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ON THE EQUIVALENCE OF TWO TRANSMISSION BOUNDARY-VALUE PROBLEMS FOR THE TIME-HARMONIC MAXWELL EQUATIONS WITHOUT DISPLACEMENT CURRENTS

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On the Equivalence of two Transmission Boundary - Value Problems for the Time - Harmonic Maxwell Equations without Displacement Currents

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We cosider two transmission boundary - value problems for the time - harmonic Maxwell equations without displacement currents. For the first problem we use the continuity of the tangential parts of the electric and magnetic fields across material discontinuities as transmission conditions. In the second case the continuity of the tangential components of the electric field E is replaced by the continuity of the normal component of the magnetization $B = \mu H$. For this problem existence of solutions is already shown in [6]. If the domains under consideration are not simply connected the solution is not unique. In this paper, we improve the regularity results obtained in [6] and then prove existence and uniqueness theorems for the first problem by extracting its solution out of the set of all solutions of the second problem. Thus we establish a connection between the solutions corresponding to the different transmission boundary conditions.

1. Introduction

A great variety of problems from electrical engineering can be reduced to a transmission boundary-value problem for the time-harmonic Maxwell equations :

Let $G^E \subset \mathbb{R}^3$ be a piece of conductive material surrounded by air. In the open complement G^L of G^E a time-harmonic current density $\tilde{J}_e(x,t) = J_e(x)\, e^{-i\omega t}$ is given (Fig. 1). We are now interested in the currents induced in G^E by $\tilde{J}_e$.

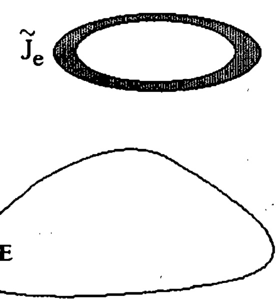

Fig. 1

The resulting *classical transmission boundary-value problem*

$$\left.\begin{aligned} \operatorname{curl} H^L &= J_e - i\omega\varepsilon^L E^L \\ \operatorname{curl} E^L &= i\omega\mu^L H^L \end{aligned}\right\} \text{ in } G^L, \qquad \left.\begin{aligned} \operatorname{curl} H^E &= (\sigma^E - i\omega\varepsilon^E)\, E^E \\ \operatorname{curl} E^E &= i\omega\mu^E H^E \end{aligned}\right\} \text{ in } G^E,$$

$$\left.\begin{array}{l} n \wedge H^E = n \wedge H^L \\ n \wedge E^E = n \wedge E^L \end{array}\right\} \quad \text{on } \Gamma = \partial G^E = \partial G^L,$$

with *Silver - Müller radiation condition*

$$H^L \wedge \frac{x}{|x|} - E^L = o\left(\frac{1}{|x|}\right)$$

and coefficients

$\omega \geq 0$	frequency,
$\varepsilon^L, \varepsilon^E > 0$	electric permittivity in G^L, G^E,
$\mu^L, \mu^E > 0$	magnetic permeability in G^L, G^E,
$\sigma^E > 0$	electric conductivity in G^E,

is well investigated. Under certain regularity assumptions this problem is uniquely solvable [5,8].

In connection with devices working at low frequencies, especially power frequencies, the above equations are usually modified in two different ways. First we neglect the displacement currents and modify the radiation conditions to get

$$\begin{array}{lll} \operatorname{curl} H^L = J_e & & \operatorname{curl} H^E = \sigma^E E^E \\ \operatorname{curl} E^L = i\omega\mu^L H^L & \text{in } G^L, & \operatorname{curl} E^E = i\omega\mu^E H^E \quad \text{in } G^E, \\ \operatorname{div} E^L = 0 & & \end{array}$$

$$\int_{\Gamma_j} n \cdot E^L \, ds = 0, \quad \text{for any connected component } \Gamma_j \text{ of } \Gamma = \partial G^E = \partial G^L, \tag{1}$$

$$H^L(x) = o(1), \qquad E^L(x) = o(1) \qquad \text{uniformly for } |x| \to \infty.$$

and

$$\left.\begin{array}{l} n \wedge H^E = n \wedge H^L \\ n \wedge E^E = n \wedge E^L \end{array}\right\} \quad \text{on } \Gamma. \tag{2}$$

In the second case, the transmission boundary conditions (2) are replaced by

$$\left.\begin{array}{l} n \wedge H^E = n \wedge H^L \\ n \cdot (\mu^E H^E) = n \cdot (\mu^L H^L) \end{array}\right\} \quad \text{on } \Gamma. \tag{3}$$

As we will see in this paper, the solutions of (1),(2) resp. (1),(3) coincide in G^E, if we require G^E to be simply connected. For multiply connected G^E, problem (1),(3) possesses a whole hyperplane of solutions, containing the unique solution of (1),(2).

Before we start with the main part, we want to give a detailed description of the assumptions on the domains G^E, G^L.

Let $C(G)$ ($C^k(G)$) denote the space of continuous (k times continuously differentiable) functions on G.

$G^E \subset \mathbb{R}^3$ is an open, bounded domain with C^2 boundary. The complement $G^L = \mathbb{R}^3 \setminus \overline{G}^E$ should be connected ($\overline{G}^E$ denotes the closure of G^E). G^E is the union of m connected components G_j^E, $j = 1,\dots,m$ having the topological genus p_j. The boundaries $\Gamma_j = \partial G_j^E$ are closed surfaces, which should be disjoint. Setting $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$ we get $\Gamma = \partial G^E = \partial G^L$.

The topological genus of G^E resp. G^L is $p = \sum_{j=1}^{m} p_j$. There exist p surfaces $\Sigma_i^E \subset G^E$ resp. $\Sigma_i^L \subset G^L$, $i = 1,\dots,p$, such that $G^E \setminus \bigcup_{i=1}^{p} \Sigma_i^E$ resp. $G^L \setminus \bigcup_{i=1}^{p} \Sigma_i^L$ are simply connected. The boundary curves $\gamma_i^L = \partial\Sigma_i^E$ and $\gamma_i^E = \partial\Sigma_i^L$ lie on Γ.

Example

Let G^E be a torus. In this case we have $m = p = 1$. The surfaces Σ_1^E, Σ_1^L and the curves γ_1^L, γ_1^E are shown in Fig. 2.

