DOKTORARBEIT

Homogenization and dimension reduction for periodic textiles made of linear elastic yarns with sliding contact

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Abstract

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This work aims to study textile structures in the frame of linear elasticity to understand how the structure and material parameters influence the macroscopic homogenized model. More precisely, we are interested in how the textile design parameters, such as the ratio between fibers' distance and cross-section width, the strength of the contact sliding between yarns, and the partial clamp on the textile boundaries determine the phenomena that one can see in shear experiments with textiles. Among others, when the warp and weft yarns change their in-plane angles first and, after reaching some critical shear angle, the textile plate comes out of the plane, and its folding starts.

The textile structure under consideration is a woven square, partially clamped on the left and bottom boundary, made of long thin fibers that cross each other in a periodic pattern. The fibers cannot penetrate each other, and in-plane sliding is allowed. This last assumption, together with the partial clamp, adds new levels of complexity to the problem due to the anisotropy in the yarn's behavior in the unclamped subdomains of the textile.

The limiting behavior and macroscopic strain fields are found by passing to the limit with respect to the yarn's thickness r and the distance between them ϵ, parameters that are asymptotically related. The homogenization and dimension reduction are done via the unfolding method, which separates the macroscopic scale from the periodicity cell. In addition to the homogenization, a dimension reduction from a 3D to a 2D problem is applied. Adapting the classical unfolding results to both the anisotropic context and to lattice grids (which are constructed starting from the center lines of the rods crossing each other) are the main tools we developed to tackle this type of model. They represent the first part of the thesis and are published in Falconi, Griso, and Orlik, 2022b and Falconi, Griso, and Orlik, 2022a.

Given the parameters mentioned above, we then proceed to classify different textile problems, incorporating the results from other works on the topic and thoroughly investigating some others. After the study is conducted, we draw conclusions and give a mathematical explanation concerning the expected approximation of the displacements, the expected solvability of the limit problems, and the phenomena mentioned above. The results can be found in "Asymptotic behavior for textiles with loose contact", which has been recently submitted.

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Ziel dieser Arbeit ist es, textile Strukturen im Rahmen der linearen Elastizität zu untersuchen, um zu verstehen, wie die Struktur- und Materialparameter das makroskopisch homogenisierte Modell beeinflussen. Genauer gesagt interessiert uns, wie die textilen Designparameter, biespielweise das Verhältnis zwischen dem Faserabstand und der Querschnittsbreite, die Stärke des Kontaktgleitens zwischen Garnen und die partielle Klemmung an den Textilrändern, die sichtbaren Phänomene bei Scherversuchen mit Textilien bestimmen. Insbesondere interessiert uns der Effekt, wenn sich der Winkel zwischen den Kett- und Schussfäden zuerst nur in der Ebene ändert und, nach Erreichen eines kritischen Scherwinkels, die Textilplatte aus der Ebene kommt und ihre Faltung beginnt.

Die betrachtete Textilstruktur ist ein Quadratgewebe, das aus langen dünnen Fasern besteht, die sich in einem periodischen Muster kreuzen und teilweise an dem linken und unteren Rand geklemmt werden. Die Fasern können nicht ineinander eindringen und ein Gleiten in der Ebene ist erlaubt. Diese letzte Annahme, zusammen mit der partiellen Klemmung, fügt dem Problem, aufgrund der Anisotropie im Verhalten des Garns in den nicht geklemmten Teilbereichen des Textils, eine neue Komplexitätsebenen hinzu.

Das Grenzverhalten und die makroskopischen Dehnungsfelder werden gefunden, indem man das assymptotische Verhalten des Gewebes in Bezug auf Garndicke r und Abstand ϵ, unter Annahme eines vorgeschriebenen Verhältnisses der beiden Parameter, untersucht. Die Homogenisierung und Dimensionsreduktion erfolgen über ein Entfaltungsverfahren, das die makroskopische Skala von der Periodizitätszelle trennt. Zusammen mit der Homogenisierung wird zusätzlich eine Dimensionsreduktion von einem 3D- auf ein 2D-Problem angewendet. Die Anpassung der klassischen Entfaltungsergebnisse sowohl an die Anisotropie, als auch an das Gitter (die kreuzende und oszillierende Balkenachsen) sind die wichtigsten Werkzeuge in der Arbeit. Sie stellen den ersten Teil der Arbeit dar und sind in Falconi, Griso, and Orlik, 2022b und Falconi, Griso, and Orlik, 2022a veröffentlicht.

Anhand der oben genannten Parameter gehen wir dann zur Klassifizierung verschiedener Textilprobleme über, wobei wir die Ergebnisse, die bereits in anderen Arbeiten zu diesem Thema erzielt wurden, einbeziehen und einige andere gründlich untersuchen. Nachdem die Studie durchgeführt wurde, ziehen wir die Schlussfolgerungen und geben eine mathematische Erklärung bezüglich der erwarteten Annäherung der Verschiebungen, der erwarteten Lösbarkeit der Grenzwertprobleme und der oben erwähnten Phänomene. Die Ergebnisse sind in "Asymptotic behavior for textiles with loose contact" zu finden, das kürzlich eingereicht wurde.

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I would like to give a great thanks to Prof. Georges Griso from the Sorbonne Université de Paris, who contributed to the creation of the mathematical tools for the homogenization of this type of problem and helped me through the whole duration of my thesis. He always had new suggestions and came up with new strategies when I reached dead ends. Much would not have been possible without him, or at least not at this level and in such a short time.

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God bless KL.

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Dedicated to the Mountain Gorilla.

The mountain gorilla (Gorilla Beringei Beringei) lives in the African forests between the Democratic Republic of Congo (RDC), Rwanda, and Uganda. Its population counts around 1000 wild exemplars, and it is one of the ten most endangered species on Earth. Not only because of the locals, from which they steal bananas but also by the nearby cobalt mines, which are used in the batteries and components for the energy transition.

As a scientist, I cannot help but believe that technology and research are one of the best ways we have to fight climate change and build a green future without carbon and fossil fuels. Nevertheless, as a non-scientist, I also believe that we need to embrace a new way of thinking that exceeds science and gives us a more humble role in this world, respecting all the biodiversity in it. Because our future consists not only of a cleaner world where our descendants can live but also a world where theirs can live as well. May their presence be the proof that we can respectfully save our planet.

Chapter 1

Introduction

This work deals with textile structures made of long thin beams, which cross each other on a periodic pattern. The aim is to find a mathematical model for the textile, which adequately describes its mechanical behavior at a small scale and which can then be homogenized to capture the textile's macroscopic behavior in the context of linear elasticity.

The homogenization is done via the unfolding method, an equivalent to the two-scale convergence. The method was first presented in Cioranescu, Damlamian, and Griso, 2002, with further development in Cioranescu, Damlamian, and Griso, 2005; Cioranescu, Donato, and Zaki, 2006; Damlamian et al., 2006; Cioranescu, Damlamian, and Griso, 2008 and extensively in Cioranescu, Damlamian, and Griso, 2018.

This homogenizing tool is well suited for these types of problems, involving periodic patterns and structures made of yarns. Indeed, it has largely found application in the homogenization of periodically perforated domains (see, e.g., Damlamian, Meunier, and Van Schaftingen, 2007; Damlamian and Meunier, 2010; Donato, Le Nguyen, and Tardieu, 2011; Ould Hammouda, 2011; Cioranescu, Damlamian, and Orlik, 2013a; Cabarrubias and Donato, 2016; Donato and Yang, 2016) and of thin structures with a periodic pattern, like periodically perforated shells (see Griso, Hauck, and Orlik, 2021), textiles made of long woven beams in strong contact (see Griso, Orlik, and Wackerle, 2020b; Griso, Orlik, and Wackerle, 2020a) and 3D lattice structures made of either beams or segments in a stable configuration (see Griso et al., 2020; Griso et al., 2021).

In order to simplify the structures we are going to investigate, a dimension reduction from three to two dimensions is also applied, so that in the limit the macroscopic behavior only depends on the in-plane variables. About dimension reduction of plates or rods, one can read, for instance, in Blanchard, Gaudiello, and Griso, 2007a; Blanchard, Gaudiello, and Griso, 2007b; Griso, 2004; Griso, 2008a; Griso, 2008b. For more information on the combination of periodic unfolding and dimension reduction, one can look into Chapter 11 of Cioranescu, Damlamian, and Griso, 2018.

The model we consider is a woven textile made of long thin rods that are not glued (so they cannot be extended to a perforated shell) but do allow for a small amount of in-plane sliding. The interest in this type of structure comes from the large number of numerical progress on the topic (among others, we would like to mention Madeo et al., 2015; Boisse et al., 2011; Orlik, Panasenko, and Shiryaev, 2016; Orlik and Shiryaev, 2016), so the aim is to give a mathematical explanation of the phenomena that arise in simulations and experiments. In particular, we are interested in how the contact between fibers and the partial clamp influence the textile behavior at a macroscopic level. In this sense, this work shows the range of possible cases and, starting from the ones already studied in Griso, Orlik, and Wackerle, 2020a, investigates the remaining ones and compares them in a qualitative manner.

The investigation of woven structures with contact sliding is able to describe more phenomena and be closer to reality but it also involves a more complex setting, and finds its limitation in the classical unfolding theorems. Hence, we decided to split the thesis into two main parts: a preparatory part, where we extend the classical unfolding results to new structures and new classes of sequences, and an investigative part, where we study different elasticity problems for the small deformations of this kind of textile structures.

1.1 First part: new tools for the periodic unfolding

The first three chapters of the thesis will furnish the necessary extensions of the classical unfolding theory and the main notions and properties concerning the N-linear and N-cubic interpolation. These results are important not only to investigate the particular periodic structures we are interested in but also all those alike.

The first section of Chapter 2 recalls the classical unfolding theory. We consider a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary and periodically paved with unitary cells $Y = [0,1]^N$ rescaled by a small parameter ε. The unfolding operator takes measurable functions on Ω and splits the functions' variable into the reference cell's position and the variable's position on the reference cell. As ε goes to zero, it splits the limit function into macroscopic behavior on Ω and microscopic behavior on the reference cell Y (see Figure 4.2). This method is very powerful in the frame of homogenization because in the limit we have separation of the microscopic cell problem from the macroscopic homogenized problem.

The unfolding operator can easily be applied to bounded sequences in L^p (which admit a weakly convergent subsequence in L^p) since its L^p norm can be bounded by the sequence's bound. From the classical unfolding theory in Section 1.4 of Cioranescu, Damlamian, and Griso, 2018, we present the unfolding for:

(i) Sequences $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla\phi_\varepsilon\|_{L^p(\Omega)} \leq C$;

(ii) Sequences $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega)$ such that $\|\phi_\varepsilon\|_{W^{1,p}(\Omega)} \leq C$;

(iii) Sequences $\{\phi_\varepsilon\}_\varepsilon \subset W^{2,p}(\Omega)$ such that $\|\phi_\varepsilon\|_{W^{2,p}(\Omega)} \leq C$.

The rest of the chapter is devoted to the properties of linear and cubic approximation of functions on a reference grid $\mathcal{G}$ that connects the vertices of the reference cell Y (and on the rescaled one εY), as well as the N-linear and N-cubic extension to the cell itself Y (and on the rescaled one). The properties will be often used throughout the whole work.

The first extension of the unfolding method is done in Chapter 3 and concerns sequences that present better estimates in some (privileged) directions with respect to others. Unlike the sequences above, whose estimates are isotropically bounded, this chapter will deal with the periodic unfolding of "anisotropically bounded" sequences.

To describe them rigorously, we consider the decomposition $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and define $x = (x', x'')$, where the variable x' corresponds to the first N_1 directions. From the unfolding with parameters of Cioranescu, Damlamian, and Griso, 2018, Chap. 7, we develop the "two-steps unfolding" and show the asymptotic behavior of the following new classes of anisotropically bounded functions:

(i)' Sequences $\{\phi_\varepsilon\}_\varepsilon \subset L^p(\Omega, \nabla_{x'})$ such that $\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla_{x'}\phi_\varepsilon\|_{L^p(\Omega)} \leq C$;

(ii)' Sequences $\{\phi_\varepsilon\}_\varepsilon \subset L^p(\Omega, \nabla_{x'})$ such that $\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla_{x'}\phi_\varepsilon\|_{L^p(\Omega)} \leq C$;

(iii)' Sequences $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla_{x'}\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla_{x''}\phi_\varepsilon\|_{L^p(\Omega)} \leq C$;

(iv)' Sequences $\{\phi_\varepsilon\}_\varepsilon \subset L^p(\Omega, \nabla_{x'})$ with $\{\nabla_{x'}\phi_\varepsilon\}_\varepsilon \subset L^p(\Omega, \nabla_{x''})$ such that

$$\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla_{x'}\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla_{x''}(\nabla_{x'}\phi_\varepsilon)\|_{L^p(\Omega)} \leq C.$$

As a direct application of this unfolding, in the last section we proceed to the complete homogenization of the following homogeneous Dirichlet problem

$$\begin{aligned}&\text{Find } u_\varepsilon \in H^1_0(\Omega) \text{ such that:}\\ &\int_\Omega A_\varepsilon \begin{bmatrix}\nabla_{x'}u_\varepsilon\\ \varepsilon\nabla_{x''}u_\varepsilon\end{bmatrix} \cdot \begin{bmatrix}\nabla_{x'}\phi\\ \varepsilon\nabla_{x''}\phi\end{bmatrix} dx = \int_\Omega f\,\phi\, dx, \quad \forall\phi \in H^1_0(\Omega),\end{aligned}$$

whose nature is anisotropic. We prove the existence and uniqueness of solutions for cell problems and macroscopic problems, the correctors, and the homogenizing operator.

Chapter 4 deals with the second type of extension of the periodic unfolding method, which is the unfolding for sequences defined on periodic lattice structures. In this sense, by "periodic lattice structure," we mean one-dimensional grids $\mathcal{S}$ defined on each ε cell and periodically repeated for each cell of Ω. For further reading on the topic of lattice structures and homogenization, we recommend Abrate, 1991; Caillerie and Moreau, 1995; Panasenko, 1998; Lenczner and Senouci-Bereksi, 1999; Casado-Diaz, Luna-Laynez, and Martin, 2001; Lenczner and Mercier, 2004.

After giving a rigorous definition of the periodic lattice $\mathcal{S}_\varepsilon \subset \mathbb{R}^N$, we define the functions on these structures. The problem of defining an unfolding operator for lattices is that the unfolding itself is done separately on each lattice direction. This means, that in the limit we obtain N different functions and we no longer know if these functions are either independent from each others, or the restriction to each line of a unique function.

To overcome this issue, we adopted the following strategy: given a sequence $\{\phi_\varepsilon\}_\varepsilon$ bounded on $W^{1,p}(\mathcal{S}_\varepsilon)$, we first uniquely decompose it into a sequence $\{\phi_{a,\varepsilon}\}_\varepsilon$, defined as an interpolation between lattice nodes, and a remainder term $\{\phi_{0,\varepsilon}\}_\varepsilon$. Concerning $\{\phi_{a,\varepsilon}\}_\varepsilon$, we can extend it by N-linear interpolation to the whole space, apply the unfolding results on $\mathbb{R}^N$ and restrict it back to the lattice itself. Concerning $\{\phi_{0,\varepsilon}\}_\varepsilon$, we can directly apply the one-dimensional unfolding since it is defined on straight segments of $\mathcal{S}_\varepsilon$. With this workaround, and due to the results of the previous chapter, we show the asymptotic behavior of sequences:

(i)″ $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\mathcal{S}_\varepsilon)$ such that $\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}$;

(ii)″ $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\mathcal{S}_\varepsilon)$ such that $\|\phi_\varepsilon\|_{W^{1,p}((\mathcal{S}_\varepsilon))} \leq C\varepsilon^{\frac{1-N}{p}}$;

(iii)″ $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\mathcal{S}_\varepsilon)$ such that $\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}'_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}''_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}$.

For sequences bounded on $W^{2,p}(\mathcal{S}_\varepsilon)$, more work is required since the N-cubic extensions of the interpolating sequence are not uniquely defined, and thus more assumptions on the bounds must be made. However, we also present another strategy, which consists of twice applying (on the functions and their partial derivatives) the results for functions bounded $W^{1,p}(\mathcal{S}_\varepsilon)$. In this sense, no other bounds are needed but at the cost of a lesser regularity of the limit fields.

At last, we again consider an application of the new results and proceed to the complete homogenization of a fourth-order Dirichlet problem defined on a lattice structure:

$$\begin{cases} \text{Find } u_\varepsilon \in H^1_0(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon) \text{ such that:} \\ \displaystyle\int_{\mathcal{S}_\varepsilon} A_\varepsilon \partial^2_{\mathbf{s}} u_\varepsilon \, \partial^2_{\mathbf{s}}\phi \, d\mathbf{s} = \int_{\mathcal{S}_\varepsilon} g_\varepsilon \, \partial_{\mathbf{s}}\phi \, d\mathbf{s} + \int_{\mathcal{S}_\varepsilon} f_\varepsilon \, \phi \, d\mathbf{s}, \quad \forall \phi \in H^1_0(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon). \end{cases}$$

1.2 Second part: classification and homogenization of textile structures made of linear elastic yarns with sliding contact

In the second part of the thesis, we investigate our actual problem, that is, the linearized elasticity problem for a textile structure made of yarns with contact sliding. A first breakthrough for this kind of problem has been made in Griso, Orlik, and Wackerle, 2020a, and we will initially consider the same setting. Then, we will apply the tools developed in the first part to extend the study to a whole new set of problems.

We dedicate Chapter 5 to the mathematical model for the structure, well-posedness of the problem, and classification according to the initial parameters.

We start by considering the simplest structure of a woven textile: a long, oscillating rod of length L with a small squared cross section of width r. From the results in Griso, 2004; Griso, 2008a; Griso, 2008b, every displacement u_ε on the rod can be decomposed according to

$$u_\varepsilon \doteq \mathcal{U}^{el}_\varepsilon + \overline{u}_\varepsilon,$$

where the $\mathcal{U}^{el}$ is the elementary displacement and consists of the middle line and rotation of the cross-section, while $\overline{u}$ is the remainder term. We improve this decomposition by showing that any rod displacement is the sum of a Bernoulli-Navier displacement and a residual term. The construction of the whole textile structure T_ε is done as depicted in Figure 1.1: we set a small parameter ε and define two beams of rods. The distance between two parallel rods is ε, and the rods of different directions cross each other in a periodic pattern (see the zoom in Figure 1.1), creating a woven canvas in the square $\Omega = (0, L)^2$. For every displacement on

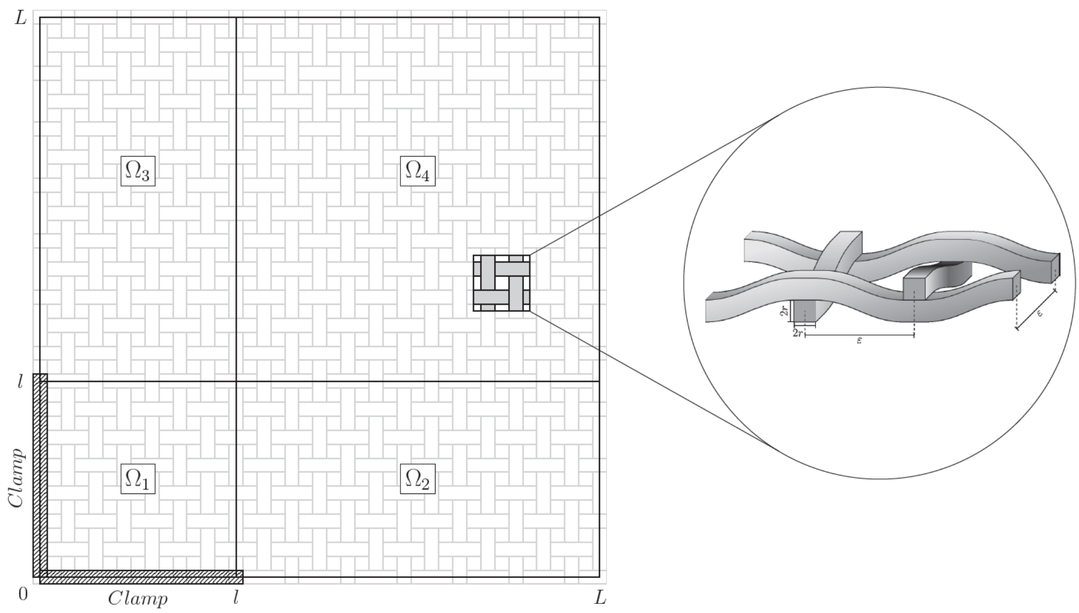

FIGURE 1.1: The textile structure. Each cell has a 2ε periodic pattern. The distance between fibers is ε, and their cross-section is $2r$. A partial clamp is set on the left and bottom boundaries.

the textile structure $u_\varepsilon \in H^1(T_\varepsilon)$ we set the following natural assumptions:

(i) clamp conditions: on a partial segment of the left and bottom boundaries, the displacements vanish;

(ii) In-plane contact conditions: in the in-plane component, the displacements are allowed to shear relative to the other in two directions up to a maximum bound given by a gap function $g_\varepsilon = \varepsilon^h g$, where $h \in N^*$ denotes the "contact strength";

(iii) Outer plane non-penetration condition: in the outer plane component, the displacements are not allowed to penetrate each other.

We define the set of admissible displacements as

$$\mathcal{X}_\varepsilon = \left\{ v_\varepsilon \in H^1(T_\varepsilon) \,\middle|\, v_\varepsilon \text{ satisfies conditions (i)-(iv)} \right\}.$$

Due to conditions (ii)-(iii), the elasticity problem is set via variational inequality, similar to in Cioranescu, Damlamian, and Orlik, 2013b; Griso, Orlik, and Wackerle, 2020a:

$$\begin{aligned} &\text{Find } u_\varepsilon \in \mathcal{X}_\varepsilon \text{ such that for every } v_\varepsilon \in \mathcal{X}_\varepsilon: \\ &\int_{T_\varepsilon} a_{ijkl,\varepsilon}\, e_{ij}(u_\varepsilon)\, e_{kl}(u_\varepsilon - v_\varepsilon)\, dx \le \int_{T_\varepsilon} f_\varepsilon \cdot (u_\varepsilon - v_\varepsilon)\, dx, \end{aligned} \tag{1.1}$$

where a_ε is the fourth order strain tensor describing the material law, and f_ε is the applied stress. The problem admits solution by Stampacchia's Lemma (see Kinderlehrer and Stampacchia, 2000), a version of Lax−Milgram for closed convex subsets of Hilbert spaces. In order to give a classification of the different textile structures, we need the estimates of all the fields and their derivatives involved in (1.1) with respect to the L^2 norm of the strain tensor. In the clamped subdomains, these estimates are obtained by the bound on their derivatives together with Poincaré's inequality. The estimates on the unclamped subdomains are proved by the results on the clamped ones together with the relations given by the contact conditions, the non-penetration condition, and the Trace theorem.

We note that the fields' estimates depend on three factors, and so does the behavior of the textile before the limit. Namely:

1. The ratio between the fibers distance ε and their cross section width r;
2. The fact that we are interested in the study of small deformations;
3. The contact strength $h \in \mathbb{N}^*$ (or friction between yarns).

Concerning the first aspect, for simplicity, we assume that $\varepsilon \sim r$. Of course, another whole study can be done without this assumption and would lead to another interesting case ($r \sim \varepsilon^2$), but given the complexity of the problem, we leave it out of the scope of this work. Concerning the second aspect, we show that the linearization for the elasticity problem is ensured if and only if the following assumption on the strain bound holds:

$$\|e(u_\varepsilon)\|_{L^2(T_\varepsilon)} \sim \varepsilon^{5/2+\delta}, \qquad \delta > 0. \tag{1.2}$$

A suitable choice of forces on the right-hand side must be made to keep the bound in such a linear regime. At last, contact strength is the parameter we are most interested in because it heavily determines the transfer of estimates from the clamped fields to the unclamped ones, influencing the final textile behavior. We spot four representative cases: textiles with almost glued fibers ($g_\varepsilon \sim \varepsilon^4 g$ or higher), with strong contact ($g_\varepsilon \sim \varepsilon^3 g$), with loose contact ($g_\varepsilon \sim \varepsilon^2 g$) and with very loose contact ($g_\varepsilon \sim \varepsilon g$). We collect all the estimates for the fields in the final Table 5.1, and draw some a priori conclusions on the displacement behaviors.

In Chapter 6, we briefly analyze the almost glued fibers, the strong contact, and the very loose contact case. The homogenization for the first two cases has already been achieved in Griso, Orlik, and Wackerle, 2020a, and we will not investigate it further. However, we will reach the same final displacement decomposition with the newly developed lattice strategy and recall the results in the conclusive chapter. The case of a very loose contact textile assumes $g_\varepsilon \sim \varepsilon g$, leading to a trivial configuration: the contact is so loose that, with the applied model, we completely lose information on the in-plane fields in the unsupported domains. Even the assumption of completely stitching the left and bottom boundary of Ω (glued conditions) does not help. Hence, a study in a woven context is of no use.

Chapter 7 deals with the loose contact case ($g_\varepsilon \sim \varepsilon^2 g$), and it can be considered the core of the work. The full homogenization is done for this case, together with the newly developed tools. We assume the gap function g_ε only in the in-plane components since it is possible to prove (see Lemma 19) that in the outer-plane direction, the estimate of the displacements' difference does not depend on the contact due to the woven behavior of the fibers crossing each other.

We start by giving sufficient forces to obtain the strain tensor bound (1.2) to stay in a linear regime. With the choice of (1.2), the ratio $r \sim \varepsilon$, and the contact strength $h = 2$, we get the explicit estimates for the displacement fields' bound.Due to compactness, the fields converge weakly in the space.

The unfolding process goes through different steps. We first show the weak convergences of the unfolded fields, using the results in Chapter 2-4. We define three operators for the textile, all in relation with each other: $\mathcal{T}_\varepsilon^{\mathfrak{G}}$ for the unfolding of the yarns' middle lines, Π_ε for the whole three-dimensional textile structure, and $\mathcal{T}_\varepsilon^{\mathbf{C}}$ for the unfolding of the contact areas, thus where the yarns are above each other. Once we find the weak limits via unfolding of the displacement fields, the form of the strain tensors, and the contact conditions, we define the limit set of admissible displacements $\mathcal{X}$.

In order to go to the limit with problem (1.1), we also need to construct suitable test functions. Namely, they must have sufficient regularity to be dense in the limit set of displacements and ensure strong convergence via unfolding, give the same limit contact conditions, limit strain tensors, and satisfy the contact conditions before the limit.

At last, we can finally go to the limit via unfolding for $\varepsilon \to 0$ and find the limit problem (7.61), whose existence is again ensured by the Stampacchia Lemma. According to the procedure in Chapter 5.6 of Cioranescu, Damlamian, and Griso, 2018, we split the microscopic scale from the macroscopic one, find the correctors of the problem, the homogenizing operator, and the macroscopic problem.

Chapter 8 is the conclusive chapter, where we give an overview of the results and do some final considerations.

Concerning the extension of the unfolding method to anisotropically bounded functions and functions defined on lattice structures, we mention their applicability to a context wider than textiles, such as structures made of beams (lattice-like in $\mathbb{R}^3$) and in an unstable configuration (anisotropic behavior).

Concerning the main object of our study, small deformation of textiles with contact sliding, from Griso, Orlik, and Wackerle, 2020a and the newly achieved results, we gather the results from the homogenized problems and the final approximation of the displacement for each case. Then, from a comparison, we draw the following qualitative considerations:

A. In all cases, the woven nature of the textile allows the displacements in the third direction to stay sufficiently close. This fact is of particular importance when the contact is loose or very loose;

B. The contact determines the linearity of the homogenized problem. In particular, with almost glued fibers, we have a linear macroscopic problem; with strong contact we have a Leray-Lions equality; with loose contact, a Leray-Lions inequality; with very loose contact, we have an in-plane separation of the problem for the two independent beams of yarns;

C. If the contact is strong or almost glued, the displacement behaves the same in the whole domain Ω, despite a partial clamp. Moreover, the fibers do not have in-plane rotation (tend to stay straight). On the other hand, if the contact is loose or very loose, the displacement behaves differently in Ω_1-Ω_4, and in-plane rotation appears in the unsupported domains. Such phenomena can be observed in reality (see Figure 1.2);

D. The macroscopic limit contact conditions give us a qualitative bound for the in-plane rotations. The maximum slide depends on the L^∞ norm of g.

In general, we can say that this work offers a detailed mathematical explanation of phenomena that involve friction between fibers and its consequences on both microscopic and macroscopic scales and ends the study of textiles made of yarns in the linear elasticity context with contact sliding. However, it gives access to further investigations concerning a different ratio between ε and r, different periodicity patterns, and other elasticity regimes, such as the nonlinear one.

For the rest of the work, the Einstein convention over repeated indexes will be used.
Moreover, if not specified, the constants C, C_0 and C_1 do not depend on the parameter ε.

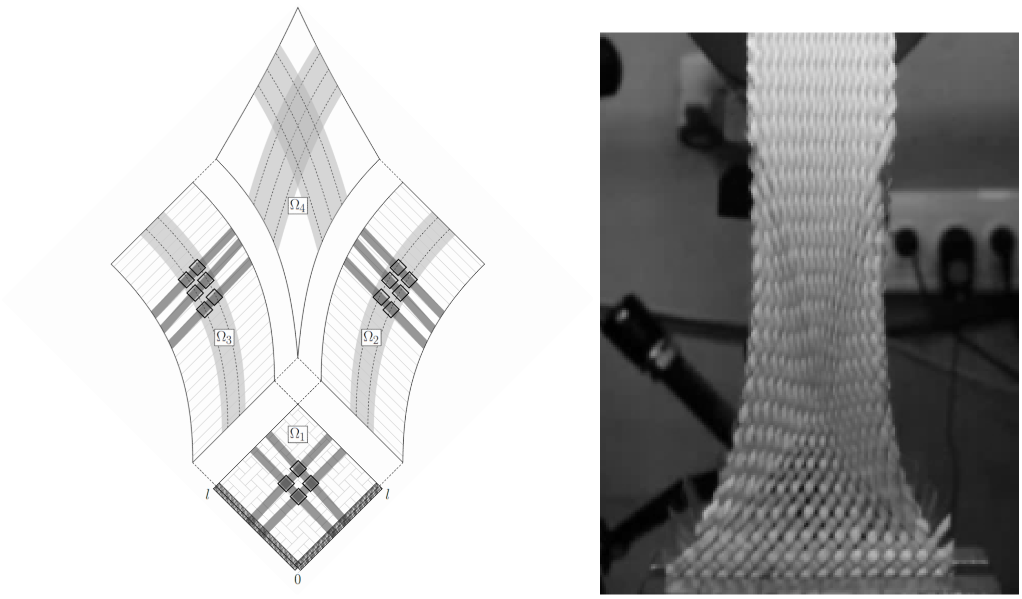

FIGURE 1.2: On the left, we have a mathematical sketch of the analysis of yarn's deformations in each textile part. On the right, we have a real experiment for textile tension with 45^o to the yarn directions.

Chapter 2

Preliminaries

In this chapter, we briefly recall some known definitions and results that often occur throughout the rest of the thesis. We can group them into two main sections. The first one concerns the classical periodic unfolding method and its main properties. The second one is focused on the N-linear and N-cubic interpolation of functions defined on a unit cell.

2.1 The periodic unfolding method

The periodic unfolding is our main homogenization tool. It takes bounded sequences on periodically paved domains and operates a scale splitting so that in the limit, we have a macroscopic behavior of the structure and a microscopic behavior, or cell behavior. Among many works that contributed to the development of this method, we will often refer to the most recent Cioranescu, Damlamian, and Griso, 2018, where most of the results are rigorously gathered.

Let $\mathbb{R}^N$ be the euclidean space with usual basis $(\mathbf{e}_1, \dots, \mathbf{e}_N)$ and $Y = (0,1)^N$ the open unit parallelotope associated with this basis. For a.e. $z \in \mathbb{R}^N$, we set the unique decomposition $z = [z]_Y + \{z\}_Y$ such that

$$[z]_Y \doteq \sum_{i=1}^{N} k_i \mathbf{e}_i, \;\; k_i \in \mathbb{Z}^N \quad \text{and} \quad \{z\}_Y \doteq z - [z]_Y \in Y.$$

In fact, instead of the grid $\mathbb{Z}^N$, we could use a more general lattice structure, but since we will not need it, we omit it for simplicity.

Let $\{\varepsilon\}$ be a sequence of strictly positive parameters going to 0. We scale our paving by ε writing

$$x = \varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\left\{\frac{x}{\varepsilon}\right\}_Y \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{2.1}$$

Let now Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary. We consider the covering

$$\Xi_\varepsilon \doteq \left\{\xi \in \mathbb{Z}^N \;\middle|\; \varepsilon(\xi + Y) \subset \Omega\right\}$$

and set (see also Figure 2.1 left)

$$\widehat{\Omega}_\varepsilon \doteq \operatorname{int}\left\{\bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\right\}, \qquad \Lambda_\varepsilon \doteq \Omega \setminus \widehat{\Omega}_\varepsilon. \tag{2.2}$$

We recall the definitions of classical unfolding operator and mean value operator from Cioranescu, Damlamian, and Griso, 2018, Definition 1.2.

Definition 1. *For every measurable function ϕ on $\widehat{\Omega}_\varepsilon$, the unfolding operator $\mathcal{T}_\varepsilon$ is defined as follows:*

$$\mathcal{T}_\varepsilon(\phi) \doteq \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{for a.e. } (x,y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for a.e. } (x,y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

It is important to note that such an operator acts on functions defined in Ω by operating on their restriction to $\widehat{\Omega}_\varepsilon$. As shown in Figure 2.1, the operator splits the function variables into reference cell number and variable position in the cell. In the limit, we obtain a split of the macroscopic scale (domain Ω) from the microscopic one (reference cell Y).

Together with the definition of unfolding operator, we have the notion of mean value operator.

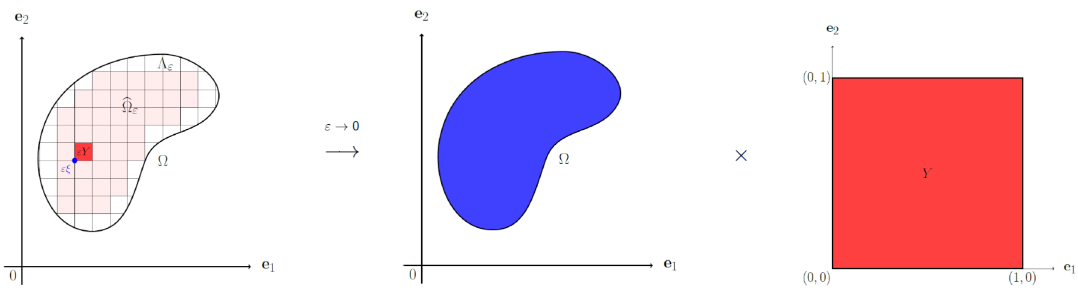

FIGURE 2.1: The unfolding via $\mathcal{T}_\varepsilon$ of the variables in $\Omega \subset \mathbb{R}^2$. One has a split of the macroscopic and microscopic scale in the limit.

ator. This operator takes unfolded functions and integrates them over the periodicity cell so that only the macroscopic part is left. We recall the definition from Cioranescu, Damlamian, and Griso, 2018, Definition 1.10

Definition 2. *For every measurable function $\widehat{\phi}$ on $L^1(\Omega \times Y)$, the mean value operator $\mathcal{M}_Y$ is defined as follows:*

$$\mathcal{M}_Y(\widehat{\phi})(x) \doteq \frac{1}{|Y|}\int_Y \widehat{\phi}(x,y)dy, \qquad \textit{for a.e. } x \in \Omega.$$

Let $p \in [1,+\infty]$ and denote by $L^p(\Omega)$ the subspace of measurable functions f such that

$$\|f\|_p \equiv \Big(\int_\Omega |f(x)|^p\,dx\Big)^{\frac{1}{p}} < +\infty.$$

From Cioranescu, Damlamian, and Griso, 2018, Propositions 1.8 and 1.11, we recall the properties of these periodic unfolding and mean value operators:

$$\begin{aligned} &\|\mathcal{T}_\varepsilon(\phi)\|_{L^p(\Omega\times Y)} \le |Y|^{\frac{1}{p}}\|\phi\|_{L^p(\Omega)} \quad \text{for every } \phi \in L^p(\Omega),\\ &\|\mathcal{M}_Y(\widehat{\phi})\|_{L^p(\Omega)} \le |Y|^{-\frac{1}{p}}\|\widehat{\phi}\|_{L^p(\Omega\times Y)} \quad \text{for every } \widehat{\phi} \in L^p(\Omega\times Y).\end{aligned} \tag{2.3}$$

At last, we recall the following definitions concerning Sobolev spaces:

$$\begin{aligned} W^{1,p}_{per}(Y) &\doteq \{\phi \in W^{1,p}(Y) \mid \phi \text{ is periodic with respect to } y_i, i \in \{1,\dots,N\}\},\\ W^{1,p}_{per,0}(Y) &\doteq \{\phi \in W^{1,p}_{per}(Y) \mid \mathcal{M}_Y(\phi) = 0\},\\ L^p(\Omega; W^{1,p}(Y)) &\doteq \{\phi \in L^p(\Omega\times Y) \mid \nabla_y\phi \in L^p(\Omega\times Y)^N\}.\end{aligned} \tag{2.4}$$

2.1.1 Asymptotic behavior of (isotropically) bounded functions

Now, we recall some known results concerning the unfolding method for the following classes of bounded functions. Namely, we consider the following:

(i) Sequences $\{\phi_\varepsilon\}_\varepsilon \in W^{1,p}(\Omega)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla\phi_\varepsilon\|_{L^p(\Omega)} \le C$;

(ii) Sequences $\{\phi_\varepsilon\}_\varepsilon \in W^{1,p}(\Omega)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla\phi_\varepsilon\|_{L^p(\Omega)} \le C$;

(iii) Sequences $\{\phi_\varepsilon\}_\varepsilon \in W^{2,p}(\Omega)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla\phi_\varepsilon\|_{L^p(\Omega)} + \|D^2\phi_\varepsilon\|_{L^p(\Omega)} \le C$.

As we can see, the notion of "isotropic bound" comes from the fact that the partial derivatives of the functions' gradients are bounded with the same order concerning all N directions.

Concerning the asymptotic behavior of sequences bounded as in (i), we recall the following proposition from Cioranescu, Damlamian, and Griso, 2018, Theorem 1.36.

Proposition 1. *Let $p \in (1, +\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\Omega)$ such that*

$$\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla\phi_\varepsilon\|_{L^p(\Omega)} \leq C.$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\phi \in L^p(\Omega)$, $\widehat{\phi} \in L^p(\Omega; W^{1,p}_{per,0}(Y))$ such that

$$\begin{aligned} &\phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in} \quad L^p(\Omega), \\ &\mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \phi + \widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(Y)), \\ &\varepsilon\mathcal{T}_\varepsilon(\nabla\phi_\varepsilon) = \nabla_y(\mathcal{T}_\varepsilon(\phi_\varepsilon)) \rightharpoonup \nabla_y\widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y)^N. \end{aligned}$$

The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Sometimes, in Proposition 1 we find convenient to replace the sum $\phi + \widehat{\phi}$, with $\phi \in L^p(\Omega)$ and $\widehat{\phi} \in L^p(\Omega; W^{1,p}_{per,0}(Y))$, by a unique function $\widehat{\phi} \in L^p(\Omega, W^{1,p}_{per}(Y))$.
Concerning the asymptotic behavior of sequences bounded as in (ii), we recall the following results from Cioranescu, Damlamian, and Griso, 2018, Corollary 1.37 and Theorem 1.41

Proposition 2. *Let $p \in (1, +\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\Omega)$ such that*

$$\phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in} \quad W^{1,p}(\Omega). \tag{2.5}$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\widehat{\phi} \in L^p(\Omega; W^{1,p}_{per,0}(Y))$ such that

$$\begin{aligned} &\mathcal{T}_\varepsilon(\phi_\varepsilon) \to \phi \quad \text{strongly in} \quad L^p(\Omega; W^{1,p}(Y)), \\ &\mathcal{T}_\varepsilon(\nabla\phi_\varepsilon) \rightharpoonup \nabla\phi + \nabla_y\widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y)^N, \\ &\frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon(\phi_\varepsilon) - \mathcal{M}_{\mathcal{Y}}(\phi_\varepsilon)\Big) \rightharpoonup y^c \cdot \nabla\phi + \widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(Y)). \end{aligned}$$

where $y^c \doteq y - \mathcal{M}_Y(y)$.
The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Sometimes, we replace hypothesis (2.5) with

$$\exists C > 0 \quad \text{such that} \quad \|\phi_\varepsilon\|_{W^{1,p}(\Omega)} \leq C,$$

which is an equivalent formulation due to compactness results.
At last, the unfolding for sequences bounded as in (iii) is treated according to the case $k = 2$ of Cioranescu, Damlamian, and Griso, 2018, Theorem 1.47. Even though such theorem holds for every sequence such that

$$\|\phi_\varepsilon\|_{W^{k,p}(\Omega)} \doteq \|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla\phi_\varepsilon\|_{L^p(\Omega)} + \|D^2\phi_\varepsilon\|_{L^p(\Omega)} + \ldots + \|D^k\phi_\varepsilon\|_{L^p(\Omega)} \leq C,$$

with $k \in \mathbb{N}^*$, we will not investigate higher orders since we do not need them.

Proposition 3. *Let $p \in (1, +\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{2,p}(\Omega)$ such that*

$$\phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in} \quad W^{2,p}(\Omega).$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\widehat{\phi} \in L^p(\Omega; W^{2,p}_{per}(Y))$ such that

$$\begin{aligned} &\mathcal{T}_\varepsilon(\phi_\varepsilon) \to \phi \quad \text{strongly in} \quad L^p(\Omega; W^{2,p}(Y)), \\ &\mathcal{T}_\varepsilon(\nabla\phi_\varepsilon) \to \nabla\phi \quad \text{strongly in} \quad L^p(\Omega; W^{1,p}(Y))^N, \\ &\mathcal{T}_\varepsilon(D^2\phi_\varepsilon) \rightharpoonup D^2\phi + D^2_y\widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N\times N}. \end{aligned}$$

The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

2.2 The approximation of functions to linear and cubic interpolates

In this section, we discuss another recurrent concept: the decomposition of functions defined on one-dimensional structures in $\mathbb{R}^N$ into interpolation on nodes and remainder term. Such approximation, which can be linear or cubic depending on the regularity of the original function, is of great importance since it can be extended from the one-dimensional structure to the whole space and vice versa.

For a one-dimensional structure in $\mathbb{R}^N$, we consider the simplest possible: the grid connecting the vertices of a unitary cell. Needless to say, all the proven results can be easily adapted to any parallelotope of fixed lengths.

Let $Y = (0,1)^N$ be the N-dimensional unit cell. We denote the set of vertices of Y by

$$\mathcal{V} \doteq \left\{ v \in \mathbb{R}^N \;\middle|\; v = \sum_{i=1}^{N} v_i \mathbf{e}_i, \quad v_i \in \{0,1\} \right\}$$

We denote $\mathcal{G}_c^{(i)}$ and $\mathcal{G}^{(i)}$ the following sets of segments whose direction is $\mathbf{e}_i$ by

$$\mathcal{G}_c^{(i)} \doteq \bigcup_{v_i=0} [v, v+\mathbf{e}_i], \qquad \mathcal{G}^{(i)} \doteq \big[(0,\dots,0), (0,\dots,0)+\mathbf{e}_i\big]$$

Hence, the one-dimensional grid constructed as the union of vertices of the cell $\overline{Y}$ is defined by

$$\mathcal{G}_c \doteq \bigcup_{i=1}^{N} \mathcal{G}_c^{(i)} \subset \overline{Y}, \qquad \mathcal{G} \doteq \bigcup_{i=1}^{N} \mathcal{G}^{(i)} \subset \overline{Y}.$$

The difference between the two grids is that one is complete (hence the letter "c"), as we can see in Figure 2.2. In these sections, we will always deal with the complete grid $\mathcal{G}_c$, even

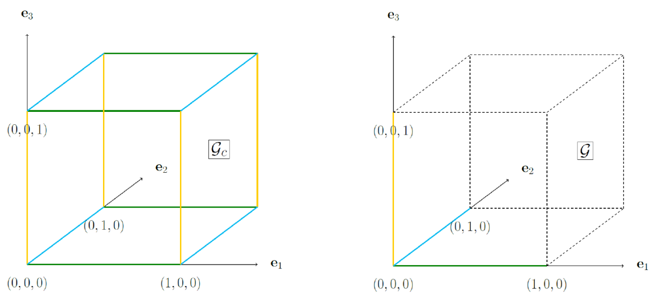

FIGURE 2.2: The complete grid $\mathcal{G}_c$ and the not complete one $\mathcal{G}$ for a reference cell $Y \subset \mathbb{R}^3$.

though this will fade later in the chapters when the considered structures consist of many rescaled reference grids $\mathcal{G}$ periodically repeated.

Now, let ε be a small parameter. We define εY, which consists of the cell Y but is rescaled by a small parameter ε. Accordingly, we rescale the grid and obtain

$$\mathcal{G}_{c,\varepsilon} \doteq \varepsilon \mathcal{G}_c \subset \varepsilon \overline{Y}, \qquad \mathcal{G}_\varepsilon \doteq \mathcal{G} \subset \varepsilon \overline{Y}.$$

Denote $\mathbf{G}$ the running point of $\mathcal{G}_c$ and $\mathbf{g}$ that of $\mathcal{G}_{c,\varepsilon}$. That gives ($i \in \{1,\dots,N\}$)

$$\mathbf{G} = v + t\mathbf{e}_i \quad \text{in } \mathcal{G}_c^{(i)}, \qquad t \in [0,1], \quad v_i = 0,$$
$$\mathbf{g} = \varepsilon v + \varepsilon t\mathbf{e}_i \quad \text{in } \mathcal{G}_{c,\varepsilon}^{(i)}, \qquad t \in [0,1], \quad v_i = 0.$$

Let $\mathcal{C}(\mathcal{G}_c)$ and $\mathcal{C}(\mathcal{G}_{c,\varepsilon})$ be the spaces of continuous functions defined on $\mathcal{G}_c$ and $\mathcal{G}_{c,\varepsilon}$ respectively. Let $i \in \{1,\dots,N\}$. We denote the spaces of functions defined on the segments in the

i-th direction and on the whole unit grid by

$$W^{1,p}(\mathcal{G}_c^{(i)}) \doteq \{\phi \in L^p(\mathcal{G}_c^{(i)}) \mid \partial_{\mathbf{G}}\phi \in L^p(\mathcal{G}_c^{(i)})\},$$
$$W^{1,p}(\mathcal{G}_c) \doteq \{\phi \in \mathcal{C}(\mathcal{G}_c) \mid \partial_{\mathbf{G}}\phi \in L^p(\mathcal{G}_c)\}.$$

and

$$W^{2,p}(\mathcal{G}_c^{(i)}) \doteq \{\phi \in W^{1,p}(\mathcal{G}_c^{(i)}) \mid \partial_{\mathbf{G}}\phi \in W^{1,p}(\mathcal{G}_c^{(i)})\},$$
$$W^{2,p}(\mathcal{G}_c) \doteq \{\phi \in \mathcal{C}(\mathcal{G}_c) \mid \partial_{\mathbf{G}}\phi_{|\mathcal{G}_c^{(j)}} \in W^{1,p}(\mathcal{G}_c^{(j)}),\ j \in \{1,\dots,N\}\}.$$

Accordingly, we define the spaces on the rescaled grid by

$$W^{1,p}(\mathcal{G}_{c,\varepsilon}^{(i)}) \doteq \{\phi \in L^p(\mathcal{G}_{c,\varepsilon}^{(i)}) \mid \partial_{\mathbf{g}}\phi \in L^p(\mathcal{G}_{c,\varepsilon}^{(i)})\},$$
$$W^{1,p}(\mathcal{G}_{c,\varepsilon}) \doteq \{\phi \in \mathcal{C}(\mathcal{G}_{c,\varepsilon}) \mid \partial_{\mathbf{g}}\phi \in L^p(\mathcal{G}_{c,\varepsilon})\}.$$

and

$$W^{2,p}(\mathcal{G}_{c,\varepsilon}^{(i)}) \doteq \{\phi \in W^{1,p}(\mathcal{G}_{c,\varepsilon}^{(i)}) \mid \partial_{\mathbf{g}}\phi \in W^{1,p}(\mathcal{G}_{c,\varepsilon}^{(i)})\},$$
$$W^{2,p}(\mathcal{G}_{c,\varepsilon}) \doteq \{\phi \in \mathcal{C}(\mathcal{G}_{c,\varepsilon}) \mid \partial_{\mathbf{g}}\phi_{|\mathcal{G}_\varepsilon^{(j)}} \in W^{1,p}(\mathcal{G}_{c,\varepsilon}^{(j)}),\ j \in \{1,\dots,N\}\}.$$

Here again, even if it is possible to extend the definition of the spaces $W^{k,p}$ to every $k \in \mathbb{N}$, we will not do it since these cases will not be considered. Hence, we omit them for the sake of simplicity.

2.2.1 The N-linear interpolation

Let f be a function belonging to $W^{1,p}(0,1)$. Denote f_a the affine function

$$f_a(t) \doteq f(0) + t\big(f(1) - f(0)\big), \qquad t \in [0,1], \tag{2.6}$$

and f_0 the reminder function vanishing at the extremities

$$f_0(t) \doteq f(t) - f_a(t), \qquad t \in [0,1].$$

Define the spaces of affine functions defined on the unit grid and the rescaled one by

$$Q^1(\mathcal{G}_c) \doteq \{\phi \in W^{1,\infty}(\mathcal{G}_c) \mid \phi \text{ is the linear interpolation between two adjacent vertices of } \mathcal{G}_c.\},$$
$$Q^1(\mathcal{G}_{c,\varepsilon}) \doteq \{\psi \in W^{1,\infty}(\mathcal{G}_{c,\varepsilon}) \mid \psi \text{ is the linear interpolation between two adjacent vertices of } \mathcal{G}_{c,\varepsilon}.\}$$

and the spaces of functions vanishing on the vertices of the unit grid and the rescaled one by

$$\mathcal{W}^{1,p}_{0,\mathcal{V}}(\mathcal{G}_c) \doteq \{\psi \in W^{1,p}(\mathcal{G}_c) \mid \psi = 0 \ \text{ on every } v \in \mathcal{V}\},$$
$$\mathcal{W}^{1,p}_{0,\mathcal{V}_\varepsilon}(\mathcal{G}_{c,\varepsilon}) \doteq \{\phi \in W^{1,p}(\mathcal{G}_{c,\varepsilon}) \mid \phi = 0 \ \text{ on every } \varepsilon v,\ v \in \mathcal{V}\}.$$

Now, since the grid $\mathcal{G}_c$ (resp. the rescaled grid $\mathcal{G}_{c,\varepsilon}$) is a union of intervals, we can decompose any function $\psi \in W^{1,p}(\mathcal{G}_c)$ (resp. $\phi \in W^{1,p}(\mathcal{G}_{c,\varepsilon})$) into an affine function, which coincides with the original one on each vertex of the grid, and a reminder function that is zero on each vertex:

$$\begin{aligned} &\psi = \psi_a + \psi_0, \qquad \psi_a \in Q^1(\mathcal{G}_c), \quad \psi_0 \in \mathcal{W}^{1,p}_{0,\mathcal{V}}(\mathcal{G}_c), \\ &\big(\text{resp. } \ \psi = \phi_a + \phi_0, \qquad \phi_a \in Q^1(\mathcal{G}_{c,\varepsilon}), \quad \phi_0 \in \mathcal{W}^{1,p}_{0,\mathcal{V}_\varepsilon}(\mathcal{G}_{c,\varepsilon})\big). \end{aligned} \tag{2.7}$$

Such decomposition is unique, and we have the following estimates.

Lemma 1. *Let $i \in \{1,\dots,N\}$ and $\psi \in W^{1,p}(\mathcal{G}_c)$. Suppose that ψ is decomposed as in (2.7)$_1$. Then, there exists a constant $C > 0$, which does not depend on ε, such that*

$$\|\partial_{\mathbf{G}}\psi_a\|_{L^p(\mathcal{G}_c^{(i)})} \le C\|\psi_a\|_{L^p(\mathcal{G}_c^{(i)})} \tag{2.8}$$

and

$$\begin{aligned} &\|\partial_{\mathbf{G}}\psi_a\|_{L^p(\mathcal{G}_c^{(i)})} \le C\|\partial_{\mathbf{G}}\psi\|_{L^p(\mathcal{G}_c^{(i)})}, \\ &\|\psi_0\|_{L^p(\mathcal{G}_c^{(i)})} + \|\partial_{\mathbf{G}}\psi_0\|_{L^p(\mathcal{G}_c^{(i)})} \le C\|\partial_{\mathbf{G}}\psi\|_{L^p(\mathcal{G}_c^{(i)})}. \end{aligned} \tag{2.9}$$

Let $\phi \in W^{1,p}(\mathcal{G}_{c,\varepsilon})$ and suppose that ϕ is decomposed as in (2.7)$_2$. Then, there exists $C > 0$, such that

$$\|\partial_{\mathbf{g}}\phi_a\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})} \le \frac{C}{\varepsilon}\|\phi_a\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})} \tag{2.10}$$

and

$$\begin{aligned} &\|\partial_{\mathbf{g}}\phi_a\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})} \le C\|\partial_{\mathbf{g}}\phi\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})}, \\ &\|\phi_0\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})} + \varepsilon\|\partial_{\mathbf{g}}\phi_0\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})} \le C\varepsilon\|\partial_{\mathbf{g}}\phi\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})}. \end{aligned} \tag{2.11}$$

Proof. A simple computation on (2.6) and the Poincaré's inequality give

$$\|\psi_a'\|^p_{L^p(0,1)} = \int_0^1 |\psi_a'(t)|^p dt = \int_0^1 |\psi_a(1) - \psi_a(0)|^p dt \le \sum_{v=0}^{1} |\psi_a(v)|^p = \|\psi_a\|^p_{L^p(0,1)}$$

and

$$\begin{aligned} &\|\psi_a'\|_{L^p(0,1)} \le \|\psi'\|_{L^p(0,1)}, \\ &\|\psi_0\|_{W^{1,p}(0,1)} \le C\|\psi_0'\|_{L^p(0,1)} \le C\|\psi' - \psi_a'\|_{W^{1,p}(0,1)} \le 2C\|\psi'\|_{L^p(0,1)}. \end{aligned}$$

Hence, estimates (2.8) and (2.9) follow by the fact that $\mathcal{G}_c^{(i)}$ is the union of a finite number of segments whose extremities belong to $\mathcal{V}$.
The proof of estimates (2.10)-(2.11) is done in the same fashion, but taking into account that now the interval rescaled of ε, thus the Poincaré's inequality becomes

$$\|\phi_0\|_{W^{1,p}(0,\varepsilon)} \le C\varepsilon\|\phi_0'\|_{L^p(0,\varepsilon)}.$$

□

The main advantage of this decomposition is that the function ψ_a, which is affine on the grid segments $\mathcal{G}_c$, can be extended by N-linear interpolation to the whole cell Y.

Definition 3. *For every function $\psi \in Q^1(\mathcal{G}_c)$ (resp. $\phi \in Q^1(\mathcal{G}_{c,\varepsilon})$), its extension $\mathfrak{Q}(\psi) \in W^{1,\infty}(Y)$ (resp. $\mathfrak{Q}(\phi) \in W^{1,\infty}(\varepsilon Y)$) is defined as the N-linear interpolation on each vertex of the cell Y (resp. of the cell εY).*

This extension is injective: a function belonging to $Q^1(\mathcal{G}_c)$ is uniquely determined by its values on the set of vertices $\mathcal{V}$ and thus can be naturally extended to a function defined on Y. We also make it surjective by defining the spaces

$$\begin{aligned} &Q^1(Y) \doteq \Big\{\Psi \in W^{1,\infty}(Y) \,\Big|\, \Psi_{|Y} \text{ is the } Q^1 \text{ interpolate of its values on the vertices of } Y\Big\}, \\ &Q^1(\varepsilon Y) \doteq \Big\{\Phi \in W^{1,\infty}(\varepsilon Y) \,\Big|\, \Phi_{|\varepsilon Y} \text{ is the } Q^1 \text{ interpolate of its values on the vertices of } \varepsilon Y\Big\}. \end{aligned}$$

Hence, the extension $\mathfrak{Q}$ is one-to-one from $Q^1(\mathcal{G}_c)$ to $Q^1(Y)$ (resp. from $Q^1(\mathcal{G}_{c,\varepsilon})$ to $Q^1(\varepsilon Y)$). Its inverse is the mere restriction of functions from the cell to the grid $_{|\mathcal{G}_c}$ (resp. $_{|\mathcal{G}_{c,\varepsilon}}$).
Below, we show the main properties of the extension operator $\mathfrak{Q}$.

Lemma 2. *Let $i \in \{1,\dots,N\}$ and $p \in [1,+\infty]$. For every $\psi \in Q^1(\mathcal{G}_c)$, there exist C_0, $C_1 > 0$ such that ($i \in \{1,\dots,N\}$)*

$$\begin{aligned} &C_0\|\mathfrak{Q}(\psi)\|_{L^p(Y)} \le \|\psi\|_{L^p(\mathcal{G}_c)} \le C_1\|\mathfrak{Q}(\psi)\|_{L^p(Y)}, \\ &C_0\|\partial_i\mathfrak{Q}(\psi)\|_{L^p(Y)} \le \|\partial_{\mathbf{G}}\psi\|_{L^p(\mathcal{G}_c^{(i)})} \le C_0\|\partial_i\mathfrak{Q}(\psi)\|_{L^p(Y)}. \end{aligned} \tag{2.12}$$

For every $\phi \in Q^1(\mathcal{G}_{c,\varepsilon})$, there exist $C_0, C_1 > 0$ such that ($i \in \{1, \dots, N\}$)

$$\begin{aligned} C_0\|\mathfrak{Q}(\phi)\|_{L^p(\varepsilon Y)} &\le C\varepsilon^{\frac{N-1}{p}}\|\phi\|_{L^p(\mathcal{G}_{c,\varepsilon})} \le C_1\|\mathfrak{Q}(\phi)\|_{L^p(\varepsilon Y)}, \\ C_0\|\partial_i\mathfrak{Q}(\phi)\|_{L^p(\varepsilon Y)} &\le C\varepsilon^{\frac{N-1}{p}}\|\partial_{\mathbf{g}}\phi\|_{L^p(\mathcal{G}^{(i)}_{c,\varepsilon})} \le C_1\|\partial_i\mathfrak{Q}(\phi)\|_{L^p(\varepsilon Y)}. \end{aligned} \tag{2.13}$$

Proof. We will only consider the case $p \in [1, +\infty)$ since the case $p = +\infty$ is trivial.
First, remind that for every function ψ defined as the N-linear interpolation of its values on the vertices of the cell Y, there exist $C_0, C_1 > 0$ such that ($i \in \{1, \dots, N\}$)

$$\begin{aligned} C_0\|\psi\|_{L^p(Y)} &\le \Big(\sum_{v\in\mathcal{V}}|\psi(v)|^p\Big)^{1/p} \le C_1\|\psi\|_{L^p(\mathcal{G})}, \\ C_0\|\partial_i\psi\|_{L^p(Y)} &\le \|\partial_{\mathbf{G}}\psi\|_{L^p(\mathcal{G}^{(i)})} \le C_1\|\partial_i\psi\|_{L^p(Y)}. \end{aligned} \tag{2.14}$$

where the constants do not depend on p. This proves (2.12).
We now prove $(2.13)_1$. For every $\phi \in Q^1(\mathcal{G}_\varepsilon)$, set $\Phi = \mathfrak{Q}(\phi)$. From $(2.14)_1$ and an affine change of variables, we easily get

$$\int_{\varepsilon Y}|\Phi(x)|^p dx = \varepsilon^N\int_Y|\Phi(\varepsilon y)|^p dy = \varepsilon^N\int_{\mathcal{G}}|\Phi(\varepsilon\mathbf{G})|^p d\mathbf{G} = \varepsilon^{N-1}\int_{\mathcal{G}_\varepsilon}|\Phi(\mathbf{g})|^p d\mathbf{g}$$

and thus $(2.13)_1$ holds since $\Phi_{|\mathcal{G}_\varepsilon} = \phi$.
We prove now $(2.13)_2$. Let i be in $\{1, \dots, N\}$. From $(2.14)_2$ and an affine change of variables, we have

$$\begin{aligned} \int_{\varepsilon Y}\Big|\frac{\partial}{\partial x_i}\Phi(x)\Big|^p dx = \varepsilon^{N-p}\int_Y\Big|\frac{\partial}{\partial y_i}\Phi(\varepsilon y)\Big|^p dy &= \varepsilon^{N-p}\int_{\mathcal{G}^{(i)}}\big|\partial_{\mathbf{G}}\Phi(\varepsilon\mathbf{G})\big|^p d\mathbf{G} \\ &= \varepsilon^{N-1}\int_{\mathcal{G}^{(i)}_\varepsilon}\big|\partial_{\mathbf{g}}\Phi(\mathbf{g})\big|^p d\mathbf{g}. \end{aligned}$$

And thus $(2.13)_2$ holds since $\Phi_{|\mathcal{G}^{(i)}_\varepsilon} = \phi_{|\mathcal{G}^{(i)}_\varepsilon}$. ☐

To conclude, we summarize what we did in this section in Figure 2.3.

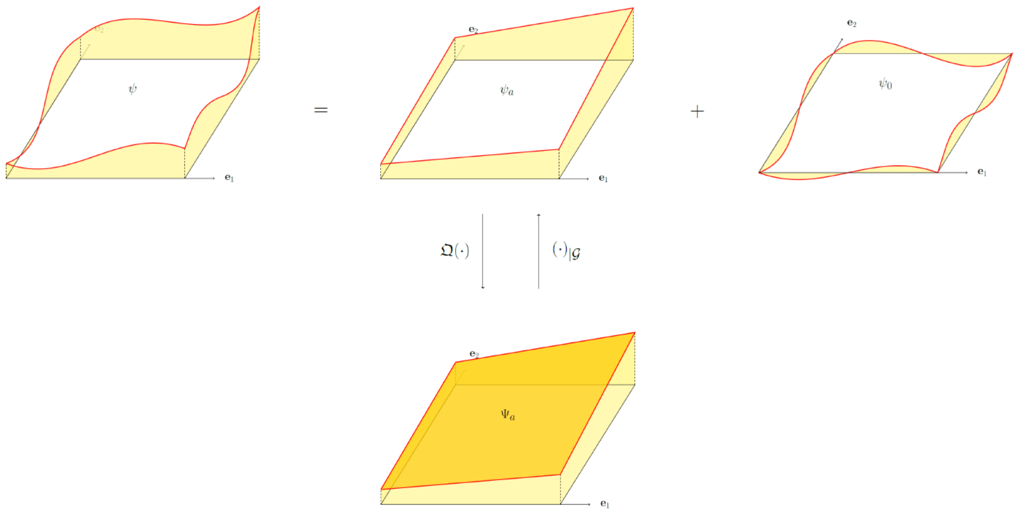

FIGURE 2.3: The decomposition of a function $\psi \in W^{1,p}(Y)$, with $Y \in \mathbb{R}^2$, into linear interpolation on the vertices of the cell and remainder. The interpolation on the vertices can be extended one-to-one to the whole domain.

2.2.2 The N-cubic interpolation

We would like now to apply the same decomposition but for functions defined on $W^{2,p}(\mathcal{G}_c)$ (and on $W^{2,p}(\mathcal{G}_{c,\varepsilon})$). As we will see, the adaptation will not be straightforward.
Let f be a function belonging to $W^{2,p}(0,1)$. Denote f_c the cubic polynomial

$$f_c(t) = f(0)(2t+1)(t-1)^2 + f(1)t^2(3-2t) + f'(0)t(t-1)^2 + f'(1)t^2(t-1), \qquad t \in [0,1]. \tag{2.15}$$

By construction, the reminder term defined by

$$f_0 \doteq f(t) - f_c(t), \qquad t \in [0,1].$$

vanishes at the extremities, as well as its first order derivatives:

$$f_0(0) = f_0(1) = f_0'(0) = f_0'(1) = 0.$$

Define the spaces of cubic polynomials defined on the unit grid and the rescaled one by

$$\begin{aligned}
\mathcal{Q}_3(\mathcal{G}_c) &\doteq \{\psi \in W^{2,\infty}(\mathcal{G}_c) \,|\, \psi \text{ is cubic interpolation between two adjacent vertices of } \mathcal{G}_c.\},\\
\mathcal{Q}_3(\mathcal{G}_{c,\varepsilon}) &\doteq \{\phi \in W^{2,\infty}(\mathcal{G}_{c,\varepsilon}) \,|\, \phi \text{ is cubic interpolation between two adjacent vertices of } \mathcal{G}_{c,\varepsilon}.\}
\end{aligned}$$

and the spaces of functions vanishing on the vertices, and with first derivative vanishing of the vertices, of the unit grid and the rescaled one by

$$\begin{aligned}
\mathcal{W}^{2,p}_{0,\mathcal{V}}(\mathcal{G}_c) &\doteq \{\psi \in W^{2,p}(\mathcal{G}_c) \;\big|\; \psi = \partial_{\mathbf{S}}\psi = 0 \;\text{ on every } v \in \mathcal{V}\},\\
\mathcal{W}^{2,p}_{0,\mathcal{V}_\varepsilon}(\mathcal{G}_{c,\varepsilon}) &\doteq \{\phi \in W^{2,p}(\mathcal{G}_{c,\varepsilon}) \;\big|\; \phi = \partial_{\mathbf{s}}\phi = 0 \;\text{ on every } \varepsilon v,\ v \in \mathcal{V}\}.
\end{aligned}$$

Similarly to the decomposition in the previous section, any $\psi \in W^{2,p}(\mathcal{G}_c)$ (resp. $\phi \in W^{1,p}(\mathcal{G}_{c,\varepsilon})$) can be decomposed as

$$\begin{aligned}
\psi &= \psi_c + \psi_0, &\quad \psi_c &\in Q^3(\mathcal{G}_c), &\quad \psi_0 &\in \mathcal{W}^{2,p}_{0,\mathcal{V}}(\mathcal{G}_c),\\
\big(\text{resp. }\ \phi &= \phi_c + \phi_0, &\quad \phi_c &\in Q^3(\mathcal{G}_{c,\varepsilon}), &\quad \phi_0 &\in \mathcal{W}^{2,p}_{0,\mathcal{V}_\varepsilon}(\mathcal{G}_{c,\varepsilon})\big).
\end{aligned} \tag{2.16}$$

Such decomposition is unique, and we have the following estimates.

Lemma 3. *Let $i \in \{1,\dots,N\}$ and $\psi \in W^{2,p}(\mathcal{G}_c)$. Suppose that ψ is decomposed as in (2.16)$_1$. Then, there exists $C > 0$ such that*

$$\begin{aligned}
&\|\partial^2_{\mathbf{G}}\psi_c\|_{L^p(\mathcal{G}_c^{(i)})} \le C\|\partial^2_{\mathbf{G}}\psi\|_{L^p(\mathcal{G}_c^{(i)})},\\
&\|\partial_{\mathbf{G}}\psi_c\|_{L^p(\mathcal{G}_c^{(i)})} \le C\|\partial_{\mathbf{G}}\psi\|_{W^{1,p}(\mathcal{G}_c^{(i)})},\\
&\|\psi_c\|_{L^p(\mathcal{G}_c^{(i)})} \le C\|\psi\|_{W^{2,p}(\mathcal{G}_c^{(i)})},\\
&\|\psi_0\|_{L^p(\mathcal{G}_c^{(i)})} + \|\partial_{\mathbf{G}}\psi_0\|_{L^p(\mathcal{G}_c^{(i)})} + \|\partial^2_{\mathbf{G}}\psi_0\|_{L^p(\mathcal{G}_c^{(i)})} \le C\|\partial^2_{\mathbf{G}}\psi\|_{L^p(\mathcal{G}_c^{(i)})}.
\end{aligned} \tag{2.17}$$

Let $\phi \in W^{2,p}(\mathcal{G}_{c,\varepsilon})$. Suppose that ϕ is decomposed as in (2.16)$_2$. Then, there exists $C > 0$ such that

$$\begin{aligned}
&\|\partial^2_{\mathbf{g}}\phi_c\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})} \le C\|\partial^2_{\mathbf{g}}\phi\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})},\\
&\|\partial_{\mathbf{g}}\phi_c\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})} \le C\|\partial_{\mathbf{g}}\phi\|_{W^{1,p}(\mathcal{G}_{c,\varepsilon}^{(i)})},\\
&\|\phi_c\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})} \le C\|\phi\|_{W^{2,p}(\mathcal{G}_{c,\varepsilon}^{(i)})},\\
&\|\phi_0\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})} + \varepsilon\|\partial_{\mathbf{g}}\phi_0\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})} + \varepsilon^2\|\partial^2_{\mathbf{g}}\phi_0\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})} \le C\varepsilon^2\|\partial^2_{\mathbf{g}}\phi\|_{L^p(\mathcal{G}_{c,\varepsilon}^{(i)})}.
\end{aligned} \tag{2.18}$$

Proof. Step 1. In this step we prove the result for $f \in W^{2,p}(0,1)$.
Let f_c be the cubic polynomial defined as in (2.15). Rewriting it differently and computing

the first and second order derivative, we get that

$$\begin{aligned}
f_c(t) &= \Big(f(1)-f(0)-\frac{1}{2}\big(f'(0)+f'(1)\big)\Big)t^2(3-2t)+\frac{1}{2}\big(f'(1)-f'(0)\big)t^2+f'(0)t+f(0),\\
f_c'(t) &= \Big(f(1)-f(0)-\frac{1}{2}\big(f'(0)+f'(1)\big)\Big)6t(1-t)+\big(f'(1)-f'(0)\big)t+f'(0),\\
f_c''(t) &= \Big(f(1)-f(0)-\frac{1}{2}\big(f'(0)+f'(1)\big)\Big)6(1-2t)+\big(f'(1)-f'(0)\big).
\end{aligned} \tag{2.19}$$

As a consequence, we have that

$$\begin{aligned}
&\|f_c''\|_{L^p(0,1)} \le C\|f''\|_{L^p(0,1)},\\
&\|f_c'\|_{L^p(0,1)} \le C\big(\|f''\|_{L^p(0,1)}+\|f'\|_{L^p(0,1)}\big),\\
&\|f_c\|_{L^p(0,1)} \le C\big(\|f''\|_{L^p(0,1)}+\|f'\|_{L^p(0,1)}+\|f\|_{L^p(0,1)}\big).
\end{aligned}$$

Moreover, from the definition of f_0, Poincaré's inequality applied twice, and the above estimates, we have that

$$\|f_0\|_{W^{2,p}(0,1)} \le C\|f_0''\|_{L^p(0,1)} \le C\|f''-f_c''\|_{L^p(0,1)} \le 2C\|f''\|_{L^p(0,1)}.$$

Step 2. We prove the statements of the lemma.

By construction, $\mathcal{G}_c^{(i)}$ is the union of a finite number of segments whose extremities belong to $\mathcal{V}$. Hence, estimates (2.17) follow from the estimates in Step 1 and an affine change of variables. The proof for estimates (2.18) is done in the same fashion, but taking into account that now the interval rescaled of ε, thus the Poincaré's inequality applied twice becomes

$$\|\phi_0\|_{W^{2,p}(0,\varepsilon)} \le C\varepsilon\|\phi_0'\|_{W^{1,p}(0,\varepsilon)} \le C\varepsilon^2\|\phi_0''\|_{L^p(0,\varepsilon)}.$$

□

Now, we would like to extend the function ϕ_c, defined on the grid segments $\mathcal{G}_c$, to the whole cell Y by N-cubic interpolation.

Definition 4. *For every function $\psi \in Q^3(\mathcal{G}_c)$ (resp. $\phi \in Q^3(\mathcal{G}_{c,\varepsilon})$), its extension $\mathfrak{Q}(\psi) \in W^{2,\infty}(Y)$ (resp. $\mathfrak{Q}(\phi) \in W^{2,\infty}(\varepsilon Y)$) is defined as the N-cubic interpolation on each vertex of the cell Y (resp. of the cell εY).*

It is clear that such extension is not surjective in the spaces:

$$\begin{aligned}
Q^3(Y) \doteq \Big\{\Psi \in W^{1,\infty}(Y)\ \Big|\ &\Psi_{|Y} \text{ is the } N\text{-cubic interpolate of its values and its partial}\\
&\text{derivatives values on the vertices of } Y\Big\},\\
Q^3(\varepsilon Y) \doteq \Big\{\Phi \in W^{1,\infty}(\varepsilon Y)\ \Big|\ &\Phi_{|\varepsilon Y} \text{ is the } N\text{-cubic interpolate of its values and its partial}\\
&\text{derivatives values on the vertices of } \varepsilon Y\Big\}.
\end{aligned}$$

Indeed, let $N=2$. In order to define the bi-cubic polynomial Ψ_c in dimension 2, we would need 16 coefficients. But from a function ψ_c defined on the grid $\mathcal{G}$, we only get 12:

- 4 coefficients are given by the function values on the vertices of the cell ($\psi_c(0,0)$, $\psi_c(1,0)$, $\psi_c(0,1)$ and $\psi_c(1,1)$);
- 4 coefficients are given by the values of the partial derivative of the function in direction $\mathbf{e}_1$ on the vertices of the cell ($\partial_1\psi_c(0,0)$, $\partial_1\psi_c(1,0)$, $\partial_1\psi_c(0,1)$ and $\partial_1\psi_c(1,1)$);
- 4 coefficients are given by the values of the partial derivative of the function in direction $\mathbf{e}_2$ on the vertices of the cell ($\partial_2\psi_c(0,0)$, $\partial_2\psi_c(1,0)$, $\partial_2\psi_c(0,1)$ and $\partial_2\psi_c(1,1)$).

The last four coefficients should be given by the mixed partial derivatives of the function on the vertices, which do not exist, since the function ψ_c is defined on the grid. As a consequence, this compromises not only the uniqueness of the N-cubic extension starting from the cubic polynomials defined on the grid $\mathcal{G}_c$, but also a bound for this function on $\mathbb{R}^N$.

To override this issue, we need to artificially construct these lacking "mixed derivatives" with the help, once again, of the linear interpolation.

Remind that for any $\psi \in W^{2,p}(\mathcal{G})$ (resp. $\phi \in W^{2,p}(\}_\varepsilon)$), its derivatives $\partial_{\mathbf{G}}\psi$ (resp. $\partial_{\mathbf{G}}\phi$) in direction $\mathbf{e}_i$ are functions belonging to $W^{1,p}(\mathcal{G}^{(i)})$ (resp. $W^{1,p}(\mathcal{G}_\varepsilon^{(i)})$), for every $i \in \{1,\dots,N\}$. As a consequence, they are defined on every node of the structure $\mathcal{G}$ (resp. $\mathcal{G}_\varepsilon$). Set

$$\mathcal{G}_c^{[i]} \doteq \bigcup_{j=1,\, j\neq i}^{N} \mathcal{G}_c^{(j)} \qquad \left(\text{resp. } \mathcal{G}_{c,\varepsilon}^{[i]} \doteq \bigcup_{j=1,\, j\neq i}^{N} \mathcal{G}_{\varepsilon,c}^{(j)}\right).$$

For every $i \in \{1,\dots,N\}$, we denote the following extensions (see also Figure 2.4)

$$\overline{\partial_i\psi} \doteq \{f \in W^{1,p}(\mathcal{G}_c^{(i)}) \times W^{1,\infty}(\mathcal{G}_c^{[i]}) \mid f_{\mathcal{G}_c^{(i)}} \text{ is extended by } N-1\text{-linear interpolation on } \mathcal{G}_c^{[i]}\},$$

$$\overline{\partial_i\phi} \doteq \{f \in W^{1,p}(\mathcal{G}_{c,\varepsilon}^{(i)}) \times W^{1,\infty}(\mathcal{G}_{c,\varepsilon}^{[i]}) \mid f_{\mathcal{G}_{c,\varepsilon}^{(i)}} \text{ is extended by } N-1\text{-linear interpolation on } \mathcal{G}_{c,\varepsilon}^{[i]}\}.$$

This allows us to uniquely determine the N-cubic extension since we artificially created the

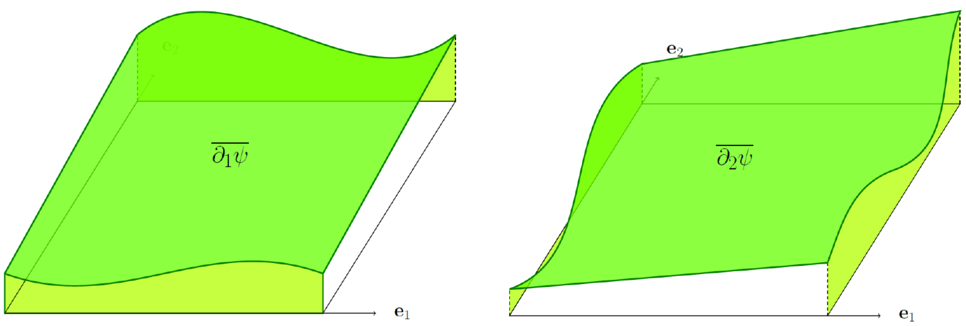

FIGURE 2.4: The extensions $\overline{\partial_1\psi}$ and $\overline{\partial_2\psi}$ for the derivatives $\partial_1\psi$ and $\partial_2\psi$ of a function $\psi \in W^2(\mathcal{G}_c)$ in dimension two.

mixed derivatives. Moreover, we can bound the interpolated function by the bound on the original function, with the additional assumption of boundedness for these derivatives.

Lemma 4. *For every $\psi \in Q^3(\mathcal{G}_c)$, one has*

$$\begin{aligned}
&\|D^2\mathfrak{Q}(\psi)\|_{L^p(Y)} \le C\sum_{i=1}^{N}\left\|\partial_{\mathbf{G}}(\overline{\partial_i\psi})\right\|_{L^p(\mathcal{G}_c)},\\
&\|\nabla\mathfrak{Q}(\psi)\|_{L^p(Y)} \le C\Big(\|\partial_{\mathbf{G}}\psi\|_{L^p(\mathcal{G}_c)} + \sum_{i=1}^{N}\left\|\partial_{\mathbf{G}}(\overline{\partial_i\psi})\right\|_{L^p(\mathcal{G}_c)}\Big),\\
&\|\mathfrak{Q}(\psi)\|_{L^p(Y)} \le C\Big(\|\psi\|_{L^p(\mathcal{G}_c)} + \|\partial_{\mathbf{G}}\psi\|_{L^p(\mathcal{G}_c)} + \sum_{i=1}^{N}\left\|\partial_{\mathbf{G}}(\overline{\partial_i\psi})\right\|_{L^p(\mathcal{G}_c)}\Big).
\end{aligned} \tag{2.20}$$

For every $\phi \in Q^3(\mathcal{G}_{c,\varepsilon})$, one has

$$\begin{aligned}
&\|D^2\mathfrak{Q}(\phi)\|_{L^p(\varepsilon Y)} \leq C\varepsilon^{\frac{N-1}{p}} \sum_{i=1}^{N} \|\partial_{\mathbf{g}}(\overline{\partial_i \phi})\|_{L^p(\mathcal{G}_{c,\varepsilon})},\\
&\|\nabla\mathfrak{Q}(\phi)\|_{L^p(\varepsilon Y)} \leq C\varepsilon^{\frac{N-1}{p}} \Big(\|\partial_{\mathbf{g}}\phi\|_{L^p(\mathcal{G}_{c,\varepsilon})} + \sum_{i=1}^{N} \|\partial_{\mathbf{g}}(\overline{\partial_i \phi})\|_{L^p(\mathcal{G}_{c,\varepsilon})}\Big),\\
&\|\mathfrak{Q}(\phi)\|_{L^p(\varepsilon Y)} \leq C\varepsilon^{\frac{N-1}{p}} \Big(\|\phi\|_{L^p(\mathcal{G}_{c,\varepsilon})} + \|\partial_{\mathbf{g}}\phi\|_{L^p(\mathcal{G}_{c,\varepsilon})} + \sum_{i=1}^{N} \|\partial_{\mathbf{g}}(\overline{\partial_i \phi})\|_{L^p(\mathcal{G}_{c,\varepsilon})}\Big).
\end{aligned} \tag{2.21}$$

Proof. We will only prove the case $N = 2$ since the extension to a higher dimension is done by an analogous argumentation.

