On $\mathbb{Q}$-factorial terminalizations of symplectic linear quotient singularities

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Abstract

Symplectic linear quotient singularities belong to the class of symplectic singularities introduced by Beauville in 2000. They are linear quotients by a group preserving a symplectic form on the vector space and are necessarily singular by a classical theorem of Chevalley–Serre–Shephard–Todd. We study $\mathbb{Q}$-factorial terminalizations of such quotient singularities, that is, crepant partial resolutions that are allowed to have mild singularities.

The only symplectic linear quotients that can possibly admit a smooth $\mathbb{Q}$-factorial terminalization are by a theorem of Verbitsky those by symplectic reflection groups. A smooth $\mathbb{Q}$-factorial terminalization is in this context referred to as a symplectic resolution and over the past two decades, there is an ongoing effort to classify exactly which symplectic reflection groups give rise to quotients that admit symplectic resolutions. We reduce this classification to finitely many, precisely 45, open cases by proving that for almost all quotients by symplectically primitive symplectic reflection groups no such resolution exists.

Concentrating on the groups themselves, we prove that a parabolic subgroup of a symplectic reflection group is generated by symplectic reflections as well. This is a direct analogue of a theorem of Steinberg for complex reflection groups.

We further study divisor class groups of $\mathbb{Q}$-factorial terminalizations of linear quotients by finite subgroups G of the special linear group and prove that such a class group is completely controlled by the symplectic reflections – or more generally junior elements – contained in G.

We finally discuss our implementation of an algorithm by Yamagishi for the computation of the Cox ring of a $\mathbb{Q}$-factorial terminalization of a linear quotient in the computer algebra system OSCAR. We use this algorithm to construct a generating system of the Cox ring corresponding to the quotient by a dihedral group of order $2d$ with d odd acting by symplectic reflections. Although our argument follows the algorithm, the proof does not logically depend on computer calculations. We are able to derive the $\mathbb{Q}$-factorial terminalization itself from the Cox ring in this case.

Zusammenfassung

Symplektische lineare Quotientensingularitäten gehören zur Klasse der symplektischen Singularitäten, die 2000 von Beauville eingeführt wurden. Es handelt sich um lineare Quotienten nach Gruppen, die eine symplektische Form auf dem Vektorraum erhalten und daher nach einem klassischen Satz von Chevalley–Serre–Shephard–Todd notwendigerweise singulär sind. Wir betrachten $\mathbb{Q}$-faktorielle Terminalisierungen solcher Quotientensingularitiäten, das heißt krepante partielle Auflösungen, die milde Singularitäten haben können.

Die einzigen symplektischen linearen Quotienten, die potentiell eine glatte $\mathbb{Q}$-faktorielle Terminalisierung haben können, sind nach einem Satz von Verbitsky jene nach symplektischen Spiegelungsgruppen. Eine glatte $\mathbb{Q}$-faktorielle Terminalisierung wird in diesem Kontext als symplektische Auflösung bezeichnet und über die vergangenen zwei

Jahrzehnte gibt es andauernde Bestrebungen genau die symplektischen Spiegelungsgruppen zu klassifizieren, die zu Quotienten führen, welche eine symplektische Auflösung zulassen. Wir reduzieren diese Klassifikation auf endlich viele, nämlich 45, offene Fälle, indem wir zeigen, dass für fast alle Quotienten nach symplektisch primitiven symplektischen Spiegelungsgruppen keine solche Auflösung existiert.

Mit dem Fokus auf die Gruppen selbst zeigen wir, dass eine parabolische Untergruppe einer symplektischen Spiegelungsgruppe wiederum von symplektischen Spiegelungen erzeugt ist. Dies entspricht einem Satz von Steinberg für komplexe Spiegelungsgruppen.

Wir betrachten außerdem Divisorenklassengruppen von $\mathbb{Q}$-faktoriellen Terminalisierungen von linearen Quotienten nach endlichen Untergruppen G der speziellen linearen Gruppe und zeigen, dass solch eine Klassengruppe vollständig von den in G enthaltenen symplektischen Spiegelungen – oder allgemeiner Juniorelementen – kontrolliert wird.

Abschließend diskutieren wir unsere Implementierung eines Algorithmus von Yamagishi zur Berechnung des Cox Rings einer $\mathbb{Q}$-faktoriellen Terminalisierung eines linearen Quotienten im Computeralgebrasystem OSCAR. Mit diesem Algorithmus konstruieren wir ein Erzeugendensystem des Cox Rings, der zu einem Quotienten nach einer Diedergruppe der Ordnung $2d$ mit d ungerade korrespondiert, die via symplektischer Spiegelungen operiert. Obwohl unsere Argumentation auf den Algorithmus aufbaut, ist der Beweis logisch unabhängig von Computerberechnungen. Wir sind in diesem Fall in der Lage die $\mathbb{Q}$-faktorielle Terminalisierung aus dem Cox Ring abzuleiten.

Acknowledgements

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On a more personal note, I want to extend a heartfelt 'Cheers, mates!' to all the other members of the working group kindergarten (occasionally referred to as 'junior researchers') for making the last years so enjoyable with good food, games, films and just pleasant company.

I thank my parents for raising me with the tenet that knowledge is important and that sometimes hard work is necessary to acquire it. Without this or without the unending love they and my brother provide, I would not be where I am now.

Somewhat unconventionally, I thank all the often invisible people providing the open-source software that was invaluable for my work on this project. This obviously includes LaTeX (with its infinite supply of packages) and the open-source computer algebra systems named throughout the text, but also Vim and the whole Linux infrastructure.

Unfortunately, not everything in life is as free as this software and some things even cost money. I therefore thank the Edinburgh Mathematical Society for financially supporting my visit to the University of Glasgow. This work is a contribution to the SFB-TRR 195 'Symbolic Tools in Mathematics and their Application' of the German Research Foundation (DFG).

In the first version of this thesis, I closed the acknowledgements by quoting Hermann Weyl lamenting the burden of writing in a foreign language. As I was since then assured that my command of the English language is adequate, I decided to replace the master of mathematics writing about the challenges of language by a master of language writing about the challenges of mathematics.

> Trouble is, just because things are obvious doesn't mean they're true.
>
> Terry Pratchett, *Wyrd Sisters*

Contents

Introduction

The central object of this thesis are linear quotients, that is, orbit spaces V/G of the action of a finite group G on a finite-dimensional complex vector space V. If the vector space is endowed with a symplectic form and G preserves this form, then the quotient also admits a symplectic structure and we call V/G a *symplectic linear quotient*. By a theorem of Chevalley, Serre, Shephard and Todd [Che55, ST54, Ser68], these quotients are necessarily singular. The singularities belong to the class of *symplectic singularities* as defined by Beauville [Bea00]. In general, it is not possible to resolve these symplectic linear quotient singularities with a crepant morphism, that is, without changing the canonical class. We are therefore interested in $\mathbb{Q}$-*factorial terminalizations*, which are projective partial resolutions that are crepant, but are allowed to have terminal singularities; they are also referred to as *minimal models* in the case of quotient singularities. If $G \leq \mathrm{SL}(V)$, there exists a $\mathbb{Q}$-factorial terminalization for the linear quotient V/G by a deep theorem of Birkar, Cascini, Hacon and McKernan in the context of the minimal model programme [BCHM10]. This notably includes the case of symplectic linear quotients.

If a crepant resolution of a symplectic linear quotient singularity exists, this is also referred to as a *symplectic resolution* as it is exactly a resolution that maintains the symplectic structure. With regard to such resolutions, there is a particular interest in quotients by finite *symplectic reflection groups*. These are finite groups $G \leq \mathrm{GL}_n(\mathbb{C})$ generated by bireflections that preserve a symplectic form; one can identify them with reflection groups over the skew field of quaternions. The reason for this special interest is a theorem of Verbitsky [Ver00], which says that if a symplectic linear quotient admits a symplectic resolution, then the group must be a symplectic reflection group. However, this theorem is not an equivalence and the question exactly which symplectic reflection groups give rise to linear quotients admitting a symplectic resolution enjoyed much interest over the past two decades. A crucial ingredient of this effort is the classification of finite symplectic reflection groups over the complex numbers by Cohen [Coh80]. Cohen in fact classifies the equivalent family of finite quaternion reflection groups; we summarize the classification in Chapter 1 by translating the results to our symplectic setting. In Chapter 2, we collect the tools employed to study $\mathbb{Q}$-factorial terminalizations and in particular the existence of symplectic resolutions, which stem from birational geometry as well as representation theory via the theory of *symplectic reflection algebras* introduced by Etingof and Ginzburg [EG02].

Stabilizer subgroups We study symplectic reflection groups and the corresponding linear quotients from different perspectives. The first perspective is group theoretic and motivated from the fact that the name 'symplectic reflection group' invites one to relate these groups to complex reflection groups. A fundamental result for the latter groups is Steinberg's fixed point theorem [Ste64], which says that any subgroup of a complex reflection group stabilizing a vector is a complex reflection group itself. The question whether the analogous theorem also holds for symplectic reflection groups was already raised by Cohen [Coh80] and in Chapter 3 we answer it in the affirmative; see Theorem 3.1.1 and Corollary 3.1.9. This part of this thesis is already published in [BST23].

Theorem. *Let V be a finite-dimensional symplectic vector space over $\mathbb{C}$, let $G \leq \mathrm{Sp}(V)$ be a finite symplectic reflection group and let U be a subset of V. Then the subgroup of G that fixes U pointwise is also a symplectic reflection group.*

Switching to the geometric perspective, we see that stabilizer subgroups of a finite group $G \leq \mathrm{GL}(V)$ are related to the singularities of the linear quotient V/G via Luna's slice theorem [Lun73]. In this context, there is the already mentioned fundamental theorem by Chevalley, Shephard and Todd [Che55, ST54] saying that the invariant ring $\mathbb{C}[V]^G$ is a polynomial algebra if and only if G is a complex reflection group. Even more amazingly, this implies that V/G is smooth if and only if G is a complex reflection group, see [Ser68]. This tells us that, if G is a symplectic reflection group, then the quotient V/G is singular as $G \leq \mathrm{SL}(V)$. Using the above theorem on stabilizer subgroups, we can prove that the singular locus of a symplectic linear quotient singularity is of pure codimension 2, see Corollary 3.2.3.

Symplectic resolutions Beauville [Bea00] defined a concept of singular varieties with a symplectic structure on the smooth locus bridging the gap between the smooth manifolds in symplectic geometry and the singularities arising in algebraic geometry. Symplectic linear quotients belong to these symplectic singularities and we consider the question whether they admit a symplectic resolution, that is, a desingularization of V/G that preserves the symplectic form. In general, the answer to this question is negative by the theorem of Verbitsky mentioned above; there might only be a symplectic resolution if the group is a symplectic reflection group. This motivates the following classification problem [Ver00, Question 1.5]: for which symplectic reflection groups does the corresponding quotient admit a symplectic resolution? The solution of this problem is ongoing work by many authors over the past two decades starting with the seminal paper by Etingof and Ginzburg [EG02]; we give a detailed overview of this classification in Section 2.2.

The groups for which this classification has so far not been completed is the family of symplectically primitive symplectic reflection groups. This family consists of infinitely many groups; we consider these groups in Chapter 4. Our results are already published in [BST22, BST23].

Theorem. *Let $G \leq \mathrm{Sp}(V)$ be a symplectically irreducible and symplectically primitive symplectic reflection group. Then the symplectic linear quotient V/G does not admit a (projective) symplectic resolution in all but possibly 45 cases.*

With our theorem, the classification is reduced to finitely many cases for the first time. All of the remaining 45 groups are of rank 4; we give the explicit list in Section 4.4. For the proof of the above theorem, we use a theorem by Etingof, Ginzburg and Kaledin [EG02, GK04] that provides a deep link between the geometry of the quotient V/G and the representation theory of symplectic reflection algebras associated to G as introduced in [EG02]. The main idea is that one studies deformations of V/G that maintain the symplectic structure. However, on the algebraic side it turns out that one should not deform the invariant ring $\mathbb{C}[V]^G$ directly, but rather the skew group ring $\mathbb{C}[V] \rtimes G$ leading to the symplectic reflection algebras. The centres of these algebras then give deformations of $\mathbb{C}[V]^G$.

Class groups One sees from the classification that symplectic resolutions are a rare phenomenon not only for symplectic linear quotients in general, but also for quotients by symplectic reflection groups. It therefore makes sense to broaden our focus by generalizing from symplectic resolutions to $\mathbb{Q}$-factorial terminalizations. From Chapter 5 onwards, we do so and also consider the bigger class of linear quotients by subgroups of $\mathrm{SL}_n(\mathbb{C})$ instead of symplectic linear quotients where possible. In fact, one can see symplectic reflection groups as a special case of subgroups of $\mathrm{SL}_n(\mathbb{C})$ generated by *junior elements* as introduced by Ito and Reid [IR96]. In this setting, we extend results by Donten-Bury, Wiśniewski and Yamagishi [DW17, Yam18] regarding the class group of a $\mathbb{Q}$-factorial terminalization of a linear quotient, see Corollary 5.4.2.

Theorem. *Let $G \leq \mathrm{SL}(V)$ be a finite group and let $H \leq G$ be the subgroup generated by the junior elements contained in G. Let $\varphi : X \to V/G$ be a $\mathbb{Q}$-factorial terminalization of V/G. Then we have*

$$\mathrm{Cl}(X) \cong \mathbb{Z}^m \oplus \mathrm{Hom}(G/H, \mathbb{C}^\times) ,$$

where m is the number of junior conjugacy classes in G.

The proof follows a general philosophy of this thesis in that we try to gain information on a $\mathbb{Q}$-factorial terminalization $X \to V/G$ via the *Cox ring* $\mathcal{R}(X)$ of X. This ring was introduced by Cox for toric varieties [Cox95] and generalized to the setting of birational geometry by Hu and Keel [HK00]. The Cox ring is graded by the class group and in our situation finitely generated. Our main tool to prove the above theorem is then a one-to-one correspondence between effective divisors on X and homogeneous elements of $\mathcal{R}(X)$ up to units, see [ADHL15, Section 1.5].

Cox rings In Chapter 6, we present an algorithm due to Yamagishi [Yam18] to compute a presentation of $\mathcal{R}(X)$ given only the finite group $G \leq \mathrm{SL}(V)$ and without constructing the $\mathbb{Q}$-factorial terminalization X itself. A key ingredient of this algorithm is the observation originally by Donten-Bury [Don16] that we can consider $\mathcal{R}(X)$ as a subring of the Laurent series ring $\mathcal{R}(V/G)[t_1^\pm, \dots, t_m^\pm]$, where m is the number of junior conjugacy classes in the group G. Here, $\mathcal{R}(V/G)$ is the Cox ring of V/G and this ring is graded-isomorphic to the invariant ring $\mathbb{C}[V]^{[G,G]}$ by a theorem of Arzhantsev and

Gaĭfullin [AG10]. We implemented Yamagishi's algorithm in the computer algebra system OSCAR [Osc23]. To the author's knowledge this is the first implementation of this kind; we give a detailed example of how one can use it in Appendix D.

For some symplectic reflection groups coming from complex reflection groups, we computed generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of the corresponding linear quotient, see Appendix C. Among these are several cases where there is no symplectic resolution by [Bel09], so the $\mathbb{Q}$-factorial terminalization is singular. In particular, the list contains the Cox rings corresponding to the symplectic reflection groups coming from the exceptional groups $G_4, \dots, G_7$ in the classification [ST54].

Guided by experiments we did with our implementation, we further arrive at a generating system for the Cox ring corresponding to a linear quotient by a dihedral group acting by symplectic reflections, see Theorem 7.3.9.

Theorem. *Let $d \in \mathbb{Z}_{\geq 3}$ be odd and let $D_d \leq \mathrm{Sp}_4(\mathbb{C})$ be the dihedral group generated by*

$$s := \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \text{ and } r := \begin{pmatrix} \zeta_d & & & \\ & \zeta_d^{-1} & & \\ & & \zeta_d^{-1} & \\ & & & \zeta_d \end{pmatrix},$$

where ζ_d is a primitive d-th root of unity. Let $X \to \mathbb{C}^4/D_d$ be a $\mathbb{Q}$-factorial terminalization of the linear quotient $\mathbb{C}^4/D_d$. Then the Cox ring $\mathcal{R}(X)$ identified with a subalgebra of $\mathbb{C}[x_1, \dots, x_4][t^{\pm}]$ is generated by:

$$\begin{aligned} &x_1x_2,\ x_3x_4,\ x_1x_3 + x_2x_4,\ (x_1x_3 - x_2x_4)t, \\ &x_1^k x_4^{d-k} + x_2^k x_3^{d-k} && (0 \leq k \leq d), \\ &(x_1^k x_4^{d-k} - x_2^k x_3^{d-k})t && (0 \leq k \leq d), \\ &t^{-2}. \end{aligned}$$

Although we use ideas from Yamagishi's algorithm for the proof, the theorem does not logically depend on computer calculations. We conjecture a generating system of the Cox ring corresponding to dihedral groups D_d with d even, see Conjecture 7.3.10. If $d > 3$, the $\mathbb{Q}$-factorial terminalization $X \to \mathbb{C}^4/D_d$ is singular by [Bel09]. Together with the computational results in Appendix C, our theorem provides the first examples of Cox rings in this situation.

Construction of $\mathbb{Q}$-factorial terminalizations Given the Cox ring, one can recover all $\mathbb{Q}$-factorial terminalizations via variation of GIT quotient, but we are not aware of a practical algorithm that is able to do this construction in general. However, for the dihedral groups D_d with d odd, the situation is rather simple and we can indeed construct the $\mathbb{Q}$-factorial terminalization from the Cox ring, see Corollary 7.3.13. This gives an explicit family of singular $\mathbb{Q}$-factorial terminalizations complementing the known examples of symplectic resolutions constructed using various techniques in [LS12, DW17, BC20].

We see this as a 'proof of concept' of the algorithmic idea to construct a $\mathbb{Q}$-factorial terminalization via its Cox ring. Concrete examples of this kind are interesting from two points of view. On one hand, we hope to gain a deeper understanding of which properties of the linear quotient or even the group itself control the (non-)existence of a symplectic resolution. All of the proofs in the classification work with ad hoc arguments and give little insight in the underlying structures, but one would hope – or maybe even expect – that deeper connections are present in light of the McKay correspondence [Rei02]. The second point of view comes from the area of higher dimensional geometry. There are very few examples of varieties of dimension 4 or higher which are available for testing conjectures in birational geometry. Linear quotient varieties are natural candidates to provide such examples as they are more accessible via the additional structure coming from the group and, again, the McKay correspondence.

Outlook We expect that our analogue of Steinberg's theorem is only the beginning of studying similarities between the families of complex reflection groups and symplectic reflection groups. For example, we know that complex reflection groups are characterized by the property that their invariant ring is a polynomial algebra and we may therefore ask whether we can also characterize symplectic reflection groups via their invariant ring. So far, we only know that this invariant ring is Gorenstein (as for all subgroups of $\mathrm{SL}_n(\mathbb{C})$), but in general not a complete intersection, see Section 3.3.

Moving on, we would hope that one can improve Yamagishi's algorithm as it often runs for several days already for groups of size around 50, see also Remark 6.4.1. For this, one might be able to generalize our ad hoc techniques involving Hilbert series from Chapter 7. Further, the algorithm can be divided into two phases as presented in Chapter 6 and in our computations we never encountered a case where the second phase was necessary. We are so far unable to give a theoretical reason for this. It would also be very interesting to see how one can algorithmically construct the $\mathbb{Q}$-factorial terminalization given its Cox ring in general, see Remark 7.3.15.

Finally, we would of course like to finish the classification of symplectic reflection groups whose corresponding linear quotient admits a symplectic resolution. Having the above algorithms at hand, one might be able to do this via brute force computations as there are only 45 groups left. However, even the smallest of these are still too large for Yamagishi's algorithm in its current form.

Conventions and notation We use basic notions from representation theory, algebraic geometry and occasionally symplectic geometry and refer the reader to the textbooks by Fulton and Harris [FH91], Hartshorne [Har77] and Cannas da Silva [Can08], respectively, for the definitions.

We always work over the field of complex numbers $\mathbb{C}$ and all rings are unital. The unit group of a ring R is denoted by $R^\times$. For a vector space V, we denote by $\mathbb{C}[V]$ the symmetric algebra on the dual space V^*; we can also see this as the coordinate ring of the affine space V. The word 'variety' means an integral separated scheme of finite type over a field. We usually abbreviate 'Weil divisor' to just 'divisor' and for a resolution of singularities $\varphi : X \to Y$, we assume that φ is a projective morphism, if not stated otherwise.

Groups are generally written multiplicatively. Given a group G acting on a set X, we denote by $g.x$ the action of $g \in G$ on $x \in X$. We further use standard abbreviations for abstract finite groups, namely C_n for the cyclic group of order n, S_n for the symmetric group on n letters, A_n for the alternating group on n letters and Q_8 for the quaternion group. Unfortunately, we also use the symbols $W(S_1)$, $W(S_2)$ and $W(S_3)$ for certain primitive symplectic reflection groups following [Coh80]; these have nothing to do with the symmetric group. For complex reflection groups, we use the notation $G(m,p,n)$ as well as $G_4, \dots, G_{37}$ coming from the classification in [ST54]. If we want to emphasize that we consider S_n as the irreducible complex reflection group of rank $n-1$, we write $\mathfrak{S}_n$.

We borrow the symbol ‖ from sheet music to mark the end of a remark or an example.

Contributions to OSCAR During the work on the project covered by this thesis, the author contributed the following implementations to OSCAR [Osc23]:(1)

- an algorithm to construct an isomorphic matrix group over a finite field for a matrix group in characteristic zero [DFO13]
- Kemper's algorithm for the computation of primary invariants (non-modular and modular case) [Kem99]
- algorithms for the computations of secondary and irreducible secondary invariants (non-modular and modular case) [KS99, Kin07]
- King's algorithm for the computation of fundamental invariants [Kin13]
- an algorithm for the computation of relations of fundamental invariants relying only on linear algebra [KS99]
- an algorithm to compute semi-invariants (also called relative invariants) with respect to a linear character [Gat96]
- large parts of the general framework for invariant theory of finite groups including methods for the computation of Molien series, Reynolds operators and bases of fixed degree components of invariants
- an algorithm to compute the Cox ring $\mathcal{R}(V/G)$ of a linear quotient V/G as described in Section 6.4
- an algorithm to compute the Cox ring $\mathcal{R}(X)$ of a $\mathbb{Q}$-factorial terminalization $X \to V/G$ [Yam18] (see also Chapter 6)

(1) See also `https://github.com/oscar-system/Oscar.jl/commits?author=joschmitt` .

1. Symplectic reflection groups

We introduce symplectic reflection groups, which are the main object of study of this thesis. After presenting the basic definitions in Section 1.1, we give an outline of the classification of these groups following [Coh80]. We summarize this classification in Figure 1.2.1.

1.1. Symplectic groups

Throughout, let V be a vector space over the field of complex numbers $\mathbb{C}$ of finite dimension $\dim V > 0$.

Recall that a bilinear form $\omega : V \times V \to \mathbb{C}$ is called *alternating* if we have $\omega(v, v) = 0$ for all $v \in V$. This is equivalent to ω being *skew-symmetric* (or *antisymmetric*), that is, $\omega(v, w) = -\omega(w, v)$ for all $v, w \in V$ by bilinearity as we are in characteristic 0.

Definition 1.1.1 (Symplectic form)**.** We call a bilinear form $\omega : V \times V \to \mathbb{C}$ on V *symplectic* if it is non-degenerate and alternating. The vector space V endowed with a symplectic form ω in this way is called a *symplectic vector space.* A morphism between two symplectic vector spaces V_1 and V_2 with symplectic forms ω_1 and ω_2, respectively, is a linear map $\varphi : V_1 \to V_2$ with $\omega_1 = \omega_2 \circ (\varphi \times \varphi)$.

Notation 1.1.2. To emphasize the symplectic form, we write (V, ω) for a symplectic vector space V with symplectic form ω.

Given a symplectic vector space (V, ω), one can find a basis $e_1, \dots, e_n, f_1, \dots, f_n \in V$ of V with $n \in \mathbb{Z}_{>0}$ such that for all $1 \leq i, j \leq n$ we have

$$\omega(e_i, e_j) = 0 = \omega(f_i, f_j) \text{ and } \omega(e_i, f_j) = \delta_{ij} ,$$

see [Can08, Theorem 1.1]. We call such a basis a *symplectic basis.* In particular, the dimension of a symplectic vector space is always even. In a symplectic basis, the symplectic form ω is given by the matrix $J_n \coloneqq \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ via

$$\omega(v, w) = v^\top J_n w$$

for all $v, w \in V$. Whenever it becomes necessary to fix a basis of (V, ω), we choose the basis such that ω is given by J_n in this way and refer to ω as the *standard symplectic form.*

Definition 1.1.3 (Symplectic group)**.** Given a symplectic vector space (V, ω), we call the group

$$\mathrm{Sp}_\omega(V) \coloneqq \{g \in \mathrm{GL}(V) \mid \omega(g.v, g.w) = \omega(v, w) \text{ for all } v, w \in V\} ,$$

that is, the group of all automorphisms of V leaving the symplectic form ω invariant, the *symplectic group* of (V, ω).

We usually omit the index and just write $\mathrm{Sp}(V)$ for the symplectic group. Identifying V with $\mathbb{C}^{2n}$ by fixing a symplectic basis gives the description

$$\mathrm{Sp}_{2n}(\mathbb{C}) := \{g \in \mathrm{GL}_{2n}(\mathbb{C}) \mid g^\top J_n g = J_n\}$$

for the symplectic group.

One sees directly from the definition that elements of $\mathrm{Sp}_\omega(V)$ must have determinant ± 1. It is more involved to prove that in fact each element has determinant $+1$, so we have $\mathrm{Sp}_\omega(V) \leq \mathrm{SL}(V)$, see [Art57, Theorem 3.25].

By abuse of language, we call any subgroup of $\mathrm{Sp}(V)$ a symplectic group as well. Throughout this thesis we are only interested in *finite* symplectic groups.

For any finite group, we can construct a representation that identifies the group with a symplectic group as demonstrated in the following example.

Example 1.1.4. Let G be a finite group and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G on a finite-dimensional complex vector space V. Recall the dual representation $\rho^* : G \to \mathrm{GL}(V^*)$ defined by $\rho^*(g) = \rho(g^{-1})^\top$, see [FH91, p. 4]. For simplicity, we denote the natural pairing between V and V^* by evaluation of maps, that is, given $v \in V$ and $f \in V^*$, we just write $f(v) \in \mathbb{C}$ for the pairing.

We endow $V \oplus V^*$ with a symplectic form ω by setting

$$\omega((v, f), (v', f')) = f'(v) - f(v') ,$$

where $v, v' \in V$ and $f, f' \in V^*$. The action of G on V^* via ρ^* is constructed in such a way that it leaves the natural pairing between V and V^* invariant. Hence, we see that the image of the representation $\rho \oplus \rho^*$ is contained in $\mathrm{Sp}_\omega(V \oplus V^*)$.

If ρ is chosen to be faithful, that is, injective, we can identify G with a symplectic group in this way. ‖

Notation 1.1.5. Let $G \leq \mathrm{GL}(V)$ be a finite group acting on a finite-dimensional complex vector space V. As described in Example 1.1.4 above, we can identify G with a subgroup of $\mathrm{Sp}(V \oplus V^*)$, which we denote by $G^{\circledast}$ (it is the direct sum $\oplus$ with the dual $*$). Notice that after fixing a basis, we can explicitly define $G^{\circledast}$ as the group

$$G^{\circledast} := \left\{ \left(\begin{smallmatrix} g & 0 \\ 0 & (g^{-1})^\top \end{smallmatrix} \right) \,\middle|\, g \in G \right\} \leq \mathrm{Sp}(V \oplus V^*) .$$

For $g \in \mathrm{GL}(V)$, we occasionally write $g^{\circledast} := \left(\begin{smallmatrix} g & 0 \\ 0 & (g^{-1})^\top \end{smallmatrix} \right) \in \mathrm{GL}(V \oplus V^*)$ as well.

Remark 1.1.6. The existence of a symplectic structure on $V \oplus V^*$ is a special case of the fact that there is a canonical symplectic structure on the cotangent bundle of any manifold, see [Can08, Chapter 2]. ‖

Remark 1.1.7. In a more rigorous way, we should think of a symplectic group $G \leq \mathrm{Sp}(V)$ as an abstract group G together with a faithful representation $\rho : G \to \mathrm{GL}(V)$, such that V is a symplectic vector space and $\rho(G) \leq \mathrm{Sp}(V)$. To simplify the notation, we always identify the abstract group G with the image $\rho(G)$ and write down the triple (V, ω, G) when we want to emphasize the structure of V as a representation of G. We occasionally refer to (V, ω, G) as a 'symplectic triple' and always imply that V is finite-dimensional and G a finite group. ‖

We are now able to introduce the main object of study of this thesis.

Definition 1.1.8 (Symplectic reflection group)**.** Let (V, ω) be a symplectic vector space. An element $g \in \mathrm{Sp}_\omega(V)$ is called a *symplectic reflection* if g is of finite order and we have $\mathrm{rk}(g-1) = 2$, where 1 denotes the identity automorphism.

We call a finite subgroup $G \leq \mathrm{Sp}_\omega(V)$ a *symplectic reflection group* if G is generated by the symplectic reflections contained in G.

Notation 1.1.9. Given any group $G \leq \mathrm{Sp}(V)$, we write $S(G) \subseteq G$ for the (possibly empty) set of symplectic reflections in G.

In general, an element $g \in \mathrm{GL}(V)$ with $\mathrm{rk}(g-1) = 2$ is called a *bireflection*. The term symplectic reflection is more common in our context and also emphasizes the fact that a symplectic reflection is an element of $\mathrm{Sp}_\omega(V)$. We introduce a different generalization of symplectic reflections – the *junior elements* – in Section 2.1.

A large class of examples of symplectic reflection groups are coming from complex reflection groups.

Example 1.1.10. Let $G \leq \mathrm{GL}(V)$ be a *complex reflection group* acting on a finite-dimensional complex vector space V, that is, let G be generated by elements $g \in G$ with $\mathrm{rk}(g-1) = 1$, the *reflections*. Then we see that $G^\circledast \leq \mathrm{Sp}(V \oplus V^*)$ is a symplectic reflection group and the (complex) reflections in G correspond one-to-one to the symplectic reflections in $G^\circledast$ by applying ${}^\circledast$. ‖

Example 1.1.11. Let $V = \mathbb{C}^2$ be endowed with the standard symplectic form induced by $J_1 = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$. In this dimension, we have $\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$ as

$$g^\top J_1 g = \begin{pmatrix} & \det g \\ -\det g & \end{pmatrix}$$

for any $g \in \mathrm{GL}_2(\mathbb{C})$. Further, all finite non-trivial subgroups of $\mathrm{SL}_2(\mathbb{C})$ are symplectic reflection groups as the condition $\mathrm{rk}(g-1) = 2$ is trivially fulfilled for all $g \in \mathrm{SL}_2(\mathbb{C})$ with $g \neq 1$.

Hence, a symplectic reflection group in $\mathrm{Sp}_2(\mathbb{C})$ is given by one of the following groups up to conjugacy in $\mathrm{Sp}_2(\mathbb{C})$:

(a) the cyclic group C_m of order m generated by

$$\begin{pmatrix} \zeta_m & \\ & \zeta_m^{-1} \end{pmatrix},$$

(b) the binary dihedral group D_m of order $4m$ generated by

$$\begin{pmatrix} \zeta_{2m} & \\ & \zeta_{2m}^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

(c) the binary tetrahedral group T of order 24 generated by

$$\frac{1}{2}\begin{pmatrix} -1-i & 1-i \\ -1-i & -1+i \end{pmatrix} \text{ and } \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

(d) the binary octahedral group O of order 48 generated by

$$\frac{1}{2}\begin{pmatrix} -1-i & 1-i \\ -1-i & -1+i \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}}\begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix},$$

(e) the binary icosahedral group I of order 120 generated by

$$\frac{1}{2}\begin{pmatrix} \tau^{-1}-\tau i & 1 \\ -1 & \tau^{-1}+\tau i \end{pmatrix} \text{ and } \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

where $\zeta_m \in \mathbb{C}$ is a primitive m-th root of unity, $i := \sqrt{-1}$ is the imaginary unit and $\tau := \frac{1}{2}(1+\sqrt{5})$. See [LT09, Theorem 5.14] and the discussion preceding it for a proof. ∎

We note that the inclusion $\mathrm{Sp}_{2n}(\mathbb{C}) \leq \mathrm{SL}_{2n}(\mathbb{C})$ is proper for $n > 1$. Indeed, we have, for example, $\begin{pmatrix} -I_2 & \\ & I_{2n-2} \end{pmatrix} \in \mathrm{SL}_{2n}(\mathbb{C}) \setminus \mathrm{Sp}_{2n}(\mathbb{C})$ if $n > 1$.

1.2. Cohen's classification

The symplectic reflection groups are classified for arbitrary dimension $2n$ by Cohen in [Coh80]. We now present an outline of this classification and refer the reader to [Coh80] for details and proofs.

Remark 1.2.1. Cohen classifies finite quaternionic reflection groups, that is, reflection groups over the skew-field of quaternions $\mathbb{H}$. However, one sees that quaternionic reflection groups and symplectic reflection groups are the same class of groups; they are linked by an explicit 'complexification' operation described in [Coh80, pp. 294, 295]. In what follows, we translate the results for the quaternions to the base field $\mathbb{C}$ whenever necessary.

In fact, we do not make use of the alternative view point provided by the 'quaternionic interpretation' of these groups and are not aware of any other source doing so. A reason for this might be that the invariant ring of a finite group, which we use heavily from Chapter 2 onwards, appears not to be well-defined if the coefficient ring is a skew field. For example, consider the quaternion group $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}$. Then the standard definition of a group action of this group on $\mathbb{H}[x]$ does in fact not give a well-defined group action; we have

$$((x^2).i).j = (-x^2).j = x^2$$

but on the other hand

$$(x^2).(ij) = (x^2).k = -x^2 \ .$$

The natural reason for this is that the evaluation of polynomials in $\mathbb{H}[x]$ is not a morphism of rings, see [Lam01, §16]. ∥

Remark 1.2.2. While Cohen's classification [Coh80] is only concerned with the case of characteristic 0, there is a classification of finite symplectic reflection groups in arbitrary characteristic by Guralnick and Saxl in [GS03, Section 10]. However, while the latter is more general, the first is more detailed and it is in particular not clear how one can obtain an explicit list of symplectic reflection groups from the results in [GS03]. As we are in need of such a list for our work in chapters 3 and 4 and only interested in characteristic 0, we do not make use of [GS03] in what follows. ∥

We now recall two basic notions from representation theory: (ir)reducibility and (im)primitivity. These characterize the classes of symplectic reflection groups in the classification.

Definition 1.2.3. Let V be a finite-dimensional vector space and let $G \leq \mathrm{GL}(V)$ be a finite group.

(a) The representation V of G is called *complex reducible* if there exists a non-trivial decomposition into G-invariant subspaces $V = V_1 \oplus V_2$. Otherwise, we call V *complex irreducible.*

(b) The representation V of G is called *complex imprimitive* if there exists a non-trivial decomposition $V = V_1 \oplus \cdots \oplus V_n$ into subspaces $V_i \leq V$ such that for any $i \in \{1, \dots, n\}$ and any $g \in G$ there exists $j \in \{1, \dots, n\}$ with $g.V_i = V_j$. In this case, we call the decomposition $V = V_1 \oplus \cdots \oplus V_n$ a *system of imprimitivity.* Otherwise, we call V *complex primitive.*

Now let (V, ω) be a symplectic vector space and let $G \leq \mathrm{Sp}_\omega(V)$ be a finite group. We define in complete analogy:

(c) The representation V of G is called *symplectically reducible* if there exists a non-trivial decomposition into G-invariant symplectic subspaces $V = V_1 \oplus V_2$. Otherwise, we call V *symplectically irreducible.*

(d) The representation V of G is called *symplectically imprimitive* if there exists a non-trivial decomposition $V = V_1 \oplus \cdots \oplus V_n$ into symplectic subspaces $V_i \leq V$ such that for any $i \in \{1, \dots, n\}$ and any $g \in G$ there exists $j \in \{1, \dots, n\}$ with $g.V_i = V_j$. Otherwise, we call V *symplectically primitive.*

We add the adjective 'complex' to the usual notions of irreducibility and imprimitivity here and in what follows to avoid confusion with the symplectic notions. By abuse of language, we often refer to reducibility and primitivity as properties of the group instead of the representation, that is, we say for example that $G \leq \mathrm{GL}(V)$ is a complex irreducible group.

From the definition, we see that complex irreducibility implies symplectic irreducibility and complex primitivity implies symplectic primitivity. However, neither notion is equivalent to the other as we see in the following example.

Example 1.2.4. Recall the subgroups of $\mathrm{Sp}_2(\mathbb{C})$ from Example 1.1.11. All of these are symplectically irreducible and all but C_m are complex irreducible. The group C_m fits in the framework of Example 1.1.10 and the representation $V \oplus V^*$ of a group G in the latter example is complex reducible. The representation $V \oplus V^*$ is furthermore symplectically irreducible if and only if V is complex irreducible.

All of the subgroups of $\mathrm{Sp}_2(\mathbb{C})$ are symplectically primitive. However, C_m and D_m are complex imprimitive. The groups T, O and I are complex primitive. ∥

We should mention that the complex irreducible (and hence symplectically irreducible) groups pass as 'proper quaternionic groups' in [Coh80].

1.2.1. Reduction to symplectically irreducible groups

In what follows, let (V, ω) be a finite-dimensional symplectic vector space over $\mathbb{C}$ and let $G \leq \mathrm{Sp}_\omega(V)$ be a finite group.

We state three easy lemmas for completeness.

Lemma 1.2.5. *Let $V = V_1 \oplus V_2$ be a decomposition in symplectic vector spaces (V_1, ω_1) and (V_2, ω_2). Then (V, ω) and $(V_1 \oplus V_2, \omega_1 \oplus \omega_2)$ are isomorphic as symplectic vector spaces.*

Proof. There are standard symplectic bases $e_1^{(i)}, \dots, e_{n_i}^{(i)}, f_1^{(i)}, \dots, f_{n_i}^{(i)} \in V_i$ for $i = 1, 2$ and with $\dim V_i = 2n_i$. Then

$$e_1^{(1)}, \dots, e_{n_1}^{(1)}, e_1^{(2)}, \dots, e_{n_2}^{(2)}, f_1^{(1)}, \dots, f_{n_1}^{(1)}, f_1^{(2)}, \dots, f_{n_2}^{(2)} \in V_1 \oplus V_2$$

is a symplectic basis for the form $\omega_1 \oplus \omega_2$. Choosing a symplectic basis for (V, ω) hence induces a change of basis, which is trivially a symplectic isomorphism. □

For a subspace $W \leq V$, we let

$$W^\perp := \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}$$

be the *symplectic complement* of W in V. One should be aware that the symplectic complement is in general not a complement in the usual sense of the word: while we always have $\dim W + \dim W^\perp = \dim V$ [Lee03, Lemma 22.3], the condition $W \cap W^\perp = \{0\}$ holds if and only if W is a symplectic subspace [Lee03, Proposition 22.5].

Lemma 1.2.6. *Let $V^G \leq V$ be the subspace of points fixed by G and $W \leq V$ the (unique) G-invariant complement to V^G in V.*

Then both V^G and W are symplectic subspaces and $W = (V^G)^\perp$ is the symplectic complement to V^G.

Proof. To prove that V^G is symplectic we need to show that ω restricts to a non-degenerate form on V^G. Let $0 \neq v \in V^G$. As ω is non-degenerate on V, there exists $w \in V$ with $\omega(v, w) \neq 0$. Then $\frac{1}{|G|}\sum_{g\in G} gw \in V^G$ and

$$\omega\Big(v, \frac{1}{|G|}\sum_{g\in G} gw\Big) = \frac{1}{|G|}\sum_{g\in G}\omega(gv, gw) = \omega(v, w) \neq 0 \;,$$

as required.

We now show that the G-invariant complement W of V^G is contained in $(V^G)^\perp$. Let $w \in W$. Then $w' := \frac{1}{|G|}\sum_{g\in G} gw \in W$ by G-invariance of W, but also $w' \in V^G$ as w' is fixed by G. Hence we must have $w' = 0$ and so

$$\omega(v, w) = \frac{1}{|G|}\sum_{g\in G}\omega(gv, gw) = \omega(v, w') = 0$$

for all $v \in V^G$ as required.

Therefore $W \leq (V^G)^\perp$ and equality follows directly for dimension reasons. In particular, W is a symplectic subspace. $\square$

Lemma 1.2.7. *Let $G \leq \mathrm{Sp}(V)$ be a symplectically reducible group leaving the decomposition $V = V_1 \oplus V_2$ into symplectic vector spaces invariant and assume $V^G = \{0\}$. Then the action of G on V_i identifies G with a subgroup $G_1 \times G_2 \leq \mathrm{Sp}(V_1) \times \mathrm{Sp}(V_2)$. Furthermore, the group G is a symplectic reflection group if and only if both $G_1 \leq \mathrm{Sp}(V_1)$ and $G_2 \leq \mathrm{Sp}(V_2)$ are symplectic reflection groups.*

Proof. The action of G on V_i gives groups $G_i \leq \mathrm{GL}(V_i)$ with $G \cong G_1 \times G_2$. We immediately have $G_i \in \mathrm{Sp}(V_i)$ by Lemma 1.2.5 as $G \leq \mathrm{Sp}(V)$.

If G_1 and G_2 are both generated by symplectic reflections, then so is G by exactly these elements. On the other hand, let G be generated by symplectic reflections. For a symplectic reflection $g \in G$, we prove that $V_i \subseteq V^g$ for either $i = 1$ or $i = 2$. Assume $V_i \not\subseteq V^g$ for $i = 1, 2$, so there are $v_1 \in V_1$ and $v_2 \in V_2$ with $g.v_i \neq v_i$. As g is a symplectic reflection, we have $\dim(V^g)^\perp = 2$ for the symplectic complement, so $\langle v_1, v_2\rangle = (V^g)^\perp$. In particular, $\omega(v_1, v_2) \neq 0$ as the symplectic complement is a symplectic space by Lemma 1.2.6. This is a contradiction to the fact that V decomposes symplectically into $V_1 \oplus V_2$ and hence $\omega(V_1, V_2) = 0$. Therefore we have $V_i \subseteq V^g$ for, say, $i = 1$ and then $(V^g)^\perp \subseteq V_1^\perp = V_2$. This means we can identify g with a symplectic reflection in G_1 or G_2. Hence, G_1 and G_2 are both symplectic reflection groups. $\square$

Lemma 1.2.7 tells us that we can restrict ourselves to the classification of symplectically irreducible symplectic reflection groups.

1.2.2. Complex reducible groups

The next facts are well-known to the experts and best described as 'folklore', see for example [BS16, Section 4.1]. We provide a proof for completeness. A subspace $W \leq V$ with $W = W^\perp$ is called *Lagrangian*, see [Lee03, p. 566].

Lemma 1.2.8. *Let $G \leq \mathrm{Sp}(V)$ be a symplectically irreducible, complex reducible symplectic reflection group. Then there is a G-invariant Lagrangian subspace $W \leq V$ such that the representation W of G is complex irreducible.*

Proof. Let $0 \lneq W \lneq V$ be a G-invariant subspace of V. We show $W = W^\perp$.

Assume there are $w, w' \in W$ with $\omega(w, w') \neq 0$. Then the subspace $W' := \langle w, w' \rangle \leq V$ is symplectic and then so is the G-invariant space $G.W'$ as G consists of symplectic morphisms. But $G.W' \leq W$ by G-invariance of W, hence $G.W' \lneq V$, a contradiction. This shows $W \leq W^\perp$. Notice that $W^\perp$ is G-invariant as well and so the same argument gives $W^\perp \leq (W^\perp)^\perp = W$.

Choose now a complex irreducible subspace $0 \lneq W \lneq V$. Then W is in particular G-invariant, hence $W = W^\perp$ and W is Lagrangian. □

Given a Lagrangian subspace $W \leq V$, there is a symplectic isomorphism $V = W \oplus W^*$, where the symplectic form on $W \oplus W^*$ is constructed as in Example 1.1.4. This gives the following proposition.

Proposition 1.2.9. *Let $G \leq \mathrm{Sp}(V)$ be a symplectic reflection group which is symplectically irreducible and complex reducible. Let $W \leq V$ be a G-invariant Lagrangian subspace as in Lemma 1.2.8. Then G identified with a subgroup of $\mathrm{GL}(W)$ is a complex reflection group, which acts on $V \cong W \oplus W^*$ as described in Example 1.1.10.*

Complex reflection groups have been classified by Shephard and Todd [ST54]. If H is a complex reflection group, then the symplectic reflection group $H^{\circledast}$ is clearly complex imprimitive. The symplectic primitivity of $H^{\circledast}$ is equivalent to the complex primitivity of H. Hence $H^{\circledast}$ is symplectically imprimitive if H belongs to the infinite series $G(m, p, n)$ with $m \geq 2$, $p \mid m$ and $n \geq 2$. If $H^{\circledast}$ is symplectically primitive, then H belongs to one of the following.

- The infinite series $G(m, 1, 1)$, $m \geq 2$, hence $H^{\circledast}$ is conjugate to C_m.
- The symmetric groups $\mathfrak{S}_n$, $n \geq 5$, acting on an $(n-1)$-dimensional space, see [LT09, Example 2.11].
- The exceptional groups $G_4, \ldots, G_{37}$, see [ST54].

1.2.3. Symplectically imprimitive groups

We now turn to symplectic reflection groups which are complex irreducible. Analogously to the classification of complex reflection groups, we use the notion of symplectic (im)primitivity to further distinguish the groups.

We note the following result for later use.

Lemma 1.2.10. *Let $G \leq \mathrm{Sp}(V)$ be a complex irreducible symplectic reflection group. If G is symplectically imprimitive with system of imprimitivity $V = V_1 \oplus \cdots \oplus V_n$, we have* $\dim V_i = 2$ *for $i = 1, \ldots, n$.*

This is explained in the proof of [Coh80, Theorem 2.9], where one has to keep in mind that Cohen works over the quaternions and hence shows $\dim V_i = 1$ as quaternionic vector spaces.

The case of symplectically imprimitive groups bears some resemblance to the situation of complex reflection groups. Let $H \leq \mathrm{SL}_2(\mathbb{C}) = \mathrm{Sp}_2(\mathbb{C})$ be a finite group and let S_n be the symmetric group for some $n \geq 1$. We have an action of S_n on H^n via

$$\sigma.(h_1, \ldots, h_n) = (h_{\sigma(1)}, \ldots, h_{\sigma(n)})$$

and we define the *wreath product* $H \wr S_n$ as the semidirect product $H^n \rtimes S_n$ with multiplication $(h, \sigma) \cdot (h', \sigma') = (h(\sigma.h'), \sigma\sigma')$. The wreath product $H \wr S_n$ gives a symplectic reflection group acting on $\mathbb{C}^{2n}$, where the elements of the group can be identified with products $D\pi$ of matrices $D, \pi \in \mathrm{GL}_{2n}(\mathbb{C})$ with a block diagonal matrix

$$D = \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix}$$

for some $h_1, \ldots, h_n \in H$ and where π is obtained from a $n \times n$ permutation matrix by replacing every entry ε by $\left(\begin{smallmatrix} \varepsilon & \\ & \varepsilon \end{smallmatrix}\right)$.

Cohen shows that any symplectically imprimitive group is conjugate to a subgroup of such an $H \wr S_n$. We summarize his results as follows.

Theorem 1.2.11. *Let $G \leq \mathrm{Sp}(V)$ be a complex irreducible, symplectically imprimitive symplectic reflection group. Then we have $\dim V \geq 4$ and G is one of the following.*

(a) In case $\dim V = 4$, there are finite subgroups $K, H \leq \mathrm{SL}_2(\mathbb{C})$ with $H \trianglelefteq K$ and an involution α of K/H such that G is conjugate to the subgroup denoted by $G(K, H, \alpha)$ of $K \wr S_2$ generated by S_2 and the cosets $(kH, \alpha(kH)) \subseteq K \times K$ for $k \in K$. See [Coh80, Table I] for the precise list of triples (K, H, α) that can occur.

(b) In case $\dim V = 2n > 4$, there are finite subgroups $K, H \leq \mathrm{SL}_2(\mathbb{C})$ with $H \trianglelefteq K$ and K not a cyclic group such that G is conjugate to the subgroup denoted by $G_n(K, H)$ of $K \wr S_n$ generated by S_n and the elements $(k_1, \ldots, k_n) \in K^n$ with $k_1 \cdots k_n \in H$.

See [Coh80, Section 2] for a proof, in particular Theorem 2.6 and Theorem 2.9.

Notice that $G_n(\mathsf{C}_{mp}, \mathsf{C}_m) \cong G(m, p, n)^{\circledast}$ is complex reducible, we hence have to exclude the case of cyclic groups K in the theorem.

1.2.4. Symplectically primitive groups

We turn to the symplectically primitive groups. Recall that such a group can be either complex imprimitive or complex primitive.

For a group $H \leq \mathrm{GL}_2(\mathbb{C})$, we define

$$E(H) := \{h^{\circledast}, h^{\circledast}s \mid h \in H\} \leq \mathrm{Sp}_4(\mathbb{C}) ,$$

where

$$s := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} .$$

This is by construction a complex imprimitive group with system of imprimitivity exactly the Lagrangian subspace and its dual space which are left invariant by H as in Lemma 1.2.8. The next theorem informs us which groups $H \leq \mathrm{GL}_2(\mathbb{C})$ lead to a symplectically primitive symplectic reflection group $E(H)$. We require some notation. For any $d \in \mathbb{Z}_{\geq 1}$, let

$$\mu_d := \left\langle \begin{pmatrix} \zeta_d & \\ & \zeta_d \end{pmatrix} \right\rangle ,$$

where $\zeta_d \in \mathbb{C}$ is a primitive d-th root of unity. Let T and O be the binary tetrahedral and binary octahedral group, respectively, as introduced in Example 1.1.11. We have $\mathsf{T} \trianglelefteq \mathsf{O}$ with $\mathsf{O}/\mathsf{T} \cong C_2$, so $\mathsf{O} = \langle \mathsf{T}, \omega \rangle$ for some $\omega \in \mathsf{O}$. We follow [Coh76, p. 392] to construct a further group OT_d for any $d \in \mathbb{Z}_{\geq 1}$ (or $(\mu_{2d} \mid \mu_d; \mathsf{O} \mid \mathsf{T})$ in Cohen's notation). For $d \in \mathbb{Z}_{\geq 1}$, let

$$\varphi : \mu_{2d}/\mu_d \to \mathsf{O}/\mathsf{T}$$

be the isomorphism defined by $\varphi\big(\overline{\zeta_{2d} I_2}\big) = \overline{\omega}$. Set

$$\mu_{2d} \times_\varphi \mathsf{O} = \{(z, g) \in \mu_{2d} \times \mathsf{O} \mid \varphi(z\mu_d) = g\mathsf{T}\} ,$$

and let OT_d denote the image of $\mu_{2d} \times_\varphi \mathsf{O}$ in $\mathrm{GL}_2(\mathbb{C})$ under the natural multiplication map. That means, we have

$$\mathsf{OT}_d = \bigcup_{\substack{k=0 \\ k \text{ even}}}^{2d-1} \zeta_{2d}^k \mathsf{T} \cup \bigcup_{\substack{k=1 \\ k \text{ odd}}}^{2d-1} \zeta_{2d}^k \omega \mathsf{T} .$$

Theorem 1.2.12. *Let $G \leq \mathrm{Sp}(V)$ be a complex irreducible, symplectically primitive, complex imprimitive symplectic reflection group. Then we either have $\dim V = 2$ and G is conjugate to the group D_m for some $m \geq 1$ or $\dim V = 4$ and G is conjugate to $E(H)$, where H is one of the following:*

(a) $\mu_d \mathsf{T}$, with d a multiple of 6,

(b) $\mu_d \mathsf{O}$, with d a multiple of 4,

(c) $\mu_d \mathsf{I}$, with d a multiple of 4, 6 or 10,

(d) OT_{2d}, with d not divisible by 4.

See [Coh80, Theorem 3.6] for a proof. We study the structure of these groups in more detail in Section 4.1.

This leaves the groups which are complex primitive (and hence symplectically primitive). This class of groups consists of only 16 groups in dimension up to 10; one might want to call these groups 'exceptional'.

Theorem 1.2.13. *Let $G \leq \mathrm{Sp}(V)$ be a complex irreducible, complex primitive symplectic reflection group. If $\dim V = 2$, then G is conjugate to one of the groups T, O or I. Otherwise, G is conjugate to one of the 13 groups listed in [Coh80, Table III].*

See [Coh80, Theorem 4.2] for a proof.

This finishes our outline of the classification of symplectic reflection groups. We provide Figure 1.2.1 as an overview of the different classes of groups.

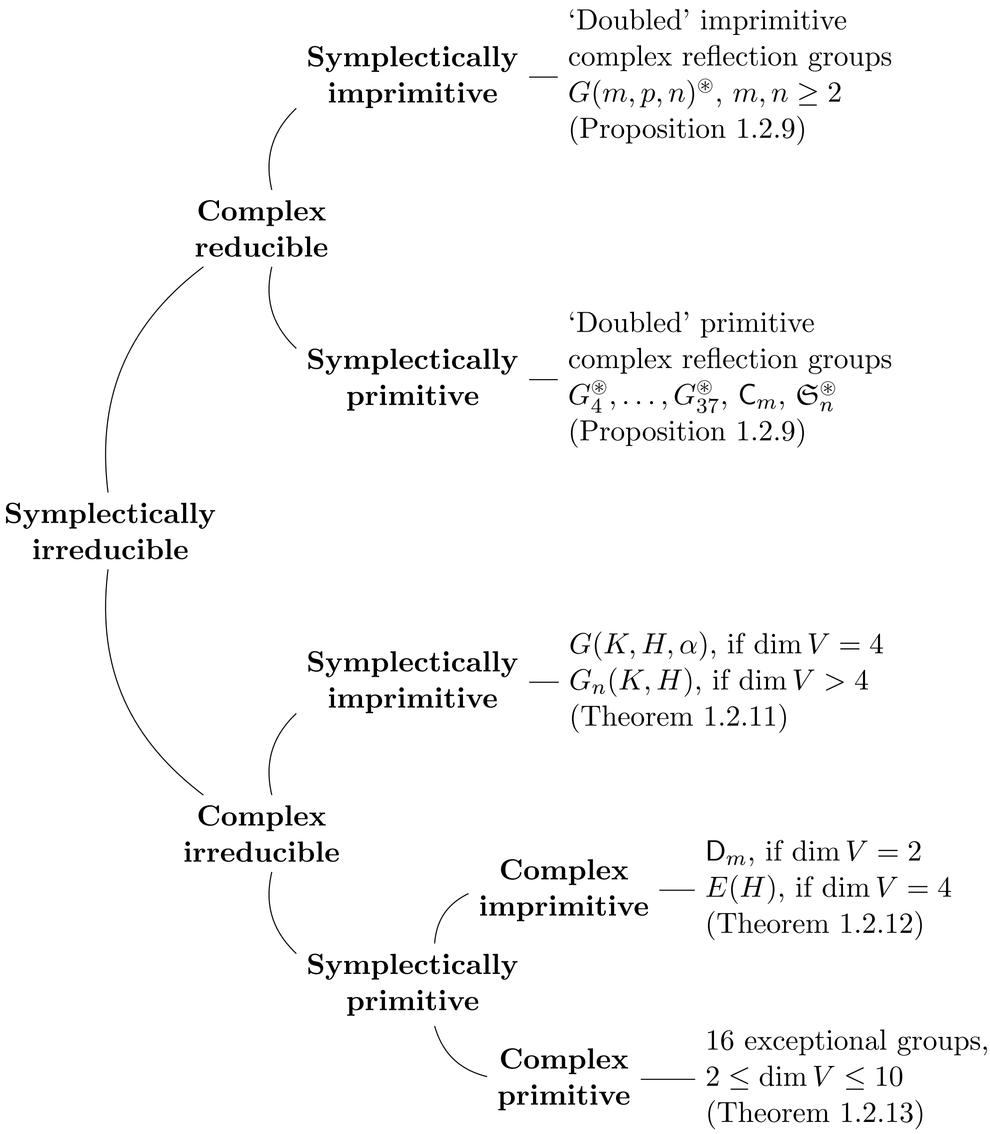


Figure 1.2.1.: The classification of symplectic reflection groups

2. Symplectic resolutions

From Chapter 4 onwards, we investigate symplectic reflection groups from a geometric point of view via their associated linear quotients and in particular via symplectic resolutions and, more general, $\mathbb{Q}$-factorial terminalizations. For this, we now introduce the necessary fundamental notions from birational geometry and – a bit surprising maybe – representation theory.

We start in Section 2.1 by collecting properties of linear quotients and present a notion of the McKay correspondence deeply connecting the $\mathbb{Q}$-factorial terminalization of such a quotient to the corresponding group. We proceed by formulating the classification problem concerning the existence of symplectic resolutions (equivalently, smooth $\mathbb{Q}$-factorial terminalizations) and summarize the state of the art of this classification (Section 2.2). In Section 2.3, we present symplectic reflection algebras and their connection to symplectic resolutions. We close the chapter by introducing Cox rings and a rough algorithmic strategy to compute the $\mathbb{Q}$-factorial terminalization of a linear quotient (Section 2.4).

In view of the breadth of the discussed material, we are not able to give our treatment the depth the topics would allow and only present the results necessary for the chapters to follow. We refer the reader looking for more in depth information to the references throughout the text and in particular to the survey articles [Fu06, Gor08, LV09, Rei02].

2.1. Birational geometry of linear quotients

2.1.1. Linear quotients

Throughout, let V be a finite-dimensional complex vector space.

Definition 2.1.1 (Invariant ring)**.** Let $G \leq \mathrm{GL}(V)$ be a finite group. We call

$$\mathbb{C}[V]^G := \{f \in \mathbb{C}[V] \mid g.f = f \text{ for all } g \in G\}$$

the *invariant ring* of G.

We require the following classical result, see [Ben93, Theorem 1.3.1].

Theorem 2.1.2 (Hilbert, Noether)**.** *Let $G \leq \mathrm{GL}(V)$ be a finite group. Then the invariant ring $\mathbb{C}[V]^G$ is a finitely generated $\mathbb{C}$-algebra.*

This allows us to define the geometric object of interest.

Definition 2.1.3 (Linear quotient)**.** Let $G \leq \mathrm{GL}(V)$ be a finite group. We denote the affine variety $\operatorname{Spec} \mathbb{C}[V]^G$ by V/G and refer to it as the *linear quotient* or the *quotient variety* of V by G.

