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DIFFEOMORPHISMS BETWEEN SETS OF LINEAR SYSTEMS

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Diffeomorphisms between sets of linear systems

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Abstract

Diffeomorphisms are given between different subsets of linear systems of fixed McMillan degree. The sets considered are the set of all systems of fixed McMillan degree, the subset of stable systems, the subset of bounded real systems, the subset of positive real systems, the subset of stable systems with Hankel singular values bounded by one. State space techniques are used in the proofs.

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1 Introduction

In such areas as system identification, time series analysis or controller design by parameter optimization, often a nonlinear search has to be performed over a specific set of linear systems of fixed McMillan degree. It is therefore important to have knowledge of the structure of such classes of systems. This motivated Brockett [2] to study the topology of the set of single-input single-output linear systems of fixed McMillan degree n. He proved that this set has $n+1$ connected components. Glover [9] showed that in the multivariable situation there is however only one connected component. The same result was established by Hanzon [12] and independently by Ober [19] for the sub-class of (asymptotically) stable systems of McMillan degree n. Results of this type suggested that there might be a close connection between these two classes of systems. Helmke [15] then showed that these two classes are homeomorphic, and Hanzon [13] used a different approach to show that they are in fact diffeomorphic.

Using different types of balanced realizations, Ober [19], [21] and Ober and McFarlane [20] derived canonical forms for several classes of linear systems of fixed McMillan degree: stable, positive real, bounded real, minimum phase and systems without constraints. An interesting aspect of these canonical forms is that they have a remarkably similar structure. This gave further evidence that there should be a strong relationship between these classes of systems. In Ober [20] it was moreover shown that all these sets of systems have identical numbers of connected components.

In this paper we will establish new diffeomorphisms between sets of systems. The diffeomorphism between the set of all systems of fixed McMillan degree and the subset of stable systems is motivated by a map that was studied in much detail in Fuhrmann-Ober [8] and by some state space formulae in Glover-McFarlane [10]. The other diffeomorphisms studied in this paper are in fact adaptations of this map. These maps also explain to a great extent why the canonical forms for the different classes of systems in Ober [21] have such a similar structure.

For single-input single-output systems it was shown in Ober [21] that each minimal, or stable, or bounded real or positive real system can be parametrized by a set of *standard parameters*:

$$
\begin{array}{cl}
\sigma_1 > \ldots > \sigma_j > \ldots > \sigma_k > 0 & \\
n_1, \ldots, n_j, \ldots, n_k & n_j \in \mathcal{N},\ \sum_{j=1}^k n_j = n; \\
s_1, \ldots, s_j, \ldots, s_k & s_j = \pm 1,\ \ 1 \le j \le k; \\
b_1, \alpha(1)_1, \ldots, \alpha(1)_j, \ldots, \alpha(1)_{n_1-1}, & b_1 > 0,\ \alpha(1)_j > 0,\ 1 \le j \le n_1 - 1; \\
\vdots & \\
b_i, \alpha(i)_1, \ldots, \alpha(i)_j, \ldots, \alpha(i)_{n_i-1}, & b_i > 0,\ \alpha(i)_j > 0,\ 1 \le j \le n_i - 1; \\
\vdots & \\
b_k, \alpha(k)_1, \ldots, \alpha(k)_j, \ldots, \alpha(k)_{n_k-1}, & b_k > 0,\ \alpha(k)_j > 0,\ 1 \le j \le n_k - 1; \\
d & d \in \Re
\end{array}
$$

In particular each system in one of the classes of systems has a unique representation, a canonical form, in terms of following 'standard system'.

The *standard system* (A, b, c, d) is then given by

1. $b = (\underbrace{b_1, 0, \ldots, 0}_{n_1}, \ldots, \underbrace{b_j, 0, \ldots, 0}_{n_j}, \ldots, \underbrace{b_k, 0, \ldots, 0}_{n_k})^T$,

2. $c = (\underbrace{s_1 b_1, 0, \ldots, 0}_{n_1}, \ldots, \underbrace{s_j b_j, 0, \ldots, 0}_{n_j}, \ldots, \underbrace{s_k b_k, 0, \ldots, 0}_{n_k})$,

3. For $A =: (A_{ij})_{1 \le i,j \le k}$ we have

a) block diagonal entries A_{jj}, $1 \leq j \leq k$:

$$A_{jj} = \begin{pmatrix} a_{jj} & \alpha(j)_1 & & & & \\ -\alpha(j)_1 & 0 & \alpha(j)_2 & & & \\ & -\alpha(j)_2 & 0 & \cdot & & 0 \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \cdot & 0 & \alpha(j)_{n_j-1} \\ & & & & -\alpha(j)_{n_j-1} & 0 \end{pmatrix}$$

with a_{jj} a function of b_j, σ_j and d.

b) off diagonal blocks A_{ij}, $1 \leq i, j \leq k$, $i \neq j$:

$$A_{ij} = \begin{pmatrix} a_{ij} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \text{ with } a_{ij} \text{ a function of } b_i, b_j, s_i, s_j, \sigma_i, \sigma_j \text{ and } d.$$

For the case of stable systems we have the following canonical form (Ober [18]). We call a minimal system stable if all its poles are in the open left half plane.

Theorem 1.1 *The following two statements are equivalent:*

(i) $g(s)$ is the transfer function of a stable minimal system over $\Re$ of McMillan degree n.

(ii) $g(s)$ has a standard n-dimensional realization (A, b, c, d) given by a standard set of parameters such that

$$a_{ij} = \frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}.$$

Moreover, the map which assigns to each stable minimal system the realization in (ii) is a canonical form.

The canonical form presented in the previous theorem is in the form of a Lyapunov balanced realization (Moore [17]).

Definition 1.1 *A stable minimal system (A, B, C, D) is called* Lyapunov balanced *if*

$$A\Sigma + \Sigma A^* = -BB^*,$$

$$A^*\Sigma + \Sigma A = -C^*C,$$

with $\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \dots, \sigma_k I_{n_k})$, $\sigma_1 > \sigma_2 > \dots > \sigma_k > 0$. The matrix Σ is called the Lyapunov grammian *of the system.*

The canonical form quoted in the previous theorem is Lyapunov balanced with Lyapunov grammian $\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \dots, \sigma_k I_{n_k})$. Another interesting property of the canonical form is that it is sign-symmetric. Indeed if

$$S = diag(s_1 \hat{I}_{n_1}, \dots, s_j \hat{I}_{n_j}, \dots, s_k \hat{I}_{n_k}),$$

where $\hat{I}_{n_j} = diag(+1, -1, +1, \ldots, (-1)^{n_j+1}) \in \Re^{n_j \times n_j}$, $j = 1, \ldots, k$, then

$$A = SA^T S, \quad c^T = Sb.$$

It should also be noted that the Cauchy index of a system is given by $trace(S)$ (see Anderson [1]).

In Section 2 we introduce a map, the so-called L-characteristic, that maps not necessarily stable minimal systems to stable minimal systems of the same McMillan degree. It will be shown that this characteristic map is in fact a bijection between the set of minimal systems of fixed McMillan degree and its subset of all stable minimal systems. This map will also be analyzed from the point of view LQG-balanced realizations and Lyapunov realizations. Sections 3-5 contain the analogous analyzes for bounded real systems, positive real systems and antistable systems. Finally in Section 6 it is shown that the bijections are in fact diffeomorphisms.

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2 Minimal systems

The aim of this section is to establish a bijection between the set of minimal systems of dimension n and the set of all stable minimal systems of dimension n. Note that we mean by a stable system a system whose poles are all in the open left half plane. This bijection which we call the L-characteristic map is in fact a map that occurred implicitly in the work by Glover and McFarlane [10] and was analyzed from an operator theoretic point of view in Fuhrmann and Ober [8].

To simplify presentation we introduce the notation,

$$A_L := A - B(I + D^*D)^{-1}D^*C,$$

for a given linear system (A, B, C, D). Note that $A_L = A$ if $D = 0$.

Definition 2.1 *Let* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *be a minimal system. Let Y be the stabilizing solution of the Riccati equation*

$$0 = A_L^*Y + YA_L - YB(I + D^*D)^{-1}B^*Y + C^*(I + DD^*)^{-1}C,$$

i.e. $A_L - B(I + D^*D)^{-1}B^*Y$ is stable, and let Z be the stabilizing solution to the Riccati equation

$$0 = A_LZ + ZA_L^* - ZC^*(I + DD^*)^{-1}CZ + B(I + D^*D)^{-1}B^*,$$

i.e. $A_L - ZC^*(I + DD^*)^{-1}C$ is stable.

Then the system

$$\chi_L\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right) := \left(\begin{array}{c|c} A_L - B(I + D^*D)^{-1}B^*Y & B(I + D^*D)^{-1/2} \\ \hline (I + DD^*)^{-1/2}C(I + ZY) & D \end{array}\right)$$

is called the L-characteristic *of the system* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$.

The following relationships are due to Bucy [3]. Since the reference is difficult to find we give a short proof.

Lemma 2.1 (Bucy relationships) *Let* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *be a minimal system. Let* Y *be a solution of the Riccati equation*

$$0 = A_L^* Y + Y A_L - Y B(I + D^* D)^{-1} B^* Y + C^*(I + DD^*)^{-1} C,$$

and let Z *be a solution of the Riccati equation*

$$0 = A_L Z + Z A_L^* - Z C^*(I + DD^*)^{-1} C Z + B(I + D^* D)^{-1} B^*,$$

then

$$[I + ZY](A_L - B(I + D^* D)^{-1} B^* Y) = (A_L - ZC^*(I + DD^*)^{-1} C)[I + ZY].$$

Proof: Consider the two Riccati equations,

$$0 = A_L^* Y + Y A_L - Y B(I + D^* D)^{-1} B^* Y + C^*(I + DD^*)^{-1} C,$$

and

$$0 = A_L Z + Z A_L^* - Z C^*(I + DD^*)^{-1} C Z + B(I + D^* D)^{-1} B^*.$$

Multiplying the first equation on the left by Z and the second equation on the right by Y, equating both equations and adding A_L to both sides we obtain

$$A_L + Z A_L^* Y + Z Y A_L - Z Y B(I + D^* D)^{-1} B^* Y + Z C^*(I + DD^*)^{-1} C$$

$$= A_L + A_L Z Y + Z A_L^* Y - Z C^*(I + DD^*)^{-1} C Z Y + B(I + D^* D)^{-1} B^* Y.$$

Canceling the term $Z A_L^* Y$ from either side and collecting terms, we obtain

$$[I + ZY](A_L - B(I + D^* D)^{-1} B^* Y) = (A_L - ZC^*(I + DD^*)^{-1} C)[I + ZY].$$

□

As a consequence of the Bucy relationships we can rewrite the L-characteristic of a system as follows,

$$\chi_L\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right) :=$$

$$\left(\begin{array}{c|c} [I + ZY]^{-1}(A_L - ZC^*(I + DD^*)^{-1} C)[I + ZY]) & B(I + D^* D)^{-1/2} \\ \hline (I + DD^*)^{-1/2} C(I + ZY) & D \end{array}\right).$$

The following Lemma shows that the L-characteristic map maps a system with no stability assumptions to a stable system of the same McMillan degree.