Denote Q_0, Q_1, dQ_0 and dQ_1 the following polynomial functions ($t \in [0,1]$)

$$\begin{aligned}
&Q_0(t) = (2t+1)(t-1)^2, && dQ_0(t) = t(t-1)^2,\\
&Q_1(t) = t^2(3-2t), && dQ_1(t) = t^2(t-1).
\end{aligned}$$

Let ψ be a function belonging to $W^{2,p}(\mathcal{G}_c)$. Denote $\Psi \in W^{2,\infty}(Y)$ its extension to the whole domain by

$$\begin{aligned}
\Psi(t) =&\psi(0,0)P_{00}(t) + \psi(0,1)P_{01}(t) + \psi(1,0)P_{10}(t) + \psi(1,1)P_{11}(t)\\
&+ \partial_1\psi(0,0)d_1P_{00}(t) + \partial_1\psi(1,0)d_1P_{10}(t) + \partial_1\psi(0,1)d_1P_{01}(t) + \partial_1\psi(1,1)d_1P_{11}(t)\\
&+ \partial_2\psi(0,0)d_2P_{00}(t) + \partial_2\psi(0,1)d_2P_{01}(t) + \partial_2\psi(1,0)d_2P_{10}(t) + \partial_2\psi(1,1)d_2P_{11}(t)
\end{aligned}$$

where for all $t = (t_1, t_2) \in [0,1]^2$:

$$\begin{aligned}
&P_{00}(t) = Q_0(t_1)Q_0(t_2), && d_1P_{00} = dQ_0(t_1)Q_0(t_2), && d_2P_{00} = Q_0(t_1)dQ_0(t_2),\\
&P_{10}(t) = Q_1(t_1)Q_0(t_2), && d_1P_{10} = dQ_1(t_1)Q_0(t_2), && d_2P_{10} = Q_1(t_1)dQ_0(t_2),\\
&P_{01}(t) = Q_0(t_1)Q_1(t_2), && d_1P_{01} = dQ_0(t_1)Q_1(t_2), && d_2P_{01} = Q_0(t_1)dQ_1(t_2),\\
&P_{11}(t) = Q_1(t_1)Q_1(t_2), && d_1P_{11} = dQ_1(t_1)Q_1(t_2), && d_2P_{11} = Q_1(t_1)dQ_1(t_2).
\end{aligned}$$

First, observe that the polynomial Ψ can be rewritten as

$$\begin{aligned}
\Psi(t) =&\big(\psi(0,0)Q_0(t_1) + \psi(1,0)Q_1(t_1) + \partial_1\psi(0,0)dQ_0(t_1) + \partial_1\psi(1,0)dQ_1(t_1)\big)Q_0(t_2)\\
&+ \big(\psi(0,1)Q_0(t_1) + \psi(1,1)Q_1(t_1) + \partial_1\psi(0,1)dQ_0(t_1) + \partial_1\psi(1,1)dQ_1(t_1)\big)Q_1(t_2)\\
&+ \big(\partial_2\psi(0,0)dQ_0(t_2) + \partial_2\psi(0,1)dQ_1(t_2)\big)Q_0(t_1)\\
&+ \big(\partial_2\psi(1,0)dQ_0(t_2) + \partial_2\psi(1,1)dQ_1(t_2)\big)Q_1(t_1).
\end{aligned}$$

Straightforward calculations lead to

$$\begin{aligned}
\|D^2\Psi\|_{L^p(Y)} \leq& C\Big(\sum_{i=1}^{2} \|\partial^2_{ii}\psi\|_{L^p(\mathcal{G}_c^{(i)})} + |\partial_2\psi(1,0) - \partial_2\psi(0,0)| + |\partial_2\psi(1,1) - \partial_2\psi(0,1)|\\
&\qquad + |\partial_1\psi(0,1) - \partial_1\psi(0,0)| + |\partial_1\psi(1,1) - \partial_1\psi(1,0)|\Big)\\
\leq& C\Big(\sum_{i=1}^{2} \|\partial^2_{ii}\psi\|_{L^p(\mathcal{G}_c^{(i)})} + \sum_{i=1}^{2} \|\partial_{\mathbf{G}}(\overline{\partial_i\psi})\|_{L^p(\mathcal{G}_c)}\Big).
\end{aligned}$$

Hence, estimate (2.20) is proven since ($i \in \{1,2\}$)

$$\|\partial^2_{ii}\Psi\|_{L^p(\mathcal{G}_c^{(i)})} \leq C\|\partial^2_{\mathbf{G}}\psi\|_{L^p(\mathcal{G}_c^{(i)})} \leq C\|\partial_{\mathbf{G}}(\overline{\partial_i\psi})\|_{L^p(\mathcal{G}_c)}.$$

On the other hand, straightforward calculations lead to

$$\|\nabla\Psi\|_{L^p(Y)} \leq C\big(\|\partial_{\mathbf{S}}\psi\|_{L^p(\mathcal{G}_c)} + \|D^2\Psi\|_{L^p(Y)}\big).$$

and to

$$\|\Psi\|_{L^p(Y)} \leq C\big(\|\psi\|_{L^p(\mathcal{G}_c)} + \|\nabla\Psi\|_{L^p(Y)}\big),$$

which ends the proof of (2.20) for $N = 2$.

Estimates (2.21) are proven in the same way as (2.14), together with an affine change of variables. $\square$

Chapter 3

New tool: periodic unfolding for anisotropically bounded functions

The entirety of this chapter is dedicated to the extension of the classic periodic unfolding described in Section 2.1 to a new class of functions: the functions "anisotropically bounded". The notion of anisotropy comes from the fact that there is a contrast in the gradient's estimates, which creates privileged directions. We will show how to apply the periodic unfolding to this type of functions and find their asymptotic behavior.

A first application of the obtained results will be given at the end of this section, where we proceed to the homogenization of a diffusion problem in an anisotropic context. Some more applications will occur in the next chapters, in the context of periodic unfolding for lattice structures, and in the homogenization of textiles with loose contact sliding.

3.1 Space partition and anisotropy of the functions

In order to show the contrast in the gradient estimates, we find convenient to set a decomposition of the Euclidean space in two sub-spaces.

Let (N_1, N_2) be in $\mathbb{N} \times \mathbb{N}^*$ and such that $N = N_1 + N_2$. Denote

$$\mathbb{R}^{N_1} = \Big\{ x' \in \mathbb{R}^N \;\Big|\; x' = \sum_{i=1}^{N_1} x_i \mathbf{e}_i, \;\; x_i \in \mathbb{R} \Big\},$$

$$\mathbb{R}^{N_2} = \Big\{ x'' \in \mathbb{R}^N \;\Big|\; x'' = \sum_{i=N_1+1}^{N} x_i \mathbf{e}_i, \;\; x_i \in \mathbb{R} \Big\},$$

and

$$Y' = \Big\{ y' \in \mathbb{R}^N \;\Big|\; y' = \sum_{i=1}^{N_1} y_i \mathbf{e}_i, \;\; y_i \in (0,1) \Big\},$$

$$Y'' = \Big\{ y'' \in \mathbb{R}^N \;\Big|\; y'' = \sum_{i=N_1+1}^{N} y_i \mathbf{e}_i, \;\; y_i \in (0,1) \Big\}$$

and

$$\mathbb{Z}^{N_1} = \mathbb{Z}\mathbf{e}_1 \oplus \ldots \oplus \mathbb{Z}\mathbf{e}_{N_1}, \qquad \mathbb{Z}^{N_2} = \mathbb{Z}\mathbf{e}_{N_1+1} \oplus \ldots \oplus \mathbb{Z}\mathbf{e}_N.$$

One has

$$\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}, \qquad Y = Y' \oplus Y'', \qquad \mathbb{Z}^N = \mathbb{Z}^{N_1} \oplus \mathbb{Z}^{N_2}.$$

For every $x \in \mathbb{R}^N$ and $y \in Y$, we write

$$x = x' + x'' \in \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}, \qquad y = y' + y'' \in Y' \oplus Y''.$$

From now on, however, we find easier to refer to such decomposition with the vectorial notation

$$x = (x', x'') \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \qquad y = (y', y'') \in Y' \times Y''.$$

Similarly to (2.1), we apply the paving to a.e. $x' \in \mathbb{R}^{N_1}$ and $x'' \in \mathbb{R}^{N_2}$ setting

$$x' = \varepsilon\Big[\frac{x'}{\varepsilon}\Big]_{Y'} + \varepsilon\Big\{\frac{x'}{\varepsilon}\Big\}_{Y'}, \quad \text{with } \Big[\frac{x'}{\varepsilon}\Big]_{Y'} \in \mathbb{Z}^{N_1}, \quad \Big\{\frac{x'}{\varepsilon}\Big\}_{Y'} \in Y',$$
$$x'' = \varepsilon\Big[\frac{x''}{\varepsilon}\Big]_{Y''} + \varepsilon\Big\{\frac{x''}{\varepsilon}\Big\}_{Y''}, \quad \text{with } \Big[\frac{x''}{\varepsilon}\Big]_{Y''} \in \mathbb{Z}^{N_2}, \quad \Big\{\frac{x''}{\varepsilon}\Big\}_{Y''} \in Y''.$$

We denote the following spaces of functions:

$$\begin{aligned}
L^p(\Omega, \nabla_{x'}) &\doteq \{\phi \in L^p(\Omega) \mid \nabla_{x'}\phi \in L^p(\Omega)^{N_1}\},\\
L^p(\Omega, \nabla_{x''}) &\doteq \{\phi \in L^p(\Omega) \mid \nabla_{x''}\phi \in L^p(\Omega)^{N_2}\},\\
L^p(\Omega, \nabla_{x'}; W^{1,p}(Y'')) &\doteq \{\widetilde{\phi} \in L^p(\Omega \times Y'') \mid \nabla_{x'}\widetilde{\phi} \in L^p(\Omega \times Y'')^{N_1},\, \nabla_{y''}\widetilde{\phi} \in L^p(\Omega \times Y'')^{N_2}\},\\
L^p(\Omega, \nabla_{x''}; W^{1,p}(Y')) &\doteq \{\widetilde{\phi} \in L^p(\Omega \times Y') \mid \nabla_{x''}\widetilde{\phi} \in L^p(\Omega \times Y')^{N_2},\, \nabla_{y'}\widetilde{\phi} \in L^p(\Omega \times Y')^{N_1}\},\\
L^p(\Omega \times Y''; W^{1,p}(Y')) &\doteq \{\widehat{\phi} \in L^p(\Omega \times Y) \mid \nabla_{y'}\widehat{\phi} \in L^p(\Omega \times Y)^{N_1}\},\\
L^p(\Omega \times Y'; W^{1,p}(Y'')) &\doteq \{\widehat{\phi} \in L^p(\Omega \times Y) \mid \nabla_{y''}\widehat{\phi} \in L^p(\Omega \times Y)^{N_2}\}.
\end{aligned}$$

We endow these spaces with the respective norms:

$$\begin{aligned}
\|\cdot\|_{L^p(\Omega,\nabla_{x'})} &\doteq \|\cdot\|_{L^p(\Omega)} + \|\nabla_{x'}(\cdot)\|_{L^p(\Omega)^{N_1}},\\
\|\cdot\|_{L^p(\Omega,\nabla_{x''})} &\doteq \|\cdot\|_{L^p(\Omega)} + \|\nabla_{x''}(\cdot)\|_{L^p(\Omega)^{N_2}},\\
\|\cdot\|_{L^p(\Omega,\nabla_{x'};W^{1,p}(Y''))} &\doteq \|\cdot\|_{L^p(\Omega\times Y'')} + \|\nabla_{x'}(\cdot)\|_{L^p(\Omega\times Y'')^{N_1}} + \|\nabla_{y''}(\cdot)\|_{L^p(\Omega\times Y'')^{N_2}},\\
\|\cdot\|_{L^p(\Omega,\nabla_{x''};W^{1,p}(Y'))} &\doteq \|\cdot\|_{L^p(\Omega\times Y')} + \|\nabla_{x''}(\cdot)\|_{L^p(\Omega\times Y')^{N_2}} + \|\nabla_{y'}(\cdot)\|_{L^p(\Omega\times Y')^{N_1}},\\
\|\cdot\|_{L^p(\Omega\times Y'';W^{1,p}(Y'))} &\doteq \|\cdot\|_{L^p(\Omega\times Y)} + \|\nabla_{y'}(\cdot)\|_{L^p(\Omega\times Y)^{N_1}},\\
\|\cdot\|_{L^p(\Omega\times Y';W^{1,p}(Y''))} &\doteq \|\cdot\|_{L^p(\Omega\times Y)} + \|\nabla_{y''}(\cdot)\|_{L^p(\Omega\times Y)^{N_2}}.
\end{aligned} \tag{3.1}$$

Since the definition of "anisotropic behavior" only denotes a contrast in the estimates with respect to the observed direction, we state here rigorously the four classes of sequences to which we are going to apply the unfolding. Namely, we have:

(i)′ Sequences $\{\phi_\varepsilon\}_\varepsilon \in L^p(\Omega, \nabla_{x'})$ such that $\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla_{x'}\phi_\varepsilon\|_{L^p(\Omega)} \le C$;

(ii)′ Sequences $\{\phi_\varepsilon\}_\varepsilon \in L^p(\Omega, \nabla_{x'})$ such that $\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla_{x'}\phi_\varepsilon\|_{L^p(\Omega)} \le C$;

(iii)′ Sequences $\{\phi_\varepsilon\}_\varepsilon \in W^{1,p}(\Omega)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla_{x'}\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla_{x''}\phi_\varepsilon\|_{L^p(\Omega)} \le C$;

(iv)′ Sequences $\{\phi_\varepsilon\}_\varepsilon \in L^p(\Omega, \nabla_{x'})$, with $\{\nabla_{x'}\phi_\varepsilon\}_\varepsilon \in L^p(\Omega, \nabla_{x''})$ and such that

$$\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla_{x'}\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla_{x''}(\nabla_{x'}\phi_\varepsilon)\|_{L^p(\Omega)} \le C.$$

As we can expect, the different amount of information we have on the sequences estimates arises a different asymptotic behavior at the limit.

3.2 The two-step unfolding

The best ready-to-use tool to tackle this kind of problems is the unfolding with parameters, which has been already developed in Cioranescu, Damlamian, and Griso, 2018, Chap. 7. It consists of unfolding only some directions of the domain, treating the variable components in the other directions as "parameters".

Here, we proceed in a similar way and define the so called "two-step unfolding". Namely, we define two partial unfolding operators with parameters. These operators are built in order to apply the unfolding only to their respective half of the domain and such that the composition of both gives the unfolding for the whole domain.

Definition 5. *For every measurable function ϕ on Ω, the unfolding operator $\mathcal{T}''_\varepsilon$ is defined as follows:*

$$\mathcal{T}''_\varepsilon(\phi)(x',x'',y'') = \begin{cases} \phi\Big(x',\varepsilon\Big[\dfrac{x''}{\varepsilon}\Big]_{Y''}+\varepsilon y''\Big) & \text{for a.e. } (x',x'',y'')\in\widehat{\Omega}_\varepsilon\times Y'', \\ 0 & \text{for a.e. } (x',x'',y'')\in\Lambda_\varepsilon\times Y''. \end{cases}$$

For every measurable function ψ on $\Omega\times Y''$, the unfolding operator $\mathcal{T}'_\varepsilon$ is defined as follows:

$$\mathcal{T}'_\varepsilon(\psi)(x',x'',y',y'') = \begin{cases} \psi\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big]_{Y'}+\varepsilon y',x'',y''\Big) & \text{for a.e. } (x',x'',y',y'')\in\widehat{\Omega}_\varepsilon\times Y, \\ 0 & \text{for a.e. } (x',x'',y',y'')\in\Lambda_\varepsilon\times Y. \end{cases}$$

Note that, in the partial unfolding operator $\mathcal{T}''_\varepsilon(\phi)$, the variable x' plays the role of a parameter, while in $\mathcal{T}'_\varepsilon(\psi)$ the role of parameters is played by the variables (x'',y'').
Accordingly, we give the definition of partial mean value operators.

Definition 6. *For every $\widehat{\phi}\in L^1(\Omega\times Y)$, the partial mean value operators are defined as follows:*

$$\mathcal{M}_{Y'}(\widehat{\phi})(x,y'') \doteq \frac{1}{|Y'|}\int_{Y'}\widehat{\phi}(x,y',y'')dy', \qquad \text{for a.e. } (x,y'')\in\Omega\times Y'',$$
$$\mathcal{M}_{Y''}(\widehat{\phi})(x,y') \doteq \frac{1}{|Y''|}\int_{Y''}\widehat{\phi}(x,y',y'')dy'', \qquad \text{for a.e. } (x,y')\in\Omega\times Y'.$$

These operators satisfy the following properties.

Lemma 5. *One has*

$$\begin{aligned} \mathcal{T}_\varepsilon &= \mathcal{T}'_\varepsilon\circ\mathcal{T}''_\varepsilon \qquad \text{a.e. in } \Omega\times Y, \\ \mathcal{M}_Y &= \mathcal{M}_{Y'}\circ\mathcal{M}_{Y''} \qquad \text{a.e. in } \Omega. \end{aligned} \tag{3.2}$$

Moreover, for every $\phi\in L^1(\Omega,\nabla_{x'})$, one has

$$\nabla_{x'}\mathcal{T}''_\varepsilon(\phi) = \mathcal{T}''_\varepsilon(\nabla_{x'}\phi) \qquad \text{a.e. in } \widehat{\Omega}_\varepsilon\times Y''. \tag{3.3}$$

Proof. Let ϕ be measurable on Ω. We have that

$$\begin{aligned} \mathcal{T}'_\varepsilon\circ\mathcal{T}''_\varepsilon(\phi)(x,y) &= \mathcal{T}'_\varepsilon\Big(\phi\Big(x',\varepsilon\Big[\frac{x''}{\varepsilon}\Big]_{Y''}+\varepsilon y''\Big)\Big) = \phi\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_{Y'}+\varepsilon y',\varepsilon\Big[\frac{x''}{\varepsilon}\Big]_{Y''}+\varepsilon y''\Big) \\ &= \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y+\varepsilon y\Big) = \mathcal{T}_\varepsilon(\phi)(x,y) \quad \text{for a.e. } (x,y)\in\widehat{\Omega}_\varepsilon\times Y. \end{aligned}$$

For $(x,y)\in\Lambda_\varepsilon\times Y$ the result is obvious.
Let $\widehat{\phi}$ be in $L^1(\Omega\times Y)$. We have

$$\begin{aligned} \mathcal{M}_{Y'}\circ\mathcal{M}_{Y''}(\widehat{\phi})(x) &= \mathcal{M}_{Y'}\Big(\frac{1}{|Y''|}\int_{Y''}\widehat{\phi}(x,y',y'')dy''\Big) \\ &= \frac{1}{|Y'||Y''|}\int_{Y'}\int_{Y''}\widehat{\phi}(x,y',y'')dy''dy' = \frac{1}{|Y|}\int_Y\widehat{\phi}(x,y)dy \\ &= \mathcal{M}_Y(\widehat{\phi})(x) \quad \text{for a.e. } x\in\Omega. \end{aligned}$$

Let now ϕ be in $L^1(\Omega,\nabla_{x'})$. We have

$$\begin{aligned} \nabla_{x'}\mathcal{T}''_\varepsilon(\phi)(x,y'') &= \nabla_{x'}\Big(\phi\Big(x',\varepsilon\Big[\frac{x''}{\varepsilon}\Big]_{Y''}+\varepsilon y''\Big)\Big) = \nabla_{x'}\phi\Big(x',\varepsilon\Big[\frac{x''}{\varepsilon}\Big]_{Y''}+\varepsilon y''\Big) \\ &= \mathcal{T}''_\varepsilon(\nabla_{x'}\phi)(x,y'') \quad \text{for a.e. } (x,y'')\in\widehat{\Omega}_\varepsilon\times Y''. \end{aligned}$$

□

3.3 Asymptotic behavior of anisotropically bounded sequences

We are now ready to proceed to the periodic unfolding for the classes of anisotropically bounded sequences defined in (i)′-(iv)′ and find their asymptotic behavior.

Lemma 6. *Let $p \in (1, +\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $L^p(\Omega, \nabla_{x'})$ satisfying*

$$\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon \|\nabla_{x'}\phi_\varepsilon\|_{L^p(\Omega)} \leq C.$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and a function $\widehat{\phi} \in L^p(\Omega \times Y''; W^{1,p}_{per}(Y'))$ such that

$$\begin{aligned} &\phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in} \quad L^p(\Omega),\\ &\mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y''; W^{1,p}(Y')), \end{aligned}$$

where $\phi = \mathcal{M}_Y(\widehat{\phi})$.
The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Proof. The proof is similar to Cioranescu, Damlamian, and Griso, 2018, Theorem 1.36. □

An analogous result holds for sequences in (ii)′, i.e. uniformly bounded in $L^p(\Omega, \nabla_{x'})$.

Lemma 7. *Let $p \in (1, +\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $L^p(\Omega, \nabla_{x'})$ satisfying*

$$\|\phi_\varepsilon\|_{L^p(\Omega, \nabla_{x'})} \leq C.$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\widetilde{\phi} \in L^p(\Omega \times Y'', \nabla_{x'})$, $\widehat{\phi} \in L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'))$ such that

$$\begin{aligned} &\phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in} \quad L^p(\Omega, \nabla_{x'}),\\ &\mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widetilde{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y''; W^{1,p}(Y')),\\ &\mathcal{T}_\varepsilon(\nabla_{x'}\phi_\varepsilon) \rightharpoonup \nabla_{x'}\widetilde{\phi} + \nabla_{y'}\widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N_1},\\ &\frac{1}{\varepsilon}\big(\mathcal{T}_\varepsilon(\phi_\varepsilon) - \mathcal{M}_{Y'} \circ \mathcal{T}_\varepsilon(\phi_\varepsilon)\big) \rightharpoonup \nabla_{x'}\widetilde{\phi} \cdot y'^c + \widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N_1} \end{aligned}$$

where $\phi = \mathcal{M}_{Y''}(\widetilde{\phi})$ and $y'^c \doteq y' - \mathcal{M}_{Y'}(y')$.
The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Proof. The proof is similar to Cioranescu, Damlamian, and Griso, 2018, Corollary 1.37 and Cioranescu, Damlamian, and Griso, 2018, Theorem 1.41. □

Now, we proceed to the unfolding of the sequences in (iii)′ and (iv)′. In these cases, the two-steps unfolding will be needed.

Lemma 8. *Let $p \in (1, +\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\Omega)$ satisfying*

$$\|\phi_\varepsilon\|_{L^p(\Omega, \nabla_{x'})} + \varepsilon \|\nabla_{x''}\phi_\varepsilon\|_{L^p(\Omega)} \leq C. \tag{3.4}$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions

$$\widetilde{\phi} \in L^p(\Omega, \nabla_{x'}; W^{1,p}_{per}(Y'')) \text{ and } \widehat{\phi} \in L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'))$$

such that

$$
\begin{aligned}
&\phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in} \quad L^p(\Omega, \nabla_{x'}),\\
&\mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widetilde{\phi} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(Y)),\\
&\mathcal{T}_\varepsilon(\nabla_{x'}\phi_\varepsilon) \rightharpoonup \nabla_{x'}\widetilde{\phi} + \nabla_{y'}\widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N_1},\\
&\varepsilon\mathcal{T}_\varepsilon(\nabla_{x''}\phi_\varepsilon) \rightharpoonup \nabla_{y''}\widetilde{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N_2},\\
&\frac{1}{\varepsilon}\big(\mathcal{T}_\varepsilon(\phi_\varepsilon) - \mathcal{M}_{Y'} \circ \mathcal{T}_\varepsilon(\phi_\varepsilon)\big) \rightharpoonup \nabla_{x'}\widetilde{\phi} \cdot y'^c + \widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N_1}
\end{aligned}
\tag{3.5}
$$

where $\phi = \mathcal{M}_{Y''}(\widetilde{\phi})$ *and* $y'^c \doteq y' - \mathcal{M}_{Y'}(y')$.
The same results hold for $p = +\infty$ *with weak topology replaced by weak-* topology in the corresponding spaces.*

Proof. From hypothesis (3.4), up to a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, one has the existence of $\phi \in L^p(\Omega, \nabla_{x'})$ such that $(3.5)_1$ holds.
Set $\{\Phi_\varepsilon\}_\varepsilon = \{\mathcal{T}''_\varepsilon(\phi_\varepsilon)\}_\varepsilon$. This sequence belongs to $L^p(\widehat{\Omega}_\varepsilon, \nabla_{x'}; W^{1,p}(Y''))$ and from estimate (3.4) and equality (3.3), it satisfies

$$
\|\Phi_\varepsilon\|_{L^p(\widehat{\Omega}_\varepsilon, \nabla_{x'}; W^{1,p}(Y''))} \le C. \tag{3.6}
$$

Up to a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, there exists functions $\widetilde{\phi} \in L^p(\Omega; W^{1,p}_{per}(Y''))$ and $\widetilde{\Phi} \in L^p(\Omega \times Y'')^{N_1}$ (the periodicity of $\widetilde{\phi}$ is proved as in Cioranescu, Damlamian, and Griso, 2018, Theorem 1.36) such that

$$
\begin{aligned}
\Phi_\varepsilon \mathbf{1}_{\widehat{\Omega}_\varepsilon \times Y''} &\rightharpoonup \widetilde{\phi} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(Y'')),\\
\nabla_{x'}\Phi_\varepsilon \mathbf{1}_{\widehat{\Omega}_\varepsilon \times Y''} &\rightharpoonup \widetilde{\Phi} \quad \text{weakly in} \quad L^p(\Omega \times Y'')^{N_1},
\end{aligned}
$$

where $\mathbf{1}_{\widehat{\Omega}_\varepsilon \times Y''}$ denotes the characteristic function of the domain $\widehat{\Omega}_\varepsilon \times Y''$.
Let g be in $\mathcal{C}^\infty_c(\Omega \times Y'')^{N_1}$. For ε sufficiently small such that $\text{supp}(g) \subset \widehat{\Omega}_\varepsilon \times Y''$, we have

$$
\begin{aligned}
&\int_{\Omega \times Y''} \nabla_{x'}\Phi_\varepsilon \mathbf{1}_{\widehat{\Omega}_\varepsilon \times Y''} \cdot g \, dxdy'' = \int_{\widehat{\Omega}_\varepsilon \times Y''} \nabla_{x'}\Phi_\varepsilon \cdot g \, dxdy''\\
&= -\int_{\widehat{\Omega}_\varepsilon \times Y''} \Phi_\varepsilon \nabla_{x'} g \, dxdy'' = -\int_{\Omega \times Y''} \Phi_\varepsilon \mathbf{1}_{\widehat{\Omega}_\varepsilon \times Y''} \nabla_{x'} g \, dxdy''.
\end{aligned}
$$

Then, passing to the limit yields

$$
\int_{\Omega \times Y''} \widetilde{\Phi} \cdot g \, dxdy'' = -\int_{\Omega \times Y''} \widetilde{\phi} \cdot \nabla_{x'} g \, dxdy'', \quad \forall g \in \mathcal{C}^\infty_c(\Omega \times Y'')^{N_1}.
$$

This means that $\widetilde{\Phi} = \nabla_{x'}\widetilde{\phi}$ a.e. in $\Omega \times Y''$, thus $\nabla_{x'}\widetilde{\phi} \in L^p(\Omega \times Y'')^{N_1}$ and therefore $\widetilde{\phi}$ belongs to the space $L^p(\Omega, \nabla_{x'}; W^{1,p}_{per}(Y''))$.
Now, we transform the sequence $\{\Phi_\varepsilon\}_\varepsilon$ using the unfolding operator $\mathcal{T}'_\varepsilon$, Y'' being a set of parameters.
From the above convergence and estimate (3.6), up to a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, Proposition 2 gives $\widehat{\phi} \in L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'))$ such that (using the rule $(3.2)_1$)

$$
\begin{aligned}
&\mathcal{T}_\varepsilon(\phi_\varepsilon) = \mathcal{T}'_\varepsilon(\Phi_\varepsilon) \rightharpoonup \widetilde{\phi} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(Y' \times Y'')),\\
&\mathcal{T}_\varepsilon(\nabla_{x'}\phi_\varepsilon) = \mathcal{T}'_\varepsilon(\nabla_{x'}\Phi_\varepsilon) \rightharpoonup \nabla_{x'}\widetilde{\phi} + \nabla_{y'}\widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y' \times Y'')^{N_1},\\
&\frac{1}{\varepsilon}\big(\mathcal{T}_\varepsilon(\phi_\varepsilon) - \mathcal{M}_{Y'}(\mathcal{T}_\varepsilon(\phi_\varepsilon))\big) = \frac{1}{\varepsilon}\big(\mathcal{T}'_\varepsilon(\Phi_\varepsilon) - \mathcal{M}_{Y'}(\mathcal{T}'_\varepsilon(\Phi_\varepsilon))\big) \rightharpoonup \nabla_{x'}\widetilde{\phi} \cdot y'^c + \widehat{\phi}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{weakly in} \quad L^p(\Omega \times Y' \times Y'').
\end{aligned}
$$

This proves convergences $(3.5)_{2,3,5}$. Moreover, from convergence $(3.5)_2$ and the unfolding properties of $\mathcal{T}_\varepsilon$ we get that

$$\varepsilon\mathcal{T}_\varepsilon(\nabla_{x''}\phi_\varepsilon) = \nabla_{y''}\mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \nabla_{y''}\widetilde{\phi} \qquad \text{weakly in} \quad L^p(\Omega\times Y'')^{N_2},$$

which proves convergence $(3.5)_4$. □

We now consider the last class of functions.

Lemma 9. *Let $p\in(1,+\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $L^p(\Omega,\nabla_{x'})$ satisfying*

$$\|\phi_\varepsilon\|_{L^p(\Omega,\nabla_{x'})} + \varepsilon\big\|\nabla_{x''}(\nabla_{x'}\phi_\varepsilon)\big\|_{L^p(\Omega)} \leq C. \tag{3.7}$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, functions

$$\widetilde{\phi}\in L^p(\Omega,\nabla_{x'};W^{1,p}_{per}(Y'')) \quad \textit{and} \quad \widehat{\Phi}\in L^p(\Omega;W^{1,p}_{per}(Y))$$

such that $\mathcal{M}_{Y'}(\widehat{\Phi})=0$ a.e. in $\Omega\times Y''$,

$$\nabla_{x'}\widetilde{\phi}\in L^p(\Omega;W^{1,p}_{per}(Y''))^{N_1}, \quad \nabla_{y'}\widehat{\Phi} \in L^p(\Omega\times Y';W^{1,p}_{per}(Y''))^{N_1}$$

and we have

$$\begin{aligned} &\phi_\varepsilon \rightharpoonup \phi \qquad \textit{weakly in} \quad L^p(\Omega,\nabla_{x'}),\\ &\mathcal{T}_\varepsilon(\phi_\varepsilon)\rightharpoonup\widetilde{\phi} \qquad \textit{weakly in} \quad L^p(\Omega;W^{1,p}(Y)),\\ &\mathcal{T}_\varepsilon(\nabla_{x'}\phi_\varepsilon)\rightharpoonup\nabla_{x'}\widetilde{\phi}+\nabla_{y'}\widehat{\Phi} \qquad \textit{weakly in} \quad L^p(\Omega\times Y';W^{1,p}(Y''))^{N_1} \end{aligned} \tag{3.8}$$

where $\phi=\mathcal{M}_{Y''}(\widetilde{\phi})$.
The same results hold for $p=+\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Proof. By estimate $(3.7)_1$ and Lemma 7, there exists a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\widetilde{\phi}\in L^p(\Omega\times Y'',\nabla_{x'})$, $\widehat{\phi}\in L^p(\Omega\times Y'';W^{1,p}_{per,0}(Y'))$ such that

$$\begin{aligned} &\phi_\varepsilon \rightharpoonup \phi \qquad \text{weakly in} \quad L^p(\Omega,\nabla_{x'}),\\ &\mathcal{T}_\varepsilon(\phi_\varepsilon)\rightharpoonup\widetilde{\phi} \qquad \text{weakly in} \quad L^p(\Omega\times Y'';W^{1,p}(Y')),\\ &\mathcal{T}_\varepsilon(\nabla_{x'}\phi_\varepsilon)\rightharpoonup\nabla_{x'}\widetilde{\phi}+\nabla_{y'}\widehat{\phi} \qquad \text{weakly in} \quad L^p(\Omega\times Y)^{N_1}. \end{aligned} \tag{3.9}$$

Set $\{\psi_\varepsilon\}_\varepsilon \doteq \{\nabla_{x'}\phi_\varepsilon\}_\varepsilon$. By estimate (3.7), this sequence satisfies

$$\|\psi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\big\|\nabla_{x''}\psi_\varepsilon\big\|_{L^p(\Omega)} \leq C,$$

where the constant does not depend on ε.
Hence, applying Lemma 6 to the above sequence (but swapping Y' and Y''), there exists a function $\widehat{\psi}\in L^p(\Omega\times Y';W^{1,p}_{per}(Y''))^{N_1}$ such that

$$\mathcal{T}_\varepsilon(\nabla_{x'}\phi_\varepsilon)=\mathcal{T}_\varepsilon(\psi_\varepsilon)\rightharpoonup\widehat{\psi} \qquad \text{weakly in} \quad L^p(\Omega\times Y';W^{1,p}(Y''))^{N_1}.$$

This, together with convergence $(3.9)_3$ implies that the quantity $\nabla_{x'}\widetilde{\phi}+\nabla_{y'}\widehat{\phi}$ belongs to $L^p(\Omega\times Y';W^{1,p}_{per}(Y''))^{N_1}$. Since $\widetilde{\phi}$ does not depend on y' and $\widehat{\phi}$ is periodic with respect to y', we have that

$$\nabla_{x'}\widetilde{\phi}=\mathcal{M}_{Y'}(\nabla_{x'}\widetilde{\phi})+\mathcal{M}_{Y'}(\nabla_{y'}\widehat{\phi})=\mathcal{M}_{Y'}(\widehat{\psi}),$$

thus $\nabla_{x'}\widetilde{\phi}\in L^p(\Omega;W^{1,p}_{per}(Y''))^{N_1}$ and therefore $\widetilde{\phi}\in L^p(\Omega,\nabla_{x'};W^{1,p}_{per}(Y''))$.
Moreover, the quantity $\nabla_{y'}\widehat{\phi}$ belongs to $L^p(\Omega\times Y';W^{1,p}_{per}(Y''))^{N_1}$ and thus, by the technical Lemma 30 in Appendix, there exists $\widehat{\boldsymbol{\phi}}\in L^p(\Omega;W^{1,p}_{per}(Y))$ with $\nabla_{y'}\widehat{\boldsymbol{\phi}}=\nabla_{y'}\widehat{\phi}$ such that $(3.8)_3$

holds. The proof follows by replacing $\widehat{\boldsymbol{\phi}}$ by the function $\widehat{\Phi} = \widehat{\boldsymbol{\phi}} - \mathcal{M}_{Y'}(\widehat{\boldsymbol{\phi}})$, which belongs to the space $L^p(\Omega; W^{1,p}_{per}(Y))$ and satisfies $\mathcal{M}_{Y'}(\widehat{\Phi}) = 0$ a.e. in $\Omega \times Y''$. $\square$

3.4 Application: homogenization of a diffusion problem in an anisotropic environment

In this last section we want to give a direct application of the periodic unfolding for anisotropically bounded sequences to a diffusion problem.

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^N$ and let $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$. Denote $M(\alpha, \beta, \mathcal{O})$ the set of $N \times N$ matrices $A = (a_{ij})_{1 \le i,j \le N}$ with coefficients in $L^\infty(\mathcal{O})$ such that for every $\lambda \in \mathbb{R}^N$ and for a.e. $x \in \mathcal{O}$, the following inequalities hold:

(i) $(A(x)\lambda, \lambda) \ge \alpha |\lambda|^2$;

(ii) $|A(x)\lambda|^2 \le \beta (A(x)\lambda, \lambda)$.

Let A be be in $\mathcal{M}(\alpha, \beta, Y)$ and let $\{A_\varepsilon\}_\varepsilon$ be the sequence of matrices belonging to $M(\alpha, \beta, \Omega)$ defined by

$$A_\varepsilon \doteq A\Big(\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big) \quad \text{a.e. } x \in \Omega. \tag{3.10}$$

For the rest of the section, let $p = 2$. From (2.4), we recall the definition of the Hilbert spaces

$$\begin{aligned} H^1_{per}(Y) &\doteq \{\phi \in H^1(Y) \mid \phi \text{ is periodic with respect to } y_i,\, i \in \{1, \dots, N\}\}, \\ H^1_{per,0}(Y) &\doteq \{\phi \in H^1_{per}(Y) \mid \mathcal{M}_Y(\phi) = 0\}. \end{aligned}$$

Let f be a function in $L^2(\Omega)$. Consider the following Dirichlet problem in variational formulation:

$$\begin{cases} \text{Find } u_\varepsilon \in H^1_0(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega A_\varepsilon \begin{bmatrix} \nabla_{x'} u_\varepsilon \\ \varepsilon \nabla_{x''} u_\varepsilon \end{bmatrix} \cdot \begin{bmatrix} \nabla_{x'} \phi \\ \varepsilon \nabla_{x''} \phi \end{bmatrix} dx = \int_\Omega f\, \phi\, dx, \quad \forall \phi \in H^1_0(\Omega), \end{cases} \tag{3.11}$$

where $\cdot$ denotes the dot product by the column vectors $A_\varepsilon \begin{bmatrix} \nabla_{x'} u_\varepsilon \\ \varepsilon \nabla_{x''} u_\varepsilon \end{bmatrix}$ and $\begin{bmatrix} \nabla_{x'} \phi_\varepsilon \\ \varepsilon \nabla_{x''} \phi_\varepsilon \end{bmatrix}$.

By the Poincaré inequality and the fact that $u_\varepsilon \in H^1_0(\Omega)$, we have that

$$\|u_\varepsilon\|_{L^2(\Omega)} \le C \|\nabla_{x'} u_\varepsilon\|_{L^2(\Omega)}.$$

Thus, problem (3.11) admits a unique solution by the Lax$-$Milgram theorem and the following inequality holds:

$$\alpha\big(\|\nabla_{x'} u_\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^2 \|\nabla_{x''} u_\varepsilon\|^2_{L^2(\Omega)}\big) \le \|f\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)} \le C \|f\|_{L^2(\Omega)} \|\nabla_{x'} u_\varepsilon\|_{L^2(\Omega)}.$$

Hence

$$\|u_\varepsilon\|_{L^2(\Omega)} + \|\nabla_{x'} u_\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\nabla_{x''} u_\varepsilon\|_{L^2(\Omega)} \le C \|f\|_{L^2(\Omega)}. \tag{3.12}$$

Set

$$\begin{aligned} \mathbf{H}^1_{0,per}(\Omega \times Y'') = \big\{\phi \in H^1(\Omega \times Y'') \mid \phi(x, y'') = 0 \quad &\text{for a.e. } (x, y'') \in \partial\Omega \times Y'' \\ &\text{and } \phi(x, \cdot) \text{ is } Y'' \text{ periodic for a.e. } x \in \Omega\big\}. \end{aligned}$$

Denote $L^2_0(\Omega, \nabla_{x'})$ (resp. $L^2_0(\Omega, \nabla_{x'}; H^1_{per}(Y''))$) the closure of $H^1_0(\Omega)$ (resp. of $\mathbf{H}^1_{0,per}(\Omega \times Y'')$) in $L^2(\Omega)$ (resp. $L^2(\Omega \times Y'')$) for the norm of $L^2(\Omega, \nabla_{x'})$ (resp. $L^2(\Omega, \nabla_{x'}; H^1_{per}(Y''))$), see Section 3.1).

Below, we give the periodic homogenization via unfolding.

Theorem 1. *Let u_ε be the solution of problem (3.11).*
There exist $\widetilde{u} \in L^2_0(\Omega, \nabla_{x'}; H^1_{per}(Y''))$ and $\widehat{u} \in L^2(\Omega \times Y''; H^1_{per,0}(Y'))$ such that

$$\begin{aligned} &u_\varepsilon \rightharpoonup \mathcal{M}_Y(\widetilde{u}) \quad \text{weakly in} \quad L^2_0(\Omega, \nabla_{x'}), \\ &\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup \widetilde{u} \quad \text{weakly in} \quad L^2(\Omega; H^1(Y)), \\ &\mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon) \to \nabla_{x'}\widetilde{u} + \nabla_{y'}\widehat{u} \quad \text{strongly in} \quad L^2(\Omega \times Y)^{N_1}, \\ &\varepsilon\mathcal{T}_\varepsilon(\nabla_{x''} u_\varepsilon) \to \nabla_{y''}\widetilde{u} \quad \text{strongly in} \quad L^2(\Omega \times Y)^{N_2}. \end{aligned} \tag{3.13}$$

The couple $(\widetilde{u}, \widehat{u})$ is the unique solution of problem

$$\begin{aligned} \int_{\Omega\times Y} A(y) \begin{bmatrix} \nabla_{x'}\widetilde{u}(x,y'') + \nabla_{y'}\widehat{u}(x,y) \\ \nabla_{y''}\widetilde{u}(x,y'') \end{bmatrix} \cdot \begin{bmatrix} \nabla_{x'}\widetilde{\phi}(x,y'') + \nabla_{y'}\widehat{\phi}(x,y) \\ \nabla_{y''}\widetilde{\phi}(x,y'') \end{bmatrix} dxdy, \\ = |Y'| \int_{\Omega\times Y''} f(x)\widetilde{\phi}(x,y'')dxdy'', \\ \forall \widetilde{\phi} \in L^2_0(\Omega, \nabla_{x'}; H^1_{per,0}(Y'')) \ \text{ and } \ \forall \widehat{\phi} \in L^2(\Omega \times Y''; H^1_{per,0}(Y')). \end{aligned} \tag{3.14}$$

Proof. Step 1. We show (3.14) and the weak formulation of convergences (3.13).
First, by the fact that $A \in M(\alpha, \beta, Y)$ by definition (3.10) and the unfolding operator properties, we immediately get that $\mathcal{T}_\varepsilon(A_\varepsilon)(x,y) = A(y)$ for a.e.$(x,y) \in \widehat{\Omega}_\varepsilon \times Y$.
Now, note that the solution u_ε of (3.11) satisfies (3.12). Hence, up to a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, Lemma 8 gives $\widetilde{u} \in L^2_0(\Omega, \nabla_{x'}; H^1_{per}(Y''))$ and $\widehat{u} \in L^2(\Omega \times Y''; H^1_{per,0}(Y'))$ such that

$$\begin{aligned} &u_\varepsilon \rightharpoonup \mathcal{M}_Y(\widetilde{u}) \quad \text{weakly in} \quad L^2_0(\Omega, \nabla_{x'}), \\ &\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup \widetilde{u} \quad \text{weakly in} \quad L^2(\Omega; H^1(Y)), \\ &\mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon) \rightharpoonup \nabla_{x'}\widetilde{u} + \nabla_{y'}\widehat{u} \quad \text{weakly in} \quad L^2(\Omega \times Y)^{N_1}, \\ &\varepsilon\mathcal{T}_\varepsilon(\nabla_{x''} u_\varepsilon) \rightharpoonup \nabla_{y''}\widetilde{u} \quad \text{weakly in} \quad L^2(\Omega \times Y'')^{N_2}. \end{aligned} \tag{3.15}$$

Now, we choose the test functions

- $\widetilde{\Phi}$ in $H^1_0(\Omega)$, $\widetilde{\phi}$ in $H^1_{per}(Y'')$,
- Φ in $\mathcal{C}^1_c(\Omega \times Y'')$,
- $\widehat{\phi}$ in $H^1_{per,0}(Y')$.

Set $\phi_\varepsilon(x) \doteq \widetilde{\Phi}(x)\widetilde{\phi}\Big(\dfrac{x''}{\varepsilon}\Big) + \varepsilon\Phi\Big(x, \dfrac{x''}{\varepsilon}\Big)\widehat{\phi}\Big(\dfrac{x'}{\varepsilon}\Big)$ for a.e. $x \in \Omega$.
Applying the unfolding operator to the sequence $\{\phi_\varepsilon\}_\varepsilon$, we get that

$$\begin{aligned} &\mathcal{T}_\varepsilon(\phi_\varepsilon) \to \widetilde{\Phi}\widetilde{\phi} \quad \text{strongly in} \quad L^2(\Omega; H^1(Y)), \\ &\mathcal{T}_\varepsilon(\nabla_{x'}\phi_\varepsilon) \to (\nabla_{x'}\widetilde{\Phi})\widetilde{\phi} + \Phi\nabla_{y'}\widehat{\phi} \quad \text{strongly in} \quad L^2(\Omega \times Y)^{N_1}, \\ &\varepsilon\mathcal{T}_\varepsilon(\nabla_{x''}\phi_\varepsilon) \to \nabla_{y''}\widetilde{\phi} \quad \text{strongly in} \quad L^2(\Omega \times Y)^{N_2}. \end{aligned}$$

Taking ϕ_ε as test function in (3.11), then transforming by unfolding and passing to the limit, it gives (3.14) with $(\widetilde{\Phi}\widetilde{\phi}, \Phi\widehat{\phi})$. Then, we extend such results for all $\widetilde{\phi} \in L^2_0(\Omega, \nabla_{x'}; H^1_{per}(Y''))$ and all $\widehat{\phi} \in L^2(\Omega \times Y''; H^1_{per,0}(Y'))$ by density argumentation. Since the solution is unique, the sequences converge to their limit.
Step 2. We prove that convergences $(3.13)_{3,4}$ are strong.
First, setting $\phi = u_\varepsilon$ in (3.11), then transforming by unfolding and using the weak lower

semicontinuity yield

$$\begin{aligned}
&\int_{\Omega\times Y} A \begin{bmatrix}\nabla_{x'}\widetilde{u}+\nabla_{y'}\widehat{u}\\ \nabla_{y''}\widetilde{u}\end{bmatrix}\cdot\begin{bmatrix}\nabla_{x'}\widetilde{u}+\nabla_{y'}\widehat{u}\\ \nabla_{y''}\widetilde{u}\end{bmatrix} dxdy\\
&\leq \liminf_{\varepsilon\to 0}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(A_\varepsilon)\begin{bmatrix}\mathcal{T}_\varepsilon(\nabla_{x'}u_\varepsilon)\\ \varepsilon\mathcal{T}_\varepsilon(\nabla_{x''}u_\varepsilon)\end{bmatrix}\cdot\begin{bmatrix}\mathcal{T}_\varepsilon(\nabla_{x'}u_\varepsilon)\\ \varepsilon\mathcal{T}_\varepsilon(\nabla_{x''}u_\varepsilon)\end{bmatrix} dxdy\\
&= \liminf_{\varepsilon\to 0}\int_{\widehat{\Omega}_\varepsilon} A_\varepsilon\begin{bmatrix}\nabla_{x'}u_\varepsilon\\ \varepsilon\nabla_{x''}u_\varepsilon\end{bmatrix}\cdot\begin{bmatrix}\nabla_{x'}u_\varepsilon\\ \varepsilon\nabla_{x''}u_\varepsilon\end{bmatrix} dxdy\\
&\leq \limsup_{\varepsilon\to 0}\int_{\widehat{\Omega}_\varepsilon} A_\varepsilon\begin{bmatrix}\nabla_{x'}u_\varepsilon\\ \varepsilon\nabla_{x''}u_\varepsilon\end{bmatrix}\cdot\begin{bmatrix}\nabla_{x'}u_\varepsilon\\ \varepsilon\nabla_{x''}u_\varepsilon\end{bmatrix} dxdy\\
&\leq \limsup_{\varepsilon\to 0}\int_{\Omega} A_\varepsilon\begin{bmatrix}\nabla_{x'}u_\varepsilon\\ \varepsilon\nabla_{x''}u_\varepsilon\end{bmatrix}\cdot\begin{bmatrix}\nabla_{x'}u_\varepsilon\\ \varepsilon\nabla_{x''}u_\varepsilon\end{bmatrix} dxdy = \limsup_{\varepsilon\to 0}\int_\Omega f\, u_\varepsilon\, dx,\\
&= \lim_{\varepsilon\to 0}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(u_\varepsilon)\, dx = \int_{\Omega\times Y} f\,\widetilde{u}\, dxdy\\
&= \int_{\Omega\times Y} A \begin{bmatrix}\nabla_{x'}\widetilde{u}+\nabla_{y'}\widehat{u}\\ \nabla_{y''}\widetilde{u}\end{bmatrix}\cdot\begin{bmatrix}\nabla_{x'}\widetilde{u}+\nabla_{y'}\widehat{u}\\ \nabla_{y''}\widetilde{u}\end{bmatrix} dxdy,
\end{aligned}$$

from which it follows that all the above inequalities are in fact equalities. Hence

$$\int_{\Lambda_\varepsilon} A_\varepsilon\begin{bmatrix}\nabla_{x'}u_\varepsilon\\ \varepsilon\nabla_{x''}u_\varepsilon\end{bmatrix}\cdot\begin{bmatrix}\nabla_{x'}u_\varepsilon\\ \varepsilon\nabla_{x''}u_\varepsilon\end{bmatrix} dxdy = 0$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega\times Y} A\begin{bmatrix}\mathcal{T}_\varepsilon(\nabla_{x'}u_\varepsilon)\\ \varepsilon\mathcal{T}_\varepsilon(\nabla_{x''}u_\varepsilon)\end{bmatrix}\cdot\begin{bmatrix}\mathcal{T}_\varepsilon(\nabla_{x'}u_\varepsilon)\\ \varepsilon\mathcal{T}_\varepsilon(\nabla_{x''}u_\varepsilon)\end{bmatrix} dxdy\\
&= \lim_{\varepsilon\to 0}\int_{\Omega} A_\varepsilon\begin{bmatrix}\nabla_{x'}u_\varepsilon\\ \varepsilon\nabla_{x''}u_\varepsilon\end{bmatrix}\cdot\begin{bmatrix}\nabla_{x'}u_\varepsilon\\ \varepsilon\nabla_{x''}u_\varepsilon\end{bmatrix} dxdy\\
&= \int_{\Omega\times Y} A \begin{bmatrix}\nabla_{x'}\widetilde{u}+\nabla_{y'}\widehat{u}\\ \nabla_{y''}\widetilde{u}\end{bmatrix}\begin{bmatrix}\nabla_{x'}\widetilde{u}+\nabla_{y'}\widehat{u}\\ \nabla_{y''}\widetilde{u}\end{bmatrix} dxdy.
\end{aligned}$$

Since the map $\Psi \in L^2(\Omega\times Y)^N \longmapsto \sqrt{\int_{\Omega\times Y} A\Psi\cdot\Psi\, dxdy}$ is a norm equivalent to the usual norm of $L^2(\Omega\times Y)^N$, we get

$$\lim_{\varepsilon\to 0}\int_{\Omega\times Y}\left|\begin{bmatrix}\mathcal{T}_\varepsilon(\nabla_{x'}u_\varepsilon)\\ \varepsilon\mathcal{T}_\varepsilon(\nabla_{x''}u_\varepsilon)\end{bmatrix}\right|^2 dxdy = \int_{\Omega\times Y}\left|\begin{bmatrix}\nabla_{x'}\widetilde{u}+\nabla_{y'}\widehat{u}\\ \nabla_{y''}\widetilde{u}\end{bmatrix}\right|^2 dxdy.$$

This, together with the fact that $(3.15)_{3,4}$ already converge weakly, ensures the strong convergences $(3.13)_{3,4}$. The proof is therefore complete. □

Now, consider the following partition of A into blocks

$$A = \begin{pmatrix} A_1 & A_2\\ A_3 & A_4\end{pmatrix},$$

where

- A_1 is a $N_1\times N_1$ matrix with entries in $L^\infty(Y)$,
- A_2 is a $N_1\times N_2$ matrix with entries in $L^\infty(Y)$,
- A_3 is a $N_2\times N_1$ matrix with entries in $L^\infty(Y)$,
- A_4 is a $N_2\times N_2$ matrix with entries in $L^\infty(Y)$.

We define the correctors $\widehat{\chi}_k$, $k \in \{1, \dots, N\}$, as the unique solutions in $L^\infty(Y'', H^1_{per,0}(Y'))$ for the cell problems

$$\int_{Y'} A_1(y) \left[\nabla_{y'} \widehat{\chi}_k(y', y'')\right] \cdot \left[\nabla_{y'} \widehat{w}(y')\right] dy' = -\int_{Y'} A(y)\, \mathbf{e}_k \cdot \begin{bmatrix} \nabla_{y'} \widehat{w}(y') \\ 0 \end{bmatrix} dy', \qquad \forall \widehat{w} \in H^1_{per,0}(Y'). \tag{3.16}$$

By the Lax–Milgram theorem applied in Hilbert space $L^2(Y'', H^1_{per,0}(Y'))$, we obtain the existence and uniqueness of the solution of (3.16) for every $k \in \{1, \dots, N\}$.
Since A belongs to $\mathcal{M}(\alpha, \beta, Y)$ we get for every $k \in \{1, \dots, N\}$:

$$\|\nabla_{y'} \widehat{\chi}_k(\cdot, y'')\|_{H^1(Y')^{N_1}} \le \frac{\beta}{\alpha}. \quad \text{for a.e. } y'' \in Y''.$$

As a consequence, $\widehat{\chi}_k$ belongs to $L^\infty(Y'', H^1_{per,0}(Y'))$[1] for every $k \in \{1, \dots, N\}$, and we have

$$\|\widehat{\chi}_k\|_{L^\infty(Y''; H^1(Y'))} \le C.$$

We can finally give the form of the homogenized problem.

Proposition 4. *The function $\widetilde{u}_0 \in L^2_0(\Omega, \nabla_{x'}; H^1_{per}(Y''))$ is the unique solution of the following homogenized problem:*

$$\int_{\Omega \times Y''} A^{hom}(y'') \begin{bmatrix} \nabla_{x'} \widetilde{u}_0(x, y'') \\ \nabla_{y''} \widetilde{u}_0(x, y'') \end{bmatrix} \cdot \begin{bmatrix} \nabla_{x'} \widetilde{\phi}(x, y'') \\ \nabla_{y''} \widetilde{\phi}(x, y'') \end{bmatrix} dxdy'' = \int_{\Omega \times Y''} f(x)\, \widetilde{\phi}(x, y'')\, dxdy'', \qquad \forall \widetilde{\phi} \in L^2_0(\Omega, \nabla_{x'}; H^1_{per}(Y'')). \tag{3.17}$$

The homogenizing operator $A^{hom} \in L^\infty(Y'')^{N \times N}$ is the matrix defined by

$$A^{hom}(y'') \doteq \frac{1}{|Y'|} \int_{Y'} \left(A + \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \nabla_{y'} \chi \right)(y', y'')\, dy', \tag{3.18}$$

where $\widehat{\chi} = \begin{pmatrix} \widehat{\chi}_1 & \widehat{\chi}_2 & \dots & \widehat{\chi}_{N_1} & \widehat{\chi}_{N_1+1} & \dots & \widehat{\chi}_N \end{pmatrix}$ and thus $\nabla_{y'} \widehat{\chi}$ is the $N_1 \times N$ matrix

$$\nabla_{y'} \widehat{\chi} \doteq \begin{pmatrix} \nabla_{y'} \widehat{\chi}_1 & \nabla_{y'} \widehat{\chi}_2 & \dots & \nabla_{y'} \widehat{\chi}_{N_1} & \nabla_{y'} \widehat{\chi}_{N_1+1} & \dots & \nabla_{y'} \widehat{\chi}_N \end{pmatrix}.$$

Note, that in such a formulation the problem mixes the macroscopic x' and microscopic variables that correspond to x''. Nevertheless, the homogenization is considered to be concluded since all the involved functions depend on such variables.
Before proceeding to the proof, we find convenient to clarify the boundary conditions for the solutions of problem (3.17) in a simple domain in two dimensions.

Remark 1. *Assume that $\Omega \doteq (0,1)^2 \subset \mathbb{R}^2$. Then, the function of $\widetilde{u}_0$ belongs to the space*

$$\{\phi \in L^2(\Omega, \partial_1; H^1_{per}(Y'')) \mid \phi(0, x_2, y_2) = \phi(1, x_2, y_2) = 0 \text{ for a.e. } (x_2, y_2) \in (0,1) \times Y''\}.$$

Proof of Proposition 4. Equation (3.14) with $\widetilde{\phi} = 0$ leads to:

$$\int_{\Omega \times Y} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} (y', y'') \begin{bmatrix} \nabla_{y'} \widehat{u}(x, y', y'') \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \nabla_{y'} \widehat{\phi}(x, y', y'') \\ 0 \end{bmatrix} dxdy'dy'' = -\int_{\Omega \times Y} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} (y', y'') \begin{bmatrix} \nabla_{x'} \widetilde{u}(x, y'') \\ \nabla_{y''} \widetilde{u}(x, y'') \end{bmatrix} \cdot \begin{bmatrix} \nabla_{y'} \widehat{\phi}(x, y', y'') \\ 0 \end{bmatrix} dxdy'dy'', \qquad \forall \widehat{\phi} \in L^2(\Omega \times Y''; H^1_{per,0}(Y')), \tag{3.19}$$

from which the form of the cell problems (3.16) follows.

[1] One can prove that $\widehat{\chi}_k$ also belongs to $L^\infty(Y)$.

By (3.19), we can write $\widehat{u}$ as

$$\widehat{u}(x,y',y'') = \sum_{k=1}^{N_1} \widehat{\chi}_k(y',y'')\partial_{x_k}\widetilde{u}(x,y'') + \sum_{k=N_1+1}^{N} \widehat{\chi}_k(y',y'')\partial_{y_k}\widetilde{u}(x,y'')$$
$$\text{for a.e. } (x,y',y'') \in \Omega \times Y' \times Y''.$$

Replacing $\widehat{u}$ by the above equality in (3.14) (note that $\widehat{\phi}$ is set to be zero since the correctors have been found) we first get

$$\begin{aligned}
&\int_{\Omega\times Y} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{bmatrix} \nabla_{x'}\widetilde{u} + \nabla_{y'}\widehat{u} \\ \nabla_{y''}\widetilde{u} \end{bmatrix} \cdot \begin{bmatrix} \nabla_{x'}\widetilde{\phi} \\ \nabla_{y''}\widetilde{\phi} \end{bmatrix} dxdy \\
&= \int_{\Omega\times Y} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{bmatrix} \nabla_{x'}\widetilde{u} \\ \nabla_{y''}\widetilde{u} \end{bmatrix} \cdot \begin{bmatrix} \nabla_{x'}\widetilde{\phi} \\ \nabla_{y''}\widetilde{\phi} \end{bmatrix} dxdy \\
&\quad + \int_{\Omega\times Y} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{bmatrix} \sum_{k=1}^{N_1} \nabla_{y'}\widehat{\chi}_k\, \partial_{x_k}\widetilde{u} + \sum_{k=N_1+1}^{N} \nabla_{y'}\widehat{\chi}_k\, \partial_{y_k}\widetilde{u} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \nabla_{x'}\widetilde{\phi} \\ \nabla_{y''}\widetilde{\phi} \end{bmatrix} dxdy.
\end{aligned}$$

Concerning the second term, straightforward calculations lead to

$$\begin{aligned}
&\int_{\Omega\times Y} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{bmatrix} \sum_{k=1}^{N_1} \nabla_{y'}\widehat{\chi}_k\, \partial_{x_k}\widetilde{u} + \sum_{k=N_1+1}^{N} \nabla_{y'}\widehat{\chi}_k\, \partial_{y_k}\widetilde{u} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \nabla_{x'}\widetilde{\phi} \\ \nabla_{y''}\widetilde{\phi} \end{bmatrix} dxdy \\
&= \int_{\Omega\times Y} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \left[\begin{pmatrix} \nabla_{y'}\widehat{\chi} \\ 0 \end{pmatrix} \begin{bmatrix} \nabla_{x'}\widetilde{u} \\ \nabla_{y''}\widetilde{u} \end{bmatrix} \right] \cdot \begin{bmatrix} \nabla_{x'}\widetilde{\phi} \\ \nabla_{y''}\widetilde{\phi} \end{bmatrix} dxdy \\
&= \int_{\Omega\times Y} \left(\begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \nabla_{y'}\widehat{\chi} \right) \begin{bmatrix} \nabla_{x'}\widetilde{u} \\ \nabla_{y''}\widetilde{u} \end{bmatrix} \cdot \begin{bmatrix} \nabla_{x'}\widetilde{\phi} \\ \nabla_{y''}\widetilde{\phi} \end{bmatrix} dxdy,
\end{aligned}$$

where we denoted $\begin{pmatrix} \nabla_{y'}\widehat{\chi} \\ 0 \end{pmatrix}$ the $N \times N$ matrix partitioned into the upper $N_1 \times N$ block $\nabla_{y'}\widehat{\chi}$ and the lower $N_2 \times N$ block with zero entrances. Hence, we get that

$$\begin{aligned}
&\int_{\Omega\times Y} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{bmatrix} \nabla_{x'}\widetilde{u} + \nabla_{y'}\widehat{u} \\ \nabla_{y''}\widetilde{u} \end{bmatrix} \cdot \begin{bmatrix} \nabla_{x'}\widetilde{\phi} \\ \nabla_{y''}\widetilde{\phi} \end{bmatrix} dxdy \\
&= \int_{\Omega\times Y} \left(\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} + \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \nabla_{y'}\widehat{\chi} \right) \begin{bmatrix} \nabla_{x'}\widetilde{u} \\ \nabla_{y''}\widetilde{u} \end{bmatrix} \cdot \begin{bmatrix} \nabla_{x'}\widetilde{\phi} \\ \nabla_{y''}\widetilde{\phi} \end{bmatrix} dxdy.
\end{aligned}$$

Gathering all the y' dependent terms, we get the form (3.18) for the operator A^{hom}.
Since $A \in L^\infty(Y)^{N\times N}$ and the $\widehat{\chi}_k$'s are in $L^\infty(Y''; H^1(Y'))$, it is clear that $A^{hom} \in L^\infty(Y'')^{N\times N}$.
We prove now that A^{hom} is coercive. Let $\boldsymbol{\xi} \doteq (\xi_1, \xi_2)$ be a vector with fixed entries in the space $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. By the construction of the homogenizing operator, straightforward calculation imply that

$$\begin{aligned}
A^{hom}[\boldsymbol{\xi}] \cdot [\boldsymbol{\xi}] &= \frac{1}{|Y'|} \int_{Y'} \left(\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} + \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \nabla_{y'}\widehat{\chi} \right) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} dy' \\
&= \frac{1}{|Y'|} \int_{Y'} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{bmatrix} \xi_1 + \nabla_{y'}\widehat{\chi}_\xi \\ \xi_2 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} dy' \\
&= \frac{1}{|Y'|} \int_{Y'} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{bmatrix} \xi_1 + \nabla_{y'}\widehat{\chi}_\xi \\ \xi_2 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 + \nabla_{y'}\widehat{\chi}_\xi \\ \xi_2 \end{bmatrix} dy' \\
&\quad - \frac{1}{|Y'|} \int_{Y'} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{bmatrix} \xi_1 + \nabla_{y'}\widehat{\chi}_\xi \\ \xi_2 \end{bmatrix} \cdot \begin{bmatrix} \nabla_{y'}\widehat{\chi}_\xi \\ 0 \end{bmatrix} dy',
\end{aligned}$$

where $\widehat{\chi}_\xi \doteq \sum_{k=1}^{N} \widehat{\chi}_k \xi_k$. Observe that by the cell problems (3.16), the second term in the last equality is equal to zero.

Now, the coercivity of the matrix A and the fact that $\widehat{\chi}_\xi \in L^\infty(Y''; H^1_{per,0}(Y'))$ imply that

$$\begin{aligned}
&A^{hom}(y'')[\xi]\cdot[\xi] \\
&= \frac{1}{|Y'|}\int_{Y'} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}(y', y'') \begin{bmatrix} \xi_1 + \nabla_{y'}\widehat{\chi}_\xi(y', y'') \\ \xi_2 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 + \nabla_{y'}\widehat{\chi}_\xi(y', y'') \\ \xi_2 \end{bmatrix} dy' \\
&\geq \alpha\big(\|\xi_1 + \nabla_{y'}\widehat{\chi}_\xi(\cdot, y'')\|^2_{L^2(Y')^{N_1}} + |\xi_2|^2\big) \\
&= \alpha\big(|\xi_1|^2 + |\xi_2|^2 + \|\nabla_{y'}\widehat{\chi}_\xi(\cdot, y'')\|^2_{L^2(Y')^{N_1}}\big) \geq \alpha|\xi|^2 \quad \text{for a.e. } y'' \in Y'',
\end{aligned}$$

which proves that A^{hom} is coercive.
Replacing the form of A^{hom} on the original problem (3.14), we get (3.17). By the boundedness and coercivity of A^{hom} and by the fact that the function $\widetilde{u}$ belongs to $L^2_0(\Omega, \nabla_{x'}; H^1_{per}(Y''))$, the above problem admits a unique solution $\widetilde{u}_0$ by the Poincaré inequality and the Lax$-$Milgram theorem. □

At last, we would like to remind that the obtaned results to this section occur not only when there is anisotropy in the displacements, but also if the contrast is present in the coefficients of the material law:

$$\begin{cases} \text{Find } u_\varepsilon \in H^1_0(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \begin{pmatrix} A_{1,\varepsilon} & \varepsilon A_{2,\varepsilon} \\ \varepsilon A_{3,\varepsilon} & \varepsilon^2 A_{4,\varepsilon} \end{pmatrix} [\nabla u_\varepsilon]\cdot[\nabla\phi]\, dx = \int_\Omega f\,\phi\, dx, \quad \forall \phi \in H^1_0(\Omega), \end{cases} \tag{3.20}$$

Indeed, this new formulation differs from (3.11) by a simple shift of contrast from the material law to the displacement. Hence, the developed method applies also to this kind of structures.

Chapter 4

New tool: periodic unfolding for functions defined on lattice structures

In this section, we developed a second tool to extend the classical results of the periodic unfolding. This time, we will not deal with a new class of functions but rather with sequences bounded on particular domains: one-dimensional lattice structures in $\mathbb{R}^N$. As we will see, this presents quite a challenge for the periodic unfolding.

4.1 The periodic lattice structure

We start by giving a rigorous definition of a one-dimensional periodic lattice structure in $\mathbb{R}^N$.

Let $i \in \{1, \dots, N\}$ and let $K_1, \dots, K_N \in \mathbb{N}^*$. Set the following subsets of $\mathbb{N}^N$ by

$$\mathbf{K} \doteq \prod_{i=1}^{N} \{0, \dots, K_i\}, \qquad \mathbf{K}_i \doteq \{k \in \mathbf{K} \mid k_i = 0\},$$
$$\widehat{\mathbf{K}} \doteq \prod_{i=1}^{N} \{0, \dots, K_i - 1\}, \qquad \widehat{\mathbf{K}}_i \doteq \{k \in \widehat{\mathbf{K}} \mid k_i = 0\}.$$

We denote $\mathcal{K}$ the set of points in the closure of the unit cell $\overline{Y}$ by

$$\mathcal{K} \doteq \Big\{ A(k) \in \mathbb{R}^N \;\Big|\; A(k) = \sum_{i=1}^{N} \frac{k_i}{K_i} \mathbf{e}_i, \quad k \in \mathbf{K} \Big\} \subset \overline{Y}.$$

In this sense, the whole unit cell $\overline{Y}$ is partitioned in a union of cells

$$\overline{Y} = \sum_{k \in \widehat{\mathbf{K}}} A(k) + \overline{Y}_K,$$

where the reference cell Y_K is defined by

$$Y_K \doteq \prod_{i=1}^{N} (0, l_i), \quad l_i = \frac{1}{K_i}.$$

We denote $\mathcal{S}_c^{(i)}$ and $\mathcal{S}^{(i)}$ the sets of segments whose direction is $\mathbf{e}_i$ by

$$\mathcal{S}_c^{(i)} \doteq \bigcup_{k \in \mathbf{K}_i} [A(k), A(k) + \mathbf{e}_i], \qquad \mathcal{S}^{(i)} \doteq \bigcup_{k \in \widehat{\mathbf{K}}_i} [A(k), A(k) + \mathbf{e}_i]$$

Hence, the lattice structure in the unit cell $\overline{Y}$ is defined by (see also Figure 4.1 right)

$$\mathcal{S}_c \doteq \bigcup_{i=1}^{N} \mathcal{S}_c^{(i)} \subset \overline{Y}, \qquad \mathcal{S} \doteq \bigcup_{i=1}^{N} \mathcal{S}^{(i)} \subset \overline{Y},$$

where again, as in Section 2.2, the letter "c" denotes the complete lattice.
Now, let $\Omega \subset \mathbb{R}^N$ be an open set. We consider its covering $\widetilde{\Omega}_\varepsilon$ defined by the ε paving

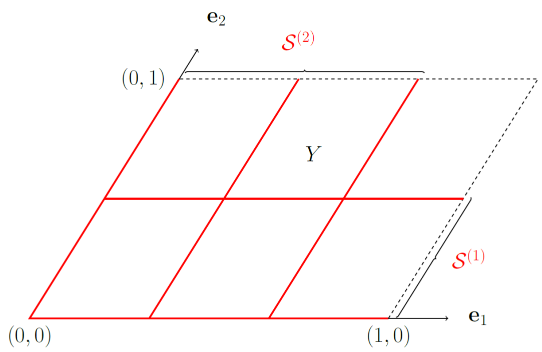

FIGURE 4.1: The lattice $\mathcal{S}$ in dimension two for $\mathbf{K} = \{0,1,2,3\} \times \{0,1,2\}$.

$$\widetilde{\Omega}_\varepsilon \doteq \text{int}\Big\{ \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} \varepsilon(\xi + \overline{Y}) \Big\}, \qquad \widetilde{\Xi}_\varepsilon \doteq \Big\{ \xi \in \mathbb{Z}^N \;\Big|\; \varepsilon(\xi + Y) \cap \Omega \neq \emptyset \Big\}.$$

From (2.2), we have (see the comparison between Figures 2.1 left and Figure 4.2 left)

$$\widehat{\Omega}_\varepsilon \subset \Omega \subset \widetilde{\Omega}_\varepsilon. \tag{4.1}$$

Note that the covering $\widetilde{\Omega}_\varepsilon$ is a connected, open set. This fact will be later crucial to get estimates of the functions defined as interpolates on lattice nodes.
The periodic lattice structure over Ω is defined by

$$\mathcal{S}_\varepsilon \doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} (\varepsilon\xi + \varepsilon\mathcal{S}) \subset \widetilde{\Omega}_\varepsilon, \qquad \mathcal{K}_\varepsilon \doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} (\varepsilon\xi + \varepsilon\mathcal{K}),$$
$$\mathcal{S}_\varepsilon^{(i)} \doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} (\varepsilon\xi + \varepsilon\mathcal{S}^{(i)}).$$

Denote $\mathbf{S}$ the running point of $\mathcal{S}$ and $\mathbf{s}$ that of $\mathcal{S}_\varepsilon$. That gives ($i \in \{1,\dots,N\}$)

$$\mathbf{S} = A(k) + t\mathbf{e}_i \quad \text{in } \mathcal{S}^{(i)}, t \in [0,1], k \in \widehat{\mathbf{K}}_i,$$
$$\mathbf{s} = \varepsilon\xi + \varepsilon A(k) + \varepsilon t\mathbf{e}_i \quad \text{in } \mathcal{S}_\varepsilon^{(i)}, t \in [0,1], k \in \widehat{\mathbf{K}}_i, \xi \in \widetilde{\Xi}_\varepsilon.$$

Let $\mathcal{C}(\mathcal{S})$ and $\mathcal{C}(\mathcal{S}_\varepsilon)$ be the spaces of continuous functions defined on $\mathcal{S}$ and $\mathcal{S}_\varepsilon$ respectively. For $p \in [1, +\infty]$, we denote the spaces of functions defined on the lattice by ($i \in \{1,\dots,N\}$)

$$W^{1,p}(\mathcal{S}^{(i)}) \doteq \{\phi \in L^p(\mathcal{S}^{(i)}) \mid \partial_\mathbf{S}\phi \in L^p(\mathcal{S}^{(i)})\},$$
$$W^{1,p}(\mathcal{S}_\varepsilon^{(i)}) \doteq \{\phi \in L^p(\mathcal{S}_\varepsilon^{(i)}) \mid \partial_\mathbf{s}\phi \in L^p(\mathcal{S}_\varepsilon^{(i)})\},$$
$$W^{1,p}(\mathcal{S}) \doteq \{\phi \in \mathcal{C}(\mathcal{S}) \mid \partial_\mathbf{S}\phi \in L^p(\mathcal{S})\},$$
$$W^{1,p}(\mathcal{S}_\varepsilon) \doteq \{\phi \in \mathcal{C}(\mathcal{S}_\varepsilon) \mid \partial_\mathbf{s}\phi \in L^p(\mathcal{S}_\varepsilon)\}$$

and

$$\begin{aligned} W^{2,p}(\mathcal{S}^{(i)}) &\doteq \{\phi \in W^{1,p}(\mathcal{S}^{(i)}) \mid \partial_{\mathbf{S}}\phi \in W^{1,p}(\mathcal{S}^{(i)})\}, \\ W^{2,p}(\mathcal{S}_\varepsilon^{(i)}) &\doteq \{\phi \in W^{1,p}(\mathcal{S}_\varepsilon^{(i)}) \mid \partial_{\mathbf{s}}\phi \in W^{1,p}(\mathcal{S}_\varepsilon^{(i)})\}, \\ W^{2,p}(\mathcal{S}) &\doteq \{\phi \in \mathcal{C}(\mathcal{S}) \mid \partial_{\mathbf{S}}\phi_{|\mathcal{S}^{(j)}} \in W^{1,p}(\mathcal{S}^{(j)}),\ j \in \{1,\dots,N\}\}, \\ W^{2,p}(\mathcal{S}_\varepsilon) &\doteq \{\phi \in \mathcal{C}(\mathcal{S}_\varepsilon) \mid \partial_{\mathbf{s}}\phi_{|\mathcal{S}_\varepsilon^{(j)}} \in W^{1,p}(\mathcal{S}_\varepsilon^{(j)}),\ j \in \{1,\dots,N\}\}. \end{aligned}$$

4.1.1 The unfolding operator for lattices

We are now in the position to give an equivalent formulation of the unfolding operator defined in 1, but for lattice structures.

Definition 7. *For every measurable function ϕ on $\mathcal{S}_\varepsilon$, the unfolding operator $\mathcal{T}_\varepsilon^{\mathcal{S}}$ is defined as follows:*

$$\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi)(x,\mathbf{S}) = \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon\mathbf{S}\Big) \quad \text{for a.e. } (x,\mathbf{S}) \in \widetilde{\Omega}_\varepsilon \times \mathcal{S}.$$

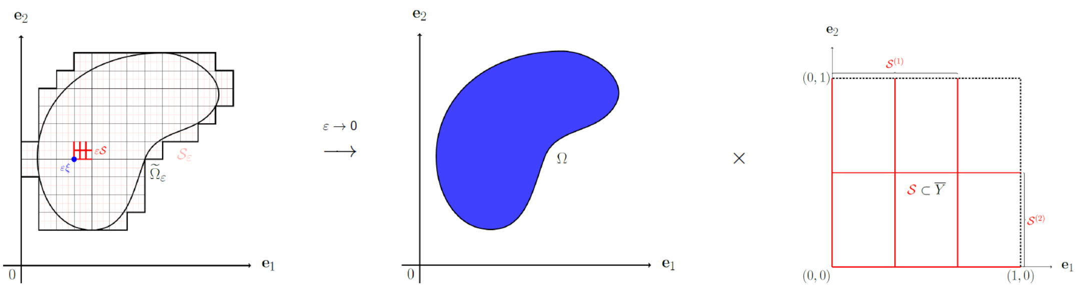

FIGURE 4.2: The unfolding via $\mathcal{T}_\varepsilon^{\mathcal{S}}$ of the variables in the periodic lattice $\mathcal{S}_\varepsilon \subset \widetilde{\Omega}_\varepsilon \subset \mathbb{R}^2$. In the limit, one has a split between the macroscopic scale and the reference lattice $\mathcal{S}$.

Observe that in the above definition of $\mathcal{T}_\varepsilon^{\mathcal{S}}$, the map from $\widetilde{\Omega}_\varepsilon \times \mathcal{S}$ into $\mathcal{S}_\varepsilon$:

$$(x,\mathbf{S}) \longmapsto \varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon\mathbf{S}$$

is almost everywhere one to one. This is not the case if we replace $\mathcal{S}$ with $\mathcal{S}_c$. Nevertheless, considerations and result for functions defined on $\mathcal{S}$ and on $\mathcal{S}_c$ are the same.
In the same way, we define the mean value operator defined in 2 but for lattice structures.

Definition 8. *For every function $\widehat{\phi}$ on $L^1(\mathcal{S}^{(i)})$, $i \in \{1,\dots,N\}$, the mean value operator $\mathcal{M}_{\mathcal{S}^{(i)}}$ on direction $\mathbf{e}_i$ is defined as follows:*

$$\mathcal{M}_{\mathcal{S}^{(i)}}(\widehat{\phi})(\mathbf{S}) \doteq \int_{A(k)}^{A(k)+\mathbf{e}_i} \widehat{\phi}(x,\mathbf{S}')d\mathbf{S}', \qquad \forall \mathbf{S} \in [A(k), A(k)+\mathbf{e}_i], \quad \forall k \in \widehat{\mathbf{K}}_i.$$

Below, we give the main property of $\mathcal{T}_\varepsilon^{\mathcal{S}}$.

Proposition 5. *Let $p \in [1,+\infty]$. For every $\phi \in L^p(\mathcal{S}_\varepsilon)$, one has*

$$\|\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi)\|_{L^p(\widetilde{\Omega}_\varepsilon \times \mathcal{S})} \le \varepsilon^{\frac{N-1}{p}} |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\mathcal{S}_\varepsilon)}.$$

Proof. We start with $p = 1$. Let ϕ be in $L^1(\mathcal{S}_\varepsilon)$. We have

$$\begin{aligned}\int_{\widetilde{\Omega}_\varepsilon\times\mathcal{S}} |\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi)(x,\mathbf{S})|dxd\mathbf{S} &= \int_{\widetilde{\Omega}_\varepsilon}\sum_{i=1}^{N}\int_{\mathcal{S}^{(i)}} |\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi)(x,\mathbf{S})|dxd\mathbf{S}\\ &= \sum_{\xi=\widetilde{\Xi}_\varepsilon} |\varepsilon\xi+\varepsilon Y|\sum_{i=1}^{N}\sum_{k\in\widehat{\mathbf{K}}_i}\int_0^1 |\phi(\varepsilon\xi+\varepsilon A(k)+\varepsilon t)|dt\\ &= \varepsilon^N|Y|\sum_{i=1}^{N}\sum_{k\in\widehat{\mathbf{K}}_i}\int_0^1 |\phi(\varepsilon\xi+\varepsilon A(k)+\varepsilon t)|dt \le \varepsilon^{N-1}|Y|\int_{\mathcal{S}_\varepsilon}|\phi(\mathbf{s})|d\mathbf{s}.\end{aligned}$$

The case $p \in (1, +\infty)$ follows by definition of L^p norm. The case $p = +\infty$ is trivial. □

4.2 Periodic unfolding for sequences defined as N-linear interpolates on the lattice nodes

Before proceeding to the actual strategy for the periodic unfolding for lattices, we dedicate this section to a useful class of functions: the sequences defined as N-linear extension from the lattice nodes to the whole domain.

The unfolding of this class of functions has two main advantages. The first is that fewer hypotheses are required for the sequences to ensure weak convergence (see property (2.10)). The second is that the convergences can be restricted to sub-spaces with lower dimensions, which will be key in the next sections.

First, since we are now working on $\widetilde{\Omega}_\varepsilon$, which contains Ω, we need to extend Definition (1) of the classical unfolding operator to functions defined in the following neighborhood of Ω:

$$\{x \in \mathbb{R}^N \mid \operatorname{dist}(x,\Omega) < \varepsilon\operatorname{diam}(Y)\}.$$

Definition 9. *For every measurable function Φ on $\widetilde{\Omega}_\varepsilon$, the unfolding operator $\mathcal{T}_\varepsilon^{ext}$ is defined as follows:*

$$\mathcal{T}_\varepsilon^{ext}(\Phi) \doteq \Phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) \quad \textit{for a.e. } (x,y)\in\widetilde{\Omega}_\varepsilon\times Y.$$

For every $\Phi \in L^p(\widetilde{\Omega}_\varepsilon)$, this operator satisfies (see also property (2.3)):

$$\|\mathcal{T}_\varepsilon^{ext}(\Phi)\|_{L^p(\widetilde{\Omega}_\varepsilon\times Y)} \le |Y|^{\frac{1}{p}}\|\Phi\|_{L^p(\widetilde{\Omega}_\varepsilon)} \quad \text{for every } \Phi\in L^p(\widetilde{\Omega}_\varepsilon).$$

Every measurable function defined on Ω can be extended to the set $\widetilde{\Omega}_\varepsilon$ by setting it to 0 on $\widetilde{\Omega}_\varepsilon \cap (\mathbb{R}^N \setminus \overline{\Omega})$. Now, let $p \in (1, +\infty)$. Assume $\{\Phi_\varepsilon\}_\varepsilon$ to be a sequence uniformly bounded in $L^p(\widetilde{\Omega}_\varepsilon)$. Then, the unfolded sequence $\{\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon)\}_\varepsilon$ is uniformly bounded in $L^p(\widetilde{\Omega}_\varepsilon \times Y)$ and thus in $L^p(\Omega \times Y)$. Hence, there exists a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\widehat{\Phi} \in L^p(\Omega \times Y)$ such that

$$\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon)_{|\Omega\times Y} \rightharpoonup \widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega\times Y).$$

For simplicity, we will omit the restriction and always write the above convergence as

$$\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon) \rightharpoonup \widehat{\Phi} \quad \textit{weakly in} \quad L^p(\Omega\times Y).$$

In this sense, we can easily transpose to this operator all the convergence results in Subsection 2.1.1 concerning the isotropically bounded sequences and in Section 3.3 concerning the anisotropically bounded ones .