Example 2.1.4. The linear quotients of $\mathbb{C}^2$ by the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ as listed in Example 1.1.11 are the well-known Kleinian singularities (also known as Du Val singularities, simple surface singularities, ADE singularities or rational double points). We have for example

$$\mathbb{C}^2/\mathsf{C}_m \cong \operatorname{Spec}\mathbb{C}[x,y,z]/\langle x^m + y^2 - z^2\rangle \ .$$

We often revisit the variety $\mathbb{C}^2/\mathsf{C}_2$ in the following examples and mostly work with the presentation $\mathbb{C}[u,v,w]/\langle uv - w^2\rangle$. This presentation is isomorphic to the former coordinate ring via the linear coordinate change

$$x \mapsto u + iv,\ y \mapsto u - iv,\ z \mapsto w\ ,$$

where $i := \sqrt{-1}$ is the imaginary unit.

See [Dur79] for many more details about these singularities. ◀

Theorem 2.1.5 (Chevalley, Serre, Shephard–Todd)**.** *Let $G \leq \mathrm{GL}(V)$ be a finite group. Then the linear quotient V/G is smooth if and only if G is a complex reflection group.*

See [Ser68, Théorème 1'] for a proof. We also refer to a singular linear quotient V/G as a *quotient singularity.*

The following corollary of Luna's slice theorem [Lun73] and the theorem above is well-known.

Lemma 2.1.6. *Let $G \leq \mathrm{GL}(V)$ be a finite group and write $\pi : V \to V/G$ for the projection morphism. For $v \in V$, the point $\pi(v)$ is a smooth point of V/G if and only if the stabilizer $G_v \leq G$ of v is a complex reflection group.*

Proof. Write $(V/G)_{\mathrm{sm}}$ for the smooth locus of V/G. By [Lun73, Lemme II.2], the map $V/G_v \to V/G$ is étale for every $v \in V$. Hence, $v \in (V/G)_{\mathrm{sm}}$ if and only if $v \in (V/G_v)_{\mathrm{sm}}$ as étale morphisms maintain regular points [Liu02, Corollary 4.3.24]. The latter happens if and only if G_v is generated by reflections by [Ser68, Théorème 1']. □

Notice that the subgroup generated by the (complex) reflections contained in a finite group $G \leq \mathrm{GL}(V)$ is normal in G as the dimension of the fixed space is invariant under conjugation.

Proposition 2.1.7. *Let $G \leq \mathrm{GL}(V)$ be a finite group and let $H \leq G$ be the subgroup generated by the reflections contained in G.*

(a) The variety V/G is normal.

(b) We have $\mathrm{Pic}(V/G) = 0$.

(c) We have $\mathrm{Cl}(V/G) \cong \mathrm{Hom}(G/H, \mathbb{C}^\times)$*. In particular, V/G is $\mathbb{Q}$-factorial.*

Statement (a) corresponds to the algebraic statement that $\mathbb{C}[V]^G$ is integrally closed, see [Ben93, Proposition 1.1.1]. For (b) and (c), we refer to [Ben93, Theorem 3.6.1] and [Ben93, Theorem 3.9.2], respectively.

Recall that a commutative Noetherian ring is called *Gorenstein*, if every localization has finite injective dimension, see [Bas63].

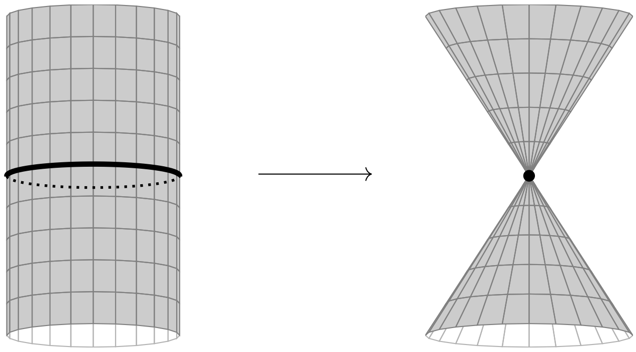


Figure 2.1.1.: A resolution of $\mathbb{C}^2/\mathsf{C}_2$ as in Example 2.1.10.

Theorem 2.1.8 (Watanabe)**.** *Let $G \leq \mathrm{GL}(V)$ be a finite group which does not contain any reflections. Then the ring $\mathbb{C}[V]^G$ is Gorenstein if and only if $G \leq \mathrm{SL}(V)$.*

See [Wat74, Theorem 1] for a proof.

In our applications, we exclusively consider subgroups $G \leq \mathrm{SL}(V)$. Theorem 2.1.8 tells us that the corresponding linear quotient V/G is Gorenstein and this implies that the canonical class $K_{V/G}$ is trivial together with Proposition 2.1.7 (b), see [Har66, Proposition V.9.3]. Further, we have that V/G is singular by Theorem 2.1.5 as a complex reflection cannot have determinant 1.

2.1.2. Minimal models

Definition 2.1.9 (Resolution)**.** Let Y be a variety. A *resolution of singularities* of Y is a proper birational morphism $\varphi : X \to Y$ from a smooth variety X which is an isomorphism outside of the singular locus of Y. If φ is furthermore a projective morphism, we call the resolution a *projective resolution*. If φ is crepant, that is, $\varphi^* K_Y = K_X$, we call the resolution a *crepant resolution*.

Example 2.1.10. The Kleinian singularities from Example 2.1.4 all admit crepant resolutions. In case $G = \mathsf{C}_2$ a resolution is given by

$$X \coloneqq \operatorname{Spec} \mathbb{C}[x, y, z]/\langle x^2 + y^2 - 1 \rangle \subseteq \mathbb{A}^3$$

with the map

$$\varphi : X \to \mathbb{C}^2/\mathsf{C}_2,\ (x, y, z) \mapsto (xz, yz, z)\ .$$

See Figure 2.1.1 for a (real) visualization of the situation. However, this resolution is certainly not projective and also not crepant. See [Liu02, Example 8.1.5] for a construction of a crepant projective resolution via blowing up. ∥

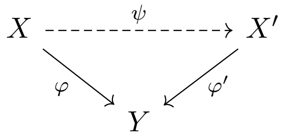
From now on, we only consider *projective resolutions* in this work and therefore usually drop the adjective projective.

While a resolution always exists, this is not true for crepant resolutions. An example is given by the quotient singularity $\mathbb{C}^4/G$ with $G = \langle -I_4 \rangle \leq \mathrm{SL}_4(\mathbb{C})$ which does not admit a crepant resolution as explained in [Gra19, Example 2.3.6]. We therefore consider $\mathbb{Q}$-factorial terminalizations, that is, crepant partial resolutions which are allowed to have mild – precisely *terminal* (see [KM98, Definition 2.12]) – singularities.

Definition 2.1.11 ($\mathbb{Q}$-factorial terminalization)**.** Let Y be a normal $\mathbb{Q}$-factorial variety. A $\mathbb{Q}$-*factorial terminalization* of Y is a projective birational morphism $\varphi : X \to Y$ such that X is a normal $\mathbb{Q}$-factorial variety with terminal singularities and φ is crepant.

In our context, $\mathbb{Q}$-factorial terminalizations are related to minimal models, see Proposition 2.1.13 and Remark 2.1.14.

Definition 2.1.12 (Minimal model)**.** Let $\varphi : X \to Y$ be a projective birational morphism of normal $\mathbb{Q}$-factorial varieties where X is smooth. A *relative minimal model* of X over Y is a projective birational morphism $\varphi' : X' \to Y$ sitting in a diagram



such that

(i) X' is a normal $\mathbb{Q}$-factorial variety with terminal singularities,

(ii) the canonical class $K_{X'}$ is φ'-nef (see [Kol13, Definition 1.4]) and

(iii) $\psi : X \dashrightarrow X'$ is a birational contraction, that is, a birational map such that the exceptional locus of ψ^{-1} has codimension at least 2.

The above definition was given following [KM98, Example 2.16] and also [Kol13, Definition 1.19]. We require the following result to be able to see that $\mathbb{Q}$-factorial terminalizations exist, see also [Gra19, Proposition 2.1.11].

Proposition 2.1.13. *Let $G \leq \mathrm{SL}(V)$ be a finite group and let $\varphi : X \to V/G$ be a projective resolution of V/G with X a $\mathbb{Q}$-factorial variety.*

If $\varphi' : X' \to V/G$ is a relative minimal model of X over V/G, then φ' is a $\mathbb{Q}$-factorial terminalization of V/G.

Proof. Let $\varphi' : X' \to V/G$ be a relative minimal model of X over V/G. We only need to prove that φ' is crepant. By Theorem 2.1.8, we have $K_{V/G} = 0$. As $K_{X'}$ is φ'-nef, we must have $-K_{X'} \geq 0$ by [KM98, Lemma 3.39]. On the other hand, V/G has canonical singularities by [Kol13, Theorem 3.21], so $K_{X'} \geq 0$ as well. We conclude $K_{X'} = 0$. □

Remark 2.1.14. In the situation of Proposition 2.1.13, one immediately sees that a $\mathbb{Q}$-factorial terminalization $X' \to V/G$ is a relative minimal model of a projective resolution $X \to V/G$ that factors through X' such that $X \dashrightarrow X'$ is a birational contraction.

In our context, a $\mathbb{Q}$-factorial terminalization $X' \to V/G$ is therefore often referred to as 'minimal model', see for example [IR96]. However, the usage of this terminology is not entirely uniform and we decided to distinguish the two notions to avoid confusion. ∥

We have the following special case of the deep result achieved in [BCHM10].

Theorem 2.1.15. *Let $G \leq \mathrm{SL}(V)$ be a finite group. There exists a $\mathbb{Q}$-factorial terminalization of V/G.*

Proof. This follows from [BCHM10, Corollary 1.4.3]. To give at least some details, we require certain notions from birational geometry, which we did not introduce before; see for example [Kol13] for definitions.

Let $\varphi : X \to V/G$ be a projective resolution of V/G. We want to apply [BCHM10, Corollary 1.4.3] to V/G and φ (or rather the log pair $(V/G, 0)$). By [Kol13, Theorem 3.21], the variety V/G has canonical singularities, so in particular $K_{V/G}$ is Kawamata log terminal, see [Kol13, p. 42]. As described in [BCHM10, p. 413], we may use [BCHM10, Corollary 1.4.3] with the set of all valuations of log discrepancy at most 1 to obtain a birational morphism $\varphi' : X' \to V/G$ where X' is $\mathbb{Q}$-factorial and has terminal singularities. In fact, we see from the proof of [BCHM10, Corollary 1.4.3] that X' is a log terminal model, see [BCHM10, Definition 3.6.7]. This means that there is a birational contraction $X \dashrightarrow X'$ turning X' into a relative minimal model of X over V/G. In particular $X' \to V/G$ is projective and $K_{X'}$ is φ'-nef. By Proposition 2.1.13, we conclude that $X' \to V/G$ is a $\mathbb{Q}$-factorial terminalization. □

2.1.3. McKay correspondence

Throughout, let $G \leq \mathrm{SL}(V)$ be a finite group. We introduce a deep connection due to Ito and Reid [IR96] between a $\mathbb{Q}$-factorial terminalization $X \to V/G$ and the group G itself.

For the following definitions, let $g \in \mathrm{GL}(V)$ be of finite order r and fix a primitive r-th root of unity ζ_r. In an eigenbasis, we can write g as a diagonal matrix

$$\begin{pmatrix} \zeta_r^{a_1} & & \\ & \ddots & \\ & & \zeta_r^{a_n} \end{pmatrix}$$

for certain integers $0 \leq a_i < r$, where $\dim V = n$.

Definition 2.1.16 (Age and junior elements)**.** We call the number

$$\mathrm{age}(g) := \frac{1}{r}\sum_{i=1}^{n} a_i$$

the *age* of g. Elements of age 1 are called *junior*.

By construction, we have that $\mathrm{age}(g)$ is an integer if $g \in \mathrm{SL}(V)$ and the junior elements in $\mathrm{SL}(V)$ are hence the non-trivial elements of minimal age 1. The age is by definition invariant under conjugacy and we refer to the conjugacy classes of a group $G \leq \mathrm{GL}(V)$ consisting (only) of junior elements as *junior conjugacy classes*.

Remark 2.1.17. We emphasize that the age as defined above depends on the choice of the root of unity ζ_r, although this is hidden in the notation. For example, the matrix $\operatorname{diag}(\varepsilon, \varepsilon, \varepsilon) \in \mathrm{SL}_3(\mathbb{C})$ with ε a primitive third root of unity has age 1, if we choose $\zeta_3 = \varepsilon$, and age 2, if we choose $\zeta_3 = \varepsilon^{-1}$. See [IR96, p. 224, Remark 3] for another (counter-)example. In [IR96], Ito and Reid circumvent this problem by defining the age not for the group G, but for the group $\Gamma := \operatorname{Hom}(\mu_{|G|}, G)$, where $\mu_{|G|}$ is the group of roots of unity of order $|G|$. On Γ, the notion of age is independent of any choices. By choosing a root of unity $\zeta \in \mathbb{C}$ of order $|G|$, the group Γ becomes isomorphic to G via $\Gamma \to G,\ \varphi \mapsto \varphi(\zeta)$.

As we are interested in explicit calculations, we require a concrete description of the age and in particular the corresponding valuation, see below, for elements of G. ‖

In contrast to the above remark, the following lemma tells us that in the symplectic case the age is independent of any choices, see also [Kal02, Lemma 2.6].

Lemma 2.1.18. *Let $g \in \mathrm{Sp}(V)$ be of finite order r and write $V^g \leq V$ for the subspace of vectors fixed by g. Then we have*

$$\operatorname{age}(g) = \frac{1}{2} \operatorname{codim} V^g \, .$$

In particular, g is a junior element if and only if g is a symplectic reflection.

Proof. By the symplectic eigenvalue theorem [AM87, Theorem 3.1.16], the eigenvalues of $g \in \mathrm{Sp}(V)$ come in pairs of the form λ, λ^{-1} for some $\lambda \in \mathbb{C}^\times$. This means we can write down the eigenvalues of g as

$$1, \ldots, 1, \zeta_r^{a_1}, \zeta_r^{r-a_1}, \ldots, \zeta_r^{a_m}, \zeta_r^{r-a_m}$$

with a primitive r-th root of unity ζ_r, $2m = \operatorname{codim} V^g$ and $0 < a_i < r$. Then

$$\operatorname{age}(g) = \frac{1}{r} \sum_{i=1}^{m} (a_i + (r - a_i)) = m = \frac{1}{2} \operatorname{codim} V^g$$

by construction. □

Definition 2.1.19 (Monomial valuation)**.** For non-negative integers $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$ with $\gcd(a_1, \ldots, a_n) = 1$, we construct a discrete valuation $v : \mathbb{C}(x_1, \ldots, x_n) \to \mathbb{Z}$ defined on $\mathbb{C}[x_1, \ldots, x_n]$ via

$$\sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \lambda_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mapsto \min_{\substack{\alpha \in \mathbb{Z}^n_{\geq 0} \\ \lambda_\alpha \neq 0}} \sum_{i=1}^{n} \alpha_i a_i \, .$$

We call v a *monomial valuation.*

This construction indeed gives a well-defined discrete valuation, see [Kal02, Definition 2.1].

Notation 2.1.20. Let $g \in \mathrm{GL}(V)$ be of finite order r written in an eigenbasis as

$$\begin{pmatrix} \zeta_r^{a_1} & & \\ & \ddots & \\ & & \zeta_r^{a_n} \end{pmatrix}$$

as above. We can always choose ζ_r in such a way that $\gcd(a_1, \dots, a_n) = 1$. Indeed, given any choice of root of unity, replacing ζ_r by $\zeta_r^{\gcd(a_1,\dots,a_n)}$ gives the desired result. We can therefore define a monomial valuation

$$v_g : \mathbb{C}(x_1, \dots, x_n) \to \mathbb{Z}$$

for g via $a_1, \dots, a_n$.

The construction of v_g again depends on the choice of a root of unity, see Remark 2.1.17 above. The valuation v_g is stable under conjugacy of g and we can hence associate valuations to conjugacy classes in G without requiring to specify a particular representative.

Theorem 2.1.21 (McKay correspondence). *Let $G \leq \mathrm{SL}(V)$ be a finite group and let $X \to V/G$ be a $\mathbb{Q}$-factorial terminalization. Then there is a one-to-one correspondence between the junior conjugacy classes of G and the irreducible exceptional divisors on X.*

More precisely, if E is a divisor corresponding to a conjugacy class of a junior element $g \in G$ of order r in this way, then $v_E = \frac{1}{r} v_g$, where v_E is the valuation of E and we identify $\mathbb{C}(X) = \mathbb{C}(V)^G$ via the birational morphism $X \to V/G$.

See [IR96, Section 2.8] for a proof.

Theorem 2.1.21 is our main tool to approach the $\mathbb{Q}$-factorial terminalization X computationally as it relates properties of X to the group G itself – an object we understand much better.

2.2. The classification of symplectic resolutions

We present a concept of singular varieties with a symplectic structure as introduced by Beauville [Bea00] with the definition of *symplectic singularities.* We refer to varieties with such singularities as symplectic varieties following [Fu06]. To be able to state the precise definition, we borrow a few notions from symplectic geometry, see [Can08, Section 1.3] or [Lee03, Chapter 22] for more details.

Definition 2.2.1 (Symplectic form on a variety). Let Y be a smooth variety. A 2-form $\omega \in \Gamma(Y, \Omega_Y^2)$ on Y is *non-degenerate at every point*, if ω restricts to a non-degenerate bilinear form on the tangent space T_pY for all $p \in Y$. We further call ω *closed*, if $\mathrm{d}\omega = 0$, where d is the exterior derivative. A *symplectic form* on Y is a closed 2-form $\omega \in \Gamma(Y, \Omega_Y^2)$ which is non-degenerate at every point.

Definition 2.2.2 (Symplectic variety). Let Y be a normal variety. We call Y a *symplectic variety*, if the smooth part Y_{sm} of Y admits a symplectic form ω such that the pull-back of ω to any (not necessarily projective) resolution $X \to Y$ extends to a 2-form on X.

We emphasize that following [Fu06] we do not require a symplectic variety to be smooth. Further, we note that the pull-back of ω to a resolution is in general not symplectic as it is closed but possibly degenerate at some points, see [Fu06, p. 211]. The singularities on a symplectic variety are exactly the symplectic singularities introduced in [Bea00, Definition 1.1].

Let (V, ω, G) be a symplectic triple, where we use the short-hand notation from Remark 1.1.7. Write $\pi : V \to V/G$ for the quotient map. Note that by Lemma 2.1.6 the smooth locus $(V/G)_{\mathrm{sm}}$ of V/G is the image under π of the elements $v \in V$ with trivial stabilizer $G_v = \{1\}$ in G. Further, the morphism π is étale at any such v by [Lun73, Lemme II.2]. Hence, we can push forward the symplectic form ω on V to obtain a symplectic form on $(V/G)_{\mathrm{sm}}$ by [Liu02, Proposition 6.2.10]. This discussion gives the following proposition.

Proposition 2.2.3. *If (V, ω, G) is a symplectic triple, then the linear quotient V/G is a symplectic variety.*

Proof. By the above discussion, $(V/G)_{\mathrm{sm}}$ admits a symplectic structure. Hence the claim follows by [Nam01, Theorem 6] using Theorem 2.1.8. □

See [Bea00, Proposition 2.4] for a proof working directly with the definition and not requiring the result in [Nam01].

If we have symplectic singularities, we should also have symplectic resolutions.

Definition 2.2.4 (Symplectic resolution)**.** Let $\varphi : X \to Y$ be a resolution of a symplectic variety Y with symplectic form ω. We call φ a *symplectic resolution*, if the pull-back $\varphi^*\omega$ can be extended to a symplectic form on X.

Proposition 2.2.5. *Let Y be a symplectic variety and let $\varphi : X \to Y$ be a (not necessarily projective) resolution of singularities. Then φ is symplectic if and only if φ is crepant.*

See [Fu06, Proposition 1.6] for a proof. The same statement restricted to Y being a quotient singularity can be found in [Ver00, Theorem 2.4].

Although a symplectic resolution is not a new concept by this proposition, we still use the term symplectic resolution in the following as it is more common in our context. As already discussed above, a crepant resolution of V/G does in general not exist. We have a more precise result for symplectic quotient singularities.

Theorem 2.2.6 (Verbitsky)**.** *Let (V, ω, G) be a symplectic triple such that the linear quotient V/G admits a (not necessarily projective) symplectic resolution. Then G is a symplectic reflection group.*

See [Ver00, Theorem 1.1] for a proof. There is a generalization of this result to subgroups $G \leq \mathrm{SL}(V)$, saying that the existence of a crepant resolution of V/G implies that G is generated by junior elements, see [Yam18, Theorem 1.1].

Theorem 2.2.6 means that for a classification of all finite subgroups of $\mathrm{Sp}(V)$ whose corresponding quotients admits a symplectic resolution we only need to look at symplectic reflection groups. We see that we can further restrict to symplectically irreducible symplectic reflection groups.

Lemma 2.2.7. *If (V, ω, G) is a symplectically reducible symplectic triple decomposing as $V = V_1 \oplus V_2$ and $G = G_1 \times G_2$ with $G_i \leq \mathrm{Sp}(V_i)$, then V/G admits a symplectic resolution if and only if both V_1/G_1 and V_2/G_2 admit symplectic resolutions.*

Proof. The decomposition $V/G = V_1/G_1 \times V_2/G_2$ implies that if each V_i/G_i admits a symplectic resolution, then so does V/G.

For the converse, assume that V/G admits a symplectic resolution. Let $v \in V_1$ be a vector with trivial stabilizer $\mathrm{Stab}_{G_1}(v) = 1$. Then $\mathrm{Stab}_G(v) = G_2$, so $V^{\mathrm{Stab}_G(v)} = V_1$. Therefore V_2/G_2 admits a symplectic resolution by [Kal03, Theorem 1.6] and so does V_1/G_1 by the analogous argument. □

We arrive at the following classification problem:

Problem. Classify all symplectically irreducible symplectic reflection groups $G \leq \mathrm{Sp}(V)$ for which V/G admits a symplectic resolution.

This classification is ongoing work by many authors over the last two decades and can be seen as almost finished with our contribution in Chapter 4. There, we reduce the classification to only finitely many open cases using the representation theory of symplectic reflection algebras as introduced in [EG02], see the next section. We now give an outline of the current state of the classification, see Figure 2.2.1 for an overview. We should emphasize that all of the results are only concerned with projective symplectic resolutions.

Complex reducible groups. The complex reducible symplectic reflection groups for which the corresponding quotient admits a symplectic resolution are the cyclic groups C_m, the symmetric groups $\mathfrak{S}_n^{\circledast}$, the wreath product groups $G(m,1,n)^{\circledast}$ and the exceptional group $G_4^{\circledast}$. The quotients by all other complex reducible groups do not admit a symplectic resolution. This part of the classification is presented in [Bel09] extending partial results in [EG02, Corollary 1.14] and [Gor03, Proposition 7.3].

Complex irreducible, symplectically imprimitive groups. The complex irreducible, symplectically imprimitive symplectic reflection groups for which the corresponding quotient admits a symplectic resolution are the wreath product groups $K \wr S_n$, for $K \leq \mathrm{SL}_2(\mathbb{C})$ a finite group, and the group $G(\mathsf{D}_2, \mathsf{C}_2, \mathrm{id}) = Q_8 \times_{\mathbb{Z}/2\mathbb{Z}} D_4$, where Q_8 is the quaternion group of order 8 and D_4 is the dihedral group of order 8. The quotients by all other symplectically imprimitive groups do not admit a symplectic resolution. The main reference for this part of the classification is [BS16] with the group $G(\mathsf{D}_2, \mathsf{C}_2, \mathrm{id})$ already treated in [BS13] and some remaining groups covered by [Yam18, Theorem 6.1].

Complex irreducible, symplectically primitive groups. The complex irreducible, symplectically primitive groups for which it is known that the corresponding quotient admits a symplectic resolution are the Kleinian groups D_m, $m \geq 1$, T, O and I. It is proven in [BS16] that there exist no symplectic resolutions for the quotients by three exceptional groups. In Chapter 4, we prove the non-existence of a symplectic resolution for the quotients by all other groups with the exception of 45 groups in rank 4 for which the classification is still open; we give the precise list in Section 4.4.

In the cases where a symplectic resolution is known to exist, these have been constructed explicitly. This is done in [BC20] for the wreath product cases using Nakajima quiver varieties. The symplectic resolutions for the quotient corresponding to $G_4^{\circledast}$ are constructed via the explicit computation of blowing-ups in [LS12]. Finally, [DW17] and [BCR$^+$21] deal with the resolutions of the quotient corresponding to $G(\mathsf{D}_2, \mathsf{C}_2, \mathrm{id})$. These resolutions are constructed in [BCR$^+$21] using Nakajima quiver varieties and in [DW17] by first constructing the Cox ring of such a resolution; we discuss the general idea of the latter approach in more detail in Section 2.4.3.

2.3. Symplectic reflection algebras

We give an overview of symplectic reflection algebras as introduced by Etingof and Ginzburg in the seminal paper [EG02]. They are a tool from representation theory that we use in Chapter 4 to prove the non-existence of symplectic resolutions for almost all quotients corresponding to symplectically primitive symplectic reflection groups.

Let (V, ω) be a symplectic vector space and let $G \leq \mathrm{Sp}(V)$ be a finite group. We want to study the quotient V/G via deformations. However, it turns out that one should not deform the ring $\mathbb{C}[V]^G$ directly, but rather the *skew group ring* $\mathbb{C}[V] \rtimes G$; see the introduction of [EG02] for some reasons for this.

Definition 2.3.1 (Skew group ring)**.** The *skew group ring* $\mathbb{C}[V] \rtimes G$ is, as a vector space, equal to $\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}G$ with the multiplication for $f, f' \in \mathbb{C}[V]$ and $g, g' \in G$ given by

$$(f \otimes g) \cdot (f' \otimes g') = f(g.f') \otimes gg'$$

and extended linearly.

The skew group ring is in general a non-commutative algebra and we observe that the centre is given by $Z(\mathbb{C}[V] \rtimes G) = \mathbb{C}[V]^G$. This means that the centre of a deformation of $\mathbb{C}[V] \rtimes G$ gives us a deformation of $\mathbb{C}[V]^G$.

We require a bit of notation for the deformations of $\mathbb{C}[V] \rtimes G$ we are particularly interested in. Recall that we write $S(G)$ for the set of symplectic reflections in the group G. For each $g \in S(G)$, we decompose $V = V^g \oplus (V^g)^\perp$, where V^g is the subspace of elements fixed by g and $(V^g)^\perp$ the symplectic complement. Let $\pi_g : V \to (V^g)^\perp$ be the projection map and define a bilinear form

$$\omega_g : V \times V \to \mathbb{C},\ (v, w) \mapsto \omega(\pi_g(v), \pi_g(w)) \ .$$

Finally, let $\mathbf{c} : S(G) \to \mathbb{C}$ be a G-conjugacy invariant function, that is, $\mathbf{c}(hgh^{-1}) = \mathbf{c}(g)$ for all $g \in S(G)$ and $h \in G$; we call $\mathbf{c}$ a *parameter*. Let $TV^* = \mathbb{C} \oplus V^* \oplus (V^* \times V^*) \oplus \cdots$ be the tensor algebra on V^*.

Definition 2.3.2 (Symplectic reflection algebra)**.** Given a symplectic triple (V, ω, G) and a G-conjugacy invariant parameter $\mathbf{c} : S(G) \to \mathbb{C}$, the *symplectic reflection algebra* $\mathsf{H}_{\mathbf{c}}(G)$ is the algebra $TV^* \rtimes G$ modulo the ideal generated by

$$v \otimes w - w \otimes v - \sum_{g \in S(G)} \mathbf{c}(g) \omega_g(v, w) g \ , \qquad \text{for all } v, w \in V^* \ .$$

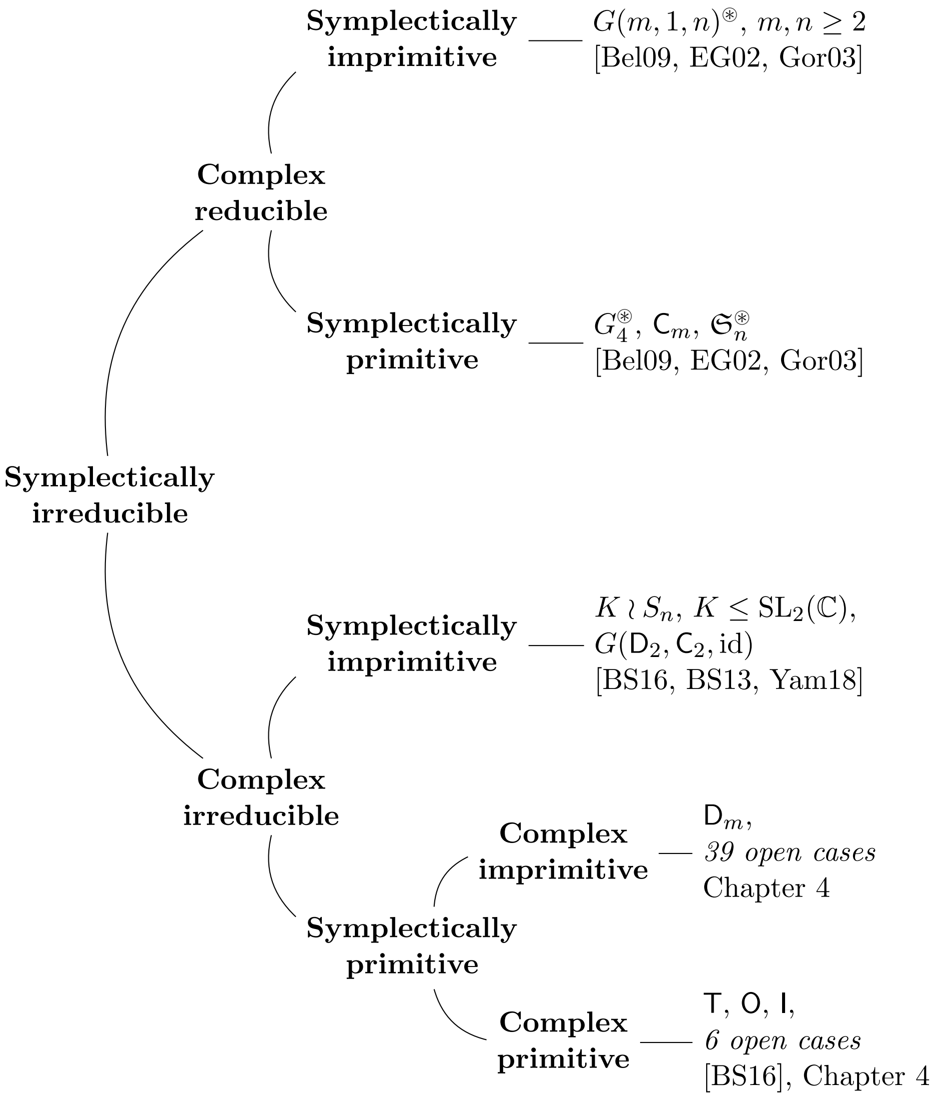


Figure 2.2.1.: Symplectic reflection groups for which the existence of a symplectic resolution of the corresponding linear quotient is known

Observe that $\mathsf{H}_0(G) = \mathbb{C}[V] \rtimes G$.

Remark 2.3.3. The general definition of a symplectic reflection algebra in [EG02] involves a further parameter $t \in \mathbb{C}$, which we omit in our definition. The research on symplectic reflection algebras splits up between the cases $t = 0$ and $t = 1$, see for example the survey [Gor08]. We only consider the case $t = 0$ in this thesis. ∥

Remark 2.3.4. Putting G in degree 0 and V in degree 1, we obtain a filtration of the symplectic reflection algebra $\mathsf{H}_{\mathbf{c}}(G)$. The associated graded algebra is then isomorphic to $\mathbb{C}[V] \rtimes G$, that is, $\mathsf{H}_{\mathbf{c}}(G)$ fulfils the Poincaré–Birkhoff–Witt property (PBW-property), see [EG02, Theorem 1.3]. In fact, [EG02, Theorem 1.3] is stronger and tells us that *all* quotients of $TV^* \rtimes G$ by a commutator relation that fulfil the PBW-property are given by symplectic reflection algebras with the additional parameter t as in Remark 2.3.3, at least if G is symplectically irreducible. ∥

Example 2.3.5. We consider the example $\mathsf{C}_2 \leq \mathrm{SL}_2(\mathbb{C})$ in this context. Write $V := \mathbb{C}^2$. The only symplectic reflection is $g := -I_2$, so we can identify a parameter $\mathbf{c}$ with an element of $\mathbb{C}$. We have $V^g = \{0\}$, so $\omega_g = \omega$. Let $x, y \in V^*$ be a basis of V^* with $\omega(x, y) = 1$. This gives the relation

$$[x, y] = \sum_{g \in S(G)} \mathbf{c}\omega(x, y)g = \mathbf{c}g \ .$$

So, the symplectic reflection algebra is given by

$$\mathsf{H}_{\mathbf{c}}(G) \cong \mathbb{C}\langle x, y, g\rangle / \langle g^2 = 1, gx = -xg, gy = -yg, [x, y] = \mathbf{c}g\rangle \ ,$$

where $\mathbb{C}\langle x, y, g\rangle$ denotes the free algebra on $\{x, y, g\}$. ∥

We want to relate the algebra $\mathsf{H}_{\mathbf{c}}(G)$ to the geometric object V/G.

Notation 2.3.6. We write $\mathsf{Z}_{\mathbf{c}}(G) := Z(\mathsf{H}_{\mathbf{c}}(G))$ for the centre of the symplectic reflection algebra $\mathsf{H}_{\mathbf{c}}(G)$.

Proposition 2.3.7 (Etingof–Ginzburg)**.** *Let (V, ω, G) be a symplectic triple and let $\mathbf{c} : S(G) \to \mathbb{C}$ be a G-conjugacy invariant function. Then the centre $\mathsf{Z}_{\mathbf{c}}(G)$ of $\mathsf{H}_{\mathbf{c}}(G)$ is a finitely generated integral $\mathbb{C}$-algebra.*

Combine [EG02, Theorem 3.1] and [EG02, Theorem 1.5 (i)] for a proof.

Definition 2.3.8 (Calogero–Moser space)**.** The affine variety $\mathsf{X}_{\mathbf{c}}(G) := \operatorname{Spec} \mathsf{Z}_{\mathbf{c}}(G)$ is called the *Calogero–Moser space* of G with parameter $\mathbf{c}$.

Example 2.3.9. Consider again $\mathsf{C}_2 \leq \mathrm{SL}_2(\mathbb{C})$ as in Example 2.3.5. The centre $\mathsf{Z}_{\mathbf{c}}(G)$ is generated by the elements $x^2, y^2, xy - \mathbf{c}g \in \mathsf{H}_{\mathbf{c}}(G)$ as can be checked using the computer algebra package CHAMP [Thi15]. This results in a presentation

$$\begin{aligned} \mathbb{C}[u, v, w]/\langle uv - w^2 + \mathbf{c}^2\rangle &\cong \mathsf{Z}_{\mathbf{c}}(G) \ , \\ u &\mapsto x^2 \ , \\ v &\mapsto y^2 \ , \\ w &\mapsto xy - \mathbf{c}g \ . \end{aligned}$$

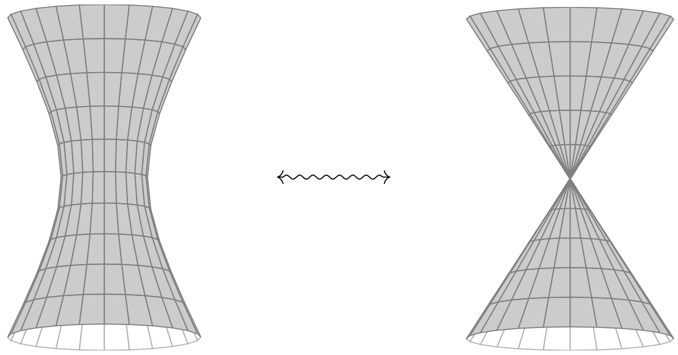


Figure 2.3.1.: A deformation of $\mathbb{C}^2/\mathsf{C}_2$ as in Example 2.3.9.

Hence, $\mathsf{X}_{\mathbf{c}}(G)$ gives deformations of $\mathbb{C}^2/\mathsf{C}_2 \cong \operatorname{Spec}\mathbb{C}[u,v,w]/\langle uv - w^2\rangle$ parametrized by $\mathbf{c}$ as expected. See Figure 2.3.1 for a (real) visualization of the situation. However, it is only a coincidence in this example that a deformation is also a resolution of $\mathbb{C}^2/\mathsf{C}_2$. ‖

We have the following connection to the classification of symplectic resolutions presented in the previous section.

Theorem 2.3.10 (Ginzburg–Kaledin, Namikawa)**.** *Let (V,ω,G) be a symplectically irreducible symplectic triple. Then the symplectic linear quotient V/G admits a symplectic resolution if and only if there is a parameter $\mathbf{c} : S(G) \to \mathbb{C}$ such that the Calogero–Moser space $\mathsf{X}_{\mathbf{c}}(G)$ is smooth.*

One implication is due to [GK04, Corollary 1.21]. See [Nam11, Corollary 5.6] for the statement that a symplectic resolution exists if and only if there is a smooth Poisson deformation of V/G. This then yields the equivalence in the theorem with [GK04, Theorem 1.18] and [GK04, Theorem 1.20], see also [Bel10, Section 4.3] for details.

The question whether $\mathsf{X}_{\mathbf{c}}(G)$ is smooth is related to the representation theory of $\mathsf{H}_{\mathbf{c}}(G)$.

Theorem 2.3.11 (Etingof–Ginzburg)**.** *Let (V,ω,G) be a symplectically irreducible symplectic triple and let $\mathbf{c} : S(G) \to \mathbb{C}$ be a parameter. If $\mathsf{X}_{\mathbf{c}}(G)$ is smooth, then the dimension of all simple $\mathsf{H}_{\mathbf{c}}(G)$-modules is equal to the order of G.*

See [EG02, Theorem 1.7] for a proof.

Combining both theorems gives the following corollary.

Corollary 2.3.12. *Let (V,ω,G) be a symplectically irreducible symplectic triple. If the symplectic linear quotient V/G admits a symplectic resolution, then there is a parameter $\mathbf{c} : S(G) \to \mathbb{C}$ such that the dimension of all simple $\mathsf{H}_{\mathbf{c}}(G)$-modules is equal to the order of G.*

2.4. Cox rings

Following [ADHL15, Section 1.4], we introduce an important invariant in birational geometry: the Cox ring of a variety. The Cox ring was originally defined by Cox [Cox95] as the *homogeneous coordinate ring* of a toric variety and transferred to a more general setting by Hu and Keel [HK00]. It is also referred to as the *total coordinate ring.*

Notation 2.4.1. We write $\mathrm{Div}(Y)$ for the group of Weil divisors on a normal variety Y and usually abbreviate 'Weil divisor' to just 'divisor'. For a divisor $D \in \mathrm{Div}(Y)$, we write $[D] \in \mathrm{Cl}(Y)$ for its class in the class group.

2.4.1. Construction

Let Y be a normal variety with finitely generated class group $\mathrm{Cl}(Y)$. Note that linear quotients fulfil this assumption, see Proposition 2.1.7. As the definition of the Cox ring is a bit involved if the class group contains torsion, we assume at first that $\mathrm{Cl}(Y)$ is a free group, to get an idea.

Recall that $\mathrm{Div}(Y)$ is the free abelian group generated by the prime divisors on Y and that $\mathrm{Cl}(Y)$ is the quotient of $\mathrm{Div}(Y)$ by the principal divisors. We can hence fix a subgroup $H \leq \mathrm{Div}(Y)$ such that the canonical map $\pi : H \to \mathrm{Cl}(Y)$ sending $D \in H$ to its class $[D] \in \mathrm{Cl}(Y)$ is an isomorphism by choosing representatives of a minimal set of generators of the free group $\mathrm{Cl}(Y)$.

Definition 2.4.2 (Cox ring)**.** Assume that $\mathrm{Cl}(Y)$ is free. The *Cox ring* associated to H is the $\mathrm{Cl}(Y)$-graded ring

$$\mathcal{R}(Y) := \bigoplus_{D \in H} \Gamma(Y, \mathcal{O}_Y(D)) \ ,$$

where the multiplication in $\mathcal{R}(Y)$ is given by multiplying homogeneous sections in the field of rational functions $\mathbb{C}(Y)$.

Given two subgroups $H, H' \leq \mathrm{Div}(Y)$ projecting isomorphically onto $\mathrm{Cl}(Y)$, their associated Cox rings are graded isomorphic, see [ADHL15, Construction 1.4.1.1]. In particular, the definition is independent of the choice of the group H.

Example 2.4.3. Let $Y = \mathbb{P}^n$ be the projective space for some $n \geq 1$ with homogeneous coordinate ring $\mathbb{C}[x_0, \ldots, x_n]$ and let $D \subseteq \mathbb{P}^n$ be the hyperplane defined by $x_0 = 0$. The class of D generates $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$ freely [Har77, Proposition II.6.4] and we choose $H = \langle D \rangle$ as system of representatives for $\mathrm{Cl}(\mathbb{P}^n)$. A section $f \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kD))$ for $k \in \mathbb{Z}$ is given by a polynomial of degree $\leq k$ in the coordinates $\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}$. In particular, $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kD))$ is trivial for $k < 0$. For $k \geq 0$, multiplying by x_0^k induces an isomorphism of $\mathbb{C}$-vector spaces

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kD)) \cong \{f \in \mathbb{C}[x_0, \ldots, x_n] \mid f \text{ homogeneous of degree } k\} \ .$$

In conclusion, the Cox ring $\mathcal{R}(\mathbb{P}^n)$ is $\mathbb{Z}$-graded isomorphic to the polynomial ring with the standard grading $\mathbb{C}[x_0, \ldots, x_n]$, which is the homogeneous coordinate ring of $\mathbb{P}^n$. ‖

We now drop the assumption that $\mathrm{Cl}(Y)$ is free. We still require that $\mathrm{Cl}(Y)$ is finitely generated and impose the additional assumption that $\Gamma(Y, \mathcal{O}_Y^\times) = \mathbb{C}^\times$, which is for example fulfilled by varieties which are projective over $\operatorname{Spec}\mathbb{C}$, but also again for linear quotients, see below. Let $H \leq \operatorname{Div}(Y)$ be a subgroup such that the canonical map $\pi : H \to \operatorname{Cl}(Y)$ sending D to its class $[D] \in \operatorname{Cl}(Y)$ is surjective. Any $D \in \ker(\pi)$ is a principal divisor, so there is $f \in \mathbb{C}(Y)^\times$ with $D = \operatorname{div}(f)$. Hence we can choose a group homomorphism $\chi : \ker(\pi) \to \mathbb{C}(Y)^\times$ with $\operatorname{div}(\chi(D)) = D$ for all $D \in \ker(\pi)$. Let

$$S := \bigoplus_{D \in H} \Gamma(Y, \mathcal{O}_Y(D))$$

be the divisorial algebra associated to H, see [ADHL15, Definition 1.3.1.1], and set

$$I := \langle 1 - \chi(D) \mid D \in \ker(\pi) \rangle \trianglelefteq S \ .$$

For a generator $1 - \chi(D)$ of I, we have 1 in degree 0 and $\chi(D)$ in degree $-D$.

Definition 2.4.4. The Cox ring of Y associated to H and χ is the quotient $\mathcal{R}(Y) := S/I$ graded by $\operatorname{Cl}(Y)$ via

$$\mathcal{R}(Y) = \bigoplus_{[D] \in \operatorname{Cl}(Y)} \mathcal{R}_{[D]}(Y) \ , \qquad \mathcal{R}_{[D]}(Y) := \rho\Big(\bigoplus_{E \in \pi^{-1}([D])} S_E \Big)$$

with the projection morphism $\rho : S \to \mathcal{R}(Y)$.

Proposition 2.4.5. *Let Y be a normal variety with $\Gamma(Y, \mathcal{O}_Y^\times) = \mathbb{C}^\times$ and finitely generated class group $\operatorname{Cl}(Y)$. Then different choices of H and χ as above give rise to Cox rings which are graded isomorphic.*

See [ADHL15, Proposition 1.4.2.2] for a proof.

Example 2.4.6. We summarize [ADHL15, Example 1.4.2.4] dealing with the affine cone $Y = \mathbb{C}^2/\mathsf{C}_2 = V(xy - z^2) \subseteq \mathbb{A}^3$. We revisit this example in Example 2.4.12 after establishing more theory. Write $f_x := x|_Y$, $f_y := y|_Y$ and $f_z := z|_Y$ for the functions on Y. We have the prime divisor $D := V_Y(f_y) = V(y, z) \cap Y$, whose class generates the divisor class group $\operatorname{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z}$ [Har77, Example II.6.5.2]. Write $H = \langle D \rangle$ and let S be the corresponding divisorial algebra as above. Grade the ring $\mathbb{C}[u, v, w^{\pm 1}]$ by H via $\deg_H(u), \deg_H(v) := D$ and $\deg_H(w) := 2D$. One can prove that there is an H-graded isomorphism

$$\begin{aligned} \mathbb{C}[u, v, w^{\pm 1}] &\to S \ , \\ u &\mapsto 1 &&\in \Gamma(Y, \mathcal{O}_Y(D)) \ , \\ v &\mapsto f_z f_y^{-1} &&\in \Gamma(Y, \mathcal{O}_Y(D)) \ , \\ w &\mapsto f_y^{-1} &&\in \Gamma(Y, \mathcal{O}_Y(2D)) \ . \end{aligned}$$

The kernel of the projection $\pi : H \to \operatorname{Cl}(Y)$ is $\ker(\pi) = \langle 2D \rangle$ and a group homomorphism $\chi : \ker(\pi) \to \mathbb{C}(Y)^\times$ as above is given by

$$\chi : \ker(\pi) \to \mathbb{C}(Y)^\times, \ 2nD \mapsto f_y^n \ .$$

By [ADHL15, Remark 1.4.3.2], the ideal $I = \langle 1 - \chi(E) \mid E \in \ker(\pi)\rangle$ is generated by the element $1 - f_y$ with 1 having degree 0 and f_y having degree $-2D$. Therefore we have

$$\mathcal{R}(Y) = S/I \cong \mathbb{C}[u, v, w^{\pm 1}]/\langle 1 - w^{-1}\rangle \cong \mathbb{C}[u, v] \ ,$$

with the $\mathrm{Cl}(Y)$-grading $\deg(u) = \deg(v) = [D]$. That means that there are two graded components in $\mathbb{C}[u, v]$; one generated as a $\mathbb{C}$-vector space by all monomials of even degree and one generated by all monomials of odd degree. ∥

2.4.2. Birational morphisms

Let X and Y be normal varieties and let $\varphi : X \to Y$ be a projective birational morphism. Assume that the class groups $\mathrm{Cl}(X)$ and $\mathrm{Cl}(Y)$ are finitely generated and that $\Gamma(X, \mathcal{O}_X^\times) = \mathbb{C}^\times = \Gamma(Y, \mathcal{O}_Y^\times)$. Write $\varphi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ for the induced isomorphism of function fields.

Recall that there is an induced push-forward morphism of groups

$$\varphi_* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$$

mapping a divisor D to $\varphi(D)$, if $\varphi(D)$ is a divisor, and to 0 otherwise. This gives rise to a morphism of class groups $\mathrm{Cl}(X) \to \mathrm{Cl}(Y)$, which we call φ_* as well by abuse of notation, see [Ful84, Theorem 1.4].

We can relate the Cox rings of X and Y as follows.

Proposition 2.4.7. *With the assumptions on X, Y and $\varphi : X \to Y$ as at the beginning of this section, there is a surjective morphism of graded rings $\varphi_* : \mathcal{R}(X) \to \mathcal{R}(Y)$ given by the isomorphism $\varphi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ on the rings and the push-forward morphism $\varphi_* : \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ on the grading groups.*

See [ADHL15, Proposition 4.1.3.1] for a proof. We only require the first statement given there and hence do not need to assume that both X and Y are complete.

2.4.3. Mori dream spaces

We give a brief overview of the theory of *Mori dream spaces* introduced by Hu and Keel [HK00] to motivate our interest in Cox rings in the context of $\mathbb{Q}$-factorial terminalizations.

Definition 2.4.8 (Mori dream space)**.** Let $X \to Y$ be a projective morphism of normal varieties X and Y. Assume that $\mathrm{Cl}(X)$ is finitely generated and that $\Gamma(X, \mathcal{O}_X^\times) = \mathbb{C}^\times$. We call X a *(relative) Mori dream space* over Y if the Cox ring $\mathcal{R}(X)$ is finitely generated.

Remark 2.4.9. There are (at least) two common ways to define Mori dream spaces: an 'algebraic definition' via the Cox ring as above or a 'geometric definition' via the structure of the nef cone of the variety X, see [HK00, Definition 1.10]. The reason for the adjective 'relative' is more obvious from the latter definition.

If we assume additionally that Y is affine, $\mathrm{Cl}(Y)$ is a torsion group and that X is $\mathbb{Q}$-factorial, then by [Gra19, Theorem 3.4.7] finite generation of $\mathcal{R}(X)$ is equivalent to a relative version of Mori dream spaces in the second sense, see [Gra19, Definition 3.4.6]. In our applications, Y is a linear quotient and $X \to Y$ a $\mathbb{Q}$-factorial terminalization, so we are always in this situation.

For a thorough presentation of relative Mori dream spaces see [Oht22], where they are called *Mori dream morphisms.* ‖

We note the following fact.

Lemma 2.4.10. *Let $X \to Y$ be a projective birational morphism of normal varieties where Y is affine. Then $\Gamma(X, \mathcal{O}_X) = \Gamma(Y, \mathcal{O}_Y)$.*

Proof. The morphism $X \to Y$ is projective, hence proper [Liu02, Theorem 3.3.30], and therefore $\Gamma(X, \mathcal{O}_X)$ is integral over $\Gamma(Y, \mathcal{O}_Y)$ by [Liu02, Proposition 3.3.18]. As Y is normal, $\Gamma(Y, \mathcal{O}_Y)$ is integrally closed in $\mathbb{C}(Y) \cong \mathbb{C}(X)$ and the claim follows. □

Let $X \to Y$ be a projective birational morphism of normal varieties where Y is affine and such that X is a relative Mori dream space over Y. We call a birational map $\varphi : X \dashrightarrow X'$ *small*, if φ defines an isomorphism of open subsets with complement of codimension 2. In other words, we require φ to not contract a divisor on X.

We can recover all normal varieties X' connected to X via a small birational map relative to Y via variation of GIT quotient. The main idea is the following. As $\mathcal{R}(X)$ is finitely generated and graded by $\mathrm{Cl}(X)$, we have the affine variety $\operatorname{Spec} \mathcal{R}(X)$ with an action by the quasitorus $\operatorname{Spec} \mathbb{C}[\mathrm{Cl}(X)]$. We now proceed by taking GIT quotients of certain linearizations of $\operatorname{Spec} \mathcal{R}(X)$ with respect to this action. We do not recall this construction here, but only give the results, see [ADHL15, Section 3.3.4] for details. In the given reference it is assumed that X is projective (over $\operatorname{Spec} \mathbb{C}$). However, the described GIT construction immediately generalizes to our relative setting as the constructed quotients are then projective over $\Gamma(X, \mathcal{O}_X)$, see [ADHL15, Proposition 3.1.2.2].

Taking GIT quotients translates to the following construction on the algebraic side. Every divisor $D \in \mathrm{Div}(X)$ on X gives rise to a positively graded algebra

$$S(D) \coloneqq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \Gamma(X, \mathcal{O}_X(kD)) \,.$$

As $\mathcal{R}(X)$ is by assumption finitely generated, so is the Veronese subalgebra $S(D)$ by [ADHL15, Proposition 1.1.2.4]. Furthermore, we have a rational map $X \dashrightarrow X(D)$ where $X(D) \coloneqq \operatorname{Proj} S(D)$. Notice that $X(D)$ is projective over $\operatorname{Spec} \Gamma(X, \mathcal{O}_X)$, so over Y by Lemma 2.4.10 in our situation. If we choose D to be ample, then $X \dashrightarrow X(D)$ is an isomorphism, that is, we can recover X (up to isomorphism) from its Cox ring, see also [ADHL15, Corollary 3.2.1.11].

For a divisor $D \in \mathrm{Div}(X)$, we call

$$\mathrm{Bs}(D) \coloneqq \bigcap_{f \in \Gamma(X, \mathcal{O}_X(D))} \mathrm{Supp}(\mathrm{div}(f) + D) \qquad \text{and} \qquad \mathbf{B}(D) \coloneqq \bigcap_{m=1}^{\infty} \mathrm{Bs}(mD)$$

the *base locus* and the *stable base locus* of D respectively. We say that D is *movable* if $\mathbf{B}(D)$ is of codimension at least 2 in X. If $D \in \mathrm{Div}(X)$ is movable, then $X \dashrightarrow X(D)$ is a small birational morphism. In fact, we can recover all small birational morphisms starting in X relative to Y in this way by the geometric definition of Mori dream spaces, see [Gra19, Definition 3.4.6] and recall Remark 2.4.9.

2.4.4. Linear quotients

Let again V be a finite-dimensional vector space and let $G \leq \mathrm{SL}(V)$ be a finite group. Recall from Proposition 2.1.7 that V/G is a normal variety with finitely generated divisor class group $\mathrm{Cl}(V/G) \cong \mathrm{Hom}(G, \mathbb{C}^\times)$ as G cannot contain any reflections. Further, we clearly have

$$\Gamma(V/G, \mathcal{O}_{V/G})^\times = (\mathbb{C}[V]^G)^\times = \mathbb{C}^\times$$

as $\mathbb{C}[V]^G \leq \mathbb{C}[V]$. This means that we may talk about the Cox ring $\mathcal{R}(V/G)$.

Let $\mathrm{Ab}(G) \coloneqq G/[G,G]$ be the abelianization of G and write $\mathrm{Ab}(G)^\vee$ for the group of irreducible (hence linear) characters of this group. By [BKZ18, Theorem 9.5], we have $\mathrm{Hom}(G, \mathbb{C}^\times) = \mathrm{Ab}(G)^\vee \cong \mathrm{Ab}(G)$. There is an action of $\mathrm{Ab}(G)$ on the ring $\mathbb{C}[V]^{[G,G]}$ induced by the action of G. This action induces a grading by $\mathrm{Ab}(G)^\vee$ by setting the graded component of a character $\chi \in \mathrm{Ab}(G)^\vee$ to be

$$\mathbb{C}[V]^{[G,G]}_\chi \coloneqq \{f \in \mathbb{C}[V]^{[G,G]} \mid \gamma.f = \chi(\gamma) f \text{ for all } \gamma \in \mathrm{Ab}(G)\} .$$

We have the following theorem.

Theorem 2.4.11 (Arzhantsev–Gaĭfullin)**.** *Let $G \leq \mathrm{SL}(V)$ be a finite group. Then there is an $\mathrm{Ab}(G)^\vee$-graded isomorphism $\mathcal{R}(V/G) \cong \mathbb{C}[V]^{[G,G]}$.*

See [AG10, Theorem 3.1] for a proof.

Example 2.4.12. We consider again Example 2.4.6 and determine $\mathcal{R}(\mathbb{C}^2/\mathsf{C}_2)$, this time by using Theorem 2.4.11. We have $[\mathsf{C}_2, \mathsf{C}_2] = 1$ as C_2 is abelian, so $\mathcal{R}(\mathbb{C}^2/\mathsf{C}_2) = \mathbb{C}[x,y]$. The group $\mathsf{C}_2^\vee = \mathrm{Hom}(\mathsf{C}_2, \mathbb{C}^\times)$ is generated by the character

$$\chi : \mathsf{C}_2 \mapsto \mathbb{C}^\times,\ \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) \mapsto 1,\ \left(\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right) \mapsto -1 .$$

We hence have the two graded components

$$\mathbb{C}[x,y]_\chi = \{f \in \mathbb{C}[x,y] \mid f \text{ involves only terms of odd degree}\}$$

and

$$\mathbb{C}[x,y]_{\chi^2} = \{f \in \mathbb{C}[x,y] \mid f \text{ involves only terms of even degree}\} .$$

Let $\varphi : X \to V/G$ be a $\mathbb{Q}$-factorial terminalization of the linear quotient V/G and let $m \in \mathbb{Z}_{\geq 0}$ be the number of junior conjugacy classes in G. Using [Har77, Proposition II.6.5 (c)] and Theorem 2.1.21, we have an exact sequence of abelian groups

$$\bigoplus_{i=1}^m \mathbb{Z}E_i \longrightarrow \mathrm{Cl}(X) \xrightarrow{\varphi_*} \mathrm{Cl}(V/G) \longrightarrow 0 ,$$

where E_i are the irreducible exceptional divisors on X. This implies that the class group $\mathrm{Cl}(X)$ is finitely generated as both $\mathrm{Cl}(V/G)$ and $\bigoplus_{i=1}^m \mathbb{Z}E_i$ are finitely generated groups. We give a precise description of the group $\mathrm{Cl}(X)$ in Corollary 5.4.2. Further, by Lemma 2.4.10, we have in particular $\Gamma(X, \mathcal{O}_X)^\times = \mathbb{C}^\times$. This means that we may again speak about the Cox ring $\mathcal{R}(X)$.

Theorem 2.4.13. *Let $G \leq \mathrm{SL}(V)$ be a finite group and let $\varphi : X \to V/G$ be a $\mathbb{Q}$-factorial terminalization. Then $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra and hence X a relative Mori dream space over V/G.*

See [Gra19, Theorem 3.4.10] for a proof. For symplectic linear quotients, this also follows from [Nam15, Main Theorem].

This makes the following algorithmic strategy feasible: to compute a $\mathbb{Q}$-factorial terminalization of a linear quotient V/G, one first computes the finitely generated algebra $\mathcal{R}(X)$ and then recovers X as a GIT quotient. The practical considerations concerning this idea are the main topic of chapters 6 and 7.

3. Parabolic subgroups of symplectic reflection groups

The name ‘symplectic reflection group’ invites one to compare these groups to complex reflection groups. A fundamental result for the latter is Steinberg’s fixed point theorem [Ste64, Theorem 1.5], which states that the parabolic subgroups of complex reflection groups are generated by the complex reflections they contain. In this chapter, we prove a symplectic analogue of this theorem; see Theorem 3.1.1 for the precise statement. In Section 3.2, we draw first consequences of this result regarding the rank of minimal and maximal parabolic subgroups as well as the codimension of the singular locus of a symplectic linear quotient. In [KW82, Theorem C] it is shown that a parabolic subgroup of a CI-group, that is, a group whose invariant ring is a complete intersection, is generated by bireflections. However, we give a short argument for why a symplectic reflection group of rank at least 6 cannot be a CI-group, see Section 3.3.

The results of this chapter are already published in [BST23].

3.1. A symplectic analogue of Steinberg’s fixed point theorem

The question of whether an analogue of Steinberg’s fixed point theorem also holds for symplectic reflection groups was posed in [Coh80, Remark (iv)] and again in [BS16, Question 9.1]. We can answer it in the affirmative.