Lemma 2.2 *The L-characteristic of a minimal system is stable and minimal. The L-characteristics of two equivalent systems are equivalent.*

Proof: Since Y is the stabilizing solution of the Riccati equation the matrix $A_L - B(I+D^*D)^{-1}B^*Y$ is stable by definition. It is easily seen that the characteristic system is reachable. The observability of the system follows by using the representation of the characteristic in which the A matrix is written in the form resulting from the Bucy relations.

Let $(A,B,C,D) \in L_n^{p,m}$. If Z is the stabilizing solution to the Riccati equation,

$$A_L Z + Z A_L^* - Z(C^*(I+DD^*)^{-1}CZ + B(I+D^*D)^{-1}B^* = 0,$$

then TZT^* is the stabilizing solution to this Riccati equation for the system $(TAT^{-1}, TB, CT^{-1}, D)$, where T is non-singular. Similarly, if Y is the stabilizing solution to

$$A_L^* Y + Y A_L - YB(I+D^*D)^{-1}B^*Y + C^*(I+DD^*)^{-1}C = 0,$$

then $T^{-*}YT^{-1}$ is the stabilizing solution to this Riccati equation for the system $(TAT^{-1}, TB, CT^{-1}, D)$. Using this fact it is easily seen that the L-characteristic of two equivalent systems are equivalent. □

The main theorem of this section will show that the L-characteristic map is in fact a bijection between the set of n-dimensional minimal systems and the set of stable n-dimensional minimal systems. We denote by $L_n^{p,m}$ the set of all minimal n-dimensional systems, with m-dimensional input and p-dimensional output space. The subset of continuous-time stable systems is denoted by $C_n^{p,m}$. Recall that we mean by a stable system what is often referred to as an asymptotically stable system, i.e. all the eigenvalues of the A matrix are in the open left half plane.

In the next definition we are going to define the so-called inverse L-characteristic map $I\chi_L : C_n^{p,m} \to L_n^{p,m}$. We will show that this map is in fact the inverse of the L-characteristic map χ_L.

Definition 2.2 *Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in C_n^{p,m}$ and let P and Q be the solutions to the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then

$$I\chi_L\left(\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D}\end{array}\right)\right) := \left(\begin{array}{c|c} \mathcal{A} + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I+PQ)^{-1} & \mathcal{B}(I+\mathcal{D}^*\mathcal{D})^{1/2} \\ \hline (I+\mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I+PQ)^{-1} & \mathcal{D}\end{array}\right)$$

is called the inverse L-characteristic system.

In order to be able to analyze the inverse L-characteristic map we need the following Lemma.

Lemma 2.3 *Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in C_n^{p,m}$ and let P and Q be such that*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*,$$

$$\mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then

$$[\mathcal{A} + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I+PQ)^{-1}][I+PQ] = [I+PQ][\mathcal{A} + (I+PQ)^{-1}(P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C}]$$

Proof: We have

$$[\mathcal{A} + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I + PQ)^{-1}][I + PQ] = \mathcal{A}(I + PQ) + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})$$

$$= \mathcal{A} + \mathcal{A}PQ + \mathcal{B}\mathcal{B}^*Q + \mathcal{B}\mathcal{D}^*\mathcal{C} = \mathcal{A} + (\mathcal{A}P + \mathcal{B}\mathcal{B}^*)Q + \mathcal{B}\mathcal{D}^*\mathcal{C}$$

$$= \mathcal{A} + (-P\mathcal{A}^*)Q + \mathcal{B}\mathcal{D}^*\mathcal{C} = \mathcal{A} - P(-Q\mathcal{A} - \mathcal{C}^*\mathcal{C}) + \mathcal{B}\mathcal{D}^*\mathcal{C} = \mathcal{A} + PQ\mathcal{A} + (P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C}$$

$$= [I + PQ][\mathcal{A} + (I + PQ)^{-1}(P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C}].$$

□

The following Lemma shows that the inverse L-characteristic system is minimal.

Lemma 2.4 *We have that*

$$I\chi_L(C_n^{p,m}) \subseteq L_n^{p,m}.$$

Proof: Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in C_n^{p,m}$. That $I\chi_L((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}))$ is reachable follows immediately from the given representation. The observability follows in a similar way after rewriting $\mathcal{A} + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I + PQ)^{-1}$ as $(I + PQ)(\mathcal{A} + (I + PQ)^{-1}(P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C})(I + PQ)^{-1}$, using Lemma 2.3. □

The following proposition shows that the characteristic map is injective. We need the following Lemma that shows how the solutions of the Riccati equations of a minimal system are related to the solutions of Lyapunov equations of its L-characteristic system.

Lemma 2.5 *Let* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *be a minimal system and let*

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) := \left(\begin{array}{c|c} A_L - B(I + D^*D)^{-1}B^*Y & B(I + D^*D)^{-1/2} \\ \hline (I + DD^*)^{-1/2}C(I + ZY) & D \end{array}\right)$$

be its L-characteristic system, with Y and Z the solutions to the respective Riccati equations. Then the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*$$

$$\mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}$$

have solutions given by

$$P = (I + ZY)^{-1}Z = Z(I + YZ)^{-1}$$

$$Q = Y(I + ZY) = (I + YZ)Y.$$

Proof: We want to show that with $P = (I + ZY)^{-1}Z = Z(I + YZ)^{-1}$ we have,

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*.$$

To do this consider

$$(I + ZY)[\mathcal{A}P + P\mathcal{A}^*](I + YZ)$$

$$= (I + ZY)[(A_L - B(I + D^*D)^{-1}B^*Y)P + P(A_L - B(I + D^*D)^{-1}B^*Y)^*](I + YZ)$$

$$= (I + ZY)(A_L - B(I + D^*D)^{-1}B^*Y)Z + Z(A_L - B(I + D^*D)^{-1}B^*Y)^*(I + YZ)$$

$$= A_L Z + ZA_L^* + Z(YA_L + A_L^*Y)Z$$

$$-2ZYB(I + D^*D)^{-1}B^*YZ - B(I + D^*D)^{-1}B^*YZ - ZYB(I + D^*D)^{-1}B^*,$$

using the two Riccati equations this gives,

$$= ZC^*(I + DD^*)^{-1}CZ - B(I + D^*D)^{-1}B^*$$

$$+Z[YB(I + D^*D)^{-1}B^*Y - C^*(I + DD^*)^{-1}C)]Z$$

$$-2ZYB(I + D^*D)^{-1}B^*YZ - B(I + D^*D)^{-1}B^*YZ - ZYB(I + D^*D)^{-1}B^*$$

$$= -(I + ZY)B(I + D^*D)^{-1}B^*(I + YZ)$$

$$= -(I + ZY)\mathcal{B}\mathcal{B}^*(I + YZ),$$

which shows the claim.

Now with $Q = Y + YZY$, we have

$$\mathcal{A}^*Q + Q\mathcal{A} = \mathcal{A}^*Y(I + ZY) + (I + YZ)Y\mathcal{A}$$

$$= (A_L^*Y - YB(I + D^*D)^{-1}B^*Y)(I + ZY) + (I + YZ)(YA_L - YB(I + D^*D)^{-1}B^*Y),$$

using the Riccati equation, we have

$$= (-YA_L - C^*(I + DD^*)^{-1}C)(I + ZY) + (I + YZ)(-A_L^*Y - C^*(I + DD^*)^{-1}C)$$

$$= -C^*(I + DD^*)^{-1}C(I + ZY) - (I + YZ)C^*(I + DD^*)^{-1}C$$

$$-A_L^*Y - YA_L - Y(A_LZ + ZA_L^*)Y$$

$$= -C^*(I + DD^*)^{-1}C(I + ZY) - (I + YZ)C^*(I + DD^*)^{-1}C$$

$$-YB(I + D^*D)^{-1}B^*Y + C^*(I + DD^*)^{-1}C - Y(ZC^*(I + DD^*)^{-1}CZ)$$

$$-B(I + D^*D)^{-1}B^*)Y$$

$$= -(I + YZ)C^*(I + DD^*)^{-1}C(I + ZY)$$

$$= -\mathcal{C}^*\mathcal{C}.$$

□

We can now prove the proposition.

Proposition 2.1 *The characteristic map χ_L is injective. More precisely, $I\chi_L \cdot \chi_L$ is the identity map on $L_n^{p,m}$.*

Proof: Let $(A,B,C,D) \in L_n^{p,m}$ and let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in C_n^{p,m}$ be its L-characteristic, i.e.

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) = \left(\begin{array}{c|c} A_L - B(I+D^*D)^{-1}B^*Y) & B(I+D^*D)^{-1/2} \\ \hline (I+DD^*)^{-1/2}C(I+ZY) & D \end{array}\right),$$

where Y and Z are the stabilizing solutions to the respective Riccati equations. We know by Lemma 2.5 that the solutions to the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}$$

are given by

$$P = (I+ZY)^{-1}Z = Z(I+YZ)^{-1},$$

$$Q = Y(I+ZY) = (I+YZ)Y.$$

Hence we can see that $PQ = ZY$. Now apply $I\chi_L$ to $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ and set $(A_1,B_1,C_1,D_1) := I\chi_L((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}))$, then $D_1 = D$ and

$$B_1 = \mathcal{B}(I+\mathcal{D}^*\mathcal{D})^{1/2} = B(I+D^*D)^{-1/2}(I+D^*D)^{1/2} = B,$$

$$C_1 = (I+\mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I+PQ)^{-1} = (I+DD^*)^{1/2}(I+DD^*)^{-1/2}C(I+ZY)(I+ZY)^{-1} = C,$$

$$\begin{aligned} A_1 &= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1} + \mathcal{B}\mathcal{D}^*\mathcal{C}(I+PQ)^{-1} \\ &= A - B(I+D^*D)^{-1}(D^*C+B^*Y) + B(I+D^*D)^{-1}B^*Y(I+ZY)(I+ZY)^{-1} \\ &\quad + B(I+D^*D)^{-1}D^*C(I+ZY)(I+ZY)^{-1} \\ &= A, \end{aligned}$$

i.e. $I\chi_L \cdot \chi_L((A,B,C,D)) = (A,B,C,D)$ for $(A,B,C,D) \in L_n^{p,m}$. □

We now need to prove that χ_L is in fact surjective, or that $\chi_L \cdot I\chi_L$ is the identity map on $C_n^{p,m}$. To do this we need the following Lemma.

Lemma 2.6 *Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in C_n^{p,m}$. Let P, Q be the positive definite solutions to the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^* = 0, \quad \mathcal{A}^*Q + Q\mathcal{A} + \mathcal{C}^*\mathcal{C} = 0.$$

Let

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := I\chi_L\left(\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)\right).$$

Then

$$0 = A_L^*Y + YA_L - YB(I+D^*D)^{-1}B^*Y + C^*(I+DD^*)^{-1}C,$$

$$0 = A_LZ + ZA_L^* - ZC^*(I+DD^*)^{-1}CZ + B(I+D^*D)^{-1}B^*,$$

with

$$Y = Q(I+PQ)^{-1} = (I+QP)^{-1}Q,$$

$$Z = P(I+QP).$$

Moreover, Y and Z are the stabilizing solutions to the Riccati equations.