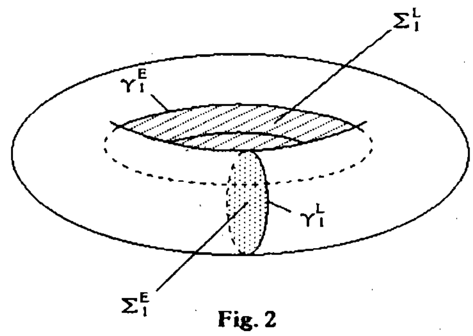

Fig. 2

G^E and G^L having topological genus p, it is well known [4], that there exist p linear independent Neumann fields Z_i^E resp. Z_i^L, $i = 1,\dots,p$, in G^E resp. G^L, fulfilling

$$\operatorname{curl} Z_i^E = 0, \qquad \operatorname{div} Z_i^E = 0 \qquad \text{in } G^E, \qquad n \cdot Z_i^E = 0 \qquad \text{on } \Gamma,$$

$$\operatorname{curl} Z_i^L = 0, \qquad \operatorname{div} Z_i^L = 0 \qquad \text{in } G^L, \qquad n \cdot Z_i^L = 0 \qquad \text{on } \Gamma,$$

$$\int_{\gamma_i^L} \tau \cdot Z_j^L \, dl = \delta_{ij}, \qquad \int_{\gamma_i^E} \tau \cdot Z_j^L \, dl = 0, \qquad \int_{\gamma_i^E} \tau \cdot Z_j^E \, dl = \delta_{ij}, \qquad \int_{\gamma_i^L} \tau \cdot Z_j^E \, dl = 0$$

and

$$Z_i^L(x) = O\left(\frac{1}{|x|^2}\right),$$

uniformly for $|x| \to \infty$. As a consequence of the regularity assumptions on G^E and G^L we get

$$Z_i^E \in C^\infty(G^E) \cap C^{0\alpha}(\overline{G}^E), \qquad Z_i^L \in C^\infty(G^L) \cap C^{0\alpha}(\overline{G}^L).$$

Additionally there exist exactly m linear independent Dirichlet fields D_k^L, $k = 1,\dots,m$, in G^L, m denoting the number of components of G^E, with

$$D_j^L \in C^\infty(G^L) \cap C^{0\alpha}(\overline{G}^L),$$

$$\operatorname{curl} D_j^L = 0, \quad \operatorname{div} D_j^L = 0, \quad \text{in } G^L, \qquad n \wedge D_j^L = 0 \quad \text{on } \Gamma,$$

$$D_j^L = O(\tfrac{1}{|x|^2}) \quad \text{uniformly for } |x| \to \infty.$$

The Dirichlet fields D_j^L can be represented as

$$D_j^L = \operatorname{grad} \varphi_j,$$

$$\Delta\varphi_j = 0 \quad \text{in } G^L, \qquad \varphi_j\big|_{\Gamma_k} = \delta_{jk}, \qquad \varphi(x) = o(1) \quad \text{uniformly for } |x| \to \infty,$$

where Γ_k denotes the boundary of the k^{th} component G_k^E of G^E.

For the prescribed data in (1) we suppose

$$J_e \in C^1(\mathbb{R}^3), \quad \operatorname{div} J_e = 0, \quad \operatorname{supp}(J_e) \subset G^J, \quad \overline{G}^J \subset G^L \text{ bounded.}$$

Moreover, we are looking for classical solutions of (1),(2) resp. (1),(3) satisfying

$$H^L, E^L \in C^1(G^L) \cap C(\overline{G}^L), \qquad H^E, E^E \in C^1(G^E) \cap C(\overline{G}^E).$$

In Chapter 2 we use the following Banach spaces:

Let $0 < \alpha < 1$

- $C^{0\alpha}(G)$, $\quad \|u\|_{0\alpha,G} = \sup\limits_{x\in G} |u(x)| + \sup\limits_{\substack{x \neq y \\ x,y\in G}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$,

  is the space of Hölder continuous functions on G.

- $V^{0\alpha}(\Gamma) = (C^{0\alpha}(\Gamma))^3$, $\quad \|a\|_{V\alpha,\Gamma} = \max\limits_{i=1,2,3}(\|a_i\|_{0\alpha,\Gamma})$,

  is the space of Hölder continuous vector fields on Γ.

- $T^{0\alpha}(\Gamma) = \{a \in V^{0\alpha}(\Gamma) \mid n \cdot a = 0\}$, $\quad \|a\|_{T\alpha,\Gamma} = \|a\|_{V\alpha,\Gamma}$,

  is the space of Hölder continuous tangential fields on Γ.

- $T_d^{0\alpha}(\Gamma) = \{a \in T^{0\alpha}(\Gamma) \mid \operatorname{Div} a \in C^{0\alpha}(\Gamma)\}$, $\quad \|u\|_{d\alpha,\Gamma} = \max(\|u\|_{T\alpha,\Gamma}, \|\operatorname{Div} u\|_{0\alpha,\Gamma})$,

  is the space of Hölder continuous tangential fields on Γ having Hölder continuous surface divergence.

The spaces $T_d^{0\alpha}(\Gamma)$ and $T^{0\alpha}(\Gamma)$ together with the nondegenrate biliniearform

$$<\varphi,\psi> = \int_\Gamma \varphi(x)\, \psi(x)\, ds(x)$$

form a dual system.

2. Uniqueness and Existence

To show existence for problem (1),(3) resp. (1),(2) we make use of the following three Lemmata:

Lemma 1

Consider $J_e \in C^1(\mathbb{R}^3)$, $\operatorname{div} J_e = 0$, $\operatorname{supp}(J_e) \subset G^J$. There exists a vector field $H^J \in C^1(\mathbb{R}^3)$ such that

$$\operatorname{curl} H^J = J_e, \qquad \operatorname{div} H^J = 0, \qquad H^J(x) = O(\tfrac{1}{|x|^2}), \quad \text{uniformly for} \quad |x| \to \infty.$$

Proof

$J_e \in C^1(\mathbb{R}^3)$ has compact support. Therefore $J_e \in C^{0\alpha}(\mathbb{R}^3)$ and together with some well known regularity results for Newtonian potentials [3], we get

$$A = \int_{G^J} J_e(y)\, \Phi_0(x,y)\, dy \quad \in \quad C^2(\mathbb{R}^3).$$

Defining H^J as $H^J = \operatorname{curl} A \in C^1(\mathbb{R}^3)$ we see that

$$\operatorname{div} H^J = 0, \qquad \operatorname{curl} H^J = \operatorname{curl}\operatorname{curl} A = (\operatorname{grad}\operatorname{div} - \Delta)\, A = J_e.$$

Moreover the components of H^J behave as $O(\frac{1}{|x|^2})$ uniformly for $|x| \to \infty$.