Theorem 3.1.1. *Let (V, ω) be a finite-dimensional symplectic vector space over $\mathbb{C}$, let $G \leq \mathrm{Sp}(V)$ be a finite symplectic reflection group and choose $v \in V$. Then the stabilizer G_v of v in G is also a symplectic reflection group.*

Specifically, the theorem says that the stabilizer of v is generated by those symplectic reflections in G that fix v. It would be interesting to see what other properties of complex reflection groups can be generalized to symplectic reflection groups.

The stabilizer G_v of a vector v is usually called a *parabolic subgroup* of G. Therefore, Theorem 3.1.1 can be rephrased as ‘every parabolic subgroup of a symplectic reflection group is a symplectic reflection group’.

The proof of Theorem 3.1.1 given below is a case-by-case analysis using the classification of symplectically irreducible symplectic reflection groups; see Proposition 3.1.6, Proposition 3.1.7 and Lemma 3.1.8.

Remark 3.1.2. By an easy induction (see Corollary 3.1.9), one can show that any subgroup of G that fixes a subset $U \subseteq V$ pointwise is also a symplectic reflection group. ∥

Remark 3.1.3. As noted in [BS16, Remark 9.2], it would be interesting to have a conceptual proof of Theorem 3.1.1 that does not rely on the classification of symplectic reflection groups. Such a proof would provide a deeper insight in the nature of symplectic reflection groups. The proofs of Steinberg's theorem for complex reflection groups given in [Ste64], [Leh04] and [Bou68, Chapter V, Exercise 8] all make use of alternative (but equivalent) characterizations of these groups. We are not aware of any similar characterization of symplectic reflection groups that would help here. ‖

3.1.1. First reductions

Throughout this chapter, let (V, ω) be a finite-dimensional symplectic vector space over $\mathbb{C}$. Let $G \leq \mathrm{Sp}(V)$ be a finite symplectic reflection group.

Lemma 3.1.4. *Theorem 3.1.1 holds trivially for triples* (V, ω, G) *with* $\dim V \leq 4$.

Proof. Let $v \in V$ and set $H := \mathrm{Stab}_G(v)$. Write $W := (V^H)^\perp$ for the symplectic complement of the subspace of points fixed by H. Since W is symplectic by Lemma 1.2.6, H is a subgroup of $\mathrm{Sp}(W)$. We must have $\dim W < \dim V$ and $\dim W$ is even.

If $\dim V = 2$ then there are no non-trivial symplectic subspaces of V and Theorem 3.1.1 is vacuous. When $\dim V = 4$, every proper symplectic subspace has dimension 2 and all finite subgroups of $\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$ are symplectic reflection groups, see Example 1.1.11. Therefore Theorem 3.1.1 holds in this case. □

Remark 3.1.5. Assume that G is symplectically reducible. If $V^G \neq \{0\}$, we may replace V by the complement $(V^G)^\perp$ and G by the corresponding (isomorphic) subgroup of $\mathrm{Sp}((V^G)^\perp)$. We are hence in the situation of Lemma 1.2.7 and have a decomposition $V = V_1 \oplus V_2$ into symplectic subspaces invariant under G such that G can be identified with a product $G_1 \times G_2 \leq \mathrm{Sp}(V_1) \times \mathrm{Sp}(V_2)$, where both G_1 and G_2 are symplectic reflection groups. Then the stabilizer $G_v = G_{v_1} \times G_{v_2}$ of a vector $v = v_1 + v_2$ in V, with $v_i \in V_i$, is a symplectic reflection group if and only if each G_{v_i} is a symplectic reflection group in G_i. ‖

From now on, we assume that G is a symplectically irreducible symplectic reflection group.

3.1.2. Complex reducible groups

Assume that G is a complex reducible group. Recall from Proposition 1.2.9 that there is a G-invariant Lagrangian subspace $W \leq V$ such that G identified with a subgroup of $\mathrm{GL}(W)$ is a complex reflection group. The following particular case of Theorem 3.1.1 was already proved as part of [BG03, Proposition 7.7]. Since our claim is weaker than the statement of the given reference, a shorter argument suffices. We give it here for the sake of completeness.

Proposition 3.1.6. *Theorem 3.1.1 holds if* G *is complex reducible.*

Proof. Let W be a G-invariant Lagrangian subspace of V and write H for the corresponding complex reflection group, that is, the image of the embedding $G \hookrightarrow \mathrm{GL}(W)$, see Proposition 1.2.9.

Let $v \in V$, so there are $v_1 \in W$ and $v_2^* \in W^*$ with $v = v_1 + v_2^*$ and we have

$$\mathrm{Stab}_G(v) = \mathrm{Stab}_H(v_1)^{\circledast} \cap \mathrm{Stab}_H(v_2^*)^{\circledast} .$$

Since W^* is the dual of the representation W, there exists a vector $v_2 \in W$ with $\mathrm{Stab}_H(v_2) = \mathrm{Stab}_H(v_2^*)$.

By Steinberg's fixed point theorem [Ste64, Theorem 1.5], the group $\mathrm{Stab}_H(v_1)$ is generated by complex reflections. We have

$$\mathrm{Stab}_H(v_1) \cap \mathrm{Stab}_H(v_2) = \mathrm{Stab}_{\mathrm{Stab}_H(v_1)}(v_2) ,$$

and a second application of Steinberg's theorem implies that this intersection is generated by complex reflections. Hence, $\mathrm{Stab}_G(v)$ is generated by symplectic reflections as claimed. □

3.1.3. Symplectically imprimitive groups

We assume from now on that G is complex irreducible.

Let G also be symplectically imprimitive. By Lemma 1.2.10, the system of imprimitivity is of the form $V = V_1 \oplus \cdots \oplus V_n$ with $\dim V_i = 2$ for $i = 1, \ldots, n$. By Lemma 3.1.4, we may assume $\dim V > 4$. That means, we are in case (b) of Theorem 1.2.11 and there are finite subgroups $K, H \leq \mathrm{SL}_2(\mathbb{C})$ with $H \trianglelefteq K$ such that G is conjugate to the group $G_n(K, H) \leq K \wr S_n$.

Notice that the transpositions in S_n act as symplectic reflections on V; they simply swap two summands in the system of imprimitivity.

Proposition 3.1.7. *Theorem 3.1.1 holds if G is complex irreducible and symplectically imprimitive.*

Proof. Let $G = G_n(K, H)$ for finite subgroups $K, H \leq \mathrm{SL}_2(\mathbb{C})$ and $n > 2$ as explained above.

Let $v = (v_1, \ldots, v_n) \in V = V_1 \oplus \cdots \oplus V_n$. Now let $\sigma \in S_n$ such that for the permutation v' of v given by $v'_j := v_{\sigma(i)}$ we have a 'block structure'

$$(v'_1, \ldots, v'_{n_0}, v'_{n_0+1}, \ldots, v'_{n_0+n_1}, \ldots, v'_{n_0+\cdots+n_{r-1}+1}, \ldots, v'_{n_0+\cdots+n_r}) ,$$

given by the condition that $Kv'_i = Kv'_j$ if and only if there exists $0 \leq s \leq r$ with $\left(\sum_{t=-1}^{s-1} n_t\right) + 1 \leq i, j \leq \sum_{t=0}^{s} n_t$, where we set $n_{-1} := 0$. That is, we permute the entries of v so that elements in the same K-orbit lie next to each other and the number of elements lying in the same orbit is given by the n_i. Without loss of generality, we may assume $v'_1 = \cdots = v'_{n_0} = 0$. After fixing representatives $w_0, \ldots, w_r$ for the occurring orbits, we can find an element $k \in K^n$ such that

$$k.v' = w = (w_0, \ldots, w_0, w_1, \ldots, w_1, \ldots, w_r, \ldots, w_r) ,$$

where $(Kw_i) \cap (Kw_j) = \emptyset$ for $i \neq j$ and $w_0 = 0$. Combining σ and k hence gives an element $g \in K \wr S_n$ with $g.v = w$.

If an element $\tau h \in G_n(K, H)$ stabilizes the vector w, then $\tau \in S_{n_0} \times S_{n_1} \times \cdots \times S_{n_r}$. Furthermore, we must have $h = (h_1, \ldots, h_{n_0}, 1, \ldots, 1)$, where $h_1, \ldots, h_{n_0} \in K$ with $h_1 \cdots h_{n_0} \in H$.

Hence, $\mathrm{Stab}_{G_n(K,H)}(v)$ is $(K \wr S_n)$-conjugate to $G_{n_0}(K, H) \times S_{n_1} \times \cdots \times S_{n_r}$, which is a (in general, symplectically reducible) symplectic reflection group. Notice that we may have $n_i = 1$ for some of the blocks, resulting in trivial factors in the above product. The claim now follows as symplectic reflections are preserved under conjugation. □

3.1.4. Symplectically primitive groups

The only remaining case is where G is complex irreducible and symplectically primitive. Once again, we may assume $\dim V > 4$ by Lemma 3.1.4. This excludes all complex imprimitive groups (see Theorem 1.2.12) and only leaves seven complex primitive groups to consider. These are given explicitly via the root systems Q to U in [Coh80, Table II] and one can check with the help of a computer that all stabilizer subgroups are indeed generated by symplectic reflections. A list of the groups occurring in this way can be found in Appendix A.

Lemma 3.1.8. *Theorem 3.1.1 holds if G is complex irreducible and symplectically primitive.*

This finishes the proof of Theorem 3.1.1. Finally, we note how Theorem 3.1.1 implies the statement in Remark 3.1.2. The proof is the same induction as in [Ste64, Section 7] (see also [LT09, Corollary 9.51]); we repeat it for the reader's convenience.

Corollary 3.1.9. *Let (V, ω) be a finite-dimensional symplectic vector space over $\mathbb{C}$, let $G \leq \mathrm{Sp}(V)$ be a finite symplectic reflection group and let U be a subset of V. Then the subgroup of G that fixes U pointwise is also a symplectic reflection group.*

Proof. As the action of G is linear, we may replace U by the linear span $\langle U \rangle$ and assume in the following that U is a subspace of V.

Let $u_1, \ldots, u_k$ be a basis of U. Recalling that $\mathrm{Stab}_G(U)$ fixes U pointwise in this discussion, we have

$$\mathrm{Stab}_G(U) = \mathrm{Stab}_G(u_1) \cap \mathrm{Stab}_G(\langle u_2, \ldots, u_k \rangle) .$$

Now

$$\mathrm{Stab}_G(u_1) \cap \mathrm{Stab}_G(\langle u_2, \ldots, u_k \rangle) = \mathrm{Stab}_{\mathrm{Stab}_G(u_1)}(\langle u_2, \ldots, u_k \rangle)$$

and $\mathrm{Stab}_G(u_1)$ is a symplectic reflection group by Theorem 3.1.1. Hence the claim follows by induction on k. □

3.2. Applications

3.2.1. Minimal and maximal parabolic subgroups

We note the following results on the rank of minimal and maximal parabolic subgroups where minimality and maximality are to be understood with respect to inclusion.

Corollary 3.2.1. *Let $G \leq \mathrm{Sp}(V)$ be a finite symplectic reflection group and let $H \leq G$ with $H \neq \{1\}$ be a minimal parabolic subgroup. Then we have* $\dim V^H = \dim V - 2$*, that is, H is of rank 2.*

Proof. By Corollary 3.1.9, the parabolic subgroup H must contain a symplectic reflection s. Set $K := \mathrm{Stab}_G(V^s)$. Then

$$\mathrm{Stab}_G(V^H + V^K) = \mathrm{Stab}_H(V^K) = H \cap K ,$$

so $H = K$ by minimality of H. Hence, $\dim V^H = \dim V^s = \dim V - 2$. □

The analogous result for maximal parabolic subgroups is easier and does not require Theorem 3.1.1.

Lemma 3.2.2. *Let $G \leq \mathrm{Sp}(V)$ be a finite symplectic reflection group with $V^G = \{0\}$ and let $H \leq G$ be a maximal parabolic subgroup. Then* $\dim V^H = 2$*, that is, H is of rank* $\dim V - 2$.

Proof. Let $S(G) \subseteq G$ be the set of symplectic reflections. Since $G = \langle S(G)\rangle$, there must exist some $s \in S(G)$ that is not in H. Then $\dim V^s = \dim V - 2$ and we know

$$\dim V^s + \dim V^H - \dim(V^s \cap V^H) = \dim V$$

as $V^s + V^H = V$. So if $\dim V^H > 2$ then $V^s \cap V^H \neq \{0\}$. If this is the case, then let $K := \mathrm{Stab}_G(V^s \cap V^H)$. Since $V^K \neq \{0\}$, we have $K \neq G$. But $\langle s, H\rangle \leq K$ so H is a proper subgroup of K. This is a contradiction. Therefore, $\dim V^H = 2$. □

3.2.2. Codimension of symplectic linear quotient singularities

As another application, we have a result on the singular locus of the symplectic linear quotient V/G.

Corollary 3.2.3. *If G is a symplectic reflection group, then the singular locus of V/G is of pure codimension 2.*

Proof. For $k \geq 0$, let

$$V_k := \{v \in V \mid \dim V - \dim V^{G_v} = 2k\} ,$$

that is, the set of all vectors $v \in V$ with stabilizer of rank $2k$, see also [Kal03, Section 4]. Let $\pi : V \to V/G$ be the projection morphism and set $Y_k := \pi(V_k)$. By Lemma 2.1.6, we have $Y_0 = (V/G)_{\mathrm{sm}}$. Let now $v \in V_k$ for some $k \geq 1$. Then there is a minimal parabolic subgroup $H \leq G$ with $v \in V^H$ and H is of rank 2 by Corollary 3.2.1. This implies that $v \in \overline{V_1}$, where the closure is taken with respect to the Zariski topology on V. Hence, we have $\overline{Y_1} = \mathrm{Sing}(V/G)$ as π is a closed map. In particular, Y_1 is dense in $\mathrm{Sing}(V/G)$.

By [Kal03, Lemma 4.1], the morphism $V_1 \to Y_1$ induced by π is étale. By construction, V_1 is of pure codimension 2, hence so is Y_1 by [Liu02, Proposition 4.3.23]. □

3.3. Complete intersections

A symplectic reflection can be viewed as a special kind of bireflection and groups generated by bireflections are related to complete intersections by [KW82, Theorem A]. More precisely, for a finite-dimensional complex vector space V and a finite group $G \leq \mathrm{GL}(V)$, [KW82, Theorem A] says that if V/G is a complete intersection, then G is generated by bireflections. A group G with V/G a complete intersection is commonly called a *CI-group*. Furthermore, [KW82, Theorem C] states that all parabolic subgroups of a CI-group are generated by bireflections. If we assume V to be symplectic and $G \leq \mathrm{Sp}(V)$ then this condition simply means that every parabolic subgroup of G must be a symplectic reflection group.

One might then expect that symplectic reflection groups give rise to a large number of CI-groups. Indeed, in dimension 2, the resulting Kleinian singularities are all hypersurfaces and hence every finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ is a CI-group. However, we have the following result.

Proposition 3.3.1. *Let (V, ω, G) be a symplectically irreducible symplectic triple with $\dim V > 4$. Then G is not a CI-group.*

Proof. We begin by noting that [KW82, Theorem A] implies that if G is a CI-group then G must be a symplectic reflection group. Assuming this, we can make use of the classification of symplectic reflection groups presented in Chapter 1 and the classification of CI-groups in [Gor86].

First, let G be complex reducible, so the action of G on the symplectic space V is induced from a complex reflection group H acting on a Lagrangian subspace W with $V = W \oplus W^*$ as in Proposition 1.2.9. If G is a CI-group, we must have $[G, G] = \{1\}$ by [Gor86, Theorem 3]. In other words, G is abelian. But this can only happen if $H = G(m, p, 1)$ in the classification [ST54]. In particular, the group H is rank 1 and hence G is rank 2.

Next, assume that G is complex irreducible and symplectically imprimitive, with system of imprimitivity $V = V_1 \oplus \cdots \oplus V_n$. Recall that this implies $\dim V_i = 2$ for all i by Lemma 1.2.10. But then G cannot be a CI-group by [Gor86, 5.2] which says that if G is a CI-group then $\dim V_i = 1$ for all i.

If G is symplectically primitive and has rank at least six then it must be complex primitive; see Theorem 1.2.12. But then it cannot be a CI-group by [Gor86, Theorem 5]. We note that this also follows from our computational results in Appendix A, together with the arguments in the first paragraph, since all of these groups contain a stabilizer of type $G(m, p, n)^{\circledast}$ with $n > 1$ which is not a CI-group; this contradicts [KW82, Theorem C]. □

We see that symplectic reflection groups do not appear to give any new examples of CI-groups in dimensions larger than 4. It remains to understand which symplectic reflection groups of rank 4 are CI-groups; in theory, one could use the classification of Gordeev and Nakajima [Gor86, NW84, Nak84, Nak85] for this, but this appears to be very difficult to do in practice.

Modulo the groups of rank four, Proposition 3.3.1 answers the first half of [Fu06, Problem 1] in the case of symplectic linear quotient singularities.

4. On symplectic resolutions of symplectically primitive quotients

After obtaining a first result on the singularities of a symplectic linear quotient V/G in Corollary 3.2.3, we now turn to the classification problem introduced in Section 2.2: for which symplectic groups does the corresponding linear quotient admit a symplectic resolution? As explained in Section 2.2, this question can be reduced to symplectically irreducible symplectic reflection groups and the only groups for which it is still unanswered are the symplectically primitive groups of rank at least 4. In this chapter, we carry out the classification for almost all of these groups and prove that all but possibly 45 linear quotients do not admit a symplectic resolution.

In sections 4.1 and 4.2, we consider the complex imprimitive groups and use the machinery of symplectic reflection algebras introduced in Section 2.3 to obtain the announced result. We turn to the complex primitive groups in Section 4.3 where the computational results in Appendix A enable us to finish the classification of symplectically primitive groups of rank at least 6. In Section 4.4 we give an explicit list of the 45 open cases.

The results of this chapter are already published in [BST22] and, with regard to Section 4.3, [BST23].

4.1. Structure of the symplectically primitive, complex imprimitive groups

Throughout this chapter, all symplectic reflection groups are assumed to be symplectically irreducible. We start with studying the structure of symplectically primitive, complex imprimitive symplectic reflection groups in more detail. Recall from Theorem 1.2.12 that these groups are of rank 2 or 4, where in rank 2 we have the well-known binary dihedral groups D_m. We now consider the groups of rank 4. By Theorem 1.2.12, such a group is conjugate to

$$E(G) = \{g^{\circledast}, g^{\circledast}s \mid g \in G\}$$

with

$$s = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

and $G \leq \mathrm{GL}_2(\mathbb{C})$ one of the following:

(a) $\mu_d\mathsf{T}$, with d a multiple of 6,

(b) $\mu_d\mathsf{O}$, with d a multiple of 4,

(c) $\mu_d\mathsf{I}$, with d a multiple of 4, 6 or 10,

(d) OT_{2d}, with d not divisible by 4.

We first analyse the possible groups G in more detail. In particular, we identify the largest complex reflection group contained in them. In the following, ζ_k denotes a primitive k-th root of unity for $k \in \mathbb{Z}_{>1}$.

4.1.1. Primitive complex reflection groups

Lemma 4.1.1. *We have $Z(\mu_d\mathsf{T}) = Z(\mu_d\mathsf{O}) = Z(\mu_d\mathsf{I}) = Z(\mathsf{OT}_d) = \mu_d$ for all even $d \in \mathbb{Z}_{\geq 1}$.*

Proof. We have $\{\pm I_2\} \subseteq \mu_d$ for even d and $Z(\mu_d\mathsf{T}) \cap \mathsf{T} \subseteq Z(\mathsf{T}) = \{\pm I_2\}$ (and analogously for O and I), which settles the first three groups.

Let $g \in Z(\mathsf{OT}_d)$. Note that $\mathsf{OT}_d \subseteq \mu_{2d}\mathsf{O}$, so for any $h \in \mathsf{OT}_d$, there exist $z \in \mu_{2d}$ and $h' \in \mathsf{O}$, such that $h = zh'$. Then $gh = hg$ implies $gzh' = zh'g$, so $gh' = h'g$. It follows $g \in Z(\mu_{2d}\mathsf{O}) = \mu_{2d}$, so $Z(\mathsf{OT}_d) \leq \mu_{2d}$. Since $\mu_{2d} \cap \mathsf{OT}_d = \mu_d$ and clearly $\mu_d \subseteq Z(\mathsf{OT}_d)$, it follows $\mu_d = Z(\mathsf{OT}_d)$. □

Lemma 4.1.2. *For any group G in Theorem 1.2.12 (a) to (d) and any $g \in G$, we have $(\det g)I_2 \in Z(G)$. More precisely, we have $\{(\det g)I_2 \mid g \in G\} = \mu_{d/2}$, if G is $\mu_d\mathsf{T}$, $\mu_d\mathsf{O}$ or $\mu_d\mathsf{I}$ and $\{(\det g)I_2 \mid g \in G\} = \mu_d$ if G is OT_d.*

Proof. Let $G = \mu_d\mathsf{T}$ with d a multiple of 6. Then the claim follows directly since $\mathsf{T} \leq \mathrm{SL}_2(\mathbb{C})$ and $(\det g)I_2 \in \mu_{d/2}$ for $g \in \mu_d$. One argues analogously for the groups $\mu_d\mathsf{O}$ and $\mu_d\mathsf{I}$.

Let $G = \mathsf{OT}_d$ with d a multiple of 2 not divisible by 8. Then $G \subseteq \mu_{2d}\mathsf{O}$, so any non-trivial determinant comes from an element $\zeta_{2d}^k g$ with a primitive $2d$-th root of unity ζ_{2d}, $g \in \mathsf{O}$ and $0 \leq k < 2d$. Then $\det(\zeta_{2d}^k g) = \zeta_d^k \in Z(G)$. For the second claim, notice that for any $0 \leq k < 2d$ either $\zeta_{2d}^k I_2 \in G$ or $\zeta_{2d}^k \omega \in G$, so we obtain indeed all elements of $Z(G)$ as determinants. □

Lemma 4.1.3. *The groups O and I are not conjugate to any subgroup of $\mu_d\mathsf{T}$ for even $d \in \mathbb{Z}_{\geq 1}$.*

Proof. Assume there is an embedding $\mathsf{O} \hookrightarrow \mu_d\mathsf{T}$ for an even d. Then we also would have an injective map $\mathsf{O}/Z(\mathsf{O}) \hookrightarrow \mu_d\mathsf{T}/Z(\mu_d\mathsf{T})$, since the preimage of $Z(\mu_d\mathsf{T})$ must be contained in $Z(\mathsf{O})$. But

$$|\mu_d\mathsf{T}/Z(\mu_d\mathsf{T})| = |\mu_d\mathsf{T}/\mu_d| = \frac{|\mathsf{T}|}{2} = 12$$

and

$$|\mathsf{O}/Z(\mathsf{O})| = \frac{|\mathsf{O}|}{2} = 24\ ,$$

so this is not possible. The same reasoning holds for I in place of O since $|\mathsf{I}/Z(\mathsf{I})| = 60$. □

Group	Shephard–Todd number	Group	Shephard–Todd number
$\mu_6\mathsf{T}$	5	$\mu_{12}\mathsf{T}$	7
$\mu_4\mathsf{O}$	13	$\mu_8\mathsf{O}$	9
$\mu_{12}\mathsf{O}$	15	$\mu_{24}\mathsf{O}$	11
$\mu_4\mathsf{I}$	22	$\mu_6\mathsf{I}$	20
$\mu_{10}\mathsf{I}$	16	$\mu_{12}\mathsf{I}$	21
$\mu_{20}\mathsf{I}$	17	$\mu_{30}\mathsf{I}$	18
$\mu_{60}\mathsf{I}$	19		
OT_2	12	OT_4	8
OT_6	14	OT_{12}	10

Table 4.1.1.: Primitive complex reflection groups

For groups $G, H \leq \mathrm{GL}_2(\mathbb{C})$, we write $H \leq_g G$ if $gHg^{-1} \leq G$ with $g \in \mathrm{GL}_2(\mathbb{C})$.

Lemma 4.1.4. *The group* O *is not conjugate to any subgroup of* OT_{2d} *for any* $d \in \mathbb{Z}_{\geq 1}$.

Proof. Assume $\mathsf{O} \leq_g \mathsf{OT}_{2d}$ for a $g \in \mathrm{GL}_2(\mathbb{C})$ and let $h \in \mathsf{O}$. By the explicit description of OT_{2d} (see the construction before Theorem 1.2.12), we may distinguish two cases.

First assume $ghg^{-1} = \zeta_{4d}^k \omega t$ for some $t \in \mathsf{T}$ and $1 \leq k < 4d$ odd. But this would imply $\det(\zeta_{4d}^k I_2) = 1$, so k must be a multiple of $2d$ in contradiction to k being odd.

Hence, we must have $ghg^{-1} = \zeta_{4d}^k t$ for some $t \in \mathsf{T}$ and $0 \leq k < 4d$ even. As this holds for all $h \in \mathsf{O}$, it follows $\mathsf{O} \leq_g \mu_{4d}\mathsf{T}$ in contradiction to Lemma 4.1.3. □

Lemma 4.1.5. *There exists* $g \in \mathrm{GL}_2(\mathbb{C})$ *with* $\mathsf{OT}_{2d} \leq_g \mathsf{OT}_{2d'}$ *for* d *and* d' *both not divisible by 4 if and only if* d *divides* d' *with* d'/d *odd.*

Proof. Assume $\mathsf{OT}_{2d} \leq_g \mathsf{OT}_{2d'}$ for some $g \in \mathrm{GL}_2(\mathbb{C})$. We have $\zeta_{4d}\omega \in \mathsf{OT}_{2d}$ so $g\zeta_{4d}\omega g^{-1} \in \mathsf{OT}_{2d'}$ and hence

$$\det(\zeta_{4d}\omega)I_2 = \zeta_{4d}^2 I_2 \in Z(\mathsf{OT}_{2d'}) = \mu_{2d'} ,$$

by Lemma 4.1.2. So $\zeta_{4d}^2 = \zeta_{2d'}^k$ for some $0 \leq k < 2d'$, which already shows $d \mid d'$. Now assume that $k = d'/d$ is even. Then the only elements of $\mathsf{OT}_{2d'}$ having determinant $\zeta_{2d'}^k$ lie in $\zeta_{4d'}^k\mathsf{T}$. But then we would have $g\zeta_{4d}\omega g^{-1} \in \mu_{4d'}\mathsf{T}$, so $g\omega g^{-1} \in \mu_{16dd'}\mathsf{T}$ in contradiction to Lemma 4.1.3. □

Every group G in Theorem 1.2.12 (a) to (d) contains a primitive complex reflection group of rank 2: following [Coh76, (3.6)], we can identify the groups in Theorem 1.2.12 (a) to (d) for 'small' values of d with the groups G_5 and G_7 to G_{22} in the classification by Shephard and Todd [ST54], see Table 4.1.1.

We now want to describe the largest complex reflection group contained in G, that is, the group $G_0 \leq G$ generated by the reflections contained in G. Let G' be any primitive complex reflection group contained in G. Then G_0 must also be primitive and of rank 2 since it contains G'. By [Coh76, Theorem 3.4], G_0 must be conjugate to one of the

group	is (conjugate to) a subgroup of
$\mu_6\mathsf{T}$	$\mu_{12}\mathsf{T}$, $\mu_d\mathsf{O}$ for $d \in \{12, 24\}$, $\mu_d\mathsf{I}$ for $d \in \{6, 12, 30, 60\}$, OT_{2d} for $d \in \{3, 6\}$
$\mu_{12}\mathsf{T}$	$\mu_{12}\mathsf{O}$, $\mu_{24}\mathsf{O}$, $\mu_{12}\mathsf{I}$, $\mu_{60}\mathsf{I}$, OT_{12}
$\mu_4\mathsf{O}$	$\mu_8\mathsf{O}$, $\mu_{12}\mathsf{O}$, $\mu_{24}\mathsf{O}$
$\mu_8\mathsf{O}$	$\mu_{24}\mathsf{O}$
$\mu_{12}\mathsf{O}$	$\mu_{24}\mathsf{O}$
$\mu_{24}\mathsf{O}$	
$\mu_4\mathsf{I}$	$\mu_{12}\mathsf{I}$, $\mu_{20}\mathsf{I}$, $\mu_{60}\mathsf{I}$
$\mu_6\mathsf{I}$	$\mu_{12}\mathsf{I}$, $\mu_{30}\mathsf{I}$, $\mu_{60}\mathsf{I}$
$\mu_{10}\mathsf{I}$	$\mu_{20}\mathsf{I}$, $\mu_{30}\mathsf{I}$, $\mu_{60}\mathsf{I}$
$\mu_{12}\mathsf{I}$	$\mu_{60}\mathsf{I}$
$\mu_{20}\mathsf{I}$	$\mu_{60}\mathsf{I}$
$\mu_{30}\mathsf{I}$	$\mu_{60}\mathsf{I}$
$\mu_{60}\mathsf{I}$	
OT_2	$\mu_d\mathsf{O}$ for $d \in \{4, 8, 12, 24\}$, OT_6
OT_4	$\mu_8\mathsf{O}$, $\mu_{24}\mathsf{O}$, OT_{12}
OT_6	$\mu_{12}\mathsf{O}$, $\mu_{24}\mathsf{O}$
OT_{12}	$\mu_{24}\mathsf{O}$

Table 4.1.2.: Subgroup relations

groups G_4 to G_{22} in [ST54]. To reduce the number of cases one has to consider in the proof of the next proposition, we computed which groups of the table are (conjugate to) a subgroup of another group using OSCAR [Osc23]. We summarize the results in Table 4.1.2. Note that the groups G_4 and G_6 do not contain any other group.

Proposition 4.1.6. *For the groups G in Theorem 1.2.12 (a) to (d), the largest complex reflection group $G_0 \leq \mathrm{GL}_2(\mathbb{C})$ contained in G is as follows:*

(a) If $G = \mu_d\mathsf{T}$ then $G_0 = \mu_{d_0}\mathsf{T}$ with $d_0 \in \{6, 12\}$ the largest number dividing d.

(b) If $G = \mu_d\mathsf{O}$ then $G_0 = \mu_{d_0}\mathsf{O}$ with $d_0 \in \{4, 8, 12, 24\}$ the largest number dividing d.

(c) If $G = \mu_d\mathsf{I}$ then $G_0 = \mu_{d_0}\mathsf{I}$ with $d_0 \in \{4, 6, 10, 12, 20, 30, 60\}$ the largest number dividing d.

(d) If $G = \mathsf{OT}_{2d}$ then $G_0 = \mathsf{OT}_{2d_0}$ with $d_0 \in \{1, 2, 3, 6\}$ the largest number dividing d, such that d/d_0 is odd.

In each case we have $G_0 \trianglelefteq G$ and $G/G_0 \cong \mu_{d'}$ with $d' := d/d_0$.

Proof. (a) Let $G = \mu_d\mathsf{T}$, d a multiple of 6. Then clearly $\mu_6\mathsf{T} \leq G$, so by the above discussion we have to consider the groups in the first row of Table 4.1.2.

The group $\mu_{12}\mathsf{T}$ is a subgroup of G if and only if d is a multiple of 12.

For any $g \in \mathrm{GL}_2(\mathbb{C})$, we cannot have $\mu_{\tilde{d}}\mathsf{O} \leq_g G$ or $\mu_{\tilde{d}}\mathsf{I} \leq_g G$ for any $\tilde{d}$ since this would imply $\mathsf{O} \leq_g G$ or $\mathsf{I} \leq_g G$ which does not hold by Lemma 4.1.3.

Assume finally $\mathsf{OT}_{2\tilde{d}} \leq_g G$ for some $g \in \mathrm{GL}_2(\mathbb{C})$. Then for all $h \in \mathsf{O}$ we have $ghg^{-1} = \zeta_d^k t$ or $g\zeta_{4\tilde{d}}hg^{-1} = \zeta_d^k t$ for some $0 \leq k < d$ and $t \in \mathsf{T}$. But then $ghg^{-1} \in \mu_{4\tilde{d}d}\mathsf{T}$, so $\mathsf{O} \leq_g \mu_{4\tilde{d}d}\mathsf{T}$ in contradiction to Lemma 4.1.3.

So the largest complex reflection group in G is $\mu_{d_0}\mathsf{T}$ with

$$d_0 := \begin{cases} 6, & d \text{ is an odd multiple of } 6, \\ 12, & d \text{ is an even multiple of } 6 \end{cases}$$

and clearly $G/G_0 \cong \mu_{d/d_0}$.

(b) Let $G = \mu_d\mathsf{O}$, d a multiple of 4. Then $\mu_4\mathsf{O} \leq G$, so $\mu_4\mathsf{O} \leq G_0$ and we only have to consider the supergroups of $\mu_4\mathsf{O}$ in Table 4.1.2. This already finishes this case.

(c) Let $G = \mu_d\mathsf{I}$, d a multiple of 4, 6 or 10. Then G certainly contains $\mu_4\mathsf{I}$, $\mu_6\mathsf{I}$ or $\mu_{10}\mathsf{I}$ and Table 4.1.2 assures us that the only possible subgroups are of the form $\mu_{d_0}\mathsf{I}$.

(d) Let $G = \mathsf{OT}_{2d}$ with d not divisible by 4. By Lemma 4.1.5, OT_{2d_0} is a subgroup of OT_{2d} if and only if d_0 divides d and d/d_0 is odd. Choosing the largest such $d_0 \in \{1, 2, 3, 6\}$ we hence obtain the largest reflection group of type OT_{2d_0} contained in OT_{2d}. Such a d_0 always exists since d is either an odd multiple of 1 or of 2. Consulting Table 4.1.2 again, it remains to prove $\mu_{\tilde{d}}\mathsf{O} \not\leq_g G$ for any $\tilde{d} \in \{4, 8, 12, 24\}$ and any $g \in \mathrm{GL}_2(\mathbb{C})$. This holds by Lemma 4.1.4.

Lastly, we prove $G/G_0 \cong \mu_{d/d_0}$. Set $d' := d/d_0$ and define $\varphi : G \to \mu_{d'}$ by $\varphi(\zeta_{4d}^k g) := \zeta_{d'}^k I_2$ for all $0 \leq k < 4d$ and $g \in \mathsf{O}$, such that $\zeta_{4d}^k g \in G$. Let $\zeta_{4d}^k g \in \ker(\varphi)$. Then $d' \mid k$, so $k = d'l$ for some $l \in \mathbb{Z}_{\geq 0}$, where l is odd if and only if k is odd, since d' is odd. Hence $\zeta_{4d}^k g = \zeta_{4d_0}^l g \in \mathsf{OT}_{2d_0}$. As φ is surjective, it follows $G/\mathsf{OT}_{2d_0} \cong \mu_{d'}$. □

4.1.2. Symplectically imprimitive symplectic reflection groups

We analyse the structure of the symplectically imprimitive symplectic reflection groups of rank 4. As before, let $G \leq \mathrm{GL}_2(\mathbb{C})$ be one of the groups in Theorem 1.2.12 and let $G_0 \leq G$ be the largest complex reflection group contained in G as in Proposition 4.1.6. Write $Z(G) = \mu_d$ as in Lemma 4.1.1, so that we have $G = \mu_d G_0$, but note that $\mu_d \cap G_0 = Z(G_0)$.

We consider certain normal subgroups of the group $E(G)$.

Lemma 4.1.7. *The subgroups $G^{\circledast}$ and $G_0^{\circledast}$ are normal subgroups of $E(G)$.*

Proof. For $g, h \in G$, we have $g^{\circledast}h^{\circledast}(g^{\circledast})^{-1} = (ghg^{-1})^{\circledast} \in G^{\circledast}$. If $h \in G_0$, then also $g^{\circledast}h^{\circledast}(g^{\circledast})^{-1} \in G_0^{\circledast}$, since either $g \in G_0$ or $g \in Z(G)$. It remains to show $sh^{\circledast}s^{-1} \in G^{\circledast}$ for $h \in G$. Here, an easy calculation shows $sh^{\circledast}s^{-1} = \big((\det h)^{-1}h\big)^{\circledast} \in G^{\circledast}$ (see Lemma 4.1.2) and the same holds for $h \in G_0$. □

Lemma 4.1.8. *The group* $D_d \coloneqq \langle \mu_d^{\circledast}, s\rangle \leq E(G)$ *is the dihedral group of order* $2d$ *and a normal subgroup of* $E(G)$*.*

Proof. By definition, D_d is generated by $r^{\circledast}$ and s, where

$$r \coloneqq \begin{pmatrix} \zeta_d & \\ & \zeta_d \end{pmatrix},$$

and the equalities

$$(r^{\circledast})^d = s^2 = (sr^{\circledast})^2 = I_4$$

hold, so D_d is indeed the dihedral group of order $2d$.

Let $t \in E(G)$, so $t = g^{\circledast}s^k$ for some $g \in G$ and $k \in \{0,1\}$. We have $tr^{\circledast}t^{-1} = r^{\circledast} \in D_d$, if $k = 0$, and $tr^{\circledast}t^{-1} = (r^{\circledast})^{-1} \in D_d$, if $k = 1$. Further, we have

$$tst^{-1} = g^{\circledast}s^k ss^{-k}(g^{\circledast})^{-1} = g^{\circledast}s(g^{\circledast})^{-1}.$$

But

$$g^{\circledast}s(g^{\circledast})^{-1} = \begin{pmatrix} & A \\ A^{-1} & \end{pmatrix}$$

with

$$A \coloneqq g \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} g^{\top} = \begin{pmatrix} & \det g \\ -\det g & \end{pmatrix}.$$

By $\det g = \zeta_d^l$ for some $0 \leq l < d$, it follows $tst^{-1} = (r^l)^{\circledast}s \in D_d$ and D_d is indeed a normal subgroup of $E(G)$. ☐

We denote by $R(G)$ the set of (complex) reflections contained in G. The symplectic reflections in $E(G)$ split up between the subgroups G and D_d.

Proposition 4.1.9. *The group* $E(G)$ *is a symplectic reflection group with symplectic reflections*

$$S \coloneqq \{g^{\circledast} \mid g \in R(G)\} \sqcup \{z^{\circledast}s \mid z \in \mu_d\}.$$

Proof. If $g \in R(G)$, then $g^{\circledast}$ is a symplectic reflection. Also, for $z = \begin{pmatrix} \zeta_d^k & \\ & \zeta_d^k \end{pmatrix} \in \mu_d$ for some $0 \leq k < d$, we have

$$z^{\circledast}s = \begin{pmatrix} & & & \zeta_d^k \\ & & -\zeta_d^k & \\ & -\zeta_d^{-k} & & \\ \zeta_d^{-k} & & & \end{pmatrix},$$

so $\operatorname{rk}(z^{\circledast}s - I_4) = 2$ and $z^{\circledast}s$ is a symplectic reflection. Hence, all elements in S are indeed symplectic reflections and $E(G)$ is a symplectic reflection group since $E(G) = \langle S\rangle$.

Now let $t \in E(G)$ be a symplectic reflection. Then either $t = g^{\circledast}$ or $t = g^{\circledast}s$ for a $g \in G$. In the first case, it directly follows $g \in R(G)$. So assume $t = g^{\circledast}s$. For ease of notation, we define

$$A \coloneqq g \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \text{ and } B \coloneqq (g^{\top})^{-1} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},$$

so that

$$t = \begin{pmatrix} & A \\ B & \end{pmatrix}.$$

From

$$I_4 - t = \begin{pmatrix} I_2 & -A \\ -B & I_2 \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ -B & I_2 - BA \end{pmatrix} \begin{pmatrix} I_2 & -A \\ 0 & I_2 \end{pmatrix}$$

it follows that $\mathrm{rk}(I_4 - t) = 2$ if and only if $BA = I_2$, so $A = B^{-1}$. A straightforward calculation shows that this requires g to be a scalar matrix, so $g \in Z(G) = \mu_d$, as all scalar matrices lie in the centre of G. Therefore, all symplectic reflections in $E(G)$ are elements of S.

Finally, note that the two given subsets of S contain matrices of different block-types, so their union is disjoint. □

Corollary 4.1.10. *All symplectic reflections in $E(G)$ lie either in $G_0^{\circledast}$ or in D_d. None of the symplectic reflections of $G_0^{\circledast}$ is conjugate in $E(G)$ to one of D_d and vice versa.*

Proof. The first part is clear since $R(G) = R(G_0)$. The second part follows from Lemma 4.1.7 and Lemma 4.1.8. □

As we want to apply the theory established in Section 2.3, we are interested in the behaviour of the symplectic reflections under conjugacy.

Lemma 4.1.11. *There are two D_d-conjugacy classes in $S(D_d)$, namely*

$$[s] \text{ and } [(\zeta_d I_2)^{\circledast} s] .$$

In case G is $\mu_d\mathsf{T}$, $\mu_d\mathsf{O}$ or $\mu_d\mathsf{I}$ these are also the $E(G)$-conjugacy classes. In case G is OT_d, there is only one $E(G)$-conjugacy class in $S(D_d)$.

Proof. For the claim about D_d-conjugacy, see [BT16, Section 8.3]. The computations in the proof of Lemma 4.1.8 show that for $g \in E(G)$ we have $gsg^{-1} = z^{\circledast}s$ with $z \in \{(\det h)I_2 \mid h \in G\}$ (and for any such z there exists a $g \in E(G)$). Hence, s and $(\zeta_d I_2)^{\circledast}s$ are conjugate in $E(G)$ if and only if there exists $h \in G$ with $\det h = \zeta_d$. By Lemma 4.1.2, this is the case if and only if $G = \mathsf{OT}_d$. □

4.2. On the non-existence of symplectic resolutions for complex imprimitive groups

We prove that the linear quotients corresponding to all but possibly 39 symplectically primitive, complex imprimitive symplectic reflection groups of rank 4 do not admit a symplectic resolution.

Let again $G \leq \mathrm{GL}_2(\mathbb{C})$ be one of the groups in Theorem 1.2.12, let G_0 be the largest complex reflection group contained in G and let $\mu_d = Z(G)$. Let $D_d := \langle \mu_d^{\circledast}, s \rangle \leq E(G)$ as before. Let $V = \mathbb{C}^4$ with standard symplectic form ω (notice that we already implicitly assumed this setting when we defined s).

We want to use Corollary 2.3.12, that is, we want to construct a simple module of the symplectic reflection algebra $\mathsf{H}_{\mathbf{c}}(E(G))$ of dimension strictly less than $|E(G)|$ to conclude that $V/E(G)$ does not admit a symplectic resolution. The main idea is to use the symplectic reflection algebras coming from the subgroups $G^{\circledast}$ and D_d of $E(G)$ identified in the previous section as we have a better understanding of the representation theory of these algebras. The keywords here are *baby Verma modules* and *rigid representations* as introduced in [Gor03] and [BT16], respectively. To be able to state the precise result, we require a bit more notation.

By Corollary 4.1.10, we may split a parameter $\mathbf{c} : S(E(G)) \to \mathbb{C}$ into two $E(G)$-invariant functions $\mathbf{c}_1 : S(E(G)) \to \mathbb{C}$ and $\mathbf{c}_2 : S(E(G)) \to \mathbb{C}$ given by

$$\mathbf{c}_1(g) = \begin{cases} \mathbf{c}(g), & g \in S(G_0^{\circledast}), \\ 0, & g \in S(D_d), \end{cases} \qquad \text{and} \qquad \mathbf{c}_2(g) = \begin{cases} 0, & g \in S(G_0^{\circledast}), \\ \mathbf{c}(g), & g \in S(D_d), \end{cases}$$

respectively, so we may think of $\mathbf{c}$ as $\mathbf{c}_1 + \mathbf{c}_2$. By abuse of notation, we also write $\mathbf{c}_1$ respectively $\mathbf{c}_2$ for the restrictions $\mathbf{c}_1|_{S(G_0^{\circledast})}$ respectively $\mathbf{c}_2|_{S(D_d)}$. We may consider the symplectic reflection algebras $\mathsf{H}_{\mathbf{c}_1}(G_0)$ and $\mathsf{H}_{\mathbf{c}_1}(G)$ (or more precisely $\mathsf{H}_{\mathbf{c}_1}(G_0^{\circledast})$ and $\mathsf{H}_{\mathbf{c}_1}(G^{\circledast})$) with the embeddings $\mathsf{H}_{\mathbf{c}_1}(G_0) \subseteq \mathsf{H}_{\mathbf{c}_1}(G) \subseteq \mathsf{H}_{\mathbf{c}_1}(E(G))$. Notice, however, that $\mathbf{c}_1$ is in general *not* a generic (or even arbitrary) parameter for $\mathsf{H}_{\mathbf{c}_1}(G_0)$, since $G_0^{\circledast}$-invariant functions are not necessarily $E(G)$-invariant.

Let $\chi_0, \dots, \chi_{d-1}$ be the irreducible characters of $Z(G) = \mu_d$, ordered such that

$$\chi_l(\zeta_d^k I_2) = \zeta_d^{kl}$$

for all $0 \leq k, l < d$ and a primitive d-th root of unity ζ_d.

Recall that d is even. We label the irreducible representations of D_d as follows. There are four 1-dimensional representations Triv, Sgn, V_1 and V_2, where

$$\mathrm{Triv}\,|_{Z(G)^{\circledast}} = \mathrm{Sgn}\,|_{Z(G)^{\circledast}} = \chi_0 \text{ and } V_1|_{Z(G)^{\circledast}} = V_2|_{Z(G)^{\circledast}} = \chi_{\frac{d}{2}}$$

(note that $Z(G)^{\circledast} \leq D_d$). Further, there are the 2-dimensional representations φ_i, $1 \leq i \leq \frac{d}{2} - 1$, for which we have

$$\varphi_i|_{Z(G)^{\circledast}} = \chi_i \oplus \chi_{d-i}\ .$$

See [BT16, Section 8.2] for more details and precise definitions of these representations.

4.2.1. Rigid representations

We say an irreducible representation φ of D_d is $\mathbf{c}_2$-*rigid*, if φ is (isomorphic to) a simple $\mathsf{H}_{\mathbf{c}_2}(D_d)$-module, see [BT16] for details. The following proposition reduces the problem of constructing $\mathsf{H}_{\mathbf{c}}(E(G))$-modules to constructing $\mathsf{H}_{\mathbf{c}_1}(G)$-modules.

Proposition 4.2.1. *Let M be a simple $\mathsf{H}_{\mathbf{c}_1}(G)$-module and set*

$$E(M) := \mathsf{H}_{\mathbf{c}_1}(E(G)) \otimes_{\mathsf{H}_{\mathbf{c}_1}(G)} M\ .$$

Then $E(M)$ is an $\mathsf{H}_{\mathbf{c}}(E(G))$-module if and only if all constituents of the restriction $E(M)|_{D_d}$ are $\mathbf{c}_2$-rigid.

Proof. By definition, $E(M)$ is an $\mathsf{H}_{\mathbf{c}_1}(E(G))$-module. We just need to show that it naturally deforms to an $\mathsf{H}_{\mathbf{c}}(E(G))$-module. The defining relations for $\mathsf{H}_{\mathbf{c}}(E(G))$ are

$$[v,w] = \sum_{g\in S(G^{\circledast})} \mathbf{c}_1(g)\omega_g(v,w)g + \sum_{g\in S(D_d)} \mathbf{c}_2(g)\omega_g(v,w)g$$

in contrast to

$$[v,w] = \sum_{g\in S(G^{\circledast})} \mathbf{c}_1(g)\omega_g(v,w)g$$

for $\mathsf{H}_{\mathbf{c}_1}(E(G))$. As $E(M)$ is an $\mathsf{H}_{\mathbf{c}_1}(E(G))$-module this means that $[v,w]$ acts as

$$\sum_{g\in S(G^{\circledast})} \mathbf{c}_1(g)\omega_g(v,w)g\ .$$

Hence, $E(M)$ is an $\mathsf{H}_{\mathbf{c}}(E(G))$-module if and only if $\sum_{g\in S(D_d)} \mathbf{c}_2(g)\omega_g(v,w)g$ acts as zero on $E(M)$ for all $v,w\in V^*$ that is, if and only if

$$\sum_{g\in S(D_d)} \mathbf{c}_2(g)\omega_g(v,w)\varphi(g) = 0$$

for any constituent φ of $E(M)|_{D_d}$. By [BT16, Lemma 4.10], this holds if and only if all constituents of $E(M)|_{D_d}$ are $\mathbf{c}_2$-rigid. □

This means that we have to understand the rigid representations of D_d.

Lemma 4.2.2. *An irreducible representation $\varphi\in\operatorname{Irr} D_d$ is $\mathbf{c}_2$-rigid for all $E(G)$-invariant functions $\mathbf{c}_2 : S(D_d)\to\mathbb{C}$ if and only if:*

(a) $\varphi=\varphi_i$ for some $1<i<(d-2)/2$, in case G is $\mu_d\mathsf{T}$, $\mu_d\mathsf{O}$ or $\mu_d\mathsf{I}$,

(b) $\varphi=\varphi_i$ for some $1<i\le(d-2)/2$ or $\varphi\in\{V_1,V_2\}$, in case G is OT_d.

Proof. By [BT16, Proposition 8.3], the representations φ_i for $1<i<(d-2)/2$ are $\mathbf{c}_2$-rigid for arbitrary parameters $\mathbf{c}_2$. By Lemma 4.1.11, the function $\mathbf{c}_2$ is determined by its values at s and $(\zeta_d I_2)^{\circledast}s$.

(a) The symplectic reflections s and $(\zeta_d I_2)^{\circledast}s$ are not $E(G)$-conjugate by Lemma 4.1.11. Hence, there exist parameters $\mathbf{c}_2$ with $\mathbf{c}_2(s)\neq\pm\mathbf{c}_2((\zeta_d I_2)^{\circledast}s)$ and all other representations are not $\mathbf{c}_2$-rigid for those parameters by [BT16, Proposition 8.3].

(b) Here, Lemma 4.1.11 states that there is only one $E(G)$-conjugacy class in $S(D_d)$. Therefore, all parameters fulfil $\mathbf{c}_2(s)=\mathbf{c}_2((\zeta_d I_2)^{\circledast}s)$ and only φ_1, Triv and Sgn are not $\mathbf{c}_2$-rigid by [BT16, Proposition 8.3]. □

Corollary 4.2.3. *Let φ be any representation of D_d. Then all constituents of φ are $\mathbf{c}_2$-rigid for all $E(G)$-invariant functions $\mathbf{c}_2 : S(D_d)\to\mathbb{C}$ if and only if:*

(a) $\chi_i \mid \varphi|_{Z(G)}$ implies $i\notin\{0,1,\frac{d}{2}-1,\frac{d}{2},\frac{d}{2}+1,d-1\}$ in case G is $\mu_d\mathsf{T}$, $\mu_d\mathsf{O}$ or $\mu_d\mathsf{I}$,

(b) $\chi_i \mid \varphi|_{Z(G)}$ implies $i\notin\{0,1,d-1\}$ in case G is OT_d.

4.2.2. Baby Verma modules

To be able to apply Proposition 4.2.1, we also need to understand the simple modules of $\mathsf{H}_{\mathbf{c}_1}(G)$. As G_0 is a reflection group, we have the *baby Verma modules* introduced in [Gor03]; we summarize the construction. The action of G_0 respectively G on V leaves a Lagrangian subspace $\mathfrak{h}$ invariant and we may identify $\mathfrak{h}$ with the reflection representation of G_0 (hence the notation $\mathfrak{h}$), see also Lemma 1.2.8. Then $\mathfrak{h} = \mathbb{C}^2$ and $\zeta_d I_2 \in \mu_d$ acts as the scalar ζ_d on $\mathfrak{h}$ and as ζ_d^{-1} on $\mathfrak{h}^*$. We may write $V = \mathfrak{h} \oplus \mathfrak{h}^*$, but note that this decomposition is of course not invariant under the action of s. Then we can define a $\mathbb{Z}$-grading on $\mathsf{H}_{\mathbf{c}_1}(G_0)$ by putting $\mathfrak{h}^*$ in degree 1, $\mathfrak{h}$ in degree -1 and G_0 in degree 0. In the same way, we obtain a $\mathbb{Z}$-grading on $\mathsf{H}_{\mathbf{c}_1}(G)$ and the inclusion $\mathsf{H}_{\mathbf{c}_1}(G_0) \subseteq \mathsf{H}_{\mathbf{c}_1}(G)$ preserves this grading.

Let

$$\overline{\mathsf{H}}_{\mathbf{c}_1}(G_0) \coloneqq \mathsf{H}_{\mathbf{c}_1}(G_0)/\big(\mathbb{C}[\mathfrak{h}]^{G_0} \otimes \mathbb{C}[\mathfrak{h}^*]^{G_0}\big)_+ \mathsf{H}_{\mathbf{c}_1}(G_0)$$

be the *restricted rational Cherednik algebra* introduced in [Gor03], where $(-)_+$ denotes the elements with no constant term. This algebra has a triangular decomposition

$$\overline{\mathsf{H}}_{\mathbf{c}_1}(G_0) \cong \mathbb{C}[\mathfrak{h}]^{\operatorname{co} G_0} \otimes \mathbb{C}G_0 \otimes \mathbb{C}[\mathfrak{h}^*]^{\operatorname{co} G_0} ,$$

see [Thi17, Corollary 2.1], where $\mathbb{C}[\mathfrak{h}]^{\operatorname{co} G_0} \coloneqq \mathbb{C}[\mathfrak{h}]/\mathbb{C}[\mathfrak{h}]_+^{G_0}\mathbb{C}[\mathfrak{h}]$ are the G_0-coinvariants. Given $\lambda \in \operatorname{Irr} G_0$, we then have the baby Verma module

$$\Delta(\lambda) \coloneqq \overline{\mathsf{H}}_{\mathbf{c}_1}(G_0) \otimes_{\mathbb{C}[\mathfrak{h}^*]^{\operatorname{co} G_0} \rtimes G_0} \lambda$$

of $\overline{\mathsf{H}}_{\mathbf{c}_1}(G_0)$ corresponding to G_0 as in [Thi17, p. 24]. The module $\Delta(\lambda)$ has a simple head $L(\lambda)$ by [Thi17, Theorem 2.3]. We may consider both of them as $\mathsf{H}_{\mathbf{c}_1}(G_0)$-modules by letting $\mathsf{H}_{\mathbf{c}_1}(G_0)$ act via the quotient morphism $\mathsf{H}_{\mathbf{c}_1}(G_0) \twoheadrightarrow \overline{\mathsf{H}}_{\mathbf{c}_1}(G_0)$. Notice that $L(\lambda)$ is also simple as $\mathsf{H}_{\mathbf{c}_1}(G_0)$-module.

Lemma 4.2.4. *Let $\lambda \in \operatorname{Irr} G$. Then*

(a) $\lambda|_{G_0} \in \operatorname{Irr} G_0$ and

(b) the $\mathsf{H}_{\mathbf{c}_1}(G_0)$-module structure on any graded quotient of $\Delta(\lambda|_{G_0})$ extends to $\mathsf{H}_{\mathbf{c}_1}(G)$. In particular, $L(\lambda|_{G_0})$ is a graded (simple) $\mathsf{H}_{\mathbf{c}_1}(G)$-module.

Proof. (a) This is [Fei82, Theorem III.2.14 (ii)] since G/G_0 is cyclic.

(b) We have to define an action of $Z(G)$ on $\Delta(\lambda|_{G_0})$. By [Thi17, Lemma 2.5], we have

$$\Delta(\lambda|_{G_0}) \cong \mathbb{C}[\mathfrak{h}]^{\operatorname{co} G_0} \otimes_{\mathbb{C}} \lambda|_{G_0}$$

as vector spaces, in particular $\Delta(\lambda|_{G_0})$ is concentrated in non-negative degree. Let $Z(G)$ act by χ on λ. By the above, $\zeta_d I_2 \in Z(G) = \mu_d$ acts by ζ_d^{-1} on $\mathfrak{h}^*$. We obtain an action of $Z(G)$ on $\Delta(\lambda|_{G_0})_k$ for any $k \geq 0$ by letting $\zeta_d I_2$ act by ζ_d^{-k} on $\mathbb{C}[\mathfrak{h}]_k^{\operatorname{co} G_0}$ and by $\chi(\zeta_d I_2)$ on $\lambda|_{G_0}$. Then this action of $Z(G)$ extends $\Delta(\lambda|_{G_0})$ to a module over $\mathsf{H}_{\mathbf{c}_1}(G)$.

Now let $M \leq \Delta(\lambda|_{G_0})$ be any graded $\mathsf{H}_{\mathbf{c}_1}(G_0)$-submodule. Since M is graded, it is stable under the action of $\mathbb{C}^\times$ induced by the action of $\mathbb{C}^\times$ on $\mathfrak{h}$. The given action of $Z(G)$ on $\mathfrak{h}^*$ is just a restriction of this action to the subgroup $\langle \zeta_d \rangle \leq \mathbb{C}^\times$. Hence this also extends M to an $\mathsf{H}_{\mathbf{c}_1}(G)$-module.

As $L(\lambda|_{G_0})$ is a graded quotient of $\Delta(\lambda|_{G_0})$, this turns $L(\lambda|_{G_0})$ into an $\mathsf{H}_{\mathbf{c}_1}(G)$-module too and $L(\lambda|_{G_0})$ is of course simple as such a module. □

4.2.3. Conclusion

We now combine the above results with Corollary 2.3.12.

Theorem 4.2.5. *If there exists a character $\lambda \in \operatorname{Irr} G$ such that $L(\lambda|_{G_0})|_{D_d}$ is $\mathbf{c}_2$-rigid for all $E(G)$-invariant functions $\mathbf{c}_2 : S(D_d) \to \mathbb{C}$ and $\dim L(\lambda|_{G_0}) < |G|$, then $V/E(G)$ does not admit a (projective) symplectic resolution.*

Proof. Since $L(\lambda|_{G_0})$ fulfils the conditions of Proposition 4.2.1, we obtain a module $E(L(\lambda|_{G_0}))$ over $\mathsf{H}_{\mathbf{c}}(E(G))$. By construction, we have

$$\dim E(L(\lambda|_{G_0})) = \dim \big(\mathsf{H}_{\mathbf{c}_1}(E(G)) \otimes_{\mathsf{H}_{\mathbf{c}_1}(G)} L(\lambda|_{G_0})\big) = 2 \dim L(\lambda|_{G_0}) < |E(G)| \,,$$

since $|E(G)| = 2|G|$. Then any simple quotient L of $E(L(\lambda|_{G_0}))$ fulfils $\dim L < |E(G)|$ as well. As this holds for arbitrary parameters $\mathbf{c}$, if follows that the variety $V/E(G)$ does not admit a symplectic resolution by Corollary 2.3.12. □

This leaves the question when a simple module $\lambda \in \operatorname{Irr} G$ as in Theorem 4.2.5 exists. We now establish theoretical bounds on d to show that this is indeed the case for almost all groups G. With the help of computer calculations, we then extend this result to some of the remaining groups.