Proof: First note that

$$\begin{aligned}
A_L &= A - B(I + D^*D)^{-1}D^*C \\
&= \mathcal{A} + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I + PQ)^{-1} - \mathcal{B}(I + \mathcal{D}^*\mathcal{D})^{1/2}(I + \mathcal{D}^*\mathcal{D})^{-1}\mathcal{D}^*(I + \mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I + PQ)^{-1} \\
&= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I + PQ)^{-1}.
\end{aligned}$$

Since,

$$\begin{aligned}
&(I + QP)[A_L^*Y + YA_L - YB(I + D^*D)^{-1}B^*Y + C^*(I + DD^*)^{-1}C](I + PQ) \\
&= (I + QP)[(\mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I + PQ)^{-1})^*Q(I + PQ)^{-1} + (I + QP)^{-1}Q(\mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I + PQ)^{-1}) \\
&\quad -(I + QP)^{-1}Q\mathcal{B}(I + \mathcal{D}^*\mathcal{D})^{1/2}(I + \mathcal{D}^*\mathcal{D})^{-1}(I + \mathcal{D}^*\mathcal{D})^{1/2}\mathcal{B}^*Q(I + PQ)^{-1} \\
&\quad +(I + PQ)^{-*}\mathcal{C}^*(I + \mathcal{D}\mathcal{D}^*)^{1/2}(I + \mathcal{D}\mathcal{D}^*)^{-1}(I + \mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I + PQ)^{-1}](I + PQ) \\
&= (I + QP)\mathcal{A}^*Q + Q\mathcal{B}\mathcal{B}^*Q + Q\mathcal{A}(I + PQ) + Q\mathcal{B}\mathcal{B}^*Q - Q\mathcal{B}\mathcal{B}^*Q + \mathcal{C}^*\mathcal{C} \\
&= \mathcal{A}^*Q + Q\mathcal{A} + \mathcal{C}^*\mathcal{C} + Q(P\mathcal{A}^* + \mathcal{A}P + \mathcal{B}\mathcal{B}^*)Q \\
&= 0,
\end{aligned}$$

we have verified the first identity. Now with $Z = P(I + QP)$ we have

$$\begin{aligned}
&A_LZ + ZA_L^* - ZC^*(I + DD^*)^{-1}CZ + B(I + D^*D)^{-1}B^* \\
&= (\mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I + PQ)^{-1})(I + PQ)P + P(I + QP)(\mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I + PQ)^{-1})^* \\
&\quad -P(I + QP)(I + PQ)^{-*}\mathcal{C}^*(I + \mathcal{D}\mathcal{D}^*)^{1/2}(I + \mathcal{D}\mathcal{D}^*)^{-1}(I + \mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I + PQ)^{-1}(I + PQ)P \\
&\quad +\mathcal{B}(I + \mathcal{D}^*\mathcal{D})^{1/2}(I + \mathcal{D}^*\mathcal{D})^{-1}(I + \mathcal{D}^*\mathcal{D})^{1/2}\mathcal{B}^* \\
&= \mathcal{A}P + \mathcal{A}PQP + \mathcal{B}\mathcal{B}^*QP + P\mathcal{A}^* + PQP\mathcal{A}^* + PQ\mathcal{B}\mathcal{B}^* - P\mathcal{C}^*\mathcal{C}P + \mathcal{B}\mathcal{B}^* \\
&= \mathcal{A}^*P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^* + (\mathcal{A}P + \mathcal{B}\mathcal{B}^*)QP - P\mathcal{C}^*\mathcal{C}P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*) \\
&= 0 - P\mathcal{A}^*QP - P\mathcal{C}^*\mathcal{C}P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*) \\
&= -P(\mathcal{A}^*Q + \mathcal{C}^*\mathcal{C})P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*) \\
&= P(Q\mathcal{A})P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*) \\
&= PQ(\mathcal{A}P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*) \\
&= 0,
\end{aligned}$$

which shows the second identity. Since

$$\begin{aligned}
&A_L - B(I + D^*D)^{-1}B^*Y \\
&= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I + PQ)^{-1} - \mathcal{B}(I + \mathcal{D}^*\mathcal{D})^{1/2}(I + \mathcal{D}^*\mathcal{D})^{-1}(I + \mathcal{D}^*\mathcal{D})^{1/2}\mathcal{B}^*(I + QP)^{-1}Q \\
&= \mathcal{A},
\end{aligned}$$

which is stable and

$$
\begin{aligned}
&A_L - ZC^*(I+DD^*)^{-1}C \\
&= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1} \\
&\quad -P(I+QP)(I+PQ)^{-*}\mathcal{C}^*(I+\mathcal{D}\mathcal{D}^*)^{1/2}(I+\mathcal{D}\mathcal{D}^*)^{-1}(I+\mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I+PQ)^{-1} \\
&= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1} - P\mathcal{C}^*\mathcal{C}(I+PQ)^{-1} \\
&= (I+PQ)[\mathcal{A}+(I+PQ)^{-1}P\mathcal{C}^*\mathcal{C}](I+PQ)^{-1} - P\mathcal{C}^*\mathcal{C}(I+PQ)^{-1} \\
&= (I+PQ)\mathcal{A}(I+PQ)^{-1},
\end{aligned}
$$

is stable, where we have used Lemma 2.3, we have shown that Z, Y are the stabilizing solutions to the Riccati equations. □

We are now in a position to state and prove the main theorem of this section. This theorem shows that the L-characteristic map is bijective.

Theorem 2.1 *The map*

$$\chi_L : L_n^{p,m} \to \mathcal{C}_n^{p,m}$$

is a bijection that preserves system equivalence. We have $\chi_L^{-1} = I\chi_L$.

Proof: In Proposition 2.1 it was shown that χ_L is injective and preserves system equivalence. Therefore it remains to show that χ_L is surjective, or more precisely that $\chi_L \cdot I\chi_L$ is the identity map. Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in \mathcal{C}_n^{p,m}$ and let

$$
\begin{aligned}
\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) &= I\chi_L\left(\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)\right) \\
&= \left(\begin{array}{c|c} \mathcal{A}+\mathcal{B}(\mathcal{B}^*Q+\mathcal{D}^*\mathcal{C})(I+PQ)^{-1} & \mathcal{B}(I+\mathcal{D}^*\mathcal{D})^{1/2} \\ \hline (I+\mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I+PQ)^{-1} & \mathcal{D} \end{array}\right).
\end{aligned}
$$

Now consider

$$
\begin{aligned}
\left(\begin{array}{c|c} \mathcal{A}_1 & \mathcal{B}_1 \\ \hline \mathcal{C}_1 & \mathcal{D}_1 \end{array}\right) &= \chi_L \cdot I\chi_L\left(\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)\right) = \chi_L\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right) \\
&= \left(\begin{array}{c|c} A_L - B(I+D^*D)^{-1}B^*Y & B(I+D^*D)^{-1/2} \\ \hline (I+DD^*)^{-1/2}C(I+ZY) & D \end{array}\right),
\end{aligned}
$$

where Z, Y are the stabilizing solutions to the Riccati equations,

$$0 = A_L^*Y + YA_L - YB(I+D^*D)^{-1}B^*Y + C^*(I+DD^*)^{-1}C,$$

$$0 = A_LZ + ZA_L^* - ZC^*(I+DD^*)^{-1}CZ + B(I+D^*D)^{-1}B^*.$$

By Lemma 2.6

$$Y = Q(I+PQ)^{-1} = (I+QP)^{-1}Q,$$

$$Z = P(I + QP),$$

with Q, P the positive definite solutions to

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Now, using that $A_L = \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1}$ (proof of Lemma 2.6), we have $\mathcal{D}_1 = \mathcal{D}$, and

$$\begin{aligned}
\mathcal{A}_1 &= A_L - B(I + D^*D)^{-1}B^*Y \\
&= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1} - \mathcal{B}(I + \mathcal{D}^*\mathcal{D})^{1/2}(I + \mathcal{D}^*\mathcal{D})^{-1}(I + \mathcal{D}^*\mathcal{D})^{1/2}\mathcal{B}Q(I+PQ)^{-1} \\
&= \mathcal{A}, \\
\mathcal{B}_1 &= B(I + D^*D)^{-1/2} = \mathcal{B}(I + \mathcal{D}^*\mathcal{D})^{1/2}(I + \mathcal{D}^*\mathcal{D})^{-1/2} \\
&= \mathcal{B}, \\
\mathcal{C}_1 &= (I + DD^*)^{-1/2}C(I + ZY) \\
&= (I + \mathcal{D}\mathcal{D}^*)^{-1/2}(I + \mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I+PQ)^{-1}(I+PQ) \\
&= \mathcal{C},
\end{aligned}$$

which shows the claim that $\chi_L \cdot I\chi_L$ is the identity map. Therefore χ_L is invertible with $\chi_L^{-1} = I\chi_L$.
□

The notion of balancing that is appropriate for minimal systems is that of LQG balancing (Jonckheere and Silverman [16]) (see Ober and McFarlane [22] for the non-strictly proper case).

Definition 2.3 *A system $(A, B, C, D) \in L_n^{p,m}$ is called LQG-balanced if*

$$A_L^*\Sigma + \Sigma A_L - \Sigma B(I + D^*D)^{-1}B^*\Sigma + C^*(I + DD^*)^{-1}C = 0,$$

$$A_L\Sigma + \Sigma A_L^* - \Sigma C^*(I + DD^*)^{-1}C\Sigma + B(I + D^*D)^{-1}B^* = 0,$$

for

$$\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k}), \quad \sigma_1 > \sigma_2 > \cdots > \sigma_k > 0.$$

The matrix Σ is called the LQG grammian *of the system.*

In Ober and McFarlane [22] (see also [21]) the following canonical form for SISO systems was given in terms of LQG balanced realizations.

Theorem 2.2 *The following two statements are equivalent:*

(i) $g(s)$ is the transfer function of a minimal system over $\Re$ of McMillan degree n.

(ii) $g(s)$ has a standard n-dimensional realization (A, b, c, d) given by a standard set of parameters such that

$$a_{ij} = \frac{-b_i b_j}{1 + d^2}\left(\frac{1 - s_i s_j \sigma_i \sigma_j}{s_i s_j \sigma_i + \sigma_j} - s_j d\right).$$

Moreover, the realization given in (ii) is LQG balanced with LQG grammian $\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k})$. The map which assigns to each minimal system the realization in (ii) is a canonical form.

For an analysis of the characteristic map from the point of view of balancing it is more appropriate to slightly change the definition of the characteristic map. This is done by performing a state-space transformation on $\chi_L((A, B, C, D))$.

Define for a system $(A, B, C, D) \in L_n^{p,m}$ the modified characteristic map

$$\tilde{\chi}_L\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right)$$

$$= \left(\begin{array}{c|c} T^{1/4}(A_L - B(I + D^*D)^{-1}B^*Y)T^{-1/4} & T^{1/4}B(I + D^*D)^{-1/2} \\ \hline (I + DD^*)^{-1/2}C(I + ZY)T^{-1/4} & D \end{array}\right),$$

where $T := (I + ZY)^*(I + ZY)$ and Z, Y are the stabilizing solutions to the two Riccati equations.