We define the spaces of interpolated functions on the lattice nodes in Y (resp. in $\widetilde{\Omega}_\varepsilon$) by

$$\begin{aligned} Q^1_{\mathcal{K}}(Y) &\doteq \Big\{\Psi \in W^{1,\infty}(Y) \,\Big|\, \Psi_{|A(k)+\overline{Y_K}} \text{ is the } Q_1 \text{ interpolate of its values} \\ &\qquad\qquad \text{on the vertices of } A(k)+\overline{Y_K},\ \forall k \in \widehat{\mathbf{K}}\Big\}, \\ Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon) &\doteq \Big\{\Phi \in W^{1,\infty}(\widetilde{\Omega}_\varepsilon) \,\Big|\, \Phi_{|\varepsilon\xi+\varepsilon A(k)+\varepsilon\overline{Y_K}} \text{ is the } Q_1 \text{ interpolate of its values} \\ &\qquad\qquad \text{on the vertices of } \varepsilon\xi+\varepsilon A(k)+\varepsilon\overline{Y_K},\ \forall k \in \widehat{\mathbf{K}},\ \forall \xi \in \widetilde{\Xi}_\varepsilon\Big\}. \end{aligned} \tag{4.2}$$

From the N-linear interpolations properties (2.10) and (2.13), for every $\Phi \in Q_1(\widetilde{\Omega}_\varepsilon)$, there exist a constant depending only on p such that

$$\|\nabla\Phi\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le \frac{C}{\varepsilon}\|\Phi\|_{L^p(\widetilde{\Omega}_\varepsilon)}. \tag{4.3}$$

Below, we give the equivalent formulation of Propositions 1, 2 and Lemma 8 but for this special class of functions.

Corollary 1. *Let $\{\Phi_\varepsilon\}_\varepsilon$ be a sequence in $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$ satisfying*

$$\|\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C.$$

Then, there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, and $\widehat{\Phi} \in L^p(\Omega)$, $\widehat{\Phi} \in L^p(\Omega; Q^1_{\mathcal{K},per,0}(Y))$ such that

$$\begin{aligned} &\Phi_{\varepsilon|\Omega} \rightharpoonup \Phi \quad \text{weakly in} \quad L^p(\Omega), \\ &\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon) \rightharpoonup \Phi + \widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega; Q^1_{\mathcal{K}}(Y)), \\ &\varepsilon\mathcal{T}^{ext}_\varepsilon(\nabla\Phi_\varepsilon) \rightharpoonup \nabla_y\widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega\times Y)^N. \end{aligned}$$

The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Proof. First, from property (4.3) on the N-linear interpolated functions, we have that

$$\|\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \varepsilon\|\nabla\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)}$$

Then, the proof is done in the same fashion as Proposition 1, together with the fact that $\{\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\}_\varepsilon \in L^p(\widetilde{\Omega}_\varepsilon; Q^1_{\mathcal{K}}(Y))$. □

Corollary 2. *Let $\{\Phi_\varepsilon\}_\varepsilon$ be a sequence in $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$ satisfying*

$$\|\Phi_\varepsilon\|_{W^{1,p}(\widetilde{\Omega}_\varepsilon)} \le C.$$

Then, there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, and $\Phi \in W^{1,p}(\Omega)$, $\widehat{\Phi} \in L^p(\Omega; Q^1_{\mathcal{K},per,0}(Y))$ such that

$$\begin{aligned} &\Phi_{\varepsilon|\Omega} \rightharpoonup \Phi \quad \text{weakly in} \quad W^{1,p}(\Omega), \\ &\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon) \rightharpoonup \Phi \quad \text{weakly in} \quad L^p(\Omega; Q^1_{\mathcal{K}}(Y)), \\ &\mathcal{T}^{ext}_\varepsilon(\nabla\Phi_\varepsilon) \rightharpoonup \nabla\Phi + \nabla_y\widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega\times Y)^N, \\ &\frac{1}{\varepsilon}\big(\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon) - \mathcal{M}_Y \circ \mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\big) \rightharpoonup \nabla\Phi \cdot y'^c + \widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega\times Y), \end{aligned}$$

where $y'^c \doteq y' - \mathcal{M}_{Y'}(y')$.
The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Proof. The proof follows from Proposition 2, together with the fact that $\{\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\}_\varepsilon \in L^p(\widetilde{\Omega}_\varepsilon; Q^1_{\mathcal{K}}(Y))$. □

Corollary 3. *Let $N_1, N_2 \in \mathbb{N}$ such that $N_1 + N_2 = N$. Let $x = (x', x'') \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $y = (y', y'') \in Y' \times Y''$. Let $\{\Phi_\varepsilon\}_\varepsilon$ be a sequence in $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$ satisfying*

$$\|\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \|\nabla_{x'}\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C,$$

where the constant does not depend on ε.
Then, there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, and functions $\widetilde{\Phi} \in L^p(\Omega, \nabla_{x'}; Q^1_{\mathcal{K},per}(Y''))$, $\widehat{\Phi} \in L^p(\Omega \times Y''; Q^1_{\mathcal{K},per,0}(Y')) \cap L^p(\Omega; Q^1_{\mathcal{K}}(Y))$ such that

$$\begin{aligned}
&\Phi_{\varepsilon|_\Omega} \rightharpoonup \Phi \quad \text{weakly in} \quad L^p(\Omega, \nabla_{x'}),\\
&\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon) \rightharpoonup \widetilde{\Phi} \quad \text{weakly in} \quad L^p(\Omega; Q^1_{\mathcal{K}}(Y)),\\
&\mathcal{T}^{ext}_\varepsilon(\nabla_{x'}\Phi_\varepsilon) \rightharpoonup \nabla_{x'}\widetilde{\Phi} + \nabla_{y'}\widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N_1},\\
&\frac{1}{\varepsilon}\big(\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon) - \mathcal{M}_{Y'} \circ \mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\big) \rightharpoonup \nabla_{x'}\widetilde{\Phi} \cdot y'^c + \widehat{\Phi} \quad \text{weakly in} \quad L^p(\Omega \times Y),
\end{aligned}$$

where $\Phi = \mathcal{M}_{Y''}(\widetilde{\Phi})$ and $y'^c \doteq y' - \mathcal{M}_{Y'}(y')$.
The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Proof. First, since the sequence $\{\Phi_\varepsilon\}_\varepsilon$ belongs to $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$, property (4.3) implies that

$$\|\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \|\nabla_{x'}\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \varepsilon\|\nabla_{x''}\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C.$$

The proof follows by Lemma 8 together with the fact that $\{\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\}_\varepsilon \subset L^p(\widetilde{\Omega}_\varepsilon; Q^1_{\mathcal{K}}(Y))$. □

Note that the above lemma, which deals with anisotropically bounded sequences, includes the results of Proposition 1 in the particular case $N_1 = 0$ and $N_2 = N$, as well as the results of Proposition 2 in the particular case $N_1 = N$ and $N_2 = 0$.

4.3 How to unfold sequences defined on periodic lattice structures

Given a function ψ defined on the lattice structure $\mathcal{S}$ (resp. ϕ on $\mathcal{S}_\varepsilon$), a direct application of the unfolding operator for lattices $\mathcal{T}^{\mathcal{S}}_\varepsilon$ would result in an independent unfolding for each of the N directions. This might break the continuity of the functions in the limit since nothing ensures that the N obtained functions coincide on the lattice nodes. Hence, we need to work around this issue.

4.3.1 A first decomposition

Recall the decomposition that we have already done in Subsection 2.2.1, but for each of the repeated cells Y_K contained in the unitary cell Y.
Since the union is finite, this does not change the results in Subsection 2.2.1, which is why we will refer many proofs to this subsection.
On the lattice structures $\mathcal{S}_\varepsilon$ (resp. $\mathcal{S}$), we define the spaces $Q^1_{\mathcal{K}}(\mathcal{S})$ and $Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$ by

$$\begin{aligned}
Q^1_{\mathcal{K}}(\mathcal{S}) &\doteq \Big\{\psi \in \mathcal{C}(\mathcal{S}) \,\Big|\, \psi \text{ is affine between two contiguous points of } \mathcal{K}\Big\},\\
Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon) &\doteq \Big\{\phi \in \mathcal{C}(\mathcal{S}_\varepsilon) \,\Big|\, \phi \text{ is affine between two contiguous points of } \mathcal{K}_\varepsilon\Big\}.
\end{aligned} \tag{4.4}$$

Then, we define the spaces of functions vanishing on the lattice nodes by ($p \in [1, +\infty]$)

$$\begin{aligned}
W^{1,p}_{0,\mathcal{K}}(\mathcal{S}) &= \{\psi \in W^{1,p}(\mathcal{S}) \mid \psi = 0 \ \text{ on every node of } \mathcal{K}\},\\
W^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon) &= \{\phi \in W^{1,p}(\mathcal{S}_\varepsilon) \mid \phi = 0 \ \text{ on every node of } \mathcal{K}_\varepsilon\}.
\end{aligned}$$

Every function ψ in $W^{1,p}(\mathcal{S})$ (resp. $\phi \in W^{1,p}(\mathcal{S}_\varepsilon)$) is defined on the set of nodes $\mathcal{K}$ (resp. $\mathcal{K}_\varepsilon$) and therefore can be decomposed as

$$\begin{aligned} \psi &= \psi_a + \psi_0, \qquad \psi_a \in Q^1_{\mathcal{K}}(\mathcal{S}), \qquad \psi_0 \in \mathcal{W}^{1,p}_{0,\mathcal{K}}(\mathcal{S}), \\ \big(\text{resp. } \phi &= \psi_a + \phi_0, \qquad \phi_a \in Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon), \qquad \phi_0 \in \mathcal{W}^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)\big), \end{aligned} \tag{4.5}$$

where ψ_a, (resp. ϕ_a) is an affine function defined as Q^1 interpolation on the nodes, and ψ_0 (resp. ϕ_0) is the remainder term, which is zero on every node.

Lemma 10. *Let $\phi \in W^{1,p}(\mathcal{S}_\varepsilon)$ be decomposed as in (4.5). Then, there exists $C > 0$ such that ($i \in \{1, \dots, N\}$)*

$$\begin{aligned} \|\partial_{\mathbf{s}}\phi_a\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_0\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} &\le C\|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)}, \\ \|\phi_0\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_0\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} &\le C\varepsilon\|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)}. \end{aligned}$$

Proof. The proof is done in the same fashion as the one in Lemma 1, but for a grid of a cell with arbitrary length Y_K, that we will call $\mathcal{G}_K$. Then, the results follow by the fact that the lattice $\mathcal{S}$ is a finite union of grids of the form $\mathcal{G}_K$, together with an affine change of variables. □

4.3.2 A commutative diagram: from lattice to R^N, and to lattice again

As we already know, a function belonging to $Q^1_{\mathcal{K}}(\mathcal{S})$ (resp. $Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$) is determined only by its values on the set of nodes $\mathcal{K}$ (resp. $\mathcal{K}_\varepsilon$), and thus we can naturally extend it to a function defined on Y (resp. on $\widetilde{\Omega}_\varepsilon$).

Definition 10. *For every function $\psi \in Q^1_{\mathcal{K}}(\mathcal{S})$, its extension $\mathfrak{Q}(\psi)$ belonging to $W^{1,\infty}(\widetilde{\Omega}_\varepsilon)$ is defined by N-linear interpolation on each parallelotope $A(k) + Y_K$ belonging to $\overline{Y}$, for every $k \in \widehat{\mathbf{K}}$.*
For every function $\phi \in Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$, its extension $\mathfrak{Q}(\phi)$ belonging to $W^{1,\infty}(\widetilde{\Omega}_\varepsilon)$ is defined by N-linear interpolation on each parallelotope $\varepsilon\xi + \varepsilon A(k) + \varepsilon\overline{Y_K}$ belonging to $\varepsilon\xi + \varepsilon\overline{Y}$, for every $\xi \in \widetilde{\Xi}_\varepsilon$ and $k \in \widehat{\mathbf{K}}$.

Now, recall the spaces (4.2). By the same argumentation done in Subsection 2.2.1 but for a finite union of cells, the extension operator $\mathfrak{Q}$ is both one to one and onto from $Q^1_{\mathcal{K}}(\mathcal{S})$ to $Q^1_{\mathcal{K}}(Y)$ (resp. from $Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$ to $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$). Its inverse is given by the restriction $_{|\mathcal{S}}$ from $Q^1_{\mathcal{K}}(Y)$ to $Q^1_{\mathcal{K}}(\mathcal{S})$ (resp. $_{|\mathcal{S}_\varepsilon}$ from $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$ to $Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$).
Below, we show the main properties of this operator.

Lemma 11. *For every $\phi \in Q^1(\mathcal{S}_\varepsilon)$, one has ($p \in [1, +\infty]$, $i \in \{1, \dots, N\}$)*

$$\|\mathfrak{Q}(\phi)\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C\varepsilon^{\frac{N-1}{p}}\|\phi\|_{L^p(\mathcal{S}_\varepsilon)}, \quad \|\partial_i\mathfrak{Q}(\phi)\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C\varepsilon^{\frac{N-1}{p}}\|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)}.$$

Proof. The proof is done in the same fashion as the one of Lemma 2 but for a cell with arbitrary length Y_K. Then, the results follow since the unitary cell Y is a finite union of cells of the form Y_K, together with an affine change of variables. □

Finally, we can apply the following strategy: given a function defined on the lattice $\phi \in W^{1,p}(\mathcal{S}_\varepsilon)$, we first decompose it as in (4.5). Then, the unfolding for the affine function $\phi_a \in Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$ is equivalent to first extending ϕ_a to $\Phi_a = \mathfrak{Q}(\phi_a)$, then applying the unfolding results for N-linear interpolates in Section 4.2 and finally restricting the convergences to the lattice again, as the following commutative diagrams show ($i \in \{1, \dots, N\}$):

$$\begin{cases} \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi) = \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{|\mathcal{S}_\varepsilon}) = \mathcal{T}^{ext}_\varepsilon(\phi)_{|\widetilde{\Omega}_\varepsilon\times\mathcal{S}}, \\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi) = \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_{|\mathcal{S}^{(i)}_\varepsilon}) = \mathcal{T}^{ext}_\varepsilon(\partial_i\phi)_{|\widetilde{\Omega}_\varepsilon\times\mathcal{S}^{(i)}}. \end{cases} \tag{4.6}$$

On the other hand, the unfolding for the remainder term $\phi_0 \in \mathcal{W}^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$ can be done using the classical unfolding results in Section 2.1, since it is only defined on segments. Hence, the final sum of the limiting unfolded fields has continuity on the nodes since it is the sum of a N-linear interpolated function restricted to the lattice and a reminder function which is zero on the nodes.

4.4 Asymptotic behavior of sequences defined on lattices with information on the first order derivatives

Finally, we can unfold sequences belonging to $W^{1,p}(\mathcal{S}_\varepsilon)$, for which we have information on the sequence itself and the gradients.

4.4.1 Sequences isotropically bounded on lattices

We start with the sequences in $W^{1,p}$ where there is a contrast between the bound on the function and the bound on their gradient. This lemma is the equivalent of Proposition 1, but for lattice structures.

Note that on the sequence bounds, a rescaling factor, which depends on the p-norm and the N-dimension, is additionally applied due to the dimension reduction.

Lemma 12. *Let $p \in (1, +\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\mathcal{S}_\varepsilon)$ satisfying*

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}. \tag{4.7}$$

Then, there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, and $\widehat{\phi} \in L^p(\Omega; W^{1,p}_{per}(\mathcal{S}))$ such that

$$\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widehat{\phi} \qquad \textit{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})).^{1} \tag{4.8}$$

The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Proof. Given $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\mathcal{S}_\varepsilon)$, we decompose it as in (4.5) and get

$$\{\phi_\varepsilon\}_\varepsilon = \{\phi_{a,\varepsilon}\}_\varepsilon + \{\phi_{0,\varepsilon}\}_\varepsilon, \qquad \{\phi_{a,\varepsilon}\}_\varepsilon \in Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon), \quad \{\phi_{0,\varepsilon}\}_\varepsilon \in \mathcal{W}^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon).$$

By Lemma 10 and hypothesis (4.7), we have

$$\|\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}+1}, \tag{4.9}$$

$$\|\phi_{a,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_{a,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} \leq \|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}} \tag{4.10}$$

We first consider the sequence $\{\phi_{0,\varepsilon}\}_\varepsilon \subset \mathcal{W}^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$. By estimate (4.9) and Proposition 1, there exist a subsequence, still denoted ε, and a function $\widehat{\phi}_0 \in L^p(\Omega; \mathcal{W}^{1,p}_{0,\mathcal{K},per}(\mathcal{S}))$ such that

$$\frac{1}{\varepsilon}\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{0,\varepsilon}) \rightharpoonup \widehat{\phi}_0 \qquad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})). \tag{4.11}$$

We consider now the sequence $\{\phi_{a,\varepsilon}\}_\varepsilon \subset Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$. We extend it to $\{\Phi_{a,\varepsilon}\}_\varepsilon = \{\mathfrak{Q}(\phi_{a,\varepsilon})\}_\varepsilon$, which belongs to $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$. By Lemma 11 and estimate (4.10), this sequence satisfies

$$\|\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \varepsilon\|\nabla\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} \leq C.$$

[1] As for $\mathcal{T}^{ext}_\varepsilon$, this convergence must be understood

$$\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon)_{|\Omega\times\mathcal{S}} \rightharpoonup \widehat{\phi} \qquad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})).$$

It will be the same for all convergences involving the unfolding operator $\mathcal{T}^{\mathcal{S}}_\varepsilon$.

By construction, $\{\Phi_\varepsilon\}_\varepsilon \in Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$ and thus $\{\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\}_\varepsilon \in L^p(\widetilde{\Omega}_\varepsilon; Q^1_{\mathcal{K}}(Y))$. Hence, Corollary 1 implies that there exist $\widehat{\Phi}_a \in L^p(\Omega; Q^1_{\mathcal{K},per}(Y))$, such that

$$\mathcal{T}^{ext}_\varepsilon(\Phi_{\varepsilon,a}) \rightharpoonup \widehat{\Phi}_a \qquad \text{weakly in} \quad L^p(\Omega; Q^1_{\mathcal{K}}(Y)).$$

Using the relations (4.6), we can restrict the above convergences from $\Omega \times Y$ to the subset $\Omega \times \mathcal{S}$. We denote $\widehat{\phi}_a \doteq \widehat{\Phi}_{a|\Omega\times\mathcal{S}}$, which then belongs to $L^p(\Omega; Q^1_{\mathcal{K},per}(\mathcal{S}))$. We have

$$\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{a,\varepsilon}) \rightharpoonup \widehat{\phi}_a \quad \text{weakly in} \quad L^p(\Omega; Q^1_{\mathcal{K}}(\mathcal{S})).$$

Hence, from the above convergence and convergence (4.11), we get

$$\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) = \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{\varepsilon,a}) + \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{\varepsilon,0}) \rightharpoonup \widehat{\phi}_a \quad \text{weakly in} \quad L^p(\Omega; Q^1_{\mathcal{K}}(\mathcal{S})),$$

which concludes the proof by setting $\phi \doteq \widehat{\phi}_a$. □

Now, we show the asymptotic behavior of uniformly bounded sequences in $W^{1,p}(\mathcal{S}_\varepsilon)$. This lemma is the equivalent of Proposition 2 but for lattice structures.

Lemma 13. *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\mathcal{S}_\varepsilon)$ satisfying ($p \in (1, +\infty)$)*

$$\|\phi_\varepsilon\|_{W^{1,p}(\mathcal{S}_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}}, \tag{4.12}$$

where the constant does not depend on ε.
Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\phi \in W^{1,p}(\Omega)$ and $\widehat{\phi} \in L^p(\Omega; W^{1,p}_{per,0}(\mathcal{S}))$ such that ($i \in \{1, \dots, N\}$)

$$\begin{aligned} &\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \to \phi \qquad \textit{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})),\\ &\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial_i\phi + \partial_{\mathbf{S}}\widehat{\phi} \qquad \textit{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Proof. Given $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\mathcal{S}_\varepsilon)$, we decompose it as in (4.5) and get

$$\{\phi_\varepsilon\}_\varepsilon = \{\phi_{a,\varepsilon}\}_\varepsilon + \{\phi_{0,\varepsilon}\}_\varepsilon, \qquad \{\phi_{a,\varepsilon}\}_\varepsilon \in Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon), \quad \{\phi_{0,\varepsilon}\}_\varepsilon \in \mathcal{W}^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon).$$

By Lemma 10 and hypothesis (4.12), we have

$$\|\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} \le C\varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}+1}, \tag{4.13}$$

$$\|\phi_{a,\varepsilon}\|_{W^{1,p}(\mathcal{S}_\varepsilon)} \le \|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}} \tag{4.14}$$

We first consider the sequence $\{\phi_{0,\varepsilon}\}_\varepsilon \subset \mathcal{W}^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$. By estimate (4.13) and Proposition 1, there exist a subsequence, still denoted $\{\varepsilon\}$, and $\phi_0 \in L^p(\Omega)$, $\widehat{\phi}_0 \in L^p(\Omega; \mathcal{W}^{1,p}_{0,\mathcal{K},per,0}(\mathcal{S}))$ such that ($i \in \{1, \dots, N\}$)

$$\begin{aligned} &\frac{1}{\varepsilon}\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{0,\varepsilon}) \rightharpoonup \phi_0 + \widehat{\phi}_0 \qquad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})),\\ &\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_{0,\varepsilon}) \rightharpoonup \partial_{\mathbf{S}}\widehat{\phi}_0 \qquad \text{weakly in} \quad L^p(\Omega; \times\mathcal{S}^{(i)}). \end{aligned} \tag{4.15}$$

We consider now the sequence $\{\phi_{a,\varepsilon}\}_\varepsilon \subset Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$. We extend it to $\{\Phi_{a,\varepsilon}\}_\varepsilon = \{\mathfrak{Q}(\phi_{a,\varepsilon})\}_\varepsilon$, which belongs to $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$. By Lemma 11 and estimate (4.14), this sequence satisfies

$$\|\Phi_{a,\varepsilon}\|_{W^{1,p}(\widetilde{\Omega}_\varepsilon)} \le C.$$

By construction, we have $\{\Phi_{a,\varepsilon}\}_\varepsilon \in Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$ and thus $\{\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\}_\varepsilon \in L^p(\widetilde{\Omega}_\varepsilon; Q^1_{\mathcal{K}}(Y))$.
Hence, Corollary 2 implies that there exist $\Phi_a \in W^{1,p}(\Omega)$ and $\widehat{\Phi}_a \in L^p(\Omega; Q^1_{\mathcal{K},per,0}(Y))$, such that

$$\begin{aligned}
&\Phi_{\varepsilon,a|\Omega} \to \Phi_a \quad \text{strongly in} \quad W^{1,p}(\Omega),\\
&\mathcal{T}^{ext}_\varepsilon(\Phi_{\varepsilon,a}) \to \Phi_a \quad \text{strongly in} \quad L^p(\Omega; Q^1_{\mathcal{K}}(Y)),\\
&\mathcal{T}^{ext}_\varepsilon(\nabla\Phi_{\varepsilon,a}) \rightharpoonup \nabla\Phi_a + \nabla_y\widehat{\Phi}_a \quad \text{weakly in} \quad L^p(\Omega\times Y)^N.
\end{aligned}$$

Using the relations (4.6), we can restrict the above convergences from $\Omega \times Y$ to the subset $\Omega \times \mathcal{S}$. We denote $\phi_a \doteq \Phi_{a|\Omega\times\mathcal{S}}$, which belongs to $W^{1,p}(\Omega)$, and $\widehat{\phi}_a \doteq \widehat{\Phi}_{a|\Omega\times\mathcal{S}}$, which then belongs to $L^p(\Omega; Q^1_{\mathcal{K},per,0}(\mathcal{S}))$. We have

$$\begin{aligned}
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{a,\varepsilon}) \to \phi_a \quad \text{strongly in} \quad L^p(\Omega; Q^1_{\mathcal{K}}(\mathcal{S})),\\
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{S}}\phi_{a,\varepsilon}) \rightharpoonup \partial_i\phi_a + \partial_{\mathbf{S}}\widehat{\phi}_a \quad \text{weakly in} \quad L^p(\Omega\times\mathcal{S}), \qquad i \in \{1,\dots,N\}.
\end{aligned}$$

Hence, the statement follows from the above convergence, convergence (4.15) and setting $\phi \doteq \phi_a \in W^{1,p}(\Omega)$ and $\widehat{\phi} \doteq \widehat{\phi}_a + \widehat{\phi}_0$, which belongs to $L^p(\Omega; W^{1,p}_{per,0}(\mathcal{S}))$. □

4.4.2 Sequences anisotropically bounded on lattices

We now consider sequences whose gradient is anisotropically bounded on the lattice.
Accordingly to Section 3.1, we apply the decomposition $\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$ and define the following partition of our lattice structure:

$$\begin{aligned}
&\mathcal{S}' \doteq \bigcup_{i=1}^{N_1} \mathcal{S}^{(i)}, \qquad \mathcal{S}'_c \doteq \bigcup_{i=1}^{N_1} \mathcal{S}^{(i)}_c, \qquad \mathcal{S}'_\varepsilon \doteq \bigcup_{\xi\in\widetilde{\Xi}_\varepsilon} (\varepsilon\xi + \varepsilon\mathcal{S}'_c),\\
&\mathcal{S}'' \doteq \bigcup_{i=N_1+1}^{N} \mathcal{S}^{(i)}, \qquad \mathcal{S}''_c \doteq \bigcup_{i=N_1+1}^{N} \mathcal{S}^{(i)}_c, \qquad \mathcal{S}''_\varepsilon \doteq \bigcup_{\xi\in\widetilde{\Xi}_\varepsilon} (\varepsilon\xi + \varepsilon\mathcal{S}''_c).
\end{aligned}$$

Accordingly to (4.2), we define the spaces $Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}'_\varepsilon)$, $Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}''_\varepsilon)$. Accordingly to and (4.4), we also define the spaces $Q^1_{\mathcal{K}}(\mathcal{S}')$, $Q^1_{\mathcal{K}}(\mathcal{S}'')$, $Q^1_{\mathcal{K},per}(\mathcal{S})$, $Q^1_{\mathcal{K},per}(\mathcal{S}')$, $Q^1_{\mathcal{K},per}(\mathcal{S}'')$ and their respective extensions $Q^1_{\mathcal{K}}(Y')$, $Q^1_{\mathcal{K}}(Y'')$, $Q^1_{\mathcal{K},per}(Y)$, $Q^1_{\mathcal{K},per}(Y')$ and $Q^1_{\mathcal{K},per}(Y'')$.

We now prove the asymptotic behavior for sequences anisotropically bounded on $W^{1,p}(\mathcal{S}_\varepsilon)$.

Lemma 14. *Let $p \in (1,+\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\mathcal{S}_\varepsilon)$ satisfying*

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{S}}\phi_\varepsilon\|_{L^p(\mathcal{S}'_\varepsilon)} + \varepsilon\|\partial_{\mathbf{S}}\phi_\varepsilon\|_{L^p(\mathcal{S}''_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}}. \tag{4.16}$$

Then, there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, $\widetilde{\phi} \in L^p(\Omega, \nabla_{x'}; W^{1,p}_{per}(\mathcal{S}''))$, and functions $\widehat{\phi} \in L^p(\Omega\times\mathcal{S}''; W^{1,p}_{per,0}(\mathcal{S}')) \cap L^p(\Omega; W^{1,p}_{per}(\mathcal{S}))$, such that ($i \in \{1,\dots,N_1\}$)

$$\begin{aligned}
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widetilde{\phi} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})),\\
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{S}}\phi_\varepsilon) \rightharpoonup \partial_i\widetilde{\phi} + \partial_{\mathbf{S}}\widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega\times\mathcal{S}^{(i)}),\\
&\frac{1}{\varepsilon}\Big(\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}}\circ\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon)\Big) \rightharpoonup \partial_i\widetilde{\phi}\,\mathbf{S}^c + \widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega\times\mathcal{S}^{(i)}),
\end{aligned} \tag{4.17}$$

where $\mathbf{S}^c \doteq \big(\mathbf{S} - \mathcal{M}_{\mathcal{S}^{(i)}}(\mathbf{S})\big)\cdot\mathbf{e}_i$[2].
The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

[2] One has $\mathbf{S} = A(k) + t\mathbf{e}_i$ in the line $[A(k), A(k)+t\mathbf{e}_i]$, $t \in [0,1]$, $k \in \widehat{\mathbf{K}}_i$. Hence $\mathbf{S}^c = t - 1/2$.

Proof. Given $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\mathcal{S}_\varepsilon)$, we decompose ϕ_ε as in (4.5) and get

$$\{\phi_\varepsilon\}_\varepsilon = \{\phi_{a,\varepsilon}\}_\varepsilon + \{\phi'_{0,\varepsilon}\}_\varepsilon + \{\phi''_{0,\varepsilon}\}_\varepsilon,$$

where

$$\{\phi_{a,\varepsilon}\}_\varepsilon \in Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon), \quad \{\phi'_{0,\varepsilon}\}_\varepsilon \in \mathcal{W}^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}'_\varepsilon), \quad \{\phi''_{0,\varepsilon}\}_\varepsilon \in \mathcal{W}^{1,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}''_\varepsilon).$$

By Lemma 10 and hypothesis (4.16), we have

$$\begin{aligned}
&\|\phi'_{0,\varepsilon}\|_{L^p(\mathcal{S}'_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi'_{0,\varepsilon}\|_{L^p(\mathcal{S}'_\varepsilon)} \le C\varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}'_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}+1},\\
&\|\phi''_{0,\varepsilon}\|_{L^p(\mathcal{S}''_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi''_{0,\varepsilon}\|_{L^p(\mathcal{S}''_\varepsilon)} \le C\varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}''_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}},\\
&\|\phi_{a,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{S}}\phi_{a,\varepsilon}\|_{L^p(\mathcal{S}'_\varepsilon)} + \varepsilon\|\partial_{\mathbf{S}}\phi_{a,\varepsilon}\|_{L^p(\mathcal{S}''_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}}.
\end{aligned} \tag{4.18}$$

By estimate $(4.18)_1$ and Proposition 1 applied on each line of $\mathcal{S}'_\varepsilon$, there exist a subsequence, still denoted $\{\varepsilon\}$, and functions $\phi'_0 \in L^p(\Omega)$, $\widehat{\phi}'_0 \in L^p(\Omega; \mathcal{W}^{1,p}_{0,\mathcal{K},per,0}(\mathcal{S}'))$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{0,\varepsilon}) \rightharpoonup \phi'_0 + \widehat{\phi}'_0 \qquad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}')),\\
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_{0,\varepsilon}) \rightharpoonup \partial_{\mathbf{S}}\widehat{\phi}'_0 \qquad \text{weakly in} \quad L^p(\Omega\times\mathcal{S}').
\end{aligned} \tag{4.19}$$

By estimate $(4.18)_2$ and Proposition 1 applied on each line of $\mathcal{S}''_\varepsilon$, there exist a subsequence, still denoted $\{\varepsilon\}$, and functions $\phi''_0 \in L^p(\Omega)$, $\widehat{\phi}''_0 \in L^p(\Omega; \mathcal{W}^{1,p}_{0,\mathcal{K},per,0}(\mathcal{S}''))$ such that

$$\begin{aligned}
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{0,\varepsilon}) \rightharpoonup \phi''_0 + \widehat{\phi}''_0 \qquad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}'')),\\
&\varepsilon\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_{0,\varepsilon}) \rightharpoonup \partial_{\mathbf{S}}\widehat{\phi}''_0 \qquad \text{weakly in} \quad L^p(\Omega\times\mathcal{S}'').
\end{aligned} \tag{4.20}$$

Now, we consider the sequence $\{\phi_{a,\varepsilon}\}_\varepsilon \in Q^1_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$ and we extend it to $\{\Phi_{a,\varepsilon}\}_\varepsilon = \{\mho(\phi_{a,\varepsilon})\}_\varepsilon$ belonging to $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$. By Lemma 11, we get

$$\|\Phi_{a,\varepsilon}\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \|\nabla_{x'}\Phi_{a,\varepsilon}\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \varepsilon\|\nabla_{x''}\Phi_{a,\varepsilon}\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C.$$

By construction, the sequence $\{\Phi_{a,\varepsilon}\}_\varepsilon$ belongs to $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$ and thus $\{\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\}_\varepsilon$ belongs to $L^p(\widetilde{\Omega}_\varepsilon; Q^1_{\mathcal{K}}(Y))$. Hence, Corollary 3 imply that there exist functions $\widetilde{\Phi}_a \in L^p(\Omega, \nabla_{x'}; Q^1_{\mathcal{K},per}(Y''))$ and $\widehat{\Phi}_a \in L^p(\Omega\times Y''; Q^1_{\mathcal{K},per,0}(Y')) \cap L^p(\Omega; Q^1_{\mathcal{K}}(Y))$ such that

$$\begin{aligned}
&\Phi_{a,\varepsilon|\Omega} \rightharpoonup \Phi_a \qquad \text{weakly in} \quad L^p(\Omega, \nabla_{x'}),\\
&\mathcal{T}^{ext}_\varepsilon(\Phi_{a,\varepsilon}) \rightharpoonup \widetilde{\Phi}_a \qquad \text{weakly in} \quad L^p(\Omega; Q^1_{\mathcal{K}}(Y)),\\
&\mathcal{T}^{ext}_\varepsilon(\nabla_{x'}\Phi_{a,\varepsilon}) \rightharpoonup \nabla_{x'}\widetilde{\Phi}_a + \nabla_{y'}\widehat{\Phi}_a \qquad \text{weakly in} \quad L^p(\Omega\times Y)^{N_1},
\end{aligned}$$

where $\Phi_a = \mathcal{M}_{Y''}(\widetilde{\Phi}_a)$.
Using the relations (4.6), we can restrict the above convergences from $\Omega\times Y$ to the subset $\Omega\times\mathcal{S}$ (and from $\Omega\times Y'$, $\Omega\times Y''$ to $\Omega\times\mathcal{S}'$, $\Omega\times\mathcal{S}''$ respectively). Setting $\widetilde{\phi}_a = \widetilde{\Phi}_{a|\Omega\times\mathcal{S}}$, we have $\widetilde{\phi}_a \in L^p(\Omega, \nabla_{x'}; Q^1_{\mathcal{K},per}(\mathcal{S}''))$. Now, let us consider $\widehat{\Phi}_{a|\Omega\times\mathcal{S}'}$, which belongs to the space $\widehat{\Phi}_a \in L^p(\Omega; Q^1_{\mathcal{K},per,0}(\mathcal{S}'))$, and we extend it as an affine function between two adjacent nodes in $\mathcal{S}''$ (see Figure 4.3). This gives $\widehat{\phi}_a \in L^p(\Omega\times\mathcal{S}''; Q^1_{\mathcal{K},per,0}(\mathcal{S}')) \cap L^p(\Omega; Q^1_{\mathcal{K},per}(\mathcal{S}))$. Hence, the following convergences hold:

$$\begin{aligned}
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{a,\varepsilon}) \rightharpoonup \widetilde{\phi}_a \qquad \text{weakly in} \quad L^p(\Omega; Q^1(\mathcal{S})),\\
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_{a,\varepsilon}) \rightharpoonup \partial_i\widetilde{\phi}_a + \partial_{\mathbf{S}}\widehat{\phi}_a \quad \text{weakly in} \quad L^p(\Omega\times\mathcal{S}^{(i)})
\end{aligned} \tag{4.21}$$

Finally, from convergences (4.19), (4.20) and (4.21), we get ($i \in \{1, \dots, N_1\}$)

$$\begin{aligned}
&\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \to \widetilde{\phi}_a \quad \text{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}')),\\
&\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \rightharpoonup \widetilde{\phi}_a + \phi_0'' + \widehat{\phi}_0'' \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}'')),\\
&\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial_i\widetilde{\phi}_a + \partial_{\mathbf{S}}(\widehat{\phi}_a + \widehat{\phi}_0') \quad \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}).
\end{aligned}$$

Setting $\widetilde{\phi} \doteq \widetilde{\phi}_a + \phi_0'' + \widetilde{\phi}_0''$, we get that $\widetilde{\phi}$ belongs to $L^p(\Omega, \nabla_{x'}; W^{1,p}_{per}(\mathcal{S}''))$. Setting $\widehat{\phi} \doteq \widehat{\phi}_a + \widehat{\phi}_0'$, this function belongs to $L^p(\Omega \times \mathcal{S}''; Q^1_{\mathcal{K},per,0}(\mathcal{S}')) \cap L^p(\Omega; Q^1_{\mathcal{K},per}(\mathcal{S}))$. Convergence $(4.17)_3$ is an immediate consequence of $(4.17)_2$. The proof is complete. □

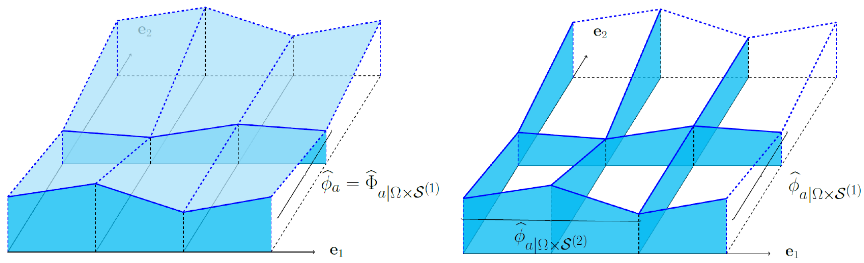

FIGURE 4.3: Construction of the microscopic variables of the periodic function $\widehat{\phi}_a$ in dimension two. On the left, the restriction to $\mathcal{S}^{(1)}$ of the 2-linear interpolate $\widehat{\Phi}_a$. On the right, the extension of $\widehat{\phi}_a$ from $\mathcal{S}^{(1)}$ to $\mathcal{S}^{(2)}$ by linear interpolation along th lattice nodes.

Again, note that the convergences in Lemma 14 include the isotropic cases in Lemma 12 for $\mathcal{S}' = \emptyset$ and $\mathcal{S}'' = \mathcal{S}$, and in Lemma 13 for $\mathcal{S}' = \mathcal{S}$ and $\mathcal{S}'' = \emptyset$.

4.5 Asymptotic behavior of sequences defined on lattices with information until the second order derivatives

We would like now to apply the same strategy but to sequences bounded in $W^{2,p}$, which have information till the second order derivative.

4.5.1 The problem of mixed derivatives

As we have seen in Subsection 2.2.2, the N-cubic extension from a cubic interpolation on the lattices to the whole domain is not uniquely defined because of the lack of mixed derivatives for the function defined on the lattice. This is because a function defined on the lattice segments can be derived twice, only in the segment directions. We overcome the problem in two different ways:

(i) We proceed as in Subsection 2.2.2 and linearly extend the N partial derivatives to adopt the same strategy above but using the N-cubic interpolation instead of the linear. This method will lead to a better regularity of the limit fields but at the cost of some artificial assumptions on the boundedness of the extended derivatives.

(ii) We adopt twice the N-linear interpolation: on the function and its partial derivatives. This method will lead to a worse regularity of the limit fields, but no further assumptions are made on the boundedness of the original sequences.

4.5.2 Unfolding via N-cubic interpolation

We proceed in the same fashion as the previous section and decompose a function into a remainder term and a cubic polynomial, extending the latter by N-cubic interpolation to the whole space.

On the lattice structures $\mathcal{S}_\varepsilon$ (resp. $\mathcal{S}$) we define the space of functions $Q_3(\mathcal{S}_\varepsilon)$ (resp. $Q_3(\mathcal{S})$) by

$$Q^3_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon) \doteq \left\{\psi \in \mathcal{C}(\mathcal{S}_\varepsilon) \,\middle|\, \psi \text{ is a cubic polynomial between two contiguous points of } \mathcal{K}_\varepsilon\right\},$$
$$Q^3_{\mathcal{K}}(\mathcal{S}) \doteq \left\{\phi \in \mathcal{C}(\mathcal{S}) \,\middle|\, \phi \text{ is a cubic polynomial between two contiguous points of } \mathcal{K}\right\}.$$

Then, we define the spaces of functions vanishing on the lattice nodes, and with derivative vanishing on the lattice nodes, by ($p \in [1, +\infty]$, $i \in \{1, \dots, N\}$)

$$\mathcal{W}^{2,p}_{0,\mathcal{K}}(\mathcal{S}) = \{\psi \in W^{2,p}(\mathcal{S}) \mid \psi = \nabla\psi = 0 \ \text{ on } \mathcal{K}\},$$
$$\mathcal{W}^{2,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon) = \{\phi \in W^{2,p}(\mathcal{S}_\varepsilon) \mid \phi = \nabla\phi = 0 \ \text{ on } \mathcal{K}_\varepsilon\}.$$

Every function $\psi \in W^{2,p}(\mathcal{S})$ (resp. $\phi \in W^{2,p}(\mathcal{S}_\varepsilon)$) is defined on the set of nodes $\mathcal{K}$ (resp. $\mathcal{K}_\varepsilon$) and therefore can be uniquely decomposed as

$$\begin{aligned} \psi &= \psi_c + \psi_0, \qquad \psi_c \in Q^3_{\mathcal{K}}(\mathcal{S}), \quad \psi_0 \in \mathcal{W}^{2,p}_{0,\mathcal{K}}(\mathcal{S}), \\ (\text{resp. } \phi &= \phi_c + \phi_0, \qquad \phi_c \in Q^3_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon), \quad \phi_0 \in \mathcal{W}^{2,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)), \end{aligned} \tag{4.22}$$

where ψ_c (resp. ϕ_c) is the cubic polynomial that coincides with the original function on the nodes (and its derivatives coincide with the original function's derivatives on the nodes), and ψ_0 (resp. ϕ_0) is the reminder term which is zero on every node (and its derivatives are zero on every node).

Lemma 15. *Let $i \in \{1, \dots, N\}$ and $\phi \in W^{2,p}(\mathcal{S}_\varepsilon)$. Suppose that ϕ is decomposed as in (4.22). Then, there exists $C > 0$ such that*

$$\begin{aligned} &\|\partial^2_{\mathbf{ss}}\phi_c\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} \le C\|\partial^2_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)}, \\ &\|\partial_{\mathbf{s}}\phi_c\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} \le C\|\partial_{\mathbf{s}}\phi\|_{W^{1,p}(\mathcal{S}^{(i)}_\varepsilon)}, \\ &\|\phi_c\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} \le C\|\phi\|_{W^{2,p}(\mathcal{S}^{(i)}_\varepsilon)}, \\ &\|\phi_0\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_0\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} + \varepsilon^2\|\partial^2_{\mathbf{s}}\phi_0\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)} \le C\varepsilon^2\|\partial^2_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)}_\varepsilon)}. \end{aligned} \tag{4.23}$$

Proof. The proof is done in the same fashion as the one in Lemma 3, but for a grid of a cell with arbitrary length Y_K, that we will call $\mathcal{G}_K$. Then, the results follow since the lattice $\mathcal{S}$ is a finite union of grids of the form $\mathcal{G}_K$, together with an affine change of variables. □

Set the spaces

$$Q^3_{\mathcal{K}}(Y) \doteq \Big\{\Psi \in W^{1,\infty}(Y) \,\Big|\, \Psi_{|A(k)+\overline{Y}_K} \text{ is } N\text{-cubic interpolate of its values and partial derivatives values on the vertices of } A(k) + \overline{Y}_K, \forall k \in \widehat{\mathbf{K}}\Big\},$$

$$Q^3_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon) \doteq \Big\{\Phi \in W^{1,\infty}(\widetilde{\Omega}_\varepsilon) \,\Big|\, \Phi_{|\varepsilon\xi+\varepsilon A(k)+\varepsilon\overline{Y}_K} \text{ is } N\text{-cubic interpolate of its values and partial derivatives values on the vertices of } \varepsilon\xi + \varepsilon A(k) + \varepsilon\overline{Y}_K, \forall k \in \widehat{\mathbf{K}}, \forall \xi \in \widetilde{\Xi}_\varepsilon\Big\}.$$

As we already did in Section 4.2, we give an equivalent formulation of Proposition 3 but for functions defined as N-cubic interpolations on the lattice nodes.

Corollary 4. *Let $p \in (1, +\infty)$ and let $\{\Phi_\varepsilon\}_\varepsilon$ be a sequence in $Q^3_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$ satisfying*

$$\|\Phi_\varepsilon\|_{W^{2,p}(\widetilde{\Omega}_\varepsilon)} \le C.$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\Phi \in W^{2,p}(\Omega)$, $\widehat{\Phi} \in L^p(\Omega; Q^3_{\mathcal{K},per}(Y))$ such that

$$\begin{aligned}
&\mathcal{T}_\varepsilon(\Phi_\varepsilon) \to \Phi \quad \textit{strongly in} \quad L^p(\Omega; W^{2,p}(Y)),\\
&\mathcal{T}_\varepsilon(\nabla\Phi_\varepsilon) \to \nabla\Phi \quad \textit{strongly in} \quad L^p(\Omega; W^{1,p}(Y))^N,\\
&\mathcal{T}_\varepsilon(D^2\Phi_\varepsilon) \rightharpoonup D^2\Phi + D^2_y\widehat{\Phi} \quad \textit{weakly in} \quad L^p(\Omega \times Y)^{N\times N}.
\end{aligned}$$

The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Proof. The proof directly follows by Proposition 3, together with the fact that $\{\mathcal{T}^{ext}_\varepsilon(\Phi_\varepsilon)\}_\varepsilon \in L^p(\widetilde{\Omega}_\varepsilon; Q^3_{\mathcal{K}}(Y))$. □

Given a function ψ_c defined on $\mathcal{S}$ (resp. ϕ_c defined on $\mathcal{S}_\varepsilon$), its extension to the whole cell Y (resp. to the whole domain $\widetilde{\Omega}_\varepsilon$) is given by any function $\Psi_c \in W^{2,\infty}(Y)$ (resp. $\Phi_c \in W^{2,\infty}(\widetilde{\Omega}_\varepsilon)$) that restricted on $\mathcal{S}$ (on $\mathcal{S}_\varepsilon$) gives back the original function.
As we already know from Subsection 2.2.2, the extension is not unique. For this reason, we set

$$\mathcal{S}^{[i]} \doteq \bigcup_{j=1,\, j\neq i}^{N} \mathcal{S}^{(j)} \qquad \left(\text{resp. } \mathcal{S}^{[i]}_\varepsilon \doteq \bigcup_{j=1,\, j\neq i}^{N} \mathcal{S}^{(j)}_\varepsilon\right). \tag{4.24}$$

For every $i \in \{1, \dots, N\}$, we denote the following extensions

$$\begin{aligned}
\overline{\partial_i\psi} &\doteq \{f \in W^{1,p}(\mathcal{S}^{(i)}) \times W^{1,\infty}(\mathcal{S}^{[i]}) \mid f_{\mathcal{S}^{(i)}_c} \text{ is extended by } N-1\text{-linear interpolation on } \mathcal{S}^{[i]}\},\\
\overline{\partial_i\phi} &\doteq \{f \in W^{1,p}(\mathcal{S}^{(i)}_\varepsilon) \times W^{1,\infty}(\mathcal{S}^{[i]}_\varepsilon) \mid f_{\mathcal{S}^{(i)}_\varepsilon} \text{ is extended by } N-1\text{-linear interpolation on } \mathcal{S}^{[i]}_\varepsilon\}.
\end{aligned} \tag{4.25}$$

These extensions not only uniquely determine the N-cubic interpolation Ψ_c: setting a bound for them allows us to bound the interpolation Ψ_c by the bounds on the lattice function ψ_c, as the following lemma shows.

Lemma 16. *Let $\Phi_c \in W^{2,p}(\widetilde{\Omega}_\varepsilon)$ be the unique cubic extension of the function $\phi_c \in W^{2,p}(\mathcal{S}_\varepsilon)$ with the derivatives extended as in (4.25). One has*

$$\|\Phi_c\|_{W^{2,p}(\widetilde{\Omega}_\varepsilon)} \leq C\varepsilon^{\frac{N-1}{p}} \Big(\|\phi\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}_\varepsilon)} + \sum_{i=1}^{N} \|\partial_{\mathbf{s}}(\overline{\partial_i\phi_c})\|_{L^p(\mathcal{S}_\varepsilon)}\Big). \tag{4.26}$$

Proof. The proof follows from Lemma 4 but for the finite union of cells εY_K of εY. □

We can finally show the asymptotic behavior of sequences bounded in $W^{2,p}(\mathcal{S}_\varepsilon)$.

Theorem 2. *Let $p \in (1, +\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{2,p}(\mathcal{S}_\varepsilon)$, satisfying*

$$\|\phi_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \sum_{i=1}^{N} \|\partial_{\mathbf{s}}(\partial_i\phi_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}. \tag{4.27}$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\phi \in W^{2,p}(\Omega)$, $\widehat{\phi} \in L^p(\Omega; W^{2,p}_{per}(\mathcal{S}))$ such that ($i \in \{1, \dots, N\}$)

$$\begin{aligned}
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \to \phi \qquad \textit{strongly in} \quad L^p(\Omega; W^{2,p}(\mathcal{S})),\\
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_\varepsilon) \to \partial_i\phi \qquad \textit{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}^{(i)})),\\
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial^2_{ii}\phi + \partial^2_{\mathbf{S}}\widehat{\phi} \quad \textit{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}).
\end{aligned}$$

Proof. Given the sequence $\{\phi_\varepsilon\}_\varepsilon \subset W^{2,p}(\mathcal{S}_\varepsilon)$, we decompose it as in (4.22) and get

$$\phi_\varepsilon = \phi_{c,\varepsilon} + \phi_{0,\varepsilon}, \quad \phi_{c,\varepsilon} \in Q^3_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon), \quad \phi_{0,\varepsilon} \in W^{2,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon).$$

We first consider the sequence $\{\phi_{0,\varepsilon}\}_\varepsilon$ belonging to $\mathcal{W}^{2,p}_{0,\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$. By estimate (4.23)$_4$ and hypothesis (4.27), we have

$$\|\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon^2\|\partial^2_{\mathbf{s}}\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} \le C\varepsilon^2\|\partial^2_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}+2}.$$

Hence, Proposition 3 implies that there exist $\widehat{\phi}_0 \in L^p(\Omega; \mathcal{W}^{2,p}_{0,\mathcal{K},per}(\mathcal{S}))$ such that

$$\frac{1}{\varepsilon^2}\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{0,\varepsilon}) \rightharpoonup \widehat{\phi}_0 \qquad \text{weakly in} \quad L^2(\Omega; W^{2,p}(\mathcal{S})). \tag{4.28}$$

Now we consider the sequence $\{\phi_{c,\varepsilon}\}_\varepsilon \in Q^3_{\mathcal{K}_\varepsilon}(\mathcal{S}_\varepsilon)$. For every $i \in \{1,\dots,N\}$, we define its derivatives extensions $\overline{\partial_i\phi_{c,\varepsilon}}$. Then, we can define the extension of the sequence to the whole domain $\{\Phi_{c,\varepsilon}\}_\varepsilon \in Q^3_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$. By estimates (4.26), we have

$$\|\Phi_{c,\varepsilon}\|_{W^{2,p}(\widetilde{\Omega}_\varepsilon)} \le C\varepsilon^{\frac{N-1}{p}}\Big(\|\phi_{c,\varepsilon}\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_{c,\varepsilon}\|_{L^2(\mathcal{S}_\varepsilon)} + \sum_{i=1}^N \|\partial_{\mathbf{s}}(\overline{\partial_i\phi_{c,\varepsilon}})\|_{L^2(\mathcal{S}_\varepsilon)}\Big) \le C.$$

Hence, Corollary 4 implies that there exist $\Phi_c \in W^{2,p}(\Omega)$ and $\widehat{\Phi}_c \in L^p(\Omega; Q^3_{\mathcal{K}}(Y))$ such that

$$\begin{aligned}
&\Phi_{c,\varepsilon|\Omega} \rightharpoonup \Phi_c \qquad \text{weakly in} \quad W^{2,p}(\Omega),\\
&\mathcal{T}_\varepsilon(\Phi_{c,\varepsilon}) \to \Phi_c \qquad \text{strongly in} \quad L^p(\Omega; W^{2,p}(Y)),\\
&\mathcal{T}_\varepsilon(\nabla\Phi_{c,\varepsilon}) \to \nabla\Phi_c \qquad \text{strongly in} \quad L^p(\Omega; W^{1,p}(Y))^N,\\
&\mathcal{T}_\varepsilon(D^2\Phi_{c,\varepsilon}) \rightharpoonup D^2\Phi_c + D^2_y\widehat{\Phi}_c \qquad \text{weakly in} \quad L^p(\Omega\times Y)^{N\times N}.
\end{aligned}$$

Note that the following relations hold ($i \in \{1,\dots,N\}$):

$$\begin{cases}
\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{c,\varepsilon}) = \mathcal{T}^{\mathcal{S}}_\varepsilon(\Phi_{c,\varepsilon|\mathcal{S}_\varepsilon}) = \mathcal{T}^{ext}_\varepsilon(\Phi_{c,\varepsilon})_{|\widetilde{\Omega}_\varepsilon\times\mathcal{S}},\\
\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_{c,\varepsilon})_{|\widetilde{\Omega}_\varepsilon\times\mathcal{S}^{(i)}} = \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\Phi_{c,\varepsilon|\mathcal{S}^{(i)}_\varepsilon}) = \mathcal{T}^{ext}_\varepsilon(\partial_i\Phi_{c,\varepsilon})_{\widetilde{\Omega}_\varepsilon\times\mathcal{S}^{(i)}},\\
\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}\phi_{c,\varepsilon})_{|\widetilde{\Omega}_\varepsilon\times\mathcal{S}^{(i)}} = \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}\Phi_{c,\varepsilon|\mathcal{S}^{(i)}_\varepsilon}) = \mathcal{T}^{ext}_\varepsilon(\partial^2_i\Phi_{c,\varepsilon})_{\widetilde{\Omega}_\varepsilon\times\mathcal{S}^{(i)}}.
\end{cases}$$

We then restrict the above convergences from $\Omega\times Y$ to the subsets $\Omega\times\mathcal{S}$ and $\Omega\times\mathcal{S}^{(i)}$, for every $i \in \{1,\dots,N\}$. Hence, there exist a function $\phi_c \doteq \Phi_{c|\Omega\times\mathcal{S}} \in W^{2,p}(\Omega)$ and a function $\widehat{\phi}_c = \widehat{\Phi}_{c|\Omega\times\mathcal{S}} \in L^p(\Omega; W^{2,p}_{per}(\mathcal{S}))$ such that ($i \in \{1,\dots,N\}$)

$$\begin{aligned}
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{c,\varepsilon}) \to \phi_c \qquad \text{strongly in} \quad L^p(\Omega; W^{2,p}(\mathcal{S})),\\
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_{c,\varepsilon}) \to \partial_i\phi_c \qquad \text{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}^{(i)})),\\
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}\phi_{c,\varepsilon}) \rightharpoonup \partial^2_{ii}\phi_c + \partial^2_{\mathbf{S}}\widehat{\phi}_c \qquad \text{weakly in} \quad L^p(\Omega\times\mathcal{S}^{(i)}).
\end{aligned}$$

Note that the strong convergences are preserved due to the polynomial character of the function $\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_{c,\varepsilon})$ with respect to the second variable.
Finally, by the above convergences and (4.28), we get ($i \in \{1,\dots,N\}$)

$$\begin{aligned}
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \to \phi_c \qquad \text{strongly in} \quad L^2(\Omega; W^{2,p}(\mathcal{S})),\\
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_\varepsilon) \to \partial_i\phi_c \qquad \text{strongly in} \quad L^2(\Omega; W^{1,p}(\mathcal{S}^{(i)})),\\
&\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial^2_{ii}\phi_c + \partial^2_{\mathbf{S}}(\widehat{\phi}_c + \widehat{\phi}_0) \qquad \text{weakly in} \quad L^2(\Omega\times\mathcal{S}^{(i)}).
\end{aligned}$$

The proof follows by setting $\phi \doteq \phi_c \in W^{2,p}(\Omega)$ and $\widehat{\phi} \doteq \widehat{\phi}_c + \widehat{\phi}_0 \in L^2(\Omega; W^{2,p}_{per}(\mathcal{S}))$. □

4.5.3 Unfolding via known results for sequences bounded in $W^{1,p}$

With this method, we consider a sequence in $W^{2,p}(\mathcal{S}_\varepsilon)$ as a sequence in $W^{1,p}(\mathcal{S}_\varepsilon)$ with partial derivatives in $W^{1,p}(\mathcal{S}_\varepsilon^{(i)})$ ($i \in \{1,\dots,N\}$), so that we can apply the results of Section 5.5.
Even though no gradient extension is needed, the limiting functions will have less regularity. Moreover, we must do some additional work to show that the N different limit functions, one for each partial derivative, are a unique function restricted to each line.
Let $p \in (1,+\infty)$. From Chapter 9 of Gilbarg and Trudinger, 1997, we recall that:

(i) If $u \in W^{1,p}(\Omega)$ satisfies $\Delta u \in L^p(\Omega)$, then $u \in W^{1,p}(\Omega) \cap W^{2,p}_{loc}(\Omega)$[3];

(ii) If Ω is a bounded domain in $\mathbb{R}^N$ with a $\mathcal{C}^{1,1}$ boundary and if $u \in W^{1,p}_0(\Omega)$ satisfies $\Delta u \in L^p(\Omega)$, then $u \in W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega)$.

Denote

$$W^{1,p}_\Delta(\Omega) \doteq \{\phi \in W^{1,p}(\Omega) \cap W^{2,p}_{loc}(\Omega) \mid \partial^2_{ii} \in L^p(\Omega) \ \text{ for every } \ i \in \{1,\dots,N\}\}.$$

We endow $W^{1,p}_\Delta(\Omega)$ with the following norm

$$\|\phi\|_{W^{1,p}_\Delta(\Omega)} \doteq \|\phi\|_{W^{1,p}(\Omega)} + \sum_{i=1}^N \|\partial^2_{ii}\phi\|_{L^p(\Omega)}.$$

Theorem 3. *Let $p \in (1,+\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{2,p}(\mathcal{S}_\varepsilon)$ satisfying*

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial^2_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}}. \tag{4.29}$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\phi \in W^{1,p}_\Delta(\Omega)$, $\widehat{\phi} \in L^p(\Omega; W^{2,p}_{per,0}(\mathcal{S}))$ such that ($i \in \{1,\dots,N\}$)

$$\begin{aligned} \mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) &\to \phi \qquad \textit{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})),\\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}}\phi_\varepsilon) &\rightharpoonup \partial_i\phi \qquad \textit{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}),\\ \mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}}\phi_\varepsilon) &\rightharpoonup \partial^2_{ii}\phi + \partial^2_{\mathbf{S}}\widehat{\phi} \quad \textit{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned} \tag{4.30}$$

The same results hold for $p = +\infty$ with weak topology replaced by weak- topology in the corresponding spaces.*

Proof. Step 1. We prove convergences $(4.30)_{1,2}$.
By estimate (4.29), the sequence $\{\phi_\varepsilon\}_\varepsilon$ satisfies

$$\|\phi_\varepsilon\|_{W^{1,p}(\mathcal{S}_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}}$$

and thus by Lemma 13, there exist $\phi \in W^{1,p}(\Omega)$ and $\widehat{\phi} \in L^p(\Omega; W^{1,p}_{per,0}(\mathcal{S}))$ such that

$$\begin{aligned} &\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \to \phi \quad \text{strongly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})),\\ &\mathcal{T}_\varepsilon(\partial_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial_i\phi + \partial_{\mathbf{S}}\widehat{\phi} \quad \text{weakly in} \quad L^p(\Omega \times \mathcal{S}^{(i)}), \qquad i \in \{1,\dots,N\}. \end{aligned} \tag{4.31}$$

Now, we consider the sequences $\{\psi^{(i)}_\varepsilon\}_\varepsilon = \{\partial_{\mathbf{s}}\phi_{\varepsilon|\mathcal{S}^{(i)}_\varepsilon}\}_\varepsilon$, $i \in \{1,\dots,N\}$. From estimate (4.29) we have

$$\|\psi^{(i)}_\varepsilon\|_{W^{1,p}(\mathcal{S}^{(i)}_\varepsilon)} \le C\varepsilon^{\frac{1-N}{p}}.$$

Recall the definition of $\mathcal{S}^{[i]}_\varepsilon$ from (4.24). Since for every $i \in \{1,\dots,N\}$, the function $\psi^{(i)}_\varepsilon$ is defined on every node of $\mathcal{S}_\varepsilon$, we extend it as in (4.25) and denote this extension $\overline{\psi}^{(i)}_\varepsilon$. It

[3] In fact, we have $\rho D^2 u \in L^p(\Omega)^{N\times N}$ where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$ for all $x \in \mathbb{R}^N$.

satisfies

$$\|\overline{\psi}_\varepsilon^{(i)}\|_{L^p(\mathcal{S}_\varepsilon)} + \big\|\partial_{\mathbf{s}}\overline{\psi}_\varepsilon^{(i)}\big\|_{L^p(\mathcal{S}_\varepsilon^{(i)})} + \varepsilon\big\|\partial_{\mathbf{s}}\overline{\psi}_\varepsilon^{(i)}\big\|_{L^p(\mathcal{S}_\varepsilon^{[i]})} \le C\varepsilon^{\frac{1-N}{p}},$$

Lemma 14 gives a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\widetilde{\psi}^{(i)} \in L^p(\Omega, \partial_i; W^{1,p}_{per}(\mathcal{S}^{[i]}))$, $\widehat{\psi}^{(i)} \in L^p(\Omega\times\mathcal{S}^{[i]}; W^{1,p}_{per,0}(\mathcal{S}^{(i)})) \cup L^p(\Omega; W^{1,p}_{per}(\mathcal{S}))$ such that $(i \in \{1,\dots,N\})$

$$\begin{aligned}
&\mathcal{T}_\varepsilon^{\mathcal{S}}(\overline{\psi}_\varepsilon^{(i)}) \rightharpoonup \widetilde{\psi}^{(i)} \qquad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S})),\\
&\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\overline{\psi}_\varepsilon^{(i)}) \rightharpoonup \partial_i\widetilde{\psi}^{(i)} + \partial_{\mathbf{S}}\widehat{\psi}^{(i)} \quad \text{weakly in} \quad L^p(\Omega\times\mathcal{S}^{(i)}).
\end{aligned}$$

The above second convergence and (4.31)$_2$ yield $(i \in \{1,\dots,N\})$

$$\partial_i\phi + \partial_{\mathbf{S}}\widehat{\phi} = \widetilde{\psi}^{(i)} \qquad \text{a.e. in } \Omega\times\mathcal{S}^{(i)}.$$

Since $\widetilde{\psi}^{(i)}$ does not depend on $\mathbf{S}$ in $\mathcal{S}^{(i)}$ and $\widehat{\phi}$ is periodic with respect to $\mathbf{S}$ in $\mathcal{S}^{(i)}$ we have $\partial_i\phi = \widetilde{\psi}^{(i)}$ and $\partial_{\mathbf{S}}\widehat{\phi} = 0$ a.e. $\Omega\times\mathcal{S}^{(i)}$ for every $i \in \{1,\dots,N\}$.
Hence, we get that $\widetilde{\psi}^{(i)}$ belongs to $L^p(\Omega,\partial_i)$ and thus that $\partial_i\phi \in L^p(\Omega,\partial_i)$. Since $\Delta\phi \in L^p(\Omega)$, we have $\phi \in W^{1,p}_\Delta(\Omega)$. Moreover, the following convergences hold:

$$\begin{aligned}
&\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \to \phi \qquad \text{strongly in} \quad L^p(\Omega; W^{2,p}(\mathcal{S})),\\
&\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial_i\phi \qquad \text{weakly in} \quad L^p(\Omega\times\mathcal{S}^{(i)}),\\
&\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial^2_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial^2_{ii}\phi + \partial_{\mathbf{S}}\widehat{\psi}^{(i)} \quad \text{weakly in} \quad L^p(\Omega\times\mathcal{S}^{(i)}),
\end{aligned}$$

and for each $i \in \{1,\dots,N\}$, we also have that

$$\frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \boldsymbol{\mathcal{M}}_{\mathcal{S}^{(i)}}\circ\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big) \rightharpoonup \partial^2_{ii}\phi\,\mathbf{S}^c + \widehat{\psi}^{(i)} \quad \text{weakly in} \quad L^p(\Omega\times\mathcal{S}^{(i)}). \tag{4.32}$$

Step 2. We prove the convergence (4.30)$_3$.
We have to prove the existence of $\widehat{\phi} \in L^p(\Omega; W^{2,p}_{per,0}(\mathcal{S}))$ such that

$$\begin{cases}
\partial_{\mathbf{S}}\widehat{\phi} = \widehat{\psi}^{(1)} \quad \text{a.e. in } \Omega\times\mathcal{S}^{(1)},\\
\vdots\\
\partial_{\mathbf{S}}\widehat{\phi} = \widehat{\psi}^{(N)} \quad \text{a.e. in } \Omega\times\mathcal{S}^{(N)}.
\end{cases}$$

A necessary and sufficient condition to get the existence of the function $\widehat{\phi}$ is (remind that $A(k+\mathbf{e}_i) = A(k) + l_i\mathbf{e}_i$)

$$\forall k \in \widehat{\mathbf{K}}, \quad \begin{aligned}
&\int_{A(k)}^{A(k+\mathbf{e}_i)} \widehat{\psi}^{(i)}(\cdot,\mathbf{S})d\mathbf{S} + \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \widehat{\psi}^{(j)}(\cdot,\mathbf{S})d\mathbf{S}\\
&= \int_{A(k)}^{A(k+\mathbf{e}_j)} \widehat{\psi}^{(j)}(\cdot,\mathbf{S})d\mathbf{S} + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \widehat{\psi}^{(i)}(\cdot,\mathbf{S})d\mathbf{S}
\end{aligned} \tag{4.33}$$

a.e. in Ω.
Since on a line belonging to $\mathcal{S}^{(i)}$, one has (see Lemma 14) $\mathbf{S}^c = t - \dfrac{1}{2}$, $t \in [0,1]$, the above equality (4.33) is equivalent to:

$$\forall k \in \widehat{\mathbf{K}}, \quad \begin{aligned}
&\int_{A(k)}^{A(k+\mathbf{e}_i)} (\partial^2_{ii}\phi\mathbf{S}^c + \widehat{\psi}^{(i)}(\cdot,\mathbf{S}))d\mathbf{S} + \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} (\partial^2_{jj}\phi\mathbf{S}^c + \widehat{\psi}^{(j)}(\cdot,\mathbf{S}))d\mathbf{S}\\
&= \int_{A(k)}^{A(k+\mathbf{e}_j)} (\partial^2_{jj}\phi\mathbf{S}^c + \widehat{\psi}^{(j)}(\cdot,\mathbf{S}))d\mathbf{S} + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} (\partial^2_{ii}\phi\mathbf{S}^c + \widehat{\psi}^{(i)}(\cdot,\mathbf{S}))d\mathbf{S}
\end{aligned} \tag{4.34}$$

a.e. in Ω.
Convergence (4.32) gives (remind that $\partial^2_{ii}\phi$ does not depends on $\mathbf{S}$)

$$\begin{aligned}
&\int_{A(k)}^{A(k+\mathbf{e}_i)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big) d\mathbf{S}\\
\forall k \in \widehat{\mathbf{K}} \quad &\to \int_{A(k)}^{A(k+\mathbf{e}_i)} \big(\partial^2_{ii}\phi\, \mathbf{S}^c + \widehat{\psi}^{(i)}\big) d\mathbf{S}\\
&= \partial^2_{ii}\phi \int_{k_i l_i}^{(k_i+1)l_i} \Big(t - \frac{1}{2}\Big) dt + \int_{A(k)}^{A(k+\mathbf{e}_i)} \widehat{\psi}^{(i)}(x, \mathbf{S}) d\mathbf{S}..
\end{aligned}$$

Similarly, one has ($j \neq i$)

$$\begin{aligned}
&\int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big) d\mathbf{S}\\
&\to \partial^2_{ii}\phi \int_{k_i l_i}^{(k_i+1)l_i} \Big(t - \frac{1}{2}\Big) dt + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \widehat{\psi}^{(i)}(x, \mathbf{S})\, d\mathbf{S}
\end{aligned}$$

and the same kind of results for the other two quantities.
Hence, to get (4.33), we have to prove that both quantities

$$\begin{aligned}
&\int_{A(k)}^{A(k+\mathbf{e}_i)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big) d\mathbf{S}\\
&+ \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(j)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big) d\mathbf{S}
\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}
&\int_{A(k)}^{A(k+\mathbf{e}_j)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(j)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big) d\mathbf{S}\\
&+ \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\Big) d\mathbf{S}.
\end{aligned} \tag{4.36}$$

admit the same limit or equivalently that the limit of their difference is 0.
First, we note that

$$\begin{aligned}
\int_{A(k)}^{A(k+\mathbf{e}_i)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} &= \frac{1}{\varepsilon}\int_{A(k)}^{A(k+\mathbf{e}_i)} \partial_{\mathbf{S}}\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) d\mathbf{S}\\
&= \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k+\mathbf{e}_i)) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k))\Big) \quad \text{a.e. in } \widetilde{\Omega}_\varepsilon.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{\varepsilon}\Big(\int_{A(k)}^{A(k+\mathbf{e}_i)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} + \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S}\Big)\\
&= \frac{1}{\varepsilon}\Big(\int_{A(k)}^{A(k+\mathbf{e}_j)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S}\Big) \quad \text{a.e. in } \widetilde{\Omega}_\varepsilon.
\end{aligned}$$

Now, recall that the function $\mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)$ is defined on $\widetilde{\Omega}_\varepsilon \times \mathcal{S}^{(i)}$ and is constant on every line of $\mathcal{S}^{(i)}$. One has a.e. in $\widetilde{\Omega}_\varepsilon$

$$\begin{aligned}
\mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) &= \int_{A(k')}^{A(k')+\mathbf{e}_i} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\, d\mathbf{S} = \frac{1}{\varepsilon}\int_{A(k')}^{A(k')+\mathbf{e}_i} \partial_{\mathbf{S}}\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) d\mathbf{S}\\
&= \frac{1}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k') + \mathbf{e}_i) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k'))\Big)
\end{aligned}$$

on $\widetilde{\Omega}_\varepsilon \times [A(k'), A(k') + \mathbf{e}_i]$, $k' \in \widehat{\mathbf{K}}_i$. Hence,

$$\int_{A(k)}^{A(k+\mathbf{e}_i)} \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} = \frac{l_i}{\varepsilon}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k') + \mathbf{e}_i) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k'))\Big) \text{ a.e. in } \widetilde{\Omega}_\varepsilon,$$

where $k' \in \widehat{\mathbf{K}}_i$ is such that $k = k' + k_i\mathbf{e}_i$. Hence, we get

$$\begin{aligned}
&\frac{1}{\varepsilon}\Big(\int_{A(k)}^{A(k+\mathbf{e}_i)} \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} - \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S}\Big)\\
&= \frac{l_i}{\varepsilon^2}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k') + \mathbf{e}_i) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k'))\\
&\quad - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k' + \mathbf{e}_j) + \mathbf{e}_i) + \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k' + \mathbf{e}_j))\Big) \text{ a.e. in } \widetilde{\Omega}_\varepsilon
\end{aligned}$$

where $k' \in \widehat{\mathbf{K}}_i$ is such that $k = k' + k_i\mathbf{e}_i$.
Now, we can apply Lemma 31 and claim that the limit of the difference of the quantities in (4.35) and (4.36) is equal to 0. This proves (4.34) for every $k \in \widehat{\mathbf{K}}$. As a consequence, there exists a unique $\widehat{\phi} \in L^p(\Omega; W^{2,p}_{per,0}(\mathcal{S}))$ such that convergence $(4.30)_3$ holds. □

4.6 Application: homogenization of a fourth 4th order homogeneous Dirichlet problem on a periodic lattice structure

Now that we concluded the unfolding for functions on lattice structures, we proceed as in Section 3.4 to the homogenization of a Dirichlet problem by the meanings of the newly developed tools.
From the rest of this section, let $p = 2$ and Ω be a bounded domain in $\mathbb{R}^N$ with a $\mathcal{C}^{1,1}$ boundary. Let $\{A_\varepsilon^{\mathcal{S}}\}_\varepsilon$ be the sequence of functions belonging to $L^\infty(\mathcal{S}_\varepsilon)$ and defined by

$$A_\varepsilon^{\mathcal{S}}(\mathbf{s}) \doteq A^{\mathcal{S}}\Big(\Big\{\frac{\mathbf{s}}{\varepsilon}\Big\}\Big) \qquad \text{for a.e. } \mathbf{s} \in \mathcal{S}_\varepsilon,$$

where $A^{\mathcal{S}} \in L^\infty(\mathcal{S})$ satisfies

$$\exists C - 0, C_1 \in (0, +\infty) \quad \text{such that} \quad C_0 \le A^{\mathcal{S}}(\mathbf{S}) \le C_1 \quad \text{for a.e. } \mathbf{S} \in \mathcal{S}. \tag{4.37}$$

Let $\{g_\varepsilon\}_\varepsilon$ and $\{f_\varepsilon\}_\varepsilon$ be sequences in $L^2(\mathcal{S}_\varepsilon)$. Set

$$H_0^1(\mathcal{S}_\varepsilon) \doteq \{\phi \in H^1(\mathcal{S}_\varepsilon) \mid \phi = 0 \text{ a.e. on } \partial\widetilde{\Omega}_\varepsilon \cap \mathcal{S}_\varepsilon\}.$$

By the Poincaré's and Poincaré−Wirtinger's inequalities, we have

$$\forall \phi \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon), \qquad \|\phi\|_{L^2(\mathcal{S}_\varepsilon)} \le C\|\partial_{\mathbf{s}}\phi\|_{L^2(\mathcal{S}_\varepsilon)} \le C\|\partial^2_{\mathbf{s}}\phi\|_{L^2(\mathcal{S}_\varepsilon)}.$$

Note also that $\mathcal{M}_{\mathcal{S}^{(i)}}(\partial_{\mathbf{s}}\phi) = 0$ for every $i \in \{1, \dots, N\}$.
We consider the fourth order homogeneous Dirichlet problem in the variational formulation:

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon) \text{ such that:} \\ \displaystyle\int_{\mathcal{S}_\varepsilon} A_\varepsilon^{\mathcal{S}} \partial^2_{\mathbf{s}} u_\varepsilon\, \partial^2_{\mathbf{s}}\phi\, d\mathbf{s} = \int_{\mathcal{S}_\varepsilon} g_\varepsilon\, \partial_{\mathbf{s}}\phi\, d\mathbf{s} + \int_{\mathcal{S}_\varepsilon} f_\varepsilon\, \phi\, d\mathbf{s}, \quad \forall \phi \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon). \end{cases} \tag{4.38}$$

By the Lax−Milgram's theorem, problem (4.38) admits a unique solution. Moreover,

$$\begin{aligned} c\|\partial^2_{\mathbf{s}} u_\varepsilon\|^2_{L^2(\mathcal{S}_\varepsilon)} &\le \|g_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}\|\partial_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|f_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}\|u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \\ &\le C\big(\|g_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|f_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}\big)\|\partial^2_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}. \end{aligned}$$

Hence,

$$\|u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial^2_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \leq C\big(\|g_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|f_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}\big). \tag{4.39}$$

Below, we give the periodic homogenization via unfolding.

Theorem 4. *Let u_ε be the solution of problem (4.38) and $\{g_\varepsilon\}_\varepsilon$, $\{f_\varepsilon\}_\varepsilon$ be such that*

$$\begin{aligned} &\varepsilon^{\frac{1-N}{2}} \mathcal{T}^{\mathcal{S}}_\varepsilon(g_\varepsilon) \to g \quad \textit{strongly in} \quad L^2(\Omega \times \mathcal{S}),\\ &\varepsilon^{\frac{1-N}{2}} \mathcal{T}^{\mathcal{S}}_\varepsilon(f_\varepsilon) \to f \quad \textit{strongly in} \quad L^2(\Omega \times \mathcal{S}). \end{aligned} \tag{4.40}$$

Then, there exist $u \in H^1_0(\Omega) \cap H^2(\Omega)$ and $\widehat{u} \in L^2(\Omega; H^2_{per,0}(\mathcal{S}))$ such that ($i \in \{1, \dots, N\}$)

$$\begin{aligned} &\mathcal{T}^{\mathcal{S}}_\varepsilon(u_\varepsilon) \to u \qquad \textit{strongly in} \quad L^2(\Omega; H^2(\mathcal{S})),\\ &\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}} u_\varepsilon) \rightharpoonup \partial_i u \qquad \textit{weakly in} \quad L^2(\Omega; H^1(\mathcal{S}^{(i)})),\\ &\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}} u_\varepsilon) \to \partial^2_{ii} u + \partial^2_{\mathbf{S}} \widehat{u} \quad \textit{strongly in} \quad L^2(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

The couple $(u, \widehat{u})$ is the unique solution of problem

$$\begin{aligned} &\sum_{i=1}^N \int_{\Omega \times \mathcal{S}^{(i)}} A^{\mathcal{S}} \left(\partial^2_{ii} u + \partial^2_{\mathbf{S}} \widehat{u}\right) \left(\partial^2_{ii} \phi + \partial^2_{\mathbf{S}} \widehat{\phi}\right) dx d\mathbf{S} = \int_\Omega G \cdot \nabla \phi \, dx + \int_\Omega F \, \phi \, dx,\\ &\qquad\qquad \forall \phi \in H^1_0(\Omega) \cap H^2(\Omega) \;\; \textit{and} \;\; \forall \widehat{\phi} \in L^2(\Omega; H^2_{per,0}(\mathcal{S})), \end{aligned} \tag{4.41}$$

where

$$G \doteq \sum_{i=1}^N \Big(\int_{\mathcal{S}^{(i)}} g(\cdot, \mathbf{S}) \, d\mathbf{S} \Big) \mathbf{e}_i, \qquad F \doteq \int_{\mathcal{S}} f(\cdot, \mathbf{S}) \, d\mathbf{S}.$$

Proof. The solution u_ε of (4.38) satisfies (4.39). Due to the convergences (4.40) we have that

$$\|u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial^2_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \leq C \varepsilon^{\frac{1-N}{2}}.$$

Hence, up to a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, Theorem 3 gives $u \in H^1_0(\Omega) \cap H^2(\Omega)$ and $\widehat{u} \in L^p(\Omega; H^2_{per,0}(\mathcal{S}))$ such that the following convergences hold ($i \in \{1, \dots, N\}$):

$$\begin{aligned} &\mathcal{T}^{\mathcal{S}}_\varepsilon(u_\varepsilon) \to u \qquad \text{strongly in} \quad L^2(\Omega; H^2(\mathcal{S})),\\ &\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}} u_\varepsilon) \rightharpoonup \partial_i u \qquad \text{weakly in} \quad L^2(\Omega; H^1(\mathcal{S}^{(i)})),\\ &\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}} u_\varepsilon) \rightharpoonup \partial^2_{ii} u + \partial^2_{\mathbf{S}} \widehat{u} \quad \text{weakly in} \quad L^2(\Omega \times \mathcal{S}^{(i)}). \end{aligned} \tag{4.42}$$

Now, we choose the test functions

- ϕ in $\mathcal{C}^\infty(\overline{\Omega}) \cap H^1_0(\Omega)$,
- Φ in $\mathcal{C}^2_c(\Omega)$,
- $\widehat{\phi}$ in $H^2_{per,0}(\mathcal{S})$.

Set

$$\phi_\varepsilon(\mathbf{s}) \doteq \varepsilon^{\frac{1-N}{2}} \Big(\phi(\mathbf{s}) + \varepsilon^2 \Phi(\mathbf{s}) \widehat{\phi}\Big(\frac{\mathbf{s}}{\varepsilon}\Big)\Big), \qquad \text{a.e. } \mathbf{s} \in \mathcal{S}_\varepsilon.$$

Applying the unfolding operator to the sequence $\{\phi_\varepsilon\}_\varepsilon$, we get ($i \in \{1, \dots, N\}$)

$$\begin{aligned} &\mathcal{T}^{\mathcal{S}}_\varepsilon(\phi_\varepsilon) \to \phi \qquad \text{strongly in} \quad L^2(\Omega; H^2(\mathcal{S})),\\ &\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial_{\mathbf{s}} \phi_\varepsilon) \to \partial_i \phi \qquad \text{strongly in} \quad L^2(\Omega; H^1(\mathcal{S}^{(i)})),\\ &\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}} \phi_\varepsilon) \to \partial^2_{ii} \phi + \Phi \partial^2_{\mathbf{S}} \widehat{\phi} \qquad \text{strongly in} \quad L^2(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

Taking ϕ_ε as test function in (4.38), then transforming by unfolding and passing to the limit give (4.41) with $(\phi, \Phi\widehat{\phi})$. By density argumentation, we extend such results to all

$\phi \in H_0^1(\Omega) \cap H^2(\Omega)$ and $\widehat{\phi} \in L^2(\Omega; H^2_{per,0}(\mathcal{S}))$. Since the solution is unique, the whole sequences converge to their limit.

