = \mu_k$$

Moreover $\frac{\pi_k}{p_k} \in H_{[k-1]}^\infty$, as p_k has exactly $k - 1$ antistable zeroes. ■

Corollary 5.1 *The polynomials π_k and p_k have no common antistable zeroes.*

Proof: Follows from the fact that $\text{rank} H_{\frac{\pi_k}{p_k}} \geq k - 1$. ■

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