■

Lemma 2

Let $H \in C^1(G^L) \cap C^{0\alpha}(\bar{G}^L)$, $H(x) = O(\frac{1}{|x|^2})$ uniformly for $|x| \to \infty$, satisfying $\operatorname{div} H = 0$ in G^L. If

$$\int_{\Gamma_j} n \cdot H\, ds = 0, \qquad j = 1,\dots,m$$

for any connected component Γ_j of Γ, there exists a field $E \in C^1(G^L) \cap C(\overline{G}^L)$, such that

$$\operatorname{curl} E = i\omega\mu^L H, \qquad \operatorname{div} E = 0 \qquad \text{in } G^L,$$

$$n \wedge E\big|_\Gamma \in T_d^{0\alpha}(\Gamma),$$

$$\int_{\Gamma_j} n \cdot E \, ds = 0, \qquad j = 1,\ldots,m,$$

$$E(x) = O\left(\frac{1}{|x|}\right), \qquad \text{uniformly for} \quad |x| \to \infty.$$

Proof

(a) In the first step we consider the interior harmonic Neumann problem

$$\Delta u = 0 \qquad \text{in } G^E,$$

$$\partial_n u = n \cdot H \qquad \text{on } \Gamma.$$

Since

$$\int_{\Gamma_j} n \cdot H \, ds = 0, \qquad j = 1,\ldots,m,$$

the solvability condition for this boundary-value problem is fulfilled [1]. Moreover from $\partial_n u = n \cdot H \in C^{0\alpha}(\Gamma)$ we get $u \in C^{0\alpha}(\overline{G}^L)$ (using an integral equation approach, as it was done in [1]) and therefore $u\big|_\Gamma \in C^{0\alpha}(\Gamma)$.

(b) Consider

$$A(x) = \operatorname{curl}_x \int_{G^L} H(y)\, \Phi_0(x,y)\, dv(y), \qquad \Phi_0(x,y) = \frac{1}{4\pi} \frac{1}{|x-y|}.$$

From [7] we get

$$A \in C^1(G^L) \cap C(\overline{G}^L), \qquad A\big|_\Gamma \in C^{0\alpha}(\Gamma),$$

$$A(x) = O\left(\frac{1}{|x|}\right), \qquad \text{uniformly for} \quad |x| \to \infty.$$

(c) Define E as

$$E(x) = i\omega\mu^L (A(x) + B(x)),$$

$$B(x) = \operatorname{curl}_x \int_\Gamma n(y)\, u(y)\, \Phi_0(x,y)\, ds(y).$$

Corresponding to [1], $B(x) = O\left(\frac{1}{|x|^2}\right)$ uniformly for $|x| \to \infty$, and since by (a) $u\big|_\Gamma \in C^{0\alpha}(\Gamma)$ we get

$$B \in C^2(G^L) \cap C^{0\alpha}(\overline{G}^L).$$

But this means

$$E \in C^1(G^L) \cap C(\overline{G}^L), \qquad n \wedge E\big|_\Gamma \in T^{0\alpha}(\Gamma),$$
$$E(x) = O(\tfrac{1}{|x|}), \qquad \text{uniformly for} \quad |x| \to \infty.$$

Moreover

$$\begin{aligned}
&\mathrm{div}_x\Big(\int_{G^L} H(y)\,\Phi_0(x,y)\,dv(y) + \int_\Gamma n(y)\,u(y)\,\Phi_0(x,y)\,ds(y)\Big)\\
&\quad = -\int_{G^L} H(y)\cdot \mathrm{grad}_y \Phi_0(x,y)\,dv(y) - \int_\Gamma u(y)\,n(y)\cdot \mathrm{grad}_y \Phi_0(x,y)\,ds(y)\\
&\quad = -\int_{G^L} \mathrm{div}_y(H(y)\,\Phi_0(x,y))\,dv(y) - \int_\Gamma u(y)\,\partial_{n_y}\Phi_0(x,y)\,ds(y)\\
&\quad = \int_\Gamma n(y)\cdot H(y)\,\Phi_0(x,y)\,ds(y) - \int_\Gamma u(y)\,\partial_{n_y}\Phi_0(x,y)\,ds(y)\\
&\quad = \int_\Gamma (\partial_{n_y}u(y)\,\Phi_0(x,y) - u(y)\,\partial_{n_y}\Phi_0(x,y))\,ds(y)\\
&\quad = 0 \qquad \text{for } x \in G^L
\end{aligned}$$

due to the representation theorem for harmonic functions. Therefore

$$\begin{aligned}
&\mathrm{curl}\,E(x)\\
&\quad = i\omega\mu^L\,\mathrm{curl}_x\,\mathrm{curl}_x\Big(\int_{G^L} H(y)\,\Phi_0(x,y)\,dv(y) + \int_\Gamma n(y)\,u(y)\,\Phi_0(x,y)\,ds(y)\Big)\\
&\quad = i\omega\mu^L\,(\mathrm{grad}_x\,\mathrm{div}_x - \Delta_x)\Big(\int_{G^L} H(y)\,\Phi_0(x,y)\,dv(y) + \int_\Gamma n(y)\,u(y)\,\Phi_0(x,y)\,ds(y)\Big)\\
&\quad = -\,i\omega\mu^L\,\Delta_x \int_{G^L} H(y)\,\Phi_0(x,y)\,dv(y)\\
&\quad = i\omega\mu^L\,H(x), \qquad x \in G^L.
\end{aligned}$$