To conclude the proof, it is left to show that the third convergence in (4.42) is, in fact, strong. Taking $\phi = u_\varepsilon$ in (4.38), then transforming by unfolding and using the weak lower semicontinuity yield

$$\begin{aligned}
&\sum_{i=1}^N \int_{\Omega\times\mathcal{S}^{(i)}} A^{\mathcal{S}} \left|\partial^2_{ii} u + \partial^2_{\mathbf{S}}\widehat{u}\right|^2 dx d\mathbf{S} \\
&\le \liminf_{\varepsilon\to 0} \sum_{i=1}^N \int_{\Omega\times\mathcal{S}} \mathcal{T}_\varepsilon(A^{\mathcal{S}}_\varepsilon) \left|\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}} u_\varepsilon)\right|^2 dx d\mathbf{S} \le \liminf_{\varepsilon\to 0} \varepsilon^{N-1} \sum_{i=1}^N \int_{\mathcal{S}_\varepsilon} A^{\mathcal{S}}_\varepsilon \left|\partial^2_{\mathbf{s}} u_\varepsilon\right|^2 d\mathbf{s} \\
&\le \limsup_{\varepsilon\to 0} \varepsilon^{N-1} \sum_{i=1}^N \int_{\mathcal{S}_\varepsilon} A^{\mathcal{S}}_\varepsilon \left|\partial^2_{\mathbf{s}} u_\varepsilon\right|^2 d\mathbf{s} = \limsup_{\varepsilon\to 0} \varepsilon^{N-1} \Big(\int_{\mathcal{S}_\varepsilon} g_\varepsilon\, \partial_{\mathbf{s}} u_\varepsilon\, d\mathbf{s} + \int_{\mathcal{S}_\varepsilon} f_\varepsilon\, u_\varepsilon\, d\mathbf{s} \Big) \\
&= |\mathcal{S}| \Big(\int_\Omega G \cdot \nabla\phi\, dx + \int_\Omega F\,\phi\, dx \Big) = \sum_{i=1}^N \int_{\Omega\times\mathcal{S}^{(i)}} A^{\mathcal{S}} \left|\partial^2_{ii} u + \partial^2_{\mathbf{S}}\widehat{u}\right|^2 dx d\mathbf{S}.
\end{aligned}$$

Also, observe that

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \sum_{i=1}^N \int_{\Omega\times\mathcal{S}} \mathcal{T}_\varepsilon(A^{\mathcal{S}}_\varepsilon) \left|\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}} u_\varepsilon)\right|^2 dx d\mathbf{S} \le \limsup_{\varepsilon\to 0} \sum_{i=1}^N \int_{\Omega\times\mathcal{S}} \mathcal{T}_\varepsilon(A^{\mathcal{S}}_\varepsilon) \left|\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}} u_\varepsilon)\right|^2 dx d\mathbf{S} \\
&\le \limsup_{\varepsilon\to 0} \varepsilon^{N-1} \sum_{i=1}^N \int_{\mathcal{S}_\varepsilon} A^{\mathcal{S}}_\varepsilon \left|\partial^2_{\mathbf{s}} u_\varepsilon\right|^2 d\mathbf{s}
\end{aligned}$$

From the above inequalities, it follows that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \sum_{i=1}^N \int_{\mathcal{S}_\varepsilon} \mathcal{T}^{\mathcal{S}}_\varepsilon(A^{\mathcal{S}}_\varepsilon) \left|\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}} u_\varepsilon)\right|^2 dx d\mathbf{S} \\
&= \lim_{\varepsilon\to 0} \sum_{i=1}^N \int_{\mathcal{S}_\varepsilon} A^{\mathcal{S}}_\varepsilon \left|\partial^2_{\mathbf{s}} u_\varepsilon\right|^2 d\mathbf{s} = \sum_{i=1}^N \int_{\Omega\times\mathcal{S}^{(i)}} A^{\mathcal{S}} \left|\partial^2_{ii} u + \partial^2_{\mathbf{S}}\widehat{u}\right|^2 dx d\mathbf{S}.
\end{aligned}$$

Since the map $\Psi \in L^2(\Omega\times\mathcal{S}) \longmapsto \sqrt{\int_{\Omega\times\mathcal{S}} A^{\mathcal{S}}\,|\Psi|^2\, dx d\mathbf{S}}$ is a norm equivalent to the usual norm of $L^2(\Omega\times\mathcal{S})$, we get

$$\lim_{\varepsilon\to 0} \int_{\Omega\times\mathcal{S}} \left|\mathcal{T}^{\mathcal{S}}_\varepsilon(\partial^2_{\mathbf{s}} u_\varepsilon)\right|^2 dx d\mathbf{S} = \int_{\Omega\times\mathcal{S}} \left|\partial^2_{ii} u + \partial^2_{\mathbf{S}}\widehat{u}\right|^2 dx d\mathbf{S}.$$

This, together with the fact that $(4.42)_3$ already converges weakly, ensures strong convergence. □

We define the correctors $\widehat{\chi}_k$, $k \in \{1, \dots, N\}$, as the unique solution in $H^2_{per,0}(\mathcal{S})$ of the cell problem

$$\int_{\mathcal{S}} A^{\mathcal{S}} \left(\mathbf{1}_{\mathcal{S}^{(k)}} + \partial^2_{\mathbf{S}}\widehat{\chi}_k\right) \partial^2_{\mathbf{S}}\widehat{w}\, d\mathbf{S} = 0, \quad \forall \widehat{w} \in H^2_{per,0}(\mathcal{S}). \tag{4.43}$$

Theorem 5. *The function $u \in H_0^1(\Omega) \cap H^2(\Omega)$ is the unique solution of the following homogenized problem:*

$$\int_\Omega A^{\mathcal{S},hom}\, \partial^2 u \cdot \partial^2\phi\, dx = \int_\Omega G\cdot\nabla\phi\, dx + \int_\Omega F\,\phi\, dx, \qquad \forall \phi \in H_0^1(\Omega)\cap H^2(\Omega), \tag{4.44}$$

where $\partial^2 u \doteq \left(\partial^2_{11} u, \dots, \partial^2_{NN} u\right)^T$ and $\partial^2\phi \doteq \left(\partial^2_{11}\phi, \dots, \partial^2_{NN}\phi\right)^T$.
In particular, the homogenized matrix $A^{\mathcal{S},hom}$ is given by ($(i,j) \in \{1,\dots,N\}^2$)

$$A^{\mathcal{S},hom}_{ij} \doteq \frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} A^{\mathcal{S}} \left(\mathbf{1}_{\mathcal{S}^{(i)}} + \partial^2_{\mathbf{S}}\widehat{\chi}_i\right)\left(\mathbf{1}_{\mathcal{S}^{(j)}} + \partial^2_{\mathbf{S}}\widehat{\chi}_j\right) d\mathbf{S}. \tag{4.45}$$

Proof. Equation (4.41) with $\phi = 0$ leads to

$$\sum_{i=1}^{N} \int_{\Omega \times \mathcal{S}^{(i)}} A^{\mathcal{S}} \left(\partial^2_{ii} u + \partial^2_{\mathbf{S}} \widehat{u} \right) \partial^2_{\mathbf{S}} \widehat{\phi} \, dx d\mathbf{S} = 0, \qquad \forall \widehat{\phi} \in L^2(\Omega; H^2_{per,0}(\mathcal{S})),$$

from which we obtain the form of the cell problems (4.43) and thus the representation of $\widehat{u}$

$$\widehat{u}(x, \mathbf{S}) = \sum_{k=1}^{N} \partial^2_{kk} u(x) \, \widehat{\chi}_k(\mathbf{S}), \qquad \text{for a.e. } (x, \mathbf{S}) \in \Omega \times \mathcal{S}.$$

Replacing the above expression of $\widehat{u}$ in (4.41) and choosing

$$\widehat{\phi}(x, \mathbf{S}) = \sum_{k=1}^{N} \partial^2_{kk} \phi(x) \, \widehat{\chi}_k(\mathbf{S}), \qquad \text{for a.e. } (x, \mathbf{S}) \in \Omega \times \mathcal{S}$$

lead to the following left hand side of (4.41):

$$\begin{aligned} &\frac{1}{|\mathcal{S}|} \int_{\Omega \times \mathcal{S}} A^{\mathcal{S}} \Big(\sum_{i=1}^{N} (\mathbf{1}_{\mathcal{S}^{(i)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_i) \partial^2_{ii} u \Big) \Big(\sum_{j=1}^{N} (\mathbf{1}_{\mathcal{S}^{(j)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_j) \partial^2_{jj} \phi \Big) \, dx d\mathbf{S} \\ &= \int_{\Omega} \sum_{i,j=1}^{N} \Big(\frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} A^{\mathcal{S}} \left(\mathbf{1}_{\mathcal{S}^{(i)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_i \right) \left(\mathbf{1}_{\mathcal{S}^{(j)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_j \right) d\mathbf{S} \Big) \partial^2_{ii} u \, \partial^2_{jj} \phi \, dx. \end{aligned}$$

Taking into account (4.43), the above expression becomes $\displaystyle\int_{\Omega} A^{\mathcal{S},hom} \, \partial^2 u \cdot \partial^2 \phi \, dx$ with the matrix $A^{\mathcal{S},hom}$ given by (4.45).

We prove now that $A^{\mathcal{S},hom}$ is coercive. Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ be a vector with fixed entries. From (4.45) we first have

$$\begin{aligned} A^{\mathcal{S},hom} \boldsymbol{\xi} \cdot \boldsymbol{\xi} &= \frac{1}{|\mathcal{S}|} \sum_{i,j=1}^{N} \int_{\mathcal{S}} A^{\mathcal{S}} \left(\mathbf{1}_{\mathcal{S}^{(i)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_i \right) \left(\mathbf{1}_{\mathcal{S}^{(j)}} + \partial^2_{\mathbf{S}} \widehat{\chi}_j \right) d\mathbf{S} \, \xi_i \, \xi_j \\ &= \frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} A^{\mathcal{S}} \left(\widetilde{\xi} + \partial^2_{\mathbf{S}} \widehat{\chi}_{\xi} \right)^2 d\mathbf{S} \end{aligned}$$

where

$$\widetilde{\xi} \doteq \sum_{i=1}^{N} \xi_i \mathbf{1}_{\mathcal{S}^{(i)}}, \qquad \widehat{\chi}_{\xi} = \sum_{k=1}^{N} \xi_k \, \widehat{\chi}_k, \quad \text{a.e. in } \mathcal{S} \ \text{ and for all } \ \boldsymbol{\xi} \in \mathbb{R}^N.$$

Then, by hypothesis (4.37) on $A^{\mathcal{S}}$, we get

$$A^{\mathcal{S},hom} \boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \frac{c}{|\mathcal{S}|} \| \widetilde{\xi} + \partial^2_{\mathbf{S}} \widehat{\chi}_{\xi} \|^2_{L^2(\mathcal{S})}.$$

By the periodicity of $\partial_{\mathbf{S}} \widehat{\chi}_{\xi}$, for every $\boldsymbol{\xi} \in \mathbb{R}^N$ we get that

$$\begin{aligned} \| \widetilde{\xi} + \partial^2_{\mathbf{S}} \widehat{\chi}_{\xi} \|^2_{L^2(\mathcal{S})} &= \| \widetilde{\xi} \|^2_{L^2(\mathcal{S})} + \| \partial^2_{\mathbf{S}} \widehat{\chi}_{\xi} \|^2_{L^2(\mathcal{S})} \geq \| \widetilde{\xi} \|^2_{L^2(\mathcal{S})} \\ &= \sum_{i=1}^{N} |\mathcal{S}^{(i)}| |\xi_i|^2 \geq \min_k |\mathcal{S}^{(k)}| \sum_{i=1}^{N} |\xi_i|^2 = \big(\min_k |\mathcal{S}^{(k)}| \big) |\boldsymbol{\xi}|^2. \end{aligned}$$

Thus the coercivity of $A^{\mathcal{S},hom}$ is proved since

$$A^{\mathcal{S},hom} \boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq c \, |\boldsymbol{\xi}|^2, \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^N.$$

By the coercivity of $A^{\mathcal{S},hom}$ and the fact that $u \in H^1_0(\Omega) \cap H^2(\Omega)$, problem (4.44) admits a unique solution. □

Chapter 5

Classification of elasticity problems for textile structures in linear regimes

In this chapter, we enter the second part of the thesis, where we investigate the asymptotic behavior of a textile canvas. The structure is modeled as a squared piece of cloth made of long and thin yarns, partially clamped on the left and bottom edges as in Figure 1.1.

Our investigation will span two main directions. The first is to determine which parameters affect the textile behavior among all those introduced to model the structure and how they do so. Different parameters lead to a range of elasticity problems to study, which are collected and classified at the end of this chapter. The second aspect is to investigate some of these problems in a linear regime (small deformations for the yarns) to understand how the different obtained displacements behave at the macroscopic level. This will be done in Chapter 6 and 7.

Before getting started, we find it convenient to give the following definitions, which will often appear throughout the rest of the work.

Symbol	Definition	Meaning
$L \in \mathbb{R}^+$	Constant	Lenght of the fibers.
$l < L \in \mathbb{R}^+$	Constant	Length of the partial clamp.
Ω	$\doteq (0, L)^2$	In-plane textile domain.
$\mathcal{Y}$	$\doteq (0, 2)^2$	In-plane reference cell.
$\varepsilon \in \mathbb{R}^+$	Small parameter	Distance between fibers.
$N_\varepsilon \in N$	$\doteq \frac{L}{2\varepsilon}$	Number of 2ε-segments in L.
$n_\varepsilon \in N$	$\doteq \frac{l}{2\varepsilon}$	Number of 2ε-segments in l.
$\mathcal{K}_\varepsilon$	$\doteq \{0, \dots, 2N_\varepsilon\}^2$	Set of nodes in $\overline{\Omega}$.
$\kappa \in [0, 1/3]$	Constant	Ratio between the fiber's distance and their cross-section.
$r \in \mathbb{R}^+$	$\doteq \kappa\varepsilon$	Width of the fiber's cross section.
ω_κ	$\doteq (-\kappa, \kappa)^2$	Reference fiber's cross section.
$\omega_r = \omega_{\kappa\varepsilon}$	$\doteq (-r, r)^2 = (-\kappa\varepsilon, \kappa\varepsilon)^2$	Rescaled fiber's cross section.
$x \in \mathbb{R}^3$	$\doteq (x_1, x_2, x_3)$	Variables in the mobile reference frame.
$z \in \mathbb{R}^3$	$\doteq (z_1, z_2, z_3)$	Variables in the straight reference frame.
$z' \in \mathbb{R}^2$	$\doteq (z_1, z_2)$	Variables restricted to the in-plane components.
∂_i	$\doteq \frac{\partial}{\partial z_i}$	Partial derivative with respect to to z_i.
$e(u)$	$\doteq \frac{1}{2}(\nabla u + (\nabla u)^T)$	Linearized strain tensor (symmetric gradient) of a displacement u.
$\alpha, \beta \in \{1, 2\}^2$	Constant	Shorten notation for direction $\mathbf{e}_1$ and $\mathbf{e}_2$.
$a, b, c \in \{0, 1\}$	Constant	Shorten notation for lines in the reference cell $\mathcal{Y}$.

5.1 Parameterization of a curved rod

In order to model a woven canvas structure, we start by modeling the basic component of which a textile consists: a long, thin, strongly oscillating rod. Then, we will define a displacement over it and its associated strain tensor. These results have already been proved in Section 3 of Griso, Orlik, and Wackerle, 2020b.

We start by considering a relaxed rod of length $L \in \mathbb{R}$ and squared cross-section $\omega_r = (-\kappa\varepsilon, \kappa\varepsilon)^2$:

$$\mathcal{P}_\varepsilon \doteq (0, L) \times \omega_r.$$

Then, we define the 2-periodic function

$$\Phi(t) \doteq \begin{cases} -\kappa & \text{if } t \in [0, \kappa], \\ \kappa\Big(6\dfrac{(t-\kappa)^2}{(1-2\kappa)^2} - 4\dfrac{(t-\kappa)^3}{(1-2\kappa)^3} - 1\Big) & \text{if } t \in [\kappa, 1-\kappa], \\ \kappa & \text{if } t \in [1-\kappa, 1], \\ \Phi(2-t) & \text{if } t \in [1, 2] \end{cases} \tag{5.1}$$

and we rescale it to a 2ε-periodic function setting $\Phi_\varepsilon(t) = \varepsilon\Phi\Big(\dfrac{t}{\varepsilon}\Big)$, which is piecewise $\mathcal{C}^2(\mathbb{R})$ and overall $\mathcal{C}^1(\mathbb{R})$. By definition, such a function satisfies

$$\varepsilon^2\|\Phi_\varepsilon''\|_{L^\infty(\mathbb{R})} + \varepsilon\|\Phi_\varepsilon'\|_{L^\infty(\mathbb{R})} + \|\Phi_\varepsilon\|_{L^\infty(\mathbb{R})} \le C\varepsilon.$$

We now define the function

$$M_\varepsilon(z_1) \doteq z_1\mathbf{e}_1 + \Phi_\varepsilon(z_1)\mathbf{e}_3, \qquad z_1 \in [0, L].$$

This curve has mean direction $\mathbf{e}_1$ and oscillations in direction $\mathbf{e}_3$. Hence, we can define the mobile reference frame $(\mathbf{t}_\varepsilon, \mathbf{e}_2, \mathbf{n}_\varepsilon)$, or so-called Frenet-Serret frame, by

$$\mathbf{t}_\varepsilon \doteq \frac{\partial_1 M_\varepsilon}{|\partial_1 M_\varepsilon|} = \frac{1}{\gamma_\varepsilon}\big(\mathbf{e}_1 + \partial_1\Phi_\varepsilon\mathbf{e}_3\big), \qquad \mathbf{n}_\varepsilon \doteq \mathbf{t}_\varepsilon \wedge \mathbf{e}_2 = \frac{1}{\gamma_\varepsilon}\big(-\partial_1\Phi_\varepsilon\mathbf{e}_1 + \mathbf{e}_3\big) \tag{5.2}$$

where $\gamma_\varepsilon \doteq \sqrt{1 + (\partial_1\Phi_\varepsilon)^2}$. We have $\mathbf{t}_\varepsilon, \mathbf{n}_\varepsilon \in \mathcal{C}^1([0, L])^3$. Their derivatives are

$$\frac{d\mathbf{t}_\varepsilon}{dz_1} = \mathbf{c}_\varepsilon\gamma_\varepsilon\mathbf{n}_\varepsilon, \qquad \frac{d\mathbf{n}_\varepsilon}{dz_1} = -\mathbf{c}_\varepsilon\gamma_\varepsilon\mathbf{t}_\varepsilon$$

where the piecewise continuous function $\mathbf{c}_\varepsilon(z_1) \doteq \dfrac{\partial_1^2\Phi_\varepsilon(z_1)}{\gamma_\varepsilon^3(z_1)}$ is the curvature. Denote by

$$\mathbf{C}_\varepsilon \doteq \big(\mathbf{t}_\varepsilon \quad \mathbf{e}_2 \quad \mathbf{n}_\varepsilon\big) \in SO(3)$$

the basis transformation matrix from the fixed frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to the mobile one $(\mathbf{t}_\varepsilon, \mathbf{e}_2, \mathbf{n}_\varepsilon)$. Now, we are ready to define our 2ε-oscillating rod:

$$\mathcal{Q}_\varepsilon \doteq \psi_\varepsilon(\mathcal{P}_\varepsilon),$$

where the function $\psi_\varepsilon : [0, L] \times \mathbb{R}^2 \to \mathbb{R}^3$ is the transition map from the straight to the oscillating rod. It is defined by

$$\psi_\varepsilon(z_1, y_2, y_3) \doteq M_\varepsilon(z_1) + y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon(z_1), \qquad (z_1, y_2, y_3) \in [0, L] \times \omega_r.$$

Note that we use the variable z to denote the yarn length while we use y to denote the cross-section.

Straightforward calculations show that

$$\nabla\psi_\varepsilon = \begin{pmatrix}\partial_1\psi_\varepsilon & \partial_{y_2}\psi_\varepsilon & \partial_{y_3}\psi_\varepsilon\end{pmatrix} = \mathbf{C}_\varepsilon\begin{pmatrix}\eta_\varepsilon & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \quad \text{and} \quad (\nabla_z\psi_\varepsilon)^{-1} = \begin{pmatrix}\frac{1}{\eta_\varepsilon} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\mathbf{C}_\varepsilon^T. \tag{5.3}$$

where η_ε is the Jacobian for the changing of coordinates

$$\eta_\varepsilon(z_1, y_2, y_3) \doteq \det\big(\nabla\psi_\varepsilon(z_1, y_2, y_3)\big) = \gamma_\varepsilon(z_1)\big(1 - y_3\mathbf{c}_\varepsilon(z_1)\big), \quad \forall(z_1, y_2, y_3) \in \overline{\mathcal{P}_\varepsilon}.$$

As it has already been shown in Remark A.1 of Griso, Orlik, and Wackerle, 2020b, if

$$\kappa \in (0, 1/3],$$

then the Jacobian η_ε of ψ_ε is bounded from below and above and therefore the transformation ψ_ε from $\mathcal{P}_\varepsilon$ onto $\mathcal{Q}_\varepsilon$ results to be a diffeomorphism. In particular, there exist two constants C_0, C_1 such that for every $\phi \in L^2(\mathcal{Q}_\varepsilon)$:

$$C_0\|\phi\circ\psi_\varepsilon\|_{L^2(\mathcal{P}_\varepsilon)} \le \|\phi\|_{L^2(\mathcal{Q}_\varepsilon)} \le C_1\|\phi\circ\psi_\varepsilon\|_{L^2(\mathcal{P}_\varepsilon)}. \tag{5.4}$$

This means that the L^2 estimates for a function computed on the straight beam and the estimates computed on the oscillating one will only differ by a constant.

From now on, we will simply denote ϕ the function $\phi\circ\psi_\varepsilon$.

5.2 Decomposition of a curved rod's displacement

Let $u \in H^1(\mathcal{Q}_\varepsilon)^3$ be a displacement. From Theorem 3.1 of Griso, 2008b and Lemma 3.2 of Griso, 2008a we have the following decomposition for a curved rod:

$$u = U_{e\ell} + \overline{u}, \qquad \text{a.e. in } \mathcal{Q}_\varepsilon \text{ or equivalently in } \mathcal{P}_\varepsilon. \tag{5.5}$$

The first quantity $U_{e\ell} \in H^1(\mathcal{P}_\varepsilon)^3$ is called elementary displacement and it is defined by

$$U_{e\ell}(z_1, y_2, y_3) \doteq \mathcal{U}(z_1) + \mathcal{R}(z_1) \wedge \big(y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon(z_1)\big),$$

where the fields $\mathcal{U}$ and $\mathcal{R}$ belong to $H^1(0, L)^3$. They represent, respectively, the rod's middle line and the rod's cross-section's rotation. The second quantity $\overline{u} \in H^1(\mathcal{P}_\varepsilon)^3$ is called warping and it consists of the remainder term of the displacement. From Griso, 2008b, it satisfies for a.e. $z_1 \in (0, L)$:

$$\int_{\omega_r} \overline{u}(z_1, y_2, y_3)dy_2dy_3 = 0, \qquad \int_{\omega_r} \overline{u}(z_1, y_2, y_3) \wedge \big(y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon(z_1)\big)dy_2dy_3 = 0.$$

Due to the equivalence of norms (5.4), the estimates with respect to the arc parameter and the straight reference frame differ from a constant. Applying this concept to the estimates for a displacement over an oscillating rod derived in Griso, 2008b, we get:

$$\|\partial_1\mathcal{R}\|_{L^2(0,L)} \le \frac{C}{\varepsilon^2}\|e_x(u)\|_{L^2(\mathcal{Q}_\varepsilon)}, \qquad \|\partial_1\mathcal{U} - \mathcal{R}\wedge\partial_1 M_\varepsilon\|_{L^2(0,L)} \le \frac{C}{\varepsilon}\|e_x(u)\|_{L^2(\mathcal{Q}_\varepsilon)}, \tag{5.6}$$

$$\|\overline{u}\|_{L^2(\mathcal{Q}_\varepsilon)} \le C\varepsilon\|e_x(u)\|_{L^2(\mathcal{Q}_\varepsilon)}, \qquad \|\nabla_x\overline{u}\|_{L^2(\mathcal{Q}_\varepsilon)} \le C\|e_x(u)\|_{L^2(\mathcal{Q}_\varepsilon)}. \tag{5.7}$$

Identically to Griso, Orlik, and Wackerle, 2020a, we find it convenient to define a more suitable decomposition for the middle line displacement $\mathcal{U}$:

$$\mathcal{U}(z_1) \doteq \mathbb{U}(z_1) + \mathcal{R}\wedge\Phi_\varepsilon(z_1)\mathbf{e}_3, \qquad z_1 \in [0, L],$$

where $\mathbb{U} \in H^1(0,L)$. In this sense, we can rewrite the elementary displacement in the following way

$$U_{e\ell}(z_1,y_2,y_3) \doteq \mathbb{U}(z_1) + \mathcal{R}(z_1) \wedge \big(\Phi_\varepsilon(z_1)\mathbf{e}_3 + y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon(z_1)\big) \tag{5.8}$$

and from Lemma 3.4 in Griso, Orlik, and Wackerle, 2020a, estimate $(5.6)_2$ becomes

$$\big\|\partial_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big\|_{L^2(0,L)} \leq \frac{C}{\varepsilon^2}\|e_x(u)\|_{L^2(\mathcal{Q}_\varepsilon)}, \tag{5.9}$$

where C only depends on κ.
We also remind that if a rod is clamped at one extremity, e.g., $z_1 = 0$, then

$$\mathcal{U}(0) = \mathbb{U}(0) = \mathcal{R}(0), \qquad \overline{u}(0,\cdot) = 0 \qquad \text{a.e. } \omega_r. \tag{5.10}$$

As we will see in the next section, the clamp is important to estimate the fields themselves starting from the estimates on their derivatives (5.6)-(5.7)-(5.9) and using the Poincaré inequality.

5.2.1 The linearized strain tensor associated with the displacement

Now that we set a suitable decomposition for the displacement, we are interested in the form of the associated strain tensor since it will later enter the left-hand side of the elasticity problem.
Note that since we are in the assumption of small deformation, we recall that the linearized strain tensor coincides with the symmetric gradient of the displacement.
Given $u \in H^1(\mathcal{Q}_\varepsilon)^3$, equality (5.3) yields

$$\begin{pmatrix}\partial_{z_1}u & \partial_{y_2}u & \partial_{y_3}u\end{pmatrix} = \nabla_x u\,\nabla\psi_\varepsilon = \nabla_x u\,\mathbf{C}_\varepsilon \begin{pmatrix}\eta_\varepsilon & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Since we will later state the problem in the straight reference frame, we want to express the symmetric gradient in such a frame. Note that the above equality implies that

$$\mathbf{C}_\varepsilon^T\,\nabla_x u\,\mathbf{C}_\varepsilon = \begin{pmatrix} \frac{1}{\eta_\varepsilon}\partial_{z_1}u\cdot\mathbf{t}_\varepsilon & \partial_{y_2}u\cdot\mathbf{t}_\varepsilon & \partial_{y_3}u\cdot\mathbf{t}_\varepsilon\\ \frac{1}{\eta_\varepsilon}\partial_{z_1}u\cdot\mathbf{e}_{y_2} & \partial_{y_2}u\cdot\mathbf{e}_{y_2} & \partial_{y_3}u\cdot\mathbf{e}_{y_2}\\ \frac{1}{\eta_\varepsilon}\partial_{z_1}u\cdot\mathbf{n}_\varepsilon & \partial_{y_2}u\cdot\mathbf{n}_\varepsilon & \partial_{y_3}u\cdot\mathbf{n}_\varepsilon \end{pmatrix},$$

which, together with the definition of symmetric gradient

$$e_x(u) \doteq \frac{1}{2}\big(\nabla_x u + (\nabla_x u)^T\big)$$

leads to the quantity we are interested in:

$$\mathbf{C}_\varepsilon^T e_x(u)\mathbf{C}_\varepsilon = \begin{pmatrix} \frac{1}{\eta_\varepsilon}\partial_{z_1}u\cdot\mathbf{t}_\varepsilon & * & *\\ \frac{1}{2}\Big(\frac{1}{\eta_\varepsilon}\partial_{z_1}u\cdot\mathbf{e}_{y_2} + \partial_{y_2}u\cdot\mathbf{t}_\varepsilon\Big) & \partial_{y_2}u\cdot\mathbf{e}_{y_2} & *\\ \frac{1}{2}\Big(\frac{1}{\eta_\varepsilon}\partial_{z_1}u\cdot\mathbf{n}_\varepsilon + \partial_{y_3}u\cdot\mathbf{t}_\varepsilon\Big) & \frac{1}{2}\Big(\partial_{y_2}u\cdot\mathbf{n}_\varepsilon + \partial_{y_3}u\cdot\mathbf{e}_{y_2}\Big) & \partial_{y_3}u\cdot\mathbf{n}_\varepsilon \end{pmatrix}.$$

By straightforward calculations on the gradient of the decomposition (5.5)-(5.8), the symmetric gradient of a rod displacement in the straight reference frame is

$$e(u) \doteq e(U_{e\ell}) + e(\overline{u}),$$

where the first quantity is the symmetric gradient of the elementary displacement

$$e(U_{e\ell}) \doteq \begin{pmatrix} \frac{1}{\eta_\varepsilon}\Big((\partial_1\mathbb{U}-\mathcal{R}\wedge\mathbf{e}_1)+\partial_1\mathcal{R}\wedge(\Phi_\varepsilon\mathbf{e}_3+y_2\mathbf{e}_2+y_3\mathbf{n}_\varepsilon)\Big)\cdot\mathbf{t}_\varepsilon & * & * \\ \frac{1}{2\eta_\varepsilon}\Big((\partial_1\mathbb{U}-\mathcal{R}\wedge\mathbf{e}_1)+\partial_1\mathcal{R}\wedge(\Phi_\varepsilon\mathbf{e}_3+y_2\mathbf{e}_2+y_3\mathbf{n}_\varepsilon)\Big)\cdot\mathbf{e}_2 & 0 & * \\ \frac{1}{2\eta_\varepsilon}\Big((\partial_1\mathbb{U}-\mathcal{R}\wedge\mathbf{e}_1)+\partial_1\mathcal{R}\wedge(\Phi_\varepsilon\mathbf{e}_3+y_2\mathbf{e}_2+y_3\mathbf{n}_\varepsilon)\Big)\cdot\mathbf{n}_\varepsilon & 0 & 0 \end{pmatrix} \tag{5.11}$$

and the second one is the symmetric gradient of the warping

$$e(\overline{u}) \doteq \begin{pmatrix} \frac{1}{\eta_\varepsilon}\partial_{z_1}\overline{u}\cdot\mathbf{t}_\varepsilon & * & * \\ \frac{1}{2}\Big(\frac{1}{\eta_\varepsilon}\partial_{z_1}\overline{u}\cdot\mathbf{e}_{y_2}+\partial_{y_2}\overline{u}\cdot\mathbf{t}_\varepsilon\Big) & \partial_{y_2}\overline{u}\cdot\mathbf{e}_{y_2} & * \\ \frac{1}{2}\Big(\frac{1}{\eta_\varepsilon}\partial_{z_1}\overline{u}\cdot\mathbf{n}_\varepsilon+\partial_{y_3}\overline{u}\cdot\mathbf{t}_\varepsilon\Big) & \frac{1}{2}\Big(\partial_{y_2}\overline{u}\cdot\mathbf{n}_\varepsilon+\partial_{y_3}\overline{u}\cdot\mathbf{e}_{y_2}\Big) & \partial_{y_3}\overline{u}\cdot\mathbf{n}_\varepsilon \end{pmatrix}. \tag{5.12}$$

5.3 A new decomposition for the displacement

Now, we would like to define a new decomposition of the displacement to simplify the form of the elementary symmetric gradient (5.11). To be sure that the new decomposition is close enough to the old one, we will use the approximation of functions by interpolations on intervals of length ε, as we have already seen in Section 2.2.

5.3.1 Properties of the interpolating functions

Let $\mathbf{A} = (A_0,\dots,A_{2N_\varepsilon})$ be a vector in $\mathbb{R}^{2N_\varepsilon+1}$. Given a function $\phi \in H^1(0,L)$, we define its linear interpolation $\phi^{[\mathbf{A}]}_{lin} \in W^{1,\infty}(0,L)$ on the ε-intervals of the segment $(0,L)$ by setting

$$\phi^{[\mathbf{A}]}_{lin}(z_1) \doteq A_{p+1}\Big(\frac{z_1-p\varepsilon}{\varepsilon}\Big)-A_p\Big(\frac{z_1-(p+1)\varepsilon}{\varepsilon}\Big),\quad \forall z_1\in[p\varepsilon,(p+1)\varepsilon],\ \forall p\in\{0,\dots,2N_\varepsilon-1\}.$$

Let $\mathbf{B} = (B_0,\dots,B_{2N_\varepsilon})$ and $\mathbf{B}' = (B'_0,\dots,B'_{2N_\varepsilon})$ be two vectors in $\mathbb{R}^{2N_\varepsilon+1}$. Given a function $\phi \in H^2(0,L)$, we define its cubic interpolation $\phi^{[\mathbf{B},\mathbf{B}']}_{cub} \in W^{2,\infty}(0,L)$ on the ε-intervals by

$$\begin{aligned}\phi^{[\mathbf{B},\mathbf{B}']}_{cub}(z_1) \doteq{}& B_p\Big(\frac{2z_1-(2p-1)\varepsilon}{\varepsilon}\Big)\Big(\frac{z_1-(p+1)\varepsilon}{\varepsilon}\Big)^2+B_{p+1}\Big(\frac{(3+2p)\varepsilon-2z_1}{\varepsilon}\Big)\Big(\frac{z_1-p\varepsilon}{\varepsilon}\Big)^2\\ &+\frac{(z_1-p\varepsilon)(z_1-(p+1)\varepsilon)}{\varepsilon^2}\big(B'_{p+1}(z_1-p\varepsilon)+B'_p(z_1-(p+1)\varepsilon)\big),\\ &\forall z_1\in[p\varepsilon,(p+1)\varepsilon],\ \forall p\in\{0,\dots,2N_\varepsilon-1\}.\end{aligned}$$

At last, let $\mathbf{D} = (D_0,\dots,D_{2N_\varepsilon})$ be a vector in $\mathbb{R}^{2N_\varepsilon+1}$. Given a function $\phi \in H^1(0,L)$, we define its "ψ"-interpolation $\psi^{[\mathbf{D}]} \in W^{1,\infty}(0,L)$ on the ε-intervals of the segment $(0,L)$ by

$$\begin{aligned}\psi^{[\mathbf{D}]}(z_1) \doteq{}& D_{p+1}\Big(\frac{z_1-p\varepsilon}{\varepsilon}\Big)-D_p\Big(\frac{z_1-(p+1)\varepsilon}{\varepsilon}\Big)\\ &+\frac{(z_1-p\varepsilon)(z_1-(p+1)\varepsilon)(2z_1-(2p+1)\varepsilon)}{\varepsilon^3}(D_{p+1}-D_p),\\ &\forall z_1\in[p\varepsilon,(p+1)\varepsilon],\ \forall p\in\{0,\dots,2N_\varepsilon-1\}.\end{aligned}$$

Below, we set the estimates for these interpolations.

Lemma 17. *For every* $\mathbf{A}\in\mathbb{R}^{2N_\varepsilon+1}$*, we have*

$$\|\phi^{[\mathbf{A}]}_{lin}\|^2_{L^2(0,L)} \le C\varepsilon\sum_{p=0}^{2N_\varepsilon}|A_p|^2,\qquad \|\partial_1\phi^{[\mathbf{A}]}_{lin}\|^2_{L^2(0,L)} \le C\varepsilon\sum_{p=0}^{2N_\varepsilon-1}\Big|\frac{A_{p+1}-A_p}{\varepsilon}\Big|^2. \tag{5.13}$$

For every $\mathbf{B}, \mathbf{B}' \in \mathbb{R}^{2N_\varepsilon+1}$, *we have*

$$\begin{aligned}
\|\phi_{cub}^{[\mathbf{B},\mathbf{B}']}\|^2_{L^2(0,L)} &\leq C\varepsilon\Big(\sum_{p=0}^{2N_\varepsilon}(|A_p|^2+\varepsilon^2|B_p|^2)+\sum_{p=0}^{2N_\varepsilon-1}\varepsilon^2\Big|\frac{A_{p+1}-A_p}{\varepsilon}-\frac{B_{p+1}+B_p}{2}\Big|^2\Big),\\
\|\partial_1\phi_{cub}^{[\mathbf{B},\mathbf{B}']}\|^2_{L^2(0,L)} &\leq C\varepsilon\Big(\sum_{p=0}^{2N_\varepsilon}|B_p|^2+\sum_{p=0}^{2N_\varepsilon-1}\Big|\frac{A_{p+1}-A_p}{\varepsilon}-\frac{B_{p+1}+B_p}{2}\Big|^2\Big),\\
\|\partial^2_{11}\phi_{cub}^{[\mathbf{B},\mathbf{B}']}\|^2_{L^2(0,L)} &\leq \frac{C}{\varepsilon}\sum_{p=0}^{2N_\varepsilon-1}\Big(|B_{p+1}-B_p|^2+\Big|\frac{A_{p+1}-A_p}{\varepsilon}-\frac{B_{p+1}+B_p}{2}\Big|^2\Big).
\end{aligned} \tag{5.14}$$

For every $\mathbf{D} \in \mathbb{R}^{2N_\varepsilon+1}$, *we have*

$$\|\psi^{[\mathbf{D}]}\|^2_{L^2(0,L)} \leq C\varepsilon\sum_{p=0}^{2N_\varepsilon}|D_p|^2, \qquad \|\partial_1\psi^{[\mathbf{D}]}\|^2_{L^2(0,L)} \leq C\varepsilon\sum_{p=0}^{2N_\varepsilon}\Big|\frac{D_{p+1}-D_p}{\varepsilon}\Big|^2. \tag{5.15}$$

Proof. The proof of (5.13) and (5.15) follows by the definition of the interpolating functions, the fact that $(0,L)=\sum_{p=0}^{2N_\varepsilon}(p\varepsilon,(p+1)\varepsilon)$ and an affine change of variables.

The proof of (5.14) follows from the same meanings, together with the particular decomposition of a cubic interpolation (2.19). □

5.3.2 The prime decomposition

In this subsection, we decompose the displacement as a sum of a Bernoulli-Navier displacement and a residual one (warping). This new fields decomposition has two main advantages:

- It contains identities that otherwise must be proven later in the limit;
- Simplifies the linearized form of the strain tensor (symmetric gradient).

Let u be a displacement in $H^1(\mathcal{P}_\varepsilon)^3$ decomposed as (5.5) and recall the 3-vector fields $\mathbb{U}$, $\mathcal{R} \in H^1(0,L)^3$. We define the new field $\mathbb{U}' \in W^{1,\infty}(0,L)\times W^{2,\infty}(0,L)^2$ by

$$\begin{aligned}
\mathbb{U}_1'(z_1) &\doteq \phi_{lin}^{[\mathbf{A}]}(z_1), && \text{with } \big(\mathbb{U}_1(0),\dots,\mathbb{U}_1(2N_\varepsilon\varepsilon)\big),\\
\mathbb{U}_2'(z_1) &\doteq \phi_{cub}^{[\mathbf{B},\mathbf{B}']}(z_1), && \text{with } \mathbf{B}=\big(\mathbb{U}_2(0),\dots,\mathbb{U}_2(2N_\varepsilon\varepsilon)\big),\quad \mathbf{B}'=-\big(\mathcal{R}_3(0),\dots,\mathcal{R}_3(2N_\varepsilon\varepsilon)\big),\\
\mathbb{U}_3'(z_1) &\doteq \phi_{cub}^{[\mathbf{B},\mathbf{B}']}(z_1), && \text{with } \mathbf{B}=\big(\mathbb{U}_3(0),\dots,\mathbb{U}_3(2N_\varepsilon\varepsilon)\big),\quad \mathbf{B}'=\big(\mathcal{R}_2(0),\dots,\mathcal{R}_2(2N_\varepsilon\varepsilon)\big),\\
& && \forall z_1\in[p\varepsilon,(p+1)\varepsilon],\quad \forall p\in\{0,\dots,2N_\varepsilon-1\},
\end{aligned}$$

and the new field $\mathcal{R}'$ by

$$\begin{aligned}
\mathcal{R}_1'(z_1) &\doteq \psi^{[\mathbf{D}]}(z_1), && \text{with } \mathbf{D}=\big(\mathcal{R}_1(0),\dots,\mathcal{R}_1(2N_\varepsilon\varepsilon)\big),\\
& && \forall z_1\in[p\varepsilon,(p+1)\varepsilon],\ \forall p\in\{0,\dots,2N_\varepsilon-1\},\\
\mathcal{R}_2'(z_1) &\doteq -\partial_1\mathbb{U}_3'(z_1), && z_1\in[0,L],\\
\mathcal{R}_3'(z_1) &\doteq \partial_1\mathbb{U}_2'(z_1), && z_1\in[0,L].
\end{aligned} \tag{5.16}$$

By construction, we get the following relation:

$$\partial_1\mathbb{U}'-\mathcal{R}'\wedge\mathbf{e}_1=\partial_1\mathbb{U}_1'\mathbf{e}_1, \qquad \text{a.e. in } (0,L). \tag{5.17}$$

Then, for a.e. $(z_1,y_2,y_3)\in(0,L)\times\omega_r$, we can define U_{BN}' and $\overline{u}'$ by

$$\begin{aligned}
U_{BN}'(z_1,y_2,y_3) &\doteq \mathbb{U}'(z_1)+\mathcal{R}'(z_1)\wedge\big(\Phi_\varepsilon(z_1)\mathbf{e}_3+y_2\mathbf{e}_2+y_3\mathbf{n}_\varepsilon(z_1)\big),\\
\overline{u}'(z_1,y_2,y_3) &\doteq u(z_1,y_2,y_3)-U_{BN}'(z_1,y_2,y_3).
\end{aligned}$$

Note that by the relation (5.17), the quantity U'_{BN} is a Bernoulli-Navier displacement.
If a rod is clamped at one extremity, e.g., $z_1 = 0$, then it still holds

$$\mathbb{U}'(0) = \mathcal{R}'(0), \qquad \overline{u}'(0,\cdot) = 0 \qquad \text{a.e. } \omega_r. \tag{5.18}$$

We have the following estimates for the fields of this new decomposition.

Theorem 6. *The fields $\mathbb{U}'$ and $\mathcal{R}'$ satisfy the following estimates:*

$$\|\partial_1 \mathcal{R}'\|_{L^2(0,L)} \le \frac{C}{\varepsilon^2}\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \quad \|\partial_1 \mathbb{U}'_1\|_{L^2(0,L)} \le \frac{C}{\varepsilon}\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \tag{5.19}$$

$$\|\partial^2_{11} \mathbb{U}'_2\|_{L^2(0,L)} + \|\partial^2_{11} \mathbb{U}'_3\|_{L^2(0,L)} \le \frac{C}{\varepsilon^2}\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}. \tag{5.20}$$

The warping term $\overline{u}'$ satisfies

$$\|\overline{u}'\|_{L^2(\mathcal{P}_\varepsilon)} \le C\varepsilon\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \qquad \|\nabla \overline{u}'\|_{L^2(\mathcal{P}_\varepsilon)} \le C\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}. \tag{5.21}$$

Proof. First, note that $\mathcal{R}'_1$ is defined as the "ψ" interpolation on the nodal values of $\mathcal{R}_1$. Hence, estimates $(5.6)_1$ and the "ψ" interpolation estimates of Lemma 17 imply that

$$\|\partial_1 \mathcal{R}'_1\|^2_{L^2(0,L)} \le \sum_{p=0}^{2N_\varepsilon-1} \varepsilon\Big|\frac{\mathcal{R}_1\big((p+1)\varepsilon\big) - \mathcal{R}_1(p\varepsilon)}{\varepsilon}\Big|^2 \le \|\partial_1 \mathcal{R}_1\|^2_{L^2(0,L)} \le \frac{C}{\varepsilon^4}\|e_x(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}.$$

Now, note that $\mathbb{U}'_1$ is defined as the linear interpolation on the nodal values of $\mathbb{U}_1$. Hence, estimates (5.9) in the first component and the linear interpolation estimates of Lemma 17 imply that

$$\|\partial_1 \mathbb{U}'_1\|^2_{L^2(0,L)} \le \sum_{p=0}^{2N_\varepsilon-1} \varepsilon\Big|\frac{\mathbb{U}_1\big((p+1)\varepsilon\big) - \mathbb{U}_1(p\varepsilon)}{\varepsilon}\Big|^2 \le \|\partial_1 \mathbb{U}_1\|^2_{L^2(0,L)} \le \frac{C}{\varepsilon^2}\|e_x(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}.$$

Now we prove the estimate for $\mathbb{U}'_2$. From the cubic interpolation estimates of Lemma 17 and estimates (5.6)-(5.9), we have

$$\begin{aligned}\|\partial^2_{11} \mathbb{U}'_2\|^2_{L^2(0,L)} \le & \frac{C}{\varepsilon^2}\Big(\sum_{p=0}^{2N_\varepsilon-1} \varepsilon\Big|\frac{\mathbb{U}_2\big((p+1)\varepsilon\big) - \mathbb{U}_2(p\varepsilon)}{\varepsilon} - \frac{1}{2}\big(\mathcal{R}_3\big((p+1)\varepsilon\big) + \mathcal{R}_3(p\varepsilon)\big)\Big|^2 \\ & + \sum_{p=0}^{2N_\varepsilon-1} \varepsilon^3\Big|\frac{\mathcal{R}_3\big((p+1)\varepsilon\big) - \mathcal{R}_3(p\varepsilon)}{\varepsilon}\Big|^2\Big) \\ \le & \frac{C}{\varepsilon^2}\big(\|(\partial_1 \mathbb{U} - \mathcal{R}\wedge \mathbf{e}_1)\cdot \mathbf{e}_2\|^2_{L^2(0,L)} + \varepsilon^2\|\partial_1 \mathcal{R}_3\|^2_{L^2(0,L)}\big) \le \frac{C}{\varepsilon^4}\|e_x(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}\end{aligned}$$

which by definition of $\mathcal{R}'_3$ it also proves the estimate for $\partial_1 \mathcal{R}'_3$. By the same argumentation we prove the estimate for $\partial_{11}\mathbb{U}'_3$ and $\partial_1 \mathcal{R}'_2$ and thus (5.19)-(5.20) are proved.
Now, we prove the warping estimates (5.21). From the clamp conditions (5.10)-(5.18) the Poincaré inequality and estimates (5.19)-(5.20) and (5.6), we obtain

$$\|\mathcal{R}' - \mathcal{R}\|_{L^2(0,L)} \le C\varepsilon\|\partial_1(\mathcal{R}' - \mathcal{R})\|_{L^2(0,L)} \le \frac{C}{\varepsilon}\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}$$

and

$$\begin{aligned}&\|\mathbb{U}'_1 - \mathbb{U}_1\|_{L^2(0,L)} \le C\varepsilon\|\partial_1(\mathbb{U}'_1 - \mathbb{U}_1)\|_{L^2(0,L)} \le C\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)},\\ &\|\mathbb{U}'_2 - \mathbb{U}_2\|_{L^2(0,L)} \le C\varepsilon\big(\|\partial_1(\mathbb{U}'_2 - \mathbb{U}_2)\|_{L^2(0,L)} \le C\varepsilon^2\big(\|\partial_{11}(\mathbb{U}'_2 - \mathbb{U}_2)\|_{L^2(0,L)} \le C\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)},\\ &\|\mathbb{U}'_3 - \mathbb{U}_3\|_{L^2(0,L)} \le C\varepsilon\big(\|\partial_1(\mathbb{U}'_3 - \mathbb{U}_3)\|_{L^2(0,L)} \le C\varepsilon^2\big(\|\partial_{11}(\mathbb{U}'_3 - \mathbb{U}_3)\|_{L^2(0,L)} \le C\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}.\end{aligned}$$

Now, note that by construction, we have

$$\overline{u} - \overline{u}' = (\mathbb{U}' - \mathbb{U}) + (\mathcal{R}' - \mathcal{R}) \wedge (\Phi_\varepsilon \mathbf{e}_3 + y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon) \quad \text{a.e. in } \mathcal{Q}_\varepsilon.$$

Hence, from the above estimates and estimates (5.7), we have

$$\begin{aligned}
\|\overline{u}'\|_{L^2(\mathcal{P}_\varepsilon)} &\leq C\varepsilon\big(\|\mathbb{U}' - \mathbb{U}\|_{L^2(0,L)} + \varepsilon\|\mathcal{R}' - \mathcal{R}\|_{L^2(0,L)}\big) + \|\overline{u}\|_{L^2(\mathcal{P}_\varepsilon)} \leq C\varepsilon\|e_x(u)\|^2_{L^2(\mathcal{P}_\varepsilon)},\\
\|\nabla\overline{u}'\|_{L^2(\mathcal{P}_\varepsilon)} &\leq C\varepsilon\big(\|\partial_1(\mathbb{U}' - \mathbb{U})\|_{L^2(0,L)} + \varepsilon\|\partial_1(\mathcal{R}' - \mathcal{R})\|_{L^2(0,L)} + \|\mathcal{R}' - \mathcal{R}\|_{L^2(0,L)}\big) + \|\nabla\overline{u}\|_{L^2(\mathcal{P}_\varepsilon)}\\
&\leq C\|e_x(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}
\end{aligned}$$

which ends the proof of estimate (5.21). □

Note that estimate (5.21) is of the same order as the classical residual displacement (5.7). This fact is important because it justifies our prime decomposition: it is more suitable for our purposes and will give the same limit fields as the classical one.

5.3.3 The linearized strain tensor associated with the prime decomposition

The definition of the fields together with equality (5.17) for the new rod decomposition leads to the following form of the symmetric gradient of the prime displacement in the straight reference frame:

$$e(u) = e(U'_{e\ell}) + e(\overline{u}'),$$

where the first quantity is the symmetric gradient of the Bernoulli-Navier displacement

$$e(U'_{e\ell}) \doteq \begin{pmatrix}
\dfrac{1}{\eta_\varepsilon}\left(\begin{pmatrix}\partial_1\mathbb{U}'_1\\0\\0\end{pmatrix} + \begin{pmatrix}\partial_1\mathcal{R}'_1\\ \partial_{11}\mathbb{U}'_3\\ -\partial_{11}\mathbb{U}'_2\end{pmatrix} \wedge (\Phi_\varepsilon\mathbf{e}_3 + y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon)\right)\cdot\mathbf{t}_\varepsilon & * & *\\
\dfrac{1}{2\eta_\varepsilon}\left(\begin{pmatrix}\partial_1\mathbb{U}'_1\\0\\0\end{pmatrix} + \begin{pmatrix}\partial_1\mathcal{R}'_1\\ \partial_{11}\mathbb{U}'_3\\ -\partial_{11}\mathbb{U}'_2\end{pmatrix} \wedge (\Phi_\varepsilon\mathbf{e}_3 + y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon)\right)\cdot\mathbf{e}_2 & 0 & *\\
\dfrac{1}{2\eta_\varepsilon}\left(\begin{pmatrix}\partial_1\mathbb{U}'_1\\0\\0\end{pmatrix} + \begin{pmatrix}\partial_1\mathcal{R}'_1\\ \partial_{11}\mathbb{U}'_3\\ -\partial_{11}\mathbb{U}'_2\end{pmatrix} \wedge (\Phi_\varepsilon\mathbf{e}_3 + y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon)\right)\cdot\mathbf{n}_\varepsilon & 0 & 0
\end{pmatrix} \tag{5.22}$$

and the second one is the symmetric gradient of the warping

$$e(\overline{u}') \doteq \begin{pmatrix}
\dfrac{1}{\eta_\varepsilon}\partial_{z_1}\overline{u}'\cdot\mathbf{t}_\varepsilon & * & *\\
\dfrac{1}{2}\Big(\dfrac{1}{\eta_\varepsilon}\partial_{z_1}\overline{u}'\cdot\mathbf{e}_{y_2} + \partial_{y_2}\overline{u}'\cdot\mathbf{t}_\varepsilon\Big) & \partial_{y_2}\overline{u}'\cdot\mathbf{e}_{y_2} & *\\
\dfrac{1}{2}\Big(\dfrac{1}{\eta_\varepsilon}\partial_{z_1}\overline{u}'\cdot\mathbf{n}_\varepsilon + \partial_{y_3}\overline{u}'\cdot\mathbf{t}_\varepsilon\Big) & \dfrac{1}{2}\Big(\partial_{y_2}\overline{u}'\cdot\mathbf{n}_\varepsilon + \partial_{y_3}\overline{u}'\cdot\mathbf{e}_{y_2}\Big) & \partial_{y_3}\overline{u}'\cdot\mathbf{n}_\varepsilon
\end{pmatrix}. \tag{5.23}$$

In comparison with (5.11)-(5.12), we have reduced the number of involved fields and incorporated some identities.

5.4 The textile structure and natural assumptions

As Figure 1.1 shows, the textile structure is defined as two beams of parallel oscillating rods that cross each other in a periodic pattern. On such a structure, we set the natural assumptions that the woven fibers should satisfy: the boundary conditions to ensure the well-posedness of the elasticity problem, the contact conditions to allow shear between rods in the areas where they are one above the other, and the non-penetration conditions not to allow fibers to penetrate one into other. These assumptions will shape the admissible set of displacements.

5.4.1 The woven structure

In Section 5.1, we studied the structure of a single curved rod in direction $\mathbf{e}_1$. Now, we do the same for a beam of parallel rods in direction $\mathbf{e}_1$ and a second beam of parallel rods in direction $\mathbf{e}_2$ to obtain a woven canvas.

We denote $\mathfrak{G}_\varepsilon$ the reference lattice structure

$$\mathfrak{G}_\varepsilon \doteq \mathfrak{G}_\varepsilon^{(1)} \cup \mathfrak{G}_\varepsilon^{(2)}, \qquad \mathfrak{G}_\varepsilon^{(1)} = \bigcup_{q=0}^{2N_\varepsilon} [0,L] \times \{q\varepsilon\}, \qquad \mathfrak{G}_\varepsilon^{(2)} = \bigcup_{p=0}^{2N_\varepsilon} \{p\varepsilon\} \times [0,L].$$

This grid represents the domain of the beam of rods' center lines in both directions. For every $(z_1, q\varepsilon) \in \mathfrak{G}_\varepsilon^{(1)}$ and every $(p\varepsilon, z_2) \in \mathfrak{G}_\varepsilon^{(2)}$ the middle lines of the beams of rods become

$$\begin{aligned} M_\varepsilon^{(1)}(z_1, q\varepsilon) &\doteq z_1\mathbf{e}_1 + q\varepsilon\mathbf{e}_2 + \Phi_\varepsilon^{(1)}(z_1, q\varepsilon)\mathbf{e}_3, \qquad \Phi_\varepsilon^{(1)}(z_1, q\varepsilon) = (-1)^{q+1}\Phi_\varepsilon(z_1),\\ M_\varepsilon^{(2)}(p\varepsilon, z_2) &\doteq p\varepsilon\mathbf{e}_1 + z_2\mathbf{e}_2 + \Phi_\varepsilon^{(2)}(p\varepsilon, z_2)\mathbf{e}_3, \qquad \Phi_\varepsilon^{(2)}(p\varepsilon, z_2) = (-1)^{p}\Phi_\varepsilon(z_2). \end{aligned}$$

Note that the quantities $(-1)^{q+1}$ and $(-1)^p$ denote the fact that the curved rods are alternate, allowing crossing between them in an alternate manner (see the zoom in Figure 1.1).

Accordingly, we denote the Frenet-Serret mobile frames derived from (5.2) in the respective direction by

$$\begin{aligned} &(\mathbf{t}_\varepsilon^{(1)}, \mathbf{e}_2, \mathbf{n}_\varepsilon^{(1)}), \quad \text{where} \quad \mathbf{t}_\varepsilon^{(1)}(z_1, q\varepsilon) = \frac{1}{\gamma_\varepsilon(z_1)}\big(\mathbf{e}_1 + \partial_1\Phi_\varepsilon^{(1)}(z_1, q\varepsilon)\mathbf{e}_3\big), \quad \mathbf{n}_\varepsilon^{(1)} \doteq \mathbf{t}_\varepsilon^{(1)} \wedge \mathbf{e}_2,\\ &(\mathbf{e}_1, \mathbf{t}_\varepsilon^{(2)}, \mathbf{n}_\varepsilon^{(2)}), \quad \text{where} \quad \mathbf{t}_\varepsilon^{(2)}(p\varepsilon, z_2) = \frac{1}{\gamma_\varepsilon(z_2)}\big(\mathbf{e}_2 + \partial_2\Phi_\varepsilon^{(2)}(p\varepsilon, z_2)\mathbf{e}_3\big), \quad \mathbf{n}_\varepsilon^{(2)} \doteq \mathbf{t}_\varepsilon^{(2)} \wedge \mathbf{e}_1. \end{aligned}$$

In these frames, the diffeomorphisms become

$$\begin{aligned} \psi_\varepsilon^{(1)}(z_1, q\varepsilon, y_2, y_3) &\doteq M_\varepsilon^{(1)}(z_1, q\varepsilon) + y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon^{(1)}(z_1, q\varepsilon), \quad \text{for a.e. } (z_1, q\varepsilon, y_2, y_3) \in\in \mathfrak{G}_\varepsilon^{(1)} \times \omega_r,\\ \psi_\varepsilon^{(2)}(p\varepsilon, z_2, y_1, y_3) &\doteq M_\varepsilon^{(2)}(p\varepsilon, z_2) + y_1\mathbf{e}_1 + y_3\mathbf{n}_\varepsilon^{(2)}(p\varepsilon, z_2), \quad \text{for a.e. } (p\varepsilon, z_2 y_1, y_3) \in \mathfrak{G}_\varepsilon^{(2)} \times \omega_r. \end{aligned}$$

Finally, the whole textile results to be

$$\mathcal{T}_\varepsilon \doteq \mathcal{T}_\varepsilon^{(1)} \cup \mathcal{T}_\varepsilon^{(2)}, \quad \text{where} \quad \mathcal{T}_\varepsilon^{(1)} \doteq \psi_\varepsilon^{(1)}\big(\mathfrak{G}_\varepsilon^{(1)} \times \omega_r\big), \quad \mathcal{T}_\varepsilon^{(2)} \doteq \psi_\varepsilon^{(2)}\big(\mathfrak{G}_\varepsilon^{(2)} \times \omega_r\big). \tag{5.24}$$

For simplicity, a function defined on $\mathfrak{G}_\varepsilon^{(\alpha)}$ is also considered as an element defined in $\mathcal{T}_\varepsilon^{(\alpha)}$ constant in the cross-sections ω_r. This is the main reason, why we name z the beam center line variables and y the cross-section variables.

Let $\mathcal{C}(\mathfrak{G}_\varepsilon)$ be the space of continuous functions defined on the lattice grid $\mathfrak{G}_\varepsilon$. We denote the spaces of functions by ($\alpha \in \{1,2\}$)

$$\begin{aligned} H^1(\mathfrak{G}_\varepsilon^{(\alpha)}) &\doteq \{\phi \in L^2(\mathfrak{G}_\varepsilon^{(\alpha)}) \,|\, \partial_\alpha\phi \in L^2(\mathfrak{G}_\varepsilon^{(\alpha)})\},\\ H^1(\mathfrak{G}_\varepsilon) &\doteq \{\phi \in \mathcal{C}(\mathfrak{G}_\varepsilon) \,|\, \partial_\alpha\phi \in L^2(\mathfrak{G}_\varepsilon^{(\alpha)}), \text{ for } \alpha \in \{1,2\}\}, \end{aligned}$$

and

$$\begin{aligned} H^2(\mathfrak{G}_\varepsilon^{(\alpha)}) &\doteq \{\phi \in H^1(\mathfrak{G}_\varepsilon^{(\alpha)}) \,|\, \partial_\alpha\phi \in H^1(\mathfrak{G}_\varepsilon^{(\alpha)})\},\\ H^2(\mathfrak{G}_\varepsilon) &\doteq \{\phi \in H^1(\mathfrak{G}_\varepsilon) \,|\, \partial_\alpha\phi \in H^1(\mathfrak{G}_\varepsilon^{(\alpha)}), \text{ for } \alpha \in \{1,2\}\}. \end{aligned}$$

We endow these spaces with the following norms:

$$\|\cdot\|_{H^1(\mathfrak{G}_\varepsilon^{(\alpha)})} \doteq \sqrt{\|\cdot\|^2_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \|\partial_\alpha(\cdot)\|^2_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})}}, \qquad \|\cdot\|_{H^1(\mathfrak{G}_\varepsilon)} \doteq \sqrt{\sum_{\alpha=1}^{2}\|\cdot\|^2_{H^1(\mathfrak{G}_\varepsilon^{(\alpha)})}},$$

$$\|\cdot\|_{H^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \doteq \sqrt{\|\cdot\|^2_{H^1(\mathfrak{G}_\varepsilon^{(\alpha)})} + \|\partial_{\alpha\alpha}(\cdot)\|^2_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})}}, \qquad \|\cdot\|_{H^2(\mathfrak{G}_\varepsilon)} \doteq \sqrt{\sum_{\alpha=1}^{2}\|\cdot\|^2_{H^2(\mathfrak{G}_\varepsilon^{(\alpha)})}}.$$

Every displacement defined on such structure is a couple $(u^{(1)}, u^{(2)})$ which belongs to the product space $H^1(T_\varepsilon^{(1)})^3 \times H^1(T_\varepsilon^{(2)})^3$ (or, due the equivalence of norms (5.4), to the product space $H^1\big(\mathfrak{G}_\varepsilon^{(1)} \times \omega_r\big)^3 \times H^1\big(\mathfrak{G}_\varepsilon^{(2)} \times \omega_r\big)^3$).

5.4.2 Boundary conditions

We set a partial clamp on the left and bottom boundary of the domain Ω. Here, the displacement is equal to zero. Given the structure (5.24), we have

$$\text{Clamp condition} \qquad \begin{cases} u^{(1)}(0, q\varepsilon, \cdot) = 0 & \text{for every } \; q \in \{0, \dots, 2n_\varepsilon\}, \\ u^{(2)}(p\varepsilon, 0, \cdot) = 0 & \text{for every } \; p \in \{0, \dots, 2n_\varepsilon\}. \end{cases} \tag{5.25}$$

As we can see in Figure 1.1, this partial clamp leads to a natural partition of the domain

$$\Omega = \operatorname{int}\big(\overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3 \cup \overline{\Omega}_4\big),$$

where the four subdomains are defined by

$$\Omega_1 \doteq (0,l)^2, \quad \Omega_2 \doteq (l,L) \times (0,l), \quad \Omega_3 \doteq (0,l) \times (l,L), \quad \Omega_4 \doteq (l,L)^2.$$

Note that even if the partial clamp that takes place on the left boundary of Ω_1 affects the behavior of the displacement $u^{(1)}$ in the whole subdomain $\Omega_1 \cup \Omega_2$ since the fibers are the same. Symmetrically, the partial clamp on the bottom boundary of Ω_1 affects the behavior of the displacement $u^{(2)}$ in the whole $\Omega_1 \cup \Omega_3$.

5.4.3 Contact and non-penetration conditions

These conditions determine how the fibers interact in the internal part of the domain. Here, we assume that they can shear one with respect to another in the in-plane and outer-plane directions and cannot penetrate each other.
The contact is restricted to the portions where the rods are right above each other. We define such contact domains in the straight reference frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by setting ($(p,q) \in \mathcal{K}_\varepsilon$)

$$\mathbf{C}_\varepsilon \doteq \bigcup_{(p,q)\in\mathcal{K}_\varepsilon} \mathbf{C}_{pq,\varepsilon}, \qquad \mathbf{C}_{pq,\varepsilon} \doteq \big(C_{pq,\varepsilon} \cap \Omega\big) \times \{0\}, \qquad C_{pq,\varepsilon} \doteq (p\varepsilon, q\varepsilon) + \omega_r. \tag{5.26}$$

In terms of the textile domain variables, these areas correspond to

$$\begin{aligned} \big(\mathfrak{G}_\varepsilon^{(1)} \times \omega_r\big)_{|C_{pq,\varepsilon}} &\doteq \big(p\varepsilon + y_1, q\varepsilon, y_2, (-1)^{p+q+1}\kappa\varepsilon\big), \qquad (y_1, y_2) \in \omega_r, \\ \big(\mathfrak{G}_\varepsilon^{(2)} \times \omega_r\big)_{|C_{pq,\varepsilon}} &\doteq \big(p\varepsilon, q\varepsilon + y_2, y_1, (-1)^{p+q}\kappa\varepsilon\big), \qquad (y_1, y_2) \in \omega_r. \end{aligned}$$

The sliding between the fibers is characterized by the non-negative gap functions g_ε. We assume

$$g_\varepsilon = \varepsilon^h g, \quad g \in \mathcal{C}(\overline{\Omega})^3,$$

where $h \in [0, \infty)$ is a parameter representing the "contact strength." Now, let $(u_\varepsilon^{(1)}, u_\varepsilon^{(2)})$ be in $H^1\big(\mathfrak{G}_\varepsilon^{(1)} \times \omega_r\big)^3 \times H^1\big(\mathfrak{G}_\varepsilon^{(2)} \times \omega_r\big)^3$ be a displacement on the textile. We define the in-plane contact conditions by setting

$$|u^{(1)}_{\alpha,\varepsilon} - u^{(2)}_{\alpha,\varepsilon}| \leq \varepsilon^h g_\alpha, \qquad \text{a.e in } \mathbf{C}_{pq,\varepsilon}, \ \forall (p,q) \in \mathcal{K}_\varepsilon. \tag{5.27}$$

From the physical point of view, these conditions mean that in the contact areas, the displacements can slide one with respect to the other from a value more than zero to a maximum given by the L^∞ norm of g in that direction.
In this sense, it becomes clear the notion of contact strength. Indeed, if $h = 0$, then we have a constant on the right-hand side, and thus no actual bound as the difference of the in-plane displacements (which still depend on ε) goes to zero. On the other hand, if $h \to \infty$, then the right-hand side goes to zero faster, and it is then equivalent to setting $g_\alpha = 0$, which would imply $u^{(1)}_{\varepsilon,r,\alpha} = u^{(2)}_{\varepsilon,r,\alpha}$ a.e. in $\mathbf{C}_{pq,\varepsilon}$, thus that the fibers are glued on the whole domain.
In the outer-plane component, we define the non-penetration and contact conditions

$$0 \leq (-1)^{p+q}\big(u^{(1)}_{3,\varepsilon} - u^{(2)}_{3,\varepsilon}\big) \leq \varepsilon^h g_3 \qquad \text{a.e in } \mathbf{C}_{pq,\varepsilon}, \ \forall (p,q) \in \mathcal{K}_\varepsilon. \tag{5.28}$$

From the physical point of view, on the left-hand side, we assume that the fibers cannot penetrate each other, while on the right-hand side, we assume an upper bound on the admissible deflection again.

5.5 Well posedness of the elasticity problem

In this section, we proceed to the definition of the elasticity problem for the small deformations of a textile structure under stress. This problem is the one we want to investigate through homogenization via the periodic unfolding method.

5.5.1 Set of admissible displacements

Given the structure, the clamp condition, the contact conditions, and the non-penetration condition, we finally define the set of admissible displacements as the closed convex set

$$\mathcal{X}_\varepsilon \doteq \Big\{ \big(u^{(1)}, u^{(2)}\big) \in H^1(\mathfrak{G}^{(1)}_\varepsilon \times \omega_r)^3 \times H^1(\mathfrak{G}^{(2)}_\varepsilon \times \omega_r)^3 \ \Big| \ \big(u^{(1)}, u^{(2)}\big) \text{ satisfies (5.25)-(5.27)-(5.28)} \Big\}.$$

We endow the product space $H^1(\mathfrak{G}^{(1)}_\varepsilon \times \omega_r)^3 \times H^1(\mathfrak{G}^{(2)}_\varepsilon \times \omega_r)^3$ with the semi-norm

$$\|u\|_{T_\varepsilon} \doteq \sqrt{\|e^{(1)}(u^{(1)})\|^2_{L^2(\mathfrak{G}^{(1)}_\varepsilon \times \omega_r)} + \|e^{(2)}(u^{(2)})\|^2_{L^2(\mathfrak{G}^{(2)}_\varepsilon \times \omega_r)}}.$$

By the clamped conditions (5.25), this semi-norm is, in fact, an equivalent norm to the usual one of the product space $H^1(\mathfrak{G}^{(1)}_\varepsilon \times \omega_r)^3 \times H^1(\mathfrak{G}^{(2)}_\varepsilon \times \omega_r)^3$. Thus, $\mathcal{X}_\varepsilon$ is a closed convex subset of $H^1(\mathfrak{G}^{(1)}_\varepsilon \times \omega_r)^3 \times H^1(\mathfrak{G}^{(2)}_\varepsilon \times \omega_r)^3$.

5.5.2 The problem in linear elasticity

Since we are interested in the small deformations for the textile yarns, we will give the linearized formulation of the elasticity problem. We state it now as an assumption, even though later, we will give sufficient stress on the right-hand side to stay in this regime.

For now, let $f^{(\alpha)}_\varepsilon \in L^2(T^{(\alpha)}_\varepsilon)^3$ simply be some applied stress and let $a^{(\alpha)}_\varepsilon$ be the fourth order strain tensor describing the elasticity of the material. Due to the contact constraints (5.27) and (5.28), we write the linearized elasticity problem in variational formulation:

$$\begin{aligned} &\text{Find } (u^{(1)}_\varepsilon, u^{(2)}_\varepsilon) \in \mathcal{X}_\varepsilon \text{ such that for every } (v^{(1)}_\varepsilon, v^{(2)}_\varepsilon) \in \mathcal{X}_\varepsilon\text{:} \\ &\sum_{\alpha=1}^{2} \int_{T^{(\alpha)}_\varepsilon} a^{(\alpha)}_{ijkl,\varepsilon}\, e_{x,ij}(u^{(\alpha)}_\varepsilon)\, e_{x,kl}(u^{(\alpha)}_\varepsilon - v^{(\alpha)}_\varepsilon)\, dx \leq \sum_{\alpha=1}^{2} \int_{T_\varepsilon} f^{(\alpha)}_\varepsilon \cdot (u^{(\alpha)}_\varepsilon - v^{(\alpha)}_\varepsilon)\, dx. \end{aligned} \tag{5.29}$$

We find it more convenient to consider this problem in the straight reference frame:

$$\begin{aligned}
&\text{Find } (u_\varepsilon^{(1)}, u_\varepsilon^{(2)}) \in \mathcal{X}_\varepsilon \text{ such that for every } (v_\varepsilon^{(1)}, v_\varepsilon^{(2)}) \in \mathcal{X}_\varepsilon\text{:}\\
&\sum_{\alpha=1}^{2} \int_{\mathfrak{G}_\varepsilon^{(\alpha)} \times \omega_r} A_{ijkl,\varepsilon}^{(\alpha)}\, e_{ij}^{(\alpha)}(u_\varepsilon^{(\alpha)})\, e_{kl}^{(\alpha)}(u_\varepsilon^{(\alpha)} - v_\varepsilon^{(\alpha)})\, \eta_\varepsilon^{(\alpha)}\, dz' dy_{3-\alpha} dy_3 \\
&\le \sum_{\alpha=1}^{2} \int_{\mathfrak{G}_\varepsilon^{(\alpha)} \times \omega_r} F_\varepsilon^{(\alpha)} \cdot (u_\varepsilon^{(\alpha)} - v_\varepsilon^{(\alpha)})\, \eta_\varepsilon^{(\alpha)}\, dz' dy_{3-\alpha} dy_3,
\end{aligned} \tag{5.30}$$

where the strain tensors are defined as the symmetric gradients (5.22)-(5.23), but for each direction.

For the material elasticity of the tensors $A_\varepsilon^{(\alpha)}$, we refer to the usual Hooke's Law, from which we derive that:

(i) $A_\varepsilon^{(\alpha)}$ is bounded: $A_{\varepsilon,r,ijkl}^{(\alpha)} \in L^\infty(\mathfrak{G}_\varepsilon^{(\alpha)} \times \omega_r)$;

(ii) $A_\varepsilon^{(\alpha)}$ is symmetric: $A_{\varepsilon,r,ijkl}^{(\alpha)} = A_{\varepsilon,jikl}^{(\alpha)} = A_{\varepsilon,r,klij}^{(\alpha)}$;

(iii) $A_\varepsilon^{(\alpha)}$ is elliptic: $\exists C_0, C_1 > 0$ such that $C_0|||\xi|||_F^2 \le A_{ijkl,\varepsilon}^{(\alpha)} \xi_{ij} \xi_{kl} \le C_1|||\xi|||_F^2$ a.e. in $\mathfrak{G}_\varepsilon^{(\alpha)} \times \omega_r$ and for all symmetric 3×3 matrix ξ (here $|||\cdot|||_F$ denotes the Frobenius norm).

The existence of solutions for the problem is ensured by Stampacchia's lemma in Kinderlehrer and Stampacchia, 2000.

However, uniqueness is not ensured everywhere in the domain. Indeed, let u_ε' and u_ε'' be both solutions of (5.30). We can first choose u_ε' as a solution and u_ε'' as a test function and vice versa. We get two inequalities, and their sum results in

$$\sum_{\alpha=1}^{2} \int_{\mathfrak{G}_\varepsilon^{(\alpha)} \times \omega_r} A_{ijkl,\varepsilon}^{(\alpha)}\, \widetilde{e}_{ij}^{(\alpha)}(u_\varepsilon^{'(\alpha)} - u_\varepsilon^{''(\alpha)})\, \widetilde{e}_{kl}^{(\alpha)}(u_\varepsilon^{'(\alpha)} - u_\varepsilon^{''(\alpha)})\, \eta_\varepsilon^{(\alpha)}\, dz' dy_{3-\alpha} dy_3 \le 0,$$

from which property (iii) of A_ε implies that $\widetilde{e}(u_\varepsilon') = \widetilde{e}(u_\varepsilon'')$, hence u_ε' and u_ε'' differ from a rigid motion. Hence, from the clamp conditions (5.25), it becomes clear that we only have

$$\begin{aligned}
&u^{'(1)}(z_1, q\varepsilon, y_2, y_3) = u^{''(1)}(z_1, q\varepsilon, y_2, y_3) \quad \forall q \in \{0, \dots, 2n_\varepsilon\}, \quad (z_1, y_2, y_3) \in (0, L) \times \omega_r,\\
&u^{'(2)}(p\varepsilon, z_2, y_1, y_3) = u^{''(2)}(p\varepsilon, z_2, y_1, y_3) \quad \forall p \in \{0, \dots, 2n_\varepsilon\}, \quad (z_2, y_1, y_3) \in (0, L) \times \omega_r,
\end{aligned}$$

while uniqueness does not hold in general in the whole domain.

5.6 Estimates for the displacements fields

Let $(u^{(1)}, u^{(2)})$ be a displacement in $\mathcal{X}_\varepsilon$. Recall the prime decomposition of section 5.3.2:

$$\begin{aligned}
u^{(1)}(z_1, q\varepsilon, y_2, y_3) &= \mathbb{U}^{'(1)}(z_1, q\varepsilon) + \mathcal{R}^{'(1)}(z_1, q\varepsilon) \wedge \left(\Phi_\varepsilon^{(1)}(z_1, q\varepsilon)\mathbf{e}_3 + y_2\mathbf{e}_2 + y_3\mathbf{n}^{(1)}(z_1, q\varepsilon)\right)\\
&\quad + \overline{u}^{'(1)}(z_1, q\varepsilon, y_2, y_3), \quad \text{for a.e. } (z_1, q\varepsilon, y_2, y_3) \in \mathfrak{G}_\varepsilon^{(1)} \times \omega_r,\\
u^{(2)}(p\varepsilon, z_2, y_1, y_3) &= \mathbb{U}^{'(2)}(p\varepsilon, z_2) + \mathcal{R}^{'(2)}(p\varepsilon, z_2) \wedge \left(\Phi_\varepsilon^{(2)}(p\varepsilon, z_2)\mathbf{e}_3 + y_1\mathbf{e}_1 + y_3\mathbf{n}^{(2)}(p\varepsilon, z_2)\right)\\
&\quad + \overline{u}^{'(2)}(p\varepsilon, z_2, y_1, y_3), \quad \text{for a.e. } (p\varepsilon, z_2, y_1, y_3) \in \mathfrak{G}_\varepsilon^{(2)} \times \omega_r.
\end{aligned} \tag{5.31}$$

In order to pass to the limit in problem (5.30), we need to bound the fields and their derivatives that appear in both the left-hand side (strain tensor (5.11)-(5.23) and its equivalent formulation in direction $\mathbf{e}_2$) and the right-hand side (fields that appear on the above displacement).

5.6.1 Fundamental estimates

From Theorem 6, the estimates for the prime decomposition's fields satisfy ($\alpha \in \{1,2\}$)

$$\begin{aligned}
&\varepsilon\|\partial_\alpha \mathcal{R}'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \|\partial_\alpha \mathbb{U}_\alpha'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon}\|u\|_{T_\varepsilon},\\
&\|\partial^2_{\alpha\alpha} \mathbb{U}_{3-\alpha}'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \|\partial^2_{\alpha\alpha} \mathbb{U}_3'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon^2}\|u\|_{T_\varepsilon},\\
&\|\overline{u}'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r)} + \varepsilon\|\nabla \overline{u}'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r)} \le C\varepsilon\|u\|_{T_\varepsilon}.
\end{aligned} \tag{5.32}$$

Moreover, from the clamp conditions (5.25), we easily derive that

$$\begin{aligned}
&\mathbb{U}'^{(1)}(0,q\varepsilon) = \mathcal{R}'^{(1)}(0,q\varepsilon) = 0, \qquad q \in \{0,\dots,2n_\varepsilon\},\\
&\mathbb{U}'^{(2)}(p\varepsilon,0) = \mathcal{R}'^{(2)}(p\varepsilon,0) = 0, \qquad p \in \{0,\dots,2n_\varepsilon\}.
\end{aligned} \tag{5.33}$$

5.6.2 Contact and non-penetration estimates

In order to have a bound for the fields, we use the bound on their derivatives, Poincaré's inequality, and the clamp conditions. However, since the domain is partially clamped, we need to transfer the bound from the fields in the clamped subdomains to those in the not clamped ones. To do so, we will use the contact and non-penetration conditions.

Starting from (5.31), we note that for a.e. $(t_1,t_2) \in \omega_r$, the displacements in the contact areas reduce to

$$\begin{aligned}
u^{(1)}(p\varepsilon+t_1, q\varepsilon, t_2, (-1)^{p+q+1}\kappa\varepsilon) &= \mathbb{U}'^{(1)}(p\varepsilon+t_1,q\varepsilon) + \mathcal{R}'^{(1)}(p\varepsilon+t_1,q\varepsilon)\wedge t_2\mathbf{e}_2\\
&\quad + \overline{u}'^{(1)}(p\varepsilon+t_1,q\varepsilon,t_2,(-1)^{p+q+1}\kappa\varepsilon),\\
u^{(2)}(p\varepsilon, q\varepsilon+t_2, t_1, (-1)^{p+q}\kappa\varepsilon) &= \mathbb{U}'^{(2)}(p\varepsilon,q\varepsilon+t_2) + \mathcal{R}'^{(2)}(p\varepsilon,q\varepsilon+t_2)\wedge t_1\mathbf{e}_1\\
&\quad + \overline{u}'^{(2)}(p\varepsilon,q\varepsilon+t_2,t_1,(-1)^{p+q}\kappa\varepsilon).
\end{aligned} \tag{5.34}$$

We start by giving the warping estimates in the contact areas.

Lemma 18. *The warping terms satisfy*

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \left(\|\overline{u}'^{(1)}\|^2_{L^2(\mathbf{C}_{pq,\varepsilon})} + \|\overline{u}'^{(2)}\|^2_{L^2(\mathbf{C}_{pq,\varepsilon})}\right) \le C\varepsilon\|u\|^2_{T_\varepsilon}. \tag{5.35}$$

Proof. It is a direct consequence of the third estimate in (5.32) of the remainder displacements $\overline{u}'^{(\alpha)}$ and the trace theorem. □

We have the following.

Lemma 19. *The in-plane contact conditions lead to the following estimate:*

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \left(|(\mathbb{U}_\alpha'^{(1)} - \mathbb{U}_\alpha'^{(2)})(p\varepsilon,q\varepsilon)|^2 + \varepsilon^2|(\mathcal{R}_3'^{(1)} - \mathcal{R}_3'^{(2)})(p\varepsilon,q\varepsilon)|^2\right) \le C\left(\varepsilon^{2h-2}\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|^2_{T_\varepsilon}\right). \tag{5.36}$$

The outer-plane non-penetration conditions lead to the following estimates:

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \left(|(\mathbb{U}_3'^{(1)} - \mathbb{U}_3'^{(2)})(p\varepsilon,q\varepsilon)|^2 + \varepsilon^2|(\mathcal{R}_\alpha'^{(1)} - \mathcal{R}_\alpha'^{(2)})(p\varepsilon,q\varepsilon)|^2\right) \le \frac{C}{\varepsilon}\|u\|^2_{T_\varepsilon}. \tag{5.37}$$

Proof. First, from the proof of Lemma 5.6 in Griso, Orlik, and Wackerle, 2020a, we have

$$\begin{aligned}
&\varepsilon^2\Big(\sum_{p,q=0}^{2N_\varepsilon} \left|\mathbb{U}'^{(1)}(p\varepsilon,q\varepsilon) - \mathbb{U}'^{(2)}(p\varepsilon,q\varepsilon)\right|^2 + \varepsilon^2\left|\mathcal{R}'^{(1)}(p\varepsilon,q\varepsilon) - \mathcal{R}'^{(2)}(q\varepsilon,p\varepsilon)\right|^2\Big)\\
&\le C\sum_{p,q=0}^{2N_\varepsilon}\left(\varepsilon^{2h+2}|g(p\varepsilon,q\varepsilon)|^2 + \|\overline{u}'^{(1)}\|^2_{L^2(\mathbf{C}_{pq,\varepsilon})} + \|\overline{u}'^{(2)}\|^2_{L^2(\mathbf{C}_{pq,\varepsilon})}\right) \le C\left(\varepsilon^{2h}\|g\|^2_{L^\infty(\Omega)} + \varepsilon\|u\|^2_{T_\varepsilon}\right),
\end{aligned}$$

which proves (5.36).
Concerning the third direction, a first upper bound is given by the above equation. However, a better bound, that is (5.37), and that does not depend on g_3, is proven in Section C in Appendix due to the long and tedious computation. □

The fact that the outer-plane direction no longer depends on the contact bound g_3 is very important. From the physical point of view, the fact that the displacement alternatively switches vertical position and the yarns cannot penetrate one into the other gives a sufficient condition to estimate the difference of the displacements in the third component.
As we will see later, by the fact that the fibers are naturally close enough, the upper bound contact function $g_{\varepsilon,3}$ in (5.28) in the limit plays a role only when the contact is very strong (namely, only if $h \geq 3$).