Lemma 4.2.6. *Let $\lambda \in \operatorname{Irr}(G)$ and let $0 \leq m, M < d$ such that m is minimal with $\chi_m \mid \lambda|_{Z(G)}$ and M is maximal with $\chi_M \mid \lambda|_{Z(G)}$. Let $k \geq 0$ be maximal such that for the graded component $L(\lambda|_{G_0})_k$ we have $L(\lambda|_{G_0})_k \neq 0$. Then $L(\lambda|_{G_0})|_{D_d}$ is $\mathbf{c}_2$-rigid for all $E(G)$-invariant functions $\mathbf{c}_2 : S(D_d) \to \mathbb{C}$ if and only if:*

(a) either $m - k > 1$ and $M < \frac{d}{2} - 1$ or $m - k > \frac{d}{2} + 1$ and $M < d - 1$ in case G is $\mu_d\mathsf{T}$, $\mu_d\mathsf{O}$ or $\mu_d\mathsf{I}$,

(b) $m - k > 1$ and $M < d - 1$ in case G is OT_d.

Proof. Let χ_{m_i}, $1 \leq i \leq s$, be the constituents of $\lambda|_{Z(G)}$. As in the proof of Lemma 4.2.4, $Z(G)$ acts on a graded component $\Delta(\lambda|_{G_0})_l$ by

$$\chi_{d-l} \otimes (\chi_{m_1} \oplus \cdots \oplus \chi_{m_s}) = \chi_{m_1 - l} \oplus \cdots \oplus \chi_{m_s - l} \,,$$

for $l \geq 0$. We have $m = \min_i\{m_i\}$ and $M = \max_i\{m_i\}$. Since $L(\lambda|_{G_0})$ is a quotient of $\Delta(\lambda|_{G_0})$ and $\Delta(\lambda|_{G_0})$ has no components in negative degree [Thi17, Lemma 2.5], this implies that if

$$[L(\lambda|_{G_0})|_{Z(G)} : \chi_i] \neq 0$$

Group	Number of reflections	Minimal value of d	Group	Number of reflections	Minimal value of d
$\mu_6\mathsf{T}$	16	39	$\mu_{12}\mathsf{T}$	22	51
$\mu_4\mathsf{O}$	18	43	$\mu_8\mathsf{O}$	30	67
$\mu_{12}\mathsf{O}$	34	75	$\mu_{24}\mathsf{O}$	46	99
$\mu_4\mathsf{I}$	30	67	$\mu_6\mathsf{I}$	40	87
$\mu_{10}\mathsf{I}$	48	103	$\mu_{12}\mathsf{I}$	70	147
$\mu_{20}\mathsf{I}$	78	163	$\mu_{30}\mathsf{I}$	88	183
$\mu_{60}\mathsf{I}$	118	243			
OT_2	12	16	OT_4	18	22
OT_6	28	32	OT_{12}	34	38

Table 4.2.1.: Number of reflections in the groups G_0

then $i \in \{m-k, \dots, M\}$. Further, we have $L(\lambda|_{G_0})_0 \cong \lambda|_{G_0} \neq 0$ by [Thi17, Lemma 2.7] and $L(\lambda|_{G_0})_k \neq 0$ by assumption. Hence,

$$[L(\lambda|_{G_0})|_{Z(G)} : \chi_{m-k}] \neq 0 \text{ and } [L(\lambda|_{G_0})|_{Z(G)} : \chi_M] \neq 0 \ ,$$

that is, the extremal values of i are achieved. Now the claim follows by Corollary 4.2.3. □

Proposition 4.2.7. *Let $N := |R(G_0)|$ be the number of reflections in G_0. The group G admits a simple module λ as in Theorem 4.2.5 if $G_0 \lneq G$ and*

(a) $2N + 6 < d$ in case G is $\mu_d\mathsf{T}$, $\mu_d\mathsf{O}$ or $\mu_d\mathsf{I}$,

(b) $N + 3 < d$ in case G is OT_d.

Proof. The coinvariant ring $\mathbb{C}[\mathfrak{h}]^{\operatorname{co} G_0}$ is a positively graded ring with $(\mathbb{C}[\mathfrak{h}]^{\operatorname{co} G_0})_l = 0$ for $l > N$, by [Kan01, Proposition 20-3A]. This implies $\Delta(\lambda)_l = 0$ for each $l > N$ or $l < 0$ and any simple G_0-module λ.

Note that $d - 2 \geq 0$ by assumption. Let $\lambda \in \operatorname{Irr} G$ be any irreducible summand of $\operatorname{Ind}_{Z(G)}^G \chi_{d-2}$, so λ restricts to a multiple of χ_{d-2} on $Z(G)$. We want to use Lemma 4.2.6. For (a), we have $2N + 6 < d$, so $N + 2 < \frac{d-2}{2}$ and

$$d - 2 - N = d - (N+2) > d - \frac{d-2}{2} = \frac{d}{2} + 1 \ .$$

For (b), $N + 3 < d$ gives

$$d - 2 - N = d + 1 - (N+3) > d + 1 - d = 1 \ .$$

Hence, $L(\lambda|_{G_0})|_{D_d}$ is $\mathbf{c}_2$-rigid for all parameters $\mathbf{c}_2$ by Lemma 4.2.6.

We have $\dim L(\lambda|_{G_0}) \leq |G_0|$ by [EG02, Theorem 1.7], hence $\dim L(\lambda|_{G_0}) < |G|$ since $G_0 \lneq G$. □

G_0	Groups containing G_0 as largest reflection group	G_0	Groups containing G_0 as largest reflection group
$\mu_6\mathsf{T}$	$\mu_d\mathsf{T}$, $d \in \{6, 18, 30\}$	$\mu_{12}\mathsf{T}$	$\mu_d\mathsf{T}$, $d \in \{12, 24, 36, 48\}$
$\mu_4\mathsf{O}$	$\mu_d\mathsf{O}$, $d \in \{4, 20, 28\}$	$\mu_8\mathsf{O}$	$\mu_d\mathsf{O}$, $d \in \{8, 16, 32, 40, 56, 64\}$
$\mu_{12}\mathsf{O}$	$\mu_d\mathsf{O}$, $d \in \{12, 36, 60\}$	$\mu_{24}\mathsf{O}$	$\mu_d\mathsf{O}$, $d \in \{24, 48, 72, 96\}$
$\mu_4\mathsf{I}$	$\mu_d\mathsf{I}$, $d \in \{4, 8, 16, 28, 32, 44, 52, 56, 64\}$	$\mu_6\mathsf{I}$	$\mu_d\mathsf{I}$, $d \in \{6, 18, 42, 54, 66, 78\}$
$\mu_{10}\mathsf{I}$	$\mu_d\mathsf{I}$, $d \in \{10, 50, 70\}$	$\mu_{12}\mathsf{I}$	$\mu_d\mathsf{I}$, $d \in \{12, 24, 36, 48, 72, 84,$
$\mu_{20}\mathsf{I}$	$\mu_d\mathsf{I}$, $d \in \{20, 40, 80, 100, 140\}$		$96, 108, 132, 144\}$
$\mu_{30}\mathsf{I}$	$\mu_d\mathsf{I}$, $d \in \{30, 90, 150\}$	$\mu_{60}\mathsf{I}$	$\mu_d\mathsf{I}$, $d \in \{60, 120, 180, 240\}$
OT_2	OT_d, $d \in \{2, 10, 14\}$	OT_4	OT_d, $d \in \{4, 20\}$
OT_6	OT_d, $d \in \{6, 18, 30\}$	OT_{12}	OT_d, $d \in \{12, 36\}$

Table 4.2.2.: Groups for which Proposition 4.2.7 does not apply

4.2.4. Sharp bounds

In Table 4.2.1 we recall the number of reflections in the possible groups G_0 from [Coh76] together with the minimal value of d fulfilling the condition in Proposition 4.2.7 (which does not mean that there exists a group G for such a d). This gives the groups G for which Proposition 4.2.7 does not apply as in Table 4.2.2. Using data computed with CHAMP [Thi15], we improve the estimates in Proposition 4.2.7. We describe the necessary computations and give a concrete example below.

As before, let G_0 be one of the complex reflection groups from Table 4.1.1 and let $(G_d)_{d\in\mathcal{D}}$ be the family of supergroups containing G_0 as subgroup generated by the reflections for a set of indices $\mathcal{D}$ determined by the conditions in Theorem 1.2.12 (a) to (d) and Proposition 4.1.6. Let $\lambda \in \operatorname{Irr} G_0$ and let $Z(G_0) = \mu_{d_0} = \langle\zeta\rangle$. Then $\lambda(\zeta) = \zeta_l I_{\dim\lambda}$ for a certain primitive l-th root of unity ζ_l with $l \mid d_0$. Hence, we can extend λ to a representation λ_d of G_d for any $d \in \mathcal{D}$ by setting $\lambda_d|_{G_0} = \lambda$ and $\lambda_d(\eta) = \zeta_{l'} I_{\dim\lambda}$, where $Z(G) = \langle\eta\rangle$ and $l' = l\frac{d}{d_0}$ (note that $l' \mid d$, since $l \mid d_0$ and $d_0\frac{d}{d_0} = d$). Here, $\zeta_{l'}$ is a primitive l'-th root of unity with $\zeta_{l'}^{d/d_0} = \zeta_l$. In particular, there may exist more than one choice for λ_d.

Now one can find, if it exists, the smallest $d_1 \in \mathcal{D}$ such that $\lambda_{d_1}(\eta) = \eta^{-m} I_{\dim\lambda}$ with $2 \leq m < \frac{d_1}{2} - 1$ respectively $2 \leq m < d_1 - 1$ if $G_0 = \mathsf{OT}_{d_0}$.

Let $k \geq 0$ be minimal such that $L(\lambda)_k = 0$ with respect to all parameters $\mathbf{c}_1$, which we can compute using CHAMP. Then by Lemma 4.2.6 the module $L(\lambda|_{G_0})$ is $\mathbf{c}_2$-rigid for all $d \in \mathcal{D}$ with $d \geq d_1$ and $d - (k-1) - m > \frac{d}{2} + 1$ respectively $d - (k-1) - m > 1$ if $G_0 = \mathsf{OT}_{d_0}$.

We give the results of our computations and in particular the best possible values for k and m for each of the families of groups in Table 4.2.3. Using those bounds for d, we obtain an improved version of Table 4.2.2, see Table 4.2.4.

Example 4.2.8. We carry out the described computations for the group $G_0 := \mu_6\mathsf{T}$. The family of supergroups is given by $G_d := \mu_d\mathsf{T}$ for $d = 12a + 6$ with $a \in \mathbb{Z}_{\geq 0}$. Let $\omega \in \mathbb{C}$ be a primitive third root of unity and set $\zeta_6 := -\omega^{-1}$. Then we may choose the matrix

G_0	Shephard–Todd	Character of λ	Number in CHAMP	k	d_1	m	Lower bound of d
$\mu_6\mathsf{T}$	G_5	$\varphi_{3,4}$	19	5	$3d_0$	2	15
$\mu_{12}\mathsf{T}$	G_7	$\varphi_{3,10}$	37	7	d_0	2	19
$\mu_4\mathsf{O}$	G_{13}	$\varphi_{2,1}$	7	3	$5d_0$	3	11
$\mu_8\mathsf{O}$	G_9	$\varphi_{4,5}$	32	7	$2d_0$	3	21
$\mu_{12}\mathsf{O}$	G_{15}	$\varphi''_{3,10}$	36	11	d_0	2	27
$\mu_{24}\mathsf{O}$	G_{11}	*No data available.*					
$\mu_4\mathsf{I}$	G_{22}	$\varphi_{4,6}$	12	1	$2d_0$	2	7
$\mu_6\mathsf{I}$	G_{20}	$\varphi'_{3,10}$	13	1	$3d_0$	2	7
$\mu_{10}\mathsf{I}$	G_{16}	$\varphi_{5,8}$	39	9	d_0	2	23
$\mu_{12}\mathsf{I}$	G_{21}	*No data available.*					
$\mu_{20}\mathsf{I}$	G_{17}	*No data available.*					
$\mu_{30}\mathsf{I}$	G_{18}	*No data available.*					
$\mu_{60}\mathsf{I}$	G_{19}	*No data available.*					
OT_2	G_{12}	$\varphi_{2,1}$	3	3	$5d_0$	3	7
OT_4	G_8	$\varphi_{4,5}$	15	7	$5d_0$	3	11
OT_6	G_{14}	$\varphi_{2,4}$	14	5	$3d_0$	2	9
OT_{12}	G_{10}	$\varphi'_{3,10}$	36	11	d_0	2	14

Table 4.2.3.: Results of the computations with CHAMP

G_0	Groups containing G_0 as largest reflection group	G_0	Groups containing G_0 as largest reflection group
$\mu_6\mathsf{T}$	$\mu_6\mathsf{T}$	$\mu_{12}\mathsf{T}$	$\mu_{12}\mathsf{T}$
$\mu_4\mathsf{O}$	$\mu_4\mathsf{O}$	$\mu_8\mathsf{O}$	$\mu_d\mathsf{O}$, $d \in \{8, 16\}$
$\mu_{12}\mathsf{O}$	$\mu_{12}\mathsf{O}$	$\mu_{24}\mathsf{O}$	$\mu_d\mathsf{O}$, $d \in \{24, 48, 72, 96\}$
$\mu_4\mathsf{I}$	$\mu_4\mathsf{I}$	$\mu_6\mathsf{I}$	$\mu_6\mathsf{I}$
$\mu_{10}\mathsf{I}$	$\mu_{10}\mathsf{I}$	$\mu_{12}\mathsf{I}$	$\mu_d\mathsf{I}$, $d \in \{12, 24, 36, 48, 72, 84, 96, 108, 132, 144\}$
$\mu_{20}\mathsf{I}$	$\mu_d\mathsf{I}$, $d \in \{20, 40, 80, 100, 140\}$		
$\mu_{30}\mathsf{I}$	$\mu_d\mathsf{I}$, $d \in \{30, 90, 150\}$	$\mu_{60}\mathsf{I}$	$\mu_d\mathsf{I}$, $d \in \{60, 120, 180, 240\}$
OT_2	OT_2	OT_4	OT_4
OT_6	OT_6	OT_{12}	OT_{12}

Table 4.2.4.: Groups for which there is no answer yet

$\zeta := \zeta_6 I_2$ as generator for $Z(G_0) = \mu_6$. Going through the representations of G_0 in the database of CHAMP, we see that the representation numbered 19 with character $\varphi_{3,4}$ maps ζ to $(-\omega - 1)I_3 = \zeta_6^{-2} I_3$. In the above notation, we hence have $m = 2$. Note that this is the best possible value of m since we require $m \geq 2$.

This gives the lower bound $m = 2 < \frac{d_1}{2} - 1$, so that $d_1 > 6$, that is, $d_1 = 18 = 3d_0$. Using CHAMP, we see that the top degree of $L(\lambda)$ is 4, hence we have $k = 5$. Therefore, we have the additional restriction

$$d - (k-1) - m = d - 6 > \frac{d}{2} + 1 ,$$

which simplifies to $d > 14$. In conclusion, we improved the lower bound for d in Proposition 4.2.7 to $d \geq 18$, leaving only the group G_0 itself. ∎

Remark 4.2.9. Studying the data from CHAMP, we see that we almost always use the smallest possible value of k, that is, of the top degree of $L(\lambda|_{G_0})$ in Table 4.2.3. The only exception is $G_0 = \mathsf{OT}_2$ where simple modules with $k = 1$ exist. As in Proposition 4.2.7, we compute that d must (independently of m) fulfil the lower bound $2k + 6 < d$, respectively $k + 3 < d$ if $G_0 = \mathsf{OT}_{d_0}$. Hence, the bounds on d (including the case OT_2) in Table 4.2.3 are sharp in the sense that we cannot find a module fulfilling the bounds in Lemma 4.2.6 for smaller values of d.

A caveat to this argument is that we did not investigate whether the restricted parameter $\mathbf{c}_1$ is generic for the group G_0. It might well be that there are different G_0-conjugacy classes of reflections which join in $E(G)$ just as for the D_d-conjugacy classes in Lemma 4.1.11. The simple modules of $\mathsf{H}_{\mathbf{c}_1}(G_0)$ for certain special parameters behave differently than in the generic case, so there could be a simple module $L(\lambda|_{G_0})$ with a smaller top degree. However, the data on simple modules for special parameters is also available in the database of CHAMP for at least some of the groups G_0 and we see that while there are cases where there is a module of smaller top degree this is never small enough to make a change on the bounds on d in Table 4.2.3. ∎

4.3. On the non-existence of symplectic resolutions for complex primitive groups

We turn to the linear quotients coming from symplectic reflection groups which are both symplectically and complex primitive. Recall from Theorem 1.2.13 that there are only 16 groups of rank up to 10 in this family. In [BS16], a theorem of Kaledin [Kal03, Theorem 1.6] is used to prove that the linear quotients by three of these groups do not admit symplectic resolutions. We now use the same argument together with the computational results in Appendix A to extend this result to four further groups.

Theorem 4.3.1. *Let G be the group $W(R)$, $W(S_1)$, $W(S_2)$ or $W(U)$ as in [Coh80, Table III] and let n be the rank of G. Then the symplectic linear quotient $\mathbb{C}^n/G$ does not admit a (projective) symplectic resolution.*

Proof. If there exists a resolution in any of these cases, then the symplectic quotient associated to every parabolic subgroup also admits a symplectic resolution by [Kal03, Theorem 1.6].

From the results described in Appendix A, we see that $W(R)$, $W(S_1)$ and $W(S_2)$ contain a parabolic subgroup conjugate to the complex reflection group $G(5,5,2)^{\circledast}$, $G(3,3,3)^{\circledast}$ and $G(3,3,3)^{\circledast}$, respectively. In all cases, the quotient by this parabolic subgroup does not admit a symplectic resolution by [Bel09]. Hence neither do the quotients by $W(R)$, $W(S_1)$ or $W(S_2)$.

Finally, $W(S_1)$ is the stabilizer of a root of $W(U)$ by [Coh80, Table III] (see also Appendix A). Therefore, this quotient cannot admit a symplectic resolution either. □

4.4. Open cases

We summarize for which symplectic reflection groups the question whether the corresponding linear quotient admits a symplectic resolution is still open. As already established in Section 2.2, all of these groups are symplectically primitive and of rank 4.

The complex imprimitive groups are as given in Table 4.2.4, that is, these are the groups $E(G) \leq \mathrm{Sp}_4(\mathbb{C})$, where G is one of the following:

(a) $\mu_d\mathsf{T}$ with $d \in \{6, 12\}$,

(b) $\mu_d\mathsf{O}$ with $d \in \{4, 8, 12, 16, 24, 48, 72, 96\}$,

(c) $\mu_d\mathsf{I}$ with $d \in \{4, 6, 10, 12, 20, 24, 30, 36, 40, 48, 60, 72, 80, 84, 90, 96, 100, 108, 120, 132, 140, 144, 150, 180, 240\}$,

(d) OT_{2d} with $d \in \{1, 2, 3, 6\}$.

However, the fact that the bounds given in Table 4.2.3 are sharp, see Remark 4.2.9, means that for the above 39 groups (besides those, for which we could not do any computations) new ideas are needed. The strategy for finding a suitable simple module used in Theorem 4.2.5 is exhausted by the equivalence in Lemma 4.2.6.

The only remaining complex primitive groups are the groups

(e) $W(O_i)$ for $i = 1, 2, 3$,

(f) $W(P_i)$ for $i = 1, 2, 3$,

from [Coh80, Table III]. All of these are of rank 4, hence the argument in Theorem 4.3.1 cannot be applied, see also the proof of Lemma 3.1.4.

With the results presented in this section, we do not expect any of the remaining linear quotients to admit a symplectic resolution. However, given the exceptional nature of the complex primitive groups, this can only be considered a wild guess. Some computational evidence supporting it can be found in Appendix B.

5. The class group of a $\mathbb{Q}$-factorial terminalization of a linear quotient

In this and the next chapter, we generalize from symplectic linear quotient singularities to linear quotients V/G by finite subgroups $G \leq \mathrm{SL}(V)$. Also, we study $\mathbb{Q}$-factorial terminalizations as a generalization of symplectic resolutions as discussed in Chapter 2.

We now describe the class group of a $\mathbb{Q}$-factorial terminalization $X \to V/G$ of a linear quotient V/G for $G \leq \mathrm{SL}(V)$. In the literature, one can find the result that the class group $\mathrm{Cl}(X)$ is free if and only if G is generated by junior elements together with $[G, G]$, see [Yam18, Proposition 4.14]. This extends [DW17, Lemma 2.11], which states that $\mathrm{Cl}(X)$ is free if X is smooth. We expand these results further and study the torsion part of $\mathrm{Cl}(X)$ to describe the class group in full detail. We prove that $\mathrm{Cl}(X)$ is completely controlled by the junior elements contained in G, see Corollary 5.4.2.

The strategy for our proof is similar to the one in [Yam18] and completely different from the approach via the Picard group in [DW17]. In fact, we follow our general philosophy that we try to gain information on X via its Cox ring $\mathcal{R}(X)$. For this, we use the correspondence of homogeneous elements in $\mathcal{R}(X)$ and effective divisors on X, which we recall in Section 5.2.

Throughout, let V be a finite-dimensional vector space over $\mathbb{C}$ and let $G \leq \mathrm{SL}(V)$ be a finite group. Let $\varphi : X \to V/G$ be a $\mathbb{Q}$-factorial terminalization of the linear quotient corresponding to V/G and let $m \in \mathbb{Z}_{\geq 0}$ be the number of junior conjugacy classes in G. We emphasize that we do *not* assume that G is generated by junior elements in this chapter if not stated otherwise. We might well have $m = 0$, but this is not an interesting case, see also Example 5.4.7.

5.1. A short exact sequence

We translate the short exact sequence in [ST88] to our setting. For this, we need the following connection to the algebraic side. For a noetherian normal domain R, we define the *class group* $\mathrm{Cl}(R)$ as the quotient of the free group on the prime ideals of codimension 1 modulo the subgroup generated by the principal fractional ideals, see [Fos73, §6]. We see immediately that $\mathrm{Cl}(R) = \mathrm{Cl}(\operatorname{Spec} R)$. We also need the following fact.

Lemma 5.1.1. *Let $S = \bigoplus_{d \geq 0} S_d$ be a noetherian normal graded domain. Then we have $\mathrm{Cl}(S) = \mathrm{Cl}(\operatorname{Proj} S)$.*

Proof. We need to show that $\mathrm{Cl}(S)$ is generated by homogeneous prime ideals of codimension 1 which do not contain the irrelevant ideal $S_+ := \bigoplus_{d \geq 1} S_d$. By [Fos73, Proposition 10.2], it suffices to consider homogeneous prime ideals in the definition of $\mathrm{Cl}(S)$.

Any prime ideal properly containing S_+ must be of codimension larger than 1, so the only case left to consider is $\operatorname{codim} S_+ = 1$ and the ideal S_+ itself.

In this case, the localization of S at the prime ideal S_+ is a regular local ring by Serre's criterion [Eis95, Theorem 11.5], hence S_{S_+} is a factorial ring by [Eis95, Theorem 19.19]. By [Fos73, Proposition 6.1], we conclude that $\operatorname{Cl}(S_{S_+}) = 0$ and this implies that $\operatorname{Cl}(S)$ is generated by the classes of prime ideals which meet $S \setminus S_+$ [Fos73, Corollary 7.2]. In conclusion, we see that $\operatorname{Cl}(S)$ is in any case generated by the classes of homogeneous prime ideals of codimension 1 which do not contain S_+, which means $\operatorname{Cl}(S) = \operatorname{Cl}(\operatorname{Proj} S)$. □

Proposition 5.1.2 (Simis–Trung)**.** *Let Y be a normal affine variety and let $\psi : \tilde{Y} \to Y$ be a blowing-up of Y along a closed subset. Then there is a short exact sequence of abelian groups*

$$0 \longrightarrow \mathbb{Z}^r \longrightarrow \operatorname{Cl}(\tilde{Y}) \xrightarrow{\psi_*} \operatorname{Cl}(Y) \longrightarrow 0\,,$$

where $r \in \mathbb{Z}_{\geq 0}$ is the number of irreducible components of the exceptional divisor of ψ and $\psi_* : \operatorname{Cl}(\tilde{Y}) \to \operatorname{Cl}(Y)$ is the induced push-forward morphism.

Proof. Since the blowing-up of a variety corresponds to a Rees algebra on the algebraic side, this is exactly [ST88, Theorem 1.1]. □

Corollary 5.1.3 (Grab)**.** *There is a short exact sequence of abelian groups*

$$0 \longrightarrow \bigoplus_{i=1}^m \mathbb{Z}E_i \longrightarrow \operatorname{Cl}(X) \xrightarrow{\varphi_*} \operatorname{Cl}(V/G) \longrightarrow 0\,,$$

where $E_i \in \operatorname{Div}(X)$ are the irreducible components of the exceptional divisor of φ and $\varphi_* : \operatorname{Cl}(X) \to \operatorname{Cl}(V/G)$ is the induced push-forward map.

Proof. By [Har77, Theorem II.7.17], the projective birational morphism φ corresponds to a blowing-up of V/G along a closed subset. Hence this follows by Proposition 5.1.2. □

See [Gra19, Proposition 4.1.3] for a more 'geometric' proof.

We write $\operatorname{Cl}(X)^{\mathrm{tors}} \leq \operatorname{Cl}(X)$ for the torsion subgroup of $\operatorname{Cl}(X)$ and $\operatorname{Cl}(X)^{\mathrm{free}}$ for the corresponding factor group, that is, $\operatorname{Cl}(X)^{\mathrm{free}} = \operatorname{Cl}(X)/\operatorname{Cl}(X)^{\mathrm{tors}}$. Denote the canonical projection by $\rho : \operatorname{Cl}(X) \to \operatorname{Cl}(X)^{\mathrm{free}}$. We note the following fact for later reference, see also [Gra19, Lemma 4.1.4].

Lemma 5.1.4. *The morphism of groups*

$$\vartheta : \operatorname{Cl}(X) \to \operatorname{Cl}(V/G) \oplus \operatorname{Cl}(X)^{\mathrm{free}},\ [D] \mapsto ([\varphi_* D], \rho([D]))$$

is injective.

Proof. This follows from the exactness of the sequence in Corollary 5.1.3 noticing that the group $\bigoplus_{i=1}^m \mathbb{Z}E_i$ embeds into $\operatorname{Cl}(X)^{\mathrm{free}}$. □

5.2. Correspondence of effective divisors and homogeneous elements

To be able to deduce information on $\mathrm{Cl}(X)$ via the ring $\mathcal{R}(X)$, we use the connection between effective divisors and *canonical sections* in the Cox ring $\mathcal{R}(V/G)$ of V/G. We recall this correspondence and adapt it to our setting.

Notation 5.2.1. For a divisor $D \in \mathrm{Div}(V/G)$, we write $\chi_{[D]} \in \mathrm{Ab}(G)^\vee$ for the character corresponding to the class $[D] \in \mathrm{Cl}(V/G)$ under the isomorphism in Proposition 2.1.7 (c).

Remark 5.2.2. Working with the ring $\mathcal{R}(V/G)$ brings two subtle problems. First of all, homogeneous elements $f \in \mathcal{R}(V/G)$ are only residue classes of elements of the function field $\mathbb{C}(V)^G$ as $\mathrm{Cl}(V/G)$ is a torsion group. We hence cannot immediately identify such elements f with a function in $\mathbb{C}(V)^G$. However, for a divisor $D \in \mathrm{Div}(V/G)$ we have an isomorphism

$$\psi_D : \Gamma(V/G, \mathcal{O}_{V/G}(D)) \to \mathcal{R}(V/G)_{[D]}$$

by [ADHL15, Lemma 1.4.3.4]. That means, once we fixed a representative of the degree of a homogeneous element $f \in \mathcal{R}(V/G)$ we can uniquely lift f to an element of $\mathbb{C}(V)^G$.

The second problem comes from the fact that we make heavy use of the graded isomorphism $\Psi : \mathcal{R}(V/G) \to \mathbb{C}[V]^{[G,G]}$ as in Theorem 2.4.11 to the extent that one might forget that the isomorphism is not an identity. This is in particular important when we work with a valuation $v : \mathbb{C}(V) \to \mathbb{Z}$. We can only use v on elements of $\mathbb{C}[V]^{[G,G]}$ and cannot apply v to elements of $\mathcal{R}(V/G)$ in a well-defined way without choosing a system of representatives for the class group. For $D \in \mathrm{Div}(V/G)$, we have an isomorphism of vector spaces

$$\tilde{\psi}_D : \Gamma(V/G, \mathcal{O}_{V/G}(D)) \to \mathbb{C}[V]^{[G,G]}_{\chi_{[D]}}$$

by setting $\tilde{\psi}_D := \Psi \circ \psi_D$. Notice that for the trivial divisor, this gives an identity as we have

$$\Gamma(V/G, \mathcal{O}_{V/G}(0)) = \mathbb{C}[V]^G = \mathbb{C}[V]^{[G,G]}_1 ,$$

where 1 denotes the trivial character. ‖

Notation 5.2.3. Let $\chi \in \mathrm{Ab}(G)^\vee$ and let $D \in \mathrm{Div}(V/G)$ with $\chi = \chi_{[D]}$. For a homogeneous element $0 \neq f \in \mathbb{C}[V]^{[G,G]}_\chi$, let $\tilde{f} \in \mathbb{C}(V)^G$ be the rational function mapping to f via the isomorphism determined by D as in Remark 5.2.2. We associate to f an effective divisor

$$\mathrm{div}_{[D]}(f) := \mathrm{div}(\tilde{f}) + D \in \mathrm{Div}(V/G) ,$$

the $[D]$-*divisor* of f. This construction is well-defined, see [ADHL15, Proposition 1.5.2.2]. In particular, the $[D]$-divisor is independent of the choice of the representative D. We have $[\mathrm{div}_{[D]}(f)] = [D]$ by definition.

The construction of a $[D]$-divisor is not limited to our setting; see [ADHL15, Construction 1.5.2.1] for more details and the general case. We point out that $f \in \mathbb{C}[V]^{[G,G]}$ is in general not an element of $\mathbb{C}(V)^G$, that is, there is no meaning in writing $\mathrm{div}(f)$.

The $[D]$-divisor behaves well with respect to the multiplication of elements.

Lemma 5.2.4. *For non-zero homogeneous elements $f \in \mathbb{C}[V]^{[G,G]}_{\chi[D_1]}$ and $g \in \mathbb{C}[V]^{[G,G]}_{\chi[D_2]}$, we have*

$$\mathrm{div}_{[D_1]+[D_2]}(fg) = \mathrm{div}_{[D_1]}(f) + \mathrm{div}_{[D_2]}(g) \ .$$

See [ADHL15, Proposition 1.5.2.2 (iii)] for a proof.

We have a converse to the construction of the $[D]$-divisor.

Proposition 5.2.5. *Let $E \in \mathrm{Div}(V/G)$ be an effective divisor. Then there exist a class $[D] \in \mathrm{Cl}(V/G)$ and an element $f \in \mathbb{C}[V]^{[G,G]}_{\chi[D]}$ with $E = \mathrm{div}_{[D]}(f)$. The element f is unique up to constants; it is called a* canonical section *of E.*

See [ADHL15, Proposition 1.5.2.2 (i)] and [ADHL15, Proposition 1.5.3.5 (ii)] for a proof.

Using the correspondence between effective divisors and homogeneous elements one can derive a precise description of the image of the strict transform of an effective divisor $D \in \mathrm{Div}(V/G)$ in the free group $\mathrm{Cl}(X)^{\mathrm{free}}$. The general idea of this argument appeared to the author's knowledge first in [DW17, Lemma 3.22]. We require a bit of notation.

Recall that by Theorem 2.1.21 we have a one-to-one correspondence between the junior conjugacy classes of G and the irreducible components of the exceptional divisor of φ. Let $\{g_1, \dots, g_m\} \in G$ be a minimal set of representatives of the junior conjugacy classes corresponding to exceptional prime divisors $E_1, \dots, E_m \in \mathrm{Div}(X)$. For each $i \in \{1, \dots, m\}$, write v_i for the monomial valuation on $\mathbb{C}(V)$ defined by g_i and recall from Theorem 2.1.21 that we have $v_{E_i} = \frac{1}{r_i} v_i$, where v_{E_i} is the divisorial valuation of E_i and r_i the order of g_i.

The following also appears in [Gra19, Proposition 4.1.9]. We present the argument from [Yam18, Lemma 4.3] for completeness.

Proposition 5.2.6. *Let $D \geq 0$ be an effective divisor on V/G and let $f \in \mathbb{C}[V]^{[G,G]}_{\chi[D]}$ be a canonical section. Write $\overline{D} \coloneqq \varphi_*^{-1}(D)$ for the strict transform of D via φ. Then we have the equality*

$$\rho([\overline{D}]) = -\sum_{i=1}^{m} \frac{1}{r_i} v_i(f) \rho([E_i])$$

in $\mathrm{Cl}(X)^{\mathrm{free}}$.

Proof. As f is homogeneous with respect to the action of $\mathrm{Ab}(G)$, there is $r \in \mathbb{Z}_{>0}$ such that $f^r \in \mathbb{C}[V]^{[G,G]}_1 = \mathbb{C}[V]^G \subseteq \mathbb{C}(V)^G$ and rD is principal. In particular, we have

$$rD = \mathrm{div}_{[rD]}(f^r) = \mathrm{div}_{[0]}(f^r) = \mathrm{div}(f^r) \ ,$$

where the first equality is by Lemma 5.2.4, the second by the independence of choice of representative and the third is by the fact that $f^r \in \mathbb{C}[V]^G$, see Remark 5.2.2. Then we have

$$\mathrm{div}(\varphi^*(f^r)) = r\overline{D} + \sum_{i=1}^{m} v_{E_i}(\varphi^*(f^r)) E_i \ .$$

Hence, we have the equality of classes

$$[r\overline{D}] = -\sum_{i=1}^{m} v_{E_i}(\varphi^*(f^r))[E_i]$$

in $\mathrm{Cl}(X)$. Now $v_{E_i}(\varphi^*(f^r)) = \frac{1}{r_i} v_i(f^r)$ by Theorem 2.1.21. Noting that v_i is a valuation on $\mathbb{C}(V)$ (and not just $\mathbb{C}(V)^G$) this yields

$$[r\overline{D}] = -\sum_{i=1}^{m} \frac{r}{r_i} v_i(f)[E_i] \ .$$

We may finally cancel r in the free group $\mathrm{Cl}(X)^{\mathrm{free}}$ giving

$$\rho([\overline{D}]) = -\sum_{i=1}^{m} \frac{1}{r_i} v_i(f)\rho([E_i]) \ .$$

□

5.3. A digression on gradings

As we want to approach the group $\mathrm{Cl}(X)$ via the ring $\mathcal{R}(X)$, which is graded by $\mathrm{Cl}(X)$, we first have to get a better understanding of the grading of $\mathbb{C}[V]^{[G,G]}$ by $\mathrm{Ab}(G)^\vee$. Unfortunately, there are a few subtle details to consider turning this into a quite technical discussion.

Again, let $g_1, \dots, g_m \in G$ be representatives of the junior conjugacy classes corresponding to the exceptional divisors $E_1, \dots, E_m \in \mathrm{Div}(X)$ of φ and write $v_1, \dots, v_m$ for the monomial valuations corresponding to the g_i.

At first, fix $i \in \{1, \dots, m\}$. We recall the construction of the valuation v_i from Section 2.1.3. In an eigenbasis, the matrix g_i is of the form

$$\begin{pmatrix} \zeta_{r_i}^{a_{i,1}} & & \\ & \ddots & \\ & & \zeta_{r_i}^{a_{i,n}} \end{pmatrix}$$

with a primitive r_i-th root of unity ζ_{r_i} and integers $0 \leq a_{i,j} < r_i$, where r_i is the order of g_i in G and $n = \dim V$. This induces a $\mathbb{Z}$-grading $\deg_i$ on $\mathbb{C}[x_1, \dots, x_n]$ by putting $\deg_i(x_j) \coloneqq a_{i,j}$. For a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$, the valuation $v_i(f)$ is then the degree of the homogeneous component of f of minimal degree with respect to $\deg_i$. Note that the grading $\deg_i$ is well-defined on $\mathbb{C}[V]$ for any basis of V, although the variables of the polynomial ring are in general not homogeneous. As we endow the same ring with gradings by different groups, we use the non-standard notation $(\mathbb{C}[V], \mathbb{Z}, \deg_i)$ for the ring $\mathbb{C}[V]$ graded by $\mathbb{Z}$ via $\deg_i$.

The group $\langle g_i \rangle$ acts on $\mathbb{C}[V]$ and hence induces a grading by $\langle g_i \rangle^\vee \cong \mathbb{Z}/r_i\mathbb{Z}$, which we denote by $\overline{\deg}_i$. Write $(\mathbb{C}[V], \mathbb{Z}/r_i\mathbb{Z}, \overline{\deg}_i)$ for the ring $\mathbb{C}[V]$ graded by $\mathbb{Z}/r_i\mathbb{Z}$ via $\overline{\deg}_i$. We directly obtain:

Lemma 5.3.1. *With the above notation, if* $f \in \mathbb{C}[V]$ *is* $\deg_i$*-homogeneous, then* f *is* $\overline{\deg}_i$*-homogeneous as well and we have*

$$\deg_i(f) \equiv \overline{\deg}_i(f) \mod r_i \ .$$

In particular, there is a graded morphism

$$(\mathbb{C}[V], \mathbb{Z}, \deg_i) \to (\mathbb{C}[V], \mathbb{Z}/r_i\mathbb{Z}, \overline{\deg}_i)$$

given by the identity on the rings and by the projection $\mathbb{Z} \to \mathbb{Z}/r_i\mathbb{Z}$ *on the grading groups.*

Observe that for every $1 \leq i \leq m$ we have an action of g_i on $\mathbb{C}[V]^{[G,G]}$. Indeed, for any $f \in \mathbb{C}[V]^{[G,G]}$ and $h \in [G,G]$, we have

$$h.(g_i.f) = (hg_i).f = (hg_i).((g_i^{-1}h^{-1}g_ih).f) = g_i.(h.f) = g_i.f \ ,$$

so $g_i.f \in \mathbb{C}[V]^{[G,G]}$ as required. Hence the grading by $\langle g_i \rangle^\vee$ descends to $\mathbb{C}[V]^{[G,G]}$. As the actions of the elements $g_1, \dots, g_m$ on $\mathbb{C}[V]^{[G,G]}$ commute, we can consider all the induced gradings at the same time and hence obtain a grading by $\mathbb{Z}/r_1\mathbb{Z} \times \dots \times \mathbb{Z}/r_m\mathbb{Z}$ on $\mathbb{C}[V]^{[G,G]}$.

The g_i do not commute with each other in general, so we cannot decompose their actions on $\mathbb{C}[V]$ into a common eigenbasis. Hence, we *cannot* put the above gradings together to obtain a grading by $\mathbb{Z}^m$ or $\mathbb{Z}/r_1\mathbb{Z} \times \dots \times \mathbb{Z}/r_m\mathbb{Z}$ on $\mathbb{C}[V]$ as there are in general no polynomials which are homogeneous with respect to all gradings at the same time.

Let $H \leq G$ be the subgroup of G generated by the junior elements contained in G. In general, the representatives $g_1, \dots, g_m$ do not suffice to generate H. Let

$$\overline{H} \coloneqq H/(H \cap [G,G]) \leq \mathrm{Ab}(G)$$

and notice that this group is generated by the residue classes $\overline{g}_1, \dots, \overline{g}_m$ modulo $[G,G]$. This gives a map

$$\langle g_1 \rangle \times \dots \times \langle g_m \rangle \to \mathrm{Ab}(G) \ ,$$

which is surjective onto $\overline{H}$. This surjection corresponds to an embedding of characters $\overline{H}^\vee \to \mathbb{Z}/r_1\mathbb{Z} \times \dots \times \mathbb{Z}/r_m\mathbb{Z}$. Further, the inclusion $\overline{H} \to \mathrm{Ab}(G)$ induces a projection of characters $\mathrm{Ab}(G)^\vee \to \overline{H}^\vee$ by restriction. We conclude:

Lemma 5.3.2. *The gradings on* $\mathbb{C}[V]^{[G,G]}$ *coming from the actions of the groups* $\mathrm{Ab}(G)$, $\overline{H}$ *and* $\langle g_1 \rangle \times \dots \times \langle g_m \rangle$ *are compatible in the sense that there is a graded morphism*

$$(\mathbb{C}[V]^{[G,G]}, \mathrm{Ab}(G)^\vee) \to (\mathbb{C}[V]^{[G,G]}, \mathbb{Z}/r_1\mathbb{Z} \times \dots \times \mathbb{Z}/r_m\mathbb{Z})$$

which factors through $(\mathbb{C}[V]^{[G,G]}, \overline{H}^\vee)$.

We state for later reference:

Lemma 5.3.3. *We have* $\mathrm{Ab}(G/H) \cong \mathrm{Ab}(G)/\overline{H}$ *and* $\overline{H}^\vee \cong \mathrm{Ab}(G)^\vee/\mathrm{Ab}(G/H)^\vee$.

Proof. For the first statement, we note that the image of $[G,G]$ under the projection $G \to G/H$ is $[G/H, G/H]$. Hence,

$$\mathrm{Ab}(G/H) \cong (G/H)/([G,G]/[G,G] \cap H) \cong G/(H[G,G])$$

and an application of the isomorphism theorems gives the claim. The second statement follows directly as $^\vee$ is a contravariant functor. □

The following three lemmas are key ingredients for our theorem on $\mathrm{Cl}(X)$.

Lemma 5.3.4. *Let $f \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous. For every $i \in \{1,\dots,m\}$, we have $v_i(f) \equiv \overline{\deg}_i(f) \mod r_i$.*

Proof. Let $f \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous. Fix an $i \in \{1,\dots,m\}$. Lemma 5.3.2 implies that f is $\overline{\deg}_i$-homogeneous. By Lemma 5.3.1, there exist $\deg_i$- and $\overline{\deg}_i$-homogeneous elements $f_{i,j} \in \mathbb{C}[V]$ such that $f = \sum_j f_{i,j}$ and $\deg_i(f_{i,j}) < \deg_i(f_{i,j'})$ whenever $j < j'$. In particular, we have $\deg_i(f_{i,1}) = v_i(f)$ and $\overline{\deg}_i(f_{i,1}) = \overline{\deg}_i(f)$. Hence, we conclude

$$v_i(f) \equiv \overline{\deg}_i(f_{i,1}) = \overline{\deg}_i(f) \mod r_i$$

by Lemma 5.3.1. □

Lemma 5.3.5. *Let $f \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous. We have $r_i \mid v_i(f)$ for all $i \in \{1,\dots,m\}$ if and only if $f \in \mathbb{C}[V]^H$, where $H \leq G$ is the subgroup generated by the junior elements contained in G.*

Proof. By Lemma 5.3.4, we have $v_i(f) \equiv \overline{\deg}_i(f) \bmod r_i$ for every i. Therefore, $r_i \mid v_i(f)$ is equivalent to $\overline{\deg}_i(f) = 0$ for every i. Equivalently, every g_i acts trivially on f. Since f is furthermore $[G,G]$-invariant, we conclude that this is the case if and only if every junior element in G leaves f invariant and hence $f \in \mathbb{C}[V]^H$. □

Lemma 5.3.6. *Let $[D] \in \mathrm{Cl}(V/G)$ be a class of divisors. Then there exists a homogeneous element in $\mathbb{C}[V]^{[G,G]}$ of degree $\chi_{[D]}$.*

Proof. This is saying that the relative invariants with respect to the linear characters of $\mathrm{Ab}(G)$ on $\mathbb{C}[V]^{[G,G]}$ are non-empty which holds by [Nak82, Lemma 2.1].

Alternatively, one lets $d \in \mathbb{Z}_{>0}$ such that $\mathbb{C}[V]^{[G,G]}$ is generated by polynomials of degree up to d. Then we can write the linear actions of any set of generators of $\mathrm{Ab}(G)$ on the vector space of polynomials in $\mathbb{C}[V]^{[G,G]}$ of degree up to d as pairwise commuting matrices. These matrices are simultaneously diagonalizable and the common eigenspaces are exactly the $\mathrm{Ab}(G)^\vee$-homogeneous components. □

Notice that the lemma also implies that we can find an effective divisor in any class of divisors in $\mathrm{Cl}(V/G)$.

5.4. The class group

We are now prepared for our theorem.

Theorem 5.4.1. *Let $G \leq \mathrm{SL}(V)$ be a finite group and let $H \leq G$ be the subgroup generated by the junior elements contained in G. Let $\varphi : X \to V/G$ be a $\mathbb{Q}$-factorial terminalization of V/G. Then we have a canonical isomorphism of abelian groups*

$$\mathrm{Cl}(X)^{\mathrm{tors}} \cong \mathrm{Ab}(G/H)^\vee = \mathrm{Hom}(G/H, \mathbb{C}^\times) ,$$

which is induced by the push-forward map $\varphi_* : \mathrm{Cl}(X) \to \mathrm{Cl}(V/G)$.

Proof. For ease of notation, we identify $\mathrm{Cl}(V/G)$ with $\mathrm{Ab}(G)^\vee$ via Proposition 2.1.7 (c) and use both groups synonymously. Notice that $\mathrm{Ab}(G/H)^\vee$ is the subgroup of $\mathrm{Ab}(G)^\vee$ consisting of those characters which take value 1 on every junior element. We claim that restricting φ_* to $\mathrm{Cl}(X)^{\mathrm{tors}}$ induces a bijection onto $\mathrm{Ab}(G/H)^\vee$.

We first show that we indeed have $\varphi_*(\mathrm{Cl}(X)^{\mathrm{tors}}) \subseteq \mathrm{Ab}(G/H)^\vee$. Let $D \in \mathrm{Div}(X)$ be a divisor on X. By Lemma 5.3.6, there is $f \in \mathbb{C}[V]^{[G,G]}$ of degree $\chi_{[\varphi_* D]}$ and we have the effective divisor $D' \coloneqq \mathrm{div}_{[\varphi_* D]}(f)$ on V/G with $[D'] = [\varphi_* D]$. Write $\overline{D'} \in \mathrm{Div}(X)$ for the strict transform of D' via φ. Then $\varphi_* \overline{D'} = D'$, hence by Corollary 5.1.3 we have

$$[\overline{D'}] = [D] + \sum_{i=1}^{m} a_i [E_i] , \tag{5.4.1}$$

with $a_i \in \mathbb{Z}$ and where $E_1, \ldots, E_m \in \mathrm{Div}(X)$ are the irreducible components of the exceptional divisor of φ. As before let $\rho : \mathrm{Cl}(X) \to \mathrm{Cl}(X)^{\mathrm{free}} \coloneqq \mathrm{Cl}(X)/\mathrm{Cl}(X)^{\mathrm{tors}}$ be the canonical projection. Applying ρ on both sides of (5.4.1) and using Proposition 5.2.6 yields

$$\rho([D]) = -\sum_{i=1}^{m} \frac{1}{r_i} v_i(f) \rho([E_i]) - \sum_{i=1}^{m} a_i \rho([E_i]) . \tag{5.4.2}$$

Assume now $[D] \in \mathrm{Cl}(X)^{\mathrm{tors}}$. Then $\rho([D]) = 0$ and we conclude by (5.4.2) that $v_i(f) = -r_i a_i$ for all i and, in particular, $r_i \mid v_i(f)$. Hence, $f \in \mathbb{C}[V]^H$ by Lemma 5.3.5 and therefore we can identify $[D'] = [\varphi_* D]$, or more precisely $\chi_{[\varphi_* D]}$, with an element of $\mathrm{Hom}(G/H, \mathbb{C}^\times)$. This means that we obtain a well-defined map

$$\psi : \mathrm{Cl}(X)^{\mathrm{tors}} \to \mathrm{Hom}(G/H, \mathbb{C}^\times),\ [D] \mapsto [\varphi_* D]$$

by restricting φ_* to $\mathrm{Cl}(X)^{\mathrm{tors}}$.

We now prove that ψ is bijective. Injectivity follows directly from the injectivity of ϑ in Lemma 5.1.4. Indeed, if we have $\psi([D]) = \psi([D'])$ for $[D], [D'] \in \mathrm{Cl}(X)^{\mathrm{tors}}$, then $\vartheta([D]) = \vartheta([D'])$ as by construction $\rho([D]) = 0 = \rho([D'])$.

Now let $\chi \in \mathrm{Hom}(G/H, \mathbb{C}^\times)$ be a character, which we identify with a class of divisors $[D] \in \mathrm{Cl}(V/G)$. By Lemma 5.3.6, there exists $0 \neq f \in \mathbb{C}[V]_\chi^{[G,G]}$ and we may assume without loss of generality that $D \in \mathrm{Div}(V/G)$ is effective and f is the canonical section of D as in Proposition 5.2.5. By the assumption on χ, we have $\frac{1}{r_i}v_i(f) \in \mathbb{Z}$ for all i by Lemma 5.3.5. Let

$$E := -\sum_{i=1}^{m} \frac{1}{r_i} v_i(f) E_i \in \mathrm{Div}(X)$$

and set $D' := \overline{D} - E$, where $\overline{D} := \varphi_*^{-1}(D)$ is the strict transform of D via φ. By Corollary 5.1.3, we have $[E] \in \ker(\varphi_*)$ and therefore $[\varphi_* D'] = [\varphi_* \overline{D}] = [D]$. Using Proposition 5.2.6, we have $\rho([\overline{D}]) = \rho([E])$, hence $\rho([D']) = 0$ and $[D'] \in \mathrm{Cl}(X)^{\mathrm{tors}}$. We conclude $\psi([D']) = [D]$ and ψ is surjective. □

Combining Theorem 2.1.21 and Theorem 5.4.1 enables us to describe the class group of X in general.

Corollary 5.4.2. *Let $G \leq \mathrm{SL}(V)$ be a finite group and let $H \leq G$ be the subgroup generated by the junior elements contained in G. Let $\varphi : X \to V/G$ be a $\mathbb{Q}$-factorial terminalization of V/G. Then we have*

$$\mathrm{Cl}(X) \cong \mathbb{Z}^m \oplus \mathrm{Ab}(G/H)^\vee ,$$

where m is the number of junior conjugacy classes in G.

Corollary 5.4.3. *Let $G \leq \mathrm{SL}(V)$ be a finite group and let $H \leq G$ be the subgroup generated by the junior elements contained in G. Let $\varphi : X \to V/G$ be a $\mathbb{Q}$-factorial terminalization of V/G. Write $\iota : \bigoplus_{i=1}^m \mathbb{Z}E_i \to \mathrm{Cl}(X)^{\mathrm{free}}$ for the canonical embedding and $\overline{H} := H/(H \cap [G,G])$ as above. Then we have $\mathrm{coker}(\iota) = \overline{H}^\vee$.*

Proof. Combining Corollaries 5.1.3 and 5.4.2 gives $\mathrm{Ab}(G)^\vee \cong \mathrm{coker}(\iota) \oplus \mathrm{Ab}(G/H)^\vee$ and then the claim follows by Lemma 5.3.3. □

Remark 5.4.4. As the isomorphism in Theorem 5.4.1 is induced by φ_*, we can see the sequence in Corollary 5.1.3 as the direct sum of the short exact sequences

$$0 \longrightarrow \textstyle\bigoplus_{i=1}^m \mathbb{Z}E_i \longrightarrow \mathrm{Cl}(X)^{\mathrm{free}} \longrightarrow \overline{H}^\vee \longrightarrow 0$$

and

$$0 \longrightarrow 0 \longrightarrow \mathrm{Cl}(X)^{\mathrm{tors}} \longrightarrow \mathrm{Ab}(G/H)^\vee \longrightarrow 0 .$$ ∥

We obtain [Yam18, Proposition 4.14] as a further corollary.

Corollary 5.4.5. *Let $G \leq \mathrm{SL}(V)$ be a finite group and let $\varphi : X \to V/G$ be a $\mathbb{Q}$-factorial terminalization of V/G. Then the class group $\mathrm{Cl}(X)$ is free if and only if G is generated by the junior elements contained in G together with $[G,G]$.*

Remark 5.4.6. Note that in Corollary 5.4.5 we cannot drop the part 'together with $[G,G]$' for the equivalence, that is, there are groups which are not generated by junior elements such that $\mathrm{Cl}(X)$ is free. For example, let $G := \mathsf{I} \times \mathsf{I} \leq \mathrm{SL}_4(\mathbb{C})$ be the group generated by two copies of the binary icosahedral group $\mathsf{I} \leq \mathrm{SL}_2(\mathbb{C})$ on the diagonal, so

$G = \{\mathrm{diag}(g, g) \mid g \in \mathsf{I}\}$. The abelianization $\mathrm{Ab}(\mathsf{I}) = \{1\}$ is trivial, so the same is true for $\mathrm{Ab}(G)$. However, every non-trivial element in I is of age 1, hence all non-trivial elements of G are of age 2 and G does not contain any junior elements. Hence, the class group of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G$ is trivial and therefore free. For an example of a non-trivially free class group, one considers the direct product of G with a group generated by junior elements. ∎

Example 5.4.7. As a 'reality check', let $G \leq \mathrm{SL}(V)$ be a group which does not contain any junior elements. Then $\mathrm{age}(g) > 1$ for every non-trivial $g \in G$, so V/G has terminal singularities by [Kol13, Theorem 3.21]. Hence, V/G is a $\mathbb{Q}$-factorial terminalization of itself and Corollary 5.4.2 gives $\mathrm{Cl}(V/G) = \mathrm{Ab}(G)^\vee$ as in Proposition 2.1.7 (c). ∎

Example 5.4.8. For a non-trivial example, we consider the group

$$G \coloneqq \left\langle \mathrm{diag}(-1, -1, -\zeta_3, -\zeta_3^2) \right\rangle \leq \mathrm{SL}_4(\mathbb{C})$$

of order 6, where ζ_3 is a primitive third root of unity. As G does not contain any reflections, we have $\mathrm{Cl}(\mathbb{C}^4/G) \cong \mathbb{Z}/6\mathbb{Z}$.

To determine the age of elements in G, we need to fix a sixth root of unity. However, the two possible choices $-\zeta_3$ and $-\zeta_3^2$ both result in the same junior elements of G, namely

$$g_1 \coloneqq \mathrm{diag}(1, 1, \zeta_3^2, \zeta_3) \text{ and } g_2 \coloneqq \mathrm{diag}(1, 1, \zeta_3, \zeta_3^2)\ .$$

As G is abelian, the conjugacy classes in G are trivial. So, the rank of the free part of the class group $\mathrm{Cl}(X)$ of a $\mathbb{Q}$-factorial terminalization $X \to \mathbb{C}^4/G$ is 2. For the torsion part, we determine that $G/H \cong C_2$ is cyclic of order 2 and we conclude

$$\mathrm{Cl}(X) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}\ .$$

Write the elements of $\mathrm{Cl}(X)$ as 3-tuples with the first two entries corresponding to the free part and the last entry corresponding to the torsion part. Then the push-forward morphism $\mathrm{Cl}(X) \to \mathrm{Cl}(\mathbb{C}^4/G)$ is given by

$$(1, 0, 0) \mapsto g_1,\ (0, 1, 0) \mapsto g_2,\ (0, 0, 1) \mapsto -I_4\ .$$ ∎

Remark 5.4.9. We make two philosophical observations. Firstly, we emphasize that by Corollary 5.4.2, the class group of a $\mathbb{Q}$-factorial terminalization is completely controlled by the group G itself. This fits well into the general framework of McKay correspondence(s), where one expects that it should be possible to give answers to questions regarding the birational geometry of V/G by only considering the action of G on V, see for example Reid's 'principle of the McKay correspondence' [Rei02, Principle 1.1].

Further, we feel that Theorem 5.4.1 mirrors Proposition 2.1.7 (c) just like Verbitsky's Theorem (Theorem 2.2.6) mirrors the Chevalley–Serre–Shephard–Todd Theorem (Theorem 2.1.5). In both cases the geometry of the linear quotient V/G is controlled by the (complex) reflections contained in G and the junior elements (or symplectic reflections) control the geometry of the $\mathbb{Q}$-factorial terminalization $X \to V/G$. Still, it appears that this picture is far from complete. Verbitsky's result on the smoothness of X is not an equivalence and also the freeness of the class group depends in a somewhat convoluted way on the junior elements, see Remark 5.4.6. ∎

6. Yamagishi's algorithm for the Cox ring of a $\mathbb{Q}$-factorial terminalization

We continue in the context of the previous chapter and let $G \leq \mathrm{SL}(V)$ be a finite group. We describe an algorithm by Yamagishi [Yam18], which computes a presentation of the Cox ring $\mathcal{R}(X)$ of a $\mathbb{Q}$-factorial terminalization $X \to V/G$ of the linear quotient given the group G. After stating the theoretical foundation of the algorithm and some preparatory comments in Section 6.1, we present the main algorithm for the computation of generators of the ring $\mathcal{R}(X)$ in full detail and prove its correctness in Section 6.2. We discuss two necessary 'subalgorithms' as well as a method to compute the relations of the computed generators in Section 6.3. These are taken from [Yam18] as well, however we present the methods in more generality and add detailed proofs of correctness. We implemented the described algorithms in the computer algebra system OSCAR [Osc23]. To the author's knowledge, this is the first implementation of this kind. In Section 6.4, we comment on this implementation.

We decided to present the algorithm from [Yam18] here in full detail for several reasons. Besides the already mentioned implementation, we differ from the presentation in [Yam18], see Remark 6.2.17 and in particular Remark 6.2.24. Further, we use the algorithm in Chapter 7 and therefore like to lay out the necessary notation here.

Throughout, let V be a finite-dimensional vector space over $\mathbb{C}$ and let $G \leq \mathrm{SL}(V)$ be a finite group. Let $\varphi : X \to V/G$ be a $\mathbb{Q}$-factorial terminalization of the linear quotient V/G and let $m \in \mathbb{Z}_{\geq 0}$ be the number of junior conjugacy classes in G. We again let $\{g_1, \dots, g_m\} \in G$ be a minimal set of representatives of the junior conjugacy classes corresponding by Theorem 2.1.21 to exceptional prime divisors $E_1, \dots, E_m \in \mathrm{Div}(X)$ of φ. For any $i \in \{1, \dots, m\}$, we write v_i for the monomial valuation on $\mathbb{C}(V)$ defined by g_i, see Section 2.1.3.

6.1. Preparations

6.1.1. Embedding the Cox ring

We describe a way of embedding the Cox ring $\mathcal{R}(X)$ into a Laurent polynomial ring over $\mathcal{R}(V/G)$. This approach was to the author's knowledge first proposed in [Don16] and further investigated in [DG16], [DW17], [Yam18] and [Gra19].