(d) E is defined as

$$E(x) = i\omega\mu^L\,\mathrm{curl}_x\Big(\int_{G^L} H(y)\,\Phi_0(x,y)\,dv(y) + \int_\Gamma n(y)\,u(y)\,\Phi_0(x,y)\,ds(y)\Big),$$

so the divergence of E vanishes. Since Γ_j, $j = 1,\ldots,m$, are closed surfaces we get

$$\int_{\Gamma_j} n\cdot E^L\,ds = 0.$$

(e) From (c) we know that $n \wedge E \in T^{0\alpha}(\Gamma)$. According to [1] there holds

$$\mathrm{Div}\,(n \wedge E) = -\,n\cdot \mathrm{curl}\,E\big|_\Gamma = -\,i\omega\mu^L\,n\cdot H\big|_\Gamma \in C^{0\alpha}(\Gamma).$$

Therefore $n \wedge E \in T_d^{0\alpha}(\Gamma)$. ∎

Lemma 3

The boundary - value problem

$$F \in C^1(G^L) \cap C(\overline{G}^L), \qquad \operatorname{curl} F \in C(\overline{G}^L), \qquad F(x) = o(1) \quad \text{uniformly for } |x| \to \infty,$$

$$\operatorname{curl} F = 0, \qquad \operatorname{div} F = 0 \qquad \text{in } G^L, \tag{4}$$

$$n \wedge F = c \in T_d^{0\alpha}(\Gamma) \quad \text{on } \Gamma, \qquad \int_{\Gamma_j} n(y) \cdot F(y)\, ds(y) = 0, \qquad j = 1,\dots,m,$$

is uniquely solvable if and only if

$$\operatorname{Div} c = 0, \qquad \int_\Gamma c(y) \cdot Z_i^L(y)\, ds(y) = 0, \qquad i = 1,\dots,p. \tag{5}$$

Proof

(a) Uniqueness

Setting $c = 0$ in (4), the field F becomes a Dirichlet field in G^L. This means

$$F = \sum_{j=1}^m \alpha_j D_j^L, \qquad \alpha_j \in \mathbb{C}, \quad j = 1,\dots,m.$$

Using the representation

$$D_j^L = \operatorname{grad} \varphi_j,$$

$$\Delta\varphi_j = 0 \quad \text{in } G^L, \qquad \varphi_j\big|_{\Gamma_k} = \delta_{jk}, \qquad \varphi(x) = o(1) \quad \text{uniformly for } |x| \to \infty,$$

and defining B^R and G^R as $B^R := \{x \mid x \in \mathbb{R}^3,\ |x| < R\}$, $G^R = G^L \cap B^R$, we get

$$\int_{\Gamma_k} n \cdot D_j^L\, ds = \int_\Gamma n \cdot D_j^L\, \overline{\varphi}_k\, ds = \lim_{R\to\infty} \Big(\int_{\partial B^R} n \cdot D_j^L\, \overline{\varphi}_k\, ds - \int_{G^R} \operatorname{div}(D_j^L\, \overline{\varphi}_k)\, dv \Big)$$
$$= -\int_{G^L} D_j^L \cdot \overline{D}_k^L\, dv.$$

where $\overline{\varphi}_k$ is the complex conjugate of φ_k. Therefore

$$0 = \int_{\Gamma_k} n \cdot F\, ds = \sum_{j=1}^m \alpha_j \int_{\Gamma_k} n \cdot D_j^L\, ds = -\sum_{j=1}^m \alpha_j \int_{G^L} D_j^L \cdot \overline{D}_k^L\, dv.$$

$$\Rightarrow \quad 0 = \sum_{k=1}^m \overline{\alpha}_k \int_{\Gamma_k} n \cdot F\, ds = -\int_{G^L} \Big(\sum_{j=1}^m \alpha_j\, D_j^L\Big) \Big(\sum_{k=1}^m \overline{\alpha}_k \overline{D}_k^L\Big)\, dv = -\int_{G^L} |F|^2\, dv.$$

$$\Rightarrow \quad F \equiv 0.$$

(b) Existence

" ⇒ "

Let F be a solution of (4). According to [1] we have

$$\operatorname{Div} c = \operatorname{Div}(n \wedge F) = -n \cdot \operatorname{rot} F\big|_{\Gamma} = 0$$

and

$$\begin{aligned}\int_\Gamma c \cdot Z_i^L \, ds &= \int_\Gamma (n \wedge F) \cdot Z_i^L \, ds = \int_\Gamma n \cdot (F \wedge Z_i^L) \, ds \\ &= \lim_{R\to\infty} \Big(\int_{\partial B^R} n \cdot (F \wedge Z_i^L) \, ds - \int_{G^R} \operatorname{div}(F \wedge Z_i^L) \, ds \Big) \\ &= -\int_{G^L} (\operatorname{curl} F \cdot Z_i^L - F \cdot \operatorname{curl} Z_i^L) \, ds \\ &= 0,\end{aligned}$$

where B^R and G^R are defined in the same way as in (a).

" ⇐ "

This direction is shown by using an integral equation approach. The corresponding ansatz for F is

$$F(x) = \operatorname{curl}_x \int_\Gamma a(y)\, \Phi_0(x,y)\, ds(y), \qquad \Phi_0(x,y) = \frac{1}{4\pi} \frac{1}{|x-y|}, \tag{6}$$

$$a \in T_d^{0\alpha}(\Gamma), \qquad \operatorname{Div} a = 0.$$

We immediately get [1]

$$F \in C^\infty(G^L) \cap C^{0\alpha}(\overline{G}^L),$$

$$\operatorname{div} F = 0 \qquad \text{in } G^L,$$

$$F(x) = o(1) \qquad \text{uniformly for } |x| \to \infty.$$

Moreover

$$\begin{aligned}\operatorname{curl} F &= \operatorname{curl}_x \operatorname{curl}_x \int_\Gamma a(y)\, \Phi_0(x,y)\, ds(y) = (\operatorname{grad}_x \operatorname{div}_x - \Delta_x) \int_\Gamma a(y)\, \Phi_0(x,y)\, ds(y) \\ &\overset{[1]}{=} \operatorname{grad}_x \int_\Gamma (\operatorname{Div} a)(y)\, \Phi_0(x,y)\, ds(y) \\ &= 0\end{aligned}$$

in G^L. Using the jump conditions for single and double layer potentials [1], the boundary condition $n \wedge F = c$ may be written as

$$c(x) = n(x) \wedge \int_\Gamma \operatorname{curl}_x(a(y)\, \Phi_0(x,y))\, ds(y) + \frac{1}{2} a(x), \qquad x \in \Gamma.$$