Corollary 2.1 *The map $\tilde{\chi}_L : L_n^{p,m} \to C_n^{p,m}$ is a bijection with the following properties,*

1. *$(A, B, C, D) \in L_n^{p,m}$ is LQG balanced with LQG grammian Σ if and only if $\tilde{\chi}_L((A, B, C, D))$ is Lyapunov balanced with Lyapunov grammian Σ.*
2. *$(A, b, c, d) \in L_n^{1,1}$ is in LQG-balanced canonical form of Theorem 2.2 if and only if $\tilde{\chi}_L((A, b, c, d))$ is in Lyapunov balanced canonical form of Theorem 1.1.*

Proof: 1.) Let $(A, B, C, D) \in L_n^{p,m}$ and let Y, Z be the stabilizing solutions to the two Riccati equations. By Lemma 2.5 $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) = \chi_L((A, B, C, D))$ is such that $P = (I + ZY)^{-1}Z$, $Q = Y(I + ZY)$ solve the Lyapunov equations,

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

If $(\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1) = \tilde{\chi}_L((A, B, C, D))$, then the positive definite solution to the Lyapunov equations,

$$\mathcal{A}_1 P_1 + P_1 \mathcal{A}_1^* = -\mathcal{B}_1\mathcal{B}_1^*, \quad \mathcal{A}_1^* Q_1 + Q_1 \mathcal{A}_1 = -\mathcal{C}_1^*\mathcal{C}_1,$$

are given by

$$P_1 = T^{1/4}PT^{1/4} = ((I + ZY)^*(I + ZY))^{1/4}(I + ZY)^{-1}Z\,((I + ZY)^*(I + ZY))^{1/4},$$

$$Q_1 = T^{-1/4}QT^{-1/4} = ((I + ZY)^*(I + ZY))^{-1/4}Y(I + ZY)((I + ZY)^*(I + ZY))^{-1/4}.$$

If (A, B, C, D) is LQG balanced then $Z = Y = \Sigma$ is diagonal and therefore

$$P_1 = Z = Y = Q_1,$$

i.e. $(\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1)$ is Lyapunov balanced with Lyapunov grammian Σ. The converse follows in the same way.

2.) This follows by straightforward verification. □

3 Bounded real systems

The next class of systems that we will consider are bounded real systems. We call a system *bounded real*, if it is stable and its transfer function satisfies,

$$I - G^*(i\omega)G(i\omega) > 0,$$

for all $\omega \in \Re \cup \{\pm\infty\}$. We denote by $B_n^{p,m} \subseteq L_n^{p,m}$ the subset of bounded real systems. We will proceed as in the previous section and construct a map of the set of bounded real systems into the set of stable systems. Whereas in the previous case the map was onto, this is not the case here. In the present case the map will be a bijection between the class of bounded real systems of McMillan degree n and the set of stable minimal system of the same McMillan degree whose Hankel singular values are less than one.

To simplify presentation, we are going to use the following notation,

$$A_B := A - B(I - D^*D)^{-1}D^*C.$$

We are now going to define the B-characteristic of a bounded real system.

Definition 3.1 *Let* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *be a minimal bounded real system. Let* Y *be the stabilizing solution of the Riccati equation*

$$0 = A_B^*Y + YA_B + YB(I - D^*D)^{-1}B^*Y + C^*(I - DD^*)^{-1}C,$$

i.e. $A_B + B(I - D^*D)^{-1}B^*Y$ *is stable, and let* Z *be the stabilizing solution to the Riccati equation*

$$0 = A_BZ + ZA_B^* + ZC^*(I - DD^*)^{-1}CZ + B(I - D^*D)^{-1}B^*,$$

i.e. $A_B + ZC^*(I - DD^*)^{-1}C$ *is stable.*

Then the system

$$\chi_B\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right) := \left(\begin{array}{c|c} A_B + B(I - D^*D)^{-1}B^*Y & B(I - D^*D)^{-1/2} \\ \hline -(I - DD^*)^{-1/2}C(I - ZY) & D \end{array}\right)$$

is called the B-characteristic *of the system* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$.

These following relations are the relations that are equivalent to the Bucy relations for the case of minimal systems. Note that standard results on the bounded real Riccati equation ([24]) imply that $I - ZY$ is non-singular, where Y and Z are the stabilizing solutions to the two bounded real Riccati equations.

Lemma 3.1 *Let* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *be a minimal bounded real system. Let* Y *be a solution of the Riccati equation*

$$0 = A_B^*Y + YA_B + YB(I - D^*D)^{-1}B^*Y + C^*(I - DD^*)^{-1}C,$$

and let Z *be a solution of the Riccati equation*

$$0 = A_BZ + ZA_B^* + ZC^*(I - DD^*)^{-1}CZ + B(I - D^*D)^{-1}B^*,$$

then

$$[I - ZY](A_B + B(I - D^*D)^{-1}B^*Y) = (A_B + ZC^*(I - DD^*)^{-1}C)[I - ZY].$$

Proof: Consider the two Riccati equations,

$$0 = A_B^* Y + Y A_B + Y B(I - D^* D)^{-1} B^* Y + C^*(I - DD^*)^{-1} C,$$

and

$$0 = A_B Z + Z A_B^* + Z C^*(I - DD^*)^{-1} C Z + B(I - D^* D)^{-1} B^*.$$

Multiplying the first equation on the left by $-Z$ and the second equation on the right by $-Y$, equating both equations and adding A_B to both sides we obtain

$$A_B - Z A_B^* Y - Z Y A_B - Z Y B(I - D^* D)^{-1} B^* Y - Z C^*(I - DD^*)^{-1} C$$

$$= A_B - A_B Z Y - Z A_B^* Y - Z C^*(I - DD^*)^{-1} C Z Y - B(I - D^* D)^{-1} B^* Y.$$

Canceling the term $Z A_B^* Y$ from either side and collecting terms, we obtain

$$[I - ZY](A_B + B(I - D^* D)^{-1} B^* Y) = (A_B + Z C^*(I - DD^*)^{-1} C)[I - ZY].$$

□

As a consequence of these Bucy type relationships we can rewrite the B-characteristic of a bounded-real system as follows,

$$\chi_B\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right) :=$$

$$\left(\begin{array}{c|c} [I - ZY]^{-1}(A_B + ZC^*(I - DD^*)^{-1}C)[I - ZY]) & B(I - D^*D)^{-1/2} \\ \hline -(I - DD^*)^{-1/2}C(I - ZY) & D \end{array}\right)$$

The following Lemma shows that the B-characteristic map maps a bounded real system to a stable system of the same McMillan degree.

Lemma 3.2 *The B-characteristic of a minimal bounded-real system is stable and minimal. The B-characteristics of two equivalent systems are equivalent.*

Proof: Since Y is the stabilizing solution of the Riccati equation the matrix $A_B + B(I - D^*D)^{-1}B^*Y$ is stable by definition. It is easily seen that the characteristic system is reachable. The observability of the system follows by using the representation of the characteristic in which the A matrix has been written in the form resulting from the Bucy type relations. That the B-characteristics of two equivalent systems are equivalent is easily verified. □

In the following Lemma we investigate the solutions of the Lyapunov equations of the B-characteristic system.

Lemma 3.3 *Let* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *be a minimal bounded-real system and let*

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) := \left(\begin{array}{c|c} A_B + B(I - D^*D)^{-1}B^*Y & B(I - D^*D)^{-1/2} \\ \hline -(I - DD^*)^{-1/2}C(I - ZY) & D \end{array}\right)$$

be its B-characteristic system, with Y and Z the stabilizing solutions to the respective Riccati equations. Then the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*$$

$$\mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}$$

have solutions given by

$$P = (I - ZY)^{-1}Z = Z(I - YZ)^{-1}$$

$$Q = Y(I - ZY) = (I - YZ)Y.$$

Proof: We first show that $\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*$. Since $I - ZY$ is invertible this follows from,

$$(I - ZY)(\mathcal{A}P + P\mathcal{A}^*)(I - YZ)$$

$$= (I - ZY)\mathcal{A}Z + Z\mathcal{A}^*(I - YZ)$$

$$= (I - ZY)[A_B + B(I - D^*D)^{-1}B^*Y]Z + Z[A_B + B(I - D^*D)^{-1}B^*Y]^*(I - YZ)$$

$$= A_BZ + ZA_B^* - Z(YA_B + A_B^*Y)Z$$

$$+(I - ZY)B(I - D^*D)^{-1}B^*YZ + ZYB(I - D^*D)^{-1}B^*(I - YZ)$$

$$= -ZC^*(I - DD^*)^{-1}CZ - B(I - D^*D)^{-1}B^*$$

$$-Z[-YB(I - D^*D)^{-1}B^*Y - C^*(I - DD^*)^{-1}C]Z$$

$$+(I - ZY)B(I - D^*D)^{-1}B^*YZ + ZYB(I - D^*D)^{-1}B^*(I - YZ)$$

$$= -B(I - D^*D)^{-1}B^* + ZYB(I - D^*D)^{-1}B^*YZ$$

$$+(I - ZY)B(I - D^*D)^{-1}B^*YZ + ZYB(I - D^*D)^{-1}B^*(I - YZ)$$

$$= -(I - ZY)B(I - D^*D)^{-1}B^*(I - YZ)$$

$$= -(I - ZY)\mathcal{B}\mathcal{B}^*(I - YZ).$$

Let now $Q = Y - YZY$. We are going to show that

$$\mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

We consider

$$\mathcal{A}^*Q + Q\mathcal{A}$$

$$= [A_B + B(I - D^*D)^{-1}B^*Y]^*Y(I - ZY) + (I - YZ)Y[A_B + B(I - D^*D)^{-1}B^*Y]$$

$$= [A_B^*Y + YB(I - D^*D)^{-1}B^*Y](I - ZY) + (I - YZ)[YA_B + YB(I - D^*D)^{-1}B^*Y],$$

using the bounded real Riccati equation this gives,

$$= [-YA_B - C^*(I - DD^*)^{-1}C](I - ZY) + (I - YZ)[-A_B^*Y - C^*(I - DD^*)^{-1}C]$$

$$= -C^*(I - DD^*)^{-1}C(I - ZY) - (I - YZ)C^*(I - DD^*)^{-1}C$$

$$
\begin{aligned}
&-YA_B - A_B^*Y + Y(A_BZ + ZA_B^*)Y \\
&= -C^*(I-DD^*)^{-1}(I-ZY) - (I-YZ)C^*(I-DD^*)^{-1}C \\
&\quad +YB(I-D^*D)^{-1}B^*Y + C^*(I-DD^*)^{-1}C \\
&\quad +Y[-ZC^*(I-DD^*)^{-1}CZ - B(I-D^*D)^{-1}B^*]Y \\
&= -(I-YZ)C^*(I-DD^*)^{-1}C(I-ZY) \\
&= -C^*C.
\end{aligned}
$$

□

A consequence of the previous Lemma is that the image of χ_B is not the whole set $C_n^{p,m}$, but the subset $UC_{n,B}^{p,m}$. A system $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in C_n^{p,m}$ is in $UC_{n,B}^{p,m}$ if

1. $\lambda_{max}(PQ) < 1$, where P, Q are the positive definite solutions to

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C},$$

2. $I - \mathcal{D}^*\mathcal{D} > 0$.

We can now show that the B-characteristic of a bounded real system is minimal and in $UC_{n,B}^{p,m}$.