Let $\mathrm{Cl}(X)^{\mathrm{free}}$ be the free part of the class group of X with canonical projection morphism $\rho : \mathrm{Cl}(X) \to \mathrm{Cl}(X)^{\mathrm{free}}$. By Corollary 5.4.2, we have $\mathrm{Cl}(X)^{\mathrm{free}} \cong \mathbb{Z}^m$. Let $\mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}] \cong \mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ be the group ring; one can also think of $\mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$ as the coordinate ring of the torus $\mathrm{Hom}(\mathrm{Cl}(X)^{\mathrm{free}}, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^m$. We consider the ring $\mathcal{R}(V/G) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$, which is graded by $\mathrm{Cl}(V/G) \oplus \mathrm{Cl}(X)^{\mathrm{free}}$ in the natural way. Recall from Proposition 2.4.7 that there is a surjective graded morphism $\varphi_* : \mathcal{R}(X) \to \mathcal{R}(V/G)$ induced by $\varphi : X \to V/G$. Let

$$\Theta : \mathcal{R}(X) \to \mathcal{R}(V/G) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$$

be the morphism of graded rings mapping a homogeneous element $f \in \mathcal{R}(X)_{[D]}$ to

$$\Theta(f) = \varphi_*(f) \otimes \rho([D]) \ ,$$

where we consider $\rho([D])$ as an element of the group ring.

Proposition 6.1.1. *The morphism Θ is injective.*

This follows directly from Lemma 5.1.4, see [Gra19, Proposition 4.1.5].

We can hence realize $\mathcal{R}(X)$ as the subring $\Theta(\mathcal{R}(X))$ of

$$\mathcal{R}(V/G) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}] \cong \big(\mathbb{C}[V]^{[G,G]}\big)[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \ .$$

Further, any set of $\mathrm{Ab}(G)^\vee$-homogeneous generators $f_1, \dots, f_k \in \mathbb{C}[V]^{[G,G]}$ gives rise to a ring $R(f_1, \dots, f_k) \leq \mathcal{R}(V/G) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}]$ as follows. Let $\chi_{[D_1]}, \dots, \chi_{[D_k]} \in \mathrm{Ab}(G)^\vee$ be the degrees of $f_1, \dots, f_k$ with divisors $D_1, \dots, D_k \in \mathrm{Div}(V/G)$. We may assume that $D_i = \mathrm{div}_{[D_i]}(f_i)$ for all $1 \leq i \leq k$. Now let $R(f_1, \dots, f_k)$ be the ring generated by the elements

$$f_i \otimes \rho([\overline{D}_i]),\ 1 \leq i \leq k, \text{ and}$$
$$1 \otimes \rho([E_j]),\ 1 \leq j \leq m,$$

where $\overline{D}_i$ is the strict transform of D_i via φ.

Lemma 6.1.2. *With the notation introduced above, we have $R(f_1, \dots, f_k) \leq \Theta(\mathcal{R}(X))$ for any set of $\mathrm{Ab}(G)^\vee$-homogeneous generators $f_1, \dots, f_k$ of $\mathbb{C}[V]^{[G,G]}$.*

Proof. Follows directly from the surjectivity of φ_* (Proposition 2.4.7). □

We now give a condition on the generators $f_1, \dots, f_k$ involving the valuations v_i ensuring that

$$R(f_1, \dots, f_k) = \Theta(\mathcal{R}(X)) \cong \mathcal{R}(X) \ ,$$

see also [Yam18, p. 610] or [Gra19, Assumption 4.1.14].

Condition 6.1.3. Let $f_1, \dots, f_k \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous generators and let $f \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous. We say that the generators $f_1, \dots, f_k$ satisfy $(*f)$, if f can be expressed as a sum of monomials $F_1, \dots, F_l$ in $f_1, \dots, f_k$ such that $v_i(f) \leq v_i(F_j)$ for every $1 \leq i \leq m$ and $1 \leq j \leq l$.

Remark 6.1.4. In the situation of Condition 6.1.3, we always have $v_i(f) \geq v_i(F_j)$ by definition of a valuation. ∥

Theorem 6.1.5. *Let $f_1, \dots, f_k \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous generators. We have $R(f_1, \dots, f_k) = \Theta(\mathcal{R}(X))$ if and only if the generators $f_1, \dots, f_k \in \mathbb{C}[V]^{[G,G]}$ satisfy $(*f)$ for every $\mathrm{Ab}(G)^\vee$-homogeneous element $f \in \mathbb{C}[V]^{[G,G]}$.*

See [Yam18, Proposition 4.4] or [Gra19, Theorem 4.1.15] for a proof.

6.1.2. Explicit representation of objects in the computer

Before we describe the algorithm given in [Yam18], which uses Theorem 6.1.5 to compute generators of $\mathcal{R}(X)$, we give a few comments to describe the involved mathematical objects more concretely as is necessary from an algorithmic point of view.

The isomorphism $\mathrm{Cl}(X)^{\mathrm{free}} \cong \mathbb{Z}^m$

Firstly, we give a more explicit description of the isomorphism $\mathrm{Cl}(X)^{\mathrm{free}} \cong \mathbb{Z}^m$. As before, let $H \leq G$ be the subgroup generated by the junior elements contained in G and write $\overline{H} := H/(H \cap [G,G])$ for the image of H in $\mathrm{Ab}(G)$. We identify the free group $\bigoplus_{i=1}^m \mathbb{Z}E_i$ with a subgroup of $\mathrm{Cl}(X)^{\mathrm{free}}$ via the canonical embedding, see Corollary 5.1.3. By Corollary 5.4.3, we have $\mathrm{Cl}(X)^{\mathrm{free}}/\bigoplus_{i=1}^m \mathbb{Z}E_i \cong \overline{H}^\vee$ and from the discussion in Section 5.3 we see that $\overline{H}^\vee$ embeds into $\mathbb{Z}/r_1\mathbb{Z} \times \cdots \mathbb{Z}/r_m\mathbb{Z}$, where the r_i are the orders of the g_i. This means that we may define an injective group morphism $\iota : \mathrm{Cl}(X)^{\mathrm{free}} \to \mathbb{Z}^m$ with the property that $\iota(\rho([E_i])) = (0, \dots, 0, -r_i, 0, \dots, 0)$ with the non-trivial entry at the i-th position. The reason for adding the negative sign here is explained below.

Remark 6.1.6. The map ι is in general not surjective. Although this is the case for the dihedral groups we consider in Chapter 7, we also have an example in the other extreme: for the binary icosahedral group $\mathsf{I} \leq \mathrm{SL}_2(\mathbb{C})$, the group $\mathrm{Ab}(\mathsf{I})$ is trivial, so we obtain $\mathrm{coker}(\iota) = \mathbb{Z}/r_1\mathbb{Z} \times \cdots \times \mathbb{Z}/r_m\mathbb{Z}$ in the above notation. The group $\mathbb{Z}^m$ containing $\bigoplus_{i=1}^m \mathbb{Z}E_i$ in the prescribed way is in down to earth terms the 'biggest' group we might need to represent elements of $\mathrm{Cl}(X)^{\mathrm{free}}$ by integers in the computer. ‖

The group morphism ι induces an embedding of rings $\mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}] \hookrightarrow \mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. The element $\rho([\overline{D}])$ corresponding to the strict transform of an effective divisor D on V/G is in this way identified with the Laurent polynomial $\prod_{i=1}^m t_i^{v_i(f)}$, where $f \in \mathbb{C}[V]_{\chi[D]}^{[G,G]}$ is a canonical section of D by Proposition 5.2.6. Note that our choice of sign for the images of the $\rho([E_i])$ results in positive exponents in this expression.

Junior elements and valuations

Recall from Section 2.1 that for a general subgroup of $\mathrm{SL}(V)$ the age of an element as well as the corresponding monomial valuation depend on a choice of root of unity, see Remark 2.1.17. For explicit computations, we therefore assume that we choose $e^{\frac{2\pi i}{r}}$ as primitive r-th root of unity, where e is Euler's number and $i := \sqrt{-1}$ the imaginary unit.

Recall further that the valuations v_i on $\mathbb{C}(V)$ corresponding to the junior elements g_i are so far only constructed with respect to an eigenbasis of g_i. Since we are going to work with m valuations simultaneously, we cannot fix one basis for V during the whole algorithm. However, the defined valuations $v_1, \dots, v_m$ easily translate to $\mathbb{C}(V)$ where V is given in any basis. Indeed, let $w_1, \dots, w_n$ be an eigenbasis for g_i and let $w_1', \dots, w_n'$ be another basis of V. Then there is a vector space automorphism $\psi : V \to V$ given by the change of basis. This induces a ring isomorphism

$$\Psi : \mathbb{C}[w_1', \dots, w_n'] \to \mathbb{C}[w_1, \dots, w_n],\ h \mapsto h(\psi(w_1'), \dots, \psi(w_n'))\ ,$$

so that we can define the valuation of $h \in \mathbb{C}[w_1', \dots, w_n']$ by $v_i(h) \coloneqq v_i(\Psi(h))$.

6.1.3. 'Negative' homogenization

A helpful tool in the algorithm is 'negative' homogenization – as we call it for lack of a better term.

Let $R \coloneqq \mathbb{C}[X_1, \dots, X_k]$ be a polynomial ring graded by a matrix $W \in \mathrm{Mat}_{r,k}(\mathbb{Z})$ of rank $r \geq 1$, that is, $\deg_W(X_j) \coloneqq (W_{1j}, \dots, W_{rj}) \in \mathbb{Z}^r$, $1 \leq j \leq k$. We introduce additional variables $t_1, \dots, t_r$ and endow the ring $R^- \coloneqq \mathbb{C}[X_1, \dots, X_k, t_1, \dots, t_r]$ with the grading induced by the matrix $W^- \coloneqq (W \mid -I_r)$, where I_r denotes the identity matrix of rank r. Analogous to the homogenization with respect to variables of degree 1, see for example [KR05, Definition 4.3.1], we define the homogenization of a polynomial $f \in R$ with respect to the variables $t_1, \dots, t_r$ of degree -1.

Definition 6.1.7 (Minimal degree and negative homogenization)**.** Let $f \in R \setminus \{0\}$ and write $f = f_1 + \dots + f_s$ with f_i the terms of f. For $j = 1, \dots, s$, let $\deg_W(f_j) = (d_{1j}, \dots, d_{rj}) \in \mathbb{Z}^r$. Moreover, for $i = 1, \dots, r$, let $\mu_i \coloneqq \min\{d_{ij} \mid j = 1, \dots, s\}$.

(a) The tuple $(\mu_1, \dots, \mu_r)$ is called the *minimal degree* of f with respect to the grading given by W and denoted by $\mathrm{mindeg}_W(f)$.

(b) The *homogenization* of f with respect to the grading given by W is the polynomial

$$f^h = \sum_{j=1}^{s} f_j t_1^{d_{1j}-\mu_1} \cdots t_r^{d_{rj}-\mu_r} \in R^- .$$

For the zero polynomial, we set $0^h = 0$.

The polynomial f^h is homogeneous of degree $\mathrm{mindeg}_W(f)$ by construction. To be able to use results from the literature concerned with homogenization in the usual sense, that is, with respect to variables of positive degree, we require the following lemma.

Lemma 6.1.8. *Let $R^+ \coloneqq \mathbb{C}[X_1, \dots, X_k, u_1, \dots, u_r]$ be a polynomial ring graded by the matrix $W^+ \coloneqq (-W \mid I_r)$. For a polynomial $f \in R \setminus \{0\}$, let f_-, respectively f_+, be the homogenization of f as an element of R^-, respectively R^+, with respect to $t_1, \dots, t_r$, respectively $u_1, \dots, u_r$. Then $f_- = f_+(X_1, \dots, X_k, t_1, \dots, t_r)$.*

Proof. Write $f = f_1 + \dots + f_s$ with f_i the terms of f. Let $\mu_+ \coloneqq \mathrm{topdeg}_{-W}(f)$ (see [KR05, Definition 4.3.1]) and $\mu_- \coloneqq \mathrm{mindeg}_W(f)$. We observe that $-\mu_+ = \mu_-$. We have

$$\begin{aligned} f_- &= \sum_{j=1}^{s} \Big(f_j \prod_{i=1}^{r} t_i^{\deg_W(f_j)_i - (\mu_-)_i} \Big) = \sum_{j=1}^{s} \Big(f_j \prod_{i=1}^{r} t_i^{-\deg_{-W}(f_j)_i + (\mu_+)_i} \Big) \\ &= f_+(X_1, \dots, X_k, t_1, \dots, t_r) , \end{aligned}$$

as required. □

Remark 6.1.9. Lemma 6.1.8 enables us to use well-known 'calculation rules' for homogenized polynomials and ideals as in [KR05, Proposition 4.3.2] and [KR05, Proposition 4.3.5], although these rules are only proven for homogenizations with respect to variables of positive degree in the given references. ∥

6.2. The algorithm

We are ready to describe the algorithm from [Yam18] for the computation of a set of generators of $\mathcal{R}(X)$.

6.2.1. A different characterization of Condition 6.1.3

Let $f_1, \dots, f_k$ be $\mathrm{Ab}(G)^\vee$-homogeneous generators of $\mathbb{C}[V]^{[G,G]}$. We want to compute additional $\mathrm{Ab}(G)^\vee$-homogeneous elements $f_{k+1}, \dots, f_{k'}$ in $\mathbb{C}[V]^{[G,G]}$ such that the elements $f_1, \dots, f_{k'}$ fulfil condition 6.1.3 for every $\mathrm{Ab}(G)^\vee$-homogeneous element $f \in \mathbb{C}[V]^{[G,G]}$. To be able to write down the algorithm we have to introduce some notation; most of it is directly taken from [Yam18].

Definition 6.2.1. Let $S = \{f_1, \dots, f_k\}$ be a set of $\mathrm{Ab}(G)^\vee$-homogeneous generators of $\mathbb{C}[V]^{[G,G]}$, let $A \subseteq \{1, \dots, m\}$ and let $f \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous. We say that S satisfies $(*A, f)$, if f can be expressed as a sum of monomials $F_1, \dots, F_l$ in the elements $f_1, \dots, f_k$ such that $v_i(f) \le v_i(F_j)$ for every $i \in A$ and $1 \le j \le l$. Further, we say that S satisfies $(*A)$, if S satisfies $(*A, f)$ for every $\mathrm{Ab}(G)^\vee$-homogeneous element $f \in \mathbb{C}[V]^{[G,G]}$.

Of course, $(*\{1, \dots, m\}, f)$ is just Condition 6.1.3 and we aim for a set of generators fulfilling $(*\{1, \dots, m\})$ as in Theorem 6.1.5.

The following corollary of the fact that $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra (Theorem 2.4.13) is necessary to ensure the termination of the algorithm after finitely many steps.

Lemma 6.2.2. *There exist* $\mathrm{Ab}(G)^\vee$*-homogeneous polynomials* $\tilde{f}_1, \dots, \tilde{f}_s \in \mathbb{C}[V]^{[G,G]}$ *such that for any set* $\{f_1, \dots, f_k\}$ *of* $\mathrm{Ab}(G)^\vee$*-homogeneous generators of* $\mathbb{C}[V]^{[G,G]}$ *and any subset* $A \subseteq \{1, \dots, m\}$*, condition* $(*A)$ *is satisfied if and only if* $(*A, \tilde{f}_j)$ *is satisfied for all* $j = 1, \dots, s$.

See [Yam18, Lemma 4.8] for a proof. The proof is non-constructive as one chooses the $\tilde{f}_j$ to be the images of generators of $\mathcal{R}(X)$ under the push-forward $\varphi_* : \mathcal{R}(X) \to \mathcal{R}(V/G)$.

We now introduce a reformulation of Condition 6.1.3. For this, we fix notation that remains in place throughout the chapter. We consider the morphism of rings

$$\alpha : \mathbb{C}[X_1, \dots, X_k] \to \mathbb{C}[V]^{[G,G]},\ X_i \mapsto f_i$$

and endow $\mathbb{C}[X_1, \dots, X_k]$ with a weighted grading $\deg_i$ for every junior element g_i, $1 \le i \le m$, by setting $\deg_i(X_j) := v_i(f_j)$ for $1 \le j \le k$. Given a polynomial

$$h = \sum_{a \in \mathbb{Z}^k_{\ge 0}} \lambda_a X_1^{a_1} \cdots X_k^{a_k} \in \mathbb{C}[X_1, \dots, X_k]\ ,$$

we let

$$\mathrm{mindeg}_i(h) := \min_{\substack{a \in \mathbb{Z}^k_{\ge 0} \\ \lambda_a \ne 0}} \deg_i \left(X_1^{a_1} \cdots X_k^{a_k} \right)$$

be the minimal degree of h with respect to $\deg_i$. Notice that $\operatorname{mindeg}_i(0) = \infty$.

Further, we lift the action of $\operatorname{Ab}(G)$ on $\mathbb{C}[V]^{[G,G]}$ to an action on $\mathbb{C}[X_1, \dots, X_k]$ via α as follows. Given $\gamma \in \operatorname{Ab}(G)$ and $1 \leq j \leq k$ there is $\lambda_j \in \mathbb{C}^\times$ with $\gamma.f_j = \lambda_j f_j$ by $\operatorname{Ab}(G)^\vee$-homogeneity so that we can define $\gamma.X_j \coloneqq \lambda_j X_j$. This gives a grading by $\operatorname{Ab}(G)^\vee$ on $\mathbb{C}[X_1, \dots, X_k]$ and α is by construction an $\operatorname{Ab}(G)^\vee$-graded morphism.

We have the following reformulation of Definition 6.2.1:

Lemma 6.2.3. *The set $\{f_1, \dots, f_k\}$ satisfies $(*A, f)$ for a subset $A \subseteq \{1, \dots, m\}$ and an $\operatorname{Ab}(G)^\vee$-homogeneous element $f \in \mathbb{C}[V]^{[G,G]}$ if and only if there exists $h \in \alpha^{-1}(f)$ with $\operatorname{mindeg}_i(h) = v_i(f)$ for all $i \in A$.*

At this point, we can already state the basic skeleton of the algorithm from [Yam18], see Algorithm 6.2.1. This relies on Algorithm 6.2.2 and Algorithm 6.2.3 discussed below, which may be seen as an 'induction start' and 'induction step', respectively. The correctness of Algorithm 6.2.1 follows directly from the correctness of Algorithm 6.2.2 and Algorithm 6.2.3, once one has convinced oneself that the nested `for`-loops ensure the conditions $(*\{1, \dots, i', i\})$ in the order

$$
\begin{aligned}
&(*\{1,2\})\\
&(*\{1,3\}), (*\{1,2,3\})\\
&(*\{1,4\}), (*\{1,2,4\}), (*\{1,2,3,4\})\\
&\qquad\vdots\\
&(*\{1,m\}), (*\{1,2,m\}), \dots, (*\{1,\dots,m\})\ .
\end{aligned}
$$

We now present the two parts of this algorithm, which we think of as 'Phase 1' (the first `for`-loop) and 'Phase 2' (the nested `for`-loops).

6.2.2. Phase 1: The case of one valuation

Fix an index $i \in \{1, \dots, m\}$ and let $\{f_1, \dots, f_k\}$ be a set of $\operatorname{Ab}(G)^\vee$-homogeneous generators of $\mathbb{C}[V]^{[G,G]}$. Following [Yam18], we prove that $(*\{i\})$ is equivalent to the equality of certain ideals.

Notation 6.2.4. Given $h \in \mathbb{C}[X_1, \dots, X_k]$ we write $\min_i(h)$ for the $\deg_i$-homogeneous part of h of minimal degree $\operatorname{mindeg}_i(h)$.

Lemma 6.2.5. *Let $h \in \mathbb{C}[X_1, \dots, X_k]$ be $\operatorname{Ab}(G)^\vee$-homogeneous. Then the $\deg_i$-minimal part $\min_i(h)$ is $\operatorname{Ab}(G)^\vee$-homogeneous as well.*

Proof. This is clear since $\min_i(h)$ is a summand of h and any monomial in $\mathbb{C}[X_1, \dots, X_k]$ is $\operatorname{Ab}(G)^\vee$-homogeneous. □

Lemma 6.2.6. *We have $\operatorname{mindeg}_i(h) \leq v_i(\alpha(h))$ for any $0 \neq h \in \mathbb{C}[X_1, \dots, X_k]$.*

Algorithm 6.2.1. Generators of $\mathbb{C}[V]^{[G,G]}$ fulfilling $(*\{1,\dots,m\})$

Input : $\mathrm{Ab}(G)^\vee$-homogeneous generators $f_1,\dots,f_k$ of $\mathbb{C}[V]^{[G,G]}$
Output: $\mathrm{Ab}(G)^\vee$-homogeneous elements $f_{k+1},\dots,f_l \in \mathbb{C}[V]^{[G,G]}$ such that $\{f_1,\dots,f_l\}$ satisfies $(*\{1,\dots,m\})$

Set $S \coloneqq \{f_1,\dots,f_k\}$
for $i = 1,\dots,m$ **do**
  Enlarge S using Algorithm 6.2.2 to ensure $(*\{i\})$
end
for $i = 2,\dots,m$ **do**
  for $i' = 1,\dots,i-1$ **do**
    Enlarge S using Algorithm 6.2.3 (with input $A = \{1,\dots,i'-1\}$) to ensure $(*\{1,\dots,i',i\})$
  end
end
return S

Proof. Write $h = \sum_{a\in\mathbb{Z}^k_{\geq 0}} \lambda_a X_1^{a_1}\cdots X_k^{a_k}$. Then we have

$$\mathrm{mindeg}_i(h) = \min_{\substack{a\in\mathbb{Z}^k_{\geq 0}\\ \lambda_a\neq 0}} v_i\big(f_1^{a_1}\cdots f_k^{a_k}\big) \leq v_i(h(f_1,\dots,f_k)) = v_i(\alpha(h)) \ .$$

□

Lemma 6.2.7. *Let* $0 \neq h \in \mathbb{C}[X_1,\dots,X_k]$. *We have* $\mathrm{mindeg}_i(h) < v_i(\alpha(h))$ *if and only if* $\deg_i(\min_i(h)) < v_i(\alpha(\min_i(h)))$.

Proof. Set $h' \coloneqq \min_i(h)$ and notice that $\deg_i(h') = \mathrm{mindeg}_i(h)$. Recall that for all $h_1, h_2 \in \mathbb{C}[V]^{[G,G]}$ we have

$$v_i(h_1) \neq v_i(h_2) \Longrightarrow v_i(h_1+h_2) = \min\{v_i(h_1), v_i(h_2)\}$$

as this holds for any valuation, see for example [Lan02, p. 481].

Assume $\mathrm{mindeg}_i(h) < v_i(\alpha(h))$ and $\deg_i(h') = v_i(\alpha(h'))$. It follows that $v_i(\alpha(h')) < v_i(\alpha(h))$ and hence $v_i(\alpha(h')) = v_i(\alpha(h-h'))$. But then

$$\mathrm{mindeg}_i(h-h') \leq v_i(\alpha(h-h')) = \deg_i(h')$$

by Lemma 6.2.6 in contradiction to $\mathrm{mindeg}_i(h-h') > \deg_i(h')$.

Assume $\deg_i(h') < v_i(\alpha(h'))$ and $\mathrm{mindeg}_i(h) = v_i(\alpha(h))$. This means that $v_i(\alpha(h)) < v_i(\alpha(h'))$ but also

$$\min\{v_i(\alpha(h')), v_i(\alpha(h-h'))\} \leq v_i(\alpha(h)) \ ,$$

so $v_i(\alpha(h')) > v_i(\alpha(h-h'))$ leading to $v_i(\alpha(h)) = v_i(\alpha(h-h'))$. This, however, implies

$$\mathrm{mindeg}_i(h-h') > \mathrm{mindeg}_i(h) = v_i(\alpha(h-h'))$$

contradicting Lemma 6.2.6. □

Notation 6.2.8. For an ideal $I \trianglelefteq \mathbb{C}[X_1, \dots, X_k]$, write

$$\min_i I \coloneqq \langle \min_i(h) \mid h \in I \rangle$$

and

$$\underline{\min}_i I \coloneqq \langle f \in I \mid f \text{ is } \mathrm{Ab}(G)^\vee\text{-homogeneous} \rangle .$$

We fix some more notation. Let $I \coloneqq \ker(\alpha)$ and notice that I is $\mathrm{Ab}(G)^\vee$-homogeneous as α is an $\mathrm{Ab}(G)^\vee$-graded morphism. Let $w_1, \dots, w_n \in V$ be an eigenbasis of V for g_i and let $\psi : \mathbb{C}[V] \to \mathbb{C}[w_1, \dots, w_n]$ be the ring isomorphism induced by the change of basis. Recall that the valuation $v_i(f)$ of $f \in \mathbb{C}[w_1, \dots, w_n]$ is the minimum of the valuations of the terms of f. Let $\min_i(f) \in \mathbb{C}[w_1, \dots, w_n]$ be the sum of the terms of minimal valuation. Define

$$\beta_i : \mathbb{C}[X_1, \dots, X_k] \to \mathbb{C}[w_1, \dots, w_n],\ X_j \mapsto \min_i(\psi(f_j))$$

and let $J_i \coloneqq \ker(\beta_i)$.

We are interested in the ideals $\min_i I$ and $\underline{\min}_i J_i$. These are well-behaved with respect to the different gradings.

Lemma 6.2.9. *The ideals* $\min_i I$ *and* $\underline{\min}_i J_i$ *are* $\deg_i$*-homogeneous and* $\mathrm{Ab}(G)^\vee$*-homogeneous.*

Proof. The $\deg_i$-homogeneity of $\min_i I$ and the $\mathrm{Ab}(G)^\vee$-homogeneity of $\underline{\min}_i J_i$ follow by construction. For the $\deg_i$-homogeneity of $\underline{\min}_i J_i$, one uses that β_i is a graded morphism with respect to $\deg_i$ on $\mathbb{C}[X_1, \dots, X_k]$ and the grading induced by v_i on $\mathbb{C}[V]$ as in Section 5.3, see [Yam18, Lemma 4.5].

It remains to show $\mathrm{Ab}(G)^\vee$-homogeneity of $\min_i I$. Let $h \in I$, giving a generator $\min_i(h) \in \min_i I$. We can write $h = \sum_{j=1}^t h_j$ with $\mathrm{Ab}(G)^\vee$-homogeneous elements $h_j \in I$ since I is an $\mathrm{Ab}(G)^\vee$-homogeneous ideal. Then the minimal parts $\min_i(h_j)$ are $\mathrm{Ab}(G)^\vee$-homogeneous as well by Lemma 6.2.5. Now $\min_i(h)$ is exactly the sum of those $\min_i(h_j)$ with $\deg_i(\min_i(h_j)) = \mathrm{mindeg}_i(h)$ and the $\mathrm{Ab}(G)^\vee$-homogeneous components of $\min_i(h)$ are hence in $\min_i I$. So, the ideal $\min_i I$ is $\mathrm{Ab}(G)^\vee$-homogeneous as well. □

The ideal $\underline{\min}_i J_i$ can be seen as the set of polynomials violating $(*\{i\})$ by the next lemma.

Lemma 6.2.10. *For an* $\mathrm{Ab}(G)^\vee$*-homogeneous polynomial* $0 \neq h \in \mathbb{C}[X_1, \dots X_k]$*, we have* $\min_i(h) \in \underline{\min}_i J_i$ *if and only if* $\mathrm{mindeg}_i(h) < v_i(\alpha(h))$*.*

Proof. We follow the argument in [Yam18, p. 611].

By Lemmas 6.2.5, 6.2.7 and 6.2.9, we may assume that h is $\deg_i$-homogeneous by replacing h by $\min_i(h)$.

Assume $h \in \underline{\min}_i J_i$. We have

$$v_i(\alpha(h)) = v_i(h(f_1, \dots, f_k)) = v_i\big(h(\psi(f_1), \dots, \psi(f_k))\big)$$

by definition of α and v_i, where ψ is the morphism to an eigenbasis of g_i as in the definition of β_i. But $h \in J_i$, so h is a non-trivial relation of the minimal parts of $\psi(f_1), \ldots, \psi(f_k)$. Hence,

$$v_i\big(h(\psi(f_1), \ldots, \psi(f_k))\big) = v_i\big(h(f'_1, \ldots, f'_k)\big) ,$$

where $f'_j \coloneqq \psi(f_j) - \min_j(\psi(f_j))$ for $1 \leq j \leq k$. Writing $h = \sum_{a \in \mathbb{Z}^k_{\geq 0}} \lambda_a X_1^{a_1} \cdots X_k^{a_k}$, we conclude

$$v_i\big(h(f'_1, \ldots, f'_k)\big) \geq \min_{\substack{a \in \mathbb{Z}^k_{\geq 0} \\ \lambda_a \neq 0}} v_i\big({f'_1}^{a_1} \cdots {f'_k}^{a_k}\big) > \min_{\substack{a \in \mathbb{Z}^k_{\geq 0} \\ \lambda_a \neq 0}} v_i\big(f_1^{a_1} \cdots f_k^{a_k}\big) = \deg_i(h) .$$

Conversely, if $\deg_i(h) < v_i(\alpha(h))$, we have with the same notation

$$\min_{\substack{a \in \mathbb{Z}^k_{\geq 0} \\ \lambda_a \neq 0}} v_i\big(f_1^{a_1} \cdots f_k^{a_k}\big) = \min_{\substack{a \in \mathbb{Z}^k_{\geq 0} \\ \lambda_a \neq 0}} v_i\big(\min_i(\psi(f_1))^{a_1} \cdots \min_i(\psi(f_k))^{a_k}\big) .$$

Further, for any two terms $\lambda_a X_1^{a_1} \cdots X_k^{a_k}$ and $\lambda_b X_1^{b_1} \cdots X_k^{b_k}$ of h we must have

$$v_i\big(\min_i(\psi(f_1))^{a_1} \cdots \min_i(\psi(f_k))^{a_k}\big) = v_i\big(\min_i(\psi(f_1))^{b_1} \cdots \min_i(\psi(f_k))^{b_k}\big)$$

by $\deg_i$-homogeneity of h. Then $v_i(\alpha(h))$ can only be properly larger than $\deg_i(h)$ if $h\big(\min_i(\psi(f_1)), \ldots, \min_i(\psi(f_k))\big)$ vanishes, so if $h \in J_i$. Since h is $\mathrm{Ab}(G)^\vee$-homogeneous, we conclude $h \in \underline{\min}_i J_i$. ☐

The ideal $\min_i I$ is a subset of $\underline{\min}_i J_i$ consisting of those 'bad' polynomials for which there exists a 'better' preimage, that is, a preimage with a higher minimal degree.

Lemma 6.2.11. *Let $0 \neq h \in \mathbb{C}[X_1, \ldots, X_k]$ be $\mathrm{Ab}(G)^\vee$-homogeneous and $\deg_i$-homogeneous. We have $h \in \min_i I$ if and only if there exists an $\mathrm{Ab}(G)^\vee$-homogeneous polynomial $\tilde{h} \in \mathbb{C}[X_1, \ldots, X_k]$ such that $h - \tilde{h} \in I$ and $\deg_i(h) < \mathrm{mindeg}_i(\tilde{h})$.*

Proof. We follow the argument in [Yam18, p. 611].

Assume $h \in \min_i I$, so $h = \min_i(h_1) + \cdots + \min_i(h_t)$ for polynomials $h_j \in I$. We may assume that the h_j are $\mathrm{Ab}(G)^\vee$-homogeneous of same degree since h is $\mathrm{Ab}(G)^\vee$-homogeneous and I is an $\mathrm{Ab}(G)^\vee$-homogeneous ideal. Set $h' \coloneqq \sum_{j=1}^t h_j \in I$. Then $\min_i(h') = h$ by $\deg_i$-homogeneity of h. Hence $\tilde{h} \coloneqq h - h'$ fulfils the requirements.

Conversely, assume that we have a polynomial $\tilde{h} \in \mathbb{C}[X_1, \ldots, X_k]$ as in the claim. Then it follows that $\min_i(h - \tilde{h}) = h$, so $h \in \min_i I$. ☐

Corollary 6.2.12. *We have $\min_i I \subseteq \underline{\min}_i J_i$.*

Proof. This follows from the above lemmas, see [Yam18, Lemma 4.5]. ☐

We are now able to relate the property $(*\{i\})$ to the ideals $\min_i I$ and $\underline{\min}_i J_i$.

Proposition 6.2.13. *The set $\{f_1, \ldots, f_k\}$ satisfies $(*\{i\})$ if and only if $\min_i I = \underline{\min}_i J_i$.*

Algorithm 6.2.2. Phase 1: ensure $(*\{i\})$

Input : $\mathrm{Ab}(G)^\vee$-homogeneous generators $f_1, \dots, f_k$ of $\mathbb{C}[V]^{[G,G]}$; $i \in \{1, \dots, m\}$

Output: $\mathrm{Ab}(G)^\vee$-homogeneous elements $f_{k+1}, \dots, f_l \in \mathbb{C}[V]^{[G,G]}$ such that $\{f_1, \dots, f_l\}$ satisfies $(*\{i\})$

Set $S \coloneqq \{f_1, \dots, f_k\}$
Compute $\min_i I$ and $\underline{\min}_i J_i$
while $\min_i I \neq \underline{\min}_i J_i$ **do**
    Write $\underline{\min}_i J_i = \min_i I + \langle h_1, \dots, h_t \rangle$ with $\deg_i$-homogeneous and $\mathrm{Ab}(G)^\vee$-homogeneous elements $h_j \notin \min_i I$
    $S \coloneqq S \cup \{\alpha(h_1), \dots, \alpha(h_t)\}$
    Update $\min_i I$ and $\underline{\min}_i J_i$
end
return S

Proof. We follow the argument in [Yam18, Proposition 4.9].

Assume that $\{f_1, \dots, f_k\}$ satisfies $(*\{i\})$. By Corollary 6.2.12, we have the inclusion $\min_i I \subseteq \underline{\min}_i J_i$ and Lemma 6.2.9 states that both ideals are $\deg_i$-homogeneous and $\mathrm{Ab}(G)^\vee$-homogeneous. Let $h \in \underline{\min}_i J_i$ be $\deg_i$-homogeneous and $\mathrm{Ab}(G)^\vee$-homogeneous. Then $\deg_i(h) < v_i(\alpha(h))$ by Lemma 6.2.10. As $\{f_1, \dots, f_k\}$ satisfies $(*\{i\})$, there must be $\tilde{h}$ with $\alpha(\tilde{h}) = \alpha(h)$ and $v_i(\alpha(\tilde{h})) = \mathrm{mindeg}_i(\tilde{h})$, by Lemma 6.2.3. Then $\tilde{h} - h \in I$ and $\mathrm{mindeg}_i(\tilde{h}) > \deg_i(h)$. Hence, $h \in \min_i I$ by Lemma 6.2.11 and $\min_i I = \underline{\min}_i J_i$ as required.

Conversely, assume $\min_i I = \underline{\min}_i J_i$ and let $f \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous. Using Lemma 6.2.3, we have to show that there is $h \in \alpha^{-1}(f)$ with $\mathrm{mindeg}_i(h) = v_i(f)$. So, let $h' \in \alpha^{-1}(f)$ be any preimage and assume we have $\mathrm{mindeg}_i(h') < v_i(f)$. We may assume that h' is $\mathrm{Ab}(G)^\vee$-homogeneous since α is a graded morphism. Hence, $h \coloneqq \min_i(h') \in \underline{\min}_i J_i$ by Lemma 6.2.10 and h is $\mathrm{Ab}(G)^\vee$-homogeneous by Lemma 6.2.5. Then by assumption $h \in \min_i I$, so there is $\tilde{h} \in \mathbb{C}[X_1, \dots, X_k]$ with $h - \tilde{h} \in I$ and $\deg_i(h) < \mathrm{mindeg}_i(\tilde{h})$ by Lemma 6.2.11. Hence, for $h'' \coloneqq h' - h + \tilde{h}$ we have $\alpha(h'') = f$ and $\mathrm{mindeg}_i(h'') > \mathrm{mindeg}_i(h')$. Increasing the minimal degree in this way we eventually obtain the desired preimage with minimal degree $v_i(f)$. ☐

Proposition 6.2.13 motivates Algorithm 6.2.2 and implies the correctness of this algorithm, assuming it terminates. We present algorithms for the computation of $\min_i I$ and $\underline{\min}_i J_i$ in Section 6.3.

Lemma 6.2.14. *Algorithm 6.2.2 terminates after finitely many steps.*

Proof. We follow the argument in [Yam18, Proposition 4.9].

By Lemma 6.2.2, there exist $\mathrm{Ab}(G)^\vee$-homogeneous elements $\tilde{f}_1, \dots, \tilde{f}_s \in \mathbb{C}[V]^{[G,G]}$, such that we only have to show that the set of generators satisfies $(*\{i\}, \tilde{f}_j)$, $j = 1, \dots, s$, after finitely many steps of the algorithm. Let $S = \{f_1, \dots, f_{k'}\}$ be the set of generators at the beginning of an iteration of the `while`-loop of the algorithm and let $f \in \{\tilde{f}_1, \dots, \tilde{f}_s\}$ be such that $(*\{i\}, f)$ is not satisfied. Choose any $h' \in \alpha^{-1}(f)$. We may assume that h' is $\mathrm{Ab}(G)^\vee$-homogeneous as α is an $\mathrm{Ab}(G)^\vee$-graded morphism. Then we have $\mathrm{mindeg}_i(h') < v_i(f)$ and hence $h := \min_i(h') \in \underline{\min}_i J_i$ by Lemma 6.2.10. By Lemma 6.2.5, h is $\mathrm{Ab}(G)^\vee$-homogeneous. Writing $\underline{\min}_i J_i = \min_i I + \langle h_1, \dots, h_t\rangle$ as in the algorithm, there are $h'' \in \min_i I$ and $a_1, \dots, a_t \in \mathbb{C}[X_1, \dots, X_{k'}]$ with $h = h'' + \sum_{j=1}^t a_j h_j$. We may assume that the polynomials h'' and $a_j h_j$ are $\deg_i$-homogeneous of same degree

$$\deg_i(h) = \deg_i(h'') = \deg_i(a_j h_j)$$

since h and the h_j are $\deg_i$-homogeneous and $\min_i I$ is a $\deg_i$-homogeneous ideal.

Write α^+, S^+, I^+ and so on for the updated instances in the next iteration of the `while`-loop. Let $X_{k'+1}, \dots, X_{k'+t}$ be the new variables corresponding to $\alpha(h_1), \dots, \alpha(h_t)$. Notice that $\alpha^+(f') = \alpha(f')$ for any $f' \in \mathbb{C}[X_1, \dots, X_{k'}]$ and in particular $I \subseteq I^+$ by considering elements of $\mathbb{C}[X_1, \dots, X_{k'}]$ as elements of $\mathbb{C}[X_1, \dots, X_{k'+t}]$ in the canonical way. We have $h_1, \dots, h_l \in \min_i I$. Indeed, for $1 \le j \le l$, by construction $h_j - X_{k+j} \in I$ and $\deg_i(X_{k+j}) > \deg_i(h_j)$, so $\min_i(h_j - X_{k+j}) = h_j \in \min_i I$ using the $\deg_i$-homogeneity of h_j. Hence, $h \in \min_i I$ and by Lemma 6.2.11 it follows that there is $\tilde{h} \in \mathbb{C}[X_1, \dots, X_k]$ with $\tilde{h} - h \in I$ and $\mathrm{mindeg}_i(\tilde{h}) > \deg_i(h)$. So, $\alpha(h' - h + \tilde{h}) = f$ and $\mathrm{mindeg}_i(h' - h + \tilde{h}) > \mathrm{mindeg}_i(h')$. In each iteration we hence find a preimage of higher degree and after finitely many steps the updated set S must satisfy $(*\{i\}, f)$. □

6.2.3. Phase 2: Combining different valuations

For the second phase of Algorithm 6.2.1, fix $i, i' \in \{1, \dots, m\}$ with $i' \neq i$. We present a condition equivalent to $(*\{i', i\})$, which is again based on the equality of certain ideals.

Remark 6.2.15. After applying Algorithm 6.2.2 for every valuation $v_1, \dots, v_m$ iteratively, we have generators $f_1, \dots, f_k \in \mathbb{C}[V]^{[G,G]}$ fulfilling $(*\{1\}), \dots, (*\{m\})$. This translates to the logical expression:

$$\forall f \in \mathbb{C}[V]^{[G,G]}\ \forall i \in \{1, \dots, m\}\ \exists h \in \alpha^{-1}(f) : \mathrm{mindeg}_i(h) = v_i(f)\ .$$

However, for $(*\{1, \dots, m\})$ we require:

$$\forall f \in \mathbb{C}[V]^{[G,G]}\ \exists h \in \alpha^{-1}(f)\ \forall i \in \{1, \dots, m\} : \mathrm{mindeg}_i(h) = v_i(f)\ .$$

In other words: so far we can ensure for $f \in \mathbb{C}[V]^{[G,G]}$ and $1 \le i' < i \le m$ that there are $h_{i'}, h_i \in \mathbb{C}[X_1, \dots, X_k]$ with $\alpha(h_{i'}) = \alpha(h_i) = f$ and $\mathrm{mindeg}_{i'}(h_{i'}) = v_{i'}(f)$ as well as $\mathrm{mindeg}_i(h_i) = v_i(f)$. But for $(*\{i', i\})$ we require a polynomial $h \in \mathbb{C}[X_1, \dots, X_k]$ that fulfils both degree conditions simultaneously. ‖

As before, let $f_1, \dots, f_k \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous generators and let I be the kernel of the morphism α. We follow [Yam18, p. 611], but are able to simplify the construction, see Remark 6.2.17. Consider the polynomial ring $\mathbb{C}[X_1, \dots, X_k, t_{i'}, t_i]$.

We extend the action of $\mathrm{Ab}(G)$ by $\gamma.t_j = t_j$, $j = i', i$, for all $\gamma \in \mathrm{Ab}(G)$ giving an $\mathrm{Ab}(G)^\vee$-grading on $\mathbb{C}[X_1, \ldots, X_k, t_{i'}, t_i]$. Further, we extend the gradings $\deg_{i'}$ and $\deg_i$ by setting $\deg_j(t_{j'}) = -\delta_{j,j'}$ for every possible choice of $j, j' \in \{i', i\}$, where $\delta_{j,j'}$ is the Kronecker delta.

Notation 6.2.16. Let $0 \neq h \in \mathbb{C}[X_1, \ldots, X_k]$ and write $h = \sum_j h_j$ with terms h_j. Then we set

$$h_{\{i',i\}} \coloneqq \sum_j \left(h_j t_{i'}^{\deg_{i'}(h_j) - \mathrm{mindeg}_{i'}(h)} t_i^{\deg_i(h_j) - \mathrm{mindeg}_i(h)}\right)$$

for the homogenization with respect to the gradings $\deg_{i'}$ and $\deg_i$, see Section 6.1.3. Notice that $h_{\{i',i\}}$ is $\deg_j$-homogeneous of degree $\mathrm{mindeg}_j(h)$ for $j = i', i$.

Remark 6.2.17. We adopt the non-standard notation $h_{\{i',i\}}$ for the homogenized polynomial from [Yam18]. However, we emphasize that we always only require to homogenize at two degrees simultaneously. In [Yam18], the homogenization is extended to an arbitrary set $A \subseteq \{1, \ldots, m\}$. The argument then works with the set $A = \{1, \ldots, i', i\}$ and we are able to drop the homogeneity with respect to $\deg_1, \ldots, \deg_{i'-1}$ in what follows. The approach in [Yam18] does not complicate the theory, but leads to a poorer performance of the algorithm in practice, as more and more variables need to be introduced. ‖

We collect some properties of the homogenization, which all follow by construction. By abuse of notation we abbreviate $h|_{t_{i'}=1, t_i=1}$ to $h|_{t=1}$.

Lemma 6.2.18. *Let $h \in \mathbb{C}[X_1, \ldots, X_k]$ and $H \in \mathbb{C}[X_1, \ldots, X_k, t_{i'}, t_i]$. Then we have:*

(a) $(h_{\{i',i\}})|_{t=1} = h$*;*

(b) if h is $\mathrm{Ab}(G)^\vee$-homogeneous then so is $h_{\{i',i\}}$;

(c) if H is $\mathrm{Ab}(G)^\vee$-homogeneous then so is $H|_{t=1}$;

(d) $\mathrm{mindeg}_j(h) = \mathrm{mindeg}_j(h_{\{i',i\}})$ *for all $j \in \{1, \ldots, m\}$;*

(e) $\mathrm{mindeg}_j(H) \leq \mathrm{mindeg}_j(H|_{t=1})$ *for all $j \in \{1, \ldots, m\}$ with equality if $j \neq i', i$ or $t_j \nmid H$;*

(f) $(H|_{t_j=0})|_{t=1} = \min_j(H|_{t=1})$ *for $j = i', i$.*

Set

$$I_{\{i',i\}} \coloneqq \langle h_{\{i',i\}} \mid h \in I \rangle \trianglelefteq \mathbb{C}[X_1, \ldots, X_k, t_{i'}, t_i]$$

and notice that this is a $\deg_j$-homogeneous ideal for $j = i', i$ by construction and also $\mathrm{Ab}(G)^\vee$-homogeneous as this is inherited from I.

Notation 6.2.19. For $i', i \in \{1, \ldots, m\}$ with $i \neq i'$, we define the ideals

$$I_{i',i} \coloneqq I_{\{i',i\}} \cap \langle t_{i'}, t_i \rangle$$

and

$$I'_{i',i} \coloneqq (I_{\{i',i\}} \cap \langle t_{i'} \rangle) + (I_{\{i',i\}} \cap \langle t_i \rangle) \ .$$

These ideals correspond to $\tilde{I}_{i',i}$ and $\tilde{I}'_{i',i}$, respectively, in [Yam18].

Lemma 6.2.20. *The ideals $I_{i',i}$ and $I'_{i',i}$ are $\mathrm{Ab}(G)^\vee$-homogeneous and $\deg_j$-homogeneous for $j = i', i$. Further, we have the inclusion $I'_{i',i} \subseteq I_{i',i}$.*

Proof. This follows by construction. □

The ideal $I_{i',i}$ has a similar function as $\underline{\min}_i J_i$ in the previous section as it is connected to the polynomials violating $(*\{i', i\})$.

Lemma 6.2.21. *Let $0 \neq h_1, h_2 \in \mathbb{C}[X_1, \ldots, X_k]$, $h_1 \neq h_2$, be $\mathrm{Ab}(G)^\vee$-homogeneous with $h_1 - h_2 \in I$. If $\mathrm{mindeg}_{i'}(h_1) > \mathrm{mindeg}_{i'}(h_2)$ and $\mathrm{mindeg}_i(h_1) < \mathrm{mindeg}_i(h_2)$, then $(h_1 - h_2)_{\{i',i\}} \in I_{i',i}$.*

Proof. By assumption, $h_1 - h_2 \in I$, so clearly $(h_1 - h_2)_{\{i',i\}} \in I_{\{i',i\}}$. For the minimal parts, we have $\min_{i'}(h_1 - h_2) = \min_{i'}(h_2)$ and $\min_i(h_1 - h_2) = \min_i(h_1)$, so every term of $(h_1 - h_2)_{\{i',i\}}$ must be divisible by $t_{i'}$ or t_i proving the claim. □

In analogy to $\min_i I$, the ideal $I'_{i',i}$ is the set of 'bad' polynomials for which we can choose a better preimage.

Lemma 6.2.22. *Let $A \subseteq \{1, \ldots, m\}$. Let $f \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous and let $h_1, h_2 \in \alpha^{-1}(f)$ be $\mathrm{Ab}(G)^\vee$-homogeneous with*

$$\mathrm{mindeg}_j(h_1) = \mathrm{mindeg}_j(h_2) = v_j(f)$$

for $j \in A$ and

$$\mathrm{mindeg}_{i'}(h_1) > \mathrm{mindeg}_{i'}(h_2) \text{ and } \mathrm{mindeg}_i(h_1) < \mathrm{mindeg}_i(h_2) .$$

If $(h_1 - h_2)_{\{i',i\}} \in I'_{i',i}$, then there exists an $\mathrm{Ab}(G)^\vee$-homogeneous element $h \in \alpha^{-1}(f)$ with

$$\mathrm{mindeg}_j(h) = v_j(f) \text{ for } j \in A$$

and

$$\mathrm{mindeg}_{i'}(h) > \mathrm{mindeg}_{i'}(h_2) \text{ and } \mathrm{mindeg}_i(h) > \mathrm{mindeg}_i(h_1) .$$

Proof. By assumption, there are $g_1 \in I_{\{i',i\}} \cap \langle t_{i'} \rangle$ and $g_2 \in I_{\{i',i\}} \cap \langle t_i \rangle$ with

$$(h_1 - h_2)_{\{i',i\}} = g_1 + g_2 .$$

In particular, $g_1 \in I_{\{i',i\}}$ and $g_1|_{t=1} \in I$ by [KR05, Proposition 4.3.5].

We claim that $h \coloneqq h_1 - g_1|_{t=1}$ has the required degrees. Clearly, $\alpha(h) = f$. For $j \in A$, we have

$$\begin{aligned} \mathrm{mindeg}_j(g_1|_{t=1}) &\geq \mathrm{mindeg}_j(g_1) \geq \mathrm{mindeg}_j((h_1 - h_2)_{\{i',i\}}) \\ &= \mathrm{mindeg}_j(h_1 - h_2) \geq v_j(f) , \end{aligned}$$

so $\mathrm{mindeg}_j(h) = v_j(f)$ by Lemma 6.2.6. We have $g_1 \in \langle t_{i'} \rangle$, so

$$\mathrm{mindeg}_{i'}(g_1|_{t=1}) > \mathrm{mindeg}_{i'}(g_1) \geq \mathrm{mindeg}_{i'}(h_2) .$$

Since also $\mathrm{mindeg}_{i'}(h_1) > \mathrm{mindeg}_{i'}(h_2)$, we conclude $\mathrm{mindeg}_{i'}(h) > \mathrm{mindeg}_{i'}(h_2)$. Further, we have $t_i \mid g_2$, so

$$\min_i(g_1|_{t=1}) = \min_i((g_1 + g_2)|_{t=1}) = \min_i(h_1)$$

and therefore $\mathrm{mindeg}_i(h) > \mathrm{mindeg}_i(h_1)$.

If h is not $\mathrm{Ab}(G)^\vee$-homogeneous, then there exists an $\mathrm{Ab}(G)^\vee$-homogeneous summand h' of h such that $h - h' \in I$ since α is an $\mathrm{Ab}(G)^\vee$-graded morphism and f is homogeneous. Then $\mathrm{mindeg}_j(h') \geq \mathrm{mindeg}_j(h)$ for all $j \in \{1, \ldots, m\}$, so h' is as required. □

We can now prove an analogue to Proposition 6.2.13. Although this is essentially [Yam18, Proposition 4.10], we prove a different statement, see Remark 6.2.24 for an explanation.

Proposition 6.2.23. *Let $i, i' \in \{1, \ldots, m\}$, $i \neq i'$ and $A \subseteq \{1, \ldots, m\}$. The set $\{f_1, \ldots, f_k\}$ satisfies $(*A \cup \{i', i\})$ if and only if it satisfies $(*A \cup \{i'\})$ and $(*A \cup \{i\})$ and we have $I_{i',i} = I'_{i',i}$.*

Proof. Assume that $\{f_1, \ldots, f_k\}$ satisfies $(*A \cup \{i', i\})$. Then this directly implies both $(*A \cup \{i'\})$ and $(*A \cup \{i\})$, so we only have to show the equality of ideals, that is, $I_{i',i} \subseteq I'_{i',i}$.

Let h' be an element of $I_{i',i}$. By Lemma 6.2.20, we may assume that h' is $\mathrm{Ab}(G)^\vee$-homogeneous and $\deg_j$-homogeneous for $j = i', i$. If $t_{i'} \mid h'$ or $t_i \mid h'$, we are done, so assume otherwise and set $h := \min_i(h'|_{t=1}) \in \min_i I$. Notice that $h'|_{t=1}$ and hence h are $\mathrm{Ab}(G)^\vee$-homogeneous by Lemma 6.2.18 and Lemma 6.2.5. Then $\deg_i(h) < v_i(\alpha(h))$ by Corollary 6.2.12 and Lemma 6.2.10. As we assume $(*A \cup \{i', i\})$, there is $\tilde{h} \in \mathbb{C}[X_1, \ldots, X_k]$ with $h - \tilde{h} \in I$ and $\mathrm{mindeg}_j(\tilde{h}) = v_j(\alpha(h))$ for all $j \in A \cup \{i', i\}$. Set $h'' = (h - \tilde{h})_{\{i',i\}} \in I_{\{i',i\}}$.

We have $t_{i'} \mid (h'|_{t_i=0})$ since $h' \in I_{i',i}$ and by assumption $t_{i'} \nmid h'$, so

$$\mathrm{mindeg}_{i'}(h'|_{t=1}) < \mathrm{mindeg}_{i'}(h) \leq \mathrm{mindeg}_{i'}(\tilde{h}) \; .$$

Hence $\mathrm{mindeg}_{i'}(h') < \mathrm{mindeg}_{i'}(h'')$. We have $(h''|_{t_i=0})|_{t_{i'}=1} = h$, so by homogeneity there is $l \in \mathbb{Z}_{>0}$ with

$$h'|_{t_i=0} = (h''|_{t_i=0}) t_{i'}^l \; .$$

Now $h'' t_{i'}^l \in I_{\{i',i\}} \cap \langle t_{i'} \rangle$ and $t_i \mid (h' - h'' t_{i'}^l)$, so $h' - h'' t_{i'}^l \in I_{\{i',i\}} \cap \langle t_i \rangle$. We conclude

$$h' = h'' t_{i'}^l + (h' - h'' t_{i'}^l) \in I'_{i',i} \; .$$

Conversely, assume that $\{f_1, \ldots, f_k\}$ satisfies both $(*A \cup \{i'\})$ and $(*A \cup \{i\})$ and that $I_{i',i} = I'_{i',i}$. Let $f \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous. Then there are $\mathrm{Ab}(G)^\vee$-homogeneous elements $h_1, h_2 \in \alpha^{-1}(f)$ with

$$\mathrm{mindeg}_j(h_1) = \mathrm{mindeg}_j(h_2) = v_j(f), \text{ for } j \in A \; ;$$
$$\mathrm{mindeg}_{i'}(h_1) = v_{i'}(f) \; ;$$
$$\mathrm{mindeg}_i(h_2) = v_i(f) \; .$$

If either $\operatorname{mindeg}_i(h_1) = v_i(f)$ or $\operatorname{mindeg}_{i'}(h_2) = v_{i'}(f)$, we are done, so we assume

$$\operatorname{mindeg}_i(h_1) < \operatorname{mindeg}_i(h_2) \text{ and}$$
$$\operatorname{mindeg}_{i'}(h_2) < \operatorname{mindeg}_{i'}(h_1) \ .$$

As $h_1 - h_2 \in I$, we have $(h_1 - h_2)_{\{i',i\}} \in I_{\{i',i\}}$. Also $(h_1 - h_2)_{\{i',i\}} \in \langle t_{i'}, t_i \rangle$ since neither h_1 nor h_2 has a term of minimal degree with respect to both $\deg_{i'}$ and $\deg_i$ by assumption. Hence $(h_1 - h_2)_{\{i',i\}} \in I_{i',i} = I'_{i',i}$. We prove at first that we can find $\tilde{h}_2 \in \alpha^{-1}(f)$ fulfilling all of the above equalities in place of h_2 and additionally $\operatorname{mindeg}_{i'}(\tilde{h}_2) > \operatorname{mindeg}_{i'}(h_2)$. We are in the situation of Lemma 6.2.22 and hence there exists $h \in \alpha^{-1}(f)$ with

$$\operatorname{mindeg}_j(h) = v_j(f), \text{ for } j \in A \ ;$$
$$\operatorname{mindeg}_{i'}(h) > \operatorname{mindeg}_{i'}(h_2) \ ;$$
$$\operatorname{mindeg}_i(h) > \operatorname{mindeg}_i(h_1) \ .$$

In case $\operatorname{mindeg}_i(h) = \operatorname{mindeg}_i(h_2)$, we set $\tilde{h}_2 := h$. Otherwise, the pair (h, h_2) fulfils the assumptions of Lemma 6.2.22, so we may assume that after applying the lemma with (h, h_2) iteratively, we arrive at a polynomial h with $\operatorname{mindeg}_i(h) = \operatorname{mindeg}_i(h_2)$ as required.

We now replace h_2 by $\tilde{h}_2$ and iterate the described process. As $\operatorname{mindeg}_{i'}(h_2)$ properly increases in every step, this terminates after finitely many steps and we arrive at a polynomial h_2 with $\operatorname{mindeg}_j(h_2) = v_j(\alpha(h_2))$ for all $j \in A \cup \{i', i\}$. □

This motivates Algorithm 6.2.3. Correctness of the algorithm is clear with Proposition 6.2.23, assuming it terminates.

Remark 6.2.24. In [Yam18, Proposition 4.10], Yamagishi claims that $(*A \cup \{i', i\})$ is equivalent to $(*A \cup \{i\})$ and $I_{i',i} = I'_{i',i}$, using our notation. However, we have to point out a mistake in the second half of the proof. There, [Yam18, Lemma 4.7] is used for the index i, where it should be used for i'. We do not see, how one could prove Yamagishi's claim.

If the result is nevertheless true, then this will lead to a more efficient algorithm as one could remove the nested `for`-loops in Algorithm 6.2.1 in favour of a single loop asserting the conditions $(*\{1,2\})$, $(*\{1,2,3\})$, $(*\{1,2,3,4\})$ and so on. ‖

Remark 6.2.25. Algorithm 6.2.3 does in fact not require a set $A \subseteq \{1, \dots, m\}$ as input as the construction of $I_{i',i}$ and $I'_{i',i}$ is independent of A. We only added this to the description of the algorithm to emphasize that it preserves the condition $(*A)$ assuming that both $(*A \cup \{i'\})$ and $(*A \cup \{i\})$ hold. ‖

The next lemma is helpful to prove termination of Algorithm 6.2.3 after finitely many steps (Proposition 6.2.27).