Defining the integral operator M_0 as

$$(M_0 a)(x) = 2\, n(x) \wedge \int_\Gamma \operatorname{curl}_x(a(y)\, \Phi_0(x,y))\, ds(y), \qquad x \in \Gamma$$

our boundary - value problem (4) is reduced to the boundary integral equation

$$(I + M_0)\, a = 2\, c. \tag{7}$$

According to [2] M_0 is compact in $T_d^{0\alpha}(\Gamma)$. Moreover,

$$M_0' : \ T^{0\alpha}(\Gamma) \to T^{0\alpha}(\Gamma), \qquad M_0' a = n \wedge (M_0(n \wedge a))$$

is compact in $T^{0\alpha}(\Gamma)$ and is the adjoint operator of M_0 with respect to the dual system $(T_d^{0\alpha}(\Gamma)\,,\, T^{0\alpha}(\Gamma)\,,\, <\cdot,\cdot>)$. For the nullspace of $I + M_0'$ we have [1]

$$N(I + M_0') = \operatorname{span}\{ v \mid v = Z_i^L|_\Gamma,\ i = 1,\dots,p \}.$$

Now the Fredholm alternative guarantees the solvability of (7) if

$$0 = <c\,,\, Z_i^L> = \int_\Gamma c \cdot Z_i^L\, ds, \qquad i = 1,\dots,p.$$

By (5), this is true for any c we consider.

Now it remains to be shown that for any solution a of (7) the surface divergence $\operatorname{Div} a$ vanishes, provided $\operatorname{Div} c = 0$. Otherwise the equation $\operatorname{rot} F = 0$ is not necessarily fulfilled for F defined through (6). Using the identity $\operatorname{curl}\operatorname{curl} = \operatorname{grad}\operatorname{div} - \Delta$ we deduce

$$\begin{aligned} 0 = \operatorname{Div} c = \operatorname{Div}((n \wedge F)|_\Gamma)) &= -\, n \cdot \operatorname{curl} F|_\Gamma \\ &= -\, n \cdot (\operatorname{curl}_x \operatorname{curl}_x \int_\Gamma a(y)\, \Phi_0(x,y)\, ds(y))|_\Gamma \\ &= -\, n \cdot (\operatorname{grad}_x \int_\Gamma (\operatorname{Div} a)(y)\, \Phi_0(x,y)\, ds(y))|_\Gamma . \end{aligned}$$

With the help of the jump conditions for the derivatives of harmonic single layer potentials we get

$$\int_\Gamma (\operatorname{Div} a)(y)\, \partial_{n_x} \Phi_0(x,y)\, ds(y) \ - \ \tfrac{1}{2} (\operatorname{Div} a)(x) \ = \ 0, \qquad x \in \Gamma,$$

resp.

$$(I - K_0')(\operatorname{Div} a) = 0,$$

$$K_0' : \ C^{0\alpha}(\Gamma) \ \to \ C^{0\alpha}(\Gamma), \qquad (K_0' \lambda)(x) = 2 \int_\Gamma \lambda(y)\, \partial_{n_x} \Phi_0(x,y)\, ds(y).$$

But since for the spectrum of K_0' there holds [1]

$$\sigma(K_0') \subset [-1\ ,1),$$

we finally have $\operatorname{Div} a = 0$, which completes the proof. ■

Theorem 1

For $J_e \in C^1(\mathbb{R}^3)$, $\operatorname{div} J_e = 0$, $\operatorname{supp}(J_e) \subset G^J$, $\overline{G}^J \subset G^L$ bounded, problem (1),(3) is solvable. E^L is not uniquely determined.

In the homogeneous case $J_e = 0$ we get exactly p linear independent solutions H^L, H^E, E^E, where p denotes the topological genus of G^E resp. G^L. The different solutions are characterized through the circulations

$$h_i^L = \int_{\gamma_i^L} \tau \cdot H^L \, dl, \qquad i = 1,\dots,p.$$

Choosing E^L in a suitable way, we have $n \wedge E^L|_\Gamma, n \wedge E^E|_\Gamma \in T_d^{0\alpha}(\Gamma)$.

Proof

Following essentially the lines of [6], we present a modified and extended version of the proof given there. We have to establish the additional regularity properties for $n \wedge E^L|_\Gamma, n \wedge E^E|_\Gamma$, which are not included in [6].

As was shown in [6], problem (1),(3) has at most one solution H^L, H^E, E^E, if the circulations h_i^L are prescribed. E^L is not unique, since by adding any non-vanishing solution of (4) to E^L we get a second vector field, which has all the properties we require of E^L.

For existence we consider H^J from Lemma 1 and

$$H^Z = \sum_{i=1}^{p} (h_i^L - h_i^J)\, Z_i^L, \qquad h_i^J = \int_{\gamma_i^L} \tau \cdot H^J \, dl, \qquad i = 1,\dots,p,$$

where the $h_i^L \in \mathbb{C}$ are the given circulations. Define c and g as

$$c = n \wedge (H^J + H^Z)|_\Gamma, \qquad g = n \cdot (\mu^L H^J)|_\Gamma.$$

From Lemma 1 and the regularity properties of the Neumann fields Z_i^L we immediately get $c \in T^{0\alpha}(\Gamma)$, $g \in C^{0\alpha}(\Gamma)$. But

$$\operatorname{Div} c = - n \cdot \operatorname{rot} (H^J + H^Z)|_\Gamma = 0$$

which means, that $c \in T_d^{0\alpha}(\Gamma)$. Therefore, the following problem is uniquely solvable [6]:

Find $\tilde{H}^L \in C^1(G^L) \cap C^{0\alpha}(\overline{G}^L)$, $H^E, E^E \in C^2(G^E) \cap C^{0\alpha}(\overline{G}^E)$, solving

$$\left.\begin{aligned} \operatorname{curl} \tilde{H}^L &= 0 \\ \operatorname{div} \tilde{H}^L &= 0 \end{aligned}\right\} \text{ in } G^L, \qquad \left.\begin{aligned} \operatorname{curl} H^E &= \sigma^E E^E \\ \operatorname{curl} E^E &= i\omega\mu^E H^E \end{aligned}\right\} \text{ in } G^E,$$

$$\left.\begin{aligned} n \wedge H^E - n \wedge \tilde{H}^L &= c \\ n \cdot (\mu^E H^E) - n \cdot (\mu^L \tilde{H}^L) &= g \end{aligned}\right\} \text{ on } \Gamma,$$

$$\int_{\gamma_i^L} \tau \cdot \tilde{H}^L \, dl = 0 \qquad i = 1,\dots,p,$$

$$\tilde{H}^L(x) = o(1) \qquad \text{uniformly for } |x| \to \infty.$$

Since $\tilde{H}^L$ is a harmonic vector field in G^L, vanishing at infinity, the field H^L, given by

$$H^L = \tilde{H}^L + H^J + H^Z,$$

has the following properties [4]:

$$H^L \in C^1(G^L) \cap C^{0\alpha}(\overline{G}^L),$$

$$H^L(x) = O(\tfrac{1}{|x|^2}), \quad \text{uniformly for } |x| \to \infty.$$

Using $\operatorname{curl} H^J = J_e$, $\operatorname{curl} H^Z = 0$, we get

$$\operatorname{curl} H^L = \operatorname{curl}(\tilde{H}^L + H^J + H^Z) = J_e \qquad \text{in } G^L.$$

Moreover

$$\left.\begin{aligned} n \wedge H^E &= n \wedge \tilde{H}^L + c = n \wedge (\tilde{H}^L + H^J + H^Z) = n \wedge H^L \\ n \cdot (\mu^E H^E) &= n \cdot (\mu^L \tilde{H}^L) + g = n \cdot (\mu^L(\tilde{H}^L + H^J + H^Z)) = n \cdot (\mu^L H^L) \end{aligned}\right\} \quad \text{on } \Gamma$$

and

$$\int_{\gamma_i^L} \tau \cdot H^L \, dl = \int_{\gamma_i^L} \tau \cdot (\tilde{H}^L + H^J + H^Z) \, dl = \int_{\gamma_i^L} \tau \cdot H^J \, dl + h_i^L - h_i^J = h_i^L, \qquad i = 1,\dots,p.$$

Applying Stokes' Theorem we conclude

$$i\omega\mu^L \int_{\Gamma_j} n \cdot H^L ds = i\omega\mu^E \int_{\Gamma_j} n \cdot H^E ds = \mu^E \int_{\Gamma_j} n \cdot \operatorname{curl} E^E ds = 0, \qquad j = 1,\dots,m,$$

because Γ_j, $j = 1,\dots,m$ are closed surfaces. But now Lemma 2 guarantees the existence of $E^L \in C^1(G^L) \cap C(\overline{G}^L)$ satisfying

$$\operatorname{curl} E^L = i\omega\mu^L H^L, \quad \operatorname{div} E^L = 0 \quad \text{in } G^L,$$

$$n \wedge E^L|_\Gamma \in T_d^{0\alpha}(\Gamma),$$

$$\int_{\Gamma_j} n \cdot E^L \, ds = 0, \qquad j = 1,\dots,m,$$

$$E^L(x) = O(\tfrac{1}{|x|}), \qquad \text{uniformly for } |x| \to \infty.$$

Therefore H^L, E^L, H^E, E^E solve (1),(3) with prescribed circulations h_i^L for H^L.

As we mentioned at the beginning of the proof, the three vector fields $H^L, H^E,$

E^E are unique for every choice of circulations h_i^L. This shows, that the homogeneous problem possesses a p-dimensional hyperplane of solutions H^L, H^E, E^E.

From the construction of E^L resp. E^E we conclude

$$n \wedge E^L\big|_\Gamma \in T_d^{0\alpha}(\Gamma), \qquad n \wedge E^E\big|_\Gamma \in T^{0\alpha}(\Gamma).$$

and

$$\mathrm{Div}\,(n \wedge E^E) = -\,n \cdot \mathrm{curl}\, E^E\big|_\Gamma = -\,i\omega n \cdot (\mu^E H^E)\big|_\Gamma \in C^{0\alpha}(\Gamma)$$

so that

$$n \wedge E^L\big|_\Gamma, \quad n \wedge E^E\big|_\Gamma \in T_d^{0\alpha}(\Gamma). \qquad \blacksquare$$

To obtain similar results for problem (1), (2), we proceed in four steps:

(i) Any solution of (1),(2) solves (1),(3).

(ii) We show, that problem (1),(2) has at most one solution. Please note, that in this case E^L is unique.

(iii) *G^E, G^L simply connected*

We consider a solution of (1),(3). According to Theorem 1, H^L, H^E, E^E are in contrast to E^L uniquely determined. With the help of Lemma 3 we are now able to choose E^L in such a way, that (1),(2) are fulfilled.

(iv) *G^E, G^L multiply connected*

Considering again solutions of (1),(3), we have now (besides the nonuniqueness of E^L) additional degrees of freedom, caused by the nontrivial solutions of the homogeneous problem mentioned in Theorem 1. Proceeding in the same way as in (iii), these degrees of freedom are compensated by the now restrictive solvability condition (5) of problem (4). Thus we are able to construct a suitable field E^L, so that (2) holds.

The steps (i) and (ii) are equivalent to the following two lemmata.

Lemma 4

Any solution of (1),(2) also solves (1),(3).

Proof

Consider a solution H^L, E^L, H^E, E^E of (1),(2). For any surface element $S \subset \Gamma$ we get

$$i\omega \int_S n \cdot (\mu^E H^E - \mu^L H^L)\, ds = \int_S n \cdot \operatorname{curl}(E^E - E^L)\, ds = \int_{\partial S} \tau \cdot (E^E - E^L)\, dl = 0.$$

Since S was arbitrary and the normal components of H^L and H^E are continuous on Γ, we get

$$n \cdot (\mu^E H^E - \mu^L H^L) = 0 \qquad \text{on } \Gamma. \qquad \blacksquare$$

Lemma 5

(1),(2) has at most one solution H^L, E^L, H^E, E^E.