Lemma 3.4 *We have*

$$\chi_B(B_n^{p,m}) \subseteq UC_{n,B}^{p,m}.$$

Proof: Let $(A,B,C,D) \in B_n^{p,m}$. It follows from standard results on the bounded real Riccati equations that $\lambda_{max}(ZY) < 1$ ([24]), where Z, Y are the stabilizing solutions to the bounded real Riccati equations. If $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) = \chi_B((A,B,C,D))$ and P, Q are the positive definite solutions to the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C},$$

then as a consequence of Lemma 3.3 we have $PQ = ZY$ and therefore that $\lambda_{max}(PQ) < 1$. Clearly, $I - \mathcal{D}^*\mathcal{D} = I - D^*D > 0$ and hence $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in UC_{n,B}^{p,m}$. □

In the next definition we are going to define the inverse B-characteristic map $I\chi_B : UC_{n,B}^{p,m} \to B_n^{p,m}$. Analogously to the case of minimal systems $I\chi_B$ will turn out to be the inverse of the B-characteristic map $\chi_B : B_n^{p,m} \to UC_{n,B}^{p,m}$.

Definition 3.2 *Let* $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in UC_{n,B}^{p,m}$ *and let* P, Q *be the solutions to the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then

$$I\chi_B\left(\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)\right) := \left(\begin{array}{c|c} \mathcal{A} - \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I-PQ)^{-1} & \mathcal{B}(I-\mathcal{D}^*\mathcal{D})^{1/2} \\ \hline -(I-\mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I-PQ)^{-1} & \mathcal{D} \end{array}\right)$$

is called the **inverse B-characteristic system.**

In order to be able to show that $I\chi_B$ maps a stable system in $UC^{p,m}_{n,B}$ to a bounded real system we need the following Lemmas.

Lemma 3.5 *Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in UC^{p,m}_{n,B}$ and let P and Q be such that*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then

$$[\mathcal{A} - \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I - PQ)^{-1}][I - PQ] = [I - PQ][\mathcal{A} - (I - PQ)^{-1}(P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C}].$$

Proof: We have

$$[\mathcal{A} - \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I - PQ)^{-1}][I - PQ]$$

$$= \mathcal{A}(I - PQ) - \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C}) = \mathcal{A} - \mathcal{A}PQ - \mathcal{B}\mathcal{B}^*Q - \mathcal{B}\mathcal{D}^*\mathcal{C} = \mathcal{A} - [\mathcal{A}P + \mathcal{B}\mathcal{B}^*]Q - \mathcal{B}\mathcal{D}^*\mathcal{C}$$

$$= \mathcal{A} - [-P\mathcal{A}^*]Q - \mathcal{B}\mathcal{D}^*\mathcal{C} = \mathcal{A} - PQ\mathcal{A} - P\mathcal{C}^*\mathcal{C} - \mathcal{B}\mathcal{D}^*\mathcal{C}$$

$$= [I - PQ][\mathcal{A} - (I - PQ)^{-1}(P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C}].$$

$\square$

Lemma 3.6 *Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in UC^{p,m}_{n,B}$. Let P, Q be the positive definite solutions to the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^* = 0, \quad \mathcal{A}^*Q + Q\mathcal{A} + \mathcal{C}^*\mathcal{C} = 0.$$

Let

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := I\chi_B\left(\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)\right).$$

Then

$$0 = A_B^*Y + YA_B + YB(I - D^*D)^{-1}B^*Y + C^*(I - DD^*)^{-1}C,$$

$$0 = A_BZ + ZA_B^* + ZC^*(I - DD^*)^{-1}CZ + B(I - D^*D)^{-1}B^*,$$

with

$$Y = Q(I - PQ)^{-1} = (I - QP)^{-1}Q,$$

$$Z = (I - PQ)P = P(I - QP).$$

Moreover, Y and Z are the stabilizing solutions to these two bounded real Riccati equations.

Proof: First note that

$$\begin{aligned}
A_B &= A - B(I - D^*D)^{-1}D^*C \\
&= \mathcal{A} - \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I - PQ)^{-1} + \mathcal{B}(I - \mathcal{D}^*\mathcal{D})^{1/2}(I - \mathcal{D}^*\mathcal{D})^{-1}\mathcal{D}^*(I - \mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I - PQ)^{-1} \\
&= \mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I - PQ)^{-1}.
\end{aligned}$$

Since $I - QP$ is invertible and

$$\begin{aligned}
&(I - QP)[A_B^*Y + YA_B + YB(I - D^*D)^{-1}B^*Y + C^*(I - DD^*)^{-1}C](I - PQ) \\
&= (I - QP)[(\mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I - PQ)^{-1})^*Q(I - PQ)^{-1} + (I - QP)^{-1}Q(\mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I - PQ)^{-1}) \\
&\quad +(I - QP)^{-1}Q\mathcal{B}(I - \mathcal{D}^*\mathcal{D})^{1/2}(I - \mathcal{D}^*\mathcal{D})^{-1}(I - \mathcal{D}^*\mathcal{D})^{1/2}\mathcal{B}^*(I - PQ)^{-1} \\
&\quad +(I - PQ)^{-*}\mathcal{C}^*(I - \mathcal{D}\mathcal{D}^*)^{1/2}(I - \mathcal{D}\mathcal{D}^*)^{-1}(I - \mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I - PQ)^{-1}](I - PQ) \\
&= (I - QP)\mathcal{A}^*Q - Q\mathcal{B}\mathcal{B}^*Q + Q\mathcal{A}(I - PQ) - Q\mathcal{B}\mathcal{B}^*Q + Q\mathcal{B}\mathcal{B}^*Q + \mathcal{C}^*\mathcal{C} \\
&= \mathcal{A}^*Q + Q\mathcal{A} + \mathcal{C}^*\mathcal{C} - Q(P\mathcal{A}^* + \mathcal{A}P + \mathcal{B}\mathcal{B}^*)Q \\
&= 0,
\end{aligned}$$

we have verified the first identity. Now with $Z = P(I - QP)$ we have

$$\begin{aligned}
&A_BZ + ZA_B^* + ZC^*(I - DD^*)^{-1}CZ + B(I - D^*D)^{-1}B^* \\
&= (\mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I - PQ)^{-1})(I - PQ)P + P(I - PQ)(\mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I - PQ)^{-1})^* \\
&\quad +P(I - QP)(I - PQ)^{-*}\mathcal{C}^*(I - \mathcal{D}\mathcal{D}^*)^{1/2}(I - \mathcal{D}\mathcal{D}^*)^{-1}(I - \mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I - PQ)^{-1}(I - PQ)P \\
&\quad +\mathcal{B}(I - \mathcal{D}^*\mathcal{D})^{1/2}(I - \mathcal{D}^*\mathcal{D})^{-1}(I - \mathcal{D}^*\mathcal{D})^{1/2}\mathcal{B}^* \\
&= \mathcal{A}P - \mathcal{A}PQP - \mathcal{B}\mathcal{B}^*QP + P\mathcal{A}^* - PQP\mathcal{A}^* - PQ\mathcal{B}\mathcal{B}^* + P\mathcal{C}^*\mathcal{C}P + \mathcal{B}\mathcal{B}^* \\
&= \mathcal{A}^*P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^* - (\mathcal{A}P + \mathcal{B}\mathcal{B}^*)QP + P\mathcal{C}^*\mathcal{C}P - PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*) \\
&= 0 + P\mathcal{A}^*QP + P\mathcal{C}^*\mathcal{C}P - PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*) \\
&= P(\mathcal{A}^*Q + \mathcal{C}^*\mathcal{C})P - PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*) \\
&= -P(Q\mathcal{A})P - PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*) \\
&= -PQ(\mathcal{A}P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*) \\
&= 0,
\end{aligned}$$

which shows the second identity. Since

$$\begin{aligned}
&A_B + B(I - D^*D)^{-1}B^*Y \\
&= \mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I - PQ)^{-1} + \mathcal{B}(I - \mathcal{D}^*\mathcal{D})^{1/2}(I - \mathcal{D}^*\mathcal{D})^{-1}(I - \mathcal{D}^*\mathcal{D})^{1/2}\mathcal{B}^*Q(I - PQ)^{-1} \\
&= \mathcal{A},
\end{aligned}$$

is stable and

$$A_B + ZC^*(I - DD^*)^{-1}C$$

$$= \mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I - PQ)^{-1}$$

$$+P(I - QP)(I - PQ)^{-*}\mathcal{C}^*(I - \mathcal{D}\mathcal{D}^*)^{1/2}(I - \mathcal{D}\mathcal{D}^*)^{-1}(I - \mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I - PQ)^{-1}$$

$$= \mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I - PQ)^{-1} + P\mathcal{C}^*\mathcal{C}(I - PQ)^{-1}$$

$$= (I - PQ)[\mathcal{A} - (I - PQ)^{-1}P\mathcal{C}^*\mathcal{C}](I + PQ)^{-1} + P\mathcal{C}^*\mathcal{C}(I + PQ)^{-1}$$

$$= (I - PQ)\mathcal{A}(I - PQ)^{-1},$$

is stable, where we have used Lemma 3.5, we have shown that Z, Y are the stabilizing solutions to the bounded real Riccati equations. □

We can now show that $I\chi_B$ maps $UC^{p,m}_{n,B}$ into $B^{p,m}_n$.

Lemma 3.7 *We have*

$$I\chi_B(UC^{p,m}_{n,B}) \subseteq B^{p,m}_n.$$

Proof: Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in UC^{p,m}_{n,B}$ and let $(A,B,C,D) = I\chi_B((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}))$. It follows immediately that (A,B,C,D) is reachable. After rewriting of (A,B,C,D) using the formula of Lemma 3.5 it follows that the system is observable. In Lemma 3.6 it was shown that the two bounded real Riccati equations for the system (A,B,C,D) have positive definite stabilizing solutions Y, Z. This together with the fact that $\lambda_{max}(YZ) = \lambda_{max}(PQ) < 1$ implies ([24]) that (A,B,C,D) is bounded real. □

We are now in a position to show that $\chi_B : B^{p,m}_n \to UC^{p,m}_{n,B}$ is a bijection.

Theorem 3.1 *The map*

$$\chi_B : B^{p,m}_n \to UC^{p,m}_{n,B}$$

is a bijection that preserves system equivalence. The inverse map is given by $\chi_B^{-1} = I\chi_B$.