Lemma 6.2.26. *In the situation of Algorithm 6.2.3, if $h \in \{h_1, \dots, h_l\}$ in some iteration of the* `while`*-loop, then $h \in I'_{i',i}$ in the next iteration.*

Algorithm 6.2.3. Phase 2: ensure $(*\{i', i\})$

Input : $i, i' \in \{1, \dots, m\}$;
$A \subseteq \{1, \dots, m\}$;
$\mathrm{Ab}(G)^\vee$-homogeneous generators $\{f_1, \dots, f_k\}$ of $\mathbb{C}[V]^{[G,G]}$ satisfying $(*A \cup \{i'\})$ and $(*A \cup \{i\})$

Output: $\mathrm{Ab}(G)^\vee$-homogeneous elements $f_{k+1}, \dots, f_l \in \mathbb{C}[V]^{[G,G]}$ such that $\{f_1, \dots, f_l\}$ satisfies $(*A \cup \{i', i\})$

Initialize $S \coloneqq \{f_1, \dots, f_k\}$
Compute $I_{i',i}$ and $I'_{i',i}$
while $I_{i',i} \neq I'_{i',i}$ **do**
  Write $I_{i',i} = I'_{i',i} + \langle h_1, \dots, h_l \rangle$ with $\deg_j$-homogeneous, $j = i', i$, and $\mathrm{Ab}(G)^\vee$-homogeneous elements $h_1, \dots, h_l \notin I'_{i',i}$
  $S \coloneqq S \cup \{\alpha(\min_i(h_1|_{t=1})), \dots, \alpha(\min_i(h_l|_{t=1}))\}$
  Update $I_{i',i}$ and $I'_{i',i}$
end
return S

Proof. Let $h \in \{h_1, \dots, h_l\}$ and let S, α, $I_{i',i}$, $I'_{i',i}$ and so on be the updated instances in the next iteration of the `while`-loop. We have $\alpha(\min_i(h|_{t=1})) \in S$ and there is a variable X with $\alpha(X) = \alpha(\min_i(h|_{t=1}))$, so $X - \min_i(h|_{t=1}) \in I$. Set $h' \coloneqq h - h|_{t_i=0}$, so $h'|_{t=1} = h|_{t=1} - \min_i(h|_{t=1})$ by Lemma 6.2.18. We have $h \in I_{\{i',i\}}$, hence $h|_{t=1} \in I$ by [KR05, Proposition 4.3.5]. Then also $X + h'|_{t=1} \in I$. Further, $\min_i(h|_{t=1}) \in \min_i I$, so

$$\mathrm{mindeg}_i(h|_{t=1}) = \deg_i(\min_i(h|_{t=1})) < \deg_i(X)$$

by Corollary 6.2.12 and Lemma 6.2.10. We have $h \in \langle t_{i'}, t_i \rangle$ and $t_i \nmid (h|_{t_i=0})$, so

$$\mathrm{mindeg}_{i'}(h|_{t=1}) < \mathrm{mindeg}_{i'}(\min_i(h|_{t=1})) \leq \deg_{i'}(X) \; .$$

By construction,

$$\mathrm{mindeg}_{i'}(h|_{t=1}) \leq \mathrm{mindeg}_{i'}(h'|_{t=1}) \; ;$$
$$\mathrm{mindeg}_{i}(h|_{t=1}) < \mathrm{mindeg}_{i}(h'|_{t=1}) \; .$$

Write $h^{(i')} \coloneqq (X - \min_i(h|_{t=1}))_{\{i',i\}} \in I_{\{i',i\}}$ and $h^{(i)} \coloneqq (X + h'|_{t=1})_{\{i',i\}} \in I_{\{i',i\}}$. Then we conclude from the above (in)equalities and Lemma 6.2.18 that

$$\mathrm{mindeg}_{i'}(h) < \mathrm{mindeg}_{i'}(h^{(i')}) \; ;$$
$$\mathrm{mindeg}_{i}(h) \leq \mathrm{mindeg}_{i}(h^{(i')}) \; ;$$
$$\mathrm{mindeg}_{i'}(h) \leq \mathrm{mindeg}_{i'}(h^{(i)}) \; ;$$
$$\mathrm{mindeg}_{i}(h) < \mathrm{mindeg}_{i}(h^{(i)}) \; .$$

Hence, there are $k_{i'}, l_i \in \mathbb{Z}_{>0}$ and $k_i, l_{i'} \in \mathbb{Z}_{\geq 0}$ such that

$$H^{(i')} := h^{(i')} t_{i'}^{k_{i'}} t_i^{k_i} \text{ and } H^{(i)} := h^{(i)} t_{i'}^{l_{i'}} t_i^{l_i}$$

are $\deg_j$-homogeneous of degree $\operatorname{mindeg}_j(h)$ for $j = i', i$. In particular, $h = H^{(i)} - H^{(i')}$ and $H^{(i')} \in \langle t_{i'} \rangle$ as well as $H^{(i)} \in \langle t_i \rangle$ by choice of $k_{i'}$ and l_i, respectively. Hence, $h \in (I_{\{i',i\}} \cap \langle t_{i'} \rangle) + (I_{\{i',i\}} \cap \langle t_i \rangle) = I'_{i',i}$ as claimed. □

Proposition 6.2.27. *Algorithm 6.2.3 terminates after finitely many steps.*

Proof. We use the notation from the algorithm. Let S be the set of generators at some iteration of the `while`-loop. By Lemma 6.2.2, there exist $\tilde{f}_1, \ldots, \tilde{f}_s \in \mathbb{C}[V]^{[G,G]}$, such that S satisfies $(*A \cup \{i', i\})$ if and only if for all $j \in \{1, \ldots, s\}$ we have $(*A \cup \{i', i\}, \tilde{f}_j)$.

Let $f \in \{\tilde{f}_1, \ldots, \tilde{f}_s\}$. As $(*A \cup \{i'\})$ and $(*A \cup \{i\})$ are satisfied, there are preimages $h_1, h_2 \in \alpha^{-1}(f)$ with $\operatorname{mindeg}_j(h_1) = \operatorname{mindeg}_j(h_2) = v_j(f)$ for all $j \in A$ and $\operatorname{mindeg}_{i'}(h_1) = v_{i'}(f)$ as well as $\operatorname{mindeg}_i(h_2) = v_i(f)$. If $\operatorname{mindeg}_i(h_1) = v_i(f)$ or $\operatorname{mindeg}_{i'}(h_2) = v_{i'}(f)$, then $(*A \cup \{i', i\}, f)$ is satisfied and we are done. Hence, we assume $\operatorname{mindeg}_i(h_1) < v_i(f)$ and $\operatorname{mindeg}_{i'}(h_2) < v_{i'}(f)$. Then $(h_1 - h_2)_{\{i',i\}} \in I_{i',i}$ by Lemma 6.2.21 and there are $h' \in I'_{i',i}$ and $a_j \in \mathbb{C}[X_1, \ldots, X_{|S|}, t_{i'}, t_i]$ with

$$(h_1 - h_2)_{\{i',i\}} = h' + \sum_{j=1}^{l} a_j h_j \,.$$

In the next iteration of the `while`-loop we then have $(h_1 - h_2)_{\{i',i\}} \in I'_{i',i}$ with the 'new' ideal $I'_{i',i}$ by Lemma 6.2.26. Now we are in the situation of Lemma 6.2.22 and by iterating this process as in Proposition 6.2.23 we eventually obtain the desired preimage after finitely many iterations. □

6.3. Subalgorithms and relations of the Cox ring

6.3.1. Computing $\min_i I$

We present an algorithm for the computation of $\min_i I$ following [Yam18, p. 632].

We state the task in a more general setting. Let $\mathbb{C}[X_1, \ldots, X_k]$ be graded by a weight vector $\mathbf{w} = (w_1, \ldots, w_k) \in \mathbb{Z}^k_{\geq 0}$, so $\deg_{\mathbf{w}}(X_i) := w_i$. Given $I \trianglelefteq \mathbb{C}[X_1, \ldots, X_k]$, we want to compute the ideal $\min_{\mathbf{w}} I := \langle \min_{\mathbf{w}}(f) \mid f \in I \rangle$, where $\min_{\mathbf{w}}(f)$ is the graded component of $f \in \mathbb{C}[X_1, \ldots, X_k]$ of minimal degree. This can be done using Algorithm 6.3.1.

Lemma 6.3.1. *Algorithm 6.3.1 is correct.*

Proof. We use the notation from the algorithm. Let $J := \langle f_1|_{t=0}, \ldots, f_s|_{t=0} \rangle$. We see that $J = \langle f|_{t=0} \mid f \in I^h \rangle$. Now let $f \in I$. Then $f^h \in I^h$ and we have $f^h = \min_{\mathbf{w}}(f) + tf'$ for some $f' \in \mathbb{C}[X_1, \ldots, X_k, t]$. Hence $\min_{\mathbf{w}}(f) = f^h|_{t=0}$. □

Algorithm 6.3.1. $\min_{\mathbf{w}} I$

Input : An ideal $I \trianglelefteq \mathbb{C}[X_1, \dots, X_k]$;
$\mathbf{w} \in \mathbb{Z}^k_{\geq 0}$
Output: $\min_{\mathbf{w}} I$

Add a variable t to $\mathbb{C}[X_1, \dots, X_k]$ of degree $\deg_{\mathbf{w}}(t) = -1$
Compute the homogenization $I^h = \langle f_1, \dots, f_s \rangle$ of I with respect to $\deg_{\mathbf{w}}$ and t
return $\langle f_1|_{t=0}, \dots, f_s|_{t=0} \rangle$

6.3.2. Computing $\underline{\min}_i J_i$

We present the algorithm in [Yam18, p. 632] for the computation of $\underline{\min}_i J_i$ in a more general setting.

Let $\mathbb{C}[X_1, \dots, X_k]$ be a polynomial ring with an action by a finite abelian group A, which induces a grading by the characters $A^\vee$ of A. For an ideal $I \trianglelefteq \mathbb{C}[X_1, \dots, X_k]$, we want to compute the ideal

$$I^A := \langle f \in I \mid f \text{ is } A^\vee\text{-homogeneous} \rangle \ .$$

For this, we first assume that $A = \langle \gamma \rangle$ is cyclic generated by some $\gamma \in A$ of order $r \in \mathbb{Z}_{>0}$. For a fixed r-th root of unity $\zeta_r \in \mathbb{C}^\times$, we then have integers $0 \leq a_i < r$ such that γ acts on X_i via $\gamma.X_i = \zeta_r^{a_i} X_i$, for $1 \leq i \leq k$. Hence, we can endow $\mathbb{C}[X_1, \dots, X_k]$ with a further grading $\deg_\gamma$ defined by $\deg_\gamma(X_i) := a_i$.

Remark 6.3.2. Note that there is a subtle difference between the properties of being homogeneous with respect to the $\deg_\gamma$-grading and being homogeneous with respect to the action of γ. If f is homogeneous in the second sense, then we require only that $\deg_\gamma(m) - \deg_\gamma(m') \equiv 0 \bmod r$ for any pair of terms m and m' of f. Hence, homogeneity with respect to $\deg_\gamma$ implies homogeneity with respect to the action of γ, but in general not vice versa. See also the related statement in Lemma 5.3.1. ‖

Algorithm 6.3.2 computes the ideal $I^{\langle \gamma \rangle}$. For an arbitrary finite abelian group A, we iteratively use Algorithm 6.3.2 with a generating system $\gamma_1, \dots, \gamma_s \in A$ to compute the ideal I^A.

Lemma 6.3.3. *Algorithm 6.3.2 is correct.*

Proof. We use the notation from the algorithm. We have to show that

$$I^{\langle \gamma \rangle} = (I^h + \langle t^r - 1 \rangle) \cap \mathbb{C}[X_1, \dots, X_k] \ .$$

Let $f \in I^{\langle \gamma \rangle}$. We may assume that f is $\langle \gamma \rangle^\vee$-homogeneous and hence we have $\deg_\gamma(m) \equiv \deg_\gamma(m') \bmod r$ for every pair m and m' of terms of f, see Remark 6.3.2. We have $f^h = \sum_{i=1}^s f_i t^{ir}$ with $f = f_1 + \dots + f_s$ and

$$f_i t^{ir} - f_i = f_i \sum_{j=1}^{i} t^{(j-1)r}(t^r - 1) \in \langle t^r - 1 \rangle \ ,$$

Algorithm 6.3.2. $I^{\langle\gamma\rangle}$

Input : An ideal $I \trianglelefteq \mathbb{C}[X_1, \ldots, X_k]$;
a cyclic finite abelian group $\langle\gamma\rangle$ acting on $\mathbb{C}[X_1, \ldots, X_k]$
Output: $I^{\langle\gamma\rangle} = \langle f \in I \mid f \text{ is } \langle\gamma\rangle^\vee\text{-homogeneous}\rangle$

Compute the order r of γ
Fix an r-th root of unity $\zeta_r \in \mathbb{C}^\times$
Determine the weights $a_1, \ldots, a_k$ giving rise to the grading $\deg_\gamma$
Add a variable t to $\mathbb{C}[X_1, \ldots, X_k]$ of degree $\deg_\gamma(t) = 1$
Compute the homogenization I^h of I with respect to $\deg_\gamma$ and t
return $(I^h + \langle t^r - 1\rangle) \cap \mathbb{C}[X_1, \ldots, X_k]$

---

so $f^h - f \in \langle t^r - 1\rangle$. Therefore $f \in I^h + \langle t^r - 1\rangle$.

Conversely, let $f \in (I^h + \langle t^r - 1\rangle) \cap \mathbb{C}[X_1, \ldots, X_k]$. Then there are polynomials $g \in I^h$ and $h \in \mathbb{C}[X_1, \ldots, X_k, t]$ with

$$f = g + h(t^r - 1) \ .$$

Write

$$g = \sum_{i=0}^{s} g_i \text{ and } h = \sum_{i=0}^{s} h_i$$

where $g_i, h_i \in \mathbb{C}[X_1, \ldots, X_k, t]$ are homogeneous with $\deg_\gamma(g_i) = i$ or $g_i = 0$ for all $0 \leq i \leq s$ and the same for the degrees of the h_i. By our assumption on f, we have $g + h(t^r - 1) \in \mathbb{C}[X_1, \ldots, X_k]$. Homogeneity with respect to $\deg_\gamma$ gives

$$g_i + h_{i-r}t^r - h_i \in \mathbb{C}[X_1, \ldots, X_k] \ ,$$

where $h_i \coloneqq 0$ for $i < 0$. For any $0 \leq i_0 < r$, we set

$$f_{i_0} \coloneqq \sum_{i=0}^{\lfloor s/r \rfloor} \big(g_{ir+i_0} + h_{ir+i_0}(t^r - 1)\big) \in \mathbb{C}[X_1, \ldots, X_k] \ .$$

We have

$$f_{i_0} = f_{i_0}|_{t=1} = \sum_{i=0}^{\lfloor t/r \rfloor} g_{ir+i_0}|_{t=1} \ .$$

Since I^h is $\deg_\gamma$-homogeneous, we have $g_i \in I^h$ for any i and therefore $g_i|_{t=1} \in I$ by [KR05, Proposition 4.3.5]. Hence $f_{i_0} \in I$. We extend the action of γ by setting $\gamma.t \coloneqq \zeta_r t$, where ζ_r is the r-th root of unity fixed at the beginning of Algorithm 6.3.2. Then we have $\gamma.f_{i_0} = \zeta_r^{i_0} f_{i_0}$, so $f_0, \ldots, f_{r-1}$ are exactly the $\langle\gamma\rangle^\vee$-homogeneous components of f. In conclusion, we have $f_{i_0} \in I^{\langle\gamma\rangle}$, so $f \in I^{\langle\gamma\rangle}$. □

Algorithm 6.3.3. Relations of the Cox ring

Input : $\mathrm{Ab}(G)^\vee$-homogeneous generators $f_1, \dots, f_k$ of $\mathbb{C}[V]^{[G,G]}$ fulfilling $(*\{1, \dots, m\})$
Output: Relations of the generators of $\mathcal{R}(X)$ corresponding to $f_1, \dots, f_k$

Compute the kernel I of the morphism $\mathbb{C}[X_1, \dots, X_k] \to \mathbb{C}[V]^{[G,G]},\ X_i \mapsto f_i$
Add variables $s_1, \dots, s_m$ of degree $\deg_i(s_j) = -\delta_{i,j}$ to $\mathbb{C}[X_1, \dots, X_k]$
Compute the homogenization I^h with respect to the gradings $\deg_1, \dots, \deg_m$ and the variables $s_1, \dots, s_m$
Pick generators $h_1, \dots, h_s \in I^h$ which are $\deg_i$-homogeneous, $1 \leq i \leq m$, and $\mathrm{Ab}(G)^\vee$-homogeneous
for $j = 1, \dots, s$ **do**
    Let d_i be maximal with $s_i^{d_i} \mid h_j$
    Set $\tilde{h}_j \coloneqq h_j/(s_1^{d_1} \cdots s_m^{d_m})$
    Substitute any occurrence of $s_i^{r_i}$ in $\tilde{h}_j$ by Y_i and denote the resulting polynomial by $\hat{h}_j$
end
return $\hat{h}_1, \dots, \hat{h}_s$

6.3.3. Relations of the Cox ring

Let $f_1, \dots, f_k \in \mathbb{C}[V]^{[G,G]}$ be $\mathrm{Ab}(G)^\vee$-homogeneous generators of $\mathbb{C}[V]^{[G,G]}$ fulfilling $(*\{1, \dots, m\})$. Recall that these give rise to generators

$$\tilde{f}_1 \coloneqq f_1 \prod_{i=1}^m t_i^{v_i(f_1)}, \dots, \tilde{f}_k \coloneqq f_k \prod_{i=1}^m t_i^{v_i(f_k)} \in \big(\mathbb{C}[V]^{[G,G]}\big)[t_1^{\pm}, \dots, t_m^{\pm}]$$

and

$$\tau_1 \coloneqq t_1^{-r_1}, \dots, \tau_m \coloneqq t_m^{-r_m} \in \big(\mathbb{C}[V]^{[G,G]}\big)[t_1^{\pm}, \dots, t_m^{\pm}]$$

of the Cox ring $\mathcal{R}(X)$, see Theorem 6.1.5 and Section 6.1.2. For a presentation of $\mathcal{R}(X)$ as an affine algebra, we require the ideal of relations of these generators, that is, the kernel of the map

$$\Psi : \mathbb{C}[X_1, \dots, X_k, Y_1, \dots, Y_m] \to \mathbb{C}[V]^{[G,G]}[t_1^{\pm}, \dots, t_m^{\pm}],\ X_i \mapsto \tilde{f}_i,\ Y_i \mapsto \tau_i\ .$$

For this, we may use Algorithm 6.3.3 following [Yam18, Section 7.2].

Lemma 6.3.4. *Algorithm 6.3.3 is correct.*

Proof. We use the notation from the algorithm. We first convince ourselves that the polynomials $\hat{h}_j$ do not involve the variables s_i. If s_i^d and s_i^e occur in $\hat{h}_j$, we must have $d - e \equiv 0 \bmod r_i$ by Lemmas 5.3.1 and 5.3.2. As $s_j \nmid \hat{h}_j$ by construction, it follows that $r_i \mid e$ for every s_i^e involved in $\hat{h}_j$. Hence, substituting these powers of s_i in $\hat{h}_j$ indeed results in a polynomial in $\mathbb{C}[X_1, \dots, X_k, Y_1, \dots, Y_m]$.

Further, note that $\hat{h}_1, \dots, \hat{h}_s$ form a generating set of I^h as I^h is saturated with respect to the variables $s_1, \dots, s_m$, see [KR05, Corollary 4.3.7].

We obtain an $\mathrm{Ab}(G)^\vee$-grading on $\mathbb{C}[X_1, \dots, X_k, Y_1, \dots, Y_m]$ by letting $\gamma \in \mathrm{Ab}(G)$ act on X_i by the same scalar as on f_i and by 1 on Y_i. Write $J := \ker(\Psi)$ and let $\tilde{J} = \langle \hat{h}_1, \dots, \hat{h}_s \rangle$. Both J and $\tilde{J}$ are $\mathrm{Ab}(G)^\vee$-homogeneous by construction.

We have

$$\hat{h}_i(\tilde{f}_1, \dots, \tilde{f}_k, \tau_1, \dots, \tau_m) = \tilde{h}_i(\tilde{f}_1, \dots, \tilde{f}_k, t_1^{-1}, \dots, t_m^{-1})$$

and for the dehomogenization we further have $\tilde{h}_i|_{s=1} \in I$ by [KR05, Proposition 4.3.5], hence $\tilde{h}_i(f_1, \dots, f_k, 1, \dots, 1) = 0$. Write $\tilde{h}_i|_{s=1} = \sum_j h_{i,j}$ with terms $h_{i,j}$. Then

$$\tilde{h}_i = (\tilde{h}_i|_{s=1})^h = \sum_j \Big(h_{i,j} \prod_{l=1}^m s_l^{\deg_l(h_{i,j}) - \mathrm{mindeg}_l(\tilde{h}_i)} \Big)$$

by definition of the homogenization and the construction of the $\tilde{h}_i$. Evaluating gives

$$\begin{aligned} \hat{h}_i(\tilde{f}_1, \dots, \tilde{f}_k, \tau_1, \dots \tau_m) &= \sum_j \Big(h_{i,j}(\tilde{f}_1, \dots, \tilde{f}_k) \prod_{l=1}^m t_l^{-(\deg_l(h_{i,j}) - \mathrm{mindeg}_l(\tilde{h}_i))} \Big) \\ &= \sum_j \Big(h_{i,j}(f_1, \dots, f_k) \big(\prod_{l=1}^m t_l^{\deg_l(h_{i,j})} \big) \big(\prod_{l=1}^m t_l^{\mathrm{mindeg}_l(\tilde{h}_i) - \deg_l(h_{i,j})} \big) \Big) \\ &= \tilde{h}_i(f_1, \dots, f_k, 1, \dots, 1) \prod_{l=1}^m t_l^{\mathrm{mindeg}_l(\tilde{h}_i)} = 0 \,, \end{aligned}$$

hence indeed $\hat{h}_i \in J$ and $\tilde{J} \subseteq J$.

Conversely, let $h \in J$ be $\mathrm{Ab}(G)^\vee$-homogeneous. Then $h(\tilde{f}_1, \dots, \tilde{f}_k, \tau_1, \dots, \tau_m) = 0$ and hence

$$h(f_1, \dots, f_k, 1, \dots, 1) = h(\tilde{f}_1, \dots, \tilde{f}_k, \tau_1, \dots, \tau_m)|_{t=1} = 0 \,,$$

therefore $h|_{s=1} \in I$. We have $(h|_{s=1})^h \in I^h$ and the variables s_i only occur as powers by multiples of r_i in $(h|_{s=1})^h$ by $\mathrm{Ab}(G)^\vee$-homogeneity. Let $\hat{h} \in \mathbb{C}[X_1, \dots, X_k, Y_1, \dots, Y_m]$ be the polynomial obtained from $(h|_{s=1})^h$ by substituting every occurrence of $s_i^{r_i}$ by Y_i. Then there are $e_1, \dots, e_m \in \mathbb{Z}_{\geq 0}$ with $h = Y_1^{e_1} \cdots Y_m^{e_m} \hat{h}$, see [KR05, Proposition 4.3.2]. In particular, $h \in \tilde{J}$. □

6.4. Implementation notes

We implemented Algorithm 6.2.1 in the computer algebra system OSCAR [Osc23]. In Appendix D, we give an overview of the available functionality by applying the algorithm to an example. Experiments with this implementation enabled us to formulate the statements contained in Chapter 7, although the proofs are then completely 'computer-free'. We further computed the Cox ring of a $\mathbb{Q}$-factorial terminalization of V/G, where $G = H^\circledast$ for certain complex reflection groups H, see Appendix C.

Remark 6.4.1. Unfortunately, the capabilities of the algorithm are still quite limited. For Appendix C, we attempted to compute the Cox rings corresponding to quotients by other fairly small symplectic reflection groups, for example, $G_{12}^{\circledast}$ or $\mathfrak{S}_4^{\circledast}$, but the computations did not finish over the course of several weeks. Likewise, we were not able to compute the Cox rings corresponding to symplectically irreducible groups, for example the groups from Section 4.4. It is not clear, what invariants of the input (that is, the group) give a meaningful 'input size' to estimate the runtime of the algorithm. Obvious candidates are the order of the group, its rank and the number of conjugacy classes of junior elements. However, these parameters cannot give the whole picture as $G_{12}^{\circledast}$ can be considered 'small' in all these categories: it is of order 48, rank 4 and has only one conjugacy class of symplectic reflections. A more sensible input size might be the cardinality of a minimal generating system of $\mathbb{C}[V]^{[G,G]}$ as this directly corresponds to the number of variables in the polynomial ring in which most of the computations have to be carried out; in case of $G_{12}^{\circledast}$ this polynomial ring already has 30 variables at the beginning of the algorithm.

We further expect that the performance of the algorithm is sensitive to the chosen representation of the group: for complex reflection groups there exist several such 'matrix models' and in our computations we used the ones from CHEVIE [GHL+96, Mic15] which result in rational generators of the invariant ring [MM10, Proposition 11.1]. For symplectically irreducible symplectic reflection groups, the situation is much less understood and the study of the invariant theory of these groups would certainly be worthwhile also from this computational perspective. ∥

We now provide some details regarding the implementation including established algorithms we made use of. Although we keep using the field $\mathbb{C}$ in our presentation, we never work over the complex numbers in practice. Instead, we work over an extension field of $\mathbb{Q}$ that contains all values of irreducible characters of G. Since G is a finite group, this is always a finite extension of $\mathbb{Q}$ by [Bra47, Theorem 1] and one can hence do exact computations in this field.

6.4.1. Constructive invariant theory

We require an $\mathrm{Ab}(G)^\vee$-homogeneous generating system of $\mathbb{C}[V]^{[G,G]}$ as input of Algorithm 6.2.1. Finding generators of an invariant ring by a finite group is a classical problem from invariant theory where minimal sets of such generators are called 'fundamental invariants'. There are two established algorithmic strategies for this problem, see also [DK15, Section 3.8]. The first approach starts with computing generators of a Noether normalization of $\mathbb{C}[V]^{[G,G]}$ (commonly called the 'primary invariants') via [Kem99] and then proceeds by finding generators of $\mathbb{C}[V]^{[G,G]}$ as a module over this subalgebra ('secondary invariants') via [KS99, Kin07]. The second approach is 'King's algorithm' – an algorithm that directly computes the fundamental invariants, see [Kin13].

All the referenced algorithms are available in OSCAR implemented by the author. If one is only interested in computing fundamental invariants, King's algorithm is expected to be more efficient. However, we are interested in a presentation of $\mathbb{C}[V]^{[G,G]}$ as an affine algebra. Therefore, we also require the relations of the fundamental invariants, that is, the kernel of the ring morphism

$$\mathbb{C}[X_1, \dots, X_k] \to \mathbb{C}[V]^{[G,G]},\ X_i \mapsto f_i$$

given the fundamental invariants $f_1, \dots, f_k$. For this, one can use standard algorithms to compute such kernels relying on the computation of a Gröbner basis, see [GP08, Section 1.8.10]. If the fundamental invariants are computed via primary and secondary invariants, there is also an algorithm relying only on linear algebra available, see [KS99, Section 17.5.5]. In our experiments, this turned out to be the more efficient way of obtaining a presentation of $\mathbb{C}[V]^{[G,G]}$.

Once fundamental invariants have been computed, one needs to transform these into $\mathrm{Ab}(G)^\vee$-homogeneous generators of $\mathbb{C}[V]^{[G,G]}$. For this, we use the algorithm described in [DK17, Construction 2.4] that relies on linear algebra computations in the vector spaces $\mathbb{C}[V]^{[G,G]}_d$ of invariants of a fixed degree $d \geq 0$ in the standard grading of the polynomial ring $\mathbb{C}[V]$. As the dimension of these vector spaces grows exponentially in d, but the polynomials we handle only involve very few monomials, we carried out these computations using the functionality for 'sparse linear algebra' available in OSCAR.

Again, we also need to take care of the relations of the $\mathrm{Ab}(G)^\vee$-homogeneous generators. Let $f_1, \dots, f_k \in \mathbb{C}[V]^{[G,G]}$ be fundamental invariants and let $\tilde{f}_1, \dots, \tilde{f}_k \in \mathbb{C}[V]^{[G,G]}$ be the $\mathrm{Ab}(G)^\vee$-homogeneous generators computed from the f_i. Since both sets of polynomials generate $\mathbb{C}[V]^{[G,G]}$, there are polynomials $F_1, \dots, F_k \in \mathbb{C}[X_1, \dots, X_k]$ with $f_i = F_i(\tilde{f}_1, \dots, \tilde{f}_k)$ for $1 \leq i \leq k$. In our implementation, we keep track of the necessary data to easily construct these polynomials F_i. We have a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}[X_1, \dots, X_k] & \xrightarrow{X_i \mapsto f_i} & \mathbb{C}[V]^{[G,G]} \\ {\scriptstyle X_i \mapsto F_i}\downarrow & & \| \\ \mathbb{C}[X_1, \dots, X_k] & \xrightarrow[X_i \mapsto \tilde{f}_i]{} & \mathbb{C}[V]^{[G,G]} . \end{array}$$

A polynomial $h \in \mathbb{C}[X_1, \dots, X_k]$ is a relation of $f_1, \dots, f_k$ if and only if $h(F_1, \dots, F_k)$ is a relation of $\tilde{f}_1, \dots, \tilde{f}_k$. In this way, we can compute the relations of the $\mathrm{Ab}(G)^\vee$-homogeneous generators from the fundamental invariants. The $\mathrm{Ab}(G)^\vee$-homogeneous presentation of $\mathbb{C}[V]^{[G,G]}$ computed in this way is exactly the Cox ring $\mathcal{R}(V/G)$ by Theorem 2.4.11.

6.4.2. Homogenization and Bayer's method

In numerous places in the algorithm, we need to compute the homogenization of an ideal, see Algorithms 6.3.1, 6.3.2, 6.3.3 and also implicitly in the construction of $I_{i',i}$ and $I'_{i',i}$ in Algorithm 6.2.3. We should therefore aim to have an efficient implementation for this fundamental operation at hand. We require some notions from the theory of Gröbner bases (or standard bases) and refer to [GP08, Chapter 1] for the basic definitions. In particular, if $>$ is a monomial ordering on a polynomial ring $\mathbb{C}[X_1, \dots, X_k]$, then following [GP08, Definition 1.2.1] we allow that $X_i < 1$, which is occasionally excluded in the definition of monomial orderings. We also adopt the terminology of only speaking of 'Gröbner bases' if the ordering is global and of 'standard bases' in general, see [GP08, Definition 1.6.1].

For the following discussion, let $\mathbb{C}[X_1, \dots, X_k]$ be graded by an integral weight vector $\mathbf{w} = (w_1, \dots, w_k) \in \mathbb{Z}^k_{\geq 0}$ via $\deg_{\mathbf{w}}(X_i) = w_i$ and let $I \trianglelefteq \mathbb{C}[X_1, \dots, X_k]$ be an ideal. We add an additional variable t to $\mathbb{C}[X_1, \dots, X_k]$ and want to compute the homogenization I^h of I with respect to the grading $\deg_{\mathbf{w}}$ and the variable t. Depending on whether we want to homogenize 'positively' or 'negatively', we set $\deg_{\mathbf{w}}(t) := 1$ or $\deg_{\mathbf{w}}(t) := -1$, respectively.

If $w_i \neq 0$ for all $1 \leq i \leq k$, there is a quite simple method to compute I^h that only involves the computation of a Gröbner basis of I with respect to the weighted degree ordering defined by $\mathbf{w}$, see [GP08, Exercise 1.7.5]. However, although the weights in our application are non-negative, they might in general be zero, so they do not give rise to a total ordering on the set of monomials and we cannot make use of this approach.

A more general idea for the computation of I^h is to homogenize a set of generators of I resulting in an ideal $\tilde{I}$ and then to compute the saturation of $\tilde{I}$ with respect to t as one has $I^h = \tilde{I} : \langle t \rangle^\infty$ by [KR05, Corollary 4.3.8]. However, a naive computation of this saturation potentially involves several expensive Gröbner basis computations as one iteratively computes ideal quotients until the result stabilizes, see [GP08, Section 1.8.9].

In our implementation, we use a more specialized approach for the saturation based on 'Bayer's method', see [Bay82, p. 120], [Sti05, Proposition 5.1.11]. In a nutshell, this means that we compute a standard basis of $\tilde{I}$ with respect to a tailored monomial ordering and then only need to divide the elements of this basis by t, see Proposition 6.4.4 for the precise statement. The core idea of Bayer's method is the following observation. Let $f \in \mathbb{C}[X_1, \dots, X_k]$ be a homogeneous polynomial with respect to the standard grading. Then the leading term $\mathrm{LT}(f)$ with respect to the degree reverse lexicographical ordering [GP08, Example 1.2.8 (1) (ii)] is divisible by X_k if and only if f is divisible by X_k. We now translate this to the grading $\deg_{\mathbf{w}}$ by considering a certain matrix ordering.

We assume that $\mathbf{w} \neq 0$, so after reordering the variables we may assume $w_k \neq 0$. If we have $\mathbf{w} = 0$, then any polynomial and hence any ideal is homogeneous, so the computation of the homogenization is trivial. Let

$$M := \begin{pmatrix} w_1 & \cdots & \cdots & w_k & \deg_{\mathbf{w}}(t) \\ 0 & \cdots & \cdots & 0 & -1 \\ 1 & \ddots & & \vdots & 0 \\ & \ddots & \ddots & \vdots & \vdots \\ 0 & & 1 & 0 & 0 \end{pmatrix} \in \mathbb{Z}^{(k+1)\times(k+1)} .$$

This is a matrix of full rank as $w_k \neq 0$ and hence induces a monomial ordering $>_M$ on the monomials of $\mathbb{C}[X_1, \dots, X_k, t]$ by multiplying the exponent vectors by M (from the left) and then using the lexicographic ordering in $\mathbb{Z}^{k+1}$, see [GP08, Remark 1.2.7].

One directly convinces oneself of the following facts.

Lemma 6.4.2. *Let $f_1 = X_1^{a_1} \cdots X_k^{a_k} t^{a_{k+1}}$ and $f_2 = X_1^{b_1} \cdots X_k^{b_k} t^{b_{k+1}}$ be two monomials. Then we have:*

(a) $X_i >_M 1$ *for* $1 \leq i \leq k$*;*

(b) if $\deg_{\mathbf{w}}(t) = 1$*, then* $t >_M 1$*, and* $1 >_M t$ *otherwise;*

(c) if $\deg_{\mathbf{w}}(f_1) > \deg_{\mathbf{w}}(f_2)$*, then* $f_1 >_M f_2$*;*

(d) if $\deg_{\mathbf{w}}(f_1) = \deg_{\mathbf{w}}(f_2)$ *and* $f_1 >_M f_2$*, then* $a_{k+1} \leq b_{k+1}$*.*

Point (d) is the direct generalization of the above mentioned 'core idea' for Bayer's method: for a $\deg_{\mathbf{w}}$-homogeneous polynomial f, we have $t \mid \mathrm{LT}_{>_M}(f)$ if and only if $t \mid f$, where $\mathrm{LT}_{>_M}$ is the leading term with respect to $>_M$. It follows from point (a) that $>_M$ is global with respect to the variables $X_1, \ldots, X_k$ and this also extends to the variable t, if $\deg_{\mathbf{w}}(t) = 1$, that is, if we homogenize positively, by point (b). However, if $\deg_{\mathbf{w}}(t) = -1$, then $t <_M 1$, so the ordering $>_M$ is local with respect to t. This second case is more challenging: in order to speak about a standard basis of $\tilde{I}$ with respect to $>_M$, we have to consider the extension of $\tilde{I}$ to the localization

$$\mathbb{C}[X_1, \ldots, X_k, t]_{>_M} \coloneqq S^{-1}\mathbb{C}[X_1, \ldots, X_k, t] \ ,$$

where

$$S \coloneqq \{u \in \mathbb{C}[X_1, \ldots, X_k, t] \setminus \{0\} \mid \mathrm{LT}_{>_M}(u) \text{ is constant}\} \ .$$

See [GP08, Section 1.5] for details. If $\deg_{\mathbf{w}}(t) = -1$, we have $S = \{h \in \mathbb{C}[t] \mid h(0) \neq 0\}$, so we may identify

$$\mathbb{C}[X_1, \ldots, X_k, t]_{>_M} = \big(\mathbb{C}[t]_{(t)}\big)[X_1, \ldots, X_k] \ .$$

In case $\deg_{\mathbf{w}}(t) = 1$, the ordering $>_M$ is global by Lemma 6.4.2, so we have $S = \mathbb{C}^\times$ by [GP08, p. 39].

Lemma 6.4.3. *With the above notation, let* $J \trianglelefteq \mathbb{C}[X_1, \ldots, X_k, t]$ *be a* $\deg_{\mathbf{w}}$*-homogeneous ideal. Then we have*

$$J = (S^{-1}J) \cap \mathbb{C}[X_1, \ldots, X_k, t] \ .$$

Proof. If $\deg_{\mathbf{w}}(t) = 1$, there is nothing to show, so let $\deg_{\mathbf{w}}(t) = -1$. For a polynomial $f \in (S^{-1}J) \cap \mathbb{C}[X_1, \ldots, X_k, t]$, there is $u \in S$ with $uf \in J$. Writing $u = \sum_j a_j t^j$ with $a_j \in \mathbb{C}$, we have $a_j t^j f \in J$ for all j since J is homogeneous. But $a_0 \neq 0$ by assumption, so $f \in J$. □

Proposition 6.4.4 (Bayer's method)**.** *Let* $J \trianglelefteq \mathbb{C}[X_1, \ldots, X_k, t]$ *be a* $\deg_{\mathbf{w}}$*-homogeneous ideal and let* $g_1, \ldots, g_s \in \mathbb{C}[X_1, \ldots, X_k, t]$ *be a standard basis of* $S^{-1}J$ *with respect to the monomial ordering* $>_M$*. Write* $g_i = t^{m_i} g_i'$ *for* $g_i' \in \mathbb{C}[X_1, \ldots, X_k, t]$ *with* $t \nmid g_i'$*. Then* $g_1', \ldots, g_s'$ *generate* $J : \langle t \rangle^\infty$*.*

Proof. By Lemma 6.4.3, we have $g_i \in J$ and hence $g_i' \in J : \langle t \rangle^\infty$.

Let $g' \in J : \langle t \rangle^\infty$. Then there is $m \geq 0$ with $g \coloneqq t^m g' \in J$, hence $g \in S^{-1}J$. So there is $i \in \{1, \ldots, s\}$ such that $\mathrm{LT}_{>_M}(g_i) \mid \mathrm{LT}_{>_M}(g)$ and therefore

$$\mathrm{LT}_{>_M}(g_i') \mid t^m \, \mathrm{LT}_{>_M}(g') \ .$$

We may assume that the g_i are $\deg_{\mathbf{w}}$-homogeneous, as replacing g_i by their homogeneous parts does not change the standard basis property. Therefore, $t \nmid \mathrm{LT}_{>_M}(g_i')$ by choice of g_i' and the properties of $>_M$. So, $\mathrm{LT}_{>_M}(g_i') \mid \mathrm{LT}_{>_M}(g')$ and this proves that $g_1', \ldots, g_s'$ is a standard basis of $S^{-1}\big(J : \langle t\rangle^\infty\big)$, so in particular a generating system [GP08, Lemma 1.6.7 (3)]. But then $g_1', \ldots, g_s'$ generate $J : \langle t\rangle^\infty$ by Lemma 6.4.3 again. □

In conclusion, to compute $I^h = \tilde{I} : \langle t\rangle^\infty$ we need to compute a standard basis for $\tilde{I}$ with respect to $>_M$ and divide the elements by t. This only involves the computation of one standard basis and proved to be quite efficient in practice compared with the computation of the saturation via iterated quotients.

6.4.3. Further comments

We close this section with a few minor comments regarding an efficient implementation. Throughout, we assume that we are given a set of generators $f_1, \ldots, f_k \in \mathbb{C}[V]^{[G,G]}$ together with the morphism

$$\alpha : \mathbb{C}[X_1, \ldots, X_k] \to \mathbb{C}[V]^{[G,G]},\ X_i \mapsto f_i$$

with kernel $I \coloneqq \ker(\alpha)$.

An application of Nakayama's Lemma

A major problem of the algorithm is the potentially large number of variables in the ring $\mathbb{C}[X_1, \ldots, X_k]$ which makes the already notorious Gröbner basis computations even slower. We therefore should aim to keep the number of variables as small as possible.

At the start of the algorithm, we have a generating system of $\mathbb{C}[V]^{[G,G]}$ of minimal cardinality (the fundamental invariants), as explained in Section 6.4.1. However, this cardinality might already be quite large, see for example Chapter 7, where we treat groups G of order $2d$, $d \in \mathbb{Z}_{\geq 3}$, for which there are already $2d+6$ fundamental invariants for the ring $\mathbb{C}[V]^{[G,G]}$. In Algorithms 6.2.2 and 6.2.3, we potentially add more generators, hence variables, but at least in the first algorithm, we can make sure that the number of new generators is minimal by the following trick.

Recall that in Algorithm 6.2.2 we have the ideals $\min_i I$ and $\underline{\min}_i J_i$ and compute polynomials $h_1, \ldots, h_t \in \underline{\min}_i J_i$ such that $\underline{\min}_i J_i = \min_i I + \langle h_1, \ldots, h_t\rangle$. These polynomials $h_1, \ldots, h_t$ then become the new generators in the next iteration of the algorithm and we hence want to find such a set of polynomials of minimal cardinality. For this, notice that the invariant ring $\mathbb{C}[V]^{[G,G]}$ inherits the standard grading of polynomials from the ring $\mathbb{C}[V]$. In particular, we may assume that the generators $f_1, \ldots, f_k$ of $\mathbb{C}[V]^{[G,G]}$ are homogeneous with respect to the standard grading as well. We can then endow the ring $\mathbb{C}[X_1, \ldots, X_k]$ with a grading by positive weights by setting $\deg(X_i) \coloneqq \deg(f_i)$. Notice that the ideals $\min_i I$ and $\underline{\min}_i J_i$ are homogeneous with respect to this grading. Therefore, we can consider $\underline{\min}_i J_i$ as a graded module of the positively graded algebra $\mathbb{C}[X_1, \ldots, X_k]/\min_i I$. In this case, the 'graded Nakayama Lemma' [DK15, Lemma 3.7.1] gives an easy criterion to find a minimal system of generators of $\underline{\min}_i J_i$ as a module in this sense. Most importantly, this criterion gives rise to an algorithm for use in practice, which is available in OSCAR.

Unfortunately, it is not clear how we can use this idea in Algorithm 6.2.3 as there appears to be no grading by positive weights for which the ideals $I_{i',i}$ and $I'_{i',i}$ are homogeneous.

Updating I

During the run of the algorithm, the set of generators $f_1, \dots, f_k$ and the ideal I change whenever we add a further generator f_{k+1}. Fortunately, we do not need to recompute I completely as a kernel of the map

$$\alpha^+ : \mathbb{C}[X_1, \dots, X_{k+1}] \to \mathbb{C}[V]^{[G,G]},\ X_i \mapsto f_i \ ,$$

but we have the following easy way of updating I. Let $h \in \mathbb{C}[X_1, \dots, X_k]$ be a polynomial with $\alpha(h) = f_{k+1}$.

Lemma 6.4.5. *With the above notation, we have* $I + \langle h - X_{k+1} \rangle = \ker(\alpha^+)$.

Proof. Clearly, $I + \langle h - X_{k+1} \rangle \subseteq \ker(\alpha^+)$. For $f \in \ker(\alpha^+)$, we can write

$$f - f(X_1, \dots, X_k, h) = h'(h - X_{k+1})$$

for some $h' \in \mathbb{C}[X_1, \dots, X_{k+1}]$. Hence, $f(X_1, \dots, X_k, h) \in \ker(\alpha^+)$, so

$$f(X_1, \dots, X_k, h) \in \ker(\alpha^+) \cap \mathbb{C}[X_1, \dots, X_k] = I \ . \qquad \square$$

One more optimization

We close this chapter with a small observation.

Lemma 6.4.6. *In the situation of Algorithm 6.2.3, we have*

$$I_{\{i',i\}} \cap \langle t_i \rangle = I_{\{i',i\}} \cdot \langle t_i \rangle$$

and analogously for the intersection with $\langle t_{i'} \rangle$.

Proof. Clearly, $I_{\{i',i\}} \cdot \langle t_i \rangle \subseteq I_{\{i',i\}} \cap \langle t_i \rangle$. For the reverse inclusion, notice that for $f \in I_{\{i',i\}} \cap \langle t_i \rangle$, there is $f' \in \mathbb{C}[X_1, \dots, X_k]$ with $f = f' t_i$. The ideal $I_{\{i',i\}}$ is saturated with respect to $\langle t_i \rangle$ as a homogenization by [KR05, Corollary 4.3.7]. Hence, $f' \in I_{\{i',i\}}$ and therefore $f \in I_{\{i',i\}} \cdot \langle t_i \rangle$. $\square$

This minor point allows us to replace the computation of an intersection of two ideals, which in general requires the computation of a Gröbner basis, see [GP08, Section 1.8.7], by a product of ideals, which can be computed by simply multiplying the generators.

7. A $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/D_d$

We apply the algorithm presented in Chapter 6 to the dihedral groups D_d of order $2d$ with $d \geq 3$ odd acting on $\mathbb{C}^4$ as symplectic reflection groups to compute generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization $X \to \mathbb{C}^4/D_d$ of the corresponding linear quotient. Using the theory from Section 2.4.3 we are able to recover X from this ring. As the proofs in this chapter are completely 'computer-free', we need to introduce new ad hoc ideas to handle the computational complexity. We explain this strategy in Section 7.1 after fixing the notation. In Section 7.2 we then construct a presentation of the Cox ring $\mathcal{R}(\mathbb{C}^4/D_d)$, from which we derive generators of $\mathcal{R}(X)$ in Section 7.3.

Although our proofs do not logically rely on computer calculations, they would not have been possible without the computer algebra system OSCAR [Osc23], which we used extensively to formulate conjectures.

7.1. Preparations

7.1.1. Notation

Let $d \geq 3$ and let $\zeta_d \in \mathbb{C}$ be a primitive d-th root of unity. Let D_d be the dihedral group of order $2d$, that is, as an abstract group, D_d is the group with the presentation $\langle s, r \mid r^d = 1, s^2 = 1, s^{-1}rs = r^{-1}\rangle$. In this chapter, we identify D_d with the symplectic reflection group $G(d,d,2)^{\circledast}$, so D_d is generated by

$$s := \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \text{ and } r := \begin{pmatrix} \zeta_d & & & \\ & \zeta_d^{-1} & & \\ & & \zeta_d^{-1} & \\ & & & \zeta_d \end{pmatrix}$$

as a subgroup of $\mathrm{Sp}_4(\mathbb{C})$ and D_d acts on $V := \mathbb{C}^4$ by symplectic reflections. Notice that the group D_3 is isomorphic to $\mathfrak{S}_3^{\circledast}$ as symplectic reflection groups.

An easy calculation shows that the commutator subgroup of D_d is generated by r if d is odd and by r^2 if d is even. Let

$$\delta := \begin{cases} d, & d \text{ odd}, \\ \frac{d}{2}, & d \text{ even}, \end{cases}$$

and write $H_\delta := [D_d, D_d]$. If d is odd, there is one conjugacy class of symplectic reflections (or junior elements) in D_d for which we choose s as a representative. If d is even, there are two such classes and we choose s and rs as representatives.

7.1.2. Strategy

As a large part of this chapter is taken up by elementary but tedious by-hand computations, we provide an outline of our strategy and highlight the main results. In principle, we use Algorithm 6.2.1 to compute generators for the Cox ring $\mathcal{R}(X)$ of a $\mathbb{Q}$-factorial terminalization $X \to V/D_d$ of V/D_d. For this, we first construct the Cox ring $\mathcal{R}(V/D_d)$, so by Theorem 2.4.11 a presentation of $\mathbb{C}[V]^{H_\delta}$ with $\mathrm{Ab}(D_d)^\vee$-homogeneous generators $f_1, \dots, f_k$, see Proposition 7.2.6.

For our main result regarding the Cox ring $\mathcal{R}(X)$, we only proceed for d odd from now on, but see also Conjecture 7.3.10. Let $\Phi_s : \mathbb{C}[V] \to \mathbb{C}[V]$ be the automorphism induced by the change into an eigenbasis of s, where s is a representative for the single conjugacy class of symplectic reflections in D_d. Let v_s be the monomial valuation on $\mathbb{C}(V)$ corresponding to s and, for $f \in \mathbb{C}[V]^{H_\delta}$, write $\min_s(\Phi_s(f))$ for the sum of the terms of minimal valuation. We have the maps

$$\alpha : \mathbb{C}[X_1, \dots, X_k] \to \mathbb{C}[V]^{H_\delta},\ X_i \mapsto f_i$$

and

$$\beta_s : \mathbb{C}[X_1, \dots, X_k] \to \mathbb{C}[V],\ X_i \mapsto \min_s(\Phi_s(f_i))$$

given by the constructed generators $f_1, \dots, f_k$ of $\mathbb{C}[V]^{H_\delta}$ and these give rise to the ideals $I := \ker(\alpha)$ and $J_s := \ker(\beta_s)$. In terms of Algorithm 6.2.1, we have to compare the ideals $\min_s I$ and $\underline{\min}_s J_s$ defined as in Section 6.2. In fact, we prove that with our chosen system of generators we have $\min_s I = \underline{\min}_s J_s$ and the algorithm terminates after one step, see Proposition 7.3.8. However, we are unable to compute the ideals $\min_s I$ and $\underline{\min}_s J_s$ by hand and instead compare their Hilbert series.

Recall that for a graded module $M = \bigoplus_{d=0}^{\infty} M_d$ with M_d finite dimensional for all d, we have the *Hilbert series* (or *Poincaré series*)

$$H(M, t) := \sum_{d=0}^{\infty} \dim(M_d) t^d \in \mathbb{Z}[[t]]$$

encoding the dimensions of the graded components of M. We can choose the generators $f_1, \dots, f_k$ to be homogeneous with respect to the standard grading on the polynomial ring $\mathbb{C}[V]$ as the action of H_δ is linear. The ideals $\min_s I$ and $\underline{\min}_s J_s$ are homogeneous with respect to the weighted grading where we put the variable X_i in degree $\deg(f_i)$. The main idea is to show that the Hilbert series $H(\min_s I, t)$ and $H(\underline{\min}_s J_s, t)$ with respect to this grading coincide. As $\min_s I \subseteq \underline{\min}_s J_s$ by Corollary 6.2.12, this implies the equality of the ideals.

For this, we prove the general fact that $H(I, t) = H(\min_s I, t)$, see Proposition 7.1.2. Further, we see that the polynomials $\min_s(\Phi_s(f_i))$ are in our situation $\mathrm{Ab}(D_d)^\vee$-homogeneous, hence $J_s = \underline{\min}_s J_s$, see Lemma 7.3.1. We conclude

$$H(\min_s I, t) = H(\underline{\min}_s J_s, t) \iff H(I, t) = H(J_s, t)\ .$$

Write $S := \mathbb{C}[X_1, \dots, X_k]$ and let $T \leq \mathbb{C}[V]$ be the $\mathbb{C}$-algebra generated by $\min_s(\Phi_s(f_i))$, $1 \leq i \leq k$, so

$$\mathbb{C}[V]^{H_\delta} \cong S/I \text{ and } T \cong S/J_s\ .$$

Both $\mathbb{C}[V]^{H_\delta}$ and T inherit the standard grading on $\mathbb{C}[V]$ and we have

$$H(I,t) = H(J_s,t) \Longleftrightarrow H(\mathbb{C}[V]^{H_\delta},t) = H(T,t)\ ,$$

see also Lemma 7.3.3. The majority of this chapter is taken up by computing the latter two Hilbert series, see Corollaries 7.2.3 and 7.3.7. In both cases, we see that the algebras are free modules over a respective Noether normalization and after computing the generators explicitly, we can read off the Hilbert series.

Remark 7.1.1. Our argument implies that T is a Cohen–Macaulay ring. It would be interesting to see, whether this is always the case for algebras generated by minimal parts in this way. ∥

Having proved the equality $\min_s I = \underline{\min}_s J_s$, we can derive a generating system of $\mathcal{R}(X)$ from the previously computed presentation of $\mathcal{R}(V/D_d)$, see Theorem 7.3.9.

7.1.3. The Hilbert series of $\min_i I$

We state a result on the Hilbert series of ideals of the form $\min_i I$ as in Chapter 6. We do this in full generality and do not restrict to the case of dihedral groups.

Let $R = \mathbb{C}[X_1,\dots,X_k]$ be a polynomial ring, graded by $\deg_d(X_i) = d_i$ with $d_i \in \mathbb{Z}_{>0}$ and let $I \trianglelefteq R$ be homogeneous with respect to this grading. Let there be another grading on R given by $\deg_e(X_i) = e_i$ with $e_i \in \mathbb{Z}_{\geq 0}$. Let us emphasize that the second grading is in general not positive.

For a polynomial $f \in R$, write $\min_e(f)$ for the homogeneous part of f with respect to $\deg_e$ of minimal degree. Let $\min_e I := \langle \min_e(f) \mid f \in I\rangle$, which is $\deg_d$-homogeneous as I is $\deg_d$-homogeneous.

Proposition 7.1.2. *With the above notation we have $H(I,t) = H(\min_e I,t)$, where we consider the grading by $\deg_d$ for the Hilbert series.*

Proof. Assume first that there is $c \in \mathbb{Q}$ with $e_i/d_i = c$ for all i, that is, the grading by $\deg_e$ is just a 'scaling' of the grading by $\deg_d$. Then it directly follows $I = \min_e I$ and the claim is trivial.

We may hence assume that after reordering the variables we have $e_{k-1}/d_{k-1} \neq e_k/d_k$. This implies that the matrix

$$M := \begin{pmatrix} d_1 & & \cdots & d_{k-1} & d_k \\ -e_1 & & \cdots & -e_{k-1} & -e_k \\ 1 & & & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ & & 1 & 0 & 0 \end{pmatrix} \in \mathbb{Z}^{k\times k}$$

is non-singular and therefore induces a monomial ordering $>_M$ on the monomials of R, see [GP08, Remark 1.2.7].

Write $\mathrm{LM}(f)$ for the leading monomial with respect to $>_M$ of a polynomial $f \in R$ and $\mathrm{L}(A)$ for the leading ideal with respect to $>_M$ of any subset $A \trianglelefteq R$, see [GP08, Definition 1.5.5]. We show that $\mathrm{L}(I) = \mathrm{L}(\min_e I)$, which implies the claim by [KR05, Theorem 5.2.18] (notice that the result directly generalizes to the case of a general positive grading, see for example [DK15, pp. 20, 21]).

First of all, let $A := \{\min_e(f) \mid f \in I\}$ be the set of all $\deg_e$-homogeneous polynomials in $\min_e I$. Then this set must contain a Gröbner basis of $\min_e I$ with respect to $>_M$ as $\min_e I$ is a $\deg_e$-homogeneous ideal, so $\mathrm{L}(A) = \mathrm{L}(\min_e I)$.

To prove $\mathrm{L}(I) = \mathrm{L}(A)$, it suffices to show that for a $\deg_d$-homogeneous polynomial $f \in R$ we have $\mathrm{LM}(f) = \mathrm{LM}(\min_e(f))$. For this, let $\mathrm{LM}(f) = X_1^{\alpha_1} \cdots X_k^{\alpha_k}$ and let $X_1^{\beta_1} \cdots X_k^{\beta_k}$ be any other monomial of f. Then $M\underline{\alpha}^\top > M\underline{\beta}^\top$, where $>$ is the lexicographical ordering on $\mathbb{Z}^k$. We have $\sum_{i=1}^k d_i\alpha_i = \sum_{i=1}^k d_i\beta_i$ by $\deg_d$-homogeneity of f, so this implies

$$-\sum_{i=1}^k e_i\alpha_i \geq -\sum_{i=1}^k e_i\beta_i \,.$$

In other words,

$$\deg_e(X_1^{\alpha_1} \cdots X_k^{\alpha_k}) \leq \deg_e(X_1^{\beta_1} \cdots X_k^{\beta_k})$$

and hence $\mathrm{LM}(f)$ must be a monomial of $\min_e(f)$. □

7.2. The Cox ring of $\mathbb{C}^4/D_d$

We give a presentation of the Cox ring $\mathcal{R}(V/D_d)$ of V/D_d, where we use Theorem 2.4.11 to identify this ring with $\mathbb{C}[V]^{H_\delta}$ graded by $\mathrm{Ab}(D_d)^\vee$. We write $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_4]$ throughout.

7.2.1. A presentation of $\mathbb{C}[V]^{H_\delta}$

We start with constructing a presentation of the invariant ring $\mathbb{C}[V]^{H_\delta}$ leaving the grading by $\mathrm{Ab}(D_d)^\vee$ aside for the moment.

Lemma 7.2.1. *The algebra $\mathbb{C}[V]^{H_\delta}$ is generated by the polynomials*

$$\begin{aligned} &f_{12} := x_1x_2,\ f_{13} := x_1x_3,\ f_{24} := x_2x_4,\ f_{34} := x_3x_4, \\ &g_k := x_1^k x_4^{\delta-k}\ (0 \leq k \leq \delta), \\ &h_k := x_2^k x_3^{\delta-k}\ (0 \leq k \leq \delta). \end{aligned}$$

Proof. The given polynomials are clearly invariants of H_δ.

Let $f \in \mathbb{C}[V]^{H_\delta}$. Then every term of f must also be invariant under the action of H_δ as H_δ only consists of diagonal matrices. Hence, we may assume $f = \prod_{i=1}^4 x_i^{a_i}$ with $a_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, 4$. Let g be the polynomial one obtains if one divides f by f_{12} 'as often as possible', that is, $\min(a_1, a_2)$ times, and then by f_{13} as often as possible. Write $g = x_1^{b_1}x_2^{b_2}x_3^{b_3}x_4^{b_4}$ and note that g is H_δ-invariant. We now have either $b_2 = b_3 = 0$ or $b_1 = 0$. In the first case, we see that $r.g = \zeta_d^{b_1+b_4}g$ (respectively $r.g = \zeta_d^{2(b_1+b_4)}g$ if d is even), so $b_1 + b_4 \equiv 0 \bmod \delta$. Hence g is a power of one of the g_k.

In the second case, we have $b_1 = 0$ and we divide further by f_{24} and f_{34} resulting in a new invariant monomial $h = x_2^{c_2} x_3^{c_3} x_4^{c_4}$ and have either $c_2 = c_3 = 0$ or $c_4 = 0$. If $c_2 = c_3 = 0$, we see that $c_4 \equiv 0 \bmod \delta$, so h is a power of g_0. In the remaining case, we have $c_2 + c_3 \equiv 0 \bmod \delta$ and h is a power of h_k for some k. □

We now rearrange the above generators to make the construction of relations between them easier. Let $Q := g_\delta + h_\delta = x_1^\delta + x_2^\delta$ and $R := g_0 + h_0 = x_3^\delta + x_4^\delta$, so that we have a generating system given by

$$f_{12},\ f_{13},\ f_{24},\ f_{34},\ Q,\ R,\ g_k\ (1 \leq k \leq \delta),\ h_k\ (0 \leq k \leq \delta - 1).$$

It is easy to see that f_{12}, f_{34}, Q and R form a system of algebraically independent polynomials. Let $P := \mathbb{C}[f_{12}, f_{34}, Q, R]$.

Lemma 7.2.2. *The $\mathbb{C}$-algebra $\mathbb{C}[V]^{H_\delta}$ is generated as a P-module by the polynomials*

$$1,\quad f_{13}^k\ (1 \leq k \leq \delta),\quad f_{24}^k\ (1 \leq k \leq \delta - 1),\quad g_k\ (1 \leq k \leq \delta),\quad h_k\ (0 \leq k \leq \delta - 1)\ .$$

In particular, $\mathbb{C}[V]^{H_\delta}$ is finite over P.

Proof. We systematically compute that any product of two of the given polynomials is again in the P-module span of the polynomials. Iteratively applying the resulting relations then yields the claim.