5.6.3 Outer-plane fields' estimates in the whole domain Ω

In this subsection we give the estimates regarding the fields $\mathbb{U}_3^{'(\alpha)}$ and $\mathcal{R}_1^{'(\alpha)}$, $\mathcal{R}_2^{'(\alpha)}$. We will use the relations in the previous sections and the Poincaré inequality to obtain them.

Proposition 6. *The outer-plane rotation fields satisfy:*

$$\|\mathcal{R}_\alpha^{'(1)}\|_{H^1(\mathfrak{G}_\varepsilon^{(1)})} + \|\mathcal{R}_\alpha^{'(2)}\|_{H^1(\mathfrak{G}_\varepsilon^{(2)})} \leq \frac{C}{\varepsilon^2}\|u\|_{T_\varepsilon}. \tag{5.38}$$

The outer-plane middle line fields satisfy

$$\|\mathbb{U}_3^{'(1)}\|_{H^2(\mathfrak{G}^{(1)})} + \|\mathbb{U}_3^{'(2)}\|_{H^2(\mathfrak{G}^{(2)})} \leq \frac{C}{\varepsilon^2}\|u\|_{T_\varepsilon}. \tag{5.39}$$

Proof. By estimate (5.32), the clamp conditions (5.33) and Poincaré's inequality, we have

$$\sum_{q=0}^{2n_\varepsilon} \|\mathcal{R}_\alpha^{'(1)}(\cdot, q\varepsilon)\|^2_{L^2(0,L)} + \sum_{p=0}^{2n_\varepsilon} \|\mathcal{R}_\alpha^{'(2)}(p\varepsilon, \cdot)\|^2_{L^2(0,L)} \leq \frac{C}{\varepsilon^4}\|u\|^2_{T_\varepsilon}.$$

Now we consider direction $\mathbf{e}_1$ and estimate $\mathcal{R}'^{(1)}$ in the non supported domain. We have

$$\sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon} \varepsilon|\mathcal{R}_\alpha^{'(2)}(p\varepsilon, q\varepsilon)|^2 \leq C\sum_{p=0}^{2n_\varepsilon}\left(\|\mathcal{R}_\alpha^{'(2)}(p\varepsilon,\cdot)\|^2_{L^2(0,L)} + \varepsilon^2\|\partial_2\mathcal{R}_\alpha^{'(2)}(p\varepsilon,\cdot)\|^2_{L^2(0,L)}\right) \leq \frac{C}{\varepsilon^4}\|u\|^2_{T_\varepsilon}.$$

Then

$$\begin{aligned}\sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon} \varepsilon|\mathcal{R}_\alpha^{'(1)}(p\varepsilon, q\varepsilon)|^2 &\leq \sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon} \varepsilon|\mathcal{R}_\alpha^{'(2)}(p\varepsilon, q\varepsilon) - \mathcal{R}_\alpha^{'(1)}(p\varepsilon, q\varepsilon)|^2 + \sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon} \varepsilon|\mathcal{R}_\alpha^{'(2)}(p\varepsilon, q\varepsilon)|^2\\ &\leq C\Big(\frac{1}{\varepsilon^4} + \frac{1}{\varepsilon^2}\Big)\|u\|^2_{T_\varepsilon} \leq \frac{C}{\varepsilon^4}\|u\|^2_{T_\varepsilon}.\end{aligned}$$

and thus

$$\|\mathcal{R}_\alpha^{'(1)}\|^2_{L^2(\mathfrak{G}_\varepsilon^{(1)})} \leq C\sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon} \varepsilon|\mathcal{R}_\alpha^{'(1)}(p\varepsilon, q\varepsilon)|^2 + C\varepsilon^2\|\partial_1\mathcal{R}_\alpha^{'(1)}\|^2_{L^2(\mathfrak{G}_\varepsilon^{(1)})} \leq \frac{C}{\varepsilon^4}\|u\|^2_{T_\varepsilon}.$$

This proves estimate (5.38) for direction $\mathbf{e}_1$.
Now we prove estimate (5.39) in direction $\mathbf{e}_1$. From (5.38) and identities (5.16), we already know that

$$\sum_{q=0}^{2n_\varepsilon} \|\partial_1\mathbb{U}_3^{'(1)}(\cdot, q\varepsilon)\|^2_{H^1(0,L)} + \sum_{q=0}^{2n_\varepsilon} \|\partial_2\mathbb{U}_3^{'(2)}(p\varepsilon, \cdot)\|^2_{H^1(0,L)} \leq \frac{C}{\varepsilon^4}\|u\|^2_{T_\varepsilon}.$$

Then, the proof follows by the same meaning as the previous estimate.

From a symmetrical argumentation, we obtain the estimate for direction $\mathbf{e}_2$. □

5.6.4 In-plane fields' estimates in the whole domain Ω

In this subsection, we give all the in-plane estimates, that are, the estimates regarding the fields $\mathbb{U}_1^{'(\alpha)}$, $\mathbb{U}_2^{'(\alpha)}$ and $\mathcal{R}_3^{'(\alpha)}$. Again, the relations and Poincaré's inequality will be sufficient. Without loss of generality, we assume a unique bound in direction $\mathbf{e}_1$ and $\mathbf{e}_2$ and set

$$\|g\|_{L^\infty(\Omega)} \doteq \|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}.$$

We have the following.

Proposition 7. *The in-plane rotation fields satisfy:*

$$\|\mathcal{R}_3^{'(1)}\|_{H^1(\mathfrak{G}_\varepsilon^{(1)})} + \|\mathcal{R}_3^{'(2)}\|_{H^1(\mathfrak{G}_\varepsilon^{(2)})} \leq C\Big(\varepsilon^{h-3/2}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|_{T_\varepsilon}\Big). \tag{5.40}$$

The in-plane middle line fields satisfy

$$\begin{aligned}
&\|\mathbb{U}_2^{'(1)}\|_{H^2(\mathfrak{G}^{(1)})} + \|\mathbb{U}_1^{'(2)}\|_{H^2(\mathfrak{G}^{(2)})} \leq C\Big(\varepsilon^{h-3/2}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|_{T_\varepsilon}\Big),\\
&\|\mathbb{U}_1^{'(1)}\|_{H^1(\mathfrak{G}^{(1)})} + \|\mathbb{U}_2^{'(2)}\|_{H^1(\mathfrak{G}^{(2)})} \leq C\Big(\varepsilon^{h-3/2}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|_{T_\varepsilon}\Big).
\end{aligned} \tag{5.41}$$

Proof. The proof is done in the same fashion as the previous one, but using estimate (5.36):

$$\begin{aligned}
\sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathcal{R}_3^{'(1)}(p\varepsilon,q\varepsilon)|^2 \leq& \sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathcal{R}_3^{'(2)}(p\varepsilon,q\varepsilon) - \mathcal{R}_3^{'(1)}(p\varepsilon,q\varepsilon)|^2 + \sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathcal{R}_3^{'(2)}(p\varepsilon,q\varepsilon)|^2\\
\leq& C\Big(\varepsilon^{2h-3}\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^4}\|u\|^2_{T_\varepsilon}\Big).
\end{aligned}$$

This proves estimate (5.40) for direction $\mathbf{e}_1$. The second direction follows by a symmetrical argumentation.

Now we prove estimate $(5.41)_1$. From (5.40) and identities (5.16), we already know that

$$\sum_{q=0}^{2n_\varepsilon}\|\partial_1\mathbb{U}_2^{'(1)}(\cdot,q\varepsilon)\|^2_{H^1(0,L)} + \sum_{q=0}^{2n_\varepsilon}\|\partial_2\mathbb{U}_1^{'(2)}(p\varepsilon,\cdot)\|^2_{H^1(0,L)} \leq C\Big(\varepsilon^{2h-3}\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^4}\|u\|^2_{T_\varepsilon}\Big).$$

Then, the proof follows in the same fashion as the proof of estimate (5.40).

Now we prove estimate $(5.41)_2$. We consider direction $\mathbf{e}_1$ and estimate $\mathbb{U}_1^{'(1)}$ in the unsupported domain. We have

$$\begin{aligned}
\sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathbb{U}_1^{'(2)}(p\varepsilon,q\varepsilon)|^2 \leq& C\sum_{p=0}^{2n_\varepsilon}\big(\|\mathbb{U}_1^{'(2)}(p\varepsilon,\cdot)\|^2_{L^2(0,L)} + \varepsilon^2\|\partial_2\mathbb{U}_1^{'(2)}(p\varepsilon,\cdot)\|^2_{L^2(0,L)}\big)\\
\leq& C\Big(\varepsilon^{2h-3}\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^4}\|u\|^2_{T_\varepsilon}\Big).
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathbb{U}_1^{'(1)}(p\varepsilon,q\varepsilon)|^2 \leq& \sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathbb{U}_1^{'(1)}(p\varepsilon,q\varepsilon) - \mathbb{U}_1^{'(2)}(p\varepsilon,q\varepsilon)|^2 + \sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathbb{U}_1^{'(2)}(p\varepsilon,q\varepsilon)|^2\\
\leq& C\Big(\varepsilon^{2h-3}\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^4}\|u\|^2_{T_\varepsilon}\Big).
\end{aligned}$$

and thus

$$\|\mathbb{U}_1^{'(1)}\|^2_{L^2(\mathfrak{G}_\varepsilon^{(1)})} \le C\sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathbb{U}_1^{'(1)}(p\varepsilon,q\varepsilon)|^2 + C\varepsilon^2\|\partial_1\mathbb{U}_1^{'(1)}\|^2_{L^2(\mathfrak{G}_\varepsilon^{(1)})}$$
$$\le C\Big(\varepsilon^{2h-3}\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^4}\|u\|^2_{T_\varepsilon}\Big),$$

which concludes the proof. □

5.6.5 Other important estimates

In this subsection, we give the last important estimates that we need to take into account: the estimates concerning the fields in the clamped subdomains, which we expect to be better than on the unsupported ones, and the estimate concerning the in-plane derivatives, which we will later use to improve the in-plane estimates for strong contacts via Korn's inequality.

Corollary 5. *One has*

$$\begin{aligned}&\|\mathbb{U}_1^{'(1)}\|_{L^2(\mathfrak{G}_\varepsilon^{(1)}\cap(\Omega_1\cup\Omega_2))} + \|\mathbb{U}_2^{'(2)}\|_{L^2(\mathfrak{G}_\varepsilon^{(2)}\cap(\Omega_1\cup\Omega_3))} \le \frac{C}{\varepsilon}\|u\|_{T_\varepsilon},\\ &\|\mathbb{U}_2^{'(1)}\|_{L^2(\mathfrak{G}_\varepsilon^{(1)}\cap(\Omega_1\cup\Omega_3))} + \|\mathbb{U}_1^{'(2)}\|_{L^2(\mathfrak{G}_\varepsilon^{(2)}\cap(\Omega_1\cup\Omega_2))} \le C\Big(\varepsilon^{h-1/2}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big)\end{aligned} \tag{5.42}$$

Proof. Estimate (5.42)$_1$ follows from estimate (5.32), the Poincaré Inequality and the clamp conditions.

Now we prove (5.42)$_2$. From (5.42)$_1$, we first have that

$$\sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathbb{U}_2^{'(2)}(p\varepsilon,q\varepsilon)|^2 \le C\sum_{p=0}^{2n_\varepsilon}\big(\|\mathbb{U}_2^{'(2)}(p\varepsilon,\cdot)\|^2_{L^2(0,L)} + \varepsilon^2\|\partial_2\mathbb{U}_2^{'(2)}(p\varepsilon,\cdot)\|^2_{L^2(0,L)}\big) \le \frac{C}{\varepsilon^2}\|u\|^2_{T_\varepsilon}.$$

Then

$$\sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathbb{U}_2^{'(1)}(p\varepsilon,q\varepsilon)|^2 \le \sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathbb{U}_2^{'(1)}(p\varepsilon,q\varepsilon) - \mathbb{U}_2^{'(2)}(p\varepsilon,q\varepsilon)|^2 + \sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathbb{U}_2^{'(2)}(p\varepsilon,q\varepsilon)|^2$$
$$\le C\Big(\varepsilon^{2h-1}\|g\|^2_{L^\infty(\Omega)} + \Big(1+\frac{1}{\varepsilon^2}\Big)\|u\|^2_{T_\varepsilon}\Big) \le C\Big(\varepsilon^{2h-1}\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|^2_{T_\varepsilon}\Big)$$

and thus

$$\sum_{q=0}^{2N_\varepsilon}\|\mathbb{U}_2^{'(1)}(\cdot,q\varepsilon)\|^2_{L^2(0,l)} \le C\sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2n_\varepsilon}\varepsilon|\mathbb{U}_2^{'(1)}(p\varepsilon,q\varepsilon)|^2 + C\varepsilon^2\|\partial_1\mathbb{U}_2^{'(1)}(\cdot,q\varepsilon)\|^2_{L^2(0,l)}$$
$$\le C\Big(\varepsilon^{2h-1}\|g\|^2_{L^\infty(\Omega)} + \Big(1+\frac{1}{\varepsilon^2}\Big)\|u\|^2_{T_\varepsilon}\Big) \le C\Big(\varepsilon^{2h-1}\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|^2_{T_\varepsilon}\Big).$$

The proof is complete. □

Corollary 6. *In the whole domain, the following estimate holds:*

$$\|\partial_1\mathbb{U}_2^{'(1)} + \partial_2\mathbb{U}_1^{'(2)}\|_{L^2(\mathfrak{G}_\varepsilon)} \le C\Big(\varepsilon^{h-3/2}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big). \tag{5.43}$$

Proof. From the fact that $(\partial_1\mathbb{U}_2^{'(1)},\partial_2\mathbb{U}_1^{'(2)}) = (-\mathcal{R}_3^{'(1)},\mathcal{R}_3^{'(2)})$, estimate (5.36) implies

$$\sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2N_\varepsilon}\big|(\partial_1\mathbb{U}_2^{'(1)} + \partial_2\mathbb{U}_1^{'(2)})(p\varepsilon,q\varepsilon)\big|^2 \le C\Big(\varepsilon^{2h-4}\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{T_\varepsilon}\Big).$$

Hence,

$$\|\partial_1 \mathbb{U}_2^{'(1)} + \partial_2 \mathbb{U}_1^{'(2)}\|^2_{L^2(\mathfrak{G}_\varepsilon)} \leq \sum_{q=0}^{2N_\varepsilon} \Big(\varepsilon^2 \|\partial_1 \mathbb{U}_2^{'(1)}(\cdot, q\varepsilon)\|^2_{L^2(0,L)}\Big) + \sum_{p=0}^{2N_\varepsilon} \Big(\varepsilon^2 \|\partial_2 \mathbb{U}_1^{'(2)}(p\varepsilon, \cdot)\|^2_{L^2(0,L)}\Big)$$
$$+ \varepsilon \sum_{q=0}^{2N_\varepsilon}\sum_{p=0}^{2N_\varepsilon} \big|(\partial_1 \mathbb{U}_2^{'(1)} + \partial_2 \mathbb{U}_1^{'(2)})(p\varepsilon, q\varepsilon)\big|^2$$
$$\leq \frac{2C}{\varepsilon^2}\|u\|^2_{T_\varepsilon} + C\varepsilon\Big(\varepsilon^{2h-4}\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{T_\varepsilon}\Big)$$
$$\leq C\Big(\varepsilon^{2h-3}\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|^2_{T_\varepsilon}\Big),$$

which concludes the proof. □

5.7 Choice of the parameters

Looking at the estimates in the previous section, we notice that three factors influence the estimates for the displacement fields: The estimates proven in the previous section depend on three main factors:

1. The ratio between the thickness of fibers r and the distance between them ε;
2. The assumption of small deformations (which gives a bound for the strain $\|u\|_{T_\varepsilon}$) ;
3. The strength of contact between the fibers h (which gives a bound for g_ε).

Concerning the first aspect, as we already know, we assume that

$$r = \kappa\varepsilon, \qquad \text{with } \kappa \in (0, 1/3],$$

so that the parameters for the fibers' cross-section and the distance between them are asymptotically related as they go to zero. We can remove this assumption, but it would require a model that does not involve the prime decomposition. Because of this, and because the problem already has a high level of difficulty, we leave the other cases out of the scope of this work.

Concerning the second aspect, we know from Friesecke, James, and Müller, 2006 that a bound for the elastic energy for the deformation of a rod $v \in H^1(\mathcal{P}_\varepsilon)^3$ (remind that $\mathcal{P}_\varepsilon \doteq (0, L) \times \omega_r$) leads to the following regimes:

$$\|\text{dist}(\nabla_x v, SO(3))\|^2_{\mathcal{P}_\varepsilon} \leq C\varepsilon^\delta, \quad \text{with} \begin{cases} \delta > 5 & \text{Linear theory;} \\ \delta = 5 & \text{Von Kármán theory;} \\ \delta \in (3,5) & \text{Linearized isometry constraint theory;} \\ \delta = 3 & \text{Bending theory;} \\ \delta < 3 & \text{Membrane theory.} \end{cases}$$

From the decomposition of a rod deformation made in Blanchard and Griso, 2009, Section II.2.2 and the associated fields estimates, it is then possible to find a bound for the elastic energy of a textile deformation

$$\frac{1}{2}\big\|(\nabla_x v)^T(\nabla_x v) - \mathbf{I}_3\big\|_{T_\varepsilon} \leq C\|\text{dist}(\nabla_x v, SO(3))\|^2_{T_\varepsilon}.$$

Hence, if we are in the linear regime, we would have that the strain tensor can be approximated to the symmetric gradient of the displacement $u = v - Id_x$ and we have

$$\|u\|^2_{T_\varepsilon} = \frac{1}{2}\big\|(\nabla_x u)^T + (\nabla_x u)\big\|_{T_\varepsilon} \leq C\varepsilon^\delta, \qquad \text{for } \delta > 5.$$

For simplicity, we will fix $\delta = 5$ and get the following bound for the strain tensor

$$\|u\|^2_{T_\varepsilon} \leq C\varepsilon^5, \tag{5.44}$$

while continuing to use the linear formulation of the elasticity problem and the linearized strain tensor (symmetric gradient of the displacement).

This is a more convenient way of proceeding, and avoids the writing of $\delta > 5$ in every future estimate. We just keep in mind that, since the estimate for the strain tensor of the elasticity problem (5.44) depends on the applied stress, we can always rescale the applied stress according to δ to remain in the context of linear elasticity.

At last, we have the contact strength. The lower bound of this value is the maximum slide we can allow such that the textile keeps a reasonable shape (for $h = 0$, there is no actual bound), while the upper bound is given by the minimum strength applied, up to which we can assume the fibers to be glued (limit case $+\infty$).

5.7.1 Different type of $r = \kappa\varepsilon$ textiles in linear regime ($\|u\|_{T_\varepsilon} \sim \varepsilon^{5/2}$)

In order to find the most representative textile structures, we first collect the estimates of Section 5.6: the global ones (5.32), the outer-plane ones (5.38), (5.39), the in-plane ones (5.40), (5.41), the ones in the clamped subdomains (5.42) and the mixed derivative ones (5.43).

Then, we need to choose the parameters in the previous subsection: we already fixed the relation $r = \kappa\varepsilon$, and in addition, we fix the gradient estimate $\|u\|_{T_\varepsilon} \sim \varepsilon^{5/2}$ to study the linear elasticity of yarns. The last parameter to fix is the contact strength. Without loss of generality, we can assume $h \in \mathbb{N}^*$ and obtain the following:

- $h > 3$: We have a textile with glued fibers;
- $h = 3$: We have a textile with strong contact;
- $h = 2$: We have a textile with loose contact;
- $h = 1$: We have a textile with very loose contact.

We collect all the explicit estimates for the cases mentioned above in Table 5.1.

This table is really important because we can derive some preliminary considerations on

	FIELDS	CONTACT ORDER		
		$h \geq 3$	$h = 2$	$h = 1$
Outer-plane	$\|\mathcal{R}'^{(1)}_{\alpha,\varepsilon}\|_{H^1(\mathfrak{G}^{(1)}_\varepsilon)} + \|\mathcal{R}'^{(2)}_{\alpha,\varepsilon}\|_{H^1(\mathfrak{G}^{(2)}_\varepsilon)}$	$\sim \sqrt{\varepsilon}$	$\sim \sqrt{\varepsilon}$	$\sim \sqrt{\varepsilon}$
	$\|\mathbb{U}'^{(1)}_{3,\varepsilon}\|_{H^2(\mathfrak{G}^{(1)})} + \|\mathbb{U}'^{(2)}_{3,\varepsilon}\|_{H^2(\mathfrak{G}^{(2)})}$	$\sim \sqrt{\varepsilon}$	$\sim \sqrt{\varepsilon}$	$\sim \sqrt{\varepsilon}$
In-plane	$\|\mathcal{R}'^{(1)}_{3,\varepsilon}\|_{H^1(\mathfrak{G}^{(1)}_\varepsilon)} + \|\mathcal{R}'^{(2)}_{3,\varepsilon}\|_{H^1(\mathfrak{G}^{(2)}_\varepsilon)}$	$\sim \sqrt{\varepsilon}$	$\sim \sqrt{\varepsilon}$	$\sim \frac{1}{\sqrt{\varepsilon}}$
	$\|\mathbb{U}'^{(1)}_{2,\varepsilon}\|_{H^2(\mathfrak{G}^{(1)})} + \|\mathbb{U}'^{(2)}_{1,\varepsilon}\|_{H^2(\mathfrak{G}^{(2)})}$	$\sim \sqrt{\varepsilon}$	$\sim \sqrt{\varepsilon}$	$\sim \frac{1}{\sqrt{\varepsilon}}$
	$\|\mathbb{U}'^{(1)}_{1,\varepsilon}\|_{H^1(\mathfrak{G}^{(1)})} + \|\mathbb{U}'^{(2)}_{2,\varepsilon}\|_{H^1(\mathfrak{G}^{(2)})}$	$\sim \sqrt{\varepsilon}$	$\sim \sqrt{\varepsilon}$	$\sim \frac{1}{\sqrt{\varepsilon}}$
In-plane	$\|\mathbb{U}'^{(1)}_{1,\varepsilon}\|_{L^2(\mathfrak{G}^{(1)}_\varepsilon \cap (\Omega_1 \cup \Omega_2))} + \|\mathbb{U}'^{(2)}_{2,\varepsilon}\|_{L^2(\mathfrak{G}^{(2)}_\varepsilon \cap (\Omega_1 \cup \Omega_3))}$	$\sim \varepsilon\sqrt{\varepsilon}$	$\sim \varepsilon\sqrt{\varepsilon}$	$\sim \varepsilon\sqrt{\varepsilon}$
clamped	$\|\mathbb{U}'^{(1)}_{2,\varepsilon}\|_{L^2(\mathfrak{G}^{(1)}_\varepsilon \cap (\Omega_1 \cup \Omega_3))} + \|\mathbb{U}'^{(2)}_{1,\varepsilon}\|_{L^2(\mathfrak{G}^{(2)}_\varepsilon \cap (\Omega_1 \cup \Omega_2))}$	$\sim \varepsilon\sqrt{\varepsilon}$	$\sim \varepsilon\sqrt{\varepsilon}$	$\sim \sqrt{\varepsilon}$
In-plane derivatives	$\|\partial_\alpha \mathbb{U}'^{(\alpha)}_{\alpha,\varepsilon}\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon)} + \|\partial_1 \mathbb{U}'^{(1)}_{2,\varepsilon} + \partial_2 \mathbb{U}'^{(2)}_{1,\varepsilon}\|_{L^2(\mathfrak{G}_\varepsilon)}$	$\sim \varepsilon\sqrt{\varepsilon}$	$\sim \sqrt{\varepsilon}$	$\sim \frac{1}{\sqrt{\varepsilon}}$

TABLE 5.1: Table of explicit estimates for the fields $r = \kappa\varepsilon$ textiles in a linear regime $\|u\|_{T_\varepsilon} \sim \varepsilon^{5/2}$ and according to the different contact strength $h \in \mathbb{N}^*$.

how the displacement behaves in the different subdomains Ω_1-Ω_4, which can help us define proper final decomposition for the displacements for each case. In particular:

A. We have an idea of how contact and strain tensor quantity interact with each other and govern the field estimates. Indeed, for $h \geq 3$, the field estimates do not change because, for such values, the contact part becomes smaller than the strain tensor one and does not influence the estimates anymore. On the other hand, as the value of h diminishes, we get worse estimates of the displacement fields because the contact part gets the upper hand.

B. The outer-plane estimates are the same for every contact case. This fact means that the outer-plane fields have a sufficiently good estimate, even if the contact is very loose or if no contact is set. Hence, the same final decomposition will apply in all cases.

C. The in-plane estimates in the clamped parts are better than in the whole domain. This is because a looser contact between fibers would compromise the transfer of information from the clamped subdomains to the not clamped ones.

D. The in-plane derivatives in the case $h \geq 3$ are of the same order as the in-plane fields in the clamped parts. This allows us to obtain in-plane estimates for the fields $\sim \varepsilon\sqrt{\varepsilon}$ due to Korn's inequality, hence the in-plane fields behave the same in the whole Ω, and a partition is no more necessary. It is not the case for $h = 2$, where the in-plane derivatives have a worse estimation, and thus the in-plane fields will have a contrast in the estimates (anisotropy) and behave differently in the four subdomains of Ω.

E. For $h = 1$, the contact is so loose that the estimates do not give a bound for the in-plane fields as $\varepsilon \to 0$. For this reason, we need to elaborate on a different strategy for this case, introducing some more assumptions.

In the next two chapters, we will delve into the investigation of these cases one by one. From now on, only the choice of $r = \kappa\varepsilon$ and the contact strength h will be fixed, while the strain tensor estimate still needs to be justified by the choice of the forces on the right-hand side of problem (5.30). Hence, the field estimates will continue to depend on the bound for the strain tensor.

Chapter 6

The cases of a textile with strong contact ($g_\varepsilon \sim \varepsilon^3$ or higher) and very loose contact ($g_\varepsilon \sim \varepsilon$)

This small chapter is dedicated to the cases of textiles with strong contact and very loose contact. The first case has already been investigated in Griso, Orlik, and Wackerle, 2020a, and here we will only show that we can reach the same final displacement decomposition with the newly developed strategy. The second case turns out to be trivial; hence no homogenization will be needed.

6.1 Textiles with very strong contact

Concerning textiles with very strong contact, for $h = 3$ the contact and non-penetration conditions (5.27)-(5.28) become:

$$\begin{cases} |u^{(1)}_{\alpha,\varepsilon} - u^{(2)}_{\alpha,\varepsilon}| \leq \varepsilon^3 g_\alpha, & \text{a.e in } \mathbf{C}_{pq,\varepsilon}, \ \forall (p,q) \in \mathcal{K}_\varepsilon, \\ 0 \leq (-1)^{p+q}(u^{(1)}_{3,\varepsilon} - u^{(2)}_{3,\varepsilon}) \leq \varepsilon^3 g_3 & \text{a.e in } \mathbf{C}_{pq,\varepsilon}, \ \forall (p,q) \in \mathcal{K}_\varepsilon. \end{cases} \tag{6.1}$$

If then the contact is even higher ($h \geq 4$), no improvement is made on the fields estimates, hence no improvements on the difference between the displacement estimates (see (5.36)-(5.37)). Hence, conditions (5.27)-(5.28) are equivalent to the following:

$$\begin{cases} |u^{(1)}_{\alpha,\varepsilon} - u^{(2)}_{\alpha,\varepsilon}| \leq 0, & \text{a.e in } \mathbf{C}_{pq,\varepsilon}, \ \forall (p,q) \in \mathcal{K}_\varepsilon, \\ 0 \leq (-1)^{p+q}(u^{(1)}_{3,\varepsilon} - u^{(2)}_{3,\varepsilon}) \leq 0 & \text{a.e in } \mathbf{C}_{pq,\varepsilon}, \ \forall (p,q) \in \mathcal{K}_\varepsilon. \end{cases} \tag{6.2}$$

This implies $u^{(1)}_{\alpha,\varepsilon} = u^{(2)}_{\alpha,\varepsilon}$ a.e. $\mathbf{C}_{pq,\varepsilon}$, hence that we can assume the fibers to be glued in the contact areas of the whole domain. In this sense, one can first extend the woven textile to a periodically perforated shell and then proceed to homogenization, as it has been done in Griso, Orlik, and Wackerle, 2020b.

The above cases have already been investigated in Griso, Orlik, and Wackerle, 2020a and have been the first breakthrough for this kind of problem. For this reason, we will reach the final decomposition for the displacement before the limit with the newly developed lattice strategy and recall the conclusions in the final chapter as a comparison with the other cases.

6.1.1 Final decomposition of the displacement in the in-plane component

Comparing the estimates for each field in Table 5.1 and the ones concerning their difference (5.36)-(5.37) for $h \geq 3$ and (5.44), we find it convenient to define the final displacements fields such that they combine both directions and that take into account the clamp conditions.

Proceeding as in Subsection 5.3.2, we define the field $\mathbb{U}_3 \in H^2(\mathfrak{G}_\varepsilon)$ by

$$\begin{aligned}
\mathbb{U}_3(z_1, q\varepsilon) &\doteq \phi_{cub}^{[\mathbf{B},\mathbf{B}']}(z_1), \quad \text{with} \quad \mathbf{B} = \frac{1}{2}\big((\mathbb{U}_3^{'(1)} + \mathbb{U}_3^{'(2)})(0, q\varepsilon), \dots, (\mathbb{U}_3^{'(1)} + \mathbb{U}_3^{'(2)})(2N_\varepsilon\varepsilon, q\varepsilon)\big),\\
&\qquad \mathbf{B}' = -\frac{1}{2}\big((\mathcal{R}_2^{'(1)} + \mathcal{R}_2^{'(1)})(0, q\varepsilon), \dots, (\mathcal{R}_2^{'(1)} + \mathcal{R}_2^{'(1)})(2N_\varepsilon\varepsilon, q\varepsilon)\big),\\
&\qquad \forall z_1 \in [p\varepsilon, (p+1)\varepsilon], \quad \forall q \in \{0, \dots, 2N_\varepsilon\},\\
\mathbb{U}_3(p\varepsilon, z_2) &\doteq \phi_{cub}^{[\mathbf{B},\mathbf{B}']}(z_2), \quad \text{with} \quad \mathbf{B} = \frac{1}{2}\big((\mathbb{U}_3^{'(1)} + \mathbb{U}_3^{'(2)})(p\varepsilon, 0), \dots, (\mathbb{U}_3^{'(1)} + \mathbb{U}_3^{'(2)})(p\varepsilon, 2N_\varepsilon\varepsilon)\big),\\
&\qquad \mathbf{B}' = \frac{1}{2}\big((\mathcal{R}_1^{'(1)} + \mathcal{R}_1^{'(1)})(p\varepsilon, 0), \dots, (\mathcal{R}_1^{'(1)} + \mathcal{R}_1^{'(1)})(2N_\varepsilon\varepsilon, q\varepsilon)\big),\\
&\qquad \forall z_2 \in [q\varepsilon, (q+1)\varepsilon], \quad \forall p \in \{0, \dots, 2N_\varepsilon\}.
\end{aligned}$$

We then define the fields $\mathcal{R}_1, \mathcal{R}_2 \in H^1(\mathfrak{G}_\varepsilon)$ by

$$\begin{aligned}
&\mathcal{R}_2(z_1, q\varepsilon) \doteq -\partial_1 \mathbb{U}_3(z_1, q\varepsilon), \qquad \forall z_1 \in [0, L], \quad \forall q \in \{0, \dots, 2N_\varepsilon\},\\
&\mathcal{R}_2(p\varepsilon, z_2) \doteq \psi_{\mathbf{B}}(z_2) \quad \text{with} \quad \mathbf{B} = \frac{1}{2}\big((\mathcal{R}_2^{'(1)} + \mathcal{R}_2^{'(2)})(p\varepsilon, 0), \dots, (\mathcal{R}_2^{'(1)} + \mathcal{R}_2^{'(2)})(p\varepsilon, 2N_\varepsilon\varepsilon)\big),\\
&\qquad \forall z_2 \in [q\varepsilon, (q+1)\varepsilon], \quad \forall p \in \{0, \dots, 2N_\varepsilon\},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{R}_1(z_1, q\varepsilon) \doteq \psi_{\mathbf{B}}(z_1) \quad \text{with} \quad \mathbf{B} = \frac{1}{2}\big((\mathcal{R}_1^{'(1)} + \mathcal{R}_1^{'(2)})(p\varepsilon, 0), \dots, (\mathcal{R}_1^{'(1)} + \mathcal{R}_1^{'(2)})(p\varepsilon, 2N_\varepsilon\varepsilon)\big),\\
&\qquad \forall z_2 \in [q\varepsilon, (q+1)\varepsilon], \quad \forall p \in \{0, \dots, 2N_\varepsilon\},\\
&\mathcal{R}_1(p\varepsilon, z_2) \doteq \partial_2 \mathbb{U}_3(p\varepsilon, z_2), \qquad \forall z_2 \in [0, L], \quad \forall p \in \{0, \dots, 2N_\varepsilon\}.
\end{aligned}$$

According to the clamp conditions (5.33), we replace

$$\begin{aligned}
&(\mathbb{U}_3^{'(1)} + \mathbb{U}_3^{'(2)})(0, q\varepsilon) \ \text{ and } \ (\mathcal{R}_2^{'(1)} + \mathcal{R}_2^{'(1)})(0, q\varepsilon) \ \text{ by } \ 0 \ \text{ if } \ q \in \{0, \dots, 2n_\varepsilon\},\\
&(\mathbb{U}_3^{'(1)} + \mathbb{U}_3^{'(2)})(p\varepsilon, 0) \ \text{ and } \ (\mathcal{R}_1^{'(1)} + \mathcal{R}_1^{'(1)})(p\varepsilon, 0) \ \text{ by } \ 0 \ \text{ if } \ p \in \{0, \dots, 2n_\varepsilon\}.
\end{aligned}$$

Note that the fields defined above vanish on the clamped points of $\mathfrak{G}_\varepsilon$ and satisfy equalities

$$\mathcal{R}_2 = -\partial_1 \mathbb{U}_3 \ \text{ a.e. in } \mathfrak{G}_\varepsilon^{(1)} \qquad \text{and} \qquad \mathcal{R}_1 = \partial_2 \mathbb{U}_3 \ \text{ a.e. in } \mathfrak{G}_\varepsilon^{(2)}.$$

Proposition 8. *The outer-plane fields satisfy the following estimates:*

$$\|\mathcal{R}_1\|_{H^1(\mathfrak{G}_\varepsilon)} + \|\mathcal{R}_2\|_{H^1(\mathfrak{G}_\varepsilon)} + \|\mathbb{U}_3\|_{H^2(\mathfrak{G})} \le \frac{C}{\varepsilon^2}\|u\|_{T_\varepsilon}.$$

Proof. Step 1. We prove the estimates of $\mathcal{R}_1$ and $\mathcal{R}_2$.
From the definitions of $\mathcal{R}_2$ and $\mathcal{R}_2^{'(1)}, \mathcal{R}_2^{'(2)}$, estimates (5.37), the clamp condition (5.33) and Lemma 17, we get

$$\|\mathcal{R}_2 - \mathcal{R}_2^{'(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \varepsilon\|\partial_\alpha(\mathcal{R}_2 - \mathcal{R}_2^{'(\alpha)})\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon}\|u\|_{T_\varepsilon}. \tag{6.3}$$

Hence, the first estimate in (5.32) and the above one lead to

$$\|\partial_1 \mathcal{R}_2\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})} + \|\partial_2 \mathcal{R}_2\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})} \le \frac{C}{\varepsilon^2}\|u\|_{T_\varepsilon}.$$

By the fact that $\mathcal{R}_2(0, q\varepsilon) = 0$ for all $q \in \{0, \dots, 2n_\varepsilon\}$, the above estimate and the Poincaré inequality imply

$$\sum_{q=0}^{2n_\varepsilon} \|\mathcal{R}_2(\cdot, q\varepsilon)\|^2_{L^2(0,L)} \le \frac{C}{\varepsilon^4}\|u\|^2_{T_\varepsilon}.$$

One has

$$\sum_{q=0}^{2n_\varepsilon}\sum_{p=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_2(p\varepsilon,q\varepsilon)|^2\le C\sum_{q=0}^{2n_\varepsilon}\big(\|\mathcal{R}_2(\cdot,q\varepsilon)\|^2_{L^2(0,L)}+\varepsilon^2\|\partial_1\mathcal{R}_2(\cdot,q\varepsilon)\|^2_{L^2(0,L)}\big)\le\frac{C}{\varepsilon^4}\|u\|_{T_\varepsilon}$$

and then

$$\|\mathcal{R}_2\|^2_{L^2(\mathfrak{G}^{(2)}_\varepsilon)}\le C\sum_{q=0}^{2n_\varepsilon}\sum_{p=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_2(p\varepsilon,q\varepsilon)|^2+C\varepsilon^2\|\partial_2\mathcal{R}_2\|^2_{L^2(\mathfrak{G}^{(2)}_\varepsilon)}\le\frac{C}{\varepsilon^4}\|u\|^2_{T_\varepsilon}.$$

By a symmetrical argumentation, we prove the above estimate in $\mathfrak{G}^{(1)}_\varepsilon$ and get that

$$\|\mathcal{R}_2\|_{H^1(\mathfrak{G}_\varepsilon)}\le\frac{C}{\varepsilon^2}\|u\|_{T_\varepsilon}.$$

We prove the estimate for $\mathcal{R}_1$ in the same fashion.

Step 2. We prove the estimates of $\mathbb{U}_3$.

First, from estimates (5.37), the clamp condition (5.33) and Lemma 17, we have

$$\|\mathbb{U}_3-\mathbb{U}_3^{'(\alpha)}\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon)}+\varepsilon\|\partial_\alpha\big(\mathbb{U}_3-\mathbb{U}_3^{'(\alpha)}\big)\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon)}+\varepsilon^2\|\partial^2_{\alpha\alpha}\big(\mathbb{U}_3-\mathbb{U}_3^{'(\alpha)}\big)\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon)}\le C\|u\|_{T_\varepsilon}. \tag{6.4}$$

Then, the second estimate in (5.32) and the above one lead to

$$\|\partial^2_{\alpha\alpha}\mathbb{U}_3^{(\alpha)}\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon)}\le\frac{C}{\varepsilon^2}\|u\|_{T_\varepsilon}.$$

In Subsection 2.2.2, we saw that the function $\mathbb{U}_3\in H^2(\mathfrak{G}^{(\alpha)}_\varepsilon)$ can be extended from the grid to a function $\mathfrak{Q}(\mathbb{U}_3)\in H^2(\Omega)$ by extending it to every small cell $Y_{pq,\varepsilon}=(p\varepsilon,q\varepsilon)+\varepsilon[0,1]^2$. The values of $\mathbb{U}_3,\mathcal{R}_1,\mathcal{R}_2$, and their derivatives at the vertices of the cell $Y_{pq,\varepsilon}$ uniquely define this extension, and we have:

$$\mathfrak{Q}(\mathbb{U}_3)_{|\mathfrak{G}_\varepsilon}=\mathbb{U}_3,\qquad\partial_1\mathfrak{Q}(\mathbb{U}_3)_{|\mathfrak{G}^{(1)}_\varepsilon}=-\mathcal{R}_2,\qquad\partial_2\mathfrak{Q}(\mathbb{U}_3)_{|\mathfrak{G}^{(2)}_\varepsilon}=\mathcal{R}_1 \tag{6.5}$$

and

$$\mathfrak{Q}(\mathbb{U}_3)=\nabla\mathfrak{Q}(\mathbb{U}_3)=0\ \text{ a.e. on }\{0\}\times(0,l)\cup(0,l)\times\{0\}.$$

Thus, applying twice Korn's inequality, we get

$$\begin{aligned}\|\mathfrak{Q}(\mathbb{U}_3)\|_{H^2(\Omega)}&\le C\|e\big(\mathfrak{Q}(\mathbb{U}_3)\big)\|_{H^1(\Omega)}\le C\|D^2\mathfrak{Q}(\mathbb{U}_3)\|_{L^2(\Omega)}\\&\le C\sqrt{\varepsilon}\big(\|\partial^2_{\alpha\alpha}\mathbb{U}_3^{(\alpha)}\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon)}+\|\mathcal{R}_1\|_{H^1(\mathfrak{G}_\varepsilon)}+\|\mathcal{R}_2\|_{H^1(\mathfrak{G}_\varepsilon)}\big)\le\frac{C}{\varepsilon\sqrt{\varepsilon}}\|u\|_{T_\varepsilon}.\end{aligned} \tag{6.6}$$

Taking the restriction to the lattice grid $\mathfrak{G}_\varepsilon$, it first gives

$$\|\mathbb{U}_3\|_{L^2(\mathfrak{G}_\varepsilon)}\le\frac{C}{\varepsilon^2}\|u\|_{T_\varepsilon}.$$

Then, we obtain

$$\|\partial_\alpha\mathbb{U}_3\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon)}\le\frac{C}{\varepsilon^2}\|u\|_{T_\varepsilon}.$$

The proof is concluded. □

6.1.2 Final decomposition of the displacement in the in-plane component

We again recall the interpolations in Subsection 5.3.2 and define the field $\mathbb{U}_1 \in H^2(\mathfrak{G}_\varepsilon)$ by

$$\begin{aligned}
\mathbb{U}_1(z_1, q\varepsilon) &\doteq \phi_{lin}^{[\mathbf{A}]}(z_1), \quad \text{with} \quad \mathbf{A} = \frac{1}{2}\big((\mathbb{U}_1'^{(1)} + \mathbb{U}_1'^{(2)})(p\varepsilon, 0), \dots, (\mathbb{U}_1'^{(1)} + \mathbb{U}_1'^{(2)})(p\varepsilon, 2N_\varepsilon\varepsilon)\big),\\
&\qquad \forall z_1 \in [p\varepsilon, (p+1)\varepsilon], \quad \forall q \in \{0, \dots, 2N_\varepsilon\},\\
\mathbb{U}_1(p\varepsilon, z_2) &\doteq \phi_{cub}^{[\mathbf{B},\mathbf{B}']}(z_2), \quad \text{with} \quad \mathbf{B} = \frac{1}{2}\big((\mathbb{U}_1'^{(1)} + \mathbb{U}_1'^{(2)})(p\varepsilon, 0), \dots, (\mathbb{U}_1'^{(1)} + \mathbb{U}_1'^{(2)})(p\varepsilon, 2N_\varepsilon\varepsilon)\big),\\
&\qquad \mathbf{B}' = \frac{1}{2}\big((\mathcal{R}_3'^{(1)} + \mathcal{R}_3'^{(2)})(p\varepsilon, 0), \dots, (\mathcal{R}_3'^{(1)} + \mathcal{R}_3'^{(2)})(p\varepsilon, 2N_\varepsilon\varepsilon)\big),\\
&\qquad \forall z_2 \in [q\varepsilon, (q+1)\varepsilon], \quad \forall p \in \{0, \dots, 2N_\varepsilon\},
\end{aligned}$$

and the field $\mathbb{U}_2 \in H^2(\mathfrak{G}_\varepsilon)$ by

$$\begin{aligned}
\mathbb{U}_2(z_1, q\varepsilon) &\doteq \phi_{cub}^{[\mathbf{B},\mathbf{B}']}(z_1), \quad \text{with} \quad \mathbf{B} = \frac{1}{2}\big((\mathbb{U}_2'^{(1)} + \mathbb{U}_2'^{(2)})(0, q\varepsilon), \dots, (\mathbb{U}_2'^{(1)} + \mathbb{U}_2'^{(2)})(2N_\varepsilon\varepsilon, q\varepsilon)\big),\\
&\qquad \mathbf{B}' = -\frac{1}{2}\big((\mathcal{R}_3'^{(1)} + \mathcal{R}_3'^{(2)})(0, q\varepsilon), \dots, (\mathcal{R}_3'^{(1)} + \mathcal{R}_3'^{(2)})(2N_\varepsilon\varepsilon, q\varepsilon)\big),\\
&\qquad \forall z_2 \in [p\varepsilon, (p+1)\varepsilon], \quad \forall q \in \{0, \dots, 2N_\varepsilon\},\\
\mathbb{U}_2(p\varepsilon, z_2) &\doteq \phi_{lin}^{[\mathbf{A}]}(z_2), \quad \text{with} \quad \mathbf{A} = \frac{1}{2}\big((\mathbb{U}_2'^{(1)} + \mathbb{U}_2'^{(2)})(0, q\varepsilon), \dots, (\mathbb{U}_2'^{(1)} + \mathbb{U}_2'^{(2)})(2N_\varepsilon\varepsilon, q\varepsilon)\big),\\
&\qquad \forall z_2 \in [q\varepsilon, (q+1)\varepsilon], \quad \forall p \in \{0, \dots, 2N_\varepsilon\}.
\end{aligned}$$

Then, we define the field $\mathcal{R}_3 \in H^1(\mathfrak{G}_\varepsilon)$ by

$$\begin{aligned}
\mathcal{R}_3(z_1, q\varepsilon) &\doteq -\partial_1 \mathbb{U}_2(z_1, q\varepsilon), \qquad \forall z_1 \in [0, L], \quad \forall q \in \{0, \dots, 2N_\varepsilon\},\\
\mathcal{R}_3(p\varepsilon, z_2) &\doteq \partial_2 \mathbb{U}_1(p\varepsilon, z_2), \qquad \forall z_2 \in [0, L], \quad \forall p \in \{0, \dots, 2N_\varepsilon\}.
\end{aligned}$$

By the clamp condition (5.33), we replace

$$\begin{aligned}
&(\mathbb{U}_1'^{(1)} + \mathbb{U}_1'^{(2)})(0, q\varepsilon) \text{ and } (\mathcal{R}_3'^{(1)} + \mathcal{R}_3'^{(1)})(0, q\varepsilon) \text{ by } 0 \text{ if } q \in \{0, \dots, 2n_\varepsilon\},\\
&(\mathbb{U}_2'^{(1)} + \mathbb{U}_2'^{(2)})(p\varepsilon, 0) \text{ and } (\mathcal{R}_3'^{(1)} + \mathcal{R}_3'^{(1)})(p\varepsilon, 0) \text{ by } 0 \text{ if } p \in \{0, \dots, 2n_\varepsilon\}.
\end{aligned}$$

Note that these fields vanish on the clamped points of $\mathfrak{G}_\varepsilon$ and satisfy equalities

$$\mathcal{R}_3 = -\partial_1 \mathbb{U}_2 \quad \text{a.e. in } \mathfrak{G}_\varepsilon^{(1)} \qquad \text{and} \qquad \mathcal{R}_3 = \partial_2 \mathbb{U}_1 \quad \text{a.e. in } \mathfrak{G}_\varepsilon^{(2)}.$$

Proposition 9. *The in-plane fields satisfy the following estimates:*

$$\begin{aligned}
&\|\mathbb{U}_1\|_{H^1(\mathfrak{G})} + \varepsilon\|\partial_{22}\mathbb{U}_1\|_{L^2(\mathfrak{G}^{(2)})} \le C\Big(\varepsilon\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big),\\
&\|\mathbb{U}_2\|_{H^1(\mathfrak{G})} + \varepsilon\|\partial_{11}\mathbb{U}_2\|_{L^2(\mathfrak{G}^{(1)})} \le C\Big(\varepsilon\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big),\\
&\|\mathcal{R}_3\|_{L^2(\mathfrak{G}_\varepsilon)} + \varepsilon\big(\|\partial_1\mathcal{R}_3\|_{H^1(\mathfrak{G}_\varepsilon^{(1)})} + \|\partial_2\mathcal{R}_3\|_{H^1(\mathfrak{G}_\varepsilon^{(1)})}\big) \le C\Big(\varepsilon\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big).
\end{aligned}$$

Proof. Step 1. We prove that the following estimate holds:

$$\|\partial_1\mathbb{U}_1\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})} + \|\partial_2\mathbb{U}_2\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})} + \|\partial_1\mathbb{U}_2 + \partial_2\mathbb{U}_1\|_{L^2(\mathfrak{G}_\varepsilon)} \le C\Big(\varepsilon\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big)$$

From the definition of the fields estimates (5.36) and Lemma 17, we get

$$
\begin{aligned}
&\|\mathbb{U}_1-\mathbb{U}_1^{'(2)}\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})}+\varepsilon\|\partial_2(\mathbb{U}_1-\mathbb{U}_1^{'(2)})\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})}+\varepsilon^2\|\partial_{22}(\mathbb{U}_1-\mathbb{U}_1^{'(2)})\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})}\\
&\qquad\leq C\Big(\varepsilon^2\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)}+\|u\|_{T_\varepsilon}\Big),\\
&\|\mathbb{U}_2-\mathbb{U}_2^{'(1)}\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})}+\varepsilon\|\partial_1(\mathbb{U}_2-\mathbb{U}_2^{'(1)})\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})}+\varepsilon^2\|\partial_{11}(\mathbb{U}_2-\mathbb{U}_2^{'(1)})\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})}\\
&\qquad\leq C\Big(\varepsilon^2\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)}+\|u\|_{T_\varepsilon}\Big)
\end{aligned}
\tag{6.7}
$$

and

$$
\begin{aligned}
&\|\mathbb{U}_1-\mathbb{U}_1^{'(1)}\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})}+\varepsilon\|\partial_1(\mathbb{U}_1-\mathbb{U}_1^{'(1)})\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})}\leq C\Big(\varepsilon^2\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)}+\|u\|_{T_\varepsilon}\Big),\\
&\|\mathbb{U}_2-\mathbb{U}_2^{'(2)}\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})}+\varepsilon\|\partial_2(\mathbb{U}_2-\mathbb{U}_2^{'(2)})\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})}\leq C\Big(\varepsilon^2\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)}+\|u\|_{T_\varepsilon}\Big).
\end{aligned}
$$

Hence, estimates in (5.32) and the above ones lead to

$$
\begin{aligned}
&\|\partial_1\mathbb{U}_1\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})}\leq\|\partial_1\mathbb{U}_1-\partial_1\mathbb{U}_1^{'(1)}\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})}+\|\partial_1\mathbb{U}_1^{'(1)}\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})}\leq C\Big(\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)}+\frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big),\\
&\|\partial_2\mathbb{U}_2\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})}\leq\|\partial_2\mathbb{U}_2-\partial_2\mathbb{U}_2^{'(2)}\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})}+\|\partial_2\mathbb{U}_2^{'(1)}\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})}\leq C\Big(\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)}+\frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big),\\
&\|\partial_1\mathbb{U}_2+\partial_2\mathbb{U}_1\|_{L^2(\mathfrak{G}_\varepsilon)}\leq\|\partial_1\mathbb{U}_2-\partial_1\mathbb{U}_2^{'(1)}\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})}+\|\partial_2\mathbb{U}_1-\partial_2\mathbb{U}_1^{'(2)}\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})}\\
&\qquad+\|\partial_1\mathbb{U}_2^{'(1)}-\partial_2\mathbb{U}_1^{'(2)}\|_{L^2(\mathfrak{G}_\varepsilon)}\leq C\Big(\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)}+\frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big).
\end{aligned}
$$

Step 2. We prove the statement of the lemma.
The estimates for $\mathbb{U}_1$ and $\mathbb{U}_2$ follow from the clamp conditions and Korn's inequality with the estimates in Step 1, while the second order derivatives follow from (6.7) and (5.32).
Concerning the estimates for $\mathcal{R}_3$, they directly follow from the construction of such function.
□

Now that we defined the final fields, the final decomposition of the Bernoulli-Navier displacements becomes

$$
U_{BN}^{(1)}(z_1,q\varepsilon,y_2,y_3)=\begin{pmatrix}\mathbb{U}_1\\ \mathbb{U}_2\\ \mathbb{U}_3\end{pmatrix}(z_1,q\varepsilon)+\begin{pmatrix}\mathcal{R}_1\\ \mathcal{R}_2\\ \mathcal{R}_3\end{pmatrix}(z_1,q\varepsilon)\wedge\Big(\Phi_\varepsilon^{(1)}(z_1,q\varepsilon)\mathbf{e}_3+y_2\mathbf{e}_2+y_3\mathbf{n}_\varepsilon^{(1)}(z_1,q\varepsilon)\Big),
$$
$$
\text{for a.e. } (z_1,q\varepsilon,y_2,y_3)\in\mathfrak{G}^{(1)}\times\omega_r,
$$
$$
U_{BN}^{(2)}(p\varepsilon,z_2,y_1,y_3)=\begin{pmatrix}\mathbb{U}_1\\ \mathbb{U}_2\\ \mathbb{U}_3\end{pmatrix}(p\varepsilon,z_2)+\begin{pmatrix}\mathcal{R}_1\\ \mathcal{R}_2\\ \mathcal{R}_3\end{pmatrix}(p\varepsilon,z_2)\wedge\Big(\Phi_\varepsilon^{(2)}(p\varepsilon,z_2)\mathbf{e}_3+y_1\mathbf{e}_1+y_3\mathbf{n}_\varepsilon^{(2)}(p\varepsilon,z_2)\Big),
$$
$$
\text{for a.e. } (p\varepsilon,z_2,y_1,y_3)\in\mathfrak{G}^{(2)}\times\omega_r.
$$

As a consequence, the residual displacements are ($\alpha\in\{1,2\}$)

$$
\overline{u}^{(\alpha)}=u^{(\alpha)}-\mathbb{U}_{BN}^{(\alpha)}\in H^1(T_\varepsilon^{(\alpha)})
$$

and due to the third estimate in (5.32), estimates (6.3), (6.4) and (**??**) they satisfy

$$
\|\overline{u}^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r)}+\varepsilon\|\nabla\overline{u}^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r)}\leq C\varepsilon\|u\|_{T_\varepsilon}.
$$

Note that this estimate is of the same order as the residual displacement of the prime decomposition (5.21) and of the classical one (5.7). This fact justifies the choice of this final decomposition, which is close enough to the classical one but incorporates all the identities and simplifications for the unfolding and homogenization.

6.1.3 A priori conclusions

The decomposition and the estimates remind us of the results obtained in Griso, Orlik, and Wackerle, 2020b. Of particular relevance is that the estimates have the same order anywhere. This is because the contact between fibers keeps them so close to each other (since it is very strong) that we can transpose the clamped subdomains' behavior to the not clamped ones without loss of information. In this sense, a domain partition is no longer necessary since the displacement will behave the same anywhere.

6.2 Textiles with very loose contact

In this section, we comment on the case of textiles with very loose contact. This means, that the contact and non-penetration conditions (5.27)-(5.28) become for $h = 1$:

$$\begin{cases} |u^{(1)}_{\alpha,\varepsilon} - u^{(2)}_{\alpha,\varepsilon}| \leq \varepsilon g_\alpha, & \text{a.e in } \mathbf{C}_{pq,\varepsilon}, \ \forall (p,q) \in \mathcal{K}_\varepsilon, \\ 0 \leq (-1)^{p+q}\big(u^{(1)}_{3,\varepsilon} - u^{(2)}_{3,\varepsilon}\big) & \text{a.e in } \mathbf{C}_{pq,\varepsilon}, \ \forall (p,q) \in \mathcal{K}_\varepsilon. \end{cases}$$

Note that in the outer-plane component, no upper bound is set. This comes from estimate (5.37), which does not depend on the norm of g. Indeed, the alternate switch of the vertical position of the fibers and the non-penetration condition gives a sufficiently good estimate (namely, $\sim \varepsilon^3$) for the difference between displacements in the outer-plane component. That is why set no bound if $h < 3$.

6.2.1 Final decomposition of the displacement in the outer-plane component

Looking at the estimates in Table 5.1 for the different contact strengths, we notice that the estimates do not change. Hence, the outer-plane component's final displacement is decomposed in the same way as in subsection 6.1.1 and gives the fields $\mathcal{R}_1$, $\mathcal{R}_2 \in H^1(\mathfrak{G}_\varepsilon)$ and $\mathbb{U}_3 \in H^2(\mathfrak{G}_\varepsilon)$.

6.2.2 New assumption: the glued conditions

From the estimates in Table (5.1), it is clear that the used lattice strategy fails. Indeed, while we have good estimates in the outer-plane direction, the in-plane ones explode to infinity as ε goes to zero. It physically means that the contact strength is so loose that the fibers in the unsupported subdomains do not inherit the estimates from the clamped ones. Hence, we have no information on the bound on those domains, leading to the textile falling apart as in Figure 6.1.

In order to avoid this behavior, we need to set more boundary conditions. Namely, we can glue the displacements in both directions on the whole left and bottom boundary of the domain:

$$\text{Glued condition} \quad \begin{cases} u^{(1)}(0, q\varepsilon, \cdot) = u^{(2)}(0, q\varepsilon, \cdot) & \text{for every } \ q \in \{0, \dots, 2N_\varepsilon\}, \\ u^{(1)}(p\varepsilon, 0, \cdot) = u^{(2)}(p\varepsilon, 0, \cdot) & \text{for every } \ p \in \{0, \dots, 2N_\varepsilon\}. \end{cases}$$

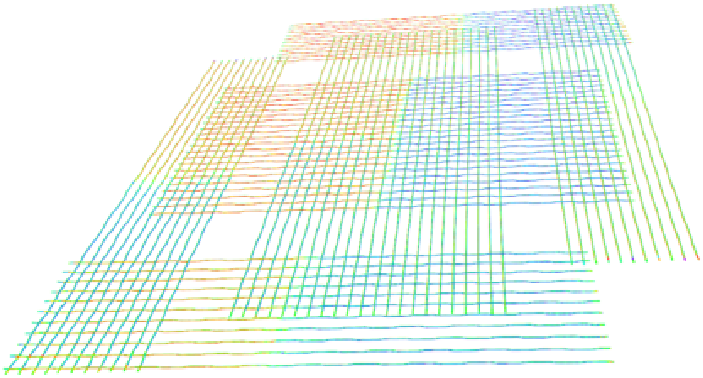

FIGURE 6.1: A textile with very loose contact. The contact is so loose that the fibers fall apart in the unsupported subdomains.

In this case, the displacements coincide but they are not zero. In the same fashion as in the proof of Lemma 19, this new assumption leads to the following in-plane estimates:

$$\begin{aligned}&\sum_{p=0}^{2N_\varepsilon}\Big(|(\mathbb{U}_\alpha^{'(1)}-\mathbb{U}_\alpha^{'(2)})(p\varepsilon,0)|^2+\varepsilon^2|(\mathcal{R}_3^{'(1)}-\mathcal{R}_3^{'(2)})(p\varepsilon,0)|^2\Big)\le\frac{C}{\varepsilon}\|u\|^2_{\mathcal{S}_\varepsilon},\\&\sum_{q=0}^{2N_\varepsilon}\Big(|(\mathbb{U}_\alpha^{'(1)}-\mathbb{U}_\alpha^{'(2)})(0,q\varepsilon)|^2+\varepsilon^2|(\mathcal{R}_3^{'(1)}-\mathcal{R}_3^{'(2)})(0,q\varepsilon)|^2\Big)\le\frac{C}{\varepsilon}\|u\|^2_{\mathcal{S}_\varepsilon}.\end{aligned}\tag{6.8}$$

Consequently, we get the following Lemma, which gives a better estimate for some of the in-plane fields. For some others, such as $\mathbb{U}_1^{'(1)}$ and $\mathbb{U}_2^{'(2)}$, there is no hope of getting a bound in the unsupported areas.

The proof is done in the same fashion as the proof of Lemma 6, but working on the boundary of the domain Ω to avoid the in-plane contact estimates and use the glued estimates (6.8) instead.

Lemma 20. *The in-plane fields satisfy*

$$\|\mathbb{U}_{3-\alpha}^{'(\alpha)}\|_{H^2(\mathfrak{S}^{(\alpha)})}+\|\mathcal{R}_3^{'(\alpha)}\|_{H^1(\mathfrak{S}_\varepsilon^{(\alpha)})}\le\frac{C}{\varepsilon^2}\|u\|_{\mathcal{S}_\varepsilon}.$$

Proof. By estimate (5.32), the clamp conditions (5.33) and Poincaré's inequality, we have

$$\|\mathcal{R}_3^{'(1)}(\cdot,0)\|^2_{L^2(0,L)}+\|\mathcal{R}_3^{'(2)}(0,\cdot)\|^2_{L^2(0,L)}\le\frac{C}{\varepsilon^4}\|u\|^2_{\mathcal{T}_\varepsilon}.$$

Now we consider direction $\mathbf{e}_1$ and estimate $\mathcal{R}^{'(1)}$ in the non supported domain. We have

$$\sum_{q=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_3^{'(2)}(0,q\varepsilon)|^2\le C\big(\|\mathcal{R}_3^{'(2)}(0,\cdot)\|^2_{L^2(0,L)}+\varepsilon^2\|\partial_2\mathcal{R}_3^{'(2)}(0,\cdot)\|^2_{L^2(0,L)}\big)\le\frac{C}{\varepsilon^4}\|u\|^2_{\mathcal{T}_\varepsilon}.$$

Then

$$\sum_{q=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_3^{'(1)}(0,q\varepsilon)|^2\le\sum_{q=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_3^{'(2)}(0,q\varepsilon)-\mathcal{R}_3^{'(1)}(0,q\varepsilon)|^2+\sum_{q=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_3^{'(2)}(0,q\varepsilon)|^2\le\frac{C}{\varepsilon^4}\|u\|^2_{\mathcal{T}_\varepsilon}.$$

and thus

$$\|\mathcal{R}_3^{'(1)}\|^2_{L^2(\mathfrak{G}_\varepsilon^{(1)})} \le C\sum_{q=0}^{2N_\varepsilon} \varepsilon|\mathcal{R}_3^{'(1)}(0,q\varepsilon)|^2 + C\varepsilon^2\|\partial_1\mathcal{R}_3^{'(1)}\|^2_{L^2(\mathfrak{G}_\varepsilon^{(1)})} \le \frac{C}{\varepsilon^4}\|u\|^2_{T_\varepsilon}.$$

This, together with (5.32)$_1$, proves the H^1 estimate for $\mathcal{R}_3^{'(1)}$ for direction $\mathbf{e}_1$. Moreover, from the above estimate and identities (5.16), we get that

$$\|\partial_1\mathbb{U}_2^{'(1)}\|^2_{H^1(\mathfrak{G}^{(1)})} + \|\partial_2\mathbb{U}_1^{'(2)}\|^2_{H^1(\mathfrak{G}^{(2)})} \le \frac{C}{\varepsilon^4}\|u\|^2_{T_\varepsilon}.$$

Then, the proof for the L^2 estimate of $\mathbb{U}_2^{'(1)}$ is done in the same fashion as for the L^2 estimate of $\mathcal{R}_3^{'(1)}$.

A symmetrical argumentation gives the estimate for direction $\mathbf{e}_2$. □

6.2.3 Final decomposition of the displacement in the in-plane component and a priori conclusions

Since the glued conditions helped to give a bound for the divergent fields, we can now define the final decomposition. Namely, we keep the in-plane fields in different directions separately and define the functions $\mathbb{U}_1^{(\alpha)}$, $\mathbb{U}_2^{(\alpha)}$ and $\mathcal{R}_3^{(\alpha)}$ by

$$\begin{aligned} &\mathbb{U}_1^{(1)} = \mathbb{U}_1^{'(1)}, \quad \mathbb{U}_2^{(1)} = \mathbb{U}_2^{'(1)}, \quad \mathcal{R}_3^{(1)} = \mathcal{R}_3^{'(1)} \quad \text{a.e. } \mathfrak{G}_\varepsilon^{(1)},\\ &\mathbb{U}_1^{(2)} = \mathbb{U}_1^{'(2)}, \quad \mathbb{U}_2^{(2)} = \mathbb{U}_2^{'(2)}, \quad \mathcal{R}_3^{(2)} = \mathcal{R}_3^{'(2)} \quad \text{a.e. } \mathfrak{G}_\varepsilon^{(2)}.\end{aligned}$$

Then, taking into account the clamping conditions (5.33), we replace

$$\begin{aligned} &\mathbb{U}_1^{'(\alpha)}(0,q\varepsilon),\ \mathbb{U}_2^{'(\alpha)}(0,q\varepsilon) \ \text{ and } \ \mathcal{R}_3^{'(\alpha)}(0,q\varepsilon) \ \text{ by } \ 0 \ \text{ if } \ q\in\{0,\dots,2n_\varepsilon\},\\ &\mathbb{U}_1^{'(\alpha)}(p\varepsilon,0),\ \mathbb{U}_2^{'(\alpha)}(p\varepsilon,0) \ \text{ and } \ \mathcal{R}_3^{'(\alpha)}(p\varepsilon,0) \ \text{ by } \ 0 \ \text{ if } \ q\in\{0,\dots,2n_\varepsilon\}.\end{aligned}$$

Note that, due to (5.16) in the respective direction, these fields vanish on the clamped points of $\mathfrak{G}_\varepsilon$ and satisfy equalities $\partial_1\mathbb{U}_2^{(1)} = \mathcal{R}_3^{(1)}$ a.e. in $\mathfrak{G}_\varepsilon^{(1)}$ and $\partial_2\mathbb{U}_1^{(2)} = -\mathcal{R}_3^{(2)}$ a.e. in $\mathfrak{G}_\varepsilon^{(2)}$.

Now note that no combined directions can be defined while going to the limit since the displacements keep a certain distance while going to the limit (see estimate (5.36) for $h=1$). Hence, the fibers no more influence each other due to the too-loose contact, and the two directions can be studied separately in the in-plane component and lead, therefore, to a trivial case.

Since the main focus is devoted to the influence of contact on woven textiles and how the behavior on clamped subdomains is transferred to the unsupported ones, we can consider this case out of scope and will not proceed to the homogenization. On the other hand, the third direction is always the same and can be, in fact, homogenized, as in the case that follows.

Chapter 7

The case of a textile with loose contact ($g_\varepsilon \sim \varepsilon^2$)

In this chapter, we give the complete study of the homogenization of textiles with loose contact. The investigation of this case is the core of the whole thesis and why we developed the tools of Chapter 3 and 4 in the first place.

The contact and non-penetration conditions (5.27)-(5.28) become for $h = 2$:

$$\begin{cases} |u^{(1)}_{\alpha,\varepsilon} - u^{(2)}_{\alpha,\varepsilon}| \leq \varepsilon^2 g_\alpha, & \text{a.e in } \mathbf{C}_{pq,\varepsilon}, \quad \forall (p,q) \in \mathcal{K}_\varepsilon, \\ 0 \leq (-1)^{p+q}(u^{(1)}_{3,\varepsilon} - u^{(2)}_{3,\varepsilon}) & \text{a.e in } \mathbf{C}_{pq,\varepsilon}, \quad \forall (p,q) \in \mathcal{K}_\varepsilon. \end{cases} \tag{7.1}$$

Again, note that we set no upper bound in the outer-plane component because of estimate (5.37), which gives a sufficiently good estimate (namely, $\sim \varepsilon^3$) concerning the difference between displacements.

Additionally, as we will later see in the construction of the test functions (to obtain the contact condition), we need the further assumption that there not exist and are in the internal part of Ω in which the fibers are glued:

$$\exists\, C_3 > 0 \quad \text{such that } g_\alpha \geq C_3 \quad \text{a.e. in } \Omega. \tag{7.2}$$

7.1 Final decomposition of the displacements

Again, comparing the estimates for each field in Table 5.1 and the ones concerning their difference (5.36)-(5.37) for $h = 2$ and (5.44), we need to be careful on how we combine the final displacement fields, especially in the in-plane components.

7.1.1 ... in the outer-plane component

Looking at the estimates in Table 5.1 for the different contact strengths, we notice that the estimates do not change. Hence, the outer-plane component's final displacement is decomposed in the same way as in subsection 6.1.1 and gives the fields $\mathcal{R}_1$, $\mathcal{R}_2 \in H^1(\mathfrak{G}_\varepsilon)$ and $\mathbb{U}_3 \in H^2(\mathfrak{G}_\varepsilon)$.

7.1.2 ... the in-plane component

Regarding the in-plane component, differently from the case $h = 3$, we know that estimate $\partial_1 \mathbb{U}^{(1)}_2 + \partial_2 \mathbb{U}^{(2)}_1$ has the same order as the estimate for the fields in the unsupported areas and therefore the Korn's inequality would not lead to an improvement of the fields estimates from the clamped subdomains to the rest of the square, as we have seen in the proof of Proposition (9).

Hence, we have a contrast in the estimates for the in-plane fields between the clamped subdomains and the not clamped ones, leading to an anisotropic behavior of the fields in Ω_1-Ω_4.

In order to find a suitable in-plane decomposition, we proceed as in the "very loose" contact case and start by considering the directions separately: we define the functions $\mathbb{U}_1^{(\alpha)}$, $\mathbb{U}_2^{(\alpha)}$ and $\mathcal{R}_3^{(\alpha)}$ by

$$\begin{aligned} &\mathbb{U}_1^{(1)} = \mathbb{U}_1^{\prime(1)}, \quad \mathbb{U}_2^{(1)} = \mathbb{U}_2^{\prime(1)}, \quad \mathcal{R}_3^{(1)} = \mathcal{R}_3^{\prime(1)} \quad \text{a.e. } \mathfrak{G}_\varepsilon^{(1)},\\ &\mathbb{U}_1^{(2)} = \mathbb{U}_1^{\prime(2)}, \quad \mathbb{U}_2^{(2)} = \mathbb{U}_2^{\prime(2)}, \quad \mathcal{R}_3^{(2)} = \mathcal{R}_3^{\prime(2)} \quad \text{a.e. } \mathfrak{G}_\varepsilon^{(2)}. \end{aligned}$$

Then, taking into account the clamping conditions (5.33), we replace

$$\begin{aligned} &\mathbb{U}_1^{\prime(\alpha)}(0,q\varepsilon),\ \ \mathbb{U}_2^{\prime(\alpha)}(0,q\varepsilon) \ \text{ and } \ \mathcal{R}_3^{\prime(\alpha)}(0,q\varepsilon) \quad \text{by } \ 0 \ \text{ if } \ q \in \{0,\dots,2n_\varepsilon\},\\ &\mathbb{U}_1^{\prime(\alpha)}(p\varepsilon,0),\ \ \mathbb{U}_2^{\prime(\alpha)}(p\varepsilon,0) \ \text{ and } \ \mathcal{R}_3^{\prime(\alpha)}(p\varepsilon,0) \quad \text{by } \ 0 \ \text{ if } \ q \in \{0,\dots,2n_\varepsilon\}. \end{aligned}$$

Note that, due to (5.16) in the respective direction, these fields vanish on the clamped points of $\mathfrak{G}_\varepsilon$ and satisfy equalities $\partial_1 \mathbb{U}_2^{(1)} = \mathcal{R}_3^{(1)}$ a.e. in $\mathfrak{G}_\varepsilon^{(1)}$ and $\partial_2 \mathbb{U}_1^{(2)} = -\mathcal{R}_3^{(2)}$ a.e. in $\mathfrak{G}_\varepsilon^{(2)}$.

Corollary 7. *The in-plane rotation fields satisfy the following:*

$$\|\mathcal{R}_3^{(1)}\|_{H^1(\mathfrak{G}_\varepsilon^{(1)})} + \|\mathcal{R}_3^{(2)}\|_{H^1(\mathfrak{G}_\varepsilon^{(2)})} \le C\Big(\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|_{T_\varepsilon}\Big). \tag{7.3}$$

The in-plane middle line fields satisfy

$$\begin{aligned} &\|\mathbb{U}_2^{(1)}\|_{H^2(\mathfrak{G}^{(1)})} + \|\mathbb{U}_1^{(2)}\|_{H^2(\mathfrak{G}^{(2)})} \le C\Big(\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|_{T_\varepsilon}\Big),\\ &\|\mathbb{U}_1^{(1)}\|_{H^1(\mathfrak{G}^{(1)})} + \|\mathbb{U}_2^{(2)}\|_{H^1(\mathfrak{G}^{(2)})} \le C\Big(\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|_{T_\varepsilon}\Big). \end{aligned} \tag{7.4}$$

Moreover, in the clamped subdomains, we have

$$\begin{aligned} &\|\mathbb{U}_1^{(1)}\|_{L^2(\mathfrak{G}_\varepsilon^{(1)}\cap(\Omega_1\cup\Omega_2))} + \|\mathbb{U}_2^{(2)}\|_{L^2(\mathfrak{G}_\varepsilon^{(2)}\cap(\Omega_1\cup\Omega_3))} \le \frac{C}{\varepsilon}\|u\|_{T_\varepsilon},\\ &\|\mathbb{U}_2^{(1)}\|_{L^2(\mathfrak{G}_\varepsilon^{(1)}\cap(\Omega_1\cup\Omega_3))} + \|\mathbb{U}_1^{(2)}\|_{L^2(\mathfrak{G}_\varepsilon^{(2)}\cap(\Omega_1\cup\Omega_2))} \le C\Big(\varepsilon\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big) \end{aligned}$$

Proof. By construction, the in-plane fields are the same as the ones of the prime decomposition. Hence, the proof follows directly from Proposition 7 and Corollary 5 for $h = 2$. □

Now that the fields are set, we can construct the Bernoulli-Navier displacements $\mathbb{U}_{BN}^{(\alpha)}$ by

$$U_{BN}^{(1)}(z_1,q\varepsilon,y_2,y_3) \doteq \begin{pmatrix} \mathbb{U}_1^{(1)}\\ \mathbb{U}_2^{(1)}\\ \mathbb{U}_3 \end{pmatrix}(z_1,q\varepsilon) + \begin{pmatrix} \mathcal{R}_1\\ \mathcal{R}_2\\ \mathcal{R}_3^{(1)} \end{pmatrix}(z_1,q\varepsilon) \wedge \big(\Phi_\varepsilon^{(1)}(z_1,q\varepsilon)\mathbf{e}_3 + y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon^{(1)}(z_1,q\varepsilon)\big),$$

$$\text{for a.e. } (z_1,q\varepsilon,y_2,y_3) \in \mathfrak{G}^{(1)} \times \omega_r,$$

$$U_{BN}^{(2)}(p\varepsilon,z_2,y_1,y_3) \doteq \begin{pmatrix} \mathbb{U}_1^{(2)}\\ \mathbb{U}_2^{(2)}\\ \mathbb{U}_3 \end{pmatrix}(p\varepsilon,z_2) + \begin{pmatrix} \mathcal{R}_1\\ \mathcal{R}_2\\ \mathcal{R}_3^{(1)} \end{pmatrix}(p\varepsilon,z_2) \wedge \big(\Phi_\varepsilon^{(2)}(p\varepsilon,z_2)\mathbf{e}_3 + y_1\mathbf{e}_1 + y_3\mathbf{n}_\varepsilon^{(2)}(p\varepsilon,z_2)\big),$$

$$\text{for a.e. } (p\varepsilon,z_2,y_1,y_3) \in \mathfrak{G}^{(1)} \times \omega_r. \tag{7.5}$$

Again, the residual displacements are

$$\overline{u}^{(\alpha)} = u^{(\alpha)} - \mathbb{U}_{BN}^{(\alpha)} \in H^1(T_\varepsilon^{(\alpha)}),$$

where the warping term satisfies

$$\|\overline{u}^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r)} + \varepsilon\|\nabla\overline{u}^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r)} \le C\varepsilon\|u\|_{T_\varepsilon}. \tag{7.6}$$

As in the previous section, this estimate justifies the choice of the final decomposition since it is of the same order as the residual displacement in the prime decomposition (5.21) and of the classical one (5.7).

7.1.3 Final split of the in-plane "centerline" displacements

In this subsection, we operate a better split of the in-plane middle line fields $\mathbb{U}_\alpha^{(1)}$ and $\mathbb{U}_\alpha^{(2)}$ in order to later better understand how they stretch and bend according to the different subdomains Ω_1-Ω_4.

We recall that a function $\phi \in H^1(\mathfrak{G}_\varepsilon^{(\alpha)})$ is defined in all the nodes of $\mathfrak{G}_\varepsilon$ and thus can be uniquely extended to a function $\Phi \in H^1(\mathfrak{G}_\varepsilon)$ by linear interpolation between two consecutive nodes of the lines in $\mathfrak{G}_\varepsilon^{(3-\alpha)}$. In this sense, from (2.10) and (2.14) for $N = p = 2$, there exist two constants $C_0, C_1 > 0$ such that

$$C_0\big(\|\Phi\|_{L^2(\mathfrak{G}_\varepsilon)} + \varepsilon\|\partial_{3-\alpha}\Phi\|_{L^2(\mathfrak{G}_\varepsilon^{(3-\alpha)})}\big) \le \sqrt{\sum_{(p,q)\in\mathcal{K}_\varepsilon} \varepsilon|\Phi(p\varepsilon,q\varepsilon)|^2} \le C_1\big(\|\Phi\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \varepsilon\|\partial_\alpha\Phi\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})}\big). \tag{7.7}$$

Now, we notice that from estimates (5.36), the definition of $\mathbb{U}_1^{(\alpha)}$ and (7.4), we have

$$\sum_{p=0}^{2N_\varepsilon}\Big(\sum_{q=0}^{2N_\varepsilon} |(\mathbb{U}_1^{(1)} - \mathbb{U}_1^{(2)})(p\varepsilon,q\varepsilon)|^2 + \varepsilon\|\mathbb{U}_1^{(2)}(p\varepsilon,\cdot)\|^2_{H^2(0,L)}\Big) \le C\Big(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{T_\varepsilon}\Big).$$

Then, there exists $\overline{p} \in \{0,\dots,2N_\varepsilon\}$ such that

$$\begin{aligned}
&\sum_{q=0}^{2N_\varepsilon} |(\mathbb{U}_1^{(1)} - \mathbb{U}_1^{(2)})(\overline{p}\varepsilon,q\varepsilon)|^2 + \varepsilon\|\mathbb{U}_1^{(2)}(\overline{p}\varepsilon,\cdot)\|^2_{H^2(0,L)}\\
&\le \frac{1}{2N_\varepsilon+1}\sum_{p=0}^{2N_\varepsilon}\Big(\sum_{q=0}^{2N_\varepsilon} |(\mathbb{U}_1^{(1)} - \mathbb{U}_1^{(2)})(p\varepsilon,q\varepsilon)|^2 + \varepsilon\|\mathbb{U}_1^{(2)}(p\varepsilon,\cdot)\|^2_{H^2(0,L)}\Big)\\
&\le C\varepsilon\Big(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{T_\varepsilon}\Big).
\end{aligned} \tag{7.8}$$

We define the following decomposition of the in-plane fields in direction $\mathbf{e}_1$:

$$\begin{aligned}
&\mathbb{U}_1(z_2) \doteq \mathbb{U}_1^{(2)}(\overline{p}\varepsilon,z_2) && \text{for a.e. } z_2 \in (0,L),\\
&\mathbb{U}_1^{(\mathbf{B})}(p\varepsilon,z_2) \doteq \begin{cases} \mathbb{U}_1^{(2)}(p\varepsilon,z_2) - \mathbb{U}_1(z_2) & \text{for a.e. } z_2 \in (0,L),\ \ p \in \{2n_\varepsilon,\dots,2N_\varepsilon\},\\ \mathbb{U}_1^{(2)}(p\varepsilon,z_2) & \text{for a.e. } z_2 \in (0,L),\ \ p \in \{0,\dots,2n_\varepsilon-1\},\end{cases}\\
&\mathbb{U}_1^{(\mathbf{S})}(z_1,q\varepsilon) \doteq \begin{cases} \mathbb{U}_1^{(1)}(z_1,q\varepsilon) - \mathbb{U}_1(q\varepsilon) & \text{for a.e. } z_1 \in (0,L),\ \ q \in \{2n_\varepsilon,\dots,2N_\varepsilon\},\\ \mathbb{U}_1^{(1)}(z_1,q\varepsilon) & \text{for a.e. } z_1 \in (0,L),\ \ q \in \{0,\dots,2n_\varepsilon-1\}.\end{cases}
\end{aligned}$$

Clearly, there exist also $\overline{q}$ such that a symmetrical formulation of (7.8) holds in the second direction. This allows us to define the in-plane fields in direction $\mathbf{e}_2$ as well:

$$\begin{aligned}
&\mathbb{U}_2(z_1) \doteq \mathbb{U}_2^{(1)}(z_1,\overline{q}\varepsilon) && \text{for a.e. } z_1 \in (0,L),\\
&\mathbb{U}_2^{(\mathbf{B})}(z_1,q\varepsilon) \doteq \begin{cases} \mathbb{U}_2^{(1)}(z_1,q\varepsilon) & \text{for a.e. } z_1 \in (0,L),\ \ q \in \{0,\dots,2n_\varepsilon-1\},\\ \mathbb{U}_2^{(1)}(z_1,q\varepsilon) - \mathbb{U}_2(z_1) & \text{for a.e. } z_1 \in (0,L),\ \ q \in \{2n_\varepsilon,\dots,2N_\varepsilon\},\end{cases}\\
&\mathbb{U}_2^{(\mathbf{S})}(p\varepsilon,z_2) \doteq \begin{cases} \mathbb{U}_2^{(2)}(p\varepsilon,z_2) & \text{for a.e. } z_2 \in (0,L),\ \ p \in \{0,\dots,2n_\varepsilon-1\},\\ \mathbb{U}_2^{(2)}(p\varepsilon,z_2) - \mathbb{U}_2(p\varepsilon) & \text{for a.e. } z_2 \in (0,L),\ \ p \in \{2n_\varepsilon,\dots,2N_\varepsilon\}.\end{cases}
\end{aligned}$$

Below, we estimate the newly defined fields.