Proof: That χ_B preserves system equivalence was established in Lemma 3.2.We are next going to show that χ_B is injective, or more precisely that $I\chi_B \cdot \chi_B$ is the identity map. Let $(A,B,C,D) \in B^{p,m}_n$, let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) = \chi_B((A,B,C,D))$ and set $(A_1,B_1,C_1,D_1) := I\chi_B((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}))$. Using Lemma 3.3 we have

$$D_1 = D,$$

$$B_1 = \mathcal{B}(I - \mathcal{D}^*\mathcal{D})^{1/2} = B(I - D^*D)^{-1/2}(I - D^*D)^{1/2} = B,$$

$$C_1 = -(I - \mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I - PQ)^{-1}$$

$$= (I - DD^*)^{-1/2}(I - DD^*)^{1/2}C(I - ZY)(I - ZY)^{-1} = C,$$

$$
\begin{aligned}
A_1 &= \mathcal{A} - \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I - PQ)^{-1} \\
&= A_B + B(I - D^*D)^{-1}B^*Y - B(I - D^*D)^{-1/2}(I - D^*D)^{-1/2}B^*Y \\
&\quad + B(I - D^*D)^{-1/2}D^*(I - DD^*)^{-1/2}C(I - ZY)(I - ZY)^{-1} \\
&= A - B(I - D^*D)^{-1}D^*C + B(I - D^*D)^{-1}D^*C \\
&= A,
\end{aligned}
$$

which shows that $I\chi_B \cdot \chi_B((A,B,C,D)) = (A,B,C,D)$ and hence that χ_B is injective. We now show that χ_B is surjective by showing that $\chi_B \cdot I\chi_B$ is the identity map. Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in UC^{p,m}_{n,B}$, let $(A,B,C,D) := I\chi_B((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}))$ and set $(\mathcal{A}_1,\mathcal{B}_1,\mathcal{C}_1,\mathcal{D}_1) := \chi_B((A,B,C,D))$. Then

$$
\begin{aligned}
\mathcal{D}_1 &= \mathcal{D}, \\
\mathcal{C}_1 &= -(I - D^*D)^{-1/2}C(I - ZY) = (I - \mathcal{D}\mathcal{D}^*)^{-1/2}(I - \mathcal{D}\mathcal{D}^*)^{1/2}\mathcal{C}(I - PQ)^{-1}(I - PQ) \\
&= \mathcal{C}, \\
\mathcal{B}_1 &= B(I - D^*D)^{-1/2} = \mathcal{B}(I - \mathcal{D}^*\mathcal{D})^{1/2}(I - \mathcal{D}^*\mathcal{D})^{-1/2} = \mathcal{B}, \\
\mathcal{A}_1 &= A_B + B(I - D^*D)^{-1}B^*Y \\
&= \mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I - PQ)^{-1} + \mathcal{B}(I - \mathcal{D}^*\mathcal{D})^{1/2}(I - \mathcal{D}^*\mathcal{D})^{-1}(I - \mathcal{D}^*\mathcal{D})^{1/2}\mathcal{B}^*Q(I - PQ)^{-1} \\
&= \mathcal{A}.
\end{aligned}
$$

This shows that χ_B is surjective. Hence we have that χ_B is bijective with inverse $\chi_B^{-1} = I\chi_B$. □

We now come to analyze the previous result from the point of view of balanced realizations. Bounded real balanced realizations were introduced by Opdenacker and Jonckheere [23].

Definition 3.3 *A system $(A,B,C,D) \in B^{p,m}_n$ is called* bounded real balanced *if*

$$
A_B^*\Sigma + \Sigma A_B + \Sigma B(I - D^*D)^{-1}B^*\Sigma + C^*(I - DD^*)^{-1}C = 0,
$$

$$
A_B\Sigma + \Sigma A_B^* + \Sigma C^*(I - DD^*)^{-1}C\Sigma + B(I - D^*D)^{-1}B^* = 0,
$$

for

$$
\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k}), \quad \sigma_1 > \sigma_2 > \cdots > \sigma_k > 0,
$$

and Σ is the stabilizing solution to both equations. The matrix Σ is called the bounded real grammian *of the system.*

In Ober [21] the following canonical form for SISO bounded real systems was given.

Theorem 3.2 *The following two statements are equivalent:*

(i) $g(s)$ is the transfer function of a bounded real system over $\Re$ of McMillan degree n.

(ii) $g(s)$ has a standard n-dimensional realization (A, b, c, d) given by a standard set of parameters such that

$$a_{ij} = \frac{-b_i b_j}{1-d^2}\left(\frac{1+s_i s_j \sigma_i \sigma_j}{s_i s_j \sigma_i + \sigma_j} + s_j d\right),$$

with $|d| < 1$, $\sigma_1 < 1$.

Moreover, the realization given in (ii) is bounded real balanced with bounded real grammian $\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k})$. The map which assigns to each bounded real system the realization in (ii) is a canonical form.

As in the case of minimal systems we now introduce a slightly modified characteristic map. Define for a system $(A, B, C, D) \in B_n^{p,m}$ the modified characteristic map

$$\tilde{\chi}_B\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right)$$

$$= \left(\begin{array}{c|c} T^{1/4}(A_B + B(I-D^*D)^{-1}B^*Y)T^{-1/4} & T^{1/4}B(I-D^*D)^{-1/2} \\ \hline -(I-DD^*)^{-1/2}C(I-ZY)T^{-1/4} & D \end{array}\right),$$

where $T := (I-ZY)^*(I-ZY)$ and Z, Y are the stabilizing solutions to the two bounded real Riccati equations. We have the following corollary.

Corollary 3.1 *The map $\tilde{\chi}_B : B_n^{p,m} \to UC_{n,B}^{p,m}$ is a bijection with the following properties,*

1. *$(A, B, C, D) \in B_n^{p,m}$ is bounded real balanced with bounded real grammian Σ if and only if $\tilde{\chi}_B((A, B, C, D))$ is Lyapunov balanced with Lyapunov grammian Σ.*

2. *$(A, b, c, d) \in B_n^{1,1}$ is in the bounded real balanced canonical form of Theorem 3.2 if and only if $\tilde{\chi}_B((A, b, c, d))$ is in Lyapunov balanced canonical form of Theorem 1.1.*

4 Positive real systems

We are now going to consider positive real systems. We call a square minimal system positive real, if it is stable and its transfer function satisfies,

$$G(i\omega) + G^*(i\omega) > 0,$$

for all $\omega \in \Re \cup \{\pm\infty\}$. We denote by P_n^m the subset of $L_n^{m,m}$ of positive real systems. We again introduce some notation to simplify the presentation. Let (A, B, C, D) be a positive real system, then set,

$$A_P := A - B(D+D^*)^{-1}C^*.$$

The P-characteristic of a positive real system is defined as follows.

Definition 4.1 *Let $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ be a minimal positive real system. Let Y be the stabilizing solution of the Riccati equation*

$$0 = A_P^*Y + YA_P + YB(D+D^*)^{-1}B^*Y + C^*(D+D^*)^{-1}C,$$

i.e. $A_P + B(D + D^*)^{-1}B^*Y$ *is stable, and let* Z *be the stabilizing solution to the Riccati equation*

$$0 = A_P Z + Z A_P^* + Z C^*(D + D^*)^{-1} C Z + B(D + D^*)^{-1} B^*,$$

i.e. $A_P + ZC^*(D + D^*)^{-1}C$ *is stable.*

Then the system

$$\chi_P\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right) := \left(\begin{array}{c|c} A_P + B(D + D^*)^{-1}B^*Y & B(D + D^*)^{-1/2} \\ \hline -(D + D^*)^{-1/2}C(I - ZY) & D \end{array}\right)$$

is called the P-characteristic *of the system* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$.

The following relations are analogous to the Bucy relations for the case of minimal systems. Note that standard results on the positive real Riccati equation (see e.g. [24]) imply that $I - ZY$ is non-singular, where Y and Z are the stabilizing solutions to the two positive real Riccati equations.

Lemma 4.1 *Let* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *be a minimal positive-real system. Let* Y *be a solution of the Riccati equation*

$$0 = A_P^* Y + Y A_P + Y B(D + D^*)^{-1} B^* Y + C^*(D + D^*)^{-1} C,$$

and let Z *be a solution of the Riccati equation*

$$0 = A_P Z + Z A_P^* + Z C^*(D + D^*)^{-1} C Z + B(D + D^*)^{-1} B^*,$$

then

$$[I - ZY](A_P + B(D + D^*)^{-1}B^*Y) = (A_P + ZC^*(D + D^*)^{-1}C)[I - ZY].$$

Proof: Consider the two Riccati equations,

$$0 = A_P^* Y + Y A_P + Y B(D + D^*)^{-1} B^* Y + C^*(D + D^*)^{-1} C,$$

and

$$0 = A_P Z + Z A_P^* + Z C^*(D + D^*)^{-1} C Z + B(D + D^*)^{-1} B^*.$$

Multiplying the first equation on the left by $-Z$ and the second equation on the right by $-Y$, equating both equations and adding A_P to both sides we obtain

$$A_P - Z A_P^* Y - Z Y A_P - Z Y B(D + D^*)^{-1} B^* Y - Z C^*(D + D^*)^{-1} C$$

$$= A_P - A_P Z Y - Z A_P^* Y - Z C^*(D + D^*)^{-1} C Z Y - B(D + D^*)^{-1} B^* Y.$$

Canceling the term $Z A_P^* Y$ from either side and collecting terms, we obtain

$$[I - ZY](A_P + B(D + D^*)^{-1}B^*Y) = (A_P + ZC^*(D + D^*)^{-1}C)[I - ZY].$$

□

As a consequence of these Bucy type relationships we can rewrite the P-characteristic of a positive real system as follows,

$$\chi_P\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right) :=$$

$$\left(\begin{array}{c|c} [I-ZY]^{-1}(A_P + ZC^*(D+D^*)^{-1}C)[I-ZY]) & B(D+D^*)^{-1/2} \\ \hline -(D+D^*)^{-1/2}C(I-ZY) & D \end{array}\right)$$

The following Lemma, that is proved in the standard way, states that the P-characteristic of a positive real system is stable and minimal.

Lemma 4.2 *The P-characteristic of a minimal positive-real system is stable and minimal. The P-characteristics of two equivalent systems are equivalent.*

We now construct the inverse map to the P-characteristic. In order to do this we first have to investigate the solutions of the Lyapunov equations of the characteristic system.

Lemma 4.3 *Let* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *be a minimal positive-real system and let*

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) := \left(\begin{array}{c|c} A_P + B(D+D^*)^{-1}B^*Y & B(D+D^*)^{-1/2} \\ \hline -(D+D^*)^{-1/2}C(I-ZY) & D \end{array}\right)$$

be its P-characteristic system, with Y and Z the stabilizing solutions to the respective Riccati equations. Then the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*$$

$$\mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}$$

have solutions given by

$$P = (I-ZY)^{-1}Z = Z(I-YZ)^{-1}$$

$$Q = Y(I-ZY) = (I-YZ)Y.$$

Proof: We first show that $\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*$. To do this note that $I - ZY$ is invertible and consider,

$$(I-ZY)(\mathcal{A}P + P\mathcal{A}^*)(I-YZ)$$

$$= (I-ZY)\mathcal{A}Z + Z\mathcal{A}^*(I-YZ)$$

$$= (I-ZY)[A_P + B(D+D^*)^{-1}B^*Y]Z + Z[A_P + B(D+D^*)^{-1}B^*Y]^*(I-YZ)$$

$$= A_PZ + ZA_P^* - Z(YA_P + A_P^*Y)Z$$

$$+(I-ZY)B(D+D^*)^{-1}B^*YZ + ZYB(D+D^*)^{-1}B^*(I-YZ)$$

$$= -ZC^*(D+D^*)^{-1}CZ - B(D+D^*)^{-1}B^*$$

$$-Z[-YB(D+D^*)^{-1}B^*Y - C^*(D+D^*)^{-1}C]Z$$

$$+(I-ZY)B(D+D^*)^{-1}B^*YZ + ZYB(D+D^*)^{-1}B^*(I-YZ)$$

$$= -B(D+D^*)^{-1}B^* + ZYB(D+D^*)^{-1}B^*YZ$$

$$+(I-ZY)B(D+D^*)^{-1}B^*YZ + ZYB(D+D^*)^{-1}B^*(I-YZ)$$

$$= -(I - ZY)B(D + D^*)^{-1}B^*(I - YZ)$$

$$= -(I - ZY)\mathcal{B}\mathcal{B}^*(I - YZ).$$

Let now $Q = Y - YZY$. We are now going to show that

$$\mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

We consider

$$\mathcal{A}^*Q + Q\mathcal{A}$$

$$= [A_P + B(D + D^*)^{-1}B^*Y]^*Y(I - ZY) + (I - YZ)Y[A_P + B(D + D^*)^{-1}B^*Y]$$

$$= [A_P^*Y + YB(D + D^*)^{-1}B^*Y](I - ZY) + (I - YZ)[YA_P + YB(D + D^*)^{-1}B^*Y],$$

using the positive real Riccati equation this gives,

$$= [-YA_P - C^*(D + D^*)^{-1}C](I - ZY) + (I - YZ)[-A_P^*Y - C^*(D + D^*)^{-1}C]$$

$$= -C^*(D + D^*)^{-1}C(I - ZY) - (I - YZ)C^*(D + D^*)^{-1}C$$

$$-YA_P - A_P^*Y + Y(A_PZ + ZA_P^*)Y$$

$$= -C^*(D + D^*)^{-1}(I - ZY) - (I - YZ)C^*(D + D^*)^{-1}C$$

$$+YB(D + D^*)^{-1}B^*Y + C^*(D + D^*)^{-1}C + Y[-ZC^*(D + D^*)^{-1}CZ - B(D + D^*)^{-1}B^*]Y$$

$$= -(I - YZ)C^*(D + D^*)^{-1}C(I - ZY)$$

$$= -\mathcal{C}^*\mathcal{C}.$$

□

A consequence of the previous Lemma is that the image of χ_P is not the whole set $C_n^{m,m}$, but the subset $UC_{n,P}^{m,m}$. A system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in C_n^{m,m}$ is in $UC_{n,P}^{m,m}$ if