Because of the equality

$$f_{13} f_{24} = f_{12} f_{34}\ ,$$

we see that products of the form $f_{13}^k f_{24}^l$ are in the P-module span of the powers of f_{13} and f_{24} respectively. The given powers of f_{13} and f_{24} are sufficient as

$$f_{13}^{\delta+1} = QRf_{13} - f_{34} Q g_1 - f_{12} R h_{\delta-1} + f_{12} f_{34} f_{24}^{\delta-1}$$

and

$$f_{24}^\delta = QR - g_\delta R - h_0 Q + f_{13}^\delta\ .$$

The products $g_k h_l$ lie in this span as well for all feasible k and l by the relation

$$g_k h_l = f_{12}^m f_{34}^{\delta-M} f_{13}^{k-m} f_{24}^{l-m}\ ,$$

where $m := \min(k, l)$ and $M := \max(k, l)$.

Let us now consider products of the form $g_k g_l$, $1 \leq k, l \leq \delta$. If $k + l \leq \delta$, we have

$$g_k g_l = (R - h_0) g_{k+l} = R g_{k+l} - f_{34}^{\delta-k-l} f_{13}^{k+l}\ ,$$

and in the other case

$$g_k g_l = (Q - h_\delta) g_{k+l-\delta} = Q g_{k+l-\delta} - f_{12}^{k+l-\delta} f_{24}^{2\delta-k-l}\ ,$$

so the product lies in the P-module span of the claimed generators in both cases. A similar argument shows the analogous claim for the products $h_k h_l$.

Fix a product of the form $f_{13}^k g_l$ for $1 \le k, l \le \delta$. If $k + l \le \delta$, we have $f_{13}^k g_l = f_{34}^k g_{k+l}$ and in the other case we obtain

$$f_{13}^k g_l = f_{34}^{\delta-l} Q f_{13}^{k+l-\delta} - f_{34}^{\delta-l} f_{12}^{k+l-\delta} h_{2\delta-k-l} \,.$$

Similar relations hold for products of the form $f_{24}^k g_l$, $f_{13}^k h_l$ and $f_{24}^k h_l$ for all feasible k and l. □

In the terminology of computational invariant theory, the module generators given in Lemma 7.2.2 are *secondary invariants* with respect to the *primary invariants* f_{12}, f_{34}, Q and R.

Corollary 7.2.3. *The Hilbert series of $\mathbb{C}[V]^{H_\delta}$ with respect to the standard grading of $\mathbb{C}[V]$ is given by*

$$H(\mathbb{C}[V]^{H_\delta}, t) = \frac{t^{2\delta} + 2\delta t^\delta + 1 + \sum_{i=1}^{\delta-1} 2t^{2i}}{(1-t^2)^2(1-t^\delta)^2} \,.$$

Proof. As f_{12}, f_{34}, Q and R are algebraically independent and $\mathbb{C}[V]^{H_\delta}$ is finite over the subalgebra generated by them, we conclude that these polynomials are a homogeneous system of parameters of $\mathbb{C}[V]^{H_\delta}$, see [DK15, Definition 2.5.6]. Using [DK15, (2.7.3)], the claim then follows directly from Lemma 7.2.2 by grouping terms of same degree together. □

Remark 7.2.4. It is possible to compute the series $H(\mathbb{C}[V]^{H_\delta}, t)$ with Molien's formula [DK15, Theorem 3.4.2] without knowing generators of $\mathbb{C}[V]^{H_\delta}$. However, we require the Hilbert series expressed in the particular form given in Corollary 7.2.3 and therefore prefer the above approach in this case. ‖

We now translate the relations showing up in the proof of Lemma 7.2.2 into relations of $\mathbb{C}[V]^{H_\delta}$ as an affine algebra.

Lemma 7.2.5. *Consider the morphism of rings*

$$\begin{aligned} \varphi : \mathbb{C}[X_{12}, X_{13}, X_{24}, X_{34}, Y_0, \dots, Y_\delta, Z_0, \dots, Z_\delta] &\to \mathbb{C}[V]^{H_\delta}, \\ X_{ij} &\mapsto f_{ij} \,, \\ Y_k &\mapsto g_k \,, \\ Z_k &\mapsto h_k \,. \end{aligned}$$

Then the kernel of φ is generated by the following polynomials:

$$X_{12}X_{34} - X_{13}X_{24}, \tag{7.2.1}$$

$$X_{12}Y_k - X_{24}Y_{k+1},\ X_{13}Y_k - X_{34}Y_{k+1},\ X_{12}Z_k - X_{13}Z_{k+1},\ X_{24}Z_k - X_{34}Z_{k+1}, \tag{7.2.2}$$

where $0 \le k \le \delta - 1$,

$$Y_k Y_{S-k} - Y_{k'} Y_{S-k'},\ Z_k Z_{S-k} - Z_{k'} Z_{S-k'}, \tag{7.2.3}$$

where $2 \le S \le 2\delta - 2,\ k := \max(0, S-\delta),\ k < k' \le \lfloor S/2 \rfloor$ *and*

$$Y_k Z_l - X_{12}^m X_{34}^{\delta-M} X_{13}^{k-m} X_{24}^{l-m}, \tag{7.2.4}$$

where $0 \le k, l \le \delta,\ m := \min(k,l),\ M := \max(k,l)$.

Proof. One directly checks that the given polynomials are indeed elements of the kernel of φ. Write I for the ideal generated by the given polynomials. To prove that $I = \ker(\varphi)$, we apply [KS99, Proposition 17.5]. This tells us that $\ker(\varphi)$ is generated by elements of the form $fg - h$ where f and g are chosen from the preimages of the module generators in Lemma 7.2.2 and h is the preimage of the representation of $\varphi(fg)$ in that module basis.

For the products of the form $X_{13}^k X_{24}^l$, it suffices to consider the products $X_{13}^k X_{24}$, respectively $X_{13} X_{24}^k$, $1 \le k \le \delta$, by [KS99, Proposition 17.5] again. From these we get the relation

$$X_{13}^k X_{24} - X_{13}^{k-1} X_{12} X_{34} = X_{13}^{k-1}(X_{13}X_{24} - X_{12}X_{34}) \in I\ ,$$

respectively

$$X_{13} X_{24}^k - X_{24}^{k-1} X_{12} X_{34} = X_{24}^{k-1}(X_{13}X_{24} - X_{12}X_{34}) \in I\ .$$

For the products $X_{13}^k Y_l$, we take the computations from the proof of Lemma 7.2.2 into account, which give us $f_{13}^k g_l = f_{34}^k g_{k+l}$ in the case $k + l \le \delta$. This gives rise to the relation

$$X_{13}^k Y_l - X_{34}^k Y_{k+l} = \sum_{i=0}^{k-1} X_{13}^{k-i-1} X_{34}^i (X_{13} Y_{l+i} - X_{34} Y_{l+i+1}) \in I\ .$$

In the case $k + l > \delta$, we obtain similarly

$$\begin{aligned}
&X_{13}^k Y_l - X_{34}^{\delta-l}(Y_\delta + Z_\delta) X_{13}^{k+l-\delta} + X_{34}^{\delta-l} X_{12}^{k+l-\delta} Z_{2\delta-k-l} \\
&= X_{13}^k Y_l - X_{34}^{\delta-l} X_{13}^{k+l-\delta} Y_\delta - X_{34}^{\delta-l}(X_{13}^{k+l-\delta} Z_\delta - X_{12}^{k+l-\delta} Z_{2\delta-k-l}) \\
&= \sum_{i=0}^{\delta-l-1} X_{13}^{k-i-1} X_{34}^i (X_{13} Y_{l+i} - X_{34} Y_{l+i+1}) \\
&\quad - X_{34}^{\delta-l} \sum_{i=0}^{k+l-\delta-1} X_{12}^{k+l-\delta-i-1} X_{13}^i (X_{12} Z_{2\delta-k-l+i} - X_{13} Z_{2\delta-k-l+i+1}) \in I\ .
\end{aligned}$$

In a similar fashion, one convinces oneself that the relations coming from the products of the form $X_{24}^k Y_l$, $X_{13}^k Z_l$ and $X_{24}^k Z_l$ are all in I.

We proceed with the products $Y_k Y_l$ for $0 \le k, l \le \delta$. Again we use the computations from the proof of Lemma 7.2.2 and treat at first the case $k + l \le \delta$. We then get the relation

$$Y_k Y_l - \varphi^{-1}(R g_{k+l} - f_{34}^{\delta-k-l} f_{13}^{k+l}) = Y_k Y_l - Y_0 Y_{k+l} - Y_{k+l} Z_0 + X_{34}^{\delta-k-l} X_{13}^{k+l} \in I\ .$$

In the case $k + l > \delta$, we obtain

$$\begin{aligned} &Y_k Y_l - \varphi^{-1}(Q g_{k+l-\delta} - f_{12}^{k+l-\delta} f_{24}^{2\delta-k-l}) \\ &= Y_k Y_l - Y_\delta Y_{k+l-\delta} - Y_{k+l-\delta} Z_\delta + X_{12}^{k+l-\delta} X_{24}^{2\delta-k-l} \in I \end{aligned}$$

and the computations for the relations coming from products of the form $Z_k Z_l$ are analogous.

This leaves us with the products $Y_k Z_l$, $0 \leq k, l \leq \delta$. Here, our previous computations give us the relations

$$Y_k Z_l - X_{12}^m X_{34}^{\delta-M} X_{13}^{k-m} X_{24}^{l-m} \in I$$

with $m := \min(k, l)$ and $M := \max(k, l)$ if $k \neq 0$ and $l \neq \delta$. So assume now $l = \delta$, but $k \neq 0$. In this case, we have

$$Y_k Z_\delta - \varphi^{-1}(g_k(Q - g_\delta)) = Y_k Z_\delta - \varphi^{-1}(f_{12}^k f_{24}^{\delta-k}) = Y_k Z_\delta - X_{12}^k X_{24}^{\delta-k} \in I \ .$$

The case $k = 0$ is treated similarly. □

7.2.2. $\mathrm{Ab}(D_d)^\vee$-homogeneous generators

We adjust the presentation of $\mathbb{C}[V]^{H_\delta}$ in Lemma 7.2.5 so that the generators are homogeneous with respect to the grading induced by the action of $\mathrm{Ab}(D_d) = D_d/H_\delta$.

In case d is odd, we have $H_\delta = \langle r \rangle$, so $\mathrm{Ab}(D_d) = \langle \overline{s} \rangle \cong C_2$, where we denote the residue class of an element $g \in D_d$ modulo $[D_d, D_d]$ by $\overline{g}$. If d is even, we have $H_\delta = \langle r^2 \rangle$, so $\mathrm{Ab}(D_d) = \langle \overline{s}, \overline{r} \rangle \cong C_2 \times C_2$. Note that the generators in Lemma 7.2.1 are already homogeneous with respect to the action of r in either case. The action of s on $\mathbb{C}[V]$ swaps the variable x_1 with x_2 and x_3 with x_4.

Let $\chi_1 : \mathrm{Ab}(D_d) \to \mathbb{C}^\times$ be the linear character defined by $\chi_1(\overline{s}) = -1$ and $\chi_1(\overline{r}) = 1$; we can see χ_1 as the determinant of the 'non-doubled' group $G(d, d, 2)$. In case d is even, let further $\chi_2 : \mathrm{Ab}(D_d) \to \mathbb{C}^\times$ be the linear character defined by $\chi_2(\overline{s}) = 1$ and $\chi_2(\overline{r}) = -1$. We have $\mathrm{Ab}(D_d)^\vee = \{1, \chi_1\}$, respectively $\mathrm{Ab}(D_d)^\vee = \{1, \chi_1, \chi_2, \chi_1\chi_2\}$, where 1 is the trivial character.

Proposition 7.2.6. *The algebra $\mathcal{R}(V/D_d) \cong \mathbb{C}[V]^{H_\delta}$ is generated by the $\mathrm{Ab}(D_d)^\vee$-homogeneous polynomials*

$p_1 := f_{12}$	*of degree* 1,
$p_2 := f_{34}$	*of degree* 1,
$p_3 := f_{13} + f_{24}$	*of degree* 1,
$p_4 := f_{13} - f_{24}$	*of degree* χ_1,
$q_k := g_k + h_k \ (0 \leq k \leq \delta)$	*of degree* 1, *respectively* χ_2,
$r_k := g_k - h_k \ (0 \leq k \leq \delta)$	*of degree* χ_1, *respectively* $\chi_1\chi_2$,

if d is odd, respectively even.

Consider the morphism of rings

$$\begin{aligned}\alpha : \mathbb{C}[U_1, \dots, U_4, V_0, \dots, V_\delta, W_0, \dots, W_\delta] &\to \mathbb{C}[V]^{H_\delta},\\ U_k &\mapsto p_k\ ,\\ V_k &\mapsto q_k\ ,\\ W_k &\mapsto r_k\ .\end{aligned}$$

Then the kernel of α is generated by the following polynomials:

$$4U_1U_2 - U_3^2 + U_4^2 \tag{7.2.5}$$

$$\begin{aligned} 2U_1V_k - U_3V_{k+1} + U_4W_{k+1}, \quad & 2U_1W_k + U_4V_{k+1} - U_3W_{k+1},\\ U_3V_k + U_4W_k - 2U_2V_{k+1}, \quad & U_4V_k + U_3W_k - 2U_2W_{k+1}, \end{aligned} \tag{7.2.6}$$

where $0 \le k \le \delta - 1$,

$$\begin{aligned} &V_kV_{S-k} + W_kW_{S-k} - V_{k'}V_{S-k'} - W_{k'}W_{S-k'},\\ &V_kW_{S-k} + W_kV_{S-k} - V_{k'}W_{S-k'} - W_{k'}V_{S-k'}, \end{aligned} \tag{7.2.7}$$

where $2 \le S \le 2\delta - 2,\ k := \max(0, S-\delta),\ k < k' \le \lfloor S/2 \rfloor$ *and*

$$\frac{1}{4}(V_k + W_k)(V_l - W_l) - U_1^mU_2^{\delta-M}\left(\frac{1}{2}U_3 + \frac{1}{2}U_4\right)^{k-m}\left(\frac{1}{2}U_3 - \frac{1}{2}U_4\right)^{l-m}, \tag{7.2.8}$$

where $0 \le k, l \le \delta,\ m := \min(k,l),\ M := \max(k,l)$.

Proof. The p_k, q_k and r_k obviously are $\mathrm{Ab}(D_d)^\vee$-homogeneous and form a system of generators. One obtains the relations by substituting the polynomials in Lemma 7.2.5 and simplifying the results by taking sums and differences, for example the first and third polynomial in (7.2.2) give rise to the first polynomial in (7.2.6) etc. □

7.3. The Cox ring of a $\mathbb{Q}$-factorial terminalization

From now on, we restrict ourselves to the case that d is odd. It follows that there is only one conjugacy class of symplectic reflections in D_d for which we may choose s as a representative.

7.3.1. Reductions

An eigenbasis of s is given by

$$\frac{1}{2}(e_1 + e_2),\ \frac{1}{2}(e_1 - e_2),\ \frac{1}{2}(e_3 + e_4),\ \frac{1}{2}(e_3 - e_4),$$

where $\{e_1, \dots, e_4\}$ is the standard basis of V. In this eigenbasis, s takes the form

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

and the change of basis into the eigenbasis induces the algebra isomorphism

$$\Phi_s : \mathbb{C}[x_1, \ldots, x_4] \to \mathbb{C}[y_1, \ldots, y_4],\ f \mapsto f(y_1 + y_2, y_1 - y_2, y_3 + y_4, y_3 - y_4)\ ,$$

where we write $\mathbb{C}[y_1, \ldots, y_4]$ for the coordinate ring of V in the eigenbasis of s for distinction.

Let v_s be the monomial valuation on $\mathbb{C}(V)$ corresponding to s and write as before $\min_s(f)$ for the minimal part of $f \in \mathbb{C}[y_1, \ldots, y_4]$. In Table 7.3.1, we give the minimal parts and valuations of the generators in Proposition 7.2.6.

The valuation v_s induces a grading on $S := \mathbb{C}[U_i, V_j, W_k]$ by weighting the variables with the valuation of their images under α as in Chapter 6. This means we have

$$\deg_s(U_1) = \deg_s(U_2) = \deg_s(U_3) = \deg_s(V_k) = 0$$

and

$$\deg_s(U_4) = \deg_s(W_k) = 1\ .$$

For $h \in S$, write $\min_s(h)$ for the $\deg_s$-homogeneous component of h of minimal $\deg_s$-degree and let $\min_s I = \langle \min_s(h) \mid h \in I \rangle$ with $I := \ker(\alpha)$.

We see from Table 7.3.1 that a $\deg_s$-homogeneous polynomial $h \in S$ is $\mathrm{Ab}(D_d)^\vee$-homogeneous, that is, the grading by $\deg_s$ is a refinement of the one by $\mathrm{Ab}(D_d)^\vee$. Let

$$\beta_s : S \to \mathbb{C}[V],\ h \mapsto \min_s(\Phi_s(\alpha(h)))\ .$$

Then β_s is by definition $\deg_s$-graded and hence $\mathrm{Ab}(D_d)^\vee$-graded. We conclude that the kernel $J_s := \ker(\beta_s)$ is $\mathrm{Ab}(D_d)^\vee$-homogeneous and therefore:

Lemma 7.3.1. *We have $J_s = \underline{\min}_s J_s$.*

Remark 7.3.2. In the situation of Chapter 6, the inclusion $\underline{\min}_i J_i \subseteq J_i$ is in general proper. This happens, for example, with the symplectic reflection group $G_6^{\circledast}$ as for this group minimal parts with respect to one grading are in general not homogeneous with respect to the other involved gradings. ‖

We claim that we have $\min_s I = \underline{\min}_s J_s$, so that the generators in Proposition 7.2.6 give rise to generators of $\mathcal{R}(X)$ by Proposition 6.2.13. However, we are not able to compute these ideals by hand, so we use the following lemma. Let T be the $\mathbb{C}$-algebra generated by the polynomials $\min_s(\Phi_s(f))$, where f runs over the generators in Proposition 7.2.6. The algebra T is graded with respect to the standard grading of polynomials on $\mathbb{C}[x_1, \ldots, x_4]$ as it is generated by homogeneous polynomials. We endow S with a further grading deg by setting $\deg(U_i) = 2$ and $\deg(V_i) = \deg(W_i) = \delta$. This turns α and β_s into graded morphisms with respect to the grading deg on S and the standard grading on $\mathbb{C}[x_1, \ldots, x_4]$. We see that I and J_s and hence $\min_s I$ and $\underline{\min}_s J_s$ are deg-homogeneous ideals. In what follows, Hilbert series of ideals in S are always with respect to the grading by deg and Hilbert series of subalgebras of $\mathbb{C}[x_1, \ldots, x_4]$ are always with respect to the standard grading.

Lemma 7.3.3. *We have $\min_s I = \underline{\min}_s J_s$ if and only if $H(\mathbb{C}[V]^{H_\delta}, t) = H(T, t)$.*

f	$\Phi_s(f)$	$\min_s(\Phi_s(f))$	$v_s(\Phi_s(f))$
$p_1 = x_1x_2$	$y_1^2 - y_2^2$	y_1^2	0
$p_2 = x_3x_4$	$y_3^2 - y_4^2$	y_3^2	0
$p_3 = x_1x_3 + x_2x_4$	$2y_1y_3 + 2y_2y_4$	$2y_1y_3$	0
$p_4 = x_1x_3 - x_2x_4$	$2y_1y_4 + 2y_2y_3$	$2y_1y_4 + 2y_2y_3$	1
$q_k = x_1^kx_4^{\delta-k} + x_2^kx_3^{\delta-k}$	$A + B$	$2y_1^ky_3^{\delta-k}$	0
$r_k = x_1^kx_4^{\delta-k} - x_2^kx_3^{\delta-k}$	$A - B$	$-2\delta y_3^{\delta-1}y_4$ if $k = 0$ $2ky_1^{k-1}y_2y_3^{\delta-k} - 2(\delta - k)y_1^ky_3^{\delta-k-1}y_4$ if $0 < k < \delta$ $2\delta y_1^{\delta-1}y_2$ if $k = \delta$	1

Table 7.3.1.: Minimal parts and valuations of the generators in Proposition 7.2.6 with $A \coloneqq (y_1 + y_2)^k(y_3 - y_4)^{\delta-k}$ and $B \coloneqq (y_1 - y_2)^k(y_3 + y_4)^{\delta-k}$.

Proof. By Corollary 6.2.12, we have $\min_s I \subseteq \underline{\min}_s J_s$, so we see that $\min_s I = \underline{\min}_s J_s$ if and only if $H(\min_s I, t) = H(\underline{\min}_s J_s, t)$. By Proposition 7.1.2, we have the equality $H(I,t) = H(\min_s I, t)$ and Lemma 7.3.1 yields $H(J_s, t) = H(\underline{\min}_s J_s)$. As α and β_s are graded morphisms, we have $H(I,t) = H(\mathbb{C}[V]^{H_\delta}, t)$ and $H(J_s, t) = H(T, t)$, giving the claim. □

7.3.2. The Hilbert series of T

To compute the Hilbert series of T, we construct generators of T as a module over a Noether normalization, so that we can read off $H(T,t)$ as in Corollary 7.2.3.

Recall that the algebra $T \leq \mathbb{C}[y_1, \dots, y_4]$ is generated by the polynomials $\min_s(\Phi_s(f))$, where f runs over the generators from Proposition 7.2.6. We label these polynomials as follows:

$$
\begin{aligned}
s_1 &:= y_1^2 \\
s_2 &:= y_3^2 \\
s_3 &:= y_1 y_3 \\
s_4 &:= y_1 y_4 + y_2 y_3 \\
t_k &:= y_1^k y_3^{\delta-k} && (0 \leq k \leq \delta) \\
u_0 &:= y_3^{\delta-1} y_4 \\
u_k &:= k y_1^{k-1} y_2 y_3^{\delta-k} - (\delta - k) y_1^k y_3^{\delta-k-1} y_4 && (1 \leq k \leq \delta - 1) \\
u_\delta &:= y_1^{\delta-1} y_2
\end{aligned}
$$

Lemma 7.3.4. *The Krull dimension of T is $\dim T = 4$.*

Proof. Since $T \leq \mathbb{C}[y_1, \dots, y_4]$, we have $\dim T \leq 4$. Consider the properly ascending chain

$$\langle 0 \rangle \subsetneq \langle y_4 \rangle \subsetneq \langle y_4, y_2 \rangle \subsetneq \langle y_4, y_2, y_1 \rangle \subsetneq \langle y_4, y_2, y_1, y_3 \rangle$$

of prime ideals in $\mathbb{C}[y_1, \dots, y_4]$. Intersecting these ideals with the subalgebra T gives an ascending chain of prime ideals in T. We need to see that all the inclusions in this chain are still proper. Indeed, we have

$$
\begin{aligned}
u_0 &\in \langle y_4 \rangle \cap T \setminus \langle 0 \rangle \\
u_\delta &\in \langle y_4, y_2 \rangle \cap T \setminus \langle y_4 \rangle \cap T \\
s_1 &\in \langle y_4, y_2, y_1 \rangle \cap T \setminus \langle y_4, y_2 \rangle \cap T \\
s_2 &\in \langle y_4, y_2, y_1, y_3 \rangle \cap T \setminus \langle y_4, y_2, y_1 \rangle \cap T\,.
\end{aligned}
$$

□

As a counterpart to Lemma 7.2.2, we have:

Lemma 7.3.5. *Let $P' := \mathbb{C}[s_1, s_4, u_0, t_0 + u_\delta]$. The algebra T is generated as a P'-module by the polynomials*

$$1,\quad s_2^k\ (1 \leq k \leq \delta - 1),\quad s_2^k s_3\ (0 \leq k \leq \delta - 1),\quad t_k\ (0 \leq k \leq \delta),\quad u_k\ (1 \leq k \leq \delta - 1)\,.$$

In particular, T is finite over P'.

Proof. We systematically compute that any product of two of the given generators is again in the P'-module span of the generators. Iteratively applying the resulting relations then yields the claim.

We see that the powers of s_2 are sufficient as

$$s_2^\delta = -s_1^{\frac{\delta-1}{2}} s_4 s_2^{\frac{\delta-1}{2}} + (t_0 + u_\delta)t_0 + u_0 t_\delta$$

and

$$s_2^\delta s_3 = (t_0 + u_\delta)^2 s_3 - s_1^{\frac{\delta-1}{2}} s_4 s_2^{\frac{\delta-1}{2}} s_3 + s_1 u_0 t_{\delta-1} - \frac{1}{\delta} s_4 (t_0 + u_\delta) t_\delta - \frac{1}{\delta} s_1 (t_0 + u_\delta) u_{\delta-1} \, .$$

Further we have

$$s_3^2 = s_1 s_2 \, ,$$

so that products $(s_2^k s_3)(s_2^l s_3)$ are in the P'-span.

Let us consider products of the form $t_k u_l$ for $0 \le k \le \delta$ and $1 \le l \le \delta - 1$. If $k + l \le \delta$, we have

$$t_k u_l = l s_1^{a_-} s_4 s_2^{d-a_+} s_3^{a_0} - \delta u_0 t_{k+l}$$

with $a_\pm := \frac{k+l}{2} \pm 1$ and $a_0 := 1$ if $k+l$ is even and $a_\pm := \frac{k+l\pm1}{2}$ and $a_0 := 0$ if $k+l$ is odd. If $k + l > \delta$, we compute

$$t_k u_l = -\delta s_1^{a_-} s_2^{\delta - a_+} s_3^{a_0} + (l - \delta) s_1^{b_-} s_4 s_2^{\delta - b_+} s_3^{b_0} + \delta (t_0 + u_\delta) t_{k+l-\delta}$$

with $a_\pm := \frac{k+l-\delta\pm1}{2}$, $a_0 := 1$, $b_\pm := \frac{k+l}{2} \pm 1$ and $b_0 := 1$ if $k+l$ is even and $a_\pm := \frac{k+l-\delta}{2}$, $a_0 := 0$, $b_\pm := \frac{k+l\pm1}{2}$ and $b_0 := 0$ if $k+l$ is odd.

For the product $t_k t_l$, we have the special case

$$t_0^2 = -s_1^{\frac{\delta-1}{2}} s_4 s_2^{\frac{\delta-1}{2}} + u_0 t_\delta + (t_0 + u_\delta) t_0$$

and for all other feasible k, l we obtain

$$t_k t_l = s_1^{a_-} s_2^{\delta - a_+} s_3^{a_0}$$

with $a_\pm := \frac{k+l}{2}$, $a_0 := 0$ if $k+l$ is even and $a_\pm := \frac{k+l\pm1}{2}$, $a_0 := 1$ if $k+l$ is odd.

We again have to distinguish several cases for products of the form $u_k u_l$ with $1 \le k, l \le \delta - 1$. If $k + l < \delta$, we have

$$u_k u_l = kl s_1^{a_-} s_4^2 s_2^{\delta - a_+} s_3^{a_0} - \delta u_0 u_{k+l}$$

with $a_\pm := \frac{k+l\pm2}{2}$ and $a_0 := 0$ if $k+l$ is even and $a_\pm := \frac{k+l\pm3}{2}$ and $a_0 := 1$ if $k+l$ is odd. If $k + l > \delta$, we have

$$\begin{aligned} u_k u_l = {} & (\delta - k)(\delta - l) s_1^{a_-} s_4^2 s_2^{\delta - a_+} s_3^{a_0} + \delta(\delta - k - l) s_1^{b_-} s_4 s_2^{\delta - b_+} s_3^{b_0} \\ & + \delta^2 u_0 t_{k+l-\delta} + \delta (t_0 + u_\delta) u_{k+l-\delta} \end{aligned}$$

with $a_\pm := \frac{k+l\pm2}{2}$, $a_0 := 0$, $b_\pm := \frac{k+l-\delta\pm1}{2}$ and $b_0 := 0$ if $k+l$ is even and $a_\pm := \frac{k+l\pm3}{2}$, $a_0 := 1$, $b_\pm := \frac{k+l-\delta\pm2}{2}$ and $b_0 := 1$ if $k+l$ is odd. In case $k+l=\delta$, we have

$$u_k u_l = kl s_1^{\frac{\delta-3}{2}} s_4^2 s_2^{\frac{\delta-3}{2}} s_3 + \delta^2 u_0 t_0 - \delta^2 u_0(t_0+u_\delta)\ .$$

We now consider products of the form $s_2^k t_l$ with $1 \le k \le \delta-1$ and $0 \le l \le \delta$. If $l-2k \ge 0$, we directly have

$$s_2^k t_l = s_1^k t_{l-2k}\ .$$

If $0 > l-2k > -\delta$, we have

$$s_2^k t_l = s_1^{a_-}(t_0+u_\delta)s_2^{k-a_+}s_3^{a_0} - \frac{2k-l}{\delta}s_1^{k-1}s_4 t_{\delta+1-2k+l} - \frac{1}{\delta}s_1^k u_{\delta-2k+l}$$

with $a_\pm := \frac{l}{2}$ and $a_0 := 0$ if l is even and $a_\pm := \frac{l\pm1}{2}$ and $a_0 := 1$ if l is odd.

If $l-2k \in \{-\delta, -\delta-1\}$, we have

$$s_2^k t_l = s_1^{a_-}(t_0+u_\delta)s_2^{k-a_+}s_3^{a_0} + s_1^{b_-}u_0 s_2^{k-b_+}s_3^{b_0} - s_1^{k-1}s_4 t_{\delta+1-2k+l}$$

with $a_\pm := \frac{l}{2}$, $a_0 := 0$, $b_\pm := \frac{\delta+l\pm1}{2}$ and $b_0 := 1$ if l is even and $a_\pm := \frac{l\pm1}{2}$, $a_0 := 1$, $b_\pm := \frac{\delta+l}{2}$ and $b_0 := 0$ if l is odd.

If $l-2k < -\delta-1$, we have

$$\begin{aligned} s_2^k t_l = {}& s_1^{a_-}(t_0+u_\delta)s_2^{k-a_+}s_3^{a_0} - s_1^{b_-}s_4(t_0+u_\delta)s_2^{k-b_+}s_3^{b_0} + s_1^{c_-}u_0 s_2^{k-c_+}s_3^{c_0} \\ & + \frac{2k-l-\delta-1}{\delta}s_1^{k-2}s_4^2 t_{2\delta+2-2k+l} + \frac{1}{\delta}s_1^{k-1}s_4 u_{2\delta+1-2k+l}\ , \end{aligned}$$

where $a_\pm := \frac{l}{2}$, $a_0 := 0$, $b_\pm := \frac{\delta\pm1+l}{2}$, $b_0 := 0$, $c_\pm := b_\pm$ and $c_0 := 1$ if l is even and $a_\pm := \frac{l\pm1}{2}$, $a_0 := 1$, $b_\pm := \frac{\delta+l}{2} \pm 1$, $b_0 := 1$, $c_\pm := \frac{\delta+l}{2}$ and $c_0 := 0$ if l is odd.

We now consider products of the form $s_2^k u_l$ with $1 \le k \le \delta-1$ and $1 \le l \le \delta-1$. If $2k-l < 0$, we have

$$s_2^k u_l = 2k s_1^{k-1}s_4 t_{l-2k+1} + s_1^k u_{l-2k}\ .$$

If $2k-l \in \{0,1\}$, then

$$s_2^k u_l = l s_4 s_1^{k-1} t_{l-2k+1} - \delta u_0 s_1^{a_-} s_2^{k-a_+} s_3^{a_0}\ ,$$

where $a_\pm := \frac{l}{2}$ and $a_0 := 0$ if l is even and $a_\pm := \frac{l\pm1}{2}$ and $a_0 := 1$ if l is odd.

If $2k-l > 1$ and $2k-l \le \delta$, we have

$$\begin{aligned} s_2^k u_l = {}& -\delta s_1^{a_-}u_0 s_2^{k-a_+}s_3^{a_0} + l s_1^{b_-}s_4(t_0+u_\delta)s_2^{k-b_+}s_3^{b_0} \\ & - \frac{l(2k-l-1)}{\delta}s_1^{k-2}s_4^2 t_{\delta-2k+l+2} - \frac{l}{\delta}s_1^{k-1}s_4 u_{\delta-2k+l+1}\ , \end{aligned}$$

where $a_\pm := \frac{l}{2}$, $a_0 := 0$, $b_\pm := \frac{l}{2} \pm 1$ and $b_0 := 1$ if l is even and $a_\pm := \frac{l\pm1}{2}$, $a_0 := 1$, $b_\pm := \frac{l\pm1}{2}$ and $b_0 := 0$ if l is odd.

If $2k - l \in \{\delta+1, \delta+2\}$, we have

$$\begin{aligned} s_2^k u_l &= -\delta s_1^{a_-} u_0 s_2^{k-a_+} s_3^{a_0} + l s_1^{b_-} s_4 (t_0 + u_\delta) s_2^{k-b_+} s_3^{b_0} \\ &\quad + l s_1^{c_-} s_4 u_0 s_2^{k-c_+} s_3^{c_0} - l s_1^{k-2} s_4^2 t_{\delta-2k+l+2} \ , \end{aligned}$$

where $a_\pm := \frac{l}{2}$, $a_0 := 0$, $b_\pm := \frac{l}{2} \pm 1$, $b_0 := 1$, $c_\pm := \frac{\delta+l\pm1}{2}$ and $c_0 := 0$ if l is even and $a_\pm := \frac{l\pm1}{2}$, $a_0 := 1$, $b_\pm := \frac{l\pm1}{2}$, $b_0 := 0$, $c_\pm := \frac{\delta+l}{2} \pm 1$ and $c_0 := 1$ if l is odd.

If $2k - l > \delta + 2$, we have

$$\begin{aligned} s_2^k u_l &= -\delta s_1^{a_-} u_0 s_2^{k-a_+} s_3^{a_0} + l s_1^{b_-} s_4 (t_0 + u_\delta) s_2^{k-b_+} s_3^{b_0} \\ &\quad + l s_1^{c_-} s_4 u_0 s_2^{k-c_+} s_3^{c_0} - l s_4^2 (t_0 + u_\delta) s_1^{d_-} s_2^{k-d_+} s_3^{d_0} \\ &\quad + \frac{l(2k-l-\delta-2)}{\delta} s_1^{k-3} s_4^3 t_{2\delta-2k+l+3} + \frac{l}{\delta} s_1^{k-2} s_4^2 u_{2\delta-2k+l+2} \ , \end{aligned}$$

where $a_\pm := \frac{l}{2}$, $a_0 := 0$, $b_\pm := \frac{l}{2} \pm 1$, $b_0 := 1$, $c_\pm := \frac{\delta+l\pm1}{2}$, $c_0 := 0$, $d_\pm := \frac{\delta+l\pm3}{2}$ and $d_0 := 1$ if l is even and $a_\pm := \frac{l\pm1}{2}$, $a_0 := 1$, $b_\pm := \frac{l\pm1}{2}$, $b_0 := 0$, $c_\pm := \frac{\delta+l}{2} \pm 1$, $c_0 := 1$, $d_\pm := \frac{\delta+l\pm2}{2}$ and $d_0 := 0$ if l is odd.

The products of the form $s_2^k s_3 t_l$ and $s_2^k s_3 u_l$ for $k \geq 1$ follow by the above relations together with $s_3^2 = s_1 s_2$. Finally, we have $s_3 t_l = s_1 t_{l-1}$ for $1 \leq l \leq \delta$ and $s_3 t_0 = (t_0 + u_\delta) s_3 - \frac{1}{\delta} s_4 t_\delta - \frac{1}{\delta} s_1 u_{\delta-1}$ as well as $s_3 u_l = s_4 t_l + s_1 u_{l-1}$ for $2 \leq l \leq \delta - 1$ and $s_3 u_1 = s_4 t_1 - \delta s_1 u_0$. □

The above lemma allows us to conclude that P' is a Noether normalization of T and that furthermore T is a Cohen–Macaulay ring.

Corollary 7.3.6. *The polynomials $s_1, s_4, u_0, t_0 + u_\delta$ form a regular sequence for T.*

Proof. By Lemma 7.3.5, the extension $P' \leq T$ is finite. Since the number of polynomials in the sequence coincides with the dimension of T by Lemma 7.3.4, we conclude that $s_1, s_4, u_0, t_0 + u_\delta$ are algebraically independent and hence they form a homogeneous system of parameters for the positively graded algebra T, see [DK15, Definition 2.5.6].

The algebraic independence of the generators of P' also implies that T is a free P'-module as the module generators given in Lemma 7.3.5 are linearly independent over P'. Hence [DK15, Proposition 2.6.3] says that T is Cohen–Macaulay and, equivalently, that every homogeneous system of parameters is T-regular. □

Corollary 7.3.7. *The Hilbert series of T is*

$$H(T,t) = \frac{t^{2\delta} + 2\delta t^\delta + 1 + \sum_{i=1}^{\delta-1} 2t^{2i}}{(1-t^2)^2(1-t^\delta)^2} \ .$$

Proof. As $s_1, s_4, u_0, t_0 + u_\delta$ is a regular sequence for the P'-module T, we have that

$$H(T/\langle s_1, s_4, u_0, t_0 + u_\delta \rangle T, t) = (1-t^2)^2(1-t^\delta)^2 H(T,t) \ ,$$

by [KR05, Corollary 5.2.17]. Further, we can directly determine the Hilbert series of $T/\langle s_1, s_4, u_0, t_0 + u_\delta\rangle T$ by counting the degrees of the generators in Lemma 7.3.5. From this we obtain

$$H(T/\langle s_1, s_4, u_0, t_0 + u_\delta\rangle T, t) = t^{2\delta} + 2\delta t^\delta + 1 + \sum_{i=1}^{\delta-1} 2t^{2i}$$

and we conclude

$$H(T, t) = \frac{t^{2\delta} + 2\delta t^\delta + 1 + \sum_{i=1}^{\delta-1} 2t^{2i}}{(1-t^2)^2(1-t^\delta)^2} .$$

□

7.3.3. Conclusion

As a reward for the tedious computations in the previous section, we can now derive the announced generating set of the Cox ring $\mathcal{R}(X)$ of a $\mathbb{Q}$-factorial terminalization $X \to V/D_d$.

Proposition 7.3.8. *We have* $\min_s I = \underline{\min}_s J_s$.

Proof. Follows from Corollaries 7.2.3 and 7.3.7 together with Lemma 7.3.3. □

Theorem 7.3.9. *Let* $d \in \mathbb{Z}_{\geq 3}$ *be odd and let* $D_d = G(d, d, 2)^{\circledast} \leq \mathrm{Sp}_4(\mathbb{C})$ *be the dihedral group generated by*

$$s := \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \text{ and } r := \begin{pmatrix} \zeta_d & & & \\ & \zeta_d^{-1} & & \\ & & \zeta_d^{-1} & \\ & & & \zeta_d \end{pmatrix},$$

where ζ_d *is a primitive* d*-th root of unity. Let* $X \to \mathbb{C}^4/D_d$ *be a* $\mathbb{Q}$*-factorial terminalization of the linear quotient* $\mathbb{C}^4/D_d$*. Then the Cox ring* $\mathcal{R}(X)$ *identified with a subalgebra of* $\mathbb{C}[x_1, \ldots, x_4][t^{\pm}]$ *is generated by:*

$$\begin{aligned}
&x_1x_2,\ x_3x_4,\ x_1x_3 + x_2x_4,\ (x_1x_3 - x_2x_4)t, \\
&x_1^k x_4^{d-k} + x_2^k x_3^{d-k} && (0 \leq k \leq d), \\
&(x_1^k x_4^{d-k} - x_2^k x_3^{d-k})t && (0 \leq k \leq d), \\
&t^{-2}.
\end{aligned}$$

Proof. Follows from Proposition 6.2.13 together with Proposition 7.3.8. □

Conjecture 7.3.10. Keep the notation of Theorem 7.3.9, but let d be even and write $\delta := \frac{d}{2}$. We conjecture that $\mathcal{R}(X)$ identified with a subalgebra of $\mathbb{C}[x_1, \ldots, x_4][t_1^{\pm}, t_2^{\pm}]$ is generated by:

$$\begin{aligned}
&x_1x_2,\ x_3x_4,\ x_1x_3 + x_2x_4,\ (x_1x_3 - x_2x_4)t_1t_2, \\
&(x_1^k x_4^{\delta-k} + x_2^k x_3^{\delta-k})t_1 && (0 \leq k \leq \delta), \\
&(x_1^k x_4^{\delta-k} - x_2^k x_3^{\delta-k})t_2 && (0 \leq k \leq \delta), \\
&t_1^{-2},\ t_2^{-2},
\end{aligned}$$

where t_1 corresponds to the symplectic reflection rs and t_2 to s.

Remark 7.3.11. The challenge in proving Conjecture 7.3.10 is that there are two conjugacy classes of symplectic reflections. We see that Proposition 7.3.8 holds in the same fashion for groups D_d with d even, but we would have to prove the analogous statement for the second conjugacy class represented by rs. Having done so, we still need to run the second phase of Algorithm 6.2.1 to combine both results.

We confirmed Conjecture 7.3.10 using a computer for $d \leq 36$. The computation for $d = 36$ alone took several days. ‖

Remark 7.3.12. Unfortunately, we are not able to give relations of the generators in Theorem 7.3.9 or Conjecture 7.3.10 to complete a presentation of $\mathcal{R}(X)$ as an affine algebra. For a fixed value of d, one can compute these relations with a computer using Algorithm 6.3.3, but doing so by hand for arbitrary d appears not to be feasible; one would need to compute the appropriate homogenization of the relations given in Proposition 7.2.6. ‖

7.3.4. Constructing the $\mathbb{Q}$-factorial terminalization

Recall from Section 2.4.3 that we can construct a $\mathbb{Q}$-factorial terminalization $X' \to V/D_d$ of V/D_d as $\operatorname{Proj} S(D)$ with the positively graded algebra

$$S(D) := \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \Gamma(X, \mathcal{O}_X(kD))$$

and D a movable divisor on X. In other words, we consider the Veronese subalgebra, see [ADHL15, Definition 1.1.2.3], of $\mathcal{R}(X)$ with respect to the monoid

$$\{[kD] \mid k \geq 0\} \leq \operatorname{Cl}(X) .$$

The linear quotient V/D_d with d odd admits a unique $\mathbb{Q}$-factorial terminalization by [BST18, Proposition 7.2] and we can choose any non-trivial element of $\operatorname{Cl}(X) \cong \mathbb{Z}$ to construct it. Hence we obtain:

Corollary 7.3.13. *With the notation of Theorem 7.3.9, let* $A \leq (\mathbb{C}[x_1, \dots, x_4])[t]$ *be the* $\mathbb{C}$*-algebra generated by*

$$\begin{aligned}
&x_1x_2,\ x_3x_4,\ x_1x_3 + x_2x_4,\ (x_1x_3 - x_2x_4)t, \\
&x_1^k x_4^{d-k} + x_2^k x_3^{d-k} && (0 \leq k \leq d), \\
&(x_1^k x_4^{d-k} - x_2^k x_3^{d-k})t && (0 \leq k \leq d).
\end{aligned}$$

Then the $\mathbb{Q}$*-factorial terminalization* $X \to \mathbb{C}^4/D_d$ *is isomorphic to* $\operatorname{Proj} A \to \mathbb{C}^4/D_d$*, where for the* Proj*-construction we endow* A *with a* $\mathbb{Z}$*-grading via the degree of the variable* t*.*

Remark 7.3.14. In [BBF+23], a symplectic partial resolution of $\mathbb{C}^4/D_d$ is constructed via an explicit blowing-up operation. The authors do not mention this, but at least in the case where d is odd, this partial resolution must in fact be a $\mathbb{Q}$-factorial terminalization.

Indeed, as the class group $\mathrm{Cl}(X)$ of a $\mathbb{Q}$-factorial terminalization is free of rank 1, the GIT fan of the Mori dream space X is the subdivision of the real line $\mathbb{R}$ into the three cones consisting of the origin and the positive and negative half-lines. The GIT quotient of $\operatorname{Spec}\mathcal{R}(X)$ corresponding to the origin is $\mathbb{C}^4/D_d$ and the quotients corresponding to the half-lines are both isomorphic to X by the uniqueness result in [BST18, Proposition 7.2]. The symplectic partial resolution constructed in [BBF+23] must be one of these GIT quotients and is hence isomorphic to X. ‖

Remark 7.3.15. For Corollary 7.3.13, we could simply find generators for the algebra A by choosing all generators of $\mathcal{R}(X)$ of non-negative degree. It is not clear, how to do this in general, that is, if $X \to V/G$ is the $\mathbb{Q}$-factorial terminalization of a linear quotient by an arbitrary finite group $G \leq \mathrm{SL}(V)$. One knows that the Veronese algebra of interest is again finitely generated as $\mathcal{R}(X)$ is finitely generated, see [ADHL15, Proposition 1.1.2.4], but it is not clear how to construct generators starting from a generating set of $\mathcal{R}(X)$. ‖

Appendix: Computational data

A. Explicit results on parabolic subgroups of symplectically primitive groups

We state the explicit results required for Lemma 3.1.8 and Theorem 4.3.1, by listing (up to conjugacy) all the maximal parabolic subgroups one finds for the symplectically and complex primitive symplectic reflection groups of rank at least 6 in Section A.2. The results of this appendix are already published in [BST23].

We outline how these groups were computed. Given a symplectically primitive symplectic reflection group, one computes the conjugacy classes of all subgroups using the computer algebra systems GAP [Gap22] or Magma [BCP97] with the command `ConjugacyClassesSubgroups` or `Subgroups` respectively. One then checks which of these subgroups are parabolic by determining their fixed space using basic linear algebra and then the stabilizer of the fixed space using the command `Stabilizer` in either GAP or Magma; if this stabilizer coincides with the group, one has found a parabolic subgroup. Let H be one of the parabolic subgroups. One can compute all symplectic reflections contained in H by computing the conjugacy classes of H (using `ConjugacyClasses` in either GAP or Magma) and checking whether the given representative is a symplectic reflection. Finally, one checks whether H is generated by the conjugacy classes of symplectic reflections determined in this way. As in Corollary 3.1.9, it is enough to consider the maximal parabolic subgroups: if $v, w \in V$, with $G_w \leq G_v$, then it suffices to check, by induction on the rank, that G_v is generated by symplectic reflections.

Identifying a parabolic subgroup with a group in Cohen's classification is an easy but tedious task using the classification and linear algebra. As the matrices generating the parabolic subgroups tend to become quite large, we do not do this in detail here; Section A.1 serves as an example for these computations.

Throughout, $i := \sqrt{-1}$ is the imaginary unit. Magma and GAP files with the necessary code to generate the symplectically primitive symplectic reflection groups can be found on the author's github page.(1)

A.1. An example: the group $W(S_1)$

As an illustration, we show that there is a parabolic subgroup H of $W(S_1)$ which is isomorphic as a symplectic reflection group to $G(3,3,3)^{\circledast}$.

The necessary computer calculations were carried out and cross-checked using the software package Hecke [FHHJ17] and the computer algebra systems GAP [Gap22] and

(1) `https://github.com/joschmitt/Parabolics`

Magma [BCP97].

The group

The group $W(S_1)$ is a subgroup of $\mathrm{Sp}_8(\mathbb{C})$ of order $2^8 \cdot 3^3 = 6912$. Like all complex primitive groups, it is given in [Coh80, Table II] by a root system. Cohen lists 36 root lines for the group. However, four are enough to generate a group of the correct order. A choice of root lines are

$$
\begin{aligned}
&(1, i, 0, 0, 0, 0, 1, -i), && (1-i, 1-i, 0, 0, 0, 0, 0, 0),\\
&(1-i, 0, 1-i, 0, 0, 0, 0, 0), && (1-i, 0, 0, 1-i, 0, 0, 0, 0).
\end{aligned}
$$

Note that these are the 'complexified' versions of the vectors over the quaternions given in [Coh80]. The group $W(S_1) \leq \mathrm{Sp}_8(\mathbb{C})$ is generated by the symplectic reflection matrices

$$
M_1 := \frac{1}{2}\begin{pmatrix}
1 & i & & & & & -1 & -i\\
-i & 1 & & & & & -i & 1\\
 & & 1 & i & 1 & i & & \\
 & & -i & 1 & i & -1 & & \\
 & & 1 & -i & 1 & -i & & \\
 & & -i & -1 & i & 1 & & \\
-1 & i & & & & & 1 & -i\\
i & 1 & & & & & i & 1
\end{pmatrix}, \qquad
M_2 := \begin{pmatrix}
 & -1 & & & & & & \\
-1 & & & & & & & \\
 & & 1 & & & & & \\
 & & & 1 & & & & \\
 & & & & & -1 & & \\
 & & & & -1 & & & \\
 & & & & & & 1 & \\
 & & & & & & & 1
\end{pmatrix},
$$

$$
M_3 := \begin{pmatrix}
 & & -1 & & & & & \\
 & 1 & & & & & & \\
-1 & & & & & & & \\
 & & & 1 & & & & \\
 & & & & & & -1 & \\
 & & & & & 1 & & \\
 & & & & -1 & & & \\
 & & & & & & & 1
\end{pmatrix}, \qquad
M_4 := \begin{pmatrix}
 & & & -1 & & & & \\
 & 1 & & & & & & \\
 & & 1 & & & & & \\
-1 & & & & & & & \\
 & & & & & & & -1\\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & -1 & & &
\end{pmatrix},
$$

through these root lines.

The parabolic subgroup

Let $v := (0, 0, 0, 1, -\alpha, \alpha, -\alpha, \alpha + 1)^\top \in \mathbb{C}^8$, where $\alpha := \frac{1}{2}(i - 1)$. Let $H \leq W(S_1)$ be the stabilizer of v. Using the command `Stabilizer` in either GAP or Magma one can compute this group:

$$
H = \langle M_2, M_3 M_1 M_3, M_4 M_1 M_4 \rangle .
$$

The space $V^H \subseteq \mathbb{C}^8$ of vectors fixed by H is generated by v and

$$
(1, -1, 1, 0, \alpha + 1, -\alpha - 1, \alpha + 1, 3\alpha)^\top .
$$

The H-invariant complement W of V^H has a basis given by the columns $w_1, \ldots, w_6 \in \mathbb{C}^8$ of the matrix

$$
\begin{pmatrix}
\zeta^2+\zeta+1 & \zeta^3+\zeta^2-\zeta-2 & \zeta & -\zeta^3+\zeta^2+\zeta-2 & \zeta^2-\zeta+1 & -\zeta^3+\zeta\\
-\zeta^3-\zeta^2+\zeta+2 & -\zeta^2-\zeta-1 & -\zeta & -\zeta^2+\zeta-1 & \zeta^3-\zeta^2-\zeta+2 & \zeta^3-\zeta\\
-\zeta^3-2\zeta^2+1 & -\zeta^3-2\zeta^2+1 & \zeta & \zeta^3-2\zeta^2+1 & \zeta^3-2\zeta^2+1 & -\zeta^3+\zeta\\
0 & 0 & -2\zeta^3+\zeta & 0 & 0 & \zeta^3+\zeta\\
-\zeta^2-\zeta+1 & -\zeta^3+\zeta^2+\zeta & \zeta^3+\zeta^2 & \zeta^3-\zeta^2-\zeta & \zeta^2+\zeta-1 & -\zeta^3+\zeta^2-1\\
\zeta^3-\zeta^2-\zeta & \zeta^2+\zeta-1 & -\zeta^3-\zeta^2 & -\zeta^2-\zeta+1 & -\zeta^3+\zeta^2+\zeta & \zeta^3-\zeta^2+1\\
\zeta^3-1 & \zeta^3-1 & \zeta^3+\zeta^2 & -\zeta^3+1 & -\zeta^3+1 & -\zeta^3+\zeta^2-1\\
0 & 0 & -\zeta^3-\zeta^2+2\zeta+2 & 0 & 0 & -\zeta^3-\zeta^2+2\zeta-1
\end{pmatrix},
$$

where $\zeta \in \mathbb{C}$ is a primitive 12-th root of unity such that $\zeta^3 = i$.

By changing the basis from $\mathbb{C}^8$ to $W \oplus V^H$ and restricting to W we may identify H with the subgroup H_W of $\mathrm{Sp}(W)$ generated by the matrices

$$\begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & 1 & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & & & & 1 \end{pmatrix}, \begin{pmatrix} & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \end{pmatrix}, \begin{pmatrix} & & \zeta^2-1 & & & \\ & 1 & & & & \\ -\zeta^2 & & & & & \\ & & & & & -\zeta^2 \\ & & & & 1 & \\ & & & \zeta^2-1 & & \end{pmatrix}.$$

The basis of W was chosen so that the symplectic form on W is, up to a constant, given by the matrix

$$\begin{pmatrix} & I_3 \\ -I_3 & \end{pmatrix}.$$

One can see directly that H_W leaves the subspace $\langle w_1, w_2, w_3 \rangle$ invariant and that this subspace is Lagrangian. Hence, H_W is a complex reducible, but symplectically irreducible, group coming from a complex reflection group in $\mathrm{GL}(\langle w_1, w_2, w_3 \rangle)$. Since the complex reflection group has rank 3 and order 54 it must be conjugate to $G(3,3,3)$ in the classification [ST54].

A.2. Maximal parabolic subgroups

We list the maximal parabolic subgroups up to conjugacy.

$W(Q)$. The group $W(Q)$ is a subgroup of $\mathrm{Sp}_6(\mathbb{C})$ of order $2^6 \cdot 3^3 \cdot 7 = 12{,}096$. It is generated by the symplectic reflections corresponding to the root lines

$$(2,0,0,0,0,0)\,, \qquad \frac{1}{2}(2i, 2i, -i+1, 0, 0, i\sqrt{5}-1)\,,$$
$$\frac{1}{2}(2i, 2, i+1, 0, 0, i+\sqrt{5})\,, \qquad \frac{1}{2}(2, 2i, i+1, 0, 0, i+\sqrt{5}).$$

The maximal parabolic subgroups are each conjugate to $H_1 := G(3,3,2)^{\circledast}$ or $H_2 := G(4,2,2)^{\circledast}$. They stabilize the following vectors:

	v
H_1	$(1,0,0,\alpha,\beta,0)$
H_2	$(1,0,1,\alpha,2\beta,\alpha)$

where $\alpha := \frac{1}{6}(i\sqrt{5} + i - \sqrt{5} + 1)$ and $\beta := \frac{1}{3}(-i + \sqrt{5})$.

$W(R)$. The group $W(R)$ is a subgroup of $\mathrm{Sp}_6(\mathbb{C})$ of order $2^8 \cdot 3^3 \cdot 5^2 \cdot 7 = 1{,}209{,}600$. It is generated by the symplectic reflections corresponding to the root lines

$$(2,0,0,0,0,0)\,, \qquad \frac{1}{2}(i+1, i-1, 0, -i\sqrt{5}+1, -i-\sqrt{5}, 0)\,,$$
$$\frac{1}{2}(0, 0, i+1, -i, 1, i+\sqrt{5})\,, \qquad \frac{1}{2}(0, 2i, i+\sqrt{5}, 2, 0, -i-1)\,.$$

The maximal parabolic subgroups are conjugate to $H_1 \coloneqq G(3,3,2)^{\circledast}$, $H_2 \coloneqq G(5,5,2)^{\circledast}$ and $H_3 \coloneqq G(\mathsf{D}_2, \mathsf{C}_2, 1)$. They stabilize the following vectors:

	v
H_1	$(0, 1, \sqrt{5}-1, \sqrt{5}-1, \frac{1}{2}(i\sqrt{5}-i+\sqrt{5}-3, 0)$
H_2	$(0, 1, \frac{1}{2}(i\sqrt{5}+i-2), \frac{1}{2}(i\sqrt{5}+i+\sqrt{5}+1), 1, \frac{1}{2}(-2i+\sqrt{5}+1))$
H_3	$(0, 1, \frac{1}{2}(\sqrt{5}+3), \frac{1}{2}(-i\sqrt{5}-i+\sqrt{5}+1), 1, \frac{1}{2}(-i\sqrt{5}-3i))$

$W(S_1)$. The group $W(S_1)$ is a subgroup of $\mathrm{Sp}_8(\mathbb{C})$ of order $2^8 \cdot 3^3 = 6912$. It is generated by the symplectic reflections corresponding to the root lines

$$(1, i, 0, 0, 0, 0, 1, -i)\,, \qquad (-i+1, -i+1, 0, 0, 0, 0, 0, 0)\,,$$
$$(-i+1, 0, -i+1, 0, 0, 0, 0, 0)\,, \qquad (-i+1, 0, 0, -i+1, 0, 0, 0, 0)\,.$$

The maximal parabolic subgroups are conjugate to $H_1 \coloneqq \mathsf{C}_2 \times \mathsf{C}_2 \times \mathsf{C}_2$, $H_2 \coloneqq G(2,2,3)^{\circledast}$ and $H_3 \coloneqq G(3,3,3)^{\circledast}$. They stabilize the following vectors:

	v
H_1	$(1, 0, 0, -1, 0, 0, 0, 0)$
H_2	$(0, 1, i, 0, 0, 0, 0, 0)$
H_2	$(1, i, i, -1, 0, 0, 0, 0)$
H_2	$(0, 0, 1, 0, 0, 0, 0, 0)$
H_2	$(0, 1, 0, 0, 0, 0, 1, 0)$
H_3	$(0, 0, 0, 1, \frac{1}{2}(1-i), \frac{1}{2}(i-1), \frac{1}{2}(1-i), \frac{1}{2}(i+1))$

Note that multiple occurrences of H_2 in the above table mean that there are distinct maximal parabolic subgroups which are conjugate in $\mathrm{GL}_8(\mathbb{C})$, but not in $W(S_1)$.