Proposition 10. *The in-plane component fields satisfy the following estimates:*

$$\begin{aligned} &\|\mathbb{U}_\alpha\|_{H^2(0,L)} \leq C\Big(\varepsilon\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon\sqrt{\varepsilon}}\|u\|_{T_\varepsilon}\Big),\\ &\|\mathbb{U}^{(\mathbf{S})}_\alpha\|_{H^1(\mathfrak{G}^{(\alpha)}_\varepsilon)} \leq C\Big(\varepsilon\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big),\\ &\|\mathbb{U}^{(\mathbf{B})}_\alpha\|_{H^2(\mathfrak{G}^{(3-\alpha)}_\varepsilon)} \leq C\Big(\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|_{T_\varepsilon}\Big). \end{aligned} \tag{7.9}$$

Moreover, we have the following improvements of some L^2 norms:

$$\begin{aligned} &\|\mathbb{U}_\alpha\|_{L^2(0,l)} \leq C\Big(\varepsilon^2\|g\|_{L^\infty(\Omega)} + \frac{1}{\sqrt{\varepsilon}}\|u\|_{T_\varepsilon}\Big),\\ &\|\mathbb{U}^{(\mathbf{B})}_\alpha\|_{L^2(\mathfrak{G}^{(3-\alpha)}_\varepsilon)} \leq C\Big(\varepsilon\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{T_\varepsilon}\Big). \end{aligned} \tag{7.10}$$

Proof. We will only prove the proposition for $\alpha = 1$. The case $\alpha = 2$ will follow by a symmetrical argumentation.
We start with estimates (7.9).
Estimate $(7.9)_1$ follows from the definition of $\mathbb{U}_1$ and estimate (7.8). From estimates (7.7), (7.8), $(7.4)_2$ and thedefinition of $\mathbb{U}^{(\mathbf{S})}_1$, we have

$$\begin{aligned} &\|\mathbb{U}^{(\mathbf{S})}_1\|^2_{H^1(\mathfrak{G}^{(1)}_\varepsilon)} \leq \|\mathbb{U}^{(1)}_1 - \mathbb{U}_1\|^2_{L^2(\mathfrak{G}^{(1)}_\varepsilon)} + \|\partial_1\mathbb{U}^{(1)}_1\|^2_{L^2(\mathfrak{G}^{(1)}_\varepsilon)}\\ &\leq C\Big(\varepsilon\sum_{p=0}^{2N_\varepsilon}\sum_{q=0}^{2N_\varepsilon}|\mathbb{U}^{(1)}_1(p\varepsilon,q\varepsilon) - \mathbb{U}_1(q\varepsilon)|^2 + \varepsilon^2\|\partial_1\mathbb{U}^{(1)}_1(\cdot,q\varepsilon)\|^2_{L^2(0,L)} + \|\partial_1\mathbb{U}^{(1)}_1(\cdot,q\varepsilon)\|^2_{L^2(0,L)}\Big)\\ &\leq C\varepsilon\Big(\sum_{p=0}^{2N_\varepsilon}\sum_{q=0}^{2N_\varepsilon}|\mathbb{U}^{(1)}_1(p\varepsilon,q\varepsilon) - \mathbb{U}^{(1)}_1(\overline{p}\varepsilon,q\varepsilon)|^2 + (2N_\varepsilon+1)\sum_{q=0}^{2N_\varepsilon}|\mathbb{U}^{(1)}_1(\overline{p}\varepsilon,q\varepsilon) - \mathbb{U}_1(q\varepsilon)|^2\Big)\\ &\qquad + C\|\partial_1\mathbb{U}^{(1)}_1(\cdot,q\varepsilon)\|^2_{L^2(0,L)}\\ &\leq C\sum_{q=0}^{2N_\varepsilon}|\mathbb{U}^{(1)}_1(\overline{p}\varepsilon,q\varepsilon) - \mathbb{U}_1(q\varepsilon)|^2 + C\|\partial_1\mathbb{U}^{(1)}_1(\cdot,q\varepsilon)\|^2_{L^2(0,L)} \leq C\varepsilon\Big(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{T_\varepsilon}\Big). \end{aligned} \tag{7.11}$$

which proves $(7.9)_2$. From (7.4) and $(7.9)_1$, we have

$$\begin{aligned} \|\mathbb{U}^{(\mathbf{B})}_1\|^2_{H^2(\mathfrak{G}^{(1)}_\varepsilon)} &\leq \|\mathbb{U}^{(2)}_1 - \mathbb{U}_1\|^2_{H^2(\mathfrak{G}^{(2)}_\varepsilon)} \leq 2\sum_{p=0}^{2N_\varepsilon}\Big(\|\mathbb{U}^{(2)}_1(p\varepsilon,\cdot)\|^2_{H^2(0,L)} + \|\mathbb{U}_1\|^2_{H^2(0,L)}\Big)\\ &\leq C\Big(\varepsilon\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^4}\|u\|^2_{T_\varepsilon}\Big) + C(2N_\varepsilon+1)\Big(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{T_\varepsilon}\Big), \end{aligned} \tag{7.12}$$

which gives $(7.9)_3$.

Now we prove estimates (7.10). From $(7.4)_1$ and the fact that$\mathbb{U}^{(1)}_1(0,q\varepsilon) = 0$ for every $q \in \{0,\dots,2n_\varepsilon\}$ gives

$$\sum_{q=0}^{2n_\varepsilon}|\mathbb{U}^{(1)}_1(\overline{p}\varepsilon,q\varepsilon)|^2 \leq L\sum_{q=0}^{2n_\varepsilon}\|\partial_1\mathbb{U}^{(1)}_1(\cdot,q\varepsilon)\|^2_{L^2(0,L)} \leq \frac{C}{\varepsilon^2}\|u\|^2_{T_\varepsilon}.$$

This, together with (7.8), yields

$$\sum_{q=0}^{2n_\varepsilon}|\mathbb{U}_1(q\varepsilon)|^2 \leq C\varepsilon\Big(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{T_\varepsilon}\Big).$$

Hence, from estimates (7.7) and $(7.9)_1$ we get

$$\|\mathbb{U}_1\|^2_{L^2(0,l)} \leq C\Big(\sum_{q=0}^{2n_\varepsilon} \varepsilon|\mathbb{U}_1(q\varepsilon)|^2 + \varepsilon^2\|\partial_2\mathbb{U}_1\|^2_{L^2(0,l)}\Big) \leq C\varepsilon^2\Big(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{T_\varepsilon}\Big),$$

which proves $(7.10)_1$. Now, note that $(\mathbb{U}_1^{(\mathbf{B})} - \mathbb{U}_1^{(\mathbf{S})})(p\varepsilon, q\varepsilon) = (\mathbb{U}_1^{(2)} - \mathbb{U}_1^{(1)})(p\varepsilon, q\varepsilon)$ for all $(p,q) \in \mathcal{K}_\varepsilon$. Hence, from the in-plane contact estimates (5.36), we obtain

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} |(\mathbb{U}_1^{(\mathbf{B})} - \mathbb{U}_1^{(\mathbf{S})})(p\varepsilon, q\varepsilon)|^2 \leq C\Big(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{T_\varepsilon}\Big).$$

Then, estimate $(7.9)_2$ and (7.7) yield

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \varepsilon|\mathbb{U}_1^{(\mathbf{S})}(p\varepsilon, q\varepsilon)|^2 \leq C\|\mathbb{U}_1^{(\mathbf{S})}\|^2_{H^1(\mathfrak{G}_\varepsilon^{(1)})} \leq C\Big(\varepsilon^3\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|^2_{T_\varepsilon}\Big).$$

So

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \varepsilon|\mathbb{U}_1^{(\mathbf{B})}(p\varepsilon, q\varepsilon)|^2 \leq C\varepsilon\Big(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{T_\varepsilon}\Big).$$

Finally, we obtain $(7.10)_2$ from (7.7) and $(7.9)_3$. □

We end this section by giving the final decomposition of the Bernoulli-Navier displacements (7.5), together with the decomposition of the in-plane fields:

$$\begin{aligned}
U^{(1)}_{BN}(z_1, q\varepsilon, y_2, y_3) &= \begin{pmatrix} \mathbb{U}_1 + \mathbb{U}_1^{(\mathbf{S})} \\ \mathbb{U}_2 + \mathbb{U}_2^{(\mathbf{B})} \\ \mathbb{U}_3 \end{pmatrix}(z_1, q\varepsilon) + \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3^{(1)} \end{pmatrix}(z_1, q\varepsilon) \wedge \big(\Phi_\varepsilon^{(1)}(z_1, q\varepsilon)\mathbf{e}_3 + y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon^{(1)}(z_1, q\varepsilon)\big), \\
&\qquad \text{for a.e. } (z_1, q\varepsilon, y_2, y_3) \in \mathfrak{G}^{(1)} \times \omega_r, \\
U^{(2)}_{BN}(p\varepsilon, z_2, y_1, y_3) &= \begin{pmatrix} \mathbb{U}_1 + \mathbb{U}_1^{(\mathbf{B})} \\ \mathbb{U}_2 + \mathbb{U}_2^{(\mathbf{S})} \\ \mathbb{U}_3 \end{pmatrix}(p\varepsilon, z_2) + \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3^{(1)} \end{pmatrix}(p\varepsilon, z_2) \wedge \big(\Phi_\varepsilon^{(2)}(p\varepsilon, z_2)\mathbf{e}_3 + y_1\mathbf{e}_1 + y_3\mathbf{n}_\varepsilon^{(2)}(p\varepsilon, z_2)\big), \\
&\qquad \text{for a.e. } (p\varepsilon, z_2, y_1, y_3) \in \mathfrak{G}^{(1)} \times \omega_r.
\end{aligned} \tag{7.13}$$

7.2 The sufficient applied stress to stay in a linear regime

As we already mentioned in Section 5.7, the assumption of linear elasticity is related to the estimate of the strain tensor of the displacement, which is given in (5.44).
Since such an estimate is determined by the stress applied to the right-hand side of the problem (5.30), we dedicate this section to the sufficient forces to apply so to obtain estimate (5.44) and stay on a linear regime.

From property (iii) of tensor $A_\varepsilon^{(\alpha)}$ applied to problem (5.29) with $v_\varepsilon^{(\alpha)} = 0$, there exists $C_0 > 0$ such that

$$\begin{aligned}
C_0\|u_\varepsilon\|^2_{T_\varepsilon} &\leq \sum_{\alpha=1}^{2} \int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} A^{(\alpha)}_{ijkl,\varepsilon}\, \widetilde{e}^{(\alpha)}_{ij}(u_\varepsilon^{(\alpha)})\, \widetilde{e}^{(\alpha)}_{kl}(u_\varepsilon^{(\alpha)})\, \eta_\varepsilon^{(\alpha)}\, dz_\alpha dy_{3-\alpha} dy_3 \\
&\leq \sum_{\alpha=1}^{2} \Big| \int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} F_\varepsilon^{(\alpha)} \cdot u_\varepsilon^{(\alpha)} \eta_\varepsilon^{(\alpha)}\, dz_\alpha dy_{3-\alpha} dy_3 \Big|.
\end{aligned} \tag{7.14}$$

Let $f^{(\alpha)} \in H^1(\Omega)^3$ and $\widetilde{f}^{(\alpha)} \in H^1(\Omega)^2$, such that

$$\widetilde{f}_1^{(\alpha)} = 0 \quad \text{a.e. in } \Omega_3 \cup \Omega_4, \qquad \widetilde{f}_2^{(\alpha)} = 0 \quad \text{a.e. in } \Omega_2 \cup \Omega_4.$$

We choose the forces for the right-hand side of problem (5.30) by setting

$$F_\varepsilon^{(1)} \doteq \varepsilon^{3/2}\begin{pmatrix} \widetilde{f}_1^{(1)} + \varepsilon f_1^{(1)} \\ \widetilde{f}_2^{(1)} + \varepsilon f_2^{(1)} \\ \varepsilon f_3^{(1)} \end{pmatrix} \quad \text{a.e. in } \mathfrak{G}_\varepsilon^{(1)}, \qquad F_\varepsilon^{(2)} \doteq \varepsilon^{3/2}\begin{pmatrix} \widetilde{f}_1^{(2)} + \varepsilon f_1^{(2)} \\ \widetilde{f}_2^{(2)} + \varepsilon f_2^{(2)} \\ \varepsilon f_3^{(2)} \end{pmatrix} \quad \text{a.e. in } \mathfrak{G}_\varepsilon^{(2)}.$$

The Hölder inequality, straightforward computation and estimates in Proposition 7 and 10 lead to

$$\begin{aligned} &\sum_{\alpha=1}^{2} \int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} |F_\varepsilon^{(\alpha)}|\,|u_\varepsilon^{(\alpha)}\eta_\varepsilon^{(\alpha)}|\,dz_\alpha dy_{3-\alpha} dy_3 \\ &\le C\varepsilon^5 \sum_{\alpha=1}^{2}\left(\|\widetilde{f}^{(\alpha)}\|_{H^1(\Omega)} + \|f^{(\alpha)}\|_{H^1(\Omega)}\right)\left(\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2\sqrt{\varepsilon}}\|u_\varepsilon\|_{T_\varepsilon}\right), \end{aligned}$$

which, together with (7.14), gives the desired estimate (5.44).

7.3 Weak convergence of the displacement fields via unfolding

We apply (5.44) to the estimates in Propositions 8 and 10 and extend the ones defined on lines to the whole grid by the meanings of (7.7) (with abuse of notation, we will call them the same way). hence, the explicit estimates for the final decomposition of the displacement (7.13) are

$$\begin{aligned} &\|\mathbb{U}_{\varepsilon,3}\|_{H^2(\mathfrak{G}_\varepsilon)} \le C\sqrt{\varepsilon}, \qquad \|\mathcal{R}_{\varepsilon,\alpha}\|_{H^1(\mathfrak{G}_\varepsilon)} + \|\mathcal{R}_{\varepsilon,3}^{(\alpha)}\|_{H^1(\mathfrak{G}_\varepsilon^{(\alpha)})} \le C\sqrt{\varepsilon}, \\ &\|\mathbb{U}_{\varepsilon,\alpha}\|_{H^2(0,L)} \le C\varepsilon, \quad \|\mathbb{U}_{\varepsilon,\alpha}\|_{L^2(0,l)} \le C\varepsilon^2, \quad \|\mathbb{U}_{\varepsilon,\alpha}^{(\mathbf{S})}\|_{H^1(\mathfrak{G}_\varepsilon^{(\alpha)})} \le C\varepsilon\sqrt{\varepsilon}, \\ &\|\mathbb{U}_{\varepsilon,\alpha}^{(\mathbf{B})}\|_{L^2(\mathfrak{G}_\varepsilon)} + \varepsilon\|\partial_\beta \mathbb{U}_{\varepsilon,\alpha}^{(\mathbf{B})}\|_{L^2(\mathfrak{G}_\varepsilon)} + \varepsilon\|\partial^2_{3-\alpha\,3-\alpha}\mathbb{U}_{\varepsilon,\alpha}^{(\mathbf{B})}\|_{L^2(\mathfrak{G}_\varepsilon^{(3-\alpha)})} \le C\varepsilon\sqrt{\varepsilon}, \end{aligned} \tag{7.15}$$

while the ones for the residual terms come from (7.6):

$$\|\overline{u}^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r)} + \varepsilon\|\nabla\overline{u}^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r)} \le C\varepsilon^3\sqrt{\varepsilon}. \tag{7.16}$$

It is known that by compactness, these fields weakly converge in the space. In the next subsections, we will introduce the unfolding operators and go to the limit via unfolding.

7.3.1 The unfolding operators for a textile with contact sliding

The convergence via unfolding is done through three different unfolding operators, all related to each other:

- The middle line unfolding operator $\mathcal{T}_\varepsilon^{\mathfrak{G}}$, which unfolds the functions defined on the one-dimensional lattice $\mathfrak{G}_\varepsilon$, given by the middle lines of the displacements;
- The global unfolding operator Π_ε, which unfolds the functions defined on the whole three-dimensional textile structure T_ε;
- The contact unfolding operator $\mathcal{T}_\varepsilon^{\mathbf{C}_{ab}}$ (for $(a,b)\in\{0,1\}^2$), which unfolds the functions defined on the four two-dimensional contact domains of the reference cell $\mathcal{Y}$.

This subsection introduces the first and most important middle line unfolding operator and its properties.
Define the reference lattice grid by ($(a,b)\in\{1,2\}^2$)

$$\mathfrak{G}^{(1)} \doteq (0,2)\times\{0,1\}, \qquad \mathfrak{G}^{(2)} \doteq \{0,1\}\times(0,2), \qquad \mathfrak{G} \doteq \mathfrak{G}^{(1)}\cup\mathfrak{G}^{(2)}.$$

Definition 11 (Middle line unfolding operator). *For every measurable function ϕ on $\mathfrak{G}_\varepsilon$, one defines the measurable function $\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi)$ in $\Omega\times\mathfrak{G}$ by*

$$\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi)(z',Y_1,Y_2)\doteq\phi\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big]+\varepsilon(Y_1\mathbf{e}_1+Y_2\mathbf{e}_2)\Big)\qquad \textit{for a.e.}\ \ (z',Y_1,Y_2)\in\Omega\times\mathfrak{G}.$$

Note that this operator is defined on the periodic grid $\mathfrak{G}_\varepsilon$, which is, in fact, a lattice in $\Omega\subset\mathbb{R}^2$. Hence, we can recall the results of Chapter 4 for this specific structure.
From Proposition 5 for $N=2$ and $p=2$, such an operator satisfies

$$\|\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi)\|_{L^2(\Omega\times\mathfrak{G})}\leq C\sqrt{\varepsilon}\|\phi\|_{L^2(\mathfrak{G}_\varepsilon)},\qquad \forall\phi\in L^2(\mathfrak{G}_\varepsilon).$$

We have the following corollaries.

Corollary 8 (Adaptation of Lemma 12). *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $H^1(\mathfrak{G}_\varepsilon)$, satisfying*

$$\|\phi_\varepsilon\|_{L^2(\mathfrak{G}_\varepsilon)}+\varepsilon\big(\|\partial_1\phi_\varepsilon\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})}+\|\partial_2\phi_\varepsilon\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})}\big)\leq\frac{C}{\sqrt{\varepsilon}}.$$

There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\widehat{\phi}\in L^2(\Omega;H^1_{per}(\mathfrak{G}))$ such that

$$\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi_\varepsilon)\rightharpoonup\widehat{\phi}\quad \textit{weakly in}\quad L^2(\Omega;H^1(\mathfrak{G})).$$

Corollary 9 (Adaptation of Lemma 13). *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $H^1(\mathfrak{G}_\varepsilon)$, satisfying*

$$\|\phi_\varepsilon\|_{H^1(\mathfrak{G}_\varepsilon)}\leq\frac{C}{\sqrt{\varepsilon}}.$$

There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\phi\in H^1(\Omega)$, and $\widehat{\phi}\in L^2(\Omega;H^1_{per,0}(\mathfrak{G}))$, such that ($\alpha\in\{1,2\}$)

$$\begin{aligned}&\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi_\varepsilon)\to\phi\quad \textit{strongly in}\quad L^2(\Omega;H^1(\mathfrak{G})),\\ &\mathcal{T}_\varepsilon^{\mathfrak{G}}(\partial_\alpha\phi_\varepsilon)\rightharpoonup\partial_\alpha\phi+\partial_{Y_\alpha}\widehat{\phi}\quad \textit{weakly in}\quad L^2(\Omega\times\mathfrak{G}^{(\alpha)}).\end{aligned}$$

Similar to the spaces defined in Section 3.1, we set (here $N_1=1$ and $N_2=1$)

$$\begin{aligned}&L^2(\Omega,\partial_\alpha;H^1_{per}(\mathfrak{G}^{(3-\alpha)}))\doteq\\ &\big\{\phi\in L^2(\Omega\times\mathfrak{G}^{(3-\alpha)})\mid\partial_\alpha\phi\in L^2(\Omega\times\mathfrak{G}^{(3-\alpha)})\text{ and }\phi\in L^2(\Omega;H^1_{per}(\mathfrak{G}^{(3-\alpha)}))\big\}.\end{aligned}$$

We have the following adaptation for the anisotropically bounded functions on lattices.

Corollary 10 (Adaptation of Lemma 14). *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $H^1(\mathfrak{G}_\varepsilon)$ and satisfying ($\alpha\in\{1,2\}$)*

$$\|\phi_\varepsilon\|_{L^2(\mathfrak{G}_\varepsilon)}+\|\partial_\alpha\phi_\varepsilon\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})}+\varepsilon\|\partial_{3-\alpha}\phi_\varepsilon\|_{L^2(\mathfrak{G}_\varepsilon^{(3-\alpha)})}\leq\frac{C}{\sqrt{\varepsilon}}.$$

There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\widetilde{\phi}\in L^2(\Omega,\partial_\alpha;H^1_{per}(\mathfrak{G}^{(3-\alpha)}))$ and $\widehat{\phi}\in L^2(\Omega\times\mathfrak{G}^{(3-\alpha)};H^1_{per,0}(\mathfrak{G}^{(\alpha)}))\cap L^2(\Omega;H^1_{per}(\mathfrak{G}))$, such that

$$\begin{aligned}&\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi_\varepsilon)\rightharpoonup\widetilde{\phi}\quad \textit{weakly in}\quad L^2(\Omega;H^1(\mathfrak{G})),\\ &\mathcal{T}_\varepsilon^{\mathfrak{G}}(\partial_\alpha\phi_\varepsilon)\rightharpoonup\partial_\alpha\widetilde{\phi}+\partial_{Y_\alpha}\widehat{\phi}\quad \textit{weakly in}\quad L^2(\Omega\times\mathfrak{G}^{(\alpha)}).\end{aligned}$$

Since we are in two dimensions, we can explicitly write the extension of the field's derivatives (4.25) in the hypothesis of the corollary below.

Corollary 11 (Adaptation of Theorem 2). *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $H^2(\mathfrak{G}_\varepsilon)$, satisfying*

$$\|\phi_\varepsilon\|^2_{H^1(\mathfrak{G}_\varepsilon)} + \sum_{p=0}^{2N_\varepsilon}\sum_{q=0}^{2N_\varepsilon-1} \varepsilon\Big|\frac{\partial_1\phi_\varepsilon(p\varepsilon, q\varepsilon+\varepsilon) - \partial_1\phi_\varepsilon(p\varepsilon, q\varepsilon)}{\varepsilon}\Big|^2 + \sum_{p=0}^{2N_\varepsilon-1}\sum_{q=0}^{2N_\varepsilon} \varepsilon\Big|\frac{\partial_2\phi_\varepsilon(p\varepsilon+\varepsilon, q\varepsilon) - \partial_2\phi_\varepsilon(p\varepsilon, q\varepsilon)}{\varepsilon}\Big|^2 \le \frac{C}{\varepsilon}.$$

There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\phi \in H^2(\Omega)$, $\widehat{\phi} \in L^2(\Omega; H^2_{per}(\mathfrak{G}))$, such that ($\alpha \in \{1,2\}$)

$$\begin{aligned}
\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi_\varepsilon) &\to \phi \quad \text{strongly in} \quad L^p(\Omega; H^2(\mathfrak{G})),\\
\mathcal{T}_\varepsilon^{\mathfrak{G}}(\partial_\alpha\phi_\varepsilon) &\to \partial_\alpha\phi \quad \text{strongly in} \quad L^2(\Omega; H^1(\mathfrak{G}^{(\alpha)})),\\
\mathcal{T}_\varepsilon^{\mathfrak{G}}(\partial_{\alpha\alpha}\phi_\varepsilon) &\rightharpoonup \partial^2_{\alpha\alpha}\phi + \partial^2_{Y_\alpha Y_\alpha}\widehat{\phi} \quad \text{weakly in} \quad L^2(\Omega\times\mathfrak{G}^{(\alpha)}).
\end{aligned}$$

Now, set the 2-periodic reference cell $\mathcal{Y} \doteq (0,2)^2$. Below, we also adapt the definition of the classical unfolding operator to this structure. Note that by construction, from (4.1), we have $\widehat{\Omega}_\varepsilon = \widetilde{\Omega}_\varepsilon = \Omega$ and $\lambda_\varepsilon = \emptyset$.

Definition 12 (Adaptation from Definition 1). *For every measurable function ϕ in Ω, one defines the measurable function $\mathcal{T}_\varepsilon(\phi)$ in $\Omega\times\mathcal{Y}$ by*

$$\mathcal{T}_\varepsilon(z', Y_1, Y_2) \doteq \phi\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big] + \varepsilon(Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2)\Big) \qquad \text{for a.e. } (z', Y_1, Y_2) \in \Omega\times\mathcal{Y}.$$

As we know from diagram (4.6) in Chapter 4, if $\phi \in H^1(\Omega)$, then

$$\mathcal{T}_\varepsilon(\phi)_{|\Omega\times\mathfrak{G}} = \mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi_{|\mathfrak{G}_\varepsilon}). \tag{7.17}$$

7.3.2 Limit displacement fields via the middle line unfolding operator

We prepare the ground for the weak convergences of the fields via the middle line unfolding operator. We first define the limit boundary condition

$$\mathbf{\Gamma} \doteq \{0\}\times(0,l)\cup(0,l)\times\{0\}.$$

Then, we set the limit spaces

$$\begin{aligned}
H^1_{\mathbf{\Gamma}}(\Omega) &\doteq \big\{\phi \in H^1(\Omega) \ \big|\ \phi = 0 \ \text{ a.e. on } \mathbf{\Gamma}\big\},\\
H^2_{\mathbf{\Gamma}}(\Omega) &\doteq \big\{\phi \in H^2(\Omega) \ \big|\ \phi = 0 \ \text{ and } \ \nabla\phi = 0 \ \text{ a.e. on } \mathbf{\Gamma}\big\}
\end{aligned} \tag{7.18}$$

and

$$\begin{aligned}
L^2_{(0,l)}(0,L) &\doteq \big\{\phi \in L^2(0,L) \ \big|\ \phi = 0 \ \text{ a.e. in } (0,l)\big\},\\
H^1_{(0,l)}(0,L) &\doteq \big\{\phi \in H^1(0,L) \ \big|\ \phi = 0 \ \text{ a.e. in } (0,l)\big\},\\
H^2_{(0,l)}(0,L) &\doteq \big\{\phi \in H^2(0,L) \ \big|\ \phi = 0 \ \text{ a.e. in } (0,l)\big\}.
\end{aligned} \tag{7.19}$$

We also define the limit spaces of anisotropic functions

$$\begin{aligned}
L^2(\Omega\times\{0,1\},\partial_\alpha) &\doteq \Big\{\phi \in L^2(\Omega\times\{0,1\}) \mid \partial_\alpha\phi \in L^2(\Omega\times\{0,1\})\Big\},\\
\mathbf{L}^2(\Omega\times\{0,1\},\partial_1) &\doteq \Big\{\phi \in L^2(\Omega\times\{0,1\},\partial_1) \mid \phi = 0 \text{ a.e. on } \{0\}\times(0,l)\Big\},\\
\mathbf{L}^2(\Omega\times\{0,1\},\partial_2) &\doteq \Big\{\phi \in L^2(\Omega\times\{0,1\},\partial_2) \mid \phi = 0 \text{ a.e. on } (0,l)\times\{0\}\Big\}.
\end{aligned} \tag{7.20}$$

We are ready to give the asymptotic behavior of our unfolded sequences.

Lemma 21. *There exist a subsequence of* $\{\varepsilon\}$, *still denoted* $\{\varepsilon\}$, *and functions* $\mathbb{U}_3 \in H^2_{\Gamma}(\Omega)$ *and* $\widehat{\mathbb{U}}_3 \in L^2(\Omega; H^2_{per}(\mathfrak{G}))$ *such that*

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\mathbb{U}_{\varepsilon,3}\big) \to \mathbb{U}_3 \ \text{ strongly in } L^2(\Omega; H^2(\mathfrak{G})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\partial_\alpha\mathbb{U}_{\varepsilon,3}\big) \to \partial_\alpha\mathbb{U}_3 \ \text{ strongly in } L^2(\Omega; H^1(\mathfrak{G}^{(\alpha)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\partial^2_{\alpha\alpha}\mathbb{U}_{\varepsilon,3}\big) \rightharpoonup \partial^2_{\alpha\alpha}\mathbb{U}_3 + \partial^2_{Y_\alpha Y_\alpha}\widehat{\mathbb{U}}_3 \ \text{ weakly in } L^2(\Omega \times \mathfrak{G}^{(\alpha)});
\end{aligned} \tag{7.21}$$

and $\widehat{\mathcal{R}}_\alpha \in L^2(\Omega; H^1_{per,0}(\mathfrak{G}))$ *such that*

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\mathcal{R}_{\varepsilon,1}\big) \to \partial_2\mathbb{U}_3 \ \text{ strongly in } L^2(\Omega; H^1(\mathfrak{G})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\partial_\alpha\mathcal{R}_{\varepsilon,1}\big) \rightharpoonup \partial_{\alpha 2}\mathbb{U}_3 + \partial_{Y_\alpha}\widehat{\mathcal{R}}_1 \ \text{ weakly in } L^2(\Omega; H^1(\mathfrak{G}^{(\alpha)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\mathcal{R}_{\varepsilon,2}\big) \to -\partial_1\mathbb{U}_3 \ \text{ strongly in } L^2(\Omega; H^1(\mathfrak{G})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\partial_\alpha\mathcal{R}_{\varepsilon,2}\big) \rightharpoonup -\partial_{\alpha 1}\mathbb{U}_3 + \partial_{Y_\alpha}\widehat{\mathcal{R}}_2 \ \text{ weakly in } L^2(\Omega; H^1(\mathfrak{G}^{(\alpha)})).
\end{aligned} \tag{7.22}$$

Moreover, we have

$$\partial_{Y_1}\widehat{\mathbb{U}}_3 = -\widehat{\mathcal{R}}_2 \ \text{ a.e. in } \Omega \times \mathfrak{G}^{(1)}, \qquad \partial_{Y_2}\widehat{\mathbb{U}}_3 = \widehat{\mathcal{R}}_1 \ \text{ a.e. in } \Omega \times \mathfrak{G}^{(2)}. \tag{7.23}$$

There exist a subsequence of $\{\varepsilon\}$, *still denoted* $\{\varepsilon\}$, *and functions* $\mathbb{U}_\alpha \in H^2_{(0,l)}\big((0,L)_{z_{3-\alpha}}\big)$, $\widehat{\mathbb{U}}_\alpha$ *in* $L^2((0,L); H^2_{per}((0,2)_{Y_{3-\alpha}}))$ *with* $\widehat{\mathbb{U}}_\alpha(z_{3-\alpha},\cdot) = 0$ *a.e. in* $(0,l) \times (0,2)$ *such that* ($\alpha \in \{1,2\}$)

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\mathbb{U}_{\varepsilon,\alpha}\big) \to \mathbb{U}_\alpha \ \text{ strongly in } L^2(\Omega; H^2(\mathfrak{G}^{(3-\alpha)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\partial_{3-\alpha}\mathbb{U}_{\varepsilon,\alpha}\big) \to \partial_{3-\alpha}\mathbb{U}_\alpha \ \text{ strongly in } L^2(\Omega; H^1(\mathfrak{G}^{(3-\alpha)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\partial^2_{3-\alpha 3-\alpha}\mathbb{U}_{\varepsilon,\alpha}\big) \rightharpoonup \partial^2_{3-\alpha 3-\alpha}\mathbb{U}_\alpha + \partial^2_{Y_{3-\alpha}Y_{3-\alpha}}\widehat{\mathbb{U}}_\alpha \ \text{ weakly in } L^2(\Omega \times \mathfrak{G}^{(3-\alpha)})
\end{aligned} \tag{7.24}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\mathcal{R}^{(1)}_{\varepsilon,3}\big) \to \partial_1\mathbb{U}_2 \ \text{ strongly in } L^2(\Omega; H^1(\mathfrak{G}^{(1)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\partial_1\mathcal{R}^{(1)}_{\varepsilon,3}\big) \rightharpoonup \partial_{11}\mathbb{U}_2 + \partial^2_{Y_1Y_1}\widehat{\mathbb{U}}_2 \ \text{ weakly in } L^2(\Omega; H^1(\mathfrak{G}^{(1)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\mathcal{R}^{(2)}_{\varepsilon,3}\big) \to -\partial_2\mathbb{U}_1 \ \text{ strongly in } L^2(\Omega; H^1(\mathfrak{G}^{(2)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\partial_2\mathcal{R}^{(2)}_{\varepsilon,3}\big) \rightharpoonup -\partial_{22}\mathbb{U}_1 - \partial^2_{Y_2Y_2}\widehat{\mathbb{U}}_1; \ \text{ weakly in } L^2(\Omega; H^1(\mathfrak{G}^{(2)})).
\end{aligned} \tag{7.25}$$

There exist a subsequence of $\{\varepsilon\}$, *still denoted* $\{\varepsilon\}$, *and functions* $\mathbb{U}^{(\mathbf{B})}_\alpha \in L^2(\Omega; H^2_{per}((0,2)_{Y_{3-\alpha}}))$, *such that*

$$\frac{1}{\varepsilon^2}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\mathbb{U}^{(\mathbf{B})}_{\varepsilon,\alpha}\big) \rightharpoonup \mathbb{U}^{(\mathbf{B})}_\alpha \ \text{ weakly in } L^2(\Omega; H^2(\mathfrak{G}^{(3-\alpha)})) \cap L^2(\Omega; H^1(\mathfrak{G})), \tag{7.26}$$

and $\mathbb{U}^{(\mathbf{S})}_\alpha \in L^2(\Omega, \partial_\alpha; H^1_{per}(\mathfrak{G}))$, $\widehat{\mathbb{U}}^{(\mathbf{S})}_\alpha \in L^2(\Omega; H^1_{per}(\mathfrak{G}))$ *such that*

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\mathbb{U}^{(\mathbf{S})}_{\varepsilon,\alpha}\big) \rightharpoonup \mathbb{U}^{(\mathbf{S})}_\alpha \ \text{ weakly in } L^2(\Omega; H^1(\mathfrak{G})),\\
&\frac{1}{\varepsilon^2}\mathcal{T}^{\mathfrak{G}}_{\varepsilon}\big(\partial_\alpha\mathbb{U}^{(\mathbf{S})}_{\varepsilon,\alpha}\big) \rightharpoonup \partial_\alpha\mathbb{U}^{(\mathbf{S})}_\alpha + \partial_{Y_\alpha}\widehat{\mathbb{U}}^{(\mathbf{S})}_\alpha \ \text{ weakly in } L^2(\Omega \times \mathfrak{G}^{(\alpha)}).
\end{aligned} \tag{7.27}$$

Proof. We organize the proof in steps.

Step 1. We prove convergences (7.21)-(7.22).
First, from the estimates concerning sequences $\{\mathcal{R}_{\varepsilon,\alpha}\}_\varepsilon$ in $(7.15)_1$ and Corollary 9, there exist $\mathcal{R}_\alpha \in H^1(\Omega)$ and $\widehat{\mathcal{R}}_\alpha \in L^2(\Omega; H^1_{per,0}(\mathfrak{G}))$ such that the following convergence hold ($(\alpha,\beta) \in \{1,2\}^2$)

$$\begin{aligned}&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon(\mathcal{R}_{\varepsilon,\alpha}) \to \mathcal{R}_\alpha \;\text{ strongly in } L^2(\Omega; H^1(\mathfrak{G})),\\ &\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon(\partial_\beta \mathcal{R}_\alpha) \rightharpoonup \partial_\beta \mathcal{R}\alpha + \partial_{Y_\beta}\widehat{\mathcal{R}}_\alpha \;\text{ weakly in } L^2(\Omega\times\mathfrak{G}^{(\beta)}).\end{aligned} \tag{7.28}$$

Now, we consider the sequence $\{\mathbb{U}_{\varepsilon,3}\}_\varepsilon \in H^2(\mathfrak{G})$. By construction, we have $(\partial_1\mathbb{U}_{\varepsilon,3}, \partial_2\mathbb{U}_{\varepsilon,3}) = (-\mathcal{R}_{\varepsilon,2}, \mathcal{R}_{\varepsilon,1})$. Hence, the derivatives $\partial_\alpha \mathbb{U}_{\varepsilon,3}$ belong to $H^1(\mathfrak{G}^{(\alpha)})$, can be naturally extended by 2-linear interpolation to the whole domain Ω and these extensions are bound by the H^1 norms of $\mathcal{R}_\alpha$. Proceeding as in Subsection 2.2.2, for every ε there exist a unique 2-cubic extension $\mathfrak{Q}(\mathbb{U}_{\varepsilon,3})$ and from estimates $(7.15)_1$, the sequence $\{\mathfrak{Q}(\mathbb{U}_{\varepsilon,3})\}_\varepsilon \in H^2(\Omega)$ satisfies

$$\|\mathfrak{Q}(\mathbb{U}_{\varepsilon,3})\|_{H^2(\Omega)} \le C\sqrt{\varepsilon}\big(\|\mathbb{U}_{\varepsilon,3}\|_{H^2(\mathfrak{G})} + \|\mathcal{R}_{\varepsilon,1}\|_{H^1(\mathfrak{G})} + \|\mathcal{R}_{\varepsilon,2}\|_{H^1(\mathfrak{G})}\big) \le C\varepsilon.$$

Hence, from the proof of Theorem 2 and the boundary conditions, there exist $\mathbb{U}_3 \in H^2_\Gamma(\Omega)$ and $\widehat{\mathfrak{U}}_3 \in L^2(\Omega; H^2_{per}(\mathcal{Y}))$ such that ($(\alpha,\beta) \in \{1,2\}^2$)

$$\begin{aligned}&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\mathfrak{Q}(\mathbb{U}_{\varepsilon,3})) \rightharpoonup \mathbb{U}_3 \quad \text{strongly in } L^2(\Omega; H^2_{per}(\mathcal{Y})),\\ &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\partial_\alpha\mathfrak{Q}(\mathbb{U}_{\varepsilon,3})) \rightharpoonup \partial_\alpha\mathbb{U}_3 \quad \text{strongly in } L^2(\Omega; H^1_{per}(\mathcal{Y})),\\ &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\partial_{\alpha\beta}\mathfrak{Q}(\mathbb{U}_{\varepsilon,3})) \rightharpoonup \partial_{\alpha\beta}\mathbb{U}_3 + \partial_{Y_\alpha Y_\beta}\widehat{\mathfrak{U}}_3 \quad \text{weakly in } L^2(\Omega\times\mathcal{Y}).\end{aligned}$$

Hence, restricting the above convergences to the lattice and setting $\widehat{\mathbb{U}}_3 \doteq \widehat{\mathfrak{U}}_{3|\Omega\times\mathfrak{G}}$, which belongs to $L^2(\Omega; H^2_{per}(\mathfrak{G}))$, we get convergences (7.21), while convergences (7.22) and identities (7.22) follow from the above convergences restricted to the lattice, (7.28) and the fact that in the limit we have $(\partial_1\mathbb{U}_3, \partial_2\mathbb{U}_3) = (-\mathcal{R}_2, \mathcal{R}_1)$.
Step 2. We prove the convergences (7.24)-(7.25)-(7.26).
From estimates $(7.15)_2$ and Proposition 3 applied to one dimension, there exist functions $\mathbb{U}_\alpha \in H^2((0,L)_{3-\alpha})$ and $\widehat{\mathbb{U}}_\alpha \in L^2(\Omega; H^2_{per}(0,2)_{Y_{3-\alpha}})$ such that convergences (7.24) hold. Moreover, again estimates $(7.15)_2$ imply that $\mathbb{U}_\alpha$ vanish on $(0,l)_{3-\alpha}$ and thus they belong to $H^2_{(0,l)}((0,L)_{z_{3-\alpha}})$.
Now, from estimates $(7.15)_1$ and the extension property (7.7), we have that

$$\|\mathcal{R}^{(\alpha)}_{\varepsilon,3}\|_{H^1(\mathfrak{G}^{(\alpha)}_\varepsilon)} + \varepsilon\|\partial_{3-\alpha}\mathcal{R}^{(\alpha)}_{\varepsilon,3}\|_{L^2(\mathfrak{G}^{(3-\alpha)}_\varepsilon)} \le C\sqrt{\varepsilon}.$$

Hence, Corollary 10 implies that there exist functions $\mathcal{R}^{(1)}_3 \in L^2(\Omega, \partial_1; H^1_{per}(\mathfrak{G}^{(2)}))$ and $\widehat{\mathcal{R}}^{(1)}_3 \in L^2(\Omega\times\mathfrak{G}^{(2)}; H^1_{per,0}(\mathfrak{G}^{(1)})) \cap L^2(\Omega; H^1_{per}(\mathfrak{G}))$ such that

$$\begin{aligned}&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon(\mathcal{R}^{(1)}_{\varepsilon,3}) \to \mathcal{R}^{(1)}_3 \;\text{ strongly in } L^2(\Omega; H^1(\mathfrak{G})),\\ &\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon(\partial_1\mathcal{R}^{(1)}_{\varepsilon,3}) \rightharpoonup \partial_1\mathcal{R}^{(1)}_3 + \partial_{Y_1}\widehat{\mathcal{R}}^{(1)}_3 \;\text{ weakly in } L^2(\Omega; H^1(\mathfrak{G}^{(1)})).\end{aligned} \tag{7.29}$$

and $\mathcal{R}^{(2)}_3 \in L^2(\Omega, \partial_2; H^1_{per}(\mathfrak{G}^{(1)}))$ and $\widehat{\mathcal{R}}^{(2)}_3 \in L^2(\Omega\times\mathfrak{G}^{(1)}; H^1_{per,0}(\mathfrak{G}^{(2)})) \cap L^2(\Omega; H^1_{per}(\mathfrak{G}))$ such that

$$\begin{aligned}&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon(\mathcal{R}^{(2)}_{\varepsilon,3}) \to \mathcal{R}^{(2)}_3 \;\text{ strongly in } L^2(\Omega; H^1(\mathfrak{G})),\\ &\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon(\partial_2\mathcal{R}^{(2)}_{\varepsilon,3}) \rightharpoonup \partial_2\mathcal{R}^{(2)}_3 + \partial_{Y_2}\widehat{\mathcal{R}}^{(2)}_3 \;\text{ weakly in } L^2(\Omega; H^1(\mathfrak{G}^{(2)})).\end{aligned} \tag{7.30}$$

From convergences $(7.15)_3$, there exist $\mathbb{U}^{(\mathbf{B})}_\alpha \in L^2(\Omega, H^1_{per}(\mathfrak{G})) \cap L^2(\Omega, H^2_{per}(\mathfrak{G}^{(3-\alpha)}))$ such that (7.26) holds.

Now, by the fact that $(\partial_1 \mathbb{U}^{(1)}_{\varepsilon,2}, -\partial_2 \mathbb{U}^{(2)}_{\varepsilon,1}) = (\mathcal{R}^{(1)}_3, \mathcal{R}^{(2)}_3)$ and the fact that the $\mathbb{U}_\alpha$ vanish on $(0,l)_{3-\alpha}$, we have that

$$\begin{cases} \partial_1 \mathbb{U}^{(\mathbf{B})}_{\varepsilon,2}(\cdot, q\varepsilon) = \mathcal{R}^{(1)}_{\varepsilon,3}(\cdot, q\varepsilon), & \text{a.e. } (0,L)\times\{0,\dots,2n_\varepsilon-1\},\\ \partial_1 \mathbb{U}_{\varepsilon,2}(\cdot) + \partial_1 \mathbb{U}^{(\mathbf{B})}_{\varepsilon,2}(\cdot, q\varepsilon) = \mathcal{R}^{(1)}_{\varepsilon,3}(\cdot, q\varepsilon) & \text{a.e. } (0,L)\times\{2n_\varepsilon,\dots,2N_\varepsilon\},\end{cases}$$
$$\begin{cases} \partial_2 \mathbb{U}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon,\cdot) = \mathcal{R}^{(2)}_{\varepsilon,3}(p\varepsilon,\cdot), & \text{a.e. } (0,L)\times\{0,\dots,2n_\varepsilon-1\},\\ \partial_2 \mathbb{U}_{\varepsilon,1}(\cdot) + \partial_2 \mathbb{U}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon,\cdot) = \mathcal{R}^{(2)}_{\varepsilon,3}(p\varepsilon,\cdot) & \text{a.e. } (0,L)\times\{2n_\varepsilon,\dots,2N_\varepsilon\}.\end{cases} \tag{7.31}$$

Hence, applying the unfolding operator, convergences (7.24)$_2$, (7.26) and (7.29)$_1$-(7.30)$_1$ imply that in the limit we get

$$\partial_1 \mathbb{U}_2 + \partial_{Y_1} \mathbb{U}^{(\mathbf{B})}_2 = \mathcal{R}^{(1)}_3 \quad \text{a.e. } \Omega\times\mathfrak{G}^{(1)} \quad \text{and} \quad \partial_2 \mathbb{U}_1 + \partial_{Y_2} \mathbb{U}^{(\mathbf{B})}_1 = \mathcal{R}^{(2)}_3 \quad \text{a.e. } \Omega\times\mathfrak{G}^{(2)}.$$

Since $\partial_1 \mathbb{U}_2$ and $\mathcal{R}^{(1)}_3$ do not depend on Y_1 and $\mathbb{U}^{(\mathbf{B})}_2$ belongs to $L^2(\Omega; H^2_{per}(\mathfrak{G}^{(1)}))$ and is therefore periodic with respect to Y_1, we get that (the same argumentation holds for $\partial_2 \mathbb{U}_1$, $\mathcal{R}^{(2)}_3$ and $\mathbb{U}^{(\mathbf{B})}_1$ with respect to Y_2)

$$\begin{aligned} &\partial_1 \mathbb{U}_2(z_1) = \mathcal{R}^{(1)}_3(z',b) \quad \text{for a.e. } (z',b)\in\Omega\times\{0,1\} \quad \text{and} \quad \partial_{Y_1}\mathbb{U}^{(\mathbf{B})}_2 = 0 \text{ a.e. in } \Omega\times\mathfrak{G}^{(1)},\\ &\partial_2 \mathbb{U}_1(z_2) = \mathcal{R}^{(2)}_3(z',a) \quad \text{for a.e. } (z',a)\in\Omega\times\{0,1\} \quad \text{and} \quad \partial_{Y_2}\mathbb{U}^{(\mathbf{B})}_1 = 0 \text{ a.e. in } \Omega\times\mathfrak{G}^{(2)}.\end{aligned}$$

As a consequence, the $\mathbb{U}^{(\mathbf{B})}_\alpha$ do not depend on Y_α (thus they belong to $L^2(\Omega; H^2_{per}((0,2)_{Y_{3-\alpha}})))$. Moreover, in the limit holds $(\partial_{11}\mathbb{U}_2, -\partial_{22}\mathbb{U}_1) = (\partial_1\mathcal{R}^{(1)}_3, \partial_2\mathcal{R}^{(2)}_3)$ and thus convergences (7.24)$_2$, (7.26) and (7.29)$_1$-(7.30)$_1$ imply that

$$\begin{aligned} \partial^2_{Y_1Y_1}\widehat{\mathbb{U}}_2 + \partial^2_{Y_1Y_1}\mathbb{U}^{(\mathbf{B})}_2 = \partial^2_{Y_1Y_1}\widehat{\mathbb{U}}_2 = \partial_{Y_1}\widehat{\mathcal{R}}^{(1)}_3 \quad \text{a.e. in } \Omega\times\mathfrak{G}^{(1)},\\ \partial^2_{Y_2Y_2}\widehat{\mathbb{U}}_1 + \partial^2_{Y_2Y_2}\mathbb{U}^{(\mathbf{B})}_1 = \partial^2_{Y_2Y_2}\widehat{\mathbb{U}}_1 = \partial_{Y_2}\widehat{\mathcal{R}}^{(2)}_3 \quad \text{a.e. in } \Omega\times\mathfrak{G}^{(2)}.\end{aligned}$$

Step 3. We prove convergence (7.27).
From estimates (7.15)$_3$ and the extension property (7.7), we have that

$$\|\mathbb{U}^{(GS)}_{\varepsilon,\alpha}\|_{H^1(\mathfrak{G}^{(\alpha)}_\varepsilon)} + \varepsilon\|\partial_{3-\alpha}\mathbb{U}^{(GS)}_{\varepsilon,\alpha}\|_{L^2(\mathfrak{G}^{(3-\alpha)}_\varepsilon)} \le C\varepsilon\sqrt{\varepsilon}.$$

Hence, Corollary 10 implies that there exist functions $\mathbb{U}^{(\mathbf{S})}_\alpha \in L^2(\Omega,\partial_\alpha; H^1_{per}(\mathfrak{G}^{(3-\alpha)}))$ and $\widehat{\mathbb{U}}^{(\mathbf{S})}_\alpha \in L^2(\Omega\times\mathfrak{G}^{(3-\alpha)}; H^1_{per,0}(\mathfrak{G}^{(\alpha)}))\cap L^2(\Omega; H^1_{per}(\mathfrak{G}))$ such that convergences (7.27) hold. Since $\mathbb{U}^{(\mathbf{S})}_\alpha$ belongs to $L^2(\Omega; H^1_{per}(\mathfrak{G}))$, it is affine with respect to $Y_{3-\alpha}$ in $\Omega\times\mathfrak{G}^{(3-\alpha)}$ and is independent of Y_α in $\Omega\times\mathfrak{G}^{(\alpha)}$, we will consider it as a function belonging to $L^2(\Omega\times\{0,1\},\partial_\alpha)$. Moreover, since $\mathbb{U}_{1,\varepsilon}(0,k\varepsilon) = 0$ (resp. $\mathbb{U}_{2,\varepsilon}(k\varepsilon,0) = 0$) for every $k\in\{0,\dots,2n_\varepsilon\}$, the function $\mathbb{U}^{(\mathbf{S})}_1$ (resp. $\mathbb{U}^{(\mathbf{S})}_2$) vanishes on $\{0\}\times(0,l)$ (resp. $(0,l)\times\{0\}$). Thus $\mathbb{U}^{(\mathbf{S})}_\alpha \in \mathbf{L}^2(\Omega\times\{0,1\},\partial_\alpha)$. □

7.3.3 Limit of the strain tensor's fields via global unfolding operator

This operator takes functions that live in the three-dimensional textile structure. It is the operator with which we will go to the limit in problem (5.30), and with which we express the form of the limit of the strain tensors.
We define the three-dimensional reference cell in the respective direction by setting

$$\begin{aligned} Cyl^{(1)} &\doteq \mathfrak{G}^{(1)}\times\omega_\kappa = (0,2)\times\{0,1\}\times(-\kappa,\kappa)^2,\\ Cyl^{(2)} &\doteq \mathfrak{G}^{(2)}\times\omega_\kappa = \{0,1\}\times(0,2)\times(-\kappa,\kappa)^2.\end{aligned}$$

Definition 13. *[Global unfolding operator] For every measurable function Φ on $\mathfrak{G}_\varepsilon^{(1)} \times \omega_r$ and Ψ on $\mathfrak{G}_\varepsilon^{(1)} \times \omega_r$, one defines the measurable functions $\Pi_\varepsilon^{(1)}(\Phi)$ on $\Omega \times Cyl^{(1)}$ and $\Pi_\varepsilon^{(2)}(\Psi)$ on $\Omega \times Cyl^{(2)}$ respectively by*

$$\Pi_\varepsilon^{(1)}(\Phi)(z', Y_1, b, Y_2, Y_3) \doteq \Phi\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big] + \varepsilon Y_1 \mathbf{e}_1 + \varepsilon b\, \mathbf{e}_2 + \varepsilon(Y_2\mathbf{e}_2 + Y_3\mathbf{e}_3)\Big)$$
$$\text{for a.e. } (z', Y_1, b, Y_2, Y_3) \in \Omega \times Cyl^{(1)},$$
$$\Pi_\varepsilon^{(2)}(\Psi)(z', a, Y_2, Y_1, Y_3) \doteq \Psi\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big] + \varepsilon a\, \mathbf{e}_1 + \varepsilon Y_2 + +\varepsilon(Y_1\mathbf{e}_1 + Y_3\mathbf{e}_3)\Big)$$
$$\text{for a.e. } (z', a, Y_2, Y_1, Y_3) \in \Omega \times Cyl^{(2)}.$$

We have the following.

Lemma 22. *For every $\phi \in L^1(\mathfrak{G}_\varepsilon^{(1)} \times \omega_r)$ and $\psi \in L^1(\mathfrak{G}_\varepsilon^{(2)} \times \omega_r)$, we have*

$$\Big|\sum_{q=0}^{2N_\varepsilon-1}\int_{(0,L)\times\omega_r} \phi(z_1, q\varepsilon, y_2, y_3)dz_1dy_2dy_3 - \frac{\varepsilon}{2}\int_\Omega\int_{Cyl^{(1)}} \Pi_\varepsilon^{(1)}(\phi)(z', Y_1, b, Y_2, Y_3)dz'dY\Big|$$
$$\le \int_{(0,L)\times\omega_r} |\phi(z_1, L, y_2, y_3)|dz_1dy_2dy_3,$$
$$\Big|\sum_{p=0}^{2N_\varepsilon-1}\int_{(0,L)\times\omega_r} \psi(p\varepsilon, z_2, y_1, y_3)dz_2dy_1dy_3 - \frac{\varepsilon}{2}\int_\Omega\int_{Cyl^{(2)}} \Pi_\varepsilon^{(2)}(\psi)(z', a, Y_2, Y_1, Y_3)dz'dY\Big|$$
$$\le \int_{(0,L)\times\omega_r} |\psi(L, z_2, y_1, y_3)|dz_2dy_1dy_3.$$

Proof. We consider the unfolding in direction $\mathbf{e}_1$. Then, the statement follows from the fact that by Definition 13, we have in the straight reference frame:

$$\sum_{q=0}^{2N_\varepsilon}\int_{(0,L)\times\omega_r} \phi(z_1, q\varepsilon, y_2, y_3)dz_1dy_2dy_3 = \frac{\varepsilon}{2}\int_{\Omega\times Cyl^{(1)}} \Pi_\varepsilon^{(1)}(\phi)(z', Y_1, b, Y_2, Y_3)dz'dY.$$

The proof in direction $\mathbf{e}_2$ is done in the same fashion. □

As a direct consequence of the above lemma, we get

$$\sum_{\alpha=1}^{2}\|\Pi_\varepsilon^{(\alpha)}(\phi)\|_{L^2(\Omega\times Cyl^{(\alpha)})} \le \frac{C}{\sqrt{\varepsilon}}\|\phi\|_{L^2(T_\varepsilon)}, \qquad \forall \phi \in L^2(T_\varepsilon). \tag{7.32}$$

For every measurable function $\boldsymbol{\phi}$ defined on $\mathfrak{G}_\varepsilon$, the middle line unfolding operator $\mathcal{T}_\varepsilon^{\mathfrak{G}}$ and the global unfolding operators $\Pi_\varepsilon^{(\alpha)}$ are related in the following way:

$$\Pi_\varepsilon^{(1)}(\boldsymbol{\phi})(z', Y_1, b, 0, 0) = \boldsymbol{\phi}\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big] + \varepsilon Y_1\mathbf{e}_1 + \varepsilon b\mathbf{e}_2\Big) = \mathcal{T}_\varepsilon^{\mathfrak{G}}(\boldsymbol{\phi})(z', Y_1, b), \ \text{ a.e. } (z', Y_1, b) \in \Omega \times \mathfrak{G}^{(1)},$$
$$\Pi_\varepsilon^{(2)}(\boldsymbol{\phi})(z', a, Y_2, 0, 0) = \boldsymbol{\phi}\Big(2\varepsilon\Big[\frac{z'}{2\varepsilon}\Big] + \varepsilon a\mathbf{e}_1 + \varepsilon Y_2\mathbf{e}_2\Big) = \mathcal{T}_\varepsilon^{\mathfrak{G}}(\boldsymbol{\phi})(z', a, Y_2), \ \text{ a.e. } (z', a, Y_2) \in \Omega \times \mathfrak{G}^{(2)}. \tag{7.33}$$

Hence, unfolding functions restricted to the beams' middle lines via $\Pi_\varepsilon^{(\alpha)}$ is equivalent to unfolding them via $\mathcal{T}_\varepsilon^{\mathfrak{G}}$. Therefore, we can use the convergence results of the previous subsection to express the strain tensor convergences on the whole structure.

Lemma 23. *The following convergences hold:*

$$\begin{aligned} &\frac{1}{\varepsilon}\Pi_\varepsilon^{(1)}(\partial_1\mathcal{R}_\varepsilon^{(1)}) \rightharpoonup \begin{pmatrix}\partial_{12}\mathbb{U}_3\\ -\partial_{11}\mathbb{U}_3\\ \partial_{11}\mathbb{U}_2\end{pmatrix} + \begin{pmatrix}\partial_{Y_1}\widehat{\mathcal{R}}_1\\ \partial_{Y_1}\widehat{\mathcal{R}}_2\\ \partial^2_{Y_1Y_1}\widehat{\mathbb{U}}_2\end{pmatrix} \quad \textit{weakly in } L^2(\Omega\times Cyl^{(1)})^3,\\ &\frac{1}{\varepsilon}\Pi_\varepsilon^{(2)}(\partial_2\mathcal{R}_\varepsilon^{(2)}) \rightharpoonup \begin{pmatrix}\partial_{22}\mathbb{U}_3\\ -\partial_{12}\mathbb{U}_3\\ -\partial_{22}\mathbb{U}_1\end{pmatrix} + \begin{pmatrix}\partial_{Y_2}\widehat{\mathcal{R}}_1\\ \partial_{Y_2}\widehat{\mathcal{R}}_2\\ -\partial^2_{Y_2Y_2}\widehat{\mathbb{U}}_1\end{pmatrix} \quad \textit{weakly in } L^2(\Omega\times Cyl^{(2)})^3 \end{aligned} \tag{7.34}$$

and ($\alpha = \{1,2\}$)

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(\alpha)}(\partial_\alpha\mathbb{U}_{\varepsilon,\alpha}^{(\alpha)}) \rightharpoonup \partial_\alpha\mathbb{U}_\alpha^{(\mathbf{S})} + \partial_{Y_\alpha}\widehat{\mathbb{U}}_\alpha^{(\mathbf{S})} \quad \textit{weakly in } L^2(\Omega\times Cyl^{(\alpha)}). \tag{7.35}$$

Proof. First, for every function $\phi \in L^2(\mathfrak{G}_\varepsilon^{(\alpha)})$, we have the following change of convergence rate:

$$\|\Pi_\varepsilon^{(\alpha)}(\phi)\|_{L^2(\Omega\times Cyl^{(\alpha)})} \le C\sqrt{\varepsilon}\|\phi\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})}.$$

Hence, convergence (7.34) follows from the above inequality, equality (7.33), and convergences (7.21), (7.23), (7.24) and (7.25). Convergence (7.35) is proven by the same meanings of (7.34), together with convergence (7.27)$_2$. □

7.3.4 Unfolded limit of the frame

In order to find the strain tensors' and displacement limit form, we need to unfold not only the fields but the reference frame as well. We do it in this subsection, and due to symmetry reasons, we will only consider direction $\mathbf{e}_1$.

We start by the unfolding of the oscillating function $\Phi_\varepsilon^{(1)}$ and we have

$$\frac{1}{\varepsilon}\Pi^{(1)}(\Phi_\varepsilon^{(1)}) \to \Phi^{(1)} \qquad \text{strongly in } \ H^2(Cyl^{(1)}),$$

where Φ is given in (5.1). Note that the convergence is strong due to the regularity of the function (see Section 5.1). As a direct consequence, straightforward calculations show that the following strong convergences hold:

$$\begin{aligned} &\Pi_\varepsilon^{(1)}(\gamma_\varepsilon) \to \gamma \doteq \sqrt{1+(\partial_1\Phi_\varepsilon^{(1)})^2}, && \varepsilon\Pi_\varepsilon^{(1)}(c_\varepsilon^{(1)}) \to \boldsymbol{c}^{(1)} \doteq \frac{\partial^2_{Y_1}\Phi^{(1)}}{\gamma^3}\\ &\Pi_\varepsilon^{(1)}(\mathbf{t}_\varepsilon^{(1)}) \to \mathbf{t}^{(1)} \doteq \frac{1}{\gamma}(\mathbf{e}_1 + \partial_{Y_1}\Phi^{(1)}\mathbf{e}_3), && \Pi_\varepsilon^{(1)}(\eta_\varepsilon^{(1)}) \to \boldsymbol{\eta}^{(1)} \doteq \gamma\,(1 - Y_3\boldsymbol{c}^{(1)}),\\ &\Pi_\varepsilon^{(1)}(\mathbf{n}_\varepsilon^{(1)}) \to \mathbf{n}^{(1)} \doteq \frac{1}{\gamma}(-\partial_{Y_1}\Phi^{(1)}\mathbf{e}_1 + \mathbf{e}_3), && \Pi_\varepsilon^{(1)}(\nabla\psi_\varepsilon^{(1)}) \to (\,\boldsymbol{\eta}^{(1)}\mathbf{t}^{(1)} \quad \mathbf{e}_2 \quad \mathbf{n}^{(1)}\,). \end{aligned} \tag{7.36}$$

7.3.5 Form of the limit strain tensors for the warping

We define the limit space of microscopic functions ($\alpha \in \{1,2\}$)

$$\mathbf{W}^{(\alpha)} \doteq \Big\{\overline{w}^{(\alpha)} \in H^1(Cyl^{(\alpha)})^3 \mid \text{2-periodic with respect to } y_\alpha\Big\}. \tag{7.37}$$

In the lemma below, we show the warping convergences.

Lemma 24 (Lemma 7.7 of Griso, Orlik, and Wackerle, 2020a)**.** *There exist a subsequence of* $\{\varepsilon\}$, *still denoted* $\{\varepsilon\}$, *and* $\overline{u}^{(1)} \in L^2(\Omega;\mathbf{W}^{(1)})$, $\overline{u}^{(2)} \in L^2(\Omega;\mathbf{W}^{(2)})$ *such that*

$$\frac{1}{\varepsilon^3}\Pi_\varepsilon^{(\alpha)}(\overline{u}_\varepsilon^{(\alpha)}) \rightharpoonup \overline{u}^{(\alpha)} \qquad \textit{weakly in} \quad L^2(\Omega;H^1(Cyl^{(\alpha)}))^3.$$

In the same fashion as in Subsection 7.3 of Griso, Orlik, and Wackerle, 2020a, by the above convergences and the convergences of the reference frame (7.36), we go to the limit with the strain tensor of the warping (5.23) for the final displacement's warping. We get

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(\alpha)}(\tilde{\mathbf{e}}(\overline{u}_\varepsilon)) \rightharpoonup \mathcal{E}_Y^{(\alpha)}(\overline{u}^{(\alpha)}) \quad \text{weakly in} \quad L^2(\Omega \times Cyl^{(\alpha)})^{3\times 3},$$

where for every $\Psi^{(1)} \in H^1(Cyl^{(1)})^3$ and every $\Psi^{(2)} \in H^1(Cyl^{(2)})^3$, we have

$$\mathcal{E}_Y^{(1)}(\Psi^{(1)}) =\doteq \begin{pmatrix} \frac{1}{\eta^{(1)}}\partial_{Y_1}\Psi^{(1)}\cdot\mathbf{t}^{(1)} & * & * \\ \frac{1}{2}\big(\frac{1}{\eta^{(1)}}\partial_{Y_1}\Psi^{(1)}\cdot\mathbf{e}_2 + \partial_{Y_2}\Psi^{(1)}\cdot\mathbf{t}^{(1)}\big) & \partial_{Y_2}\Psi^{(1)}\cdot\mathbf{e}_2 & * \\ \frac{1}{2}\big(\frac{1}{\eta^{(1)}}\partial_{Y_1}\Psi^{(1)}\cdot\mathbf{n}^{(1)} + \partial_{Y_3}\Psi^{(1)}\cdot\mathbf{t}^{(1)}\big) & \frac{1}{2}\big(\partial_{Y_2}\Psi^{(1)}\cdot\mathbf{n}^{(1)} + \partial_{Y_3}\Psi^{(1)}\cdot\mathbf{e}_2\big) & \partial_{Y_3}\Psi^{(1)}\cdot\mathbf{n}^{(1)} \end{pmatrix} \tag{7.38}$$

and

$$\mathcal{E}_Y^{(2)}(\Psi^{(2)}) \doteq \begin{pmatrix} \partial_{Y_1}\Psi^{(2)}\cdot\mathbf{e}_1 & * & * \\ \frac{1}{2}\big(\partial_{Y_1}\Psi^{(2)}\cdot\mathbf{t}^{(2)} + \frac{1}{\eta^{(2)}}\partial_{Y_2}\Psi^{(2)}\cdot\mathbf{e}_1\big) & \frac{1}{\eta^{(2)}}\partial_{Y_2}\Psi^{(2)}\cdot\mathbf{t}^{(2)} & * \\ \frac{1}{2}\big(\partial_{Y_1}\Psi^{(2)}\cdot\mathbf{n}^{(2)} + \partial_{Y_3}\Psi^{(2)}\cdot\mathbf{e}_1\big) & \frac{1}{2}\big(\frac{1}{\eta^{(2)}}\partial_{Y_2}\Psi^{(2)}\cdot\mathbf{n}^{(2)} + \partial_{Y_3}\Psi^{(2)}\cdot\mathbf{t}^{(2)}\big) & \partial_{Y_3}\Psi^{(2)}\cdot\mathbf{n}^{(2)} \end{pmatrix}. \tag{7.39}$$

7.3.6 Form of the limit strain tensors

We denote by $\mathbf{\Theta}$ (and its derivative by $\mathbf{\Theta}'$) the following function belonging to $W^{1,\infty}_{per}(0,1)$:

$$\mathbf{\Theta}(t) = \frac{1}{2}\begin{cases} t^2 & \text{if } t \in [0,\kappa], \\ \kappa^2 & \text{if } t \in [\kappa, 1-\kappa], \\ (t-1)^2 & \text{if } t \in [1-\kappa, 1], \end{cases} \qquad \mathbf{\Theta}'(t) = \begin{cases} t & \text{if } t \in [0,\kappa), \\ 0 & \text{if } t \in (\kappa, 1-\kappa), \\ t-1 & \text{if } t \in (1-\kappa, 1]. \end{cases}$$

We define the notation for the strain tensors' form in the limit. Let $X = (X_0, X_{00}, X_1, X_2, X_3)$ in $\mathbb{R}^5$. For $\alpha = \{1,2\}$, we define the functions $\mathcal{E}^{(\alpha)}$ by

$$\mathcal{E}^{(1)}(X) = \begin{pmatrix} \frac{1}{\eta^{(1)}}\mathfrak{F}^{(1)}(X)\cdot\mathbf{t}^{(1)} & * & * \\ \frac{1}{2\eta^{(1)}}\mathfrak{F}^{(1)}(X)\cdot\mathbf{e}_2 & 0 & * \\ \frac{1}{2\eta^{(1)}}\mathfrak{F}^{(1)}(X)\cdot\mathbf{n}^{(1)} & 0 & 0 \end{pmatrix}, \qquad \mathcal{E}^{(2)}(X) = \begin{pmatrix} 0 & * & * \\ \frac{1}{2\eta^{(2)}}\mathfrak{F}^{(2)}(X)\cdot\mathbf{e}_1 & \frac{1}{\eta^{(2)}}\mathfrak{F}^{(2)}(X)\cdot\mathbf{t}^{(2)} & * \\ 0 & \frac{1}{2\eta^{(2)}}\mathfrak{F}^{(2)}(X)\cdot\mathbf{n}^{(2)} & 0 \end{pmatrix}, \tag{7.40}$$

where $\mathfrak{F}^{(\alpha)}$ are the functions from $\mathbb{R}^5$ into $\mathbb{R}^3 \times \mathfrak{G}^{(\alpha)}$ respectively defined by

$$\mathfrak{F}^{(1)}(X) \doteq \begin{cases} \begin{pmatrix} X_0 \\ 0 \\ -\mathbf{\Theta}'X_2 \end{pmatrix} + \begin{pmatrix} X_1 \\ -X_2 \\ X_3 \end{pmatrix} \wedge \big(\Phi^{(1)}\mathbf{e}_3 + Y_2\mathbf{e}_2 + Y_3\mathbf{n}^{(1)}\big) & \text{if } b = 0, \\ \begin{pmatrix} X_{00} \\ 0 \\ -\mathbf{\Theta}'X_2 \end{pmatrix} + \begin{pmatrix} X_1 \\ -X_2 \\ X_3 \end{pmatrix} \wedge \big(\Phi^{(1)}\mathbf{e}_3 + Y_2\mathbf{e}_2 + Y_3\mathbf{n}^{(1)}\big) & \text{if } b = 1, \end{cases}$$

and

$$\mathfrak{F}^{(2)}(X) \doteq \begin{cases} \begin{pmatrix} 0 \\ X_0 \\ -\mathbf{\Theta}'X_1 \end{pmatrix} + \begin{pmatrix} X_1 \\ -X_2 \\ -X_3 \end{pmatrix} \wedge \big(\Phi^{(2)}\mathbf{e}_3 + Y_1\mathbf{e}_1 + Y_3\mathbf{n}^{(2)}\big) & \text{if } a = 0, \\ \begin{pmatrix} 0 \\ X_{00} \\ -\mathbf{\Theta}'X_1 \end{pmatrix} + \begin{pmatrix} X_1 \\ -X_2 \\ -X_3 \end{pmatrix} \wedge \big(\Phi^{(2)}\mathbf{e}_3 + Y_1\mathbf{e}_1 + Y_3\mathbf{n}^{(2)}\big) & \text{if } a = 1. \end{cases}$$

Before going to the limit, we must prove that the unfolded strain tensor is bounded. From the change of convergence rate (7.32) and (5.44), we have

$$\Big\|\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(\alpha)}\big(\widetilde{\mathbf{e}}(u_\varepsilon)\big)\Big\|_{L^2(\Omega\times Cyl^{(\alpha)})} \le \frac{1}{\varepsilon^{5/2}}\|u_\varepsilon\|_{T_\varepsilon} \le C.$$

We first consider the direction $\mathbf{e}_1$. Due to the representation of the strain tensors (5.11)-(5.12), the convergences in Lemmas 23-24 and the frame convergences (7.36), we obtain

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(1)}\big(\widetilde{\mathbf{e}}(u_\varepsilon)\big) \rightharpoonup \mathcal{E}^{(1)}(\partial\mathbf{U}^{(1)}) + \mathcal{E}_Y^{(1)}(\widehat{u}^{(1)}) \qquad \text{weakly in } L^2(\Omega; H^1(Cyl^{(1)}))^{3\times 3}, \tag{7.41}$$

where the first quantity is given as in $(7.40)_1$, but with X replaced by

$$\partial\mathbf{U}^{(1)} = \big(\partial_1\mathbb{U}_1^{(\mathbf{S})}(\cdot,0), \partial_1\mathbb{U}_1^{(\mathbf{S})}(\cdot,1), \partial_{12}\mathbb{U}_3, \partial_{11}\mathbb{U}_3, \partial_{11}\mathbb{U}_2\big),$$

and where the second quantity is given by (7.38) and is the symmetric gradient of the displacement

$$\begin{aligned}\widehat{u}^{(1)} \doteq \widehat{\mathbb{U}}_1^{(\mathbf{S})}\mathbf{e}_1 + \mathbb{U}_2^{(\mathbf{B})}\mathbf{e}_2 + (\widehat{\mathbb{U}}_3 + \mathbf{\Theta}\partial_{11}^2\mathbb{U}_3)\mathbf{e}_3 + \big(\widehat{\mathcal{R}}_1\mathbf{e}_1 + \widehat{\mathcal{R}}_2\mathbf{e}_2 + \partial_{Y_1}\widehat{\mathbb{U}}_2\mathbf{e}_3\big)\\ \wedge\big(\Phi^{(1)}\mathbf{e}_3 + Y_3\mathbf{n}^{(1)} + Y_2\mathbf{e}_2\big) + \overline{u}^{(1)}.\end{aligned} \tag{7.42}$$

We have $\widehat{u}^{(1)} \in L^2(\Omega; \mathbf{W}^{(1)})$.
Concerning direction $\mathbf{e}_2$, the same argumentation applies and the limit strain tensor becomes

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(2)}\big(\widetilde{\mathbf{e}}(u_\varepsilon)\big) \rightharpoonup \mathcal{E}^{(2)}(\partial\mathbf{U}^{(2)}) + \mathcal{E}_Y^{(2)}(\widehat{u}^{(2)}) \qquad \text{weakly in } L^2(\Omega; H^1(Cyl^{(2)}))^{3\times 3}, \tag{7.43}$$

where again the first quantity is given by $(7.40)_2$, but with X replaced by

$$\partial\mathbf{U}^{(2)} = \big(\partial_2\mathbb{U}_2^{(\mathbf{S})}(\cdot,0), \partial_2\mathbb{U}_2^{(\mathbf{S})}(\cdot,1), \partial_{22}\mathbb{U}_3, \partial_{12}\mathbb{U}_3, \partial_{22}\mathbb{U}_1\big),$$

and where the second quantity is given by (7.39) and is the symmetric gradient of the displacement

$$\begin{aligned}\widehat{u}^{(2)} \doteq \mathbb{U}_1^{(\mathbf{B})}\mathbf{e}_1 + \widehat{\mathbb{U}}_2^{(\mathbf{S})}\mathbf{e}_2 + (\widehat{\mathbb{U}}_3 + \mathbf{\Theta}\partial_{22}^2\mathbb{U}_3)\mathbf{e}_3 + \big(\widehat{\mathcal{R}}_1\mathbf{e}_1 + \widehat{\mathcal{R}}_2\mathbf{e}_2 - \partial_{Y_2}\widehat{\mathbb{U}}_1\mathbf{e}_3\big)\\ \wedge\big(\Phi^{(2)}\mathbf{e}_3 + Y_3\mathbf{n}^{(2)} + Y_1\mathbf{e}_1\big) + \overline{u}^{(2)}.\end{aligned} \tag{7.44}$$

We have $\widehat{u}^{(2)} \in L^2(\Omega; \mathbf{W}^{(2)})$.
Note that in the expressions of $\widehat{u}^{(1)}$ and $\widehat{u}^{(2)}$ given above, the terms $\mathbf{\Theta}\partial_{11}^2\mathbb{U}_3$ and $\mathbf{\Theta}\partial_{22}^2\mathbb{U}_3$ do come neither from the asymptotic behavior of the strain tensors nor from other displacement fields' weak convergences. These terms have been added to simplify the non-penetration limit condition in the next section (see Lemma 26).

7.3.7 Unfold of the contact conditions via contact unfolding operator

The main purpose of this unfolding operator is to unfold functions on the two-dimensional contact areas $\mathbf{C}_\varepsilon$, defined in (5.26), in order to find out the unfolded limit contact conditions.
We define the limit reference contact domains by

$$\mathbf{C}_{ab} \doteq \big((a,b) + \omega_\kappa\big) \cap \Omega, \qquad \text{for } (a,b) \in \{0,1\}^2.$$

Definition 14 (Contact unfolding operator)**.** *For every measurable function ϕ in $\mathbf{C}_{ab}$, we define the four measurable functions $T_\varepsilon^{\mathbf{C}_{ab}}(\phi)$ in $\Omega \times \omega_\kappa$ by ($(a,b) \in \{0,1\}^2$)*

$$T_\varepsilon^{\mathbf{C}_{ab}}(\phi)(z', Y_1, Y_2) \doteq \phi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon\big(a\mathbf{e}_1 + b\mathbf{e}_2\big) + \varepsilon(Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2)\right) \quad \textit{for a.e. } (z', Y_1, Y_2) \in \Omega \times \omega_\kappa.$$

Note that for every $\phi \in L^2(\mathfrak{G}_\varepsilon^{(1)})$ (resp. $\psi \in L^2(\mathfrak{G}_\varepsilon^{(2)})$), the unfolding perator $T_\varepsilon^{\mathbf{C}_{ab}}(\phi)$ (resp. $T_\varepsilon^{\mathbf{C}_{ab}}(\psi)$) is given by

$$T_\varepsilon^{\mathbf{C}_{ab}}(\phi)(z',Y_1,0) \doteq \phi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right]+\varepsilon\big(a\mathbf{e}_1+b\mathbf{e}_2\big)+\varepsilon Y_1\mathbf{e}_1\right) \qquad \text{for a.e. } (z',Y_1)\in\Omega\times(-\kappa,\kappa),$$
$$\text{(resp. } T_\varepsilon^{\mathbf{C}_{ab}}(\psi)(z',0,Y_2) \doteq \phi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right]+\varepsilon\big(a\mathbf{e}_1+b\mathbf{e}_2\big)+\varepsilon Y_2\mathbf{e}_2\right) \qquad \text{for a.e. } (z',Y_2)\in\Omega\times(-\kappa,\kappa)).$$

Let ϕ be in $L^2(\mathfrak{G}_\varepsilon^{(1)})$. The operator $\mathcal{T}_\varepsilon^{\mathbf{C}_{ab}}$ is related to $\mathcal{T}_\varepsilon^{\mathfrak{G}}$ via the following relations:

$$T_\varepsilon^{\mathbf{C}_{0b}}(\phi)(z',Y_1,0)=\begin{cases}\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi)(z',Y_1,b) & \text{for a.e. } (z',Y_1,b)\in\Omega\times(0,\kappa)\times\{0,1\},\\ \mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi)(z'-2\varepsilon\mathbf{e}_1,2+Y_1,b) & \text{for a.e. } (z',Y_1,b)\in\big(\Omega\cap(\Omega+2\varepsilon\mathbf{e}_1)\big)\\ & \qquad\times(-\kappa,0)\times\{0,1\},\end{cases}$$
$$T_\varepsilon^{\mathbf{C}_{1b}}(\phi)(z',Y_1,0)=\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi)(z',1+Y_1,b) \qquad \text{for a.e. } (z',Y_1,b)\in\Omega\times(-\kappa,\kappa)\times\{0,1\}, \tag{7.45}$$

One can easily give similar equalities if $\psi\in L^2(\mathfrak{G}_\varepsilon^{(2)})$, or $\Phi\in L^2(T_\varepsilon)$. We have

$$\begin{aligned}&\|T_\varepsilon^{\mathbf{C}_{ab}}(\phi)\|_{L^2(\Omega\times\omega_\kappa)}\le C\sqrt{\varepsilon}\|\phi\|_{L^2(\mathfrak{G}_\varepsilon)}, && \forall\phi\in L^2(\mathfrak{G}_\varepsilon),\\ &\|T_\varepsilon^{\mathbf{C}_{ab}}(\psi)\|_{L^2(\Omega\times\omega_\kappa)}\le\frac{C}{\sqrt{\varepsilon}}\big(\|\psi\|_{L^2(T_\varepsilon^{(\alpha)})}+\varepsilon\|\nabla\psi\|_{L^2(T_\varepsilon^{(\alpha)})}\big), && \forall\psi\in H^1(T_\varepsilon^{(\alpha)}).\end{aligned} \tag{7.46}$$

Now, recall the form of the final displacements (7.13) and restrict it to the contact areas. For a.e. (t_1,t_2) in ω_r (or equivalently, in $\omega_{\kappa\varepsilon}$), we have

$$\begin{aligned}u^{(1)}(p\varepsilon+t_1,q\varepsilon,t_2,(-1)^{p+q+1}\kappa\varepsilon)&=\begin{pmatrix}\mathbb{U}_1+\mathbb{U}_1^{(\mathbf{S})}\\ \mathbb{U}_2+\mathbb{U}_2^{(\mathbf{B})}\\ \mathbb{U}_3\end{pmatrix}(p\varepsilon+t_1,q\varepsilon)+\begin{pmatrix}\mathcal{R}_1\\ \mathcal{R}_2\\ \mathcal{R}_3^{(1)}\end{pmatrix}(p\varepsilon+t_1,q\varepsilon)\wedge t_2\mathbf{e}_2\\ &\quad+\overline{u}^{(1)}(p\varepsilon+t_1,q\varepsilon,t_2,(-1)^{p+q+1}\kappa\varepsilon),\\ u^{(2)}(p\varepsilon,q\varepsilon+t_2,t_1,(-1)^{p+q}\kappa\varepsilon)&=\begin{pmatrix}\mathbb{U}_1+\mathbb{U}_1^{(\mathbf{B})}\\ \mathbb{U}_2+\mathbb{U}_2^{(\mathbf{S})}\\ \mathbb{U}_3\end{pmatrix}(p\varepsilon,q\varepsilon+t_2)+\begin{pmatrix}\mathcal{R}_1\\ \mathcal{R}_2\\ \mathcal{R}_3^{(1)}\end{pmatrix}(p\varepsilon,q\varepsilon+t_2)\wedge t_1\mathbf{e}_1\\ &\quad+\overline{u}^{(2)}(p\varepsilon,q\varepsilon+t_2,t_1,(-1)^{p+q}\kappa\varepsilon).\end{aligned} \tag{7.47}$$

We start with the in-plane components: due to the contact conditions (7.1), we have the following bound for the difference between the displacements in the contact areas:

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\|u_{\varepsilon,\alpha}^{(1)}-u_{\varepsilon,\alpha}^{(2)}\|^2_{L^2(\mathbf{C}_{pq,\varepsilon})}\le C\varepsilon^4.$$

Hence, the unfolded sequence $\{u_{\varepsilon,\alpha}^{(1)}-u_{\varepsilon,\alpha}^{(2)}\}_\varepsilon$ is bounded, and we can go to the limit in the in-plane components.