Proof

Let H^L, E^L, H^E, E^E be a solution of the homogeneous problem (1),(2). Using the transmission condition (2), we have

$$\int_\Gamma n \cdot (\overline{H}^L \wedge E^L)\, ds = \int_\Gamma n \cdot (\overline{H}^E \wedge E^E)\, ds. \tag{8}$$

Applying the Gaussian theorem to the right hand side we deduce

$$\int_\Gamma n \cdot (\overline{H}^E \wedge E^E)\, ds = \int_{G^E} (\sigma^E E^E \cdot \overline{E}^E - i\omega\mu^E H^E \cdot \overline{H}^E)\, dv \tag{9}$$

Let us now consider $G^R = G^L \cap B^R$, $B^R := \{x \mid x \in \mathbb{R}^3, |x| < R\}$. For large enough R, we get by using again the Gaussian theorem

$$\int_{\partial B^R} \frac{x}{|x|} \cdot (\overline{H}^L \wedge E^L)\, ds - \int_\Gamma n \cdot (\overline{H}^L \wedge E^L)\, ds = \int_{\partial G^R} n' \cdot (\overline{H}^L \wedge E^L)\, ds = -i\omega\mu^L \int_{G^R} H^L \cdot \overline{H}^L\, dv,$$

n' being the outer normal to G^R.

Together with (8) and (9) this means

$$-i\omega\mu^L \int_{G^R} H^L \cdot \overline{H}^L\, ds + \int_{G^E} (\sigma^E E^E \cdot \overline{E}^E - i\omega\mu^E H^E \cdot \overline{H}^E)\, dv = \int_{\partial B^R} \frac{x}{|x|} \cdot (\overline{H}^L \wedge E^L)\, ds. \tag{10}$$

For H^L we have

$$\operatorname{curl} H^L = 0, \qquad \operatorname{div} H^L = 0 \qquad \text{in } G^L, \qquad H^L(x) = o(1) \quad \text{uniformly for } |x| \to \infty.$$

Therefore H^L is a harmonic vectorfield in G^L tending to 0 for $|x| \to \infty$ and thus [4]

$$H^L(x) = O(\frac{1}{|x|^2}), \qquad \text{uniformly for } |x| \to \infty.$$

From $E^L(x) = o(1)$ uniformly for $|x| \to \infty$ we get

$$\left|\frac{x}{|x|} \cdot (\bar{H}^L \wedge E^L)\right| = o(\frac{1}{R^2}) \qquad \text{uniformly on } \partial B^R \quad \text{for} \quad R \to \infty$$

and

$$\int_{\partial B^R} \frac{x}{|x|} \cdot (\bar{H}^L \wedge E^L)\, ds = o(1) \qquad \text{for} \quad R \to \infty.$$

Taking the limit $R \to \infty$, equation (10) yields

$$\sigma^E \int_{G^E} E^E \cdot \bar{E}^E\, dv - i\omega\,(\mu^E \int_{G^E} H^E \cdot \bar{H}^E\, dv + \mu^L \int_{G^L} H^L \cdot \bar{H}^L\, dv) = 0.$$

Since $\omega, \sigma^E, \mu^L, \mu^E$ are real and positive constants, we conclude

$$H^L \equiv 0, \qquad E^E \equiv 0, \qquad H^E \equiv 0$$

and therefore

$$\operatorname{curl} E^L = 0, \qquad \operatorname{div} E^L = 0 \quad \text{in } G^L, \qquad E^L = o(1) \quad \text{uniformly for } |x| \to \infty,$$

$$n \wedge E^L = 0 \quad \text{on } \Gamma, \qquad \int_{\Gamma_j} n \cdot E^L\, ds = 0, \qquad j = 1,\dots,m,$$

so that E^L is a solution of the homogeneous problem (4). According to Lemma 3, E^L vanishes in G^L. ■

***Theorem** 2*

Let G^L, G^E be simply connected. Then (1), (2) possesses a unique solution. Moreover (1), (3) is solvable with H^L, H^E, E^E uniquely determined. The fields H^L, H^E, E^E of both solutions coincide.

Proof

Theorem 1 states the solvability of (1), (3) together with

$$n \wedge E^L|_\Gamma, \quad n \wedge E^E|_\Gamma \in T_d^{0\alpha}(\Gamma).$$

and the required uniqueness. Setting

$$c = n \wedge E^E - n \wedge E^L \in T_d^{0\alpha}(\Gamma)$$

we get

$$\operatorname{Div} c = \operatorname{Div}(n \wedge (E^E - E^L)) = -n \cdot \operatorname{curl}(E^E - E^L)\big|_\Gamma = -i\omega n \cdot (\mu^E H^E - \mu^L H^L)\big|_\Gamma = 0$$

and therefore, by Lemma 3, the boundary-value problem

$$F \in C^1(G^L) \cap C(\overline{G}^L), \qquad \operatorname{curl} F \in C(\overline{G}^L), \qquad F(x) = o(1) \quad \text{uniformly for } |x| \to \infty,$$

$$\operatorname{curl} F = 0, \qquad \operatorname{div} F = 0 \qquad \text{in } G^L,$$

$$n \wedge F = c \in T_d^{0\alpha}(\Gamma) \quad \text{on } \Gamma, \qquad \int_{\Gamma_j} n(y) \cdot F(y)\, ds(y) = 0, \qquad j = 1,\dots,m,$$

is uniquely solvable.

Defining $\tilde{E}^L$ by $\tilde{E}^L = E^L + F$, the fields $H^L, \tilde{E}^L, H^E, E^E$ solve (1), (2). According to Lemma 5, the solution is unique. The fields H^L, H^E, E^E remained unaltered.

■

Theorem 3

Let G^L, G^E be multiply connected with topological genus p. Then (1), (2) is uniquely solvable. Moreover there exists a unique choice of circulations h_i^L for problem (1), (3), such that the fields H^L, H^E, E^E of the corresponding solution of (1), (3) are the same as those obtained from (1), (2).