$W(S_2)$. The group $W(S_2)$ is a subgroup of $\mathrm{Sp}_8(\mathbb{C})$ of order $2^{10} \cdot 3^4 = 82{,}944$. It is generated by the symplectic reflections corresponding to the root lines

$$(1, i, 0, 0, 0, 0, 1, -i)\,, \qquad (-i+1, -i+1, 0, 0, 0, 0, 0, 0)\,,$$
$$(-i+1, 0, -i+1, 0, 0, 0, 0, 0)\,, \qquad (2, 0, 0, 0, 0, 0, 0, 0)\,.$$

The maximal parabolic subgroups are conjugate to $H_1 \coloneqq \mathsf{C}_2 \times G(3,3,2)^{\circledast}$, $H_2 \coloneqq G(2,2,3)^{\circledast}$, $H_3 \coloneqq G(2,1,3)^{\circledast}$, $H_4 \coloneqq G(3,3,3)^{\circledast}$ and $H_5 \coloneqq G(4,4,3)^{\circledast}$. They stabilize the following vectors:

	v
H_1	$(1, -1, -1, 0, 0, 0, 0, 0)$
H_2	$(1, -1, 0, 0, 0, 0, i-1, 0)$
H_3	$(0, 1, 0, 0, 0, 0, 0, 0)$
H_4	$(1, 0, 1, 0, -1, i-1, -i, 0)$
H_5	$(1, -i, 0, 0, 0, 0, 0, 0)$

$W(S_3)$. The group $W(S_3)$ is a subgroup of $\mathrm{Sp}_8(\mathbb{C})$ of order $2^{13} \cdot 3^4 \cdot 5 = 3{,}317{,}760$. It is generated by the symplectic reflections corresponding to the root lines

$$(1, i, 0, 0, 0, 0, 1, -i)\ , \qquad (-i+1, -i+1, 0, 0, 0, 0, 0, 0)\ ,$$
$$(-i+1, 0, -i+1, 0, 0, 0, 0, 0)\ , \qquad (2, 0, 0, 0, 0, 0, 0, 0)\ ,$$
$$(-i+1, 0, 0, 0, 0, -i+1, 0, 0)\ .$$

The maximal parabolic subgroups are conjugate to $H_1 \coloneqq \mathsf{C}_2 \times G(3,3,2)^{\circledast}$, $H_2 \coloneqq G(2,2,3)^{\circledast}$, $H_3 \coloneqq G(3,3,3)^{\circledast}$ and $H_4 \coloneqq G_3(\mathsf{D}_2, \mathsf{C}_2)$. They stabilize the following vectors:

	v
H_1	$(1, -i, 0, 0, 0, 0, 3, i)$
H_2	$(0, 0, 2, 0, -1, i, i, 1)$
H_3	$(0, 1, 0, -1, -1, i-1, i, 0)$
H_4	$(1, 0, 0, 1, 0, 0, 0, 0)$

$W(T)$. The group $W(T)$ is a subgroup of $\mathrm{Sp}_8(\mathbb{C})$ of order $2^8 \cdot 3^4 \cdot 5^3 = 2{,}592{,}000$. It is generated by the symplectic reflections corresponding to the root lines

$$(-\zeta^3+\zeta^2+1, \zeta^3-\zeta^2, -1, 0, 0, 0, 0, 0)\ , \qquad (1, 0, 0, 0, 0, 0, 0, 0)\ ,$$
$$(1, 1, 1, 1, 0, 0, 0, 0)\ , \qquad (1, i, 0, 0, 0, 0, -1, i)\ ,$$

where ζ is a primitive 10-th root of unity. The maximal parabolic subgroups are conjugate to $H_1 \coloneqq \mathsf{C}_2 \times G(3,3,2)^{\circledast}$, $H_2 \coloneqq \mathsf{C}_2 \times G(5,5,2)^{\circledast}$, $H_3 \coloneqq G(2,2,3)^{\circledast}$, $H_4 \coloneqq G(3,3,3)^{\circledast}$, $H_5 \coloneqq G_{23}^{\circledast}$ and $H_6 \coloneqq G(5,5,3)^{\circledast}$. They stabilize the following vectors:

	v
H_1	$(0, 0, 0, 0, 1, \frac{1}{2}(-\zeta^3+\zeta^2+1), -\zeta^3+\zeta^2+\frac{1}{2}, \frac{1}{2}(\zeta^3-\zeta^2))$
H_2	$(0, 0, 0, 0, 1, -\zeta^3+\zeta^2+1, -\zeta^3+\zeta^2, 2\zeta^3-2\zeta^2-2)$
H_3	$(1, 0, \zeta^3+\zeta^2+1, -3\zeta^3+3\zeta^2+4, i\zeta^3-i\zeta^2-\zeta^3+\zeta^2+1,$ $-2i\zeta^3-2i\zeta^2-2i-2\zeta^3+2\zeta^2+4, i+\zeta^3-\zeta^2, -i\zeta^3+i\zeta^2+i+1)$
H_3	$(0, 0, 0, 0, 1, -\zeta^3+\zeta^2+2, 0, \zeta^3-\zeta^2-3)$
H_4	$(1, 0, -\zeta^3+\zeta^2+1, \zeta^3-\zeta^2, -i, i\zeta^3-i\zeta^2-i+\zeta^3-\zeta^2-1,$ $i\zeta^3-i\zeta^2+1, \zeta^3-\zeta^2)$
H_5	$(0, 1, \frac{1}{5}(4i\zeta^3-4i\zeta^2-2i+3\zeta^3-3\zeta^2-4), (2i\zeta^3-2i\zeta^2-6i-\zeta^3+\zeta^2+3),$ $\frac{1}{5}(4i\zeta^3-4i\zeta^2-2i-2\zeta^3+2\zeta^2+6), (3i\zeta^3-3i\zeta^2-4i+\zeta^3-\zeta^2-3),$ $\frac{1}{5}(i\zeta^3-i\zeta^2-3i+2\zeta^3-2\zeta^2-1))$
H_6	$(0, 1, \zeta^3-\zeta^2, \zeta^3-\zeta^2+1, -2i\zeta^3+2i\zeta^2, -i\zeta^3+i\zeta^2-\zeta^3+\zeta^2+1,$ $-i\zeta^3+i\zeta^2+i-1, i-\zeta^3+\zeta^2)$

Note that there are two distinct maximal parabolic subgroups which are conjugate in $\mathrm{GL}_8(\mathbb{C})$, but not in $W(T)$.

$W(U)$. The group $W(U)$ is a subgroup of $\mathrm{Sp}_{10}(\mathbb{C})$ of order $2^{11} \cdot 3^5 \cdot 5 \cdot 11 = 27{,}371{,}520$. It is generated by the symplectic reflections corresponding to the root lines

$$\begin{aligned}&(2,0,0,0,0,0,0,0,0,0)\ ,\\&(0,2,i-1,i-1,2,0,0,-i+1,-i+1,0)\ ,\\&(0,2,i-1,-i-1,2i,0,0,i-1,-i-1,0)\ ,\\&(0,2,-i-1,i-1,0,0,0,i+1,i-1,2)\ ,\\&(2,i-1,i-1,2,0,0,-i+1,-i+1,0,0)\ .\end{aligned}$$

The maximal parabolic subgroups are conjugate to $H_1 \coloneqq \mathsf{C}_2 \times G(2,2,3)^{\circledast}$, $H_2 \coloneqq \mathsf{C}_2 \times G(3,3,3)^{\circledast}$, $H_3 \coloneqq \mathfrak{S}_5^{\circledast}$, $H_4 \coloneqq G(3,3,4)^{\circledast}$ and $H_5 \coloneqq W(S_1)$. They stabilize the following vectors:

	v
H_1	$(2,0,i-1,0,-i+3,2i+2,-2,-i-1,0,i-1)$
H_2	$(0,2,i-1,0,-i-1,-6i,0,i+1,0,-i+1)$
H_3	$(1,0,-2i+1,i+1,-i+1,-i,2i,i,-2i,0)$
H_4	$(2,0,0,-i-1,-i-1,0,-i+1,-2i,-i-1,i+1)$
H_5	$(2,0,i+1,0,i-1,0,-2i,i-1,0,i+1)$

B. Some results on the remaining groups in the classification of symplectic resolutions

We present computational evidence suggesting that the linear quotients of some of the groups for which the existence of a symplectic resolution is still open (see Section 4.4) do in fact not admit such a resolution. This is based on the following proposition.

Proposition B.1. *Let $\mathsf{H}_{\mathbf{c}}(G)$ be a symplectic reflection algebra associated to $G \leq \mathrm{Sp}(V)$. Let $g \in G$, $x \in V^g$ and $y \in V$. Then the element $g[x,y] \in \mathsf{H}_{\mathbf{c}}(G)$ lies in $\mathbb{C}[G]$ and is annihilated by all characters of finite-dimensional representations of $\mathsf{H}_{\mathbf{c}}(G)$.*

See [BS13, Proposition 1.3.1] for a proof.

This gives a criterion to check if V/G admits a symplectic resolution.

Corollary B.2. *If every character χ of G that annihilates all $g[x,y]$ for $g \in G$, $x \in V^g$ and $y \in V$ contains the regular character as a direct summand, then V/G admits a symplectic resolution.*

See [BS13] for details and an application.

We use this as follows. Given a symplectic reflection group G, we determine computationally which direct summands of the regular character of G annihilate all of the above commutator expressions. If there are none, we can conclude that V/G must admit a symplectic resolution. If we find a proper summand of the regular character, we have some evidence encouraging us that no such resolution exists. However, in the latter case we do not have a definite answer, see [BS13, Remark 1.3.2].

We now formalize and simplify the computational problem. Let $\chi_1, \dots, \chi_m$ be the irreducible characters of G. We need to find all $0 \leq z_i \leq \chi_i(1)$ such that for $\chi = \sum_i z_i \chi_i$ we have: for all $g \in G$, for all $x \in V^g$ and all $y \in V$: $\chi(g[x,y]) = 0$. Recall that in $\mathsf{H}_{\mathbf{c}}(G)$ we have the commutator relations

$$[x, y] = \sum_{s \in S(G)} \mathbf{c}(s) \omega_s(x, y) s \ ,$$

where ω_s is defined as in Section 2.3. First, we get rid of the parameters in these equations. Write

$$[x, y]_{[s]} \coloneqq \sum_{t \in [s]} \omega_t(x, y) t \ ,$$

where $[s]$ is the conjugacy class of s in G. Let $s_1, \dots, s_r$ be a system of representatives of the conjugacy classes in $S(G)$. We then have $[x, y] = \sum_i \mathbf{c}(s_i)[x, y]_{[s_i]}$. As $\chi(g[x, y])$ must vanish for generic parameters, we obtain the following system of equations since characters are linear functions. We need to find all $0 \leq z_i \leq \chi_i(1)$ such that for $\chi = \sum_i z_i \chi_i$ we have: for all s_j, for all $g \in G$, for all $x \in V^g$ and all $y \in V$: $\chi(g[x, y]_{[s_j]}) = 0$.

We can reduce the number of equations as follows. Let $g, h \in G$. For $t \in S(G)$ and $x, y \in V$, we have $\omega_{hth^{-1}}(x, y) = \omega_t(h^{-1}x, h^{-1}y)$: Indeed,

$$(\mathrm{id} - hth^{-1})(x) = x - hth^{-1}x = h(h^{-1}x - th^{-1}x) = h(\mathrm{id} - t)(h^{-1}x) \ .$$

And hence

$$\begin{aligned} \omega_{hth^{-1}}(x, y) &= \omega((\mathrm{id} - hth^{-1})(x), (\mathrm{id} - hth^{-1})(y)) \\ &= \omega(h(\mathrm{id} - t)(h^{-1}x), h(\mathrm{id} - t)(h^{-1}y)) \\ &= \omega((\mathrm{id} - t)(h^{-1}x), (\mathrm{id} - t)(h^{-1}y)) \\ &= \omega_t(h^{-1}x, h^{-1}y) \end{aligned}$$

as h leaves ω invariant. Therefore we compute:

$$\begin{aligned} hgh^{-1}[x, y]_{[s_i]} &= \sum_{t \in [s_i]} \omega_t(x, y) hgh^{-1} t = \sum_{t \in [s_i]} \omega_{hth^{-1}}(x, y) hgh^{-1} hth^{-1} \\ &= \sum_{t \in [s_i]} \omega_t(h^{-1}x, h^{-1}y) hgth^{-1} \ . \end{aligned}$$

Assume now that there is a character χ of G such that for all $x \in V^g$, $y \in V$ and $i \in \{1, \dots, r\}$ we have $\chi(g[x, y]_{[s_i]}) = 0$. Let $x' \in V^{hgh^{-1}}$, $y' \in V$ and $j \in \{1, \dots, r\}$ be arbitrary. Then

$$\begin{aligned} \chi(hgh^{-1}[x', y']_{[s_j]}) &= \chi\big(\sum_{t \in [s_j]} \omega_t(h^{-1}x', h^{-1}y') hgth^{-1}\big) \\ &= \chi(g[h^{-1}x', h^{-1}y']_{[s_j]}) = 0 \end{aligned}$$

Group	number of integer points	Group	number of integer points
$W(O_1)$	1424	$E(\mathsf{O}_4)$	174
$W(O_2)$	149,347,520	$E(\mathsf{O}_8)$	6090
$W(O_3)$	5,078,468	$E(\mathsf{OT}_2)$	154
$W(P_1)$	75,810	$E(\mathsf{OT}_4)$	672
$W(P_2)$	59,019,794	$E(\mathsf{OT}_6)$	8334
$E(\mathsf{T}_6)$	302	$E(\mathsf{I}_4)$	6,245,746
$E(\mathsf{T}_{12})$	19,024		

Table B.1.: Number of integer points in the polyhedra arising from some of the groups in Section 4.4

as χ is a class function and $h^{-1}x' \in V^g$ and $h^{-1}y' \in V$. Hence, it suffices to consider representatives of conjugacy classes in G.

We arrive at the following system of equations. Let $s_1, \dots, s_r$ be a system of representatives of the conjugacy classes in $S(G)$, let $g_1, \dots, g_s$ be a system of representatives of the conjugacy classes in G and let $y_1, \dots, y_n$ be a basis of V. For any $k \in \{1, \dots, s\}$, let $x_1^k, \dots, x_{n_k}^k$ be a basis of V^{g_k}. Write

$$A_{i,j,k,l,m} \coloneqq \chi_i(g_k[x_l^k, y_m]_{[s_j]}) \; .$$

By the above discussion and the linearity of characters we need to find all $z \in \mathbb{Z}^r$ with $0 \leq z_i \leq \chi_i(1)$ such that for all $1 \leq j \leq r$, $1 \leq k \leq s$, $1 \leq l \leq n_k$, $1 \leq m \leq n$ we have

$$\sum_{i=1}^{r} z_i A_{i,j,k,l,m} = 0 \; .$$

Fixing any ordering of the tuples (j, k, l, m) we write the entries $A_{i,j,k,l,m}$ in the i-th row of a matrix A. Then we have to search for all integer vectors $z \in \mathbb{Z}^r$ with

$$zA = 0 \text{ and } 0 \leq z_i \leq \chi_i(1) \; . \tag{B.1}$$

Note that the entries of A live in a cyclotomic extension K of $\mathbb{Q}$. By considering K as a $\mathbb{Q}$-vector space we can transform A into a rational matrix, say B. Set

$$c \coloneqq \begin{pmatrix} 0 & \cdots & 0 & \chi_1(1) & \cdots & \chi_r(1) \end{pmatrix}^\top \in \mathbb{Q}^{2r}$$

and consider the polyhedron

$$P \coloneqq \{ w \in \mathbb{Q}^r \mid Bw = 0, \left(\begin{smallmatrix} -I_r \\ I_r \end{smallmatrix}\right) w \leq c \} \; .$$

Then a vector $z \in \mathbb{Z}^r$ is a solution to (B.1) if and only if $z \in P$. This means we have to find all the integer points contained in P, which can be done using the function `lattice_points` in OSCAR [Osc23]. However, these computations are quite involved as the polyhedra quickly get large. We could only compute the integer points for some of the groups.

Notice that P always contains at least two integer points corresponding to the vector $0 \in \mathbb{Z}^r$ and the regular character of G. In Table B.1, we list the number of integer points we computed. This number is always much higher than 2, so we cannot apply Corollary B.2 immediately, but rather see this as evidence that the corresponding linear quotients do not admit a symplectic resolution.

C. Generators of Cox rings related to some complex reflection groups

We list generators of the Cox ring $\mathcal{R}(X)$ of a $\mathbb{Q}$-factorial terminalization $X \to \mathbb{C}^{2n}/G^{\circledast}$ for some complex reflection groups G of rank n. The computations were carried out with our implementation of Algorithm 6.2.1 in the computer algebra system OSCAR [Osc23]. We use the matrix models from CHEVIE [GHL+96, Mic15] for the complex reflection groups. As always, we consider the Cox ring $\mathcal{R}(X)$ as a subring of $\mathbb{C}[x_1, \dots, x_{2n}][t_1^{\pm}, \dots, t_k^{\pm}]$, where n is the rank of the complex reflection group G and k the number of conjugacy classes of reflections in G. We do not repeat our results on the dihedral groups $G(d, d, 2)^{\circledast}$ with $d \geq 3$, see Chapter 7. Also, for the Cox rings corresponding to the cyclic groups $\mathsf{C}_d \cong G(de, e, 1)^{\circledast}$ for $d > 1$ and $e \geq 1$ we refer to [FGL11, Don16, Yam18]. In the following lists, we mark the generators with which we started Algorithm 6.2.1 with $\circ$. For $d \in \mathbb{Z}_{>0}$, we denote a primitive d-th root of unity by ζ_d.

Remark C.1. We do not give the relations of the generators of the Cox rings in the following simply for reasons of space. These relations can be computed in all cases using Algorithm 6.3.3. We have been suggested to feed these presentations into the software package [HK15]. However, as far as we know, many algorithms implemented in [HK15] require a system of pairwise non-associated $\mathrm{Cl}(X)$-prime generators of $\mathcal{R}(X)$, see already the definition of a 'bunched ring' in [Kei14, Definition 1.3.5]. We do not know whether our generators fulfil this condition; we hope that we can return to this question in subsequent work.

On the other hand, much of the 'polyhedral' data regarding the structure of the GIT fan, which one might want to compute with [HK15], is already known in the case of complex reflection groups by [BST18]. ∥

$G(2,1,2) = C_2 \wr S_2$. The group $G(2,1,2)$ is of order 8 and rank 2 generated by

$$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

There are two conjugacy classes of reflections, both with representatives of order 2. The commutator subgroup of $G(2,1,2)$ is generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have $[G(2,1,2), G(2,1,2)] \cong C_2$ and $\mathrm{Ab}(G(2,1,2)) \cong C_2^2$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G(2,1,2)^{\circledast}$ are given by:

- $(-x_1x_3 + 2x_1x_4 + x_2x_4)t_2$
- $(2x_1x_2 + x_2^2)t_2$
- $(x_3^2 - 2x_3x_4)t_2$
- $x_1x_3 + x_2x_4$
- $2x_1^2 + 2x_1x_2 + x_2^2$
- $x_3^2 - 2x_3x_4 + 2x_4^2$
- $(-x_1x_3 - x_2x_3 + 2x_1x_4 + x_2x_4)t_1t_2$
- $(-x_1x_3 - x_2x_3 + x_2x_4)t_1$
- $(x_1^2 + x_1x_2)t_1$
- $(-x_3x_4 + x_4^2)t_1$

t_1^{-2}

t_2^{-2}

Remark C.2. The group $G(2,1,2)^{\circledast}$ is also treated in [Gra19, Section 7.4], where a different representation is used. This representation is isomorphic to the representation above and the induced isomorphism of invariant rings then leads immediately to a graded isomorphism of the Cox ring given above and the one in [Gra19, Theorem 7.4.9]. If we label the generators for the Cox ring given above as $f_1, \ldots, f_{12}$, then this isomorphism is defined by

$$\begin{array}{llllll} f_1 \mapsto -w_{21}, & f_2 \mapsto w_{22}, & f_3 \mapsto -w_{23}, & f_4 \mapsto -w_{03}, & f_5 \mapsto w_{01}, & f_6 \mapsto w_{02}, \\ f_7 \mapsto w_3, & f_8 \mapsto w_{11}, & f_9 \mapsto w_{12}, & f_{10} \mapsto -w_{13}, & f_{11} \mapsto s, & f_{12} \mapsto t, \end{array}$$

where w_{ij}, s and t are as in [Gra19, Theorem 7.4.9] and $w_3 \coloneqq \varphi_3 t_1 t_2$ with φ_3 as in [Gra19, Proposition 7.4.7]. ‖

$G(4,2,2)$. The group $G(4,2,2)$ is of order 16 and rank 2 generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\zeta_4 \\ \zeta_4 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There are three conjugacy classes of reflections, all with representatives of order 2. The commutator subgroup of $G(4,2,2)$ is generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have $[G(4,2,2), G(4,2,2)] \cong C_2$ and $\mathrm{Ab}(G(4,2,2)) \cong C_2^3$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G(4,2,2)^{\circledast}$ are given by:

- $x_1x_3 + x_2x_4$

- $(-x_1x_3 + x_2x_4)t_2t_3$
- $(-x_1^2 + x_2^2)t_2$
- $(-x_3^2 + x_4^2)t_2$
- $(x_2x_3 - x_1x_4)t_1t_2$
- $(x_1^2 + x_2^2)t_3$
- $(x_3^2 + x_4^2)t_3$
- $(x_2x_3 + x_1x_4)t_1t_3$
- $x_1x_2t_1$
- $x_3x_4t_1$

  t_1^{-2}
  t_2^{-2}
  t_3^{-2}

$C_3 \wr S_2 = G(3,1,2)$. The group $G(3,1,2)$ is of order 18 and rank 2 generated by

$$\begin{pmatrix} \zeta_3 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There are three conjugacy classes of reflections, all with representatives of order 2. The commutator subgroup of $G(3,1,2)$ is generated by

$$\begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}.$$

We have $[G(3,1,2), G(3,1,2)] \cong C_3$ and $\mathrm{Ab}(G(3,1,2)) \cong C_6$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G(3,1,2)^{\circledast}$ are given by:

- $(-x_1x_3 + x_2x_4)t_1$
- $x_3x_4t_2t_3^2$
- $x_1x_3 + x_2x_4$
- $x_1x_2t_2^2t_3$
- $(x_1^3 - x_2^3)t_1$
- $(x_3^3 - x_4^3)t_1$
- $(x_2x_3^2 - x_1x_4^2)t_1t_2^2t_3$
- $(x_2^2x_3 + x_1^2x_4)t_2t_3^2$
- $(x_2^2x_3 - x_1^2x_4)t_1t_2t_3^2$
- $x_1^3 + x_2^3$
- $x_3^3 + x_4^3$
- $(x_2x_3^2 + x_1x_4^2)t_2^2t_3$

t_1^{-2}
t_2^{-3}
t_3^{-3}

$G(6,3,2)$. The group $G(6,3,2)$ is of order 24 and rank 2 generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\zeta_3 \\ \zeta_3 + 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There are two conjugacy classes of reflections, both with representatives of order 2. The commutator subgroup of $G(6,3,2)$ is generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}.$$

We have $[G(6,3,2), G(6,3,2)] \cong C_6$ and $\mathrm{Ab}(G(6,3,2)) \cong C_2^2$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G(6,3,2)^{\circledast}$ are given by:

- $(-x_1x_3 + x_2x_4)t_2$
- $x_1x_3 + x_2x_4$
- $x_1x_2t_1$
- $x_3x_4t_1$
- $(x_2^4x_3^2 - x_1^4x_4^2)t_1^2t_2$
- $(x_2^2x_3^4 - x_1^2x_4^4)t_1^2t_2$
- $(x_1^6 - x_2^6)t_2$
- $(x_3^6 - x_4^6)t_2$
- $(x_2^5x_3 - x_1^5x_4)t_1t_2$
- $(x_2^3x_3^3 - x_1^3x_4^3)t_1^3t_2$
- $(x_2x_3^5 - x_1x_4^5)t_1t_2$
- $(x_2^4x_3^2 + x_1^4x_4^2)t_1^2$
- $(x_2^2x_3^4 + x_1^2x_4^4)t_1^2$
- $x_1^6 + x_2^6$
- $x_3^6 + x_4^6$
- $(x_2^5x_3 + x_1^5x_4)t_1$
- $(x_2^3x_3^3 + x_1^3x_4^3)t_1^3$
- $(x_2x_3^5 + x_1x_4^5)t_1$

t_1^{-2}
t_2^{-2}

$C_4 \wr S_2 = G(4,1,2)$. The group $G(4,1,2)$ is of order 32 and rank 2 generated by

$$\begin{pmatrix} \zeta_4 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There are four conjugacy classes of reflections, two with representatives of order 2 and two with representatives of order 4. The commutator subgroup of $G(4,1,2)$ is generated by

$$\begin{pmatrix} \zeta_4 & 0 \\ 0 & -\zeta_4 \end{pmatrix}.$$

We have $[G(4,1,2), G(4,1,2)] \cong C_4$ and $\mathrm{Ab}(G(4,1,2)) \cong C_2 \times C_4$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G(4,1,2)^{\circledast}$ are given by:

- $x_3x_4t_1t_3t_4^3$
- $x_1x_2t_1t_3^3t_4$
- $(-x_1x_3 + x_2x_4)t_2$
- $x_1x_3 + x_2x_4$
- $(x_2^3x_3 + x_1^3x_4)t_1t_3t_4^3$
- $(x_2x_3^3 + x_1x_4^3)t_1t_3^3t_4$
- $(x_2^2x_3^2 - x_1^2x_4^2)t_1^2t_2t_3^2t_4^2$
- $(x_1^4 - x_2^4)t_2$
- $(x_3^4 - x_4^4)t_2$
- $x_1^4 + x_2^4$
- $x_3^4 + x_4^4$
- $(x_2^2x_3^2 + x_1^2x_4^2)t_1^2t_3^2t_4^2$
- $(x_2^3x_3 - x_1^3x_4)t_1t_2t_3t_4^3$
- $(x_2x_3^3 - x_1x_4^3)t_1t_2t_3^3t_4$

t_1^{-2}
t_2^{-2}
t_3^{-4}
t_4^{-4}

$G(8,4,2)$. The group $G(8,4,2)$ is of order 32 and rank 2 generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_8^{-1} \\ \zeta_8 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There are three conjugacy classes of reflections, all with representatives of order 2. The commutator subgroup of $G(8,4,2)$ is generated by

$$\begin{pmatrix} \zeta_4 & 0 \\ 0 & -\zeta_4 \end{pmatrix}.$$

We have $[G(8,4,2), G(8,4,2)] \cong C_4$ and $\mathrm{Ab}(G(8,4,2)) \cong C_2^3$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G(8,4,2)^{\circledast}$ are given by:

- $x_1x_3 + x_2x_4$
- $(-x_1x_3 + x_2x_4)t_2t_3$
- $x_1x_2t_1$
- $x_3x_4t_1$
- $(x_2^3x_3 - x_1^3x_4)t_1t_2$
- $(x_2x_3^3 - x_1x_4^3)t_1t_2$
- $(x_2^3x_3 + x_1^3x_4)t_1t_3$
- $(x_2x_3^3 + x_1x_4^3)t_1t_3$
- $(x_2^2x_3^2 + x_1^2x_4^2)t_1^2t_3$
- $(x_1^4 + x_2^4)t_3$
- $(x_3^4 + x_4^4)t_3$
- $(x_2^2x_3^2 - x_1^2x_4^2)t_1^2t_2$
- $(x_1^4 - x_2^4)t_2$
- $(x_3^4 - x_4^4)t_2$

t_1^{-2}
t_2^{-2}
t_3^{-2}

$G(6,2,2)$. The group $G(6,2,2)$ is of order 36 and rank 2 generated by

$$\begin{pmatrix} \zeta_3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\zeta_3 \\ \zeta_3+1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There are four conjugacy classes of reflections, two with representatives of order 2 and two with representatives of order 3. The commutator subgroup of $G(6,2,2)$ is generated by

$$\begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{pmatrix}.$$

We have $[G(6,2,2), G(6,2,2)] \cong C_3$ and $\mathrm{Ab}(G(6,2,2)) \cong C_2 \times C_6$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G(6,2,2)^{\circledast}$ are given by:

- $x_1x_3 + x_2x_4$
- $(-x_1x_3 + x_2x_4)t_1t_2$
- $x_1x_2t_3^2t_4$
- $x_3x_4t_3t_4^2$

- $(x_2^2x_3 - x_1^2x_4)t_2t_3t_4^2$
- $(x_2x_3^2 + x_1x_4^2)t_1t_3^2t_4$
- $(x_2^2x_3 + x_1^2x_4)t_1t_3t_4^2$
- $(x_1^3 + x_2^3)t_1$
- $(x_3^3 + x_4^3)t_1$
- $(x_1^3 - x_2^3)t_2$
- $(x_3^3 - x_4^3)t_2$
- $(x_2x_3^2 - x_1x_4^2)t_2t_3^2t_4$

  t_1^{-2}

  t_2^{-2}

  t_3^{-3}

  t_4^{-3}

$G(12,6,2)$. The group $G(12,6,2)$ is of order 48 and rank 2 generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\zeta_{12}^3 + \zeta_{12} \\ \zeta_{12} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There are three conjugacy classes of reflections, all with representatives of order 2. The commutator subgroup of $G(12,6,2)$ is generated by

$$\begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix}.$$

We have $[G(12,6,2), G(12,6,2)] \cong C_6$ and $\mathrm{Ab}(G(12,6,2)) \cong C_2^3$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G(12,6,2)^{\circledast}$ are given by:

- $x_1x_3 + x_2x_4$
- $(-x_1x_3 + x_2x_4)t_2t_3$
- $x_1x_2t_1$
- $x_3x_4t_1$
- $(x_2^5x_3 - x_1^5x_4)t_1t_2$
- $(x_2^3x_3^3 - x_1^3x_4^3)t_1^3t_2$
- $(x_2x_3^5 - x_1x_4^5)t_1t_2$
- $(x_2^5x_3 + x_1^5x_4)t_1t_3$
- $(x_2^3x_3^3 + x_1^3x_4^3)t_1^3t_3$
- $(x_2x_3^5 + x_1x_4^5)t_1t_3$
- $(x_2^4x_3^2 + x_1^4x_4^2)t_1^2t_3$

- $(x_2^2x_3^4 + x_1^2x_4^4)t_1^2t_3$
- $(x_1^6 + x_2^6)t_3$
- $(x_3^6 + x_4^6)t_3$
- $(x_2^4x_3^2 - x_1^4x_4^2)t_1^2t_2$
- $(x_2^2x_3^4 - x_1^2x_4^4)t_1^2t_2$
- $(x_1^6 - x_2^6)t_2$
- $(x_3^6 - x_4^6)t_2$

  t_1^{-2}

  t_2^{-2}

  t_3^{-2}

$C_5 \wr S_2 = G(5,1,2)$. The group $G(5,1,2)$ is of order 50 and rank 2 generated by

$$\begin{pmatrix} \zeta_5 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There are five conjugacy classes of reflections, one with representative of order 2 and four with representatives of order 5. The commutator subgroup of $G(5,1,2)$ is generated by

$$\begin{pmatrix} \zeta_5 & 0 \\ 0 & \zeta_5^{-1} \end{pmatrix}.$$

We have $[G(5,1,2), G(5,1,2)] \cong C_5$ and $\mathrm{Ab}(G(5,1,2)) \cong C_{10}$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G(5,1,2)^{\circledast}$ are given by:

- $x_3x_4t_2t_3^2t_4^3t_5^4$
- $x_1x_2t_2^4t_3^3t_4^2t_5$
- $(-x_1x_3 + x_2x_4)t_1$
- $x_1x_3 + x_2x_4$
- $(x_2^2x_3^3 + x_1^2x_4^3)t_2^3t_3^6t_4^4t_5^2$
- $(x_2x_3^4 - x_1x_4^4)t_1t_2^4t_3^3t_4^2t_5$
- $(x_2^3x_3^2 - x_1^3x_4^2)t_1t_2^2t_3^4t_4^6t_5^3$
- $(x_2^3x_3^2 + x_1^3x_4^2)t_2^2t_3^4t_4^6t_5^3$
- $(x_2^4x_3 + x_1^4x_4)t_2t_3^2t_4^3t_5^4$
- $(x_2^4x_3 - x_1^4x_4)t_1t_2t_3^2t_4^3t_5^4$
- $(x_2x_3^4 + x_1x_4^4)t_2^4t_3^3t_4^2t_5$
- $(x_1^5 - x_2^5)t_1$
- $(x_3^5 - x_4^5)t_1$

- $(x_2^2x_3^3 - x_1^2x_4^3)t_1t_2^3t_3^6t_4^4t_5^2$
- $x_1^5 + x_2^5$
- $x_3^5 + x_4^5$

  t_1^{-2}
  t_2^{-5}
  t_3^{-5}
  t_4^{-5}
  t_5^{-5}

G_4. The group G_4 is of order 24 and rank 2 generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix} \text{ and } \frac{1}{3}\begin{pmatrix} 2\zeta_3+1 & \zeta_3-1 \\ 2\zeta_3-2 & \zeta_3+2 \end{pmatrix}.$$

There are two conjugacy classes of reflections, both with representatives of order 3. The commutator subgroup of G_4 is generated by

$$\frac{1}{3}\begin{pmatrix} -2\zeta_3-1 & -\zeta_3-2 \\ -2\zeta_3+2 & 2\zeta_3+1 \end{pmatrix} \text{ and } \frac{1}{3}\begin{pmatrix} -\zeta_3-1 & -\zeta_3+1 \\ -2\zeta_3-4 & 2\zeta_3+1 \end{pmatrix}.$$

We have $[G_4, G_4] \cong Q_8$ and $\mathrm{Ab}(G_4) \cong C_3$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G_4^{\circledast}$ are given by:

- $x_1x_3 + x_2x_4$
- $(x_3^3x_4 + x_4^4)t_1t_2^2$
- $(x_2^2x_3^2 - 4x_1^2x_3x_4 + 4x_1x_2x_4^2)t_1t_2^2$
- $(x_1x_2^2x_3 + \frac{4}{3}x_1^3x_4 - \frac{1}{3}x_2^3x_4)t_1t_2^2$
- $(x_2x_3^3 + 6x_1x_3x_4^2 - 2x_2x_4^3)t_1^2t_2$
- $(x_1x_2x_3^2 - x_2^2x_3x_4 - 2x_1^2x_4^2)t_1^2t_2$
- $(x_1^3x_2 + \frac{1}{8}x_2^4)t_1^2t_2$
- $x_3^4 - 8x_3x_4^3$
- $x_1x_3^3 - 3x_2x_3^2x_4 + 4x_1x_4^3$
- $x_1^3x_3 + \frac{1}{2}x_2^3x_3 - 3x_1^2x_2x_4$
- $x_1^4 - x_1x_2^3$
- $(x_2^2x_3^4 + 4x_1^2x_3^3x_4 - 12x_1x_2x_3^2x_4^2 + 4x_2^2x_3x_4^3 - 8x_1^2x_4^4)t_1t_2^2$
- $(x_1^3x_2^2x_3 - \frac{1}{10}x_2^5x_3 - \frac{4}{5}x_1^5x_4 - x_1^2x_2^3x_4)t_1t_2^2$

- $(x_2x_3^5 - 10x_1x_3^3x_4^2 + 10x_2x_3^2x_4^3 + 8x_1x_4^5)t_1^2t_2$
- $(x_1^3x_2x_3^2 - \frac{1}{4}x_2^4x_3^2 - 3x_1^2x_2^2x_3x_4 + 2x_1^4x_4^2 + x_1x_2^3x_4^2)t_1^2t_2$
- $x_3^6 + 20x_3^3x_4^3 - 8x_4^6$
- $4x_1^3x_3^3 + \frac{1}{2}x_2^3x_3^3 + 18x_1^2x_2x_3^2x_4 + 9x_1x_2^2x_3x_4^2 + 4x_1^3x_4^3 + 5x_2^3x_4^3$
- $x_1^6 + \frac{5}{2}x_1^3x_2^3 - \frac{1}{8}x_2^6$

  $(x_2^3x_3^3 + 12x_1^2x_2x_3^2x_4 - 6x_1x_2^2x_3x_4^2 + 8x_1^3x_4^3 + 2x_2^3x_4^3)t_1^3t_2^3$

  t_1^{-3}

  t_2^{-3}

G_5. The group G_5 is of order 72 and rank 2 generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix} \text{ and } \frac{1}{3}\begin{pmatrix} \zeta_3+2 & -\zeta_3+1 \\ -2\zeta_3+2 & 2\zeta_3+1 \end{pmatrix}.$$

There are four conjugacy classes of reflections, all with representatives of order 3. The commutator subgroup of G_5 is generated by

$$\frac{1}{3}\begin{pmatrix} -2\zeta_3-1 & -\zeta_3-2 \\ -2\zeta_3+2 & 2\zeta_3+1 \end{pmatrix} \text{ and } \frac{1}{3}\begin{pmatrix} -2\zeta_3-1 & 2\zeta_3+1 \\ 4\zeta_3+2 & 2\zeta_3+1 \end{pmatrix}.$$

We have $[G_5, G_5] \cong Q_8$ and $\mathrm{Ab}(G_5) \cong C_3^2$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G_5^{\circledast}$ are given by:

- $x_1x_3 + x_2x_4$
- $(x_1x_2^2x_3 + \frac{4}{3}x_1^3x_4 - \frac{1}{3}x_2^3x_4)t_1t_2^2$
- $(x_3^3x_4 + x_4^4)t_1t_2^2$
- $(x_1^3x_2 + \frac{1}{8}x_2^4)t_1^2t_2$
- $(x_2x_3^3 + 6x_1x_3x_4^2 - 2x_2x_4^3)t_1^2t_2$
- $(x_1x_3^3 - 3x_2x_3^2x_4 + 4x_1x_4^3)t_3^2t_4$
- $(x_1^4 - x_1x_2^3)t_3^2t_4$
- $(x_2^2x_3^2 - 4x_1^2x_3x_4 + 4x_1x_2x_4^2)t_1t_2^2t_3^2t_4$
- $(x_1x_2x_3^2 - x_2^2x_3x_4 - 2x_1^2x_4^2)t_1^2t_2t_3t_4^2$
- $(x_1^3x_3 + \frac{1}{2}x_2^3x_3 - 3x_1^2x_2x_4)t_3t_4^2$
- $(x_3^4 - 8x_3x_4^3)t_3t_4^2$
- $(x_1^3x_2^2x_3 - \frac{1}{10}x_2^5x_3 - \frac{4}{5}x_1^5x_4 - x_1^2x_2^3x_4)t_1t_2^2t_3t_4^2$

- $(x_2^2x_3^4 + 4x_1^2x_3^3x_4 - 12x_1x_2x_3^2x_4^2 + 4x_2^2x_3x_4^3 - 8x_1^2x_4^4)t_1t_2^2t_3t_4^2$
- $x_1^3x_3^3 - x_2^3x_3^3 - 9x_1^2x_2x_3^2x_4 + 9x_1x_2^2x_3x_4^2 - 8x_1^3x_4^3 - x_2^3x_4^3$
- $x_1^6 + \frac{5}{2}x_1^3x_2^3 - \frac{1}{8}x_2^6$
- $x_3^6 + 20x_3^3x_4^3 - 8x_4^6$
- $(x_1^3x_2x_3^2 - \frac{1}{4}x_2^4x_3^2 - 3x_1^2x_2^2x_3x_4 + 2x_1^4x_4^2 + x_1x_2^3x_4^2)t_1^2t_2t_3^2t_4$
- $(x_2x_3^5 - 10x_1x_3^3x_4^2 + 10x_2x_3^2x_4^3 + 8x_1x_4^5)t_1^2t_2t_3^2t_4$
  $(x_2^3x_3^3 + 12x_1^2x_2x_3^2x_4 - 6x_1x_2^2x_3x_4^2 + 8x_1^3x_4^3 + 2x_2^3x_4^3)t_1^3t_2^3$
  $(2x_1^3x_3^3 - x_2^3x_3^3 - 6x_1^2x_2x_3^2x_4 + 12x_1x_2^2x_3x_4^2 - 8x_1^3x_4^3)t_3^3t_4^3$
  t_1^{-3}
  t_2^{-3}
  t_3^{-3}
  t_4^{-3}

G_6. The group G_6 is of order 48 and rank 2 generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta_{12}^2 - 1 \end{pmatrix} \text{ and } \frac{1}{3}\begin{pmatrix} -\zeta_{12}^3 + 2\zeta_{12} & -\zeta_{12}^3 + 2\zeta_{12} \\ -2\zeta_{12}^3 + 4\zeta_{12} & \zeta_{12}^3 - 2\zeta_{12} \end{pmatrix}.$$

There are three conjugacy classes of reflections, two with representatives of order 3 and one with representative of order 2. The commutator subgroup of G_6 is generated by

$$\frac{1}{3}\begin{pmatrix} 2\zeta_{12}^2 - 1 & -\zeta_{12}^2 + 2 \\ -2\zeta_{12}^2 - 2 & -2\zeta_{12}^2 + 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2\zeta_{12}^2 - 1 & 2\zeta_{12}^2 - 1 \\ 4\zeta_{12}^2 - 2 & -2\zeta_{12}^2 + 1 \end{pmatrix}.$$

We have $[G_6, G_6] \cong Q_8$ and $\mathrm{Ab}(G_6) \cong C_6$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G_6^{\circledast}$ are given by:

- $x_1x_3 + x_2x_4$
- $(x_1x_2x_3^2 - x_2^2x_3x_4 + 2x_1^2x_4^2)t_1^2t_2$
- $(x_1^3x_2 - \frac{1}{8}x_2^4)t_1^2t_2$
- $(x_2^2x_3^2 + 4x_1^2x_3x_4 - 4x_1x_2x_4^2)t_1t_2^2$
- $(x_3^3x_4 - x_4^4)t_1t_2^2$
- $(x_1x_2^2x_3 - \frac{4}{3}x_1^3x_4 - \frac{1}{3}x_2^3x_4)t_1t_2^2t_3$
- $(x_2x_3^3 - 6x_1x_3x_4^2 + 2x_2x_4^3)t_1^2t_2t_3$
- $(x_1^3x_3 - \frac{1}{2}x_2^3x_3 - 3x_1^2x_2x_4)t_3$
- $(x_1x_3^3 - 3x_2x_3^2x_4 - 4x_1x_4^3)t_3$

- $x_1^4 + x_1x_2^3$
- $x_3^4 + 8x_3x_4^3$
- $(x_2x_3^5 + 10x_1x_3^3x_4^2 - 10x_2x_3^2x_4^3 + 8x_1x_4^5)t_1^2t_2$
- $(x_1^3x_2^2x_3 + \frac{1}{10}x_2^5x_3 + \frac{4}{5}x_1^5x_4 - x_1^2x_2^3x_4)t_1t_2^2$
- $(x_2^2x_3^4 - 4x_1^2x_3^3x_4 + 12x_1x_2x_3^2x_4^2 - 4x_2^2x_3x_4^3 - 8x_1^2x_4^4)t_1t_2^2t_3$
- $(x_1^3x_2x_3^2 + \frac{1}{4}x_2^4x_3^2 - 3x_1^2x_2^2x_3x_4 - 2x_1^4x_4^2 + x_1x_2^3x_4^2)t_1^2t_2t_3$
- $(x_1^6 - \frac{5}{2}x_1^3x_2^3 - \frac{1}{8}x_2^6)t_3$
- $(x_3^6 - 20x_3^3x_4^3 - 8x_4^6)t_3$
- $(x_1^3x_3^3 + x_2^3x_3^3 - 9x_1^2x_2x_3^2x_4 + 9x_1x_2^2x_3x_4^2 + 8x_1^3x_4^3 - x_2^3x_4^3)t_3^2$
  $(-x_2^3x_3^3 + 12x_1^2x_2x_3^2x_4 - 6x_1x_2^2x_3x_4^2 - 8x_1^3x_4^3 + 2x_2^3x_4^3)t_1^3t_2^3$
  t_1^{-3}
  t_2^{-3}
  t_3^{-2}

G_7. The group G_7 is of order 144 and rank 2 generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A := \frac{1}{2}\begin{pmatrix} -\zeta_{12}^3 + \zeta_{12}^2 + \zeta_{12} & -\zeta_{12}^3 + \zeta_{12}^2 + \zeta_{12} \\ -\zeta_{12}^3 - \zeta_{12}^2 + \zeta_{12} & \zeta_{12}^3 + \zeta_{12}^2 - \zeta_{12} \end{pmatrix} \text{ and } A^\top.$$

There are five conjugacy classes of reflections, one with representative of order 2 and four with representatives of order 2. The commutator subgroup of G_7 is generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \zeta_{12}^3 \\ \zeta_{12}^3 & 0 \end{pmatrix}.$$

We have $[G_7, G_7] \cong Q_8$ and $\mathrm{Ab}(G_7) \cong C_3 \times C_6$. Generators of the Cox ring of a $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/G_7^{\circledast}$ are given by:

- $x_1x_3 + x_2x_4$
- $((1 - 2\zeta_{12}^2)x_2x_3^3 + 3x_1x_3^2x_4 - 3x_2x_3x_4^2 + (-1 + 2\zeta_{12}^2)x_1x_4^3)t_1t_3^2t_4$
- $((-1 + 2\zeta_{12}^2)x_3^4 + 6x_3^2x_4^2 + (-1 + 2\zeta_{12}^2)x_4^4)t_2t_5^2$
- $((-1 + 2\zeta_{12}^2)x_1^4 + 6x_1^2x_2^2 + (-1 + 2\zeta_{12}^2)x_2^4)t_3^2t_4$
- $(3x_1^2x_2x_3 + (1 - 2\zeta_{12}^2)x_2^3x_3 + (-1 + 2\zeta_{12}^2)x_1^3x_4 - 3x_1x_2^2x_4)t_1t_3t_4^2$
- $(3x_1^2x_2x_3 + (-1 + 2\zeta_{12}^2)x_2^3x_3 + (1 - 2\zeta_{12}^2)x_1^3x_4 - 3x_1x_2^2x_4)t_1t_2t_5^2$
- $((1 - 2\zeta_{12}^2)x_1^4 + 6x_1^2x_2^2 + (1 - 2\zeta_{12}^2)x_2^4)t_2^2t_5$
- $(x_1^2x_3^2 + (-1 + 2\zeta_{12}^2)x_2^2x_3^2 - 4x_1x_2x_3x_4 + (-1 + 2\zeta_{12}^2)x_1^2x_4^2 + x_2^2x_4^2)t_2^2t_3t_4^2t_5$
- $((1 - 2\zeta_{12}^2)x_3^4 + 6x_3^2x_4^2 + (1 - 2\zeta_{12}^2)x_4^4)t_3t_4^2$

- $(x_1^2x_3^2 + (1-2\zeta_{12}^2)x_2^2x_3^2 - 4x_1x_2x_3x_4 + (1-2\zeta_{12}^2)x_1^2x_4^2 + x_2^2x_4^2)t_2t_3^2t_4t_5^2$
- $((-1+2\zeta_{12}^2)x_2x_3^3 + 3x_1x_3^2x_4 - 3x_2x_3x_4^2 + (1-2\zeta_{12}^2)x_1x_4^3)t_1t_2^2t_5$
- $(2x_1x_2^3x_3^2 + x_1^4x_3x_4 - x_2^4x_3x_4 - 2x_1^3x_2x_4^2)t_1t_2^2t_3^2t_4t_5$
- $(x_1x_2x_3^4 - 2x_2^2x_3^3x_4 + 2x_1^2x_3x_4^3 - x_1x_2x_4^4)t_1t_2t_3t_4^2t_5^2$
- $(x_1x_2^2x_3^3 - x_2^3x_3^2x_4 - x_1^3x_3x_4^2 + x_1^2x_2x_4^3)t_1^2$
- $(-x_1x_3^5 + 5x_2x_3^4x_4 + 5x_1x_3x_4^4 - x_2x_4^5)t_2^2t_3^2t_4t_5$
- $(x_1^5x_2 - x_1x_2^5)t_1$
- $(x_3^5x_4 - x_3x_4^5)t_1$
- $(x_1^5x_3 - 5x_1x_2^4x_3 - 5x_1^4x_2x_4 + x_2^5x_4)t_2t_3t_4^2t_5^2$

$(x_1^3x_3^3 + (-3+6\zeta_{12}^2)x_1x_2^2x_3^3 + 3x_1^2x_2x_3^2x_4 + (3-6\zeta_{12}^2)x_2^3x_3^2x_4$
$\quad + (3-6\zeta_{12}^2)x_1^3x_3x_4^2 + 3x_1x_2^2x_3x_4^2 + (-3+6\zeta_{12}^2)x_1^2x_2x_4^3 + x_2^3x_4^3)t_2^3t_5^3$
$((-1+2\zeta_{12}^2)x_1^3x_3^3 + 9x_1x_2^2x_3^3 + (-3+6\zeta_{12}^2)x_1^2x_2x_3^2x_4 - 9x_2^3x_3^2x_4 - 9x_1^3x_3x_4^2$
$\quad + (-3+6\zeta_{12}^2)x_1x_2^2x_3x_4^2 + 9x_1^2x_2x_4^3 + (-1+2\zeta_{12}^2)x_2^3x_4^3)t_3^3t_4^3$
t_1^{-2}
t_2^{-3}
t_3^{-3}
t_4^{-3}
t_5^{-3}

D. Yamagishi's algorithm in OSCAR: an example

We give an example of how one can use our implementation of Algorithm 6.2.1 in OSCAR [Osc23]. See Chapter 6 and in particular Section 6.4 for references of the implemented algorithms. OSCAR is a software package written in the programming language Julia [BEKS17]; see the website

`https://www.oscar-system.org/install/`

for installation instructions. Our implementation and the related functionality is still under development and we cannot guarantee that the interface remains completely the same in the future. Further, the relevant code is so far only contained in a developer version of OSCAR. In order to reproduce our presentation below, the same version of OSCAR that was used to write this section can be installed by entering

```
julia> Pkg.add(url="https://github.com/oscar-system/Oscar.jl",
               rev="c58ae9abd999507d95d325d525dd7ad877831bbc")
```

instead of `Pkg.add("Oscar")` during the installation. After the installation, OSCAR can be loaded by:

```
julia> using Oscar
```

In the following, we compute the Cox ring of a $\mathbb{Q}$-factorial terminalization of the linear quotient by the symplectic reflection group $G_4^{\circledast}$. This group can be defined over a cyclotomic extension of $\mathbb{Q}$ of order 3, but in our implementation, we require that the field contains an e-th root of unity, where e is the exponent of the group. In case of $G_4^{\circledast}$, we hence need to work over a cyclotomic extension of order 12.

```
julia> K, a = cyclotomic_field(12, "a");
```

We now enter the generators for the group over this field.

```
julia> g1 = matrix(K, 4, 4, [ 1 0 0 0;
                              0 a^4 0 0;
                              0 0 1 0;
                              0 0 0 a^-4 ]);

julia> g2 = 1//3*matrix(K, 4, 4, [ 2*a^4 + 1 a^4 - 1 0 0;
                                    2*a^4 - 2 a^4 + 2 0 0;
                                    0 0 -2*a^4 - 1 -2*a^4 - 4;
                                    0 0 -a^4 - 2 -a^4 + 1 ]);
```

We set up the group and the corresponding linear quotient as follows.

```
julia> G = matrix_group(g1, g2);

julia> L = linear_quotient(G);
```

Note that the latter command does not do any computations, but only sets up a 'container'. We now ask for representatives of the conjugacy classes of junior elements in $G_4^{\circledast}$. Recall that the property of being junior is defined with respect to a fixed root of unity. In OSCAR, there is the function `Oscar.fixed_root_of_unity(L)`, which returns a fixed e-th root of unity stored in `L`, where e is the exponent of $G_4^{\circledast}$, to ensure consistency of the computations. Since these functions are still under development, they are not 'exported' in OSCAR, so we have to write '`Oscar.`' in front.

```
julia> zeta = Oscar.fixed_root_of_unity(L);
julia> Oscar.representatives_of_junior_elements(G, zeta)
2-element Vector{MatrixGroupElem{nf_elem,
                 AbstractAlgebra.Generic.MatSpaceElem{nf_elem}}}:
 [1 0 0 0; 0 -a^2 0 0; 0 0 1 0; 0 0 0 a^2-1]
 [1 0 0 0; 0 a^2-1 0 0; 0 0 1 0; 0 0 0 -a^2]
```

We see that there are two conjugacy classes of junior elements in $G_4^{\circledast}$. We can also check whether $\mathbb{C}^4/G_4^{\circledast}$ has canonical or terminal singularities. This uses the Reid–Tai criterion [Kol13, Theorem 3.21] in the background.

```
julia> has_canonical_singularities(L)
true

julia> has_terminal_singularities(L)
false
```

We compute the Cox ring of the linear quotient $\mathbb{C}^4/G_4^{\circledast}$ itself:

```
julia> RVG, RVGtoR = cox_ring(L);
```

This returns the Cox ring $\mathcal{R}(\mathbb{C}^4/G_4^{\circledast})$ as an affine algebra `RVG` and a map `RVGtoR` from this ring to the polynomial ring $\mathbb{C}[x_1, \ldots, x_4]$, that is, the coordinate ring of the vector space $\mathbb{C}^4$. The latter is relevant to obtain an explicit description of the generators of the Cox ring:

```
julia> map(RVGtoR, gens(RVG))
18-element Vector{MPolyDecRingElem{nf_elem,
                  AbstractAlgebra.Generic.MPoly{nf_elem}}}:
 x[1]*x[3] + x[2]*x[4]
 x[3]^3*x[4] + x[4]^4
 x[2]^2*x[3]^2 - 4*x[1]^2*x[3]*x[4] + 4*x[1]*x[2]*x[4]^2
 x[1]*x[2]^2*x[3] + 4//3*x[1]^3*x[4] - 1//3*x[2]^3*x[4]
 x[2]*x[3]^3 + 6*x[1]*x[3]*x[4]^2 - 2*x[2]*x[4]^3
 x[1]*x[2]*x[3]^2 - x[2]^2*x[3]*x[4] - 2*x[1]^2*x[4]^2
 x[1]^3*x[2] + 1//8*x[2]^4
 x[3]^4 - 8*x[3]*x[4]^3
 x[1]*x[3]^3 - 3*x[2]*x[3]^2*x[4] + 4*x[1]*x[4]^3
 x[1]^3*x[3] + 1//2*x[2]^3*x[3] - 3*x[1]^2*x[2]*x[4]
 x[1]^4 - x[1]*x[2]^3
 x[2]^2*x[3]^4 + 4*x[1]^2*x[3]^3*x[4] - 12*x[1]*x[2]*x[3]^2*x[4]^2
  + 4*x[2]^2*x[3]*x[4]^3 - 8*x[1]^2*x[4]^4
 x[1]^3*x[2]^2*x[3] - 1//10*x[2]^5*x[3] - 4//5*x[1]^5*x[4]
  - x[1]^2*x[2]^3*x[4]
 x[2]*x[3]^5 - 10*x[1]*x[3]^3*x[4]^2 + 10*x[2]*x[3]^2*x[4]^3
  + 8*x[1]*x[4]^5
 x[1]^3*x[2]*x[3]^2 - 1//4*x[2]^4*x[3]^2 - 3*x[1]^2*x[2]^2*x[3]*x[4]
  + 2*x[1]^4*x[4]^2 + x[1]*x[2]^3*x[4]^2
 x[3]^6 + 20*x[3]^3*x[4]^3 - 8*x[4]^6
 x[1]^3*x[3]^3 + 1//8*x[2]^3*x[3]^3 + 9//2*x[1]^2*x[2]*x[3]^2*x[4]
  + 9//4*x[1]*x[2]^2*x[3]*x[4]^2 + x[1]^3*x[4]^3 + 5//4*x[2]^3*x[4]^3
 x[1]^6 + 5//2*x[1]^3*x[2]^3 - 1//8*x[2]^6
```

The Cox ring is graded by the class group of $\mathbb{C}^4/G_4^{\circledast}$ and we can ask for the degree of a generator. This makes use of the functionality of graded rings provided by OSCAR.

```
julia> grading_group(RVG)
```

```
GrpAb: Z/3

julia> degree(gen(RVG, 1))
Element of
GrpAb: Z/3
with components [0]
```

Finally, we compute the Cox ring of a $\mathbb{Q}$-factorial terminalization $X \to \mathbb{C}^4/G_4^{\circledast}$. Again, this returns the ring as an affine algebra and a 'structure morphism', which maps to the Laurent polynomial ring $\mathbb{C}[x_1, \ldots, x_4][t_1^{\pm}, t_2^{\pm}]$. Note that the next command might compute for a few minutes.

```
julia> RX, RXtoRt = Oscar.cox_ring_of_qq_factorial_terminalization(L);
```

To obtain the generators as elements of $\mathbb{C}[x_1, \ldots, x_4][t_1^{\pm}, t_2^{\pm}]$, we do:

```
julia> map(RXtoRt, gens(RX))
21-element Vector{...}:
 x[1]*x[3] + x[2]*x[4]
 (x[3]^3*x[4] + x[4]^4)*t1*t2^2
 (x[2]^2*x[3]^2 - 4*x[1]^2*x[3]*x[4] + 4*x[1]*x[2]*x[4]^2)*t1*t2^2
 (x[1]*x[2]^2*x[3] + 4//3*x[1]^3*x[4] - 1//3*x[2]^3*x[4])*t1*t2^2
 (x[2]*x[3]^3 + 6*x[1]*x[3]*x[4]^2 - 2*x[2]*x[4]^3)*t1^2*t2
 (x[1]*x[2]*x[3]^2 - x[2]^2*x[3]*x[4] - 2*x[1]^2*x[4]^2)*t1^2*t2
 (x[1]^3*x[2] + 1//8*x[2]^4)*t1^2*t2
 x[3]^4 - 8*x[3]*x[4]^3
 x[1]*x[3]^3 - 3*x[2]*x[3]^2*x[4] + 4*x[1]*x[4]^3
 x[1]^3*x[3] + 1//2*x[2]^3*x[3] - 3*x[1]^2*x[2]*x[4]
 x[1]^4 - x[1]*x[2]^3
 (x[2]^2*x[3]^4 + 4*x[1]^2*x[3]^3*x[4] - 12*x[1]*x[2]*x[3]^2*x[4]^2
  + 4*x[2]^2*x[3]*x[4]^3 - 8*x[1]^2*x[4]^4)*t1*t2^2
 (x[1]^3*x[2]^2*x[3] - 1//10*x[2]^5*x[3] - 4//5*x[1]^5*x[4]
  - x[1]^2*x[2]^3*x[4])*t1*t2^2
 (x[2]*x[3]^5 - 10*x[1]*x[3]^3*x[4]^2 + 10*x[2]*x[3]^2*x[4]^3
  + 8*x[1]*x[4]^5)*t1^2*t2
 (x[1]^3*x[2]*x[3]^2 - 1//4*x[2]^4*x[3]^2 - 3*x[1]^2*x[2]^2*x[3]*x[4]
  + 2*x[1]^4*x[4]^2 + x[1]*x[2]^3*x[4]^2)*t1^2*t2
 x[3]^6 + 20*x[3]^3*x[4]^3 - 8*x[4]^6
 x[1]^3*x[3]^3 + 1//8*x[2]^3*x[3]^3 + 9//2*x[1]^2*x[2]*x[3]^2*x[4]
  + 9//4*x[1]*x[2]^2*x[3]*x[4]^2 + x[1]^3*x[4]^3 + 5//4*x[2]^3*x[4]^3
 x[1]^6 + 5//2*x[1]^3*x[2]^3 - 1//8*x[2]^6
 (-1//8*x[2]^3*x[3]^3 - 3//2*x[1]^2*x[2]*x[3]^2*x[4]
  + 3//4*x[1]*x[2]^2*x[3]*x[4]^2 - x[1]^3*x[4]^3
  - 1//4*x[2]^3*x[4]^3)*t1^3*t2^3
 t1^-3
 t2^-3
```

The ring $\mathcal{R}(X)$ is graded by the class group $\mathrm{Cl}(X) \cong \mathbb{Z}^2$ via the degrees of the variables t_1 and t_2.

```
julia> grading_group(RX)
GrpAb: Z^2

julia> degree(gen(RX, 1))
graded by [0 0]

julia> degree(gen(RX, 2))
graded by [1 2]
```

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