Lemma 25. *Let $(a,b)\in\{0,1\}^2$. For a.e. $z'\in\Omega$, the in-plane limit contact conditions are*

$$\begin{aligned}&|\mathbb{U}_1^{(\mathbf{S})}(z',b)-\mathbb{U}_1^{(\mathbf{B})}(z',a)|+\kappa|\partial_2\mathbb{U}_1(z_2)+\partial_1\mathbb{U}_2(z_1)|\le g_1(z'),\\ &|\mathbb{U}_2^{(\mathbf{S})}(z',a)-\mathbb{U}_2^{(\mathbf{B})}(z',b)|+\kappa|\partial_2\mathbb{U}_1(z_2)+\partial_1\mathbb{U}_2(z_1)|\le g_2(z').\end{aligned} \tag{7.48}$$

Proof. We prove only the first inequality in (7.48), since the second one follows the same lines. We split the proof into two steps.

Step 1. A preliminary convergence.
For a.e. (z',Y_2) in $\Omega\times(-\kappa,\kappa)$, we define the function ($(a,b)\in\{0,1\}^2$)

$$\ddot{\mathbb{U}}_{\varepsilon,1}(z',Y_2,a,b)\doteq T_\varepsilon^{\mathbf{C}_{ab}}(\mathbb{U}_{\varepsilon,1})(z',0,0)-T_\varepsilon^{\mathbf{C}_{ab}}(\mathbb{U}_{\varepsilon,1})(z',0,Y_2),$$

which does not depend on z_1 by definition of $\mathbb{U}_{\varepsilon,1}$. It belongs to $L^2(\Omega; H^1(-\kappa,\kappa))$ and the following relation holds:

$$\partial_{Y_2}\ddot{\mathbb{U}}_{\varepsilon,1} = -\varepsilon T_\varepsilon^{\mathbf{C}_{ab}}(\partial_2 \mathbb{U}_{\varepsilon,1}).$$

From the Poincaré inequality, the first inequality in (7.46) and estimates (7.15), we have

$$\|\ddot{\mathbb{U}}_{\varepsilon,1}\|_{L^2(\Omega;H^1(-\kappa,\kappa))} \le C\varepsilon^2.$$

This, together with the third convergence in (7.24) and (7.45), imply that there exist a function $\ddot{\mathbb{U}}_1 \in L^2(\Omega; H^1(-\kappa,\kappa))$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\ddot{\mathbb{U}}_{\varepsilon,1} \rightharpoonup \ddot{\mathbb{U}}_1 \ \text{ weakly in } L^2(\Omega; H^1(-\kappa,\kappa)),\\
&\frac{1}{\varepsilon^2}\partial_{Y_2}\ddot{\mathbb{U}}_{\varepsilon,1} \rightharpoonup \partial_{Y_2}\ddot{\mathbb{U}}_2 = -\partial_2\mathbb{U}_1 \ \text{ weakly in } L^2(\Omega\times(-\kappa,\kappa)).
\end{aligned}$$

As a consequence, we get the equality $\ddot{\mathbb{U}}_1(z',Y_2,a,b) = -Y_2\partial_2\mathbb{U}_1(z_2)$ a.e. in $\Omega\times(-\kappa,\kappa)$.

Step 2. We prove the first statement of the lemma.

By the form of the final displacement in the contact areas (7.47), we go to the limit for the following expressions ($(p,q)\in\{0,\dots,2N_\varepsilon\}\times\{2n_\varepsilon,\dots,2N_\varepsilon\}$)

$$\begin{aligned}
\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^{\mathbf{C}_{ab}}\Big(\mathbb{U}^{(\mathbf{S})}_{\varepsilon,1}(p\varepsilon+y_1,q\varepsilon) - \mathbb{U}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon,q\varepsilon+y_2) - y_2\mathcal{R}^{(1)}_{\varepsilon,3}(p\varepsilon+y_1,q\varepsilon) + \overline{u}^{(1)}_{\varepsilon,1} - \overline{u}^{(2)}_{\varepsilon,1}\\
+\mathbb{U}_{\varepsilon,1}(q\varepsilon) - \mathbb{U}_{\varepsilon,1}(q\varepsilon+y_2)\Big)
\end{aligned}$$

and ($(p,q)\in\{0,\dots,2N_\varepsilon\}\times\{0,\dots,2n_\varepsilon\}$)

$$\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^{\mathbf{C}_{ab}}\Big(\mathbb{U}^{(\mathbf{S})}_{\varepsilon,1}(p\varepsilon+y_1,q\varepsilon) - \mathbb{U}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon,q\varepsilon+y_2) - y_2\mathcal{R}^{(1)}_{\varepsilon,3}(p\varepsilon+y_1,q\varepsilon) + \overline{u}^{(1)}_{\varepsilon,1} - \overline{u}^{(2)}_{\varepsilon,1}\Big)$$

Concerning the warping terms, from estimate (7.16) and the second inequality in (7.46), we obtain

$$\big\|T_\varepsilon^{\mathbf{C}_{ab}}(\overline{u}^{(1)}_{\varepsilon,\alpha} - \overline{u}^{(2)}_{\varepsilon,\alpha})\big\|_{L^2(\Omega\times\omega_\kappa)} \le \frac{C}{\sqrt{\varepsilon}}\|\overline{u}^{(1)}_{\varepsilon,\alpha} - \overline{u}^{(2)}_{\varepsilon,\alpha}\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon\times\omega_r)} \le C\varepsilon^3.$$

Hence, applying the contact unfolding operator to the in-plane warping quantity leads to

$$\frac{1}{\varepsilon^2}T_\varepsilon^{\mathbf{C}_{ab}}(\overline{u}^{(1)}_{\varepsilon,\alpha} - \overline{u}^{(2)}_{\varepsilon,\alpha}) \to 0 \quad \text{strongly in } \ L^2(\Omega\times\omega_\kappa). \tag{7.49}$$

Using convergences (7.25),(7.27), (7.26), (7.49) and the ones in Step 1 we get

$$\begin{aligned}
\frac{1}{\varepsilon^2}T_\varepsilon^{\mathbf{C}_{ab}}\Big(\big(\mathbb{U}^{(\mathbf{S})}_{\varepsilon,1}(p\varepsilon+y_1,q\varepsilon) - \mathbb{U}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon,q\varepsilon+y_2) - y_2\mathcal{R}^{(1)}_{\varepsilon,3}(p\varepsilon+y_1,q\varepsilon)\big) + \big(\overline{u}^{(1)}_{\varepsilon,1} - \overline{u}^{(2)}_{\varepsilon,1}\big)\Big)\\
+\frac{1}{\varepsilon^2}\ddot{\mathbb{U}}_{\varepsilon,1}(\cdot,Y_1,a,b)\\
\rightharpoonup \mathbb{U}^{(\mathbf{S})}_1(\cdot,b) - \mathbb{U}^{(\mathbf{B})}_1(\cdot,a) - Y_2\partial_1\mathbb{U}_2 - Y_2\partial_2\mathbb{U}_1 \ \text{ weakly in } \ L^2\big((\Omega_3\cup\Omega_4)\times(-\kappa,\kappa)\big)^2
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\varepsilon^2}T_\varepsilon^{\mathbf{C}_{ab}}\Big(\big(\mathbb{U}^{(\mathbf{S})}_{\varepsilon,1}(p\varepsilon+y_1,q\varepsilon) - \mathbb{U}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon,q\varepsilon+y_2) - y_2\mathcal{R}^{(1)}_{\varepsilon,3}(p\varepsilon+y_1,q\varepsilon)\big) + \big(\overline{u}^{(1)}_{\varepsilon,1} - \overline{u}^{(2)}_{\varepsilon,1}\big)\Big)\\
\rightharpoonup \mathbb{U}^{(\mathbf{S})}_1(\cdot,b) - \mathbb{U}^{(\mathbf{B})}_1(\cdot,a) - Y_2\partial_1\mathbb{U}_2\\
\text{weakly in } \ L^2\big((\Omega_1\cup\Omega_2)\times(-\kappa,\kappa)\big)^2.
\end{aligned}$$

So, for a.e $(z',Y_2)\in\Omega\times(-\kappa,\kappa)$ we get

$$|\mathbb{U}^{(\mathbf{S})}_1(z',b) - \mathbb{U}^{(\mathbf{B})}_1(z',a) - Y_2(\partial_2\mathbb{U}_1(z_2) + \partial_1\mathbb{U}_2(z_1))| \le g_1(z').$$

The statement follows by the admissible choice of $Y_2 = \pm\kappa$. □

Now, we look at the outer-plane component. From equalities (5.34) and estimates (5.35), (5.37) (a consequence of the non-penetration condition) lead to the following estimate in the contact areas (see also Griso, Orlik, and Wackerle, 2020a):

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|u^{(1)}_{\varepsilon,3} - u^{(2)}_{\varepsilon,3}\|^2_{L^2(\mathbf{C}_{pq,\varepsilon})} \le C\varepsilon^6.$$

We are ready to go to the limit in the outer-plane component.

Lemma 26. *Let $(a,b)\in\{0,1\}^2$. For a.e. $\Omega\times\omega_\kappa$, the outer-plane limit contact conditions are*

$$0 \le (-1)^{a+b}\Big(\widehat{u}^{(1)}_3(\cdot, a+Y_1, b, Y_2, (-1)^{a+b+1}\kappa) - \widehat{u}^{(2)}_3(\cdot, a, b+Y_2, Y_1, (-1)^{a+b}\kappa)\Big). \quad (7.50)$$

Proof. We split the proof into two steps.
Step 1. Preliminary convergences.
For a.e. (z',Y_1) in $\Omega\times(-\kappa,\kappa)$ we consider the function ($(a,b)\in\{0,1\}^2$)

$$\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3}(z',Y_1,a,b) \doteq T^{\mathbf{C}_{ab}}_\varepsilon(\mathbb{U}_{\varepsilon,3})(z',Y_1,0) - T^{\mathbf{C}_{ab}}_\varepsilon(\mathbb{U}_{\varepsilon,3})(z',0,0) - \varepsilon Y_1 T^{\mathbf{C}_{ab}}_\varepsilon(\partial_1\mathbb{U}_{\varepsilon,3})(z',0,0).$$

This function belongs to $L^2(\Omega; H^2(-\kappa,\kappa))$ and we have

$$\partial^2_{Y_1Y_1}\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3} = \varepsilon^2 T^{\mathbf{C}_{ab}}_\varepsilon(\partial^2_{11}\mathbb{U}_{\varepsilon,3}).$$

From the Poincaré inequality, the first inequality in (7.46) and estimates (7.15), we obtain

$$\|\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3}\|_{L^2(\Omega;H^2(-\kappa,\kappa))} \le C\varepsilon^3.$$

This, together with the third convergence in (7.21) and equalities (7.45), imply that there exists a function $\ddot{\mathbb{U}}^{(1)}_3 \in L^2(\Omega; H^2(-\kappa,\kappa))$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon^3}\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3} \rightharpoonup \ddot{\mathbb{U}}^{(1)}_3 \quad \text{weakly in } L^2(\Omega; H^2(-\kappa,\kappa)),\\
&\frac{1}{\varepsilon^3}\partial^2_{Y_1Y_1}\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3} \rightharpoonup \partial^2_{Y_1Y_1}\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3} = \partial_{11}\mathbb{U}_3 + \partial^2_{Y_1Y_1}\widehat{\mathbb{U}}_3(\cdot, a+Y_1, b) \quad \text{weakly in } L^2(\Omega\times(-\kappa,\kappa)).
\end{aligned}$$

As a consequence, we get a.e. in $\Omega\times\omega_\kappa$

$$\ddot{\mathbb{U}}^{(1)}_3(z',Y_1,a,b) = \frac{1}{2}Y_1^2\partial_{11}\mathbb{U}_3(z') + \widehat{\mathbb{U}}_3(z',a+Y_1,b) - \widehat{\mathbb{U}}_3(z',a,b) - Y_1\partial_{Y_1}\widehat{\mathbb{U}}_3(z',a,b).$$

Now, for a.e. (z',Y_1) in $\Omega\times(-\kappa,\kappa)$, we consider the function ($(a,b)\in\{0,1\}^2$)

$$\ddot{\mathcal{R}}^{(1)}_{\varepsilon,1}(z',Y_1,a,b) \doteq T^{\mathbf{C}_{ab}}_\varepsilon(\mathcal{R}_{\varepsilon,1})(z',Y_1,0) - T^{\mathbf{C}_{ab}}_\varepsilon(\mathcal{R}_{\varepsilon,1})(z',0,0).$$

This function belongs to $L^2(\Omega; H^1(-\kappa,\kappa))$. Proceeding as in the proof of Lemma 25, we show that there exists a function $\ddot{\mathcal{R}}^{(1)}_1 \in L^2(\Omega; H^1(-\kappa,\kappa))$, such that

$$\frac{1}{\varepsilon^2}\ddot{\mathcal{R}}^{(1)}_{\varepsilon,1} \rightharpoonup \ddot{\mathcal{R}}^{(1)}_1 \quad \text{weakly in } L^2(\Omega; H^1(-\kappa,\kappa)),$$

where

$$\ddot{\mathcal{R}}^{(1)}_1(z',Y_1,a,b) = Y_1\partial_{12}\mathbb{U}_3(z') + \widehat{\mathcal{R}}_1(z',a+Y_1,b) - \widehat{\mathcal{R}}_1(z',a,b) \quad \text{for a.e. } (z',Y_1) \text{ in } \Omega\times(-\kappa,\kappa).$$

Regarding direction $\mathbf{e}_2$, we set for a.e. (z',Y_2) in $\Omega\times(-\kappa,\kappa)$ the functions

$$\begin{aligned}
&\ddot{\mathbb{U}}^{(2)}_{\varepsilon,3}(z',Y_2,a,b) \doteq T^{\mathbf{C}_{ab}}_\varepsilon(\mathbb{U}_{\varepsilon,3})(z',0,Y_2) - T^{\mathbf{C}_{ab}}_\varepsilon(\mathbb{U}_{\varepsilon,3})(z',0,0) - \varepsilon Y_2 T^{\mathbf{C}_{ab}}_\varepsilon(\partial_1\mathbb{U}_{\varepsilon,3})(z',0,0),\\
&\ddot{\mathcal{R}}^{(2)}_{\varepsilon,2}(z',Y_2,a,b) \doteq T^{\mathbf{C}_{ab}}_\varepsilon(\mathcal{R}_{\varepsilon,2})(z',0,Y_2) - T^{\mathbf{C}_{ab}}_\varepsilon(\mathcal{R}_{\varepsilon,2})(z',0,0).
\end{aligned}$$

We have

$$\frac{1}{\varepsilon^3}\ddot{\mathbb{U}}^{(2)}_{\varepsilon,3} \rightharpoonup \ddot{\mathbb{U}}^{(2)}_3 \quad \text{weakly in } L^2(\Omega; H^2(-\kappa,\kappa)),$$
$$\frac{1}{\varepsilon^2}\ddot{\mathcal{R}}^{(2)}_{\varepsilon,2} \rightharpoonup \ddot{\mathcal{R}}^{(2)}_2 \quad \text{weakly in } L^2(\Omega; H^1(-\kappa,\kappa)),$$

where

$$\ddot{\mathbb{U}}^{(2)}_3(z',Y_2,a,b) = \frac{1}{2}Y_2^2\partial_{22}\mathbb{U}_3(z') + \widehat{\mathbb{U}}_3(z',a,b+Y_2) - \widehat{\mathbb{U}}_3(z',a,b) - Y_2\partial_{Y_2}\widehat{\mathbb{U}}_3(z',a,b),$$
$$\ddot{\mathcal{R}}^{(2)}_2(z',Y_2,a,b) = -Y_2\partial_{12}\mathbb{U}_3(z') + \widehat{\mathcal{R}}_2(z',a,b+Y_2) - \widehat{\mathcal{R}}_2(z',a,b) \quad \text{a.e. } (z',Y_2) \text{ in } \Omega\times(-\kappa,\kappa).$$

Step 2. We prove the statement.
We consider the difference (7.47) in the third direction. Using $\mathcal{T}^{\mathbf{C}_{ab}}_\varepsilon$ and taking into account the functions introduced in the first step, that gives

$$\frac{1}{\varepsilon^3}\mathcal{T}^{\mathbf{C}_{ab}}_\varepsilon\big(u^{(1)}_{\varepsilon,3} - u^{(2)}_{\varepsilon,3}\big) = \frac{1}{\varepsilon^3}\big(\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3} - \ddot{\mathbb{U}}^{(2)}_{\varepsilon,3} + \varepsilon Y_1\ddot{\mathcal{R}}^{(2)}_{\varepsilon,2} + \varepsilon Y_2\ddot{\mathcal{R}}^{(1)}_{\varepsilon,1} + \mathcal{T}^{\mathbf{C}_{ab}}_\varepsilon\big(\overline{u}^{(1)}_{\varepsilon,3} - \overline{u}^{(2)}_{\varepsilon,3}\big)\big) \rightharpoonup \Delta \quad \text{weakly in } L^2(\Omega\times\omega_\kappa),$$

where

$$\begin{aligned}\Delta(z',Y_1,Y_2,a,b) &= \frac{1}{2}Y_1^2\partial_{11}\mathbb{U}_3(z') + \widehat{\mathbb{U}}_3(z',a+Y_1,b) - \widehat{\mathbb{U}}_3(z',a,b) - Y_1\partial_{Y_1}\widehat{\mathbb{U}}_3(z',a,b)\\ &\quad -\Big(\frac{1}{2}Y_2^2\partial_{22}\mathbb{U}_3(z') + \widehat{\mathbb{U}}_3(z',a,b+Y_2) - \widehat{\mathbb{U}}_3(z',a,b) - Y_2\partial_{Y_2}\widehat{\mathbb{U}}_3(z',a,b)\Big)\\ &\quad +Y_1\big(-Y_2\partial_{12}\mathbb{U}_3(z') + \widehat{\mathcal{R}}_2(z',a,b+Y_2) - \widehat{\mathcal{R}}_2(z',a,b)\big)\\ &\quad +Y_2\big(Y_1\partial_{12}\mathbb{U}_3(z') + \widehat{\mathcal{R}}_1(z',a+Y_1,b) - \widehat{\mathcal{R}}_1(z',a,b)\big)\\ &\quad +\overline{u}^{(1)}_3\big(z',a+Y_1,b,Y_2,(-1)^{a+b+1}\kappa\big) - \overline{u}^{(2)}_3\big(z',a,b+Y_2,Y_1,(-1)^{a+b}\kappa\big).\end{aligned}$$

Taking into account the expressions of $\widehat{u}^{(1)}$ and $\widehat{u}^{(2)}$ given by (7.42)-(7.44) and equalities (7.23), we have

$$\partial_{Y_1}\widehat{\mathbb{U}}_3(z',a,b) = -\widehat{\mathcal{R}}_2(z',a,b), \qquad \partial_{Y_2}\widehat{\mathbb{U}}_3(z',a,b) = \widehat{\mathcal{R}}_1(z',a,b).$$

Hence, the outer-plane contact condition (7.50) is proved. □

7.3.8 The displacements limit set

Now, since all the fields involved in the limit strain tensor, limit displacement, and limit contact conditions have been found, we can finally define the limit set of admissible displacements.
From (7.18),(7.19),(7.20) and (7.37), we set

- $\mathcal{X}_M \doteq H^2_{(0,l)}\big((0,L)_{z_2}\big) \times H^2_{(0,l)}\big((0,L)_{z_1}\big) \times H^2_\Gamma(\Omega)$ the space of macroscopic functions;
- $\mathcal{X}_S \doteq \mathbf{L}^2(\Omega\times\{0,1\},\partial_1) \times \mathbf{L}^2(\Omega\times\{0,1\},\partial_2)$ the space of the relative macroscopic stretching functions;
- $\mathcal{X}_B \doteq L^2(\Omega\times\{0,1\})^2$ the space of the relative macroscopic bending functions;
- $\mathcal{X}_m \doteq L^2(\Omega;\mathbf{W}^{(1)}) \times L^2(\Omega;\mathbf{W}^{(2)})$ the space of all the microscopic functions.

In particular, the functions belonging to their respective spaces are defined by

$$\begin{aligned}&\mathbb{V} \doteq (\mathbb{V}_1,\mathbb{V}_2,\mathbb{V}_3) \in \mathcal{X}_M, && \mathbb{V}^{(\mathbf{S})} \doteq \big(\mathbb{V}^{(\mathbf{S})}_1,\mathbb{V}^{(\mathbf{S})}_2\big) \in \mathcal{X}_S,\\ &\mathbb{V}^{(\mathbf{B})} \doteq \big(\mathbb{V}^{(\mathbf{B})}_1,\mathbb{V}^{(\mathbf{B})}_2\big) \in \mathcal{X}_B, && \widehat{v} \doteq \big(\widehat{v}^{(1)},\widehat{v}^{(2)}\big) \in \mathcal{X}_m.\end{aligned} \tag{7.51}$$

Including the limit contact conditions (7.48) and (7.50), the limit set of admissible displacements is defined by

$$\begin{aligned}\mathcal{X} \doteq \Big\{ (\mathbb{V}, \mathbb{V}^{(\mathbf{S})}, \mathbb{V}^{(\mathbf{B})}, \widehat{v}) \in \mathcal{X}_M \times \mathcal{X}_S \times \mathcal{X}_B \times \mathcal{X}_m \;\Big|\; & \\ |\mathbb{V}_1^{(\mathbf{S})}(\cdot, b) - \mathbb{V}_1^{(\mathbf{B})}(\cdot, a)| + \kappa|\partial_2 \mathbb{V}_1 + \partial_1 \mathbb{V}_2| \leq g_1 \text{ a.e. in } \Omega, & \\ |\mathbb{V}_2^{(\mathbf{S})}(\cdot, a) - \mathbb{V}_2^{(\mathbf{B})}(\cdot, b)| + \kappa|\partial_2 \mathbb{V}_1 + \partial_1 \mathbb{V}_2| \leq g_2 \text{ a.e. in } \Omega, & \\ 0 \leq (-1)^{a+b} \Big(\widehat{v}_3^{(1)}(\cdot, a + Y_1, b, Y_2, (-1)^{a+b+1}\kappa) - \widehat{v}_3^{(2)}(\cdot, a, b + Y_2, Y_1, (-1)^{a+b}\kappa) \Big) & \\ \text{a.e. in } \Omega \times \omega_\kappa,\ (a, b) \in \{0, 1\}^2 \Big\}. &\end{aligned}$$

Note that $\mathcal{X}$ is a closed convex subset of the Hilbert space $\mathcal{X}_M \times \mathcal{X}_S \times \mathcal{X}_B \times \mathcal{X}_m$, endowed with the product norm. We set

$$\begin{aligned}\partial \mathbf{V}^{(1)} &= (\partial_1 \mathbb{V}_1^{(\mathbf{S})}(\cdot, 0), \partial_1 \mathbb{V}_1^{(\mathbf{S})}(\cdot, 1), \partial_{12} \mathbb{V}_3, \partial_{11} \mathbb{V}_3, \partial_{11} \mathbb{V}_2), \\ \partial \mathbf{V}^{(2)} &= (\partial_2 \mathbb{V}_2^{(\mathbf{S})}(\cdot, 0), \partial_2 \mathbb{V}_2^{(\mathbf{S})}(\cdot, 1), \partial_{22} \mathbb{V}_3, \partial_{12} \mathbb{V}_3, \partial_{22} \mathbb{V}_1).\end{aligned} \tag{7.52}$$

7.4 Strong convergence of the test functions via unfolding

We construct the test functions with sufficient regularity to belong to a dense subset of $\mathcal{X}$ and ensure strong convergence via unfolding. In addition, they must have the same strain tensor as in the limit and match the contact condition before and after the limit.

7.4.1 Construction of the test functions

Consider the spaces

$$\begin{aligned}\mathcal{C}_M &\doteq \mathcal{C}^3(\overline{\Omega})^3 \cap \mathcal{X}_M, & \mathcal{C}_S &\doteq \mathcal{C}^2(\overline{\Omega} \times \{0,1\})^2 \cap \mathcal{X}_S, \\ \mathcal{C}_B &\doteq \mathcal{C}_c^2(\Omega \times \{0,1\})^2 \cap \mathcal{X}_B, & \mathcal{C}_m &\doteq \mathcal{C}_c^1(\Omega; \mathbf{W}^{(1)}) \times \mathcal{C}_c^1(\Omega; \mathbf{W}^{(2)}).\end{aligned}$$

Accordingly to (7.51), we take $(\mathbb{V}, \mathbb{V}^{(\mathbf{S})}, \mathbb{V}^{(\mathbf{B})}, \widehat{v}) \in \mathcal{C}_M \times \mathcal{C}_S \times \mathcal{C}_B \times \mathcal{C}_m$.
First, we define the vectors of the test function for the combined directions

$$\begin{pmatrix} \mathbb{V}_1(q\varepsilon) \\ \mathbb{V}_2(z_1) \\ \mathbb{V}_3(z_1, q\varepsilon) \end{pmatrix} \text{ a.e. } (z_1, q\varepsilon) \in \mathfrak{G}^{(1)} \quad \text{and} \quad \begin{pmatrix} \mathbb{V}_1(z_2) \\ \mathbb{V}_2(p\varepsilon) \\ \mathbb{V}_3(p\varepsilon, z_2) \end{pmatrix} \text{ a.e. } (p\varepsilon, z_2) \in \mathfrak{G}^{(2)}.$$

Then, we define the test functions for stretching $\mathbb{V}_{\varepsilon,1}^{(\mathbf{S})}$, $\mathbb{V}_{\varepsilon,2}^{(\mathbf{S})}$ and bending $\mathbb{V}_{\varepsilon,1}^{(\mathbf{B})}$, $\mathbb{V}_{\varepsilon,2}^{(\mathbf{B})}$ by

$$\begin{aligned}&\left.\begin{aligned}\mathbb{V}_{\varepsilon,1}^{(\mathbf{S})}(z_1, q\varepsilon) &\doteq \mathbb{V}_1^{(\mathbf{S})}\Big(z_1, 2\Big[\frac{q}{2}\Big]\varepsilon, 2\Big\{\frac{q}{2}\Big\}\Big), \\ \mathbb{V}_{\varepsilon,2}^{(\mathbf{B})}(z_1, q\varepsilon) &\doteq \mathbb{V}_2^{(\mathbf{B})}\Big(z_1, 2\Big[\frac{q}{2}\Big]\varepsilon, 2\Big\{\frac{q}{2}\Big\}\Big),\end{aligned}\right\} \text{ a.e. } (z_1, q\varepsilon) \in \mathfrak{G}^{(1)}, \\ &\left.\begin{aligned}\mathbb{V}_{\varepsilon,2}^{(\mathbf{S})}(p\varepsilon, z_2) &\doteq \mathbb{V}_2^{(\mathbf{S})}\Big(2\Big[\frac{p}{2}\Big]\varepsilon, z_2, 2\Big\{\frac{p}{2}\Big\}\Big), \\ \mathbb{V}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon, z_2) &\doteq \mathbb{V}_1^{(\mathbf{B})}\Big(2\Big[\frac{p}{2}\Big]\varepsilon, z_2, 2\Big\{\frac{p}{2}\Big\}\Big),\end{aligned}\right\} \text{ a.e. } (p\varepsilon, z_2) \in \mathfrak{G}^{(2)}\end{aligned}$$

At last, we define the warping test functions $\widehat{v}^{(1)}_\varepsilon$, $\widehat{v}^{(2)}_\varepsilon$ by

$$\widehat{v}^{(1)}_\varepsilon(z_1,q\varepsilon,y_2,y_3) \doteq \begin{cases} \widehat{v}^{(1)}\Big(p\varepsilon, 2\Big[\dfrac{q}{2}\Big]\varepsilon, 2\Big\{\dfrac{z_1}{2\varepsilon}\Big\}, 2\Big\{\dfrac{q}{2}\Big\}, \dfrac{y_2}{\varepsilon}, \dfrac{y_3}{\varepsilon}\Big) & \text{if } z_1 \in [p\varepsilon - r, p\varepsilon + r], \\ \text{linear interpolated with respect to } z_1 & \text{if } z_1 \in [p\varepsilon + r, (p+1)\varepsilon - r], \end{cases}$$
$$\text{a.e.} \quad (z_1, q\varepsilon, y_2, y_3) \in \mathfrak{G}^{(1)} \times \omega_r,$$

$$\widehat{v}^{(2)}_\varepsilon(p\varepsilon,z_2,y_1,y_3) \doteq \begin{cases} \widehat{v}^{(2)}\Big(2\Big[\dfrac{p}{2}\Big]\varepsilon, q\varepsilon, 2\Big\{\dfrac{p}{2}\Big\}, 2\Big\{\dfrac{z_2}{2\varepsilon}\Big\}, \dfrac{y_1}{\varepsilon}, \dfrac{y_3}{\varepsilon}\Big) & \text{if } z_2 \in [q\varepsilon - r, q\varepsilon + r], \\ \text{linear interpolated with respect to } z_2 & \text{if } z_2 \in [q\varepsilon + r, (q+1)\varepsilon - r], \end{cases}$$
$$\text{a.e.} \quad (p\varepsilon, z_2, y_1, y_3) \in \mathfrak{G}^{(2)} \times \omega_r.$$

The final test displacements v_ε in directions $\mathbf{e}_1$ and $\mathbf{e}_2$ result to be ($\alpha \in \{1,2\}$)

$$v^{(\alpha)}_\varepsilon = V^{(\alpha)}_{\varepsilon,BN} + \varepsilon^3 \widehat{v}^{(\alpha)}_\varepsilon, \tag{7.53}$$

where the Bernoulli-Navier displacements $V^{(1)}_{\varepsilon,BN}$, $V^{(2)}_{\varepsilon,BN}$ are given by

$$V^{(1)}_{\varepsilon,BN}(z_1,q\varepsilon,y_2,y_3) \doteq \begin{pmatrix} \varepsilon\mathbb{V}_1(q\varepsilon) + \varepsilon^2\mathbb{V}^{(\mathbf{S})}_{\varepsilon,1}(z_1,q\varepsilon) \\ \varepsilon\mathbb{V}_2(z_1) + \varepsilon^2\mathbb{V}^{(\mathbf{B})}_{\varepsilon,2}(z_1,q\varepsilon) \\ \varepsilon\mathbb{V}_3(z_1,q\varepsilon) \end{pmatrix} + \begin{pmatrix} \varepsilon\partial_2\mathbb{V}_{\varepsilon,3}(z_1,q\varepsilon) \\ -\varepsilon\partial_1\mathbb{V}_{\varepsilon,3}(z_1,q\varepsilon) \\ \varepsilon\partial_1\mathbb{V}_{\varepsilon,2}(z_1) + \varepsilon^2\partial_1\mathbb{V}^{(\mathbf{B})}_{\varepsilon,2}(z_1,q\varepsilon) \end{pmatrix}$$
$$\wedge\Big(\varepsilon(-1)^{q+1}\Phi\Big(2\Big\{\frac{z_1}{2\varepsilon}\Big\}\Big)\mathbf{e}_3 + y_2\mathbf{e}_2 + y_3(-1)^{q+1}\mathbf{n}\Big(2\Big\{\frac{z_1}{2\varepsilon}\Big\}\Big)\Big),$$

and

$$V^{(2)}_{\varepsilon,BN}(p\varepsilon,z_2,y_1,y_3) \doteq \varepsilon\begin{pmatrix} \varepsilon\mathbb{V}_1(z_2) + \varepsilon^2\mathbb{V}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon,z_2) \\ \varepsilon\mathbb{V}_2(p\varepsilon) + \varepsilon^2\mathbb{V}^{(\mathbf{S})}_{\varepsilon,2}(p\varepsilon,z_2) \\ \varepsilon\mathbb{V}_3(p\varepsilon,z_2) \end{pmatrix} + \begin{pmatrix} \varepsilon\partial_2\mathbb{V}_{\varepsilon,3}(p\varepsilon,z_2) \\ -\varepsilon\partial_1\mathbb{V}_{\varepsilon,3}(p\varepsilon,z_2) \\ -\varepsilon\partial_2\mathbb{V}_{\varepsilon,1}(z_2) - \varepsilon^2\partial_2\mathbb{V}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon,z_2) \end{pmatrix}$$
$$\wedge\Big(\varepsilon(-1)^{p}\Phi\Big(2\Big\{\frac{z_2}{2\varepsilon}\Big\}\Big)\mathbf{e}_3 + y_1\mathbf{e}_1 + y_3(-1)^{p}\mathbf{n}\Big(2\Big\{\frac{z_2}{2\varepsilon}\Big\}\Big)\Big).$$

7.4.2 Limit strain tensors for the test functions

The limit of the unfolded strain tensor is an immediate consequence of (5.22) and (5.23) for the final displacement, the unfolding operator properties, and the regularity of the test functions (see also Lemma 8.1 in Griso, Orlik, and Wackerle, 2020a). We obtain

$$\frac{1}{\varepsilon^2}\Pi^{(\alpha)}\big(\widetilde{\mathbf{e}}(v^{(\alpha)}_\varepsilon)\big) \to \mathcal{E}^{(\alpha)}(\partial\mathbf{V}^{(\alpha)}) + \mathcal{E}^{(\alpha)}_Y(\widehat{v}^{(\alpha)}) \quad \text{strongly in } L^2(\Omega \times Cyl^{(\alpha)})^{3\times 3}, \tag{7.54}$$

where $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ are respectively given by (7.40) with the fields $\partial\mathbf{V}^{(1)}$, $\partial\mathbf{V}^{(2)}$ given as in definition (7.52).

7.4.3 Contact conditions for the test functions

First, the clamping conditions are satisfied by the construction of the test displacements. Now, we check the in-plane contact conditions (5.27). We set

$$\mathbf{N} \doteq \sum_{\alpha=1}^{2}\big(\|\nabla\mathbb{V}^{(\mathbf{S})}_\alpha\|_{L^\infty(\Omega\times\{0,1\})} + \|\nabla^2\mathbb{V}^{(\mathbf{B})}_\alpha\|_{L^\infty(\Omega\times\{0,1\})} + \|\widehat{v}^{(\alpha)}\|_{L^\infty(\Omega\times Cyl^{(\alpha)})}\big).$$

Below, we replace the components $v^{(1)}_{\varepsilon,\alpha}$ and $v^{(2)}_{\varepsilon,\alpha}$ of the test displacement by $\lambda^*_\varepsilon v^{(1)}_{\varepsilon,\alpha}$ and $\lambda^*_\varepsilon v^{(2)}_{\varepsilon,\alpha}$ in order to satisfy the contact conditions (5.27). We will choose $\lambda^*_\varepsilon \doteq 1 - C^*\varepsilon$, where C^* is a non-negative constant that will be assigned later.
We start with the first component. Taking the difference between the displacements, we have

(remind that $\kappa < 1$)

$$\begin{aligned}
&v^{(1)}_{\varepsilon,1}(y_1+p\varepsilon, q\varepsilon, y_2, (-1)^{a+b+1}\kappa\varepsilon) - v^{(2)}_{\varepsilon,1}(p\varepsilon, q\varepsilon+y_2, y_1, (-1)^{a+b}\kappa\varepsilon)\\
&= \varepsilon\big(\mathbb{V}_1(q\varepsilon) - \mathbb{V}_1(q\varepsilon+y_2) + y_2\partial_1\mathbb{V}_2(p\varepsilon+y_1)\big)\\
&\quad + \varepsilon^2\big(\mathbb{V}^{(\mathbf{S})}_{\varepsilon,1}(p\varepsilon+y_1, q\varepsilon) - \mathbb{V}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon, q\varepsilon+y_2) + y_2\partial_1\mathbb{V}^{(\mathbf{B})}_{\varepsilon,2}(p\varepsilon+y_1, q\varepsilon)\big)\\
&\quad + \varepsilon^3\big(\widehat{v}^{(1)}_{\varepsilon,1}(y_1+p\varepsilon, q\varepsilon, y_2, (-1)^{a+b+1}\kappa\varepsilon) - \widehat{v}^{(2)}_{\varepsilon,1}(p\varepsilon, q\varepsilon+y_2, y_1, (-1)^{a+b}\kappa\varepsilon)\big).
\end{aligned}$$

Besides, for a.e. $(y_1, y_2) \in \omega_r$ we have

$$|\mathbb{V}_1(q\varepsilon) - \mathbb{V}_1(q\varepsilon+y_2) + y_2\partial_2\mathbb{V}_1(q\varepsilon+y_2)| \le \kappa^2\varepsilon^2\|\partial^2_{22}\mathbb{V}_1\|_{L^\infty(\Omega)}$$

and

$$\begin{aligned}
&\big|\big(\mathbb{V}^{(\mathbf{S})}_{\varepsilon,1}(p\varepsilon+y_1, q\varepsilon) - \mathbb{V}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon, q\varepsilon+y_2) + y_2\partial_1\mathbb{V}^{(\mathbf{B})}_{\varepsilon,2}(p\varepsilon+y_1, q\varepsilon)\big)\\
&\qquad - \big(\mathbb{V}^{(\mathbf{S})}_{\varepsilon,1}(p\varepsilon+y_1, q\varepsilon+y_2) - \mathbb{V}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon+y_1, q\varepsilon+y_2)\big)\big|\\
&\qquad \le \kappa\varepsilon\big(\|\nabla\mathbb{V}^{(\mathbf{S})}_1\|_{L^\infty(\Omega)} + \|\nabla^2\mathbb{V}^{(\mathbf{B})}_1\|_{L^\infty(\Omega\times\{0,1\})}\big)\\
&\Big|\widehat{v}^{(1)}_{\varepsilon,1}(y_1+p\varepsilon, q\varepsilon, y_2, (-1)^{a+b+1}\kappa\varepsilon) - \widehat{v}^{(1)}_1\Big(p\varepsilon+y_1, q\varepsilon+y_2, 2\Big\{\frac{q}{2}\Big\}, 2\Big\{\frac{p\varepsilon+y_1}{2\varepsilon}\Big\}, \frac{y_2}{\varepsilon}, (-1)^{a+b+1}\kappa\Big)\Big|\\
&\qquad \le C\|\partial_2\widehat{v}^{(1)}_1\|_{L^\infty(\Omega\times Cyl^{(1)})},\\
&\Big|\widehat{v}^{(2)}_{\varepsilon,1}(p\varepsilon, q\varepsilon+y_2, y_1, (-1)^{a+b}\kappa\varepsilon) - \widehat{v}^{(2)}_1\Big(p\varepsilon+y_1, q\varepsilon+y_2, 2\Big\{\frac{p}{2}\Big\}, 2\Big\{\frac{q\varepsilon+y_2}{2\varepsilon}\Big\}, \frac{y_1}{\varepsilon}, (-1)^{a+b}\kappa\Big)\Big|\\
&\qquad \le C\|\partial_2\widehat{v}^{(1)}_1\|_{L^\infty(\Omega\times Cyl^{(1)})}.
\end{aligned}$$

Hence, we have a.e. in C_{pq} that

$$\begin{aligned}
&\Big|\big(v^{(1)}_{\varepsilon,1}(z_1, q\varepsilon, y_2, (-1)^{a+b+1}\kappa) - v^{(2)}_{\varepsilon,1}(p\varepsilon, z_2, y_1, (-1)^{a+b}\kappa)\big)\\
&\qquad - \varepsilon^2\Big(\mathbb{V}^{(\mathbf{S})}_1(z', b) - \mathbb{V}^{(\mathbf{B})}_1(z', a) - \frac{y_2}{\varepsilon}\big(\partial_2\mathbb{V}_1(z_2) + \partial_1\mathbb{V}_2(z_1)\big)\Big)\Big| \le C^\diamond\varepsilon^3\mathbf{N},
\end{aligned}\tag{7.55}$$

where $C^\diamond$ does not depend on ε. So a.e. in $C_{pq,\varepsilon}$, we have

$$|v^{(1)}_{\varepsilon,1} - v^{(2)}_{\varepsilon,1}| \le \varepsilon^2 g_1 + C^\diamond\varepsilon^3\mathbf{N}. \tag{7.56}$$

Taking into account the property (7.2) of g_α, we take the value $C^* = C^\diamond\mathbf{N}/C_3$. Hence, the in-plane contact conditions (5.27) with $h = 2$ are satisfied, since

$$\begin{aligned}
|\lambda^*_\varepsilon v^{(1)}_{\varepsilon,1} - \lambda^*_\varepsilon v^{(2)}_{\varepsilon,1}| &\le \varepsilon^2\lambda^*_\varepsilon g_1 + \lambda^*_\varepsilon C^\diamond\varepsilon^3\mathbf{N}\\
&\le \varepsilon^2 g_1 - C^*\varepsilon^3 g_1 + C^\diamond\varepsilon^3\mathbf{N} \le \varepsilon^2 g_1 - \varepsilon^3(C^*C_3 - C^\diamond\mathbf{N}) = \varepsilon^2 g_1.
\end{aligned}$$

This proves that the contact conditions are satisfied before the limit for the first component. To prove that they hold in the limit, we first note that $\lim_{\varepsilon\to 0}\lambda^*_\varepsilon = 1$. Hence, going to the limit via unfolding with (7.55) and (7.56), we get the limit contact condition $(7.48)_1$.

The second component follows by analogous argumentation, while the test displacement in the outer-plane component is constructed in the same way as in Section 8.1 of Griso, Orlik, and Wackerle, 2020a and, by the meanings of such section, they satisfy the non-penetration conditions before and in the limit.

7.5 Study of the limit problem

In this section, we employ all the results developed in the previous ones to go to the limit for problem (5.30). We will then proceed to its investigation.

Before going to the limit, we provide a couple of preliminary lemmas for the section.

Lemma 27. *Let* $X = (X_0, X_{00}, X_1, X_2, X_3)$ *be in* $\mathbb{R}^5$ *and* $\widehat{v}^{(\alpha)} \in \mathbf{W}^{(\alpha)}$ *satisfying*

$$\mathcal{E}^{(\alpha)}(X) + \mathcal{E}_Y^{(\alpha)}(\widehat{v}^{(\alpha)}) = 0. \tag{7.57}$$

Then $X = 0$ *and* $\widehat{v}^{(\alpha)}$ *are periodic rigid displacements.*
Moreover, there exist two strictly positive constants C_0, C_1 *such that for every* $X \in \mathbb{R}^4$ *and every* $\widehat{v}^{(\alpha)} \in \mathbf{W}^{(\alpha)}$,

$$\begin{aligned} C_0\big(|X|^2 + \|\mathcal{E}_Y^{(\alpha)}(\widehat{v}^{(\alpha)})\|^2_{L^2(Cyl^{(\alpha)})}\big) &\le \|\mathcal{E}^{(\alpha)}(X) + \mathcal{E}_Y^{(\alpha)}(\widehat{v}^{(\alpha)})\|^2_{L^2(Cyl^{(\alpha)})} \\ &\le C_1\big(|X|^2 + \|\mathcal{E}_Y^{(\alpha)}(\widehat{v}^{(\alpha)})\|^2_{L^2(Cyl^{(\alpha)})}\big). \end{aligned} \tag{7.58}$$

Proof. We prove the statement for $\alpha = 1$.
The solution of the equation (7.57) is given by

$$\widehat{v}^{(1)} = \mathcal{A}^{(1)} + \mathcal{B}^{(1)} \wedge \big(\Phi^{(1)}\mathbf{e}_3 + Y_2\mathbf{e}_2 + Y_3\mathbf{n}^{(1)}\big), \quad \mathcal{A}^{(1)},\ \mathcal{B}^{(1)} \in H^1(\mathfrak{G}^{(1)})^3$$

with first (see (7.38)-(7.40)) $\partial_{Y_1}\mathcal{B}^{(1)} = X_1\mathbf{e}_1 - X_2\mathbf{e}_2 + X_3\mathbf{e}_3$. Since $\mathcal{B}^{(1)}$ is periodic, this gives $X_1 = X_2 = X_3 = 0$ and $\mathcal{B}^{(1)}(Y_1, b) = \mathbf{B}^{(1)}(b)$ for a.e. $(Y_1, b) \in \mathfrak{G}^{(1)}$. Then, we get $\partial_{Y_1}\mathcal{A}^{(1)}(Y_1, 0) = \mathbf{B}^{(1)}(0) \wedge \mathbf{e}_1 - X_0\mathbf{e}_1$ (resp. $\partial_{Y_1}\mathcal{A}^{(1)}(Y_1, 1) = \mathbf{B}^{(1)}(1) \wedge \mathbf{e}_1 - X_{00}\mathbf{e}_1$), again since $\mathcal{A}^{(1)}$ is periodic, this gives $X_0 = X_{00} = 0$ and $\mathbf{B}^{(1)} = \mathbf{b}^{(1)}\mathbf{e}_1$. Hence $\widehat{v}^{(1)}$ is a rigid periodic displacement

$$\widehat{v}^{(1)}(Y_1, b, Y_2, Y_3) = \mathbf{A}^{(1)}(b) + \mathbf{b}^{(1)}(b)\mathbf{e}_1 \wedge \big(\Phi^{(1)}\mathbf{e}_3 + Y_2\mathbf{e}_2 + Y_3\mathbf{n}^{(1)}\big), \quad \mathbf{A}^{(1)}(b),\ \mathbf{b}^{(1)}(b) \in \mathbb{R}^3.$$

The inequality in the right-hand side of (7.58) is obvious. The left-hand side inequality is proven by contradiction. $\square$

The lemma below concerns the integration of the reference frame over the limit reference cells $Cyl^{(\alpha)}$.

Lemma 28. *One has the following values for the integrals* ($\alpha \in \{1, 2\}$)

$$\begin{aligned} &\int_{Cyl^{(\alpha)}} \boldsymbol{\eta}^{(\alpha)} dY = 4\kappa^2 \int_0^2 \gamma\, dt, \\ &\int_{Cyl^{(\alpha)}} \big(\Phi^{(\alpha)}\mathbf{e}_3 + Y_{3-\alpha}\mathbf{e}_{3-\alpha} + Y_3\mathbf{n}^{(\alpha)}\big)\boldsymbol{\eta}^{(\alpha)} dY = 4\kappa^2\Big(\int_0^2 \gamma\, \Phi\, dt\Big)\mathbf{e}_3. \end{aligned} \tag{7.59}$$

Proof. We will just prove the statement for direction $\mathbf{e}_1$, since the second one follows by an analogous argumentation.
From the definition of $\boldsymbol{\eta}^{(1)}$ and the symmetries of the cross-section with respect to the lines $Y_2 = 0$ and $Y_3 = 0$, equality $(7.59)_1$ holds.
Concerning $(7.59)_1$, we first note that the symmetries of the cross-section with respect to the lines $Y_2 = 0$ and $Y_3 = 0$ lead to

$$\begin{aligned} &\int_{Cyl^{(1)}} \big(\Phi^{(1)}(Y_1)\mathbf{e}_3 + Y_2\mathbf{e}_2 + Y_3\mathbf{n}^{(1)}(Y_1)\big)\boldsymbol{\eta}^{(1)}(Y_1, Y_3) dY \\ &= 4\kappa^2\Big(\int_0^2 \Phi^{(1)}\gamma dY_1\Big)\mathbf{e}_3 + \frac{4\kappa^4}{3}\Big(\int_0^2 \partial_{Y_1}\Phi^{(1)}\boldsymbol{c}^{(1)} dY_1\Big)\mathbf{e}_1 - \frac{4\kappa^4}{3}\Big(\int_0^2 \boldsymbol{c}^{(1)} dY_1\Big)\mathbf{e}_3. \end{aligned}$$

Then, we note that the second and third integral vanish since Φ is 2-periodic with respect to Y_1 and satisfies $\partial_{Y_1}\Phi(0) = \partial_{Y_1}\Phi(1) = \partial_{Y_1}\Phi(2) = 0$. $\square$

7.5.1 The unfolded limit problem

We are now ready to show the limit elasticity problem.

Theorem 7. *Let $u_\varepsilon \in \mathcal{X}_\varepsilon$ be a solution of problem (5.30) and let $f^{(\alpha)} \in H^1(\Omega)^3$, $\widetilde{f}^{(\alpha)} \in H^1(\Omega)^3$ be as in Subsection 7.2. Assume that there exist $A^{(\alpha)} \in L^\infty(Cyl^{(\alpha)})^{6\times 6}$ such that*

$$\Pi_\varepsilon^{(\alpha)}\Big(\mathbf{A}_\varepsilon^{(\alpha)}\Big(\frac{\cdot}{\varepsilon}\Big)\Big)(z',Y) \to A^{(\alpha)}(Y) \quad \textit{for a.e.} \ \ (z',Y)\in\Omega\times Cyl^{(\alpha)}. \tag{7.60}$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $(\mathbb{U},\mathbb{U}^{(\mathbf{S})},\mathbb{U}^{(\mathbf{B})},\widehat{u}) \in \mathcal{X}$ such that a solution $(u_\varepsilon^{(1)},u_\varepsilon^{(2)})$ of problem (5.30) converges. The unfolded limit problem admits solutions and has the following formulation:

Find $(\mathbb{U},\mathbb{U}^{(\mathbf{S})},\mathbb{U}^{(\mathbf{B})},\widehat{u}) \in \mathcal{X}$ such that for every $(\mathbb{V},\mathbb{V}^{(\mathbf{B})},\mathbb{V}^{(\mathbf{B})},\widehat{v}) \in \mathcal{X}$:

$$\begin{aligned}
&\sum_{\alpha=1}^{2}\int_{\Omega\times Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl}\big(\mathcal{E}^{(\alpha)}_{ij}(\partial\mathbf{U}^{(\alpha)})+\mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{u}^{(\alpha)})\big)\big(\mathcal{E}^{(\alpha)}_{kl}(\partial\mathbf{U}^{(\alpha)}-\partial\mathbf{V}^{(\alpha)})+\mathcal{E}^{(\alpha)}_{Y,kl}(\widehat{u}^{(\alpha)}-\widehat{v}^{(\alpha)})\big)\boldsymbol{\eta}^{(\alpha)}dz'dY\\
&\le \mathbf{C}_0(\kappa)\Big(\sum_{\alpha,\beta=1}^{2}\int_\Omega f_\alpha^{(\beta)}\,(\mathbb{U}_\alpha-\mathbb{V}_\alpha)+f_3^{(\beta)}\,(\mathbb{U}_3-\mathbb{V}_3)dz'\\
&\qquad+\sum_{\alpha=1}^{2}\int_\Omega \widetilde{f}_\alpha^{(\alpha)}\,(\mathbb{U}_\alpha^{(\mathbf{S})}-\mathbb{V}_\alpha^{(\mathbf{S})})+\widetilde{f}_\alpha^{(3-\alpha)}\,(\mathbb{U}_\alpha^{(\mathbf{B})}-\mathbb{V}_\alpha^{(\mathbf{B})})\,dz'\Big)\\
&\quad-\mathbf{C}_1(\kappa)\sum_{\alpha,\beta=1}^{2}\int_\Omega \widetilde{f}_\alpha^{(\beta)}(\partial_\alpha\mathbb{U}_3-\partial_\alpha\mathbb{V}_3)dz',
\end{aligned}\tag{7.61}$$

where $\partial\mathbf{U}^{(\alpha)}$ and $\partial\mathbf{V}^{(\alpha)}$ are defined in (7.52) and where

$$\mathbf{C}_0(\kappa)\doteq 4\kappa^2\int_0^2\gamma(t)dt,\qquad \mathbf{C}_1(\kappa)\doteq 4\kappa^2\int_0^2\Phi(t)\gamma(t)dt.$$

Proof. First, from the weak convergence of the strain tensors (7.41)-(7.43) the strong convergence of the test functions (7.54), convergence (7.60) and Corollary 2.12 of Cioranescu, Damlamian, and Griso, 2008, we get that

$$\begin{aligned}
&\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} A^{(\alpha)}_{ijkl,\varepsilon}\widetilde{\mathbf{e}}_{ij}(u_\varepsilon^{(\alpha)})\widetilde{\mathbf{e}}_{kl}(v_\varepsilon^{(\alpha)})\ \eta^{(\alpha)}\,dz\\
&\to \int_{\Omega\times Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl}\big(\mathcal{E}^{(\alpha)}_{ij}(\partial\mathbf{U}^{(\alpha)})+\mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{u}^{(\alpha)})\big)\big(\mathcal{E}^{(\alpha)}_{kl}(\partial\mathbf{V}^{(\alpha)})+\mathcal{E}^{(\alpha)}_{Y,kl}(\widehat{v}^{(\alpha)})\big)\boldsymbol{\eta}^{(\alpha)}\,dz'dY,
\end{aligned}\tag{7.62}$$

where $(u_\varepsilon^{(1)},u_\varepsilon^{(2)})$ is a solution of (5.30) and $(v_\varepsilon^{(1)},v_\varepsilon^{(2)})$ is the test function defined in (7.53). By the weak convergences (7.41)-(7.43), the weak lower semicontinuity of the convex functionals and the definition of problem (5.30) we have

$$\begin{aligned}
&\sum_{\alpha=1}^{2}\int_{\Omega\times Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl}\big(\mathcal{E}^{(\alpha)}_{ij}(\partial\mathbf{U}^{(\alpha)})+\mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{u}^{(\alpha)})\big)\big(\mathcal{E}^{(\alpha)}_{kl}(\partial\mathbf{U}^{(\alpha)})+\mathcal{E}^{(\alpha)}_{Y,kl}(\widehat{u}^{(\alpha)})\big)\boldsymbol{\eta}^{(\alpha)}\,dz'dY\\
&\le \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} A^{(\alpha)}_{ijkl,\varepsilon}\widetilde{\mathbf{e}}_{ij}(u_\varepsilon^{(\alpha)})\widetilde{\mathbf{e}}_{kl}(u_\varepsilon^{(\alpha)})\ \eta^{(\alpha)}dz \le \liminf_{\varepsilon\to 0}\sum_{\alpha=1}^{2}\int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} F_\varepsilon^{(\alpha)}\cdot u_\varepsilon^{(\alpha)}\ \eta^{(\alpha)}dz.
\end{aligned}\tag{7.63}$$

We prove now that the last term in (7.63) converges. By the strong convergence of the applied forces, the definition of displacement (7.13) together with the weak convergences in Lemma

21, the reference frame convergences (7.59), we obtain

$$\begin{aligned}&\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} F_\varepsilon^{(\alpha)}\cdot u_\varepsilon^{(\alpha)}\,\eta^{(\alpha)}\,dz'\\ &\to \mathbf{C}_0(\kappa)\sum_{\alpha,\beta=1}^{2}\Big(\int_\Omega f_\alpha^{(\beta)}\,\mathbb{U}_\alpha+f_3^{(\beta)}\,\mathbb{U}_3dz'+\int_\Omega \widetilde{f}_\alpha^{(\alpha)}\,\mathbb{U}_\alpha^{(\mathbf{S})}+\widetilde{f}_\alpha^{(3-\alpha)}\mathbb{U}_\alpha^{(\mathbf{B})}\,dz'\Big)\\ &\qquad\qquad -\mathbf{C}_1(\kappa)\sum_{\alpha,\beta=1}^{2}\int_\Omega \widetilde{f}_\alpha^{(\beta)}\partial_\alpha\mathbb{U}_3dz'.\end{aligned}\tag{7.64}$$

At last, we get the limit of

$$\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} F_\varepsilon^{(\alpha)}\cdot v_\varepsilon^{(\alpha)}\,\eta^{(\alpha)}\,dz$$

by replacing in (7.64) the functions $(\mathbb{U},\mathbb{U}^{(\mathbf{S})},\mathbb{U}^{(\mathbf{B})})$ by the functions $(\mathbb{V},\mathbb{V}^{(\mathbf{S})},\mathbb{V}^{(\mathbf{B})})$. Hence, inequality (7.61) follows due to (7.62), (7.63) and (7.64). A density argument gives (7.61) for any test function in $\mathcal{X}$.

The existence of solutions for problem (7.61) is a direct consequence of the bilinearity, boundedness, and coercivity of $A^{(\alpha)}$ (inherited from the properties (i)-(iii) of $A_\varepsilon^{(\alpha)}$ in Subsection 5.5.2 through convergence (7.60)) and the Stampacchia's Lemma. □

7.5.2 The microscopic cell problem

Now that the limit problem has been found, we can proceed to the split of the microscopic scale from the macroscopic one. In this subsection, we investigate the microscopic problem, or cell problem, whose solution is the correctors that will later form the homogenizing operator in the macroscopic scale.

We first define $\mathbf{W}$ as the the convex subset of $\mathbf{W}^{(1)}\times\mathbf{W}^{(2)}$ by

$$\begin{aligned}\mathbf{W}\doteq\Big\{&(\widehat{w}^{(1)},\widehat{w}^{(2)})\in\mathbf{W}^{(1)}\times\mathbf{W}^{(2)}\ \Big|\\ &0\le(-1)^{a+b}\Big(\widehat{w}_3^{(1)}(a+Y_1,b,Y_2,(-1)^{a+b+1}\kappa)-\widehat{w}_3^{(2,a)}(a,b+Y_2,Y_1,(-1)^{a+b}\kappa)\Big)\\ &\qquad\qquad\text{a.e. on }\omega_\kappa,\ (a,b)\in\{0,1\}^2\Big\}.\end{aligned}$$

Now, we introduce the correctors' problem. For every $X\in\mathbb{R}^9$, we denote

$$X^{(1)}=(X_1,X_2,X_5,X_6,X_7),\qquad X^{(2)}=(X_3,X_4,X_8,X_5,X_9).$$

We consider the following microscopic cell problems:

$$\begin{aligned}&\text{For each }X^{(\alpha)}\in\mathbb{R}^5,\text{ find }\widehat{\chi}\in\mathbf{W}\text{ such that for every }\widehat{v}\in\mathbf{W}:\\ &\sum_{\alpha=1}^{2}\int_{Cyl^{(\alpha)}}A_{ijkl}^{(\alpha)}\,(\mathcal{E}_{ij}^{(\alpha)}(X^{(\alpha)})+\mathcal{E}_{Y,ij}^{(\alpha)}(\widehat{\chi})\mathcal{E}_{Y,kl}^{(\alpha)}(\widehat{\chi}-\widehat{v})\,\boldsymbol{\eta}^{(\alpha)}\,dY\le 0.\end{aligned}\tag{7.65}$$

The existence of solutions follows by Stampacchia's Lemma.

Now, if $\widehat{\chi}$ and $\widetilde{\chi}$ are both solutions of (7.65), then we can first consider problem (7.65) with $\widehat{\chi}$ as solution and $\widetilde{\chi}$ as test-function and then vice versa. Summing up both inequalities leads to

$$\sum_{\alpha=1}^{2}\int_{Cyl^{(\alpha)}}A_{ijkl}^{(\alpha)}\,\mathcal{E}_{ij,Y}^{(\alpha)}(\widetilde{\chi}-\widehat{\chi})\,\mathcal{E}_{kl,Y}^{(\alpha)}(\widetilde{\chi}-\widehat{\chi})\,\boldsymbol{\eta}^{(\alpha)}\,dY\le 0,$$

from where we get that $\mathcal{E}_Y^{(\alpha)}(\widehat{\chi}) = \mathcal{E}_Y^{(\alpha)}(\widetilde{\chi})$, since by coercivity the above quantity is also non-negative. Hence, Lemma 27 implies that there exist rigid displacements $\mathbf{r}^{(\alpha)} \in \mathbf{W}$ such that

$$\widehat{\chi} = \mathbf{r}^{(\alpha)} + \widetilde{\chi}, \qquad \text{a.e. in} \quad Cyl^{(\alpha)}.$$

As the strain tensors of the solutions of (7.65) are uniquely determined, we will henceforth denote them $\mathcal{E}_Y^{(\alpha)}(\widehat{\chi}(X^{(\alpha)},\cdot))$.

7.5.3 The homogenizing operator and the macroscopic cell problem

Now that problem (7.65) has been investigated, we can define the homogenizing operators by integrating over the solutions of the cell problems.

Definition 15. *We define the homogenizing operator A_{hom} by*

$$\forall X \in \mathbb{R}^9, \qquad A_{hom,n}(X) \doteq \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} \left(\mathcal{E}^{(\alpha)}_{ij}(X^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{\chi}(X^{(\alpha)},Y))\right)\mathcal{E}^{(\alpha)}_{kl}(\mathbf{e}^{(\alpha)}_n)\,\boldsymbol{\eta}^{(\alpha)}\,dY,$$

where $\widehat{\chi}(X^{(\alpha)},\cdot)$ are a solution of problem (7.65) and $\left(\mathbf{e}_1,\dots,\mathbf{e}_9\right)$ the usual basis of $\mathbb{R}^9$.

Now, to ensure the existence of solutions for the macroscopic problem, we need to prove some properties of the homogenizing operator to apply the Stampacchia Lemma.

Proposition 11. *The operator A_{hom} is continuous (and thus of Caratheodory type), bounded, monotone and coercive.*

Proof. Step 1. We show that the map $X^{(1)} \in \mathbb{R}^5 \longmapsto \mathcal{E}_Y(\widehat{\chi}(X^{(1)},\cdot))$ is Lipschitz continuous for the strong topology of $L^2(Cyl^{(1)})^6$.
We will only prove the statement for $\alpha = 1$, since the proof for $\alpha = 2$ is analogous.
Let $X^{(1)}, Z^{(1)}$ be two vectors in $\mathbb{R}^5$ and $\widehat{\chi}(X^{(1)},\cdot), \widehat{\chi}(Z^{(1)},\cdot)$ be the associated solutions given by the cell problem (7.65). By the coercivity of the tensor $A^{(1)}$, we have

$$\begin{aligned}
&\|\mathcal{E}_Y^{(1)}(\widehat{\chi}(X^{(1)},\cdot)) - \mathcal{E}_Y^{(1)}(\widehat{\chi}(Z^{(1)},\cdot))\|^2_{L^2(Cyl^{(1)})} \\
&\le \int_{Cyl^{(1)}} A^{(1)}_{ijkl}\left(\mathcal{E}^{(1)}_{Y,ij}(\widehat{\chi}(X^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,ij}(\widehat{\chi}(Z^{(1)},\cdot))\right)\left(\mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(X^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(Z^{(1)},\cdot))\right)\boldsymbol{\eta}^{(1)}\,dY \\
&\le \int_{Cyl^{(1)}} A^{(1)}_{ijkl}\,\mathcal{E}^{(1)}_{Y,ij}(\widehat{\chi}(X^{(1)},\cdot))\left(\mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(X^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(Z^{(1)},\cdot))\right)\boldsymbol{\eta}^{(1)}\,dY \\
&\quad + \int_{Cyl^{(1)}} A^{(1)}_{ijkl}\,\mathcal{E}^{(1)}_{Y,ij}(\widehat{\chi}(Z^{(1)},\cdot))\left(\mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(Z^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(X^{(1)},\cdot))\right)\boldsymbol{\eta}^{(1)}\,dY \\
&\le -\int_{Cyl^{(1)}} A^{(1)}_{ijkl}\,\mathcal{E}^{(1)}_{ij}(X^{(1)})\left(\mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(X^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(Z^{(1)},\cdot))\right)\boldsymbol{\eta}^{(1)}\,dY \\
&\quad -\int_{Cyl^{(1)}} A^{(1)}_{ijkl}\,\mathcal{E}^{(1)}_{ij}(Z^{(1)})\left(\mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(Z^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(X^{(1)},\cdot))\right)\boldsymbol{\eta}^{(1)}\,dY \\
&\le \int_{Cyl^{(1)}} A^{(1)}_{ijkl}\,\mathcal{E}^{(1)}_{ij}(X^{(1)} - Z^{(1)})\left(\mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(X^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\widehat{\chi}(Z^{(1)},\cdot))\right)\boldsymbol{\eta}^{(1)}\,dY \\
&\le C\|\mathcal{E}^{(1)}(X^{(1)} - Z^{(1)})\|_{L^2(Cyl^{(1)})}\|\mathcal{E}_Y^{(1)}(\widehat{\chi}(X^{(1)},\cdot)) - \mathcal{E}_Y^{(1)}(\widehat{\chi}(Z^{(1)},\cdot)))\|_{L^2(Cyl^{(1)})}.
\end{aligned}$$

Hence, the Lipschitz continuity is proven since

$$\|\mathcal{E}_Y^{(1)}(\widehat{\chi}(X^{(1)},\cdot)) - \mathcal{E}_Y^{(1)}(\widehat{\chi}(Z^{(1)},\cdot))\|_{L^2(Cyl^{(1)})} \le C\|\mathcal{E}^{(1)}(X^{(1)} - Z^{(1)})\|_{L^2(Cyl^{(1)})} \le C|X^{(1)} - Z^{(1)}|.$$

So, the statement of this step is proved. As a consequence the map $X \in \mathbb{R}^9 \longmapsto A_{hom}(X) \in \mathbb{R}^9$ is continuous.
Step 2.We prove that A_{hom} is monotone.

Let X and Z be two vectors in $\mathbb{R}^9$. By the coercivity of the tensor $A^{(\alpha)}$, we have

$$\begin{aligned}
&\big(A_{hom}(X) - A_{hom}(Z)\big) \cdot (X - Z) \\
&= \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} \left(\mathcal{E}^{(\alpha)}_{ij}(X^{(\alpha)} - Z^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{\chi}(X^{(\alpha)},\cdot) - \widehat{\chi}(Z^{(\alpha)},\cdot))\right) \mathcal{E}^{(\alpha)}_{kl}(X^{(\alpha)} - Z^{(\alpha)})\, \boldsymbol{\eta}^{(\alpha)}\, dY \\
&\geq C \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} |\mathcal{E}^{(\alpha)}_{ij}(X^{(\alpha)} - Z^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{\chi}(X^{(\alpha)},\cdot) - \widehat{\chi}(Z^{(\alpha)},\cdot))|^2 \boldsymbol{\eta}^{(\alpha)}\, dY \\
&\quad - \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} (\mathcal{E}^{(\alpha)}_{ij}(X^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{\chi}(X^{(\alpha)},\cdot))\, \mathcal{E}^{(\alpha)}_{Y,kl}(\widehat{\chi}(X^{(\alpha)},\cdot) - \widehat{\chi}(Z^{(\alpha)},\cdot))\, \boldsymbol{\eta}^{(\alpha)}\, dY \\
&\quad - \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} (\mathcal{E}^{(\alpha)}_{ij}(Z^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{\chi}(Z^{(\alpha)},\cdot)))\, \mathcal{E}^{(\alpha)}_{Y,kl}(\widehat{\chi}(Z^{(\alpha)},\cdot) - \widehat{\chi}(X^{(\alpha)},\cdot))\, \boldsymbol{\eta}^{(\alpha)}\, dY \geq 0,
\end{aligned}$$

where the last passage follows from the fact that the first integral is non-negative, while the second and third integrals are non-negative by definition of problem (7.65) with the choice of test functions $\widehat{\chi}(X^{(1)},\cdot)$ and $\widehat{\chi}(Z^{(1)},\cdot)$ respectively. Thus the monotonicity of A_{hom} is proved.
Step 3. We prove that A_{hom} is coercive.
From the first inequality of (7.58) we have ($\alpha \in \{1,2\}$)

$$\int_{Cyl^{(\alpha)}} |\mathcal{E}^{(\alpha)}(X^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y}(\widehat{\chi}(X^{(\alpha)},\cdot))|^2 \boldsymbol{\eta}^{(\alpha)}\, dY \geq C_0 |X^{(\alpha)}|^2 \qquad \forall X^{(\alpha)} \in \mathbb{R}^5. \tag{7.66}$$

Hence, for every X in $\mathbb{R}^9$ we get

$$\begin{aligned}
A_{hom}(X)\cdot X &= \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} \left(\mathcal{E}^{(\alpha)}_{ij}(X^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{\chi}(X^{(\alpha)},\cdot))\right) \mathcal{E}^{(\alpha)}_{kl}(X^{(\alpha)})\, \boldsymbol{\eta}^{(\alpha)}\, dY \\
&= \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} \left(\mathcal{E}^{(\alpha)}_{ij}(X^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{\chi}(X^{(\alpha)},\cdot))\right) \left(\mathcal{E}^{(\alpha)}_{kl}(X^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,kl}(\widehat{\chi}(X^{(\alpha)},\cdot))\right) \boldsymbol{\eta}^{(\alpha)}\, dY \\
&\quad - \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} \left(\mathcal{E}^{(\alpha)}_{ij}(X^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{\chi}(X^{(\alpha)},\cdot))\right) \mathcal{E}^{(\alpha)}_{Y,kl}(\widehat{\chi}(X^{(\alpha)},\cdot))\, \boldsymbol{\eta}^{(\alpha)}\, dY \geq C_0 |X|^2,
\end{aligned}$$

where the last passage follows from inequality (7.66) and the fact that the second integral is non-negative by the definition of problem (7.65) with the choice of a zero test function. Hence, the coercivity of A_{hom} is proved. □

We can finally write the macroscopic problem. Set

$$\begin{aligned}
\mathcal{X}^H \doteq \Big\{ (\mathbb{V}, \mathbb{V}^{(\mathbf{S})}, \mathbb{V}^{(\mathbf{B})}) \in \mathcal{X}_M \times \mathcal{X}_S \times \mathcal{X}_B \;\Big|\; & \\
& |\mathbb{V}_1^{(\mathbf{S})}(\cdot, b) - \mathbb{V}_1^{(\mathbf{B})}(\cdot, a)| + \kappa |\partial_2 \mathbb{V}_1 + \partial_1 \mathbb{V}_2| \leq g_1 \ \text{ a.e. in } \ \Omega, \\
& |\mathbb{V}_2^{(\mathbf{S})}(\cdot, a) - \mathbb{V}_2^{(\mathbf{B})}(\cdot, b)| + \kappa |\partial_2 \mathbb{V}_1 + \partial_1 \mathbb{V}_2| \leq g_2 \ \text{ a.e. in } \ \Omega, (a,b) \in \{0,1\}^2 \Big\},
\end{aligned}$$

which consists of the original space $\mathcal{X}$ without the microscopic functions.

Theorem 8. *The macroscopic homogenized problem has the following formulation:*

$$\begin{cases} \text{Find } \big(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}, \mathbb{U}^{(\mathbf{B})}\big) \in \mathcal{X}^H \text{ such that for every } \big(\mathbb{V}, \mathbb{V}^{(\mathbf{S})}, \mathbb{V}^{(\mathbf{B})}\big) \in \mathcal{X}^H: \\ \displaystyle\int_\Omega A_{hom}(\partial \mathbf{U}) \cdot (\partial \mathbf{U} - \partial \mathbf{V})\, dz' \leq \mathbf{C}_0(\kappa) \sum_{\beta=1}^{2} \Big(\sum_{\alpha=1}^{2} \int_\Omega f_\alpha^{(\beta)} (\mathbb{U}_\alpha - \mathbb{V}_\alpha)\, dz' + \int_\Omega f_3^{(\beta)} (\mathbb{U}_3 - \mathbb{V}_3) dz' \Big) \\ \displaystyle + \frac{\mathbf{C}_0(\kappa)}{2} \int_\Omega \sum_{c=0}^{1} \big(\widetilde{f}_\alpha^{(\alpha)} \, (\mathbb{U}_\alpha^{(\mathbf{S})} - \mathbb{V}_\alpha^{(\mathbf{S})})(\cdot, c) + \widetilde{f}_\alpha^{(3-\alpha)} (\mathbb{U}_\alpha^{(\mathbf{B})} - \mathbb{V}_\alpha^{(\mathbf{B})})(\cdot, c)\big)\, dz' \\ \displaystyle - \mathbf{C}_1(\kappa) \sum_{\alpha=1,\beta=1}^{2} \int_\Omega \widetilde{f}_\alpha^{(\beta)} (\partial_\alpha \mathbb{U}_3 - \partial_\alpha \mathbb{V}_3) dz', \quad \forall \big(\mathbb{V}, \mathbb{V}^{(\mathbf{S})}, \mathbb{V}^{(\mathbf{B})}\big) \in \mathcal{X}_M \times \mathcal{X}_S \times \mathcal{X}_B. \end{cases} \tag{7.67}$$

It admits solutions, but in general, the solution is not unique.

Proof. The existence of solutions to problem (7.67) is a direct consequence of the properties of the homogenizing operator A_{hom} given in Proposition 11 together with the Stampacchia's Lemma. □

The operator structure of the homogenized problem is known as the Leray−Lions operator.

Starting from the form of the final decomposition of the displacement (7.13) and going to the limit, the cell problem (7.65) and the macroscopic problem (7.67) give the approximation of the limit displacements in the direction of beams $\mathbf{e}_1$ and $\mathbf{e}_2$, that are a.e. $z' \in \Omega$:

$$\begin{aligned} u^{(1)}(z_1, q\varepsilon, y_2, y_3) \approx & \underbrace{\begin{pmatrix} \varepsilon \mathbb{U}_1(q\varepsilon) + \varepsilon^2 \mathbb{U}_1^{(\mathbf{S})}(z_1, q\varepsilon, b) \\ \varepsilon \mathbb{U}_2(z_1) + \varepsilon^2 \mathbb{U}_2^{(\mathbf{B})}(z_1, q\varepsilon, b) \\ \varepsilon \mathbb{U}_3(z_1, q\varepsilon) \end{pmatrix} + \begin{pmatrix} \varepsilon \partial_2 \mathbb{U}_3(z_1, q\varepsilon) \\ -\varepsilon \partial_1 \mathbb{U}_3(z_1, q\varepsilon) \\ \varepsilon \partial_1 \mathbb{U}_2(z_1) \end{pmatrix} \wedge \Phi_\varepsilon^{(1)}(z_1) \mathbf{e}_3}_{\text{middle line displacement}} \\ & + \underbrace{\begin{pmatrix} \varepsilon \partial_2 \mathbb{U}_3(z_1, q\varepsilon) \\ -\varepsilon \partial_1 \mathbb{U}_3(z_1, q\varepsilon) \\ \varepsilon \partial_1 \mathbb{U}_2(z_1) \end{pmatrix}}_{\text{cross-section rotation}} \wedge \big(y_2 \mathbf{e}_2 + y_3 \mathbf{n}_\varepsilon^{(1)}(z_1)\big), \\ u^{(2)}(p\varepsilon, z_2, y_1, y_3) \approx & \begin{pmatrix} \varepsilon \mathbb{U}_1(z_2) + \varepsilon^2 \mathbb{U}_2^{(\mathbf{B})}(p\varepsilon, z_2, a) \\ \varepsilon \mathbb{U}_2(p\varepsilon) + \varepsilon^2 \mathbb{U}_2^{(\mathbf{S})}(p\varepsilon, z_2) \\ \varepsilon \mathbb{U}_3(p\varepsilon, z_2) \end{pmatrix} + \begin{pmatrix} \varepsilon \partial_2 \mathbb{U}_3(p\varepsilon, z_2) \\ -\varepsilon \partial_1 \mathbb{U}_3(p\varepsilon, z_2) \\ \varepsilon \partial_2 \mathbb{U}_1(z_2) \end{pmatrix} \wedge \Phi_\varepsilon^{(2)}(z_2) \mathbf{e}_3 \\ & + \begin{pmatrix} \varepsilon \partial_2 \mathbb{U}_3(p\varepsilon, z_2) \\ -\varepsilon \partial_1 \mathbb{U}_3(p\varepsilon, z_2) \\ \varepsilon \partial_2 \mathbb{U}_1(z_2) \end{pmatrix} \wedge \big(y_1 \mathbf{e}_1 + y_3 \mathbf{n}_\varepsilon^{(2)}(z_2)\big). \end{aligned} \tag{7.68}$$

Chapter 8

Conclusions

In this chapter, we gather all the obtained results throughout the thesis to have a final overview of our achievements, their physical meaning, and their application.

8.1 Results of the extension of the unfolding method

Concerning the periodic unfolding for anisotropically bounded sequences, the results have been crucial to find the convergences of fields in the unsupported subdomains in Chapter 7. However, the theoretical results exceed the frame of textile structures and can be applied to many other contexts. Among others, we mention Griso, Khilkova, and Orlik, 2022, where structures made of beams are considered, and the same contrast on the gradient estimates appears on the unstable oscillating thin straits. Moreover, the homogenization of problem (3.11), and its equivalent formulation (3.20), where the anisotropy is shifted to the material coefficients, can be found Griso, Migunova, and Orlik, 2017 and Griso, Migunova, and Orlik, 2016.

Concerning the periodic unfolding for lattice structures, it is a very powerful tool when dealing with thin periodic structures made from lattices. In this context, we would like to cite again lattice structures made of beams in stable (see Griso et al., 2020; Griso et al., 2021) and unstable configuration (see again Griso, Khilkova, and Orlik, 2022, together with anisotropic behaviors). More generally, such a tool can be applied to many other problems related to partial differential equations on domains involving periodic grids, lattices, thin frames, and glued fiber structures.

8.2 Macroscopic behavior of $r = \kappa\varepsilon$ textiles with linear elastic yarns according to the contact strength ε^h.

Concerning the second part of the thesis (Chapter 5-7), we now give an overview of the results concerning the macroscopic behavior of our square of woven elastic yarns, with particular attention to the role the contact strength between yarns plays in the supported and unsupported parts of the domain, and on the displacement behavior.

For the sake of completeness, to the results obtained in Section 6.2 for $h = 1$ and in Section 7.5 for $h = 2$, we also recall the main results in Griso, Orlik, and Wackerle, 2020a concerning textiles with strong contact ($h = 3$) or almost glued fibers ($h \geq 4$).

8.2.1 Results (known) for a textile with contact $g_\varepsilon \sim \varepsilon^4 g$ or higher

As we know from Section 6.1, the estimates for the displacement fields are the same for $h \geq 3$ in the whole domain Ω. However, the contact conditions are not (see (6.1) and (6.2)), and so they are not on the homogenized problem.