1. $\lambda_{max}(PQ) < 1$, where P, Q are the positive definite solutions to

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C},$$

2. $\mathcal{D} + \mathcal{D}^* > 0$.

We can now show that the P-characteristic of a positive real system is minimal and in $UC_{n,P}^{m,m}$.

Lemma 4.4 *We have*

$$\chi_B(P_n^m) \subseteq UC_{n,P}^{m,m}.$$

Proof: The proof is analogous to the proof of Lemma 3.4. □

In the next definition we are going to define the inverse P-characteristic map $I\chi_P : UC_{n,P}^{m,m} \to P_n^m$. Analogously to the case of minimal and bounded real systems $I\chi_P$ will turn out to be the inverse of the P-characteristic map $\chi_P : P_n^m \to UC_{n,P}^{m,m}$.

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Definition 4.2 *Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in UC^{m,m}_{n,P}$ and let P, Q be the solutions to the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then

$$I\chi_P\left(\left(\begin{array}{c|c}\mathcal{A} & \mathcal{B}\\ \hline \mathcal{C} & \mathcal{D}\end{array}\right)\right) := \left(\begin{array}{c|c}\mathcal{A} - \mathcal{B}(\mathcal{B}^*Q + \mathcal{C})(I - PQ)^{-1} & \mathcal{B}(\mathcal{D}^* + \mathcal{D})^{1/2}\\ \hline -(\mathcal{D} + \mathcal{D}^*)^{1/2}\mathcal{C}(I - PQ)^{-1} & \mathcal{D}\end{array}\right)$$

is called the inverse P-characteristic system.

Lemma 4.5 *Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in UC^{m,m}_{n,P}$ and let P and Q be such that*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then,

$$[\mathcal{A} - \mathcal{B}(\mathcal{B}^*Q + \mathcal{C})(I - PQ)^{-1}][I - PQ] = [I - PQ][\mathcal{A} - (I - PQ)^{-1}(P\mathcal{C}^* + \mathcal{B})\mathcal{C}].$$

Proof: We have

$$\begin{aligned}
&[\mathcal{A} - \mathcal{B}(\mathcal{B}^*Q + \mathcal{C})(I - PQ)^{-1}][I - PQ]\\
&= \mathcal{A}(I - PQ) - \mathcal{B}(\mathcal{B}^*Q + \mathcal{C}) = \mathcal{A} - \mathcal{A}PQ - \mathcal{B}\mathcal{B}^*Q - \mathcal{B}\mathcal{C} = \mathcal{A} - [\mathcal{A}P + \mathcal{B}\mathcal{B}^*]Q - \mathcal{B}\mathcal{C}\\
&= \mathcal{A} - [-P\mathcal{A}^*]Q - \mathcal{B}\mathcal{C} = \mathcal{A} - PQ\mathcal{A} - P\mathcal{C}^*\mathcal{C} - \mathcal{B}\mathcal{C}\\
&= [I - PQ][\mathcal{A} - (I - PQ)^{-1}(P\mathcal{C}^* + \mathcal{B})\mathcal{C}].
\end{aligned}$$

□

In order to be able to show that $I\chi_P$ maps a stable system in $UC^{m,m}_{n,P}$ to a positive real system was need the following Lemma.

Lemma 4.6 *Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in UC^{m,m}_{n,P}$. Let P, Q be the positive definite solutions to the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^* = 0, \quad \mathcal{A}^*Q + Q\mathcal{A} + \mathcal{C}^*\mathcal{C} = 0.$$

Let

$$\left(\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right) := I\chi_P\left(\left(\begin{array}{c|c}\mathcal{A} & \mathcal{B}\\ \hline \mathcal{C} & \mathcal{D}\end{array}\right)\right).$$

Then,

$$0 = A_P^*Y + YA_P + YB(D + D^*)^{-1}B^*Y + C^*(D + D^*)^{-1}C,$$

$$0 = A_PZ + ZA_P^* + ZC^*(D + D^*)^{-1}CZ + B(D + D^*)^{-1}B^*,$$

with

$$Y = Q(I - PQ)^{-1} = (I - QP)^{-1}Q,$$

$$Z = (I - PQ)P = P(I - QP).$$

Moreover, Y and Z are the stabilizing solutions to these two positive real Riccati equations.

Proof: The proof is completely analogous to the proof of Lemma 3.6. □

We now have that $I\chi_P$ maps $UC^{m,m}_{n,P}$ into P^m_n.

Lemma 4.7 *We have*

$$I\chi_P(UC^{m,m}_{n,B}) \subseteq P^m_n.$$

Proof: The proof is analogous to the proof of Lemma 3.7 . □

We are now in a position to show that $\chi_P : P^m_n \to UC^{m,m}_{n,P}$ is a bijection.

Theorem 4.1 *The map*

$$\chi_P : P^{m,m}_n \to UC^{m,m}_{n,P}$$

is a bijection that preserves system equivalence.
The inverse map $\chi_P^{-1} : UC^{m,m}_{n,P} \to P^m_n$ is given by $\chi_P^{-1} = I\chi_P$.

Proof: The proof is a straightforward verification and analogous to the bounded real case. □

Balancing for positive real systems has been introduced by Desai and Pal [7] (see also [14], [11]).

Definition 4.3 *A system $(A,B,C,D) \in P^m_n$ is called* positive real balanced *if*

$$A_P^*\Sigma + \Sigma A_P + \Sigma B(D+D^*)^{-1}B^*\Sigma + C^*(D+D^*)^{-1}C = 0,$$

$$A_P\Sigma + \Sigma A_P^* + \Sigma C^*(D+D^*)^{-1}C\Sigma + B(D+D^*)^{-1}B^* = 0,$$

for

$$\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k}), \quad \sigma_1 > \sigma_2 > \cdots > \sigma_k > 0,$$

and Σ is the stabilizing solution to both equations. The matrix Σ is called the positive real grammian *of the system.*

In Ober [21] the following canonical form for SISO positive real systems was given.

Theorem 4.2 *The following two statements are equivalent:*

(i) $g(s)$ is the transfer function of a positive real system over $\Re$ of McMillan degree n.

(ii) $g(s)$ has a standard n-dimensional realization (A,b,c,d) given by a standard set of parameters such that

$$a_{ij} = \frac{-b_i b_j(1-s_i\sigma_i)(1-s_j\sigma_j)}{2d(s_i s_j \sigma_i + \sigma_j)},$$

with $d > 0$, $\sigma_1 < 1$.

Moreover, the realization given in (ii) is positive real balanced with positive real grammian $\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \dots, \sigma_k I_{n_k})$. The map which assigns to each positive real system the realization in (ii) is a canonical form.

As in the case of minimal systems we now introduce a slightly modified characteristic map. Define for a system $(A, B, C, D) \in P_n^m$ the modified P-characteristic map

$$\tilde{\chi}_P\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right) = \left(\begin{array}{c|c} T^{1/4}(A_P + B(D+D^*)^{-1}B^*Y)T^{-1/4} & T^{1/4}B(D+D^*)^{-1/2} \\ \hline -(D+D^*)^{-1/2}C(I-ZY)T^{-1/4} & D \end{array}\right),$$

where $T := (I - ZY)^*(I - ZY)$ and Z, Y are the stabilizing solutions to the two positive real Riccati equations. We have the following corollary.

Corollary 4.1 *The map $\tilde{\chi}_P : P_n^m \to UC_{n,P}^{m,m}$ is a bijection with the following properties,*

1. *$(A, B, C, D) \in P_n^{m,m}$ is positive real balanced with positive real grammian Σ if and only if $\tilde{\chi}_P((A, B, C, D))$ is Lyapunov balanced with Lyapunov grammian Σ.*
2. *$(A, b, c, d) \in P_n^1$ is in positive real balanced canonical form of Theorem 4.2 if and only if $\tilde{\chi}_P((A, b, c, d))$ is in Lyapunov balanced canonical form of Theorem 1.1.*

5 Antistable systems

The last class of systems that we will consider is the class of antistable functions. We call a system antistable whose eigenvalues are all in the open left half plane. In this section we are going to study a map from the set of antistable systems of fixed McMillan degree to the set of stable systems of McMillan degree n. There are of course a number of obvious such maps. This map here however appears to be different.

Definition 5.1 *Let $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ be a minimal antistable system. Let Y be the stabilizing solution of the Riccati equation*

$$A^*Y + YA - YBB^*Y = 0,$$

i.e. $A - BB^*Y$ is stable, and let Z be the stabilizing solution to the Riccati equation

$$AZ + ZA^* - ZC^*CZ = 0,$$

i.e. $A - ZC^*C$ is stable.
Then the system

$$\chi_S\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right) := \left(\begin{array}{c|c} A - BB^*Y & B \\ \hline CZY & D \end{array}\right)$$

is called the S-characteristic *of the system* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$.

These following relations are the relations that are equivalent to the Bucy relations for the case of minimal systems.

Lemma 5.1 *Let* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *be a minimal antistable system. Let Y be a solution of the Riccati equation*

$$0 = A * Y + YA - YBB^*Y,$$

and let Z be a solution of the Riccati equation

$$0 = AZ + ZA^* - ZC^*CZ,$$

then

$$ZY(A - BB^*Y) = (A - ZC^*C)ZY.$$

Proof: Consider the two Riccati equations,

$$0 = A^*Y + YA - YBB^*Y$$

and

$$0 = AZ + ZA^* - ZC^*CZ.$$

Multiplying the first equation on the left by Z and the second equation on the right by Y, equating both equations, we obtain

$$ZA^*Y + ZYA - ZYBB^*Y = AZY + ZA^*Y - ZC^*CZY.$$

Canceling the term ZA^*Y from either side and collecting terms, we obtain

$$ZY(A - BB^*Y) = (A - ZC^*C)ZY.$$

□

As a consequence of these Bucy type relationships we can rewrite the characteristic of a system as follows,

$$\chi_P\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right) := \left(\begin{array}{c|c} [ZY]^{-1}(A - ZC^*C)[ZY] & B \\ \hline CZY & D \end{array}\right).$$

The following Lemma shows that the S-characteristic maps antistable systems to stable minimal systems. The proof is by now standard.