Proof

Denote by $H_0^L, E_0^L, H_0^E, E_0^E$ a solution of (1), (3) with homogeneous circulations $h_{0i}^L = 0$, $i = 1,\dots,p$. Let $H_k^L, E_k^L, H_k^E, E_k^E$, $k = 1,\dots,p$, be nontrivial solutions of the homogeneous problem (1), (3) ($J_e = 0$) with circulations $h_{ki}^L = \delta_{ki}$, $i,k = 1,\dots,p$. According to Theorem 1, the set of all solutions $H^L, \tilde{E}^L, H^E, E^E$ of (1), (3) can be characterized by

$$\begin{aligned} H^L &= H_0^L + \sum_{k=1}^{p} h_k^L H_k^L \\ H^E &= H_0^E + \sum_{k=1}^{p} h_k^L H_k^E \qquad\qquad h_k^L \in \mathbb{C}, \quad k = 1,\dots,p, \\ E^E &= E_0^E + \sum_{k=1}^{p} h_k^L E_k^E \end{aligned} \tag{11}$$

resp.

$$\begin{aligned} E^L &= E_0^L + \sum_{k=1}^{p} h_k^L E_k^L \\ \tilde{E}^L &= E^L + F, \end{aligned} \tag{12}$$

where the (smooth enough) field F represents the additional degree of freedom for the electric fields in G^L, which is caused by the nonuniqueness mentioned in

Theorem 1. Due to the special choice of circulations for H_0^L and H_k^L the coefficients h_k^L are the circulations of H^L along γ_k^L.

From Lemma 4 we know, that any solution of (1), (2) automatically solves (1), (3). Therefore, if there exists a solution of (1), (2), it can be written in the form (11), (12), for some fixed $h_k^L \in \mathbb{C}$ and fixed F.

For the existence proof we use (11), (12), determine the coefficients h_k^L and modify E^L by F, so that the transmission condition for the tangential components of $\tilde{E}^L, E^E$ holds.

To correct E^L in this way, F has to be a solution of the boundary-value problem (4) with boundary-value $c = n \wedge E^E - n \wedge E^L \in T_d^{0\alpha}(\Gamma)$. But this problem is solvable if and only if

$$\operatorname{Div} c = 0, \qquad \int_\Gamma c(y) \cdot Z_j^L(y)\, ds(y) = 0, \qquad j = 1,\dots,p.$$

c automatically satisfies the first condition, because H^L, E^L, H^E, E^E are solutions of (1), (3) and

$$\operatorname{Div} c = \operatorname{Div}(n \wedge (E^E - E^L)) = -\, n \cdot \operatorname{curl}(E^E - E^L)\big|_\Gamma = -\, i\omega\, n \cdot (\mu^E H^E - \mu^L H^L)\big|_\Gamma = 0.$$

From the second condition, we get

$$\begin{aligned} 0 = \int_\Gamma c \cdot Z_j^L\, ds &= \int_\Gamma (n \wedge (E^E - E^L)) \cdot Z_j^L\, ds \\ &= \int_\Gamma (n \wedge (E_0^E - E_0^L)) \cdot Z_j^L\, ds + \sum_{k=1}^p h_k^L \int_\Gamma (n \wedge (E_k^E - E_k^L)) \cdot Z_j^L\, ds \end{aligned}$$

which is a system of linear equations for the coefficients h_k^L of the form

$$A h = b, \tag{13}$$

$$\begin{aligned} &A = (a_{jk}), && a_{jk} = \int_\Gamma (n \wedge (E_k^E - E_k^L)) \cdot Z_j^L\, ds, && j,k = 1,\dots,p, \\ &h = (h_k^L), && && k = 1,\dots,p, \\ &b = (b_j), && b_j = -\int_\Gamma (n \wedge (E_0^E - E_0^L)) \cdot Z_j^L\, ds && j = 1,\dots,p. \end{aligned}$$

Therefore, the boundary-value problem (4) for F is solvable, if and only if (13) possesses a solution. This is obviously true, if A is a nonsingular matrix.

To show this, suppose that $Ah = 0$. By definition of $H_k^L, E_k^L, H_k^E, E_k^E$,

$$\hat{H}^L = \sum_{k=1}^p h_k^L H_k^L, \qquad \hat{E}^L = \sum_{k=1}^p h_k^L E_k^L, \qquad \hat{H}^E = \sum_{k=1}^p h_k^L H_k^E, \qquad \hat{E}^E = \sum_{k=1}^p h_k^L E_k^E,$$

solve the homogeneous problem (1), (3) with circulations

$$\int_{\gamma_k^L} \tau \cdot \hat{H}^L\, dl = h_k^L.$$

and $Ah = 0$ is equivalent to

$$\int_{\Gamma} (n \wedge (\hat{E}^E - \hat{E}^L)) \cdot Z_j^L \, ds = 0, \qquad j = 1, \ldots, p.$$

Therefore, the boundary-value problem (4) with $c = n \wedge \hat{E}^E - n \wedge \hat{E}^L \in T_d^{0\alpha}(\Gamma)$, has a unique solution $\hat{F}$ and $\hat{H}^L, \hat{E}^L + \hat{F}, \hat{H}^E, \hat{E}^E$ solve the homogeneous problem (1), (2). Lemma 5 states the uniqueness of (1), (2), so that $\hat{H}^L = 0$ and

$$h_k^L = \int_{\gamma_k^L} \tau \cdot \hat{H}^L \, dl = 0.$$

Thus A is injective and (13) is uniquely solvable for all right hand sides b.

Let h now be the unique solution of (13). Defining H^L, E^L, H^E, E^E by using h in (11), (12), we get the existence of the auxiliary field F, which is needed to correct E^L. Now $H^L, \tilde{E}^L = E^L + F, H^E, E^E$ solve (1), (2). The uniqueness of this solution was already shown in Lemma 5. ■

Conclusion

In the simply connected case, the induced fields in G^E are the same for both transmission boundary-value problems (1), (2) and (1), (3).

For multiply connected domains G^L, G^E there exists exactly one choice of circulations

$$h_k^L = \int_{\gamma_k^L} \tau \cdot H^L \, dl$$

for the solutions of (1), (3) so that the induced fields in G^E coincide with those of (1), (2), which are unique.

Proof

This is a direct consequence of the last two theorems. ■

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