Recall the space definitions (7.18) and let $\mathbf{U} \doteq (\mathbb{U}_1, \mathbb{U}_2, \mathbb{U}_3) \in H^1(\Omega)^2_\Gamma \times H^2(\Omega)_\Gamma$. Set

$$e_{\alpha\beta}(\mathbb{U}) \doteq \frac{1}{2}\big(\partial_\alpha \mathbb{U}_\beta + \partial_\beta \mathbb{U}_\alpha\big), \qquad (\alpha, \beta) \in \{1,2\}^2.$$

The macroscopic homogenized problem has the following formulation:

$$\begin{cases} \text{Find } \mathbf{U} \in H^1(\Omega)^2_\Gamma \times H^2(\Omega)_\Gamma \text{ such that for every } \mathbf{V} \in H^1(\Omega)^2_\Gamma \times H^2(\Omega)_\Gamma\text{:} \\ \int_\Omega A_{hom,lin}(\partial \mathbf{U}) \cdot \partial \mathbf{V}\, dz' = \mathbf{C}_0(\kappa) \int_\Omega f^{(\alpha)} \cdot \mathbf{V}\, dz', \end{cases} \tag{8.1}$$

where

$$\begin{aligned} \partial \mathbf{U} &\doteq \big(e_{11}(\mathbb{U}), e_{12}(\mathbb{U}), e_{22}(\mathbb{U}), \partial_{11}\mathbb{U}_3, \partial_{22}\mathbb{U}_3, \partial_{12}\mathbb{U}_3\big), \\ \partial \mathbf{V} &\doteq \big(e_{11}(\mathbb{V}), e_{12}(\mathbb{V}), e_{22}(\mathbb{V}), \partial_{11}\mathbb{V}_3, \partial_{22}\mathbb{V}_3, \partial_{12}\mathbb{V}_3\big). \end{aligned} \tag{8.2}$$

The homogenizing operator $A_{hom,lin}$ is the bilinear function from $\mathbb{R}^6$ to $\mathbb{R}^6$ defined by

$$A_{hom,lin}(X_m, X_n) \doteq X_m X_n \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} \left(\mathcal{E}^{(\alpha)}_{ij}(\mathbf{e}_m) + \mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{\chi}_m)\right)\mathcal{E}^{(\alpha)}_{kl}(\mathbf{e}_n)\, \boldsymbol{\eta}^{(\alpha)}\, dY.$$

For every $X \in \mathbb{R}^6$, the macroscopic strain tensors $\mathcal{E}^{(1)}(X)$ and $\mathcal{E}^{(2)}(X)$ are defined as in (7.40), but with

$$\begin{aligned} \mathfrak{F}^{(1)}(X) &\doteq \left(\begin{pmatrix} X_1 \\ X_2 \\ -\mathbf{\Theta}' X_4 \end{pmatrix} + \begin{pmatrix} X_6 \\ -X_4 \\ 0 \end{pmatrix} \wedge \left(\Phi^{(1)}\mathbf{e}_3 + Y_2\mathbf{e}_2 + Y_3\mathbf{n}^{(1)}\right)\right), \\ \mathfrak{F}^{(2)}(X) &\doteq \left(\begin{pmatrix} X_2 \\ X_3 \\ -\mathbf{\Theta}' X_5 \end{pmatrix} + \begin{pmatrix} X_5 \\ -X_6 \\ 0 \end{pmatrix} \wedge \left(\Phi^{(2)}\mathbf{e}_3 + Y_1\mathbf{e}_1 + Y_3\mathbf{n}^{(2)}\right)\right) \end{aligned} \tag{8.3}$$

The correctors $\widehat{\chi}_1, \dots, \widehat{\chi}_6$ belong to the convex set $\mathbf{W}$ defined by

$$\begin{aligned} \mathbf{W}_{lin} \doteq \Big\{ &(\widehat{w}^{(1)}, \widehat{w}^{(2)}) \in \mathbf{W}^{(1)} \times \mathbf{W}^{(2)} \;\Big| \\ &\big|(Y_1 - a)X_1 - (Y_2 - b)X_2 + \widehat{w}^{(1)}_1(a + Y_1, b, Y_2, (-1)^{a+b+1}) - \widehat{w}^{(2)}_1(a, b + Y_2, Y_1, (-1)^{a+b})\big| = 0, \\ &\big|(Y_1 - a)X_2 - (Y_2 - b)X_3 + \widehat{w}^{(1)}_2(a + Y_1, b, Y_2, (-1)^{a+b+1}) - \widehat{w}^{(2)}_2(a, b + Y_2, Y_1, (-1)^{a+b})\big| = 0 \\ &\widehat{w}^{(1)}_1(a + Y_1, b, Y_2, (-1)^{a+b+1}) - \widehat{w}^{(2)}_1(a, b + Y_2, Y_1, (-1)^{a+b}) = 0, \text{a.e. on } \omega_\kappa,\; (a,b) \in \{0,1\}^2 \Big\}. \end{aligned}$$

Furthermore, they are the solution of the microscopic cell problems:

$$\begin{cases} \text{For each } X_i \in \mathbb{R}^6\text{, find } \widehat{\chi}_i \in \mathbf{W}_{lin} \text{ such that for every } \widehat{v} \in \mathbf{W}_{lin} : \\ \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} \left(\mathcal{E}^{(\alpha)}_{ij}(X^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{\chi}) \mathcal{E}^{(\alpha)}_{Y,kl}(\widehat{v})\, \boldsymbol{\eta}^{(\alpha)}\, dY = 0. \end{cases}$$

The displacements behave the same in the whole domain Ω. In particular, their approximation with the limit fields as solutions of the homogenized problem (8.1):

$$\begin{aligned}
u^{(1)}(z_1, q\varepsilon, y_2, y_3) &\approx \underbrace{\begin{pmatrix} \varepsilon^2 \mathbb{U}_1 \\ \varepsilon^2 \mathbb{U}_2 \\ \varepsilon \mathbb{U}_3 \end{pmatrix}(z_1, q\varepsilon) + \begin{pmatrix} \varepsilon\partial_2 \mathbb{U}_3 \\ -\varepsilon\partial_1 \mathbb{U}_3 \\ 0 \end{pmatrix}(z_1, q\varepsilon) \wedge \Phi_\varepsilon^{(1)}(z_1)\mathbf{e}_3}_{\text{middle line displacement}} \\
&\quad + \underbrace{\begin{pmatrix} \varepsilon\partial_2 \mathbb{U}_3 \\ -\varepsilon\partial_1 \mathbb{U}_3 \\ 0 \end{pmatrix}}_{\text{cross-section rotation}}(z_1, q\varepsilon) \wedge \big(y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon^{(1)}(z_1)\big), \\
u^{(2)}(p\varepsilon, z_2, y_1, y_3) &\approx \begin{pmatrix} \varepsilon^2 \mathbb{U}_1 \\ \varepsilon^2 \mathbb{U}_2 \\ \varepsilon \mathbb{U}_3 \end{pmatrix}(p\varepsilon, z_2) + \begin{pmatrix} \varepsilon\partial_2 \mathbb{U}_3 \\ -\varepsilon\partial_1 \mathbb{U}_3 \\ 0 \end{pmatrix}(p\varepsilon, z_2) \wedge \Phi_\varepsilon^{(2)}(z_2)\mathbf{e}_3 \\
&\quad + \begin{pmatrix} \varepsilon\partial_2 \mathbb{U}_3 \\ -\varepsilon\partial_1 \mathbb{U}_3 \\ 0 \end{pmatrix}(p\varepsilon, z_2) \wedge \big(y_1\mathbf{e}_1 + y_3\mathbf{n}_\varepsilon^{(2)}(z_2)\big).
\end{aligned} \tag{8.4}$$

Concerning this elasticity problem, we note that in the definition of the microscopic space $\mathbf{W}_{lin}$, the contact is so strong that the gap function vanishes on the right-hand side, leaving linear conditions in the three components. This fact leads to linear cell problems, a bi-linear homogenizing operator, and thus a fully linear problem.

This case could have been achieved by gluing all the fibers in all the contact domains of Ω, namely $g = 0$ in (5.27)-(5.28). The problem could have been studied as in Griso, Orlik, and Wackerle, 2020b by extending the woven textile to a periodically perforated domain.

From (8.4), the displacement is expected to behave the same in all the four subdomains $\Omega_1 - \Omega_4$, and a partition is unnecessary. This means that the contact is so strong that even if a partial clamp is set, the fibers inherit all the properties from the clamped ones.

We also note that the limit displacements (8.4) have the third component of the cross-section rotation equal to zero. This translates into an absence of in-plane rotation for the fibers: the fibers tend to stay straight for small deformations.

8.2.2 Results (known) for a textile with contact $g_\varepsilon \sim \varepsilon^3 g$

As in the previous case, the macroscopic homogenized problem has the following formulation:

$$\begin{cases} \text{Find } \mathbf{U} \in H^1(\Omega)^2_{\mathbf{\Gamma}} \times H^2(\Omega)_{\mathbf{\Gamma}} \text{ such that for every } \mathbf{V} \in H^1(\Omega)^2_{\mathbf{\Gamma}} \times H^2(\Omega)_{\mathbf{\Gamma}}\text{:} \\ \displaystyle\int_\Omega A_{hom,lin}(\partial\mathbf{U}) \cdot \partial\mathbf{V}\, dz' = \mathbf{C}_0(\kappa) \int_\Omega f^{(\alpha)} \cdot \mathbf{V}\, dz', \end{cases} \tag{8.5}$$

where $\partial\mathbf{U}$ and $\partial\mathbf{V}$ are defined as in (8.2).

The homogenizing operator $A_{hom,lin}$ is function from $\mathbb{R}^6$ to $\mathbb{R}^6$ defined by

$$A_{hom}(z', X_m) \cdot X_n \doteq X_n \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A_{ijkl}^{(\alpha)} \big(\mathcal{E}_{ij}^{(\alpha)}(X_m) + \mathcal{E}_{Y,ij}^{(\alpha)}(\widehat{\chi}_m(z', Y))\big) \mathcal{E}_{kl}^{(\alpha)}(\mathbf{e}_n^{(\alpha)})\, \boldsymbol{\eta}^{(\alpha)}\, dY.$$

For every $X \in \mathbb{R}^6$, the macroscopic strain tensors $\mathcal{E}^{(1)}(X)$ and $\mathcal{E}^{(2)}(X)$ are defined as in (7.40), but with $\mathfrak{F}^{(1)}(X)$, $\mathfrak{F}^{(2)}(X)$ replaced by (8.3).

The correctors $\widehat{\chi}_1(z,\cdot),\dots,\widehat{\chi}_6(z,\cdot)$ belong to the convex set $\mathbf{W}$ defined by

$$\begin{aligned}\mathbf{W}_z \doteq \Big\{&(\widehat{w}^{(1)},\widehat{w}^{(2)}) \in \mathbf{W}^{(1)} \times \mathbf{W}^{(2)} \mid \\ &\big|(Y_1-a)X_1-(Y_2-b)X_2+\widehat{w}_1^{(1)}(a+Y_1,b,Y_2,(-1)^{a+b+1})-\widehat{w}_1^{(2)}(a,b+Y_2,Y_1,(-1)^{a+b})\big| \le g_1,\\ &\big|(Y_1-a)X_2-(Y_2-b)X_3+\widehat{w}_2^{(1)}(a+Y_1,b,Y_2,(-1)^{a+b+1})-\widehat{w}_2^{(2)}(a,b+Y_2,Y_1,(-1)^{a+b})\big| \le g_2\\ &0 \le (-1)^{a+b}\big(\widehat{w}_1^{(1)}(a+Y_1,b,Y_2,(-1)^{a+b+1})-\widehat{w}_1^{(2)}(a,b+Y_2,Y_1,(-1)^{a+b})\big) \le g_3,\\ &\text{a.e. on } \omega_\kappa,\ (a,b)\in\{0,1\}^2\Big\}.\end{aligned}$$

Furthermore, they are the solution of the microscopic cell problems:

$$\begin{cases}\text{For each } (z',X_i) \in \overline{\Omega}\times\mathbb{R}^6, \text{ find } \widehat{\chi}_i \in \mathbf{W}_z \text{ such that for every } \widehat{v}\in\mathbf{W}_z:\\ \displaystyle\sum_{\alpha=1}^{2}\int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl}\left(\mathcal{E}^{(\alpha)}_{ij}(X_i)+\mathcal{E}^{(\alpha)}_{Y,ij}(\widehat{\chi}_i(z',Y)\right)\mathcal{E}^{(\alpha)}_{Y,kl}(\widehat{\chi}_i(z',Y)-\widehat{v}(z',Y))\,\boldsymbol{\eta}^{(\alpha)}\,dY \le 0.\end{cases}$$

The displacements' approximation by the limit fields as solutions of the homogenized problem (8.5) is given by (8.4).

In this case, we make some new considerations. Looking at the definition of the microscopic space $\mathbf{W}_z$ the function g does not vanish on all three components and maintains the macro-micro inequality, leading to non-linear cell problems with field coupling. Consequently, the solutions are non-linear correctors that still depend on the macroscopic variable z'.

The homogenizing operator also depends non-linearly on the macroscopic fields, but the absence of only macroscopic conditions leads to a linear homogenized problem.

The displacement is the same as in the previous case: it behaves the same in all four subdomains $\Omega_1-\Omega_4$ due to the strong contact order, and we expect no in-plane rotations.

8.2.3 New results for a textile with contact $g_\varepsilon \sim \varepsilon^2 g$

For this case, we refer to the results rigorously proved in Section 7.5 and more generally in Chapter 7.

We note that in the definition of the microscopic space $\mathbf{W}$, only one inequality appears in the third direction. It involves the macro-micro remainders $\widehat{w}$, and an upper bound given by g_3 is no more present. Nevertheless, the inequality is maintained in the cell problems and leads to the presence of non-linear correctors, as in the strong contact case. But differently from this case, the absence of fully macroscopic fields in $\mathbf{W}$ implies that the correctors do not depend on z', and so does not the homogenizing operator A_{hom}.

On the other hand, the in-plane macroscopic fields become a constraint for the homogenized problem in the in-plane components (see the definition of $\mathcal{X}^H$), and therefore inequality is also maintained in the macroscopic scale.

Concerning the approximation of the displacement, we note that due to the definition of the fields $\mathbb{U}_1$ (which vanishes in $\Omega_1\cup\Omega_2$) and $\mathbb{U}_2$ (which vanishes in $\Omega_1\cup\Omega_3$), the displacement (7.68) is different in the four subdomains Ω_1-Ω_4. Moreover, their presence is responsible for the displacement's in-plane rotations (see the comparison between the third component in the rotation cross-section of (7.68) and the ones of (8.4). In this sense, the presence of these partially vanishing fields is one of the biggest results of the study of this type of textile.

The limit contact conditions give another crucial aspect of this case: they bind not only the distance between stretching and bending in the contact areas but also give a maximum bound on the in-plane rotations.

Delving into this last point in more detail, we find it convenient to restrict the limit contact conditions to the respective subdomain Ω_1-Ω_4. Since the function $\mathbb{U}_1$ vanishes by definition in $\Omega_1 \cup \Omega_2$ and $\mathbb{U}_2$ vanishes by definition in $\Omega_1 \cup \Omega_3$, we have:

$$\begin{cases} |\mathbb{U}_1^{(\mathbf{S})}(z',b) - \mathbb{U}_1^{(\mathbf{B})}(z',a)| \le g_1(z'), \\ |\mathbb{U}_2^{(\mathbf{S})}(z',a) - \mathbb{U}_2^{(\mathbf{B})}(z',b)| \le g_2(z'), \end{cases} \quad \text{a.e. } z' \in \Omega_1,$$

$$\begin{cases} |\mathbb{U}_1^{(\mathbf{S})}(z',b) - \mathbb{U}_1^{(\mathbf{B})}(z',a)| + \kappa|\partial_1 \mathbb{U}_2(z_1)| \le g_1(z'), \\ |\mathbb{U}_2^{(\mathbf{S})}(z',a) - \mathbb{U}_2^{(\mathbf{B})}(z',b)| + \kappa|\partial_1 \mathbb{U}_2(z_1)| \le g_2(z'), \end{cases} \quad \text{a.e. } z' \in \Omega_2$$

$$\begin{cases} |\mathbb{U}_1^{(\mathbf{S})}(z',b) - \mathbb{U}_1^{(\mathbf{B})}(z',a)| + \kappa|\partial_2 \mathbb{U}_1(z_2)| \le g_1(z'), \\ |\mathbb{U}_2^{(\mathbf{S})}(z',a) - \mathbb{U}_2^{(\mathbf{B})}(z',b)| + \kappa|\partial_2 \mathbb{U}_1(z_2)| \le g_2(z'), \end{cases} \quad \text{a.e. } z' \in \Omega_3$$

$$\begin{cases} |\mathbb{U}_1^{(\mathbf{S})}(z',b) - \mathbb{U}_1^{(\mathbf{B})}(z',a)| + \kappa|\partial_2 \mathbb{U}_1(z_2) + \partial_1 \mathbb{U}_2(z_1)| \le g_1(z'), \\ |\mathbb{U}_2^{(\mathbf{S})}(z',a) - \mathbb{U}_2^{(\mathbf{B})}(z',b)| + \kappa|\partial_2 \mathbb{U}_1(z_2) + \partial_1 \mathbb{U}_2(z_1)| \le g_2(z'), \end{cases} \quad \text{a.e. } z' \in \Omega_4.$$

In the subdomain Ω_1, the fields $\mathbb{U}_1$ and $\mathbb{U}_2$ both vanish, and thus the displacements (7.68) have no common directions, and their difference is bounded by the in-plane contact functions g_1 and g_2. In Ω_2, the field $\mathbb{U}_2$ appears as a common direction in the second component of the displacements and as an in-plane rotation of the displacement $u^{(1)}$. hence, the yarns in direction $\mathbf{e}_1$ have an in-plane rotation with an angle given by $\varepsilon\partial_1\mathbb{U}_2$. This angle is bounded by the macroscopic in-plane constraint $\kappa|\partial_1\mathbb{U}_2| \le g_2$. A symmetrical equivalent appears in Ω_3 due to the presence of $\mathbb{U}_1$. In Ω_4, both fields are present: the displacements have an in-plane common direction and in-plane rotations. These rotations are bounded by the macroscopic contact conditions $\kappa|\partial_2\mathbb{U}_1 + \partial_1\mathbb{U}_2| \le \min\{g_1, g_2\}$. This behavior is represented in Figure 8.1 (in the drawing, the rotations are exaggerated for the sake of understanding, we still consider small deformations).

8.2.4 The trivial case of a textile with contact $g_\varepsilon \sim \varepsilon g$

The displacement in the third direction is expected to be homogenized and behave as in the previous cases. Concerning the in-plane limit displacements, the contact is so loose that no interaction between the yarns in direction $\mathbf{e}_1$ and in direction $\mathbf{e}_2$ takes place (and the beams are free to have in-plane rotations $\mathcal{R}_3^{(1)}$ and $\mathcal{R}_3^{(2)}$, which do not depend on each other).

Since the in-plane behavior consists of the homogenization of each direction independently, this last case is of little interest for the initial task of a woven textile and especially for the role of yarns' contact.

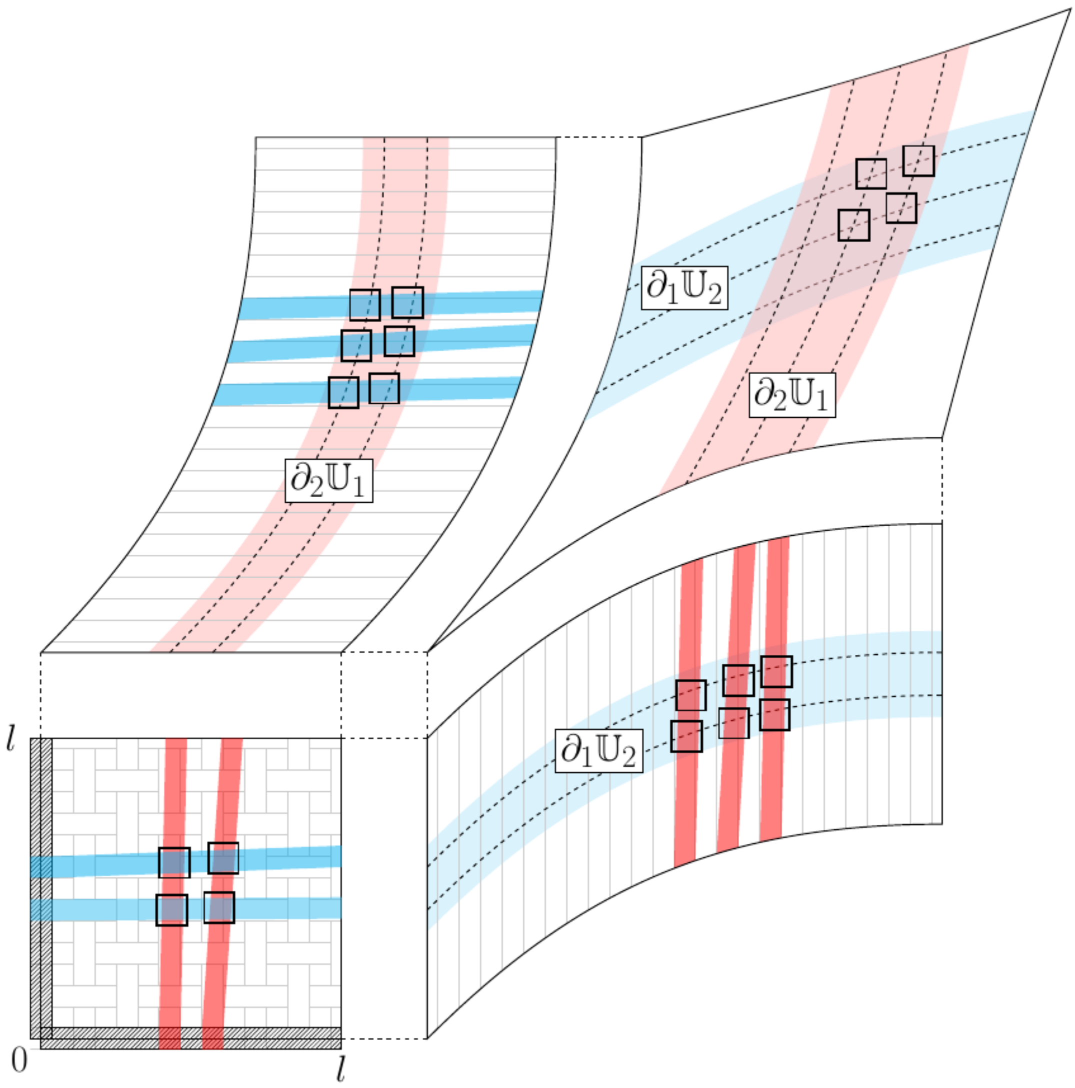

FIGURE 8.1: The expected displacement behavior in the different parts of the domain for a textile with loose contact. The black rectangles denote the admissible in-plane sliding allowed by the contact function g.

Appendix A

Technical lemmas

In this appendix, we present the technical lemmas that furnished a decisive theoretical breakthrough for the proofs of quite some of the propositions through the whole draft. Georges Griso has done the formulation of these Lemmas.

A.1 ...about the periodic unfolding for anisotropically bounded sequences

Lemma 29. *Let $p \in (1, +\infty)$ and let u be in $L^p(Y''; W^{1,p}(Y'))$ such that*

$$\nabla_{y'} u \in L^p(Y'; W^{1,p}(Y''))^{N_1}.$$

Then $\mathfrak{u} = u - \mathcal{M}_{Y'}(u)$ belongs to $W^{1,p}(Y)$. It satisfies

$$\nabla_{y'} \mathfrak{u} = \nabla_{y'} u \qquad a.e.\ in\ Y \tag{A.1}$$

and

$$\|\mathfrak{u}\|_{W^{1,p}(Y)} \le C\big(\|\nabla_{y'} u\|_{L^p(Y'\times Y'')} + \|\nabla_{y''}(\nabla_{y'} u)\|_{L^p(Y'\times Y'')}\big). \tag{A.2}$$

Proof. Step 1. We prove the statement for $u \in \mathcal{C}^2(\overline{Y})$.
Set $\mathfrak{u} = u - \mathcal{M}_{Y'}(u)$. It is clear that (A.1) is satisfied. We prove now the estimate (A.2) of $\mathfrak{u}$. By definition of $\mathfrak{u}$, equality (A.1) and the Poincaré-Wirtinger Inequality we have

$$\begin{aligned} &\|\mathfrak{u}\|_{L^p(Y'\times Y'')} = \|u - \mathcal{M}_{Y'}(u)\|_{L^p(Y'\times Y'')} \le C\|\nabla_{y'} u\|_{L^p(Y'\times Y'')},\\ &\|\nabla_{y'}\mathfrak{u}\|_{L^p(Y'\times Y'')} = \|\nabla_{y'} u\|_{L^p(Y'\times Y'')}. \end{aligned} \tag{A.3}$$

Observe that $\mathcal{M}_{Y'}(\nabla_{y''}\mathfrak{u}) = \nabla_{y''}\mathcal{M}_{Y'}(\mathfrak{u}) = 0$.
Then, again by equality (A.1) and the Poincaré-Wirtinger Inequality, we get

$$\begin{aligned} \|\nabla_{y''}\mathfrak{u}\|_{L^p(Y'\times Y'')} &= \|\nabla_{y''}\mathfrak{u} - \mathcal{M}_{Y'}(\nabla_{y''}\mathfrak{u})\|_{L^p(Y'\times Y'')} \le C\|\nabla_{y'}(\nabla_{y''}\mathfrak{u})\|_{L^p(Y'\times Y'')}\\ &= C\|\nabla_{y''}(\nabla_{y'}\mathfrak{u})\|_{L^p(Y'\times Y'')} = C\|\nabla_{y''}(\nabla_{y'} u)\|_{L^p(Y'\times Y'')}. \end{aligned} \tag{A.4}$$

Hence, by estimates (A.3)-(A.4), we obtain (A.2).
Step 2. We prove the statement of the lemma
Suppose $u \in L^p(Y''; W^{1,p}(Y'))$ and $\nabla_{y'} u \in L^p(Y'; W^{1,p}(Y''))^{N_1}$. Since $\mathcal{C}^2(\overline{Y})$ is dense in this subspace of $L^p(Y''; W^{1,p}(Y'))$, there exists a sequence of functions $u_n \in \mathcal{C}^2(\overline{Y})$ such that

$$\begin{aligned} &u_n \to u && \text{strongly in } L^p(Y''; W^{1,p}(Y'))),\\ &\mathcal{M}_{Y'}(u_n) \to \mathcal{M}_{Y'}(u) && \text{strongly in } L^p(Y''),\\ &\nabla_{y'} u_n \to \nabla_{y'} u && \text{strongly in } L^p(Y'; W^{1,p}(Y''))^{N_1}. \end{aligned}$$

The corresponding sequence $\{\mathfrak{u}_n\}$ (given by Step 1) satisfies $\nabla_{y'}\mathfrak{u}_n = \nabla_{y'} u_n$, moreover it belongs to $\mathcal{C}^2(\overline{Y})$ and is bounded in $W^{1,p}(Y)$ (from (A.2)). Passing to the limit, this gives

$\mathfrak{u} \in W^{1,p}(Y)$ such that

$$\mathfrak{u}_n \to \mathfrak{u} \quad \text{strongly in } W^{1,p}(Y), \qquad \nabla_{y'}\mathfrak{u} = \nabla_{y'} u \quad \text{a.e. in } Y.$$

Finally, observe that $\mathfrak{u} = u - \mathcal{M}_{Y'}(u)$. □

Lemma 30. *Let $p \in (1, +\infty)$ and let u be in $L^p(Y''; W^{1,p}_{per}(Y'))$ such that*

$$\nabla_{y'} u \in L^p(Y'; W^{1,p}_{per}(Y''))^{N_1}.$$

Then, there exists $\mathfrak{w} \in W^{1,p}_{per}(Y)$ such that

$$\nabla_{y'}\mathfrak{w} = \nabla_{y'} u \qquad \text{a.e. in } Y. \tag{A.5}$$

Proof. Since u in $L^p(Y''; W^{1,p}_{per}(Y'))$ and $\nabla_{y'} u \in L^p(Y'; W^{1,p}_{per}(Y''))^{N_1}$, Lemma 29 shows that the function $\mathfrak{u} = u - \mathcal{M}_{Y'}(u)$ belongs to $W^{1,p}(Y)$. It is obvious that $\mathfrak{u}$ is periodic with respect to the variables $y_1, \dots, y_{N_1}$. One also has $\nabla_{y'}\mathfrak{u} = \nabla_{y'} u \in L^p(Y'; W^{1,p}_{per}(Y''))^{N_1}$. Denote

$$\begin{aligned} Y_i &= \{y \in \overline{Y} \mid y_i = 0,\ y_j \in (0,1),\ j \in \{1, \dots, N\}\ \ j \neq i\},\ i \in \{1, \dots, N\},\\ Y_i'' &= \{y \in \overline{Y''} \mid y_i = 0,\ y_j \in (0,1),\ j \in \{N_1+1, \dots, N\}\ \ j \neq i\},\ i \in \{N_1+1, \dots, N\}. \end{aligned}$$

Since $\nabla_{y'}\mathfrak{u} = \nabla_{y'} u$ and is y_j periodic, $j \in \{N_1+1, \dots, N\}$, one gets

$$\nabla_{y'}\mathfrak{u}_{|Y_j + \mathbf{e}_j} - \nabla_{y'}\mathfrak{u}_{|Y_j} = \nabla_{y'} u_{|Y_j + \mathbf{e}_j} - \nabla_{y'} u_{|Y_j} = 0 \qquad \text{a.e. in } Y_j.$$

Hence

$$\mathfrak{u}_{|Y_j + \mathbf{e}_j} - \mathfrak{u}_{|Y_j} \in W^{1-1/p,p}(Y_j''), \qquad j \in \{N_1+1, \dots, N\}.$$

Besides, one has

$$\mathfrak{u}_{|Y_j + \mathbf{e}_j} - \mathfrak{u}_{|Y_j} = 0, \qquad j \in \{1, \dots, N_1\}.$$

Then, following the same lines of the proofs of Cioranescu, Damlamian, and Griso, 2018, Proposition 13.34 and Lemmas 13.35-13.36, there exits $\mathfrak{w} \in W^{1,p}_{per}(Y)$ such that

$$\mathfrak{w} - \mathfrak{u} \in W^{1,p}(Y'')$$

and we have

$$\begin{aligned} \|\mathfrak{w} - \mathfrak{u}\|_{W^{1,p}(Y)} \leq& C \sum_{j=N_1+1}^{N} \|\mathfrak{u}_{|Y_j + \mathbf{e}_j} - \mathfrak{u}_{|Y_j}\|_{W^{1-1/p,p}(Y_j'')}\\ \leq& C\big(\|\nabla_{y'} u\|_{L^p(Y)} + \|\nabla_{y'} u\|_{L^p(Y'; W^{1,p}(Y''))}\big). \end{aligned}$$

The function $\mathfrak{w}$ satisfies (A.5). □

A.2 ...about the periodic unfolding for lattices

Lemma 31. *Let $p \in (1, +\infty)$ and let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $W^{2,p}(\mathcal{S}_\varepsilon)$ satisfying*

$$\|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial^2_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}.$$

For every $k' \in \widehat{\mathbf{K}}$ and $(i,j) \in \{1,\dots,N\}^2$, $i \neq j$, we define in $\widetilde{\Omega}_\varepsilon \times \widehat{\mathbf{K}}_i$ the piecewise constant function $\Phi_\varepsilon^{(i,j)}$ by

$$\Phi_\varepsilon^{(i,j)}(\cdot,k') \doteq \begin{cases} \dfrac{l_i}{\varepsilon^2}\Big(\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot,A(k')+\mathbf{e}_i\big) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot,A(k')\big) \\ \qquad - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot,A(k'+\mathbf{e}_j)+\mathbf{e}_i\big) + \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\big(\cdot,A(k'+\mathbf{e}_j)\big)\Big) & \text{a.e. in } \widetilde{\Omega}_\varepsilon \times \widehat{\mathbf{K}}_i, \\ 0 & \text{a.e. in } \big(\mathbb{R}^N \setminus \overline{\widetilde{\Omega}_\varepsilon}\big) \times \widehat{\mathbf{K}}_i. \end{cases}$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and a function ϕ in $W^{1,p}(\Omega) \cap W^{2,p}_{loc}(\Omega)$ such that ($(i,j) \in \{1,\dots,N\}^2$, $i \neq j$, $k' \in \widehat{\mathbf{K}}_i$)

$$\begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) &\to \phi \qquad \text{strongly in} \quad L^p(\Omega; W^{2,p}(\mathcal{S})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) &\rightharpoonup \partial_j\phi \qquad \text{weakly in} \quad L^p(\Omega; W^{1,p}(\mathcal{S}^{(j)})), \\ \Phi_\varepsilon^{(i,j)}(\cdot,k') &\rightharpoonup -l_il_j\partial^2_{ij}\phi \qquad \text{weakly in} \quad W^{-1,p}(\mathbb{R}^N). \end{aligned} \tag{A.6}$$

Proof. There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and a function ϕ in the space $W^{1,p}(\Omega) \cap W^{2,p}_{loc}(\Omega)$ such that convergences (A.6)$_{1,2}$ hold (see Theorem 3).
Now, let ψ be in $W^{1,p'}(\mathbb{R}^N)$, one has

$$\begin{aligned} &\int_\Omega \psi(x)\,\Phi_\varepsilon^{(i,j)}(x,k')\,dx \\ &= \varepsilon^N \sum_{\xi\in\mathbb{Z}^N} \mathcal{M}_Y(\psi)(\varepsilon\xi)\frac{l_i}{\varepsilon^2}\Big(\phi_\varepsilon\big(\varepsilon\xi+\varepsilon A(k')+\varepsilon\mathbf{e}_i\big) - \phi_\varepsilon\big(\varepsilon\xi+\varepsilon A(k')\big) \\ &\qquad\qquad - \phi_\varepsilon\big(\varepsilon\xi+\varepsilon A(k'+\varepsilon\mathbf{e}_j)+\varepsilon\mathbf{e}_i\big) + \phi_\varepsilon\big(\varepsilon\xi+\varepsilon A(k'+\mathbf{e}_j)\big)\Big) \\ &= \varepsilon^N l_i \sum_{\xi\in\mathbb{Z}^N} \frac{\mathcal{M}_Y(\psi)(\varepsilon\xi-\varepsilon\mathbf{e}_i) - \mathcal{M}_Y(\psi)(\varepsilon\xi)}{\varepsilon} \\ &\qquad\qquad \cdot \frac{\phi_\varepsilon\big(\varepsilon\xi+\varepsilon A(k')\big) - \phi_\varepsilon\big(\varepsilon\xi+\varepsilon A(k'+\mathbf{e}_j)\big)}{\varepsilon} \\ &= l_i\int_\Omega \frac{\psi-\psi(\cdot-\varepsilon\mathbf{e}_i)}{\varepsilon}\Big(\int_{A(k')}^{A(k'+\mathbf{e}_j)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)\,d\mathbf{S}\Big)dx. \end{aligned}$$

Then, due to convergences (A.6)$_2$, we get

$$\lim_{\varepsilon\to 0}\int_\Omega \psi(x)\,\Phi_\varepsilon^{(i,j)}(x,k')\,dx = l_i\int_\Omega \partial_i\psi\Big(\int_{A(k')}^{A(k'+\mathbf{e}_j)} \partial_j\phi\,d\mathbf{S}\Big)dx = l_il_j\int_\Omega \partial_i\psi\partial_j\phi\,dx.$$

Hence, (A.6)$_3$ is proved. □

Appendix B

Complements of the periodic unfolding for anisotropically bounded functions

This appendix is dedicated to the extension of Lemma 11.11 of Cioranescu, Damlamian, and Griso, 2018 to the anisotropic case. We initially gave it interest by the crucial role it plays in the unfolding of periodic structures made of beams (see Griso, Hauck, and Orlik, 2021) and yarns (seeGriso, Orlik, and Wackerle, 2020b), and the original plan was to extend it to the anisotropic case to homogenize the textile with loose contact.
However, the change of strategy that involved the periodic unfolding of lattice structures made it superfluous for this purpose. Nevertheless, we give here the results since they can be useful for the study of other structures.

We start by giving its original formulation.

Lemma 32. *Let $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$ be a sequence converging weakly to (u, v) in $W^{1,p}(\Omega) \times W^{1,p}(\Omega)^N$, $p \in (1, +\infty)$. Moreover, assume that there exist $\mathcal{Z} \in L^p(\Omega)^N$ and $\widehat{v} \in L^p(\Omega; W^{1,p}_{per,0}(Y))^N$ such that*

$$\begin{aligned} &\frac{1}{\varepsilon}(\nabla u_\varepsilon + v_\varepsilon) \rightharpoonup \mathcal{Z} \quad \text{weakly in} \quad L^p(\Omega)^N, \\ &\mathcal{T}_\varepsilon(\nabla v_\varepsilon) \rightharpoonup \nabla v + \nabla_y \widehat{v} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N\times N}. \end{aligned}$$

Then, u belongs to $W^{2,p}(\Omega)$. Moreover, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\mathfrak{u} \in L^p(\Omega; W^{1,p}_{per,0}(Y))$ such that

$$\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\nabla u_\varepsilon + v_\varepsilon) \rightharpoonup \mathcal{Z} + \nabla_y \mathfrak{u} + \widehat{v} \quad \text{weakly in} \quad L^p(\Omega \times Y)^N.$$

As a direct consequence, we get the following.

Corollary 12. *Let $\mathcal{O}$ be an open set in $\mathbb{R}^M$, $M \geq 1$. Let $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$ be a sequence converging weakly to (u, v) in $L^p(\mathcal{O}; W^{1,p}(\Omega)) \times L^p(\mathcal{O}; W^{1,p}(\Omega))^N$, $p \in (1, +\infty)$. Moreover, assume that there exist $\mathcal{Z} \in L^p(\mathcal{O} \times \Omega)^N$ and $\widehat{v} \in L^p(\mathcal{O} \times \Omega; W^{1,p}_{per,0}(Y))^N$ such that*

$$\begin{aligned} &\frac{1}{\varepsilon}(\nabla u_\varepsilon + v_\varepsilon) \rightharpoonup \mathcal{Z} \quad \text{weakly in} \quad L^p(\mathcal{O} \times \Omega)^N, \\ &\mathcal{T}_\varepsilon(\nabla v_\varepsilon) \rightharpoonup \nabla v + \nabla_y \widehat{v} \quad \text{weakly in} \quad L^p(\mathcal{O} \times \Omega \times Y)^{N\times N}. \end{aligned}$$

Then, u belongs to $L^p(\mathcal{O}; W^{2,p}(\Omega))$. Furthermore, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\mathfrak{u} \in L^p(\mathcal{O} \times \Omega; W^{1,p}_{per,0}(Y))$ such that:

$$\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\nabla u_\varepsilon + v_\varepsilon) \rightharpoonup \mathcal{Z} + \nabla_y \mathfrak{u} + \widehat{v} \quad \text{weakly in} \quad L^p(\mathcal{O} \times \Omega \times Y)^N.$$

Define the spaces

$$L^p(\Omega\times Y'',D^2_{x'}) \doteq \{\widetilde{\phi}\in L^p(\Omega\times Y'')\,|\,\nabla_{x'}\widetilde{\phi}\in L^p(\Omega\times Y'')^{N_1}\ \text{and}\ (\nabla_{x'}\otimes\nabla_{x'})\widetilde{\phi}\in L^p(\Omega\times Y'')^{N_1\times N_1}\},$$
$$L^p(\Omega,D^2_{x'};W^{1,p}_{per}(Y'')) \doteq \{\widetilde{\phi}\in L^p(\Omega;W^{1,p}_{per}(Y''))\,|\,\nabla_{x'}\widetilde{\phi}\in L^p(\Omega;W^{1,p}_{per}(Y''))^{N_1}\ \text{and}\ (\nabla_{x'}\otimes\nabla_{x'})\widetilde{\phi}\in L^p(\Omega;W^{1,p}_{per}(Y''))^{N_1\times N_1}\},$$

where $(\nabla_{x'}\otimes\nabla_{x'})\widetilde{\phi}$ denotes the first $N_1\times N_1$ entries of the Hessian matrix of ϕ.
We endow such spaces with the respective norms:

$$\|\cdot\|_{L^p(\Omega\times Y'',D^2_{x'})} \doteq \|\cdot\|_{L^p(\Omega\times Y'')}+\|\nabla_{x'}(\cdot)\|_{L^p(\Omega\times Y'')}+\|D^2_{x'}(\cdot)\|_{L^p(\Omega\times Y'')},$$
$$\|\cdot\|_{L^p(\Omega,D^2_{x'};W^{1,p}(Y''))} \doteq \|\cdot\|_{L^p(\Omega\times Y'',D^2_{x'})}+\|\nabla_{y''}(\cdot)\|_{L^p(\Omega\times Y'')}.$$

We are ready to extend Lemma 32 to the class of anisotropically bounded sequences.

Lemma 33. *Let $\{(u_\varepsilon,v_\varepsilon)\}_\varepsilon$ be a sequence in the space $L^p(\Omega,\nabla_{x'})\times L^p(\Omega,\nabla_{x'})^{N_1}$, $p\in(1,+\infty)$, satisfying*

$$\|u_\varepsilon\|_{L^p(\Omega,\nabla_{x'})}\le C,\qquad \|v_\varepsilon\|_{L^p(\Omega,\nabla_{x'})}\le C, \tag{B.1}$$

where the constant does not depend on ε.
Moreover, assume that there exist $\mathcal{Z}\in L^p(\Omega)^{N_1}$ such that

$$\frac{1}{\varepsilon}(\nabla_{x'}u_\varepsilon+v_\varepsilon)\rightharpoonup\mathcal{Z}\quad\textit{weakly in}\quad L^p(\Omega)^{N_1}. \tag{B.2}$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and

$$\widetilde{\mathcal{Z}}\in L^p(\Omega\times Y'')^{N_1}\ \textit{with}\ \mathcal{M}_{Y''}(\widetilde{\mathcal{Z}})=\mathcal{Z},$$
$$\widetilde{u}\in L^p(\Omega\times Y'',D^2_{x'}),$$
$$\mathfrak{u}\in L^p(\Omega\times Y'';W^{1,p}_{per,0}(Y')),$$
$$\widehat{v}\in L^p(\Omega\times Y'';W^{1,p}_{per,0}(Y'))^{N_1}$$

such that

$$\begin{aligned}&\mathcal{T}_\varepsilon(\nabla_{x'}v_\varepsilon)\rightharpoonup -D^2_{x'}\widetilde{u}+\nabla_{y'}\widehat{v}\quad\textit{weakly in}\quad L^p(\Omega\times Y)^{N_1\times N_1},\\&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\nabla_{x'}u_\varepsilon+v_\varepsilon)\rightharpoonup\widetilde{\mathcal{Z}}+\nabla_{y'}\mathfrak{u}+\widehat{v}\quad\textit{weakly in}\quad L^p(\Omega\times Y)^{N_1}.\end{aligned} \tag{B.3}$$

Proof. We first apply the unfolding operator $\mathcal{T}_\varepsilon$ to both sequences $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$. By Lemma 7 and estimates (B.1), there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, and functions $\widetilde{u}\in L^p(\Omega\times Y'',\nabla_{x'})$, $\widetilde{v}\in L^p(\Omega\times Y'',\nabla_{x'})^{N_1}$, $\widehat{u}\in L^p(\Omega\times Y'';W^{1,p}_{per,0}(Y'))$ and a function $\widehat{v}\in L^p(\Omega\times Y'';W^{1,p}_{per,0}(Y'))^{N_1}$ such that

$$\begin{aligned}\mathcal{T}_\varepsilon(u_\varepsilon)&\rightharpoonup\widetilde{u}\quad\text{weakly in}\quad L^p(\Omega\times Y'';W^{1,p}(Y')),\\\mathcal{T}_\varepsilon(\nabla_{x'}u_\varepsilon)&\rightharpoonup\nabla_{x'}\widetilde{u}+\nabla_{y'}\widehat{u}\quad\text{weakly in}\quad L^p(\Omega\times Y)^{N_1},\\\mathcal{T}_\varepsilon(v_\varepsilon)&\rightharpoonup\widetilde{v}\quad\text{weakly in}\quad L^p(\Omega\times Y'';W^{1,p}(Y'))^{N_1},\\\mathcal{T}_\varepsilon(\nabla_{x'}v_\varepsilon)&\rightharpoonup\nabla_{x'}\widetilde{v}+\nabla_{y'}\widehat{v}\quad\text{weakly in}\quad L^p(\Omega\times Y)^{N_1\times N_1}.\end{aligned} \tag{B.4}$$

By convergence (B.2), there exist a subsequence of $\{\varepsilon\}$, denoted $\{\varepsilon\}$, and $\widehat{\mathcal{Z}}\in L^p(\Omega\times Y)^{N_1}$ with $\mathcal{M}_Y(\widehat{\mathcal{Z}})=\mathcal{Z}$ such that

$$\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\nabla_{x'}u_\varepsilon+v_\varepsilon)\rightharpoonup\widehat{\mathcal{Z}}\quad\text{weakly in}\quad L^p(\Omega\times Y)^{N_1}. \tag{B.5}$$

From convergences (B.4)$_{2,3}$ and (B.5) we get

$$\nabla_{x'}\widetilde{u} + \nabla_{y'}\widehat{u} + \widetilde{v} = 0 \quad \text{a.e. in} \quad \Omega \times Y.$$

Applying $\mathcal{M}_{Y'}$ to the above equality and since $\widehat{u} \in L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'))$, while $\widetilde{u} \in L^p(\Omega \times Y'', \nabla_{x'})$, $\widetilde{v} \in L^p(\Omega \times Y'', \nabla_{x'})^{N_1}$, we get that $\nabla_{x'}\widetilde{u} + \widetilde{v} = 0$ a.e. in $\Omega \times Y''$. Hence, $\nabla_{y'}\widehat{u} = 0$ and thus $\widehat{u} = 0$ because it belongs to $L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'))$. As a consequence, one has

$$\widetilde{u} \in L^p(\Omega \times Y'', D^2_{x'}).$$

Set $U_\varepsilon = \mathcal{T}''_\varepsilon(u_\varepsilon)$, $V_\varepsilon = \mathcal{T}''_\varepsilon(v_\varepsilon)$. Again by convergence (B.2), there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\widetilde{\mathcal{Z}} \in L^p(\Omega \times Y'')^{N_1}$ such that

$$\frac{1}{\varepsilon}\nabla_{x'}U_\varepsilon + V_\varepsilon \rightharpoonup \widetilde{\mathcal{Z}} \quad \text{weakly in} \quad L^p(\Omega \times Y'')^{N_1}.$$

Then, due to convergence (B.5) we have $\widetilde{\mathcal{Z}} = \mathcal{M}_{Y'}(\widehat{\mathcal{Z}})$.
Now, let ω' and ω'' be two open sets such that

$$\omega' \subset \mathbb{R}^{N_1}, \quad \omega'' \subset \mathbb{R}^{N_2} \quad \text{and} \quad \overline{\omega' \times \omega''} \subset \Omega. \tag{B.6}$$

First, observe that

$$U_\varepsilon \in L^p(\omega'' \times Y''; W^{1,p}(\omega')), \qquad V_\varepsilon \in L^p(\omega'' \times Y''; W^{1,p}(\omega'))^{N_1}.$$

By the above convergence and (B.4)$_4$, one has

$$\frac{1}{\varepsilon}\nabla_{x'}U_\varepsilon + V_\varepsilon \rightharpoonup \widetilde{\mathcal{Z}} \quad \text{weakly in} \quad L^p(\omega' \times \omega'' \times Y'')^{N_1},$$
$$\mathcal{T}_\varepsilon(\nabla_{x'}v_\varepsilon) = \mathcal{T}'_\varepsilon(\nabla_{x'}V_\varepsilon) \rightharpoonup \nabla_{x'}\widetilde{v} + \nabla_{y'}\widehat{v} \quad \text{weakly in} \quad L^p(\omega' \times \omega'' \times Y' \times Y'')^{N_1 \times N_1}.$$

Lemma 12 claims that up to a subsequence, there exists $\mathfrak{u}_{\omega' \times \omega''} \in L^p(\omega' \times \omega'' \times Y''; W^{1,p}_{per,0}(Y'))$, such that the following convergence holds:

$$\frac{1}{\varepsilon}\mathcal{T}'_\varepsilon(\nabla_{x'}U_\varepsilon + V_\varepsilon) \rightharpoonup \widetilde{\mathcal{Z}} + \nabla_{y'}\mathfrak{u}_{\omega' \times \omega''} + \widehat{v} \quad \text{weakly in} \quad L^p(\omega' \times \omega'' \times Y' \times Y'')^{N_1}.$$

Taking into account convergence (B.5) we get

$$\widehat{\mathcal{Z}} = \widetilde{\mathcal{Z}} + \nabla_{y'}\mathfrak{u}_{\omega' \times \omega''} + \widehat{v} \quad \text{in} \quad \omega' \times \omega'' \times Y.$$

Since one can cover Ω by a countable family of open subsets $\omega' \times \omega''$ satisfying (B.6), there exists $\mathfrak{u}$ in $L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'))$ such that $\widehat{\mathcal{Z}} - \widetilde{\mathcal{Z}} - \widehat{v} = \nabla_{y'}\mathfrak{u}$. The proof of (B.3) is therefore complete. □

With some more assumptions, we can improve the regularity of the limit functions.

Lemma 34. *Let $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$ be a sequence in $L^p(\Omega, \nabla_{x'}) \times L^p(\Omega, \nabla_{x'})^{N_1}$, with $p \in (1, +\infty)$, satisfying the assumptions in Lemma 33. Moreover, assume that*

$$\big\|\nabla_{x''}(\nabla_{x'}u_\varepsilon + v_\varepsilon)\big\|_{L^p(\Omega)} + \varepsilon\big\|\nabla_{x''}(\nabla_{x'}v_\varepsilon)\big\|_{L^p(\Omega)} \le C, \tag{B.7}$$

where the constant does not depend on ε.
Then, there exist a subsequence of $\{\varepsilon\}$*, still denoted* $\{\varepsilon\}$*, such that*

$$\begin{aligned}
&\widetilde{\mathbf{Z}} \in L^p(\Omega; W^{1,p}_{per}(Y''))^{N_1},\\
&\widetilde{u} \in L^p(\Omega, D^2_{x'}; W^{1,p}_{per}(Y'')),\\
&\mathfrak{w} \in L^p(\Omega; W^{1,p}_{per,0}(Y)),\\
&\widehat{\mathbf{v}} \in L^p(\Omega; W^{1,p}_{per,0}(Y))^{N_1}
\end{aligned}$$

such that

$$\begin{aligned}
&\mathcal{T}_\varepsilon(\nabla_{x'} v_\varepsilon) \rightharpoonup -D^2_{x'}\widetilde{u} + \nabla_{y'}\widehat{\mathbf{v}} \quad \text{weakly in} \quad L^p(\Omega\times Y)^{N_1\times N_1},\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon + v_\varepsilon) \rightharpoonup \widetilde{\mathbf{Z}} + \nabla_{y'}\mathfrak{w} + \widehat{\mathbf{v}} \quad \text{weakly in} \quad L^p(\Omega\times Y)^{N_1}.
\end{aligned}$$

Proof. From Lemma 33, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\widetilde{\mathcal{Z}} \in L^p(\Omega\times Y'')^{N_1}$, $\mathfrak{u} \in L^p(\Omega\times Y''; W^{1,p}_{per,0}(Y'))$, $\widetilde{u} \in L^p(\Omega\times Y'', D^2_{x'})$ and $\widehat{v} \in L^p(\Omega\times Y''; W^{1,p}_{per,0}(Y'))^{N_1}$ such that

$$\begin{aligned}
&\mathcal{T}_\varepsilon(\nabla_{x'} v_\varepsilon) \rightharpoonup -D^2_{x'}\widetilde{u} + \nabla_{y'}\widehat{v} \quad \text{weakly in} \quad L^p(\Omega\times Y)^{N_1\times N_1},\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon + v_\varepsilon) \rightharpoonup \widetilde{\mathcal{Z}} + \nabla_{y'}\mathfrak{u} + \widehat{v} \quad \text{weakly in} \quad L^p(\Omega\times Y)^{N_1}.
\end{aligned}$$

By hypothesis (B.7), Lemma 6 (swapping Y' and Y'') and the proof of Lemma 9 one has

$$\begin{aligned}
&\mathcal{T}_\varepsilon(\nabla_{x'} v_\varepsilon) \rightharpoonup -D^2_{x'}\widetilde{u} + \nabla_{y'}\widehat{\mathbf{v}} \quad \text{weakly in} \quad L^p(\Omega\times Y)^{N_1\times N_1},\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon + v_\varepsilon) \rightharpoonup \widetilde{\mathbf{Z}} + \nabla_{y'}\mathfrak{u} + \widehat{\mathbf{v}} \in L^p(\Omega\times Y'; W^{1,p}_{per}(Y''))^{N_1}
\end{aligned}$$

with $\widetilde{\mathbf{Z}} \in L^p(\Omega\times Y'')^{N_1}$, $\widetilde{u} \in L^p(\Omega, D^2_{x'}; W^{1,p}_{per}(Y''))$ and $\widehat{\mathbf{v}} \in L^p(\Omega; W^{1,p}_{per}(Y))^{N_1}$ satisfying $\mathcal{M}_{Y'}(\widehat{\mathbf{v}}) = 0$ a.e. in $\Omega\times Y''$.
Since, $\widehat{\mathbf{v}}$ satisfies $\mathcal{M}_{Y'}(\widehat{\mathbf{v}}) = 0$ a.e. in $\Omega\times Y''$ and $\mathcal{M}_{Y'}(\nabla_{y'}\mathfrak{u}) = 0$ a.e. in $\Omega\times Y''$ by periodicity of $\mathfrak{u}$, we obtain

$$\widetilde{\mathbf{Z}} = \mathcal{M}_{Y'}(\widetilde{\mathbf{Z}}) \in L^p(\Omega; W^{1,p}_{per}(Y''))^{N_1}.$$

Hence $\nabla_{y'}\mathfrak{u}$ lies in $L^p(\Omega\times Y'; W^{1,p}_{per}(Y''))^{N_1}$. Lemma 30 in Appendix gives a function $\mathfrak{w} \in L^p(\Omega; W^{1,p}_{per,0}(Y))$ such that $\nabla_{y'}\mathfrak{w} = \nabla_{y'}\mathfrak{u}$. The proof is complete. □

Appendix C

Proof of a better bound for estimate (5.37) in Lemma 19

The idea behind the proof is to take the difference between the displacements in the contact areas and set as remainders the terms with a "sufficiently good" estimate, namely $\sim \varepsilon \|u\|^2_{\mathcal{S}_\varepsilon}$. Then, the remaining terms are paired, taking into account the oscillating manner, which again gives remainders with a sufficiently good estimate $\sim \varepsilon \|u\|^2_{\mathcal{S}_\varepsilon}$. Iterating this procedure, we get that all the terms paired have an estimate $\sim \varepsilon \|u\|^2_{\mathcal{S}_\varepsilon}$.

Proof of estimate (5.37) of Lemma 19. First, to shorten the notation, a.e. (t_1, t_2) in $\omega_{\kappa\varepsilon}$, we set

$$\mathrm{u}^{(1)}_{pq}(t_1,t_2) = u^{(1)}(t_1 + p\varepsilon, q\varepsilon, t_2, (-1)^{p+q+1}\kappa\varepsilon), \quad \mathrm{u}^{(2)}_{pq}(t_1,t_2) = u^{(2)}(p\varepsilon, t_2 + q\varepsilon, t_1, (-1)^{p+q}\kappa\varepsilon),$$
$$\bar{\mathrm{u}}'^{(1)}_{pq}(t_1,t_2) = \bar{u}'^{(1)}(t_1 + p\varepsilon, q\varepsilon, t_2, (-1)^{p+q+1}\kappa\varepsilon), \quad \bar{\mathrm{u}}'^{(2)}_{pq}(t_1,t_2) = \bar{u}'^{(2)}(p\varepsilon, t_2 + q\varepsilon, t_1, (-1)^{p+q}\kappa\varepsilon).$$

From (5.34), the displacements become

$$\begin{aligned} \mathrm{u}^{(1)}_{pq}(t_1,t_2) &= \mathbb{U}'^{(1)}(p\varepsilon + t_1, q\varepsilon) + \mathcal{R}'^{(1)}(p\varepsilon + t_1, q\varepsilon) \wedge t_2\mathbf{e}_2 + \bar{\mathrm{u}}'^{(1)}_{pq}(t_1,t_2), \\ \mathrm{u}^{(2)}_{pq}(t_1,t_2) &= \mathbb{U}'^{(2)}(p\varepsilon, q\varepsilon + t_2) + \mathcal{R}'^{(2)}(p\varepsilon, q\varepsilon + t_2) \wedge t_1\mathbf{e}_1 + \bar{\mathrm{u}}'^{(2)}_{pq}(t_1,t_2). \end{aligned} \tag{C.1}$$

We then organize the proof in steps.

Step 1. We rewrite the displacements in the contact areas as (for a.e. (t_1, t_2) in $\omega_{\kappa\varepsilon}$)

$$\begin{aligned} \mathrm{u}^{(1)}_{pq}(t_1,t_2) &= \mathbb{U}'^{(1)}(p\varepsilon, q\varepsilon) + \mathcal{R}'^{(1)}(p\varepsilon, q\varepsilon) \wedge (t_1\mathbf{e}_1 + t_2\mathbf{e}_2) + Q^{(1)}_{pq}(t_1,t_2), \\ \mathrm{u}^{(2)}_{pq}(t_1,t_2) &= \mathbb{U}'^{(2)}(p\varepsilon, q\varepsilon) + \mathcal{R}'^{(2)}(q\varepsilon, p\varepsilon) \wedge (t_1\mathbf{e}_1 + t_2\mathbf{e}_2) + Q^{(2)}_{pq}(t_1,t_2), \end{aligned} \tag{C.2}$$

where the remainder terms $Q^{(\alpha)}_{pq}$ are estimated by

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|Q^{(\alpha)}_{pq}\|^2_{L^2(\omega_{\kappa\varepsilon})} \le C\varepsilon \|u\|^2_{\mathcal{S}_\varepsilon}. \tag{C.3}$$

From the form of the displacement in the contact areas (C.1), for a.e. (t_1, t_2) in $\omega_{\kappa\varepsilon}$ the remainder terms $Q^{(\alpha)}_{pq}$ are defined by

$$\begin{aligned} Q^{(1)}_{pq}(t_1,t_2) \doteq& \big(\mathbb{U}'^{(1)}(p\varepsilon + t_1, q\varepsilon) - \mathbb{U}'^{(1)}(p\varepsilon, q\varepsilon) - \mathcal{R}'^{(1)}(p\varepsilon, q\varepsilon) \wedge t_1\mathbf{e}_1\big) \\ &+ \big(\mathcal{R}'^{(1)}(p\varepsilon + t_1, q\varepsilon) - \mathcal{R}'^{(1)}(p\varepsilon, q\varepsilon)\big) \wedge t_2\mathbf{e}_2 + \bar{\mathrm{u}}'^{(1)}_{pq}(t_1,t_2), \\ Q^{(2)}_{pq}(t_1,t_2) \doteq& \big(\mathbb{U}'^{(2)}(p\varepsilon, q\varepsilon + t_2) - \mathbb{U}'^{(2)}(p\varepsilon, q\varepsilon) - \mathcal{R}'^{(2)}(p\varepsilon, q\varepsilon) \wedge t_2\mathbf{e}_2\big) \\ &+ \big(\mathcal{R}'^{(2)}(p\varepsilon, q\varepsilon + t_2) - \mathcal{R}'^{(2)}(p\varepsilon, q\varepsilon)\big) \wedge t_1\mathbf{e}_1 + \bar{\mathrm{u}}'^{(2)}_{pq}(t_1,t_2). \end{aligned}$$

We want now to prove (C.3) and due to the symmetrical behavior, we will only estimate $Q^{(1)}_{pq}$. We first have that

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|Q^{(1)}_{pq}\|^2_{L^2(\omega_{\kappa\varepsilon})} = \sum_{(p,q)\in\mathcal{K}_\varepsilon} \Big(\int_{\omega_{\kappa\varepsilon}} \Big| \int_0^{t_1} \partial_1 \mathbb{U}'^{(1)}(p\varepsilon+s, q\varepsilon) - \mathcal{R}'^{(1)}(p\varepsilon, q\varepsilon)\wedge \mathbf{e}_1 ds\Big|^2 dt_1 dt_2 + \int_{\omega_{\kappa\varepsilon}} t_2^2 \Big| \int_0^{t_1} \partial_1 \mathcal{R}'^{(1)}(p\varepsilon+s, q\varepsilon)ds\Big|^2 dt_1 dt_2 \Big) + \sum_{(p,q)\in\mathcal{K}_\varepsilon} \|\overline{\mathrm{u}}^{(1)}\|^2_{L^2(\omega_{\kappa\varepsilon})}.$$

Using Jensen's inequality on each term in the parenthesis and equality (5.17), we get

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \int_{\omega_{\kappa\varepsilon}} t_2^2 \Big| \int_0^{t_1} \partial_1 \mathcal{R}'^{(1)}(p\varepsilon+s, q\varepsilon)ds\Big|^2 dt_1 dt_2 \le C\varepsilon^5 \sum_{q=0}^{2N_\varepsilon-1} \|\partial_1 \mathcal{R}'^{(1)}(\cdot, q\varepsilon)\|^2_{L^2(0,L)},$$
$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \int_{\omega_{\kappa\varepsilon}} \Big| \int_0^{t_1} \partial_1 \mathbb{U}'^{(1)}(p\varepsilon+s, q\varepsilon) - \mathcal{R}'^{(1)}(p\varepsilon, q\varepsilon)\wedge \mathbf{e}_1 dt\Big|^2 dt_1 dt_2 \le C \sum_{q=0}^{2N_\varepsilon-1} \Big(\varepsilon^3 \|\partial_1 \mathbb{U}_1'^{(1)}(\cdot, q\varepsilon)\|^2_{L^2(0,L)} + \varepsilon^5 \|\partial_1 \mathcal{R}'^{(1)}(\cdot, q\varepsilon)\|^2_{L^2(0,L)^3} \Big).$$

By the first line of estimates in (5.32) and Lemma 18, we get (C.3).

Step 2. By the non penetration condition (5.28) in the contact parts of the cell $(p\varepsilon, q\varepsilon)+\varepsilon Y$, for a.e. (t_1,t_2) in $\omega_{\kappa\varepsilon}$ we show that

$$\begin{aligned} 0 \le &(-1)^{p+q}\Big[\big(\mathrm{u}^{(1)}_{pq,3} - \mathrm{u}^{(2)}_{pq,3}\big) + \big(\mathrm{u}^{(1)}_{(p+1)(q+1),3} - \mathrm{u}^{(2)}_{(p+1)(q+1),3}\big) + \big(\mathrm{u}^{(2)}_{(p+1)q,3} - \mathrm{u}^{(1)}_{(p+1)q,3}\big) \\ &+ \big(\mathrm{u}^{(2)}_{p(q+1),3} - \mathrm{u}^{(1)}_{p(q+1),3}\big)\Big](t_1,t_2) \\ = &(-1)^{p+q}\Big[\varepsilon\Big(\mathcal{R}^{(1)}_2(p\varepsilon, q\varepsilon) - \mathcal{R}^{(2)}_1(p\varepsilon+\varepsilon, q\varepsilon) - \mathcal{R}^{(1)}_2(p\varepsilon, q\varepsilon+\varepsilon) + \mathcal{R}^{(2)}_1(p\varepsilon, q\varepsilon)\Big) \\ &+ \Big(R^{(1)}_{pq} + R^{(2)}_{(p+1)q} + R^{(1)}_{p(q+1)} + R^{(2)}_{pq}\Big)(t_1,t_2)\Big], \end{aligned} \tag{C.4}$$

where the four remainder terms $R^{(\alpha)}_{pq}$, $R^{(1)}_{p(q+1)}$ and $R^{(2)}_{(p+1)q}$ are estimated by

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|R^{(\alpha)}_{pq}\|^2_{L^2(\omega_{\kappa\varepsilon})} + \|R^{(1)}_{p(q+1)}\|^2_{L^2(\omega_{\kappa\varepsilon})} + \|R^{(2)}_{(p+1)q}\|^2_{L^2(\omega_{\kappa\varepsilon})} \le C\varepsilon \|u\|^2_{\mathcal{S}_\varepsilon}. \tag{C.5}$$

Indeed, by the non-penetration condition (5.28) on the vertices of the cell $(p\varepsilon, q\varepsilon)+\varepsilon Y$ and pairing the involved terms differently, we get a.e. (t_1,t_2) in $\omega_{\kappa\varepsilon}$ that

$$\begin{aligned} 0 \le &(-1)^{p+q}\Big(\big(\mathrm{u}^{(1)}_{pq,3} - \mathrm{u}^{(2)}_{pq,3}\big) + \big(\mathrm{u}^{(1)}_{(p+1)(q+1),3} - \mathrm{u}^{(2)}_{(p+1)(q+1),3}\big) + \big(\mathrm{u}^{(2)}_{(p+1)q,3} - \mathrm{u}^{(1)}_{(p+1)q,3}\big) \\ &+ \big(\mathrm{u}^{(2)}_{p(q+1),3} - \mathrm{u}^{(1)}_{p(q+1),3}\big)\Big)(t_1,t_2) \\ = &(-1)^{p+q}\Big(\big(\mathrm{u}^{(1)}_{pq,3} - \mathrm{u}^{(1)}_{(p+1)q,3}\big) + \big(\mathrm{u}^{(2)}_{(p+1)q,3} - \mathrm{u}^{(2)}_{(p+1)(q+1),3}\big) + \big(\mathrm{u}^{(1)}_{(p+1)(q+1),3} - \mathrm{u}^{(1)}_{p(q+1),3}\big) \\ &+ \big(\mathrm{u}^{(2)}_{p(q+1),3} - \mathrm{u}^{(2)}_{pq,3}\big)\Big)(t_1,t_2). \end{aligned}$$

Then, the right-hand side of the above equality is rewritten in the following way:

$$\begin{aligned} \big(\mathrm{u}^{(1)}_{pq,3} - \mathrm{u}^{(1)}_{(p+1)q,3}\big)(t_1,t_2) &= \varepsilon\mathcal{R}^{(1)}_2(p\varepsilon, q\varepsilon) + R^{(1)}_{pq}(t_1,t_2), \\ \big(\mathrm{u}^{(2)}_{(p+1)q,3} - \mathrm{u}^{(2)}_{(p+1)(q+1),3}\big)(t_1,t_2) &= -\varepsilon\mathcal{R}^{(2)}_1(p\varepsilon+\varepsilon, q\varepsilon) + R^{(2)}_{(p+1)q}(t_1,t_2) \\ \big(\mathrm{u}^{(1)}_{(p+1)(q+1),3} - \mathrm{u}^{(1)}_{p(q+1),3}\big)(t_1,t_2) &= -\varepsilon\mathcal{R}^{(1)}_2(p\varepsilon, q\varepsilon+\varepsilon) + R^{(1)}_{p(q+1)}(t_1,t_2) \\ \big(\mathrm{u}^{(2)}_{p(q+1),3} - \mathrm{u}^{(2)}_{pq,3}\big)(t_1,t_2) &= \varepsilon\mathcal{R}^{(2)}_1(p\varepsilon, q\varepsilon) + R^{(2)}_{pq}(t_1,t_2), \end{aligned}$$

where $R^{(1)}_{pq}(t_1,t_2)$ and $R^{(2)}_{pq}(t_1,t_2)$ are defined by

$$R^{(1)}_{pq} \doteq \big(\mathbb{U}^{'(1)}_3(p\varepsilon,q\varepsilon) - \mathbb{U}^{'(1)}_3(p\varepsilon+\varepsilon,q\varepsilon) - \varepsilon\mathcal{R}^{'(1)}_2(p\varepsilon,q\varepsilon)\big) + \big(\mathcal{R}^{'(1)}_1(p\varepsilon,q\varepsilon) - \mathcal{R}^{'(1)}_1(p\varepsilon+\varepsilon,q\varepsilon)\big)t_2 \\ - \big(\mathcal{R}^{'(1)}_2(p\varepsilon,q\varepsilon) - \mathcal{R}^{'(1)}_2(p\varepsilon+\varepsilon,q\varepsilon)\big)t_1 + Q^{(1)}_{pq} - Q^{(1)}_{(p+1)q},$$

$$R^{(2)}_{pq} \doteq \big(\mathbb{U}^{'(2)}_3(p\varepsilon,q\varepsilon) - \mathbb{U}^{'(2)}_3(p\varepsilon,q\varepsilon+\varepsilon) + \varepsilon\mathcal{R}^{'(2)}_1(p\varepsilon,q\varepsilon)\big) + \big(\mathcal{R}^{'(2)}_1(p\varepsilon,q\varepsilon) - \mathcal{R}^{'(2)}_1(p\varepsilon,q\varepsilon+\varepsilon)\big)t_2 \\ - \big(\mathcal{R}^{'(2)}_2(p\varepsilon,q\varepsilon) - \mathcal{R}^{'(2)}_2(p\varepsilon,q\varepsilon+\varepsilon)\big)t_1 + Q^{(2)}_{pq} - Q^{(2)}_{p(q+1)}$$

and $R^{(2)}_{(p+1)q}$, $R^{(1)}_{p(q+1)}$ are referred from the above defined. It is now left to prove estimate (C.5) and due to the symmetrical behavior, we will only estimate $R^{(1)}_{pq}$. We first have

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|R^{(1)}_{pq}\|^2_{L^2(\omega_{\kappa\varepsilon})} = \sum_{(p,q)\in\mathcal{K}_\varepsilon}\Big(\int_{\omega_{\kappa\varepsilon}}\Big|\int_0^\varepsilon \partial_1\mathbb{U}^{'(1)}_3(p\varepsilon+s,q\varepsilon) - \mathcal{R}^{'(1)}_2(p\varepsilon,q\varepsilon)ds\Big|^2 dt_1dt_2 \\ + \int_{\omega_{\kappa\varepsilon}} t_2^2\Big|\int_0^\varepsilon \partial_1\mathcal{R}^{'(1)}_1(p\varepsilon+s,q\varepsilon)ds\Big|^2 dt_1dt_2 + \int_{\omega_{\kappa\varepsilon}} t_1^2\Big|\int_0^\varepsilon \partial_1\mathcal{R}^{'(1)}_2(p\varepsilon+s,q\varepsilon)ds\Big|^2 dt_1dt_2\Big) \\ + \sum_{(p,q)\in\mathcal{K}_\varepsilon}\|Q^{(1)}_{pq}\|^2_{L^2(\omega_{\kappa\varepsilon})} + \sum_{(p,q)\in\mathcal{K}_\varepsilon}\|Q^{(1)}_{(p+1)q}\|^2_{L^2(\omega_{\kappa\varepsilon})}.$$

Using Jensen's inequality on each term in the parenthesis and equality (5.17), we get

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\int_{\omega_{\kappa\varepsilon}} t_2^2\Big|\int_0^\varepsilon \partial_1\mathcal{R}^{'(1)}_1(p\varepsilon+s,q\varepsilon)ds\Big|^2 dt_1dt_2 \le C\varepsilon^5\sum_{q=0}^{2N_\varepsilon-1}\|\partial_1\mathcal{R}^{'(1)}_1(\cdot,q\varepsilon)\|^2_{L^2(0,L)},$$

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\int_{\omega_{\kappa\varepsilon}} t_1^2\Big|\int_0^\varepsilon \partial_1\mathcal{R}^{'(1)}_2(p\varepsilon+s,q\varepsilon)ds\Big|^2 dt_1dt_2 \le C\varepsilon^5\sum_{q=0}^{2N_\varepsilon-1}\|\partial_1\mathcal{R}^{'(1)}_2(\cdot,q\varepsilon)\|^2_{L^2(0,L)},$$

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\int_{\omega_{\kappa\varepsilon}}\Big|\int_0^\varepsilon \partial_1\mathbb{U}^{'(1)}_3(p\varepsilon+s,q\varepsilon) - \mathcal{R}^{'(1)}_2(p\varepsilon,q\varepsilon)ds\Big|^2 dt_1dt_2 \le C\varepsilon^5\sum_{q=0}^{2N_\varepsilon-1}\|\partial_1\mathcal{R}^{'(1)}_2(\cdot,q\varepsilon)\|^2_{L^2(0,L)}.$$

By the first estimates in (5.32) and estimate (C.3) in Step 1, we get estimate (C.5) for $R^{(1)}_{pq}$.

Step 3. In this step we prove that for a.e. $(t_1,t_2)\in\omega_{\kappa\varepsilon}$

$$\sum_{k=p}^{p+1}\sum_{\ell=q}^{q+1}\big|\big(\mathbb{U}^{'(1)}_3 - \mathbb{U}^{'(2)}_3\big)(k\varepsilon,\ell\varepsilon) - t_1\big(\mathcal{R}^{'(1)}_2 - \mathcal{R}^{'(2)}_2\big)(k\varepsilon,\ell\varepsilon) + t_2\big(\mathcal{R}^{'(1)}_1 - \mathcal{R}^{'(1)}_1\big)(k\varepsilon,\ell\varepsilon)\big| \\ \le (-1)^{p+q}\varepsilon\Big(\mathcal{R}^{(1)}_2(p\varepsilon,q\varepsilon) - \mathcal{R}^{(2)}_1(p\varepsilon+\varepsilon,q\varepsilon) - \mathcal{R}^{(1)}_2(p\varepsilon,q\varepsilon+\varepsilon) + \mathcal{R}^{(2)}_1(p\varepsilon,q\varepsilon)\Big) + S_{pq}(t_1,t_2), \tag{C.6}$$

where the remainder term S_{pq} *is estimated by*

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\|S_{p,q}\|^2_{L^2(\omega_{\kappa\varepsilon})} \le C\varepsilon\|u\|^2_{\mathcal{S}_\varepsilon}. \tag{C.7}$$

We first note that in (C.4), the left-hand side is positive. Hence, we replace the left-hand side with (C.2) and take the modulus. Applying Step 1 on the left-hand side and Step 2 on the right-hand side, we get a.e $(t_1, t_2) \in \omega_{\kappa\varepsilon}$ that

$$\begin{aligned}
&\sum_{k=p}^{p+1}\sum_{\ell=q}^{q+1}\Big|\big(\mathbb{U}_3^{'(1)} - \mathbb{U}_3^{'(2)}\big)(k\varepsilon,\ell\varepsilon) - t_1\big(\mathcal{R}_2^{'(1)} - \mathcal{R}_2^{'(2)}\big)(k\varepsilon,\ell\varepsilon) + t_2\big(\mathcal{R}_1^{'(1)} - \mathcal{R}_1^{'(1)}\big)(k\varepsilon,\ell\varepsilon)\\
&\qquad + \big(Q^{(1)}_{k\ell,3} - Q^{(2)}_{k\ell,3}\big)(t_1,t_2)\Big|\\
&= (-1)^{p+q}\Big[\varepsilon\Big(\mathcal{R}_2^{(1)}(p\varepsilon,q\varepsilon) - \mathcal{R}_1^{(2)}(p\varepsilon+\varepsilon,q\varepsilon) - \mathcal{R}_2^{(1)}(p\varepsilon,q\varepsilon+\varepsilon) + \mathcal{R}_1^{(2)}(p\varepsilon,q\varepsilon)\Big)\\
&\qquad + \Big(R^{(1)}_{pq} + R^{(2)}_{(p+1)q} + R^{(1)}_{p(q+1)} + R^{(2)}_{pq}\Big)(t_1,t_2)\Big].
\end{aligned}$$

Then, the above equation can be rewritten in the form (C.6) with S_{pq} defined by

$$S_{p,q} \doteq (-1)^{p+q}\Big(R^{(1)}_{p,q} + R^{(2)}_{p+1,q} - R^{(1)}_{p,q+1} - R^{(2)}_{p,q}\Big) + \sum_{k=p}^{p+1}\sum_{\ell=q}^{q+1}\Big|\big(Q^{(1)}_{k\ell,3} - Q^{(2)}_{k\ell,3}\big)\Big|.$$

Step 4. In this step, we prove the statement, i.e., estimate (5.37).
Starting from inequality (C.6) of Step 3, we replace (p,q) by $(2p,2q)$, $(2p+1,2q)$, $(2p,2q+1)$ and $(2p+1,2q+1)$. For a.e $(t_1,t_2) \in \omega_{\kappa\varepsilon}$, we obtain

$$\begin{aligned}
&\sum_{k=2p}^{2p+1}\sum_{\ell=2q}^{2q+1}\Big|\big(\mathbb{U}_3^{'(1)} - \mathbb{U}_3^{'(2)}\big)(k\varepsilon,\ell\varepsilon) - t_1\big(\mathcal{R}_2^{'(1)} - \mathcal{R}_2^{'(2)}\big)(k\varepsilon,\ell\varepsilon) + t_2\big(\mathcal{R}_1^{'(1)} - \mathcal{R}_1^{'(2)}\big)(k\varepsilon,\ell\varepsilon)\Big|\\
&+ \sum_{k=2p}^{2p+1}\sum_{\ell=2q+1}^{2q+2}\Big|\big(\mathbb{U}_3^{'(1)} - \mathbb{U}_3^{'(2)}\big)(k\varepsilon,\ell\varepsilon) - t_1\big(\mathcal{R}_2^{'(1)} - \mathcal{R}_2^{'(2)}\big)(k\varepsilon,\ell\varepsilon) + t_2\big(\mathcal{R}_1^{'(1)} - \mathcal{R}_1^{'(2)}\big)(k\varepsilon,\ell\varepsilon)\Big|\\
&+ \sum_{k=2p+1}^{2p+2}\sum_{\ell=2q}^{2q+1}\Big|\big(\mathbb{U}_3^{'(1)} - \mathbb{U}_3^{'(2)}\big)(k\varepsilon,\ell\varepsilon) - t_1\big(\mathcal{R}_2^{'(1)} - \mathcal{R}_2^{'(2)}\big)(k\varepsilon,\ell\varepsilon) + t_2\big(\mathcal{R}_1^{'(1)} - \mathcal{R}_1^{'(2)}\big)(k\varepsilon,\ell\varepsilon)\Big|\\
&+ \sum_{k=2p+1}^{2p+2}\sum_{\ell=2q+1}^{2q+2}\Big|\big(\mathbb{U}_3^{'(1)} - \mathbb{U}_3^{'(2)}\big)(k\varepsilon,\ell\varepsilon) - t_1\big(\mathcal{R}_2^{'(1)} - \mathcal{R}_2^{'(2)}\big)(k\varepsilon,\ell\varepsilon) + t_2\big(\mathcal{R}_1^{'(1)} - \mathcal{R}_1^{'(2)}\big)(k\varepsilon,\ell\varepsilon)\Big|\\
&\le \varepsilon\sum_{k=2p}^{2p+1}\sum_{\ell=2q}^{2q+1}(-1)^{k+\ell}\Big(\mathcal{R}_2^{'(1)}(k\varepsilon,\ell\varepsilon) - \mathcal{R}_1^{'(2)}(k\varepsilon+\varepsilon,\ell\varepsilon) - \mathcal{R}_2^{'(1)}(k\varepsilon,\ell\varepsilon+\varepsilon) + \mathcal{R}_1^{'(2)}(k\varepsilon,\ell\varepsilon)\Big)\\
&\quad + \Big(S_{(2p)(2q)} + S_{(2p+1)(2q)} + S_{(2p)(2q+1)} + S_{(2p+1)(2q+1)}\Big)(t_1,t_2).
\end{aligned} \tag{C.8}$$

We set

$$T_{pq}(t_1,t_2) \doteq \varepsilon\sum_{k=2p}^{2p+1}\sum_{\ell=2q}^{2q+1}(-1)^{k+\ell}\Big(\mathcal{R}_2^{(1)}(k\varepsilon,\ell\varepsilon) - \mathcal{R}_1^{(2)}(k\varepsilon+\varepsilon,\ell\varepsilon) - \mathcal{R}_2^{(1)}(k\varepsilon,\ell\varepsilon+\varepsilon) + \mathcal{R}_1^{(2)}(k\varepsilon,\ell\varepsilon)\Big)$$

and we want to prove that this term has a sufficiently good estimate

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\|T_{pq}\|^2_{L^2(\omega_{\kappa\varepsilon})} \le C\varepsilon\|u\|^2_{\mathcal{S}_\varepsilon}. \tag{C.9}$$

Indeed, by writing down the sum and pairing the terms, we get that

$$\begin{aligned}
&T_{pq}(t_1,t_2)\\
&= \varepsilon\Big(\big(\mathcal{R}_2^{(1)}(2p\varepsilon,2q\varepsilon) - \mathcal{R}_2^{(1)}(2p\varepsilon+\varepsilon,2q\varepsilon)\big) + \big(\mathcal{R}_2^{(1)}(2p\varepsilon+\varepsilon,2q\varepsilon+\varepsilon) - \mathcal{R}_2^{(1)}(2p\varepsilon,2q\varepsilon+\varepsilon)\big)\\
&+ \big(\mathcal{R}_2^{(1)}(2p\varepsilon+\varepsilon,2q\varepsilon+\varepsilon) - \mathcal{R}_2^{(1)}(2p\varepsilon,2q\varepsilon+\varepsilon)\big) + \big(\mathcal{R}_2^{(1)}(2p\varepsilon,2q\varepsilon+2\varepsilon) - \mathcal{R}_2^{(1)}(2p\varepsilon+\varepsilon,2q\varepsilon+2\varepsilon)\big)\\
&- \big(\mathcal{R}_1^{(2)}(2p\varepsilon+\varepsilon,2q\varepsilon) - \mathcal{R}_1^{(2)}(2p\varepsilon+\varepsilon,2q\varepsilon+\varepsilon)\big) - \big(\mathcal{R}_1^{(2)}(2p\varepsilon+2\varepsilon,2q\varepsilon+\varepsilon) - \mathcal{R}_1^{(2)}(2p\varepsilon+2\varepsilon,2q\varepsilon)\big)\\
&- \big(\mathcal{R}_1^{(2)}(2p\varepsilon,2q\varepsilon+\varepsilon) - \mathcal{R}_1^{(2)}(2p\varepsilon,2q\varepsilon)\big) - \big(\mathcal{R}_1^{(2)}(2p\varepsilon+\varepsilon,2q\varepsilon) - \mathcal{R}_1^{(2)}(2p\varepsilon+\varepsilon,2q\varepsilon+\varepsilon)\big)\Big).
\end{aligned}$$

Hence, estimate (C.9) follows from first estimates in (5.32) and the fact that

$$\begin{aligned}
&\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|T_{pq}\|^2_{L^2(\omega_{\kappa\varepsilon})} \\
&= \varepsilon^2 \sum_{(p,q)\in\mathcal{K}_\varepsilon} \Big(\int_{\omega_{\kappa\varepsilon}} \Big| \int_0^\varepsilon -\partial_1 \mathcal{R}_2^{'(1)}(2p\varepsilon+s,2q\varepsilon) + 2\partial_1\mathcal{R}_2^{(1)}(2p\varepsilon+s,2q\varepsilon+\varepsilon) - \partial_1\mathcal{R}_2^{'(1)}(2p\varepsilon+s,2q\varepsilon+2\varepsilon)ds\Big|^2 dt_1dt_2 \\
&\qquad + \int_{\omega_{\kappa\varepsilon}} \Big| \int_0^\varepsilon -\partial_2 \mathcal{R}_1^{'(2)}(2p\varepsilon,2q\varepsilon+s) + 2\partial_2\mathcal{R}_1^{'(2)}(2p\varepsilon+\varepsilon,2q\varepsilon+s) - \partial_2\mathcal{R}_1^{'(2)}(2p\varepsilon+2\varepsilon,2q\varepsilon+s)ds\Big|^2 dt_1dt_2\Big) \\
&\le C\varepsilon^5 \Big(\sum_{q=0}^{2N_\varepsilon-1} \|\partial_1\mathcal{R}_2^{'(1)}(\cdot,q\varepsilon)\|^2_{L^2(0,L)} + \sum_{p=0}^{2N_\varepsilon-1} \|\partial_2\mathcal{R}_1^{'(2)}(p\varepsilon,\cdot)\|^2_{L^2(0,L)}\Big) \le C\varepsilon\|u\|^2_{\mathcal{S}_\varepsilon}.
\end{aligned}$$

Taking the L^2 norm in the left-hand side of (C.8) and applying (C.7)-(C.9) on the right hand side, we finally obtain

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \Big(\varepsilon^2\big|\big(\mathbb{U}_3^{(1)} - \mathbb{U}_3^{(2)}\big)(p\varepsilon,q\varepsilon)\big|^2 + \varepsilon^4\big|\big(\mathcal{R}_\alpha^{(1)} - \mathcal{R}_\alpha^{(2)}\big)(p\varepsilon,q\varepsilon)\big|^2\Big) \le C\varepsilon\|u\|^2_{\mathcal{S}_\varepsilon}$$

which divided by ε^2, gives estimate (5.37). □

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