Lemma 5.2 *The S-characteristic of a minimal anti-stable system is stable and minimal. The S-characteristics of two equivalent systems are equivalent.*

In the following definition we are going to introduce the candidate map for the inverse of the characteristic map. We denote by $AS_n^{p,m} \subseteq L_n^{p,m}$ the subset of antistable systems.

Definition 5.2 *Let* $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in C_n^{p,m}$ *and let P, Q be the solutions to the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then

$$I\chi_S\left(\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)\right) := \left(\begin{array}{c|c} \mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1} & \mathcal{B} \\ \hline \mathcal{C}(PQ)^{-1} & \mathcal{D} \end{array}\right)$$

is called the inverse S-characteristic system.

To show that $I\chi_S$ maps a stable minimal system to an antistable minimal system we need the following Lemma.

Lemma 5.3 *Let* $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in C_n^{p,m}$ *and let* P *and* Q *be such that*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then

$$[\mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1}]PQ = PQ[\mathcal{A} + Q^{-1}\mathcal{C}^*\mathcal{C}]$$

Proof: We have

$$[\mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1}]PQ = \mathcal{A}PQ + \mathcal{B}\mathcal{B}^*Q = [\mathcal{A}P + \mathcal{B}\mathcal{B}^*]Q = [-P\mathcal{A}^*]Q = -P[-\mathcal{C}^*\mathcal{C} - Q\mathcal{A}]$$
$$= PQ\mathcal{A} + P\mathcal{C}^*\mathcal{C} = PQ[\mathcal{A} + Q^{-1}\mathcal{C}^*\mathcal{C}].$$

□

Lemma 5.4 *We have*

$$I\chi_S(C_n^{p,m}) \subseteq AS_n^{p,m}.$$

Proof: Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in C_n^{p,m}$. The minimality of $(A,B,C,D) = I\chi_S((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}))$ follows in the standard way. To show that A is antistable let P be the positive definite solution to the Lyapunov equation,

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*.$$

Now consider,

$$AP + PA^* - BB^* = (\mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1})P + P(\mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1})^* - \mathcal{B}\mathcal{B}^* = \mathcal{A}P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*$$
$$= 0,$$

which implies that A is antistable. Hence $(A,B,C,D) \in AS_n^{p,m}$. □

In order to prove the main result of this section we will need to again establish connections between solutions to Lyapunov equations and solutions to Riccati equations.

Lemma 5.5 *Let* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *be a minimal antistable system and let*

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) := \left(\begin{array}{c|c} A - BB^*Y & B \\ \hline CZY & D \end{array}\right)$$

be its S-characteristic system, with Y and Z the stabilizing solutions to the respective Riccati equations. Then the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*$$
$$\mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}$$

have solutions given by

$$P = Y^{-1}$$
$$Q = YZY.$$

Proof: We first show that $\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*$. To do this consider

$$Y(\mathcal{A}P + P\mathcal{A}^*)Y = Y\mathcal{A} + \mathcal{A}^*Y = A^*Y + YA - YBB^*Y - YBB^*Y,$$

using the Riccati equation this gives,

$$= YBB^*Y - 2YBB^*Y = -Y\mathcal{B}\mathcal{B}Y,$$

which implies the claim. Now, let $Q = YZY$ and consider,

$$\mathcal{A}^*Q + Q\mathcal{A} = (A - BB^*Y)^*YZY + YZY(A - BB^*Y),$$

using the Bucy type relations, this gives

$$= [(ZY)^{-1}(A - ZC^*C)ZY]^*YZY + YZY(ZY)^{-1}(A - ZC^*C)ZY$$

$$= YZ(A - ZC^*C)^*Y + Y(A - ZC^*C)ZY$$

$$= Y(AZ + ZA^*)Y - 2YZC^*CZY,$$

using the Riccati equation in Z, we obtain,

$$= YZC^*CZY - 2YZC^*CY = -YZC^*CZY = -C^*C,$$

which implies the result. □

Lemma 5.6 *Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in \mathcal{C}_n^{p,m}$. Let P, Q be the positive definite solutions to the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^* = 0, \quad \mathcal{A}^*Q + Q\mathcal{A} + \mathcal{C}^*\mathcal{C} = 0.$$

Let

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := I_{\chi P}\left(\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)\right).$$

Then,

$$0 = A^*Y + YA - YBB^*Y,$$

$$0 = A_PZ + ZA_P^* - ZC^*CZ,$$

with

$$Y = P^{-1},$$

$$Z = PQP.$$

Moreover, Y and Z are the stabilizing solutions to these two Riccati equations.

Proof: Since,

$$QP[A^*Y + YA - YBB^*Y]PQ$$

$$= QP[(\mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1})^*P^{-1} + P^{-1}(\mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1}) - P^{-1}\mathcal{B}\mathcal{B}^*P^{-1}]PQ$$

$$= QP\mathcal{A}^*Q + Q\mathcal{B}\mathcal{B}^*Q + Q\mathcal{A}PQ + Q\mathcal{B}\mathcal{B}^*Q - Q\mathcal{B}\mathcal{B}^*Q$$

$$= Q(P\mathcal{A}^* + \mathcal{A}P + \mathcal{B}\mathcal{B}^*)Q$$

$$= 0,$$

we have verified the first identity. Now with $Z = PQP$ we have

$$AZ + ZA^* - ZC^*CZ$$

$$= (\mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1})PQP + PQP(\mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1})^* - PQP(PQ)^{-*}\mathcal{C}^*\mathcal{C}(PQ)^{-1}PQP$$

$$= \mathcal{A}PQP + \mathcal{B}\mathcal{B}^*QP + PQP\mathcal{A}^* + PQ\mathcal{B}\mathcal{B}^* - P\mathcal{C}^*\mathcal{C}P$$

$$= (\mathcal{A}P + \mathcal{B}\mathcal{B}^*)QP - P\mathcal{C}^*\mathcal{C}P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)$$

$$= -P\mathcal{A}^*QP - P\mathcal{C}^*\mathcal{C}P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)$$

$$= -P(\mathcal{A}^*Q + \mathcal{C}^*\mathcal{C})P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)$$

$$= P(Q\mathcal{A})P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)$$

$$= PQ(\mathcal{A}P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)$$

$$= 0,$$

which shows the second identity. Since

$$A - BB^*Y = \mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1} - \mathcal{B}\mathcal{B}^*P^{-1} = \mathcal{A},$$

which is stable and

$$A - ZC^*C = \mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1} - PQP(PQ)^{-*}\mathcal{C}^*\mathcal{C}(PQ)^{-1}$$

$$= \mathcal{A} + \mathcal{B}\mathcal{B}^*P^{-1} - P\mathcal{C}^*\mathcal{C}(PQ)^{-1}$$

$$= PQ[\mathcal{A} + Q^{-1}\mathcal{C}^*\mathcal{C}](PQ)^{-1} - P\mathcal{C}^*\mathcal{C}(PQ)^{-1}$$

$$= (PQ)\mathcal{A}(PQ)^{-1},$$

is stable, where we have used Lemma 5.3, we have shown that Z, Y are the stabilizing solutions to the Riccati equations. □

We now state the main theorem of this section which shows that the $\mathcal{S}$-characteristic map is a bijection.

Theorem 5.1 *The map*

$$\chi_S : AS_n^{p,m} \to C_n^{p,m}$$

is a bijection that preserves system equivalence. The inverse map $\chi_S^{-1} : C_n^{p,m} \to AS_n^{p,m}$ *is given by* $\chi_S^{-1} = I\chi_S$.

Proof: The proof is analogous to the proof of Theorem 3.1. □

Following the examples of the previous sections we now introduce a balancing scheme for anti-stable systems.

Definition 5.3 *A system* $(A, B, C, D) \in AS_n^{p,m}$ *is called* anti-stable balanced *if*

$$A^*\Sigma + \Sigma A - \Sigma BB^*\Sigma = 0,$$

$$A\Sigma + \Sigma A^* - \Sigma C^*C\Sigma = 0,$$

for

$$\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k}), \quad \sigma_1 > \sigma_2 > \cdots > \sigma_k > 0,$$

and Σ *is the stabilizing solution to both equations. The matrix* Σ *is called the* anti-stable grammian *of the system.*

We are again going to define a modified characteristic map. Let $(A, B, C, D) \in AS_n^{p,m}$ then define

$$\tilde{\chi}_S\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\right) = \left(\begin{array}{c|c} T^{1/4}(A - BB^*Y)T^{-1/4} & T^{1/4}B \\ \hline CZYT^{-1/4} & D \end{array}\right),$$

where $T := (ZY)^*(ZY)$ and Z, Y are the stabilizing solutions to the two Riccati equations. We have the following corollary.

Corollary 5.1 *The map* $\tilde{\chi}_S : AS_n^{p,m} \to C_n^{p,m}$ *is a bijection such that,* $(A, B, C, D) \in AS_n^{p,m}$ *is antistable balanced with anti-stable grammian* Σ *if and only if* $\tilde{\chi}_S((A, B, C, D))$ *is Lyapunov balanced with Lyapunov grammian* Σ.

6 Diffeomorphisms

In the previous sections we have studied bijections between various sets of linear systems. We are now going to show that these maps are in fact diffeomorphisms.

We will need a new notation. To indicate the subsets of strictly proper systems we append the additional subscript 0, e.g. $C_{n,0}^{p,m}$ denotes the strictly proper systems in $C_n^{p,m}$. To indicate the subsets with identity feedthrough term we append the subscript I, e.g. $UC_{n,I}^{m,m}$ denotes the subset of $UC_{n,P}^{m,m}$ with $D = I$.

These sets can be endowed with a topology by embedding them in a natural way in Euclidean space. If we denote by $\sim$ the equivalence relation given by system equivalence then we consider the quotient spaces $L_{n,0}^{p,m}/\sim$, $C_{n,0}^{p,m}/\sim$, ..., to be endowed with the quotient topology.

The key to our way of proceeding is a result by Delchamps [4], [5],[6] that states that stabilizing solutions to Riccati equations are differentiable functions. Similarly, the positive definite solutions to Lyapunov equations are differentiable function of the system matrices. This implies that the bijections that we have constructed are in fact diffeomorphisms. In the same series of papers Delchamps also showed that $B_{n,0}^{p,m}$ and $C_{n,0}^{p,m}$ are diffeomorphic.

From this discussion we immediately have the following result.

Theorem 6.1 *We have that*

$$C^{p,m}_{n,0}/\sim,\ B^{p,m}_{n,0}/\sim,\ L^{p,m}_{n,0}/\sim,\ UC^{p,m}_{n,0}/\sim,\ AS^{p,m}_{p,0}/\sim$$

are diffeomorphic. Moreover,

$$C^{m,m}_{n,0}/\sim,\ B^{m,m}_{n,0}/\sim,\ L^{m,m}_{n,0}/\sim,\ AS^{m,m}_{p,0}/\sim,\ UC^{m,m}_{n,0}/\sim, UC^{m,m}_{n,I}/\sim, P^{m}_{n,I}/\sim$$

are diffeomorphic.

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