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EXACT SOLUTIONS OF DISCRETE KINETIC MODELS AND STATIONARY PROBLEMS FOR THE PLANE BROADWELL MODEL

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To Joachim Wick

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Introduction

In 1982 J.Wick published a simple class of exact solutions to the Carleman discrete velocity model of the Boltzmann equation [1]. A more general class of solutions to the same model was published in 1983 [2] by the present author in a one-page abstract without any derivation. After that J.Wick published in 1984 his own elegant derivation of these solutions based on the idea to consider the space-time plane (x,t) as a complex plane of the variable $z=t+ix$ [3].

Two years later this solution appeared to be one of starting points for a large number of H.Cornille papers; we refer here only some first publications [4,5] and the review [6], where one can find other references. H.Cornille constructed the first (after the original paper of J.E.Broadwell [7]) non-trivial solutions for the Broadwell model and also exact solutions to some more general discrete models. It should be mentioned for completeness the interesting paper of F.Golse [8], where an other class of solutions was considered for the Broadwell model. The Cornille approach was generalized and formalizedby H.Cabannes and T.H.Tiem [9].

It is remarkable that almost all known exact solutions have just the same form as the first solutions [2,3] for the Carleman model. More than that this very special form is the main *ansatz* in the Cornille method of construction of exact solutions. Such an approach seems to be very artificial, therefore it is desirable to clarify the specific role of these solutions and to find a more natural method of its derivation. The first part of the paper (Sec. 1-4) is directed just to this end. We discuss three different approaches to the constuction of exact solutions and also some specific properties of these solutions.

The second part of the paper (Sec 4.-8) is devoted to some stationary problems for the plane Broadwell model. We consider these problems in direct connection with the above mentioned exact solutions of the Carleman model. The matter is that the stationary two-dimensional Broadwell model for a partial class of its solutions (similar to incompressible fluid) appears to be exactly equivalent to the Carleman model [10]. It is a formal equivalence because we consider this model now for two space (not space-time) variables, so the statement of boundary (not initial-boundary) value problems is quite different. The Wick transition to complex variables [3] seems to be very natural due to this equivalence. We consider in Sec.5-6 some related problems and discuss the physical meaning (flows in channels) of this class of solutions. In Sec. 7 it is shown that a wide class of stationary solutions can be explicitly expressed in terms of the degenerate hypergeometric function. In Sec.8 we consider the Navier-Stokes approximation for the stationary Broadwell model. We show here that the approximate equations are in this case more complicated than the exact equations and their validity is not clear. At the end of the paper we discuss some open problems.

I. Three approaches to the construction of exact solutions for discrete velocity models

1. The approach based on the Poincare normal form method (Carleman model)

We consider here the Carleman equations in dimensionless form

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)f_1=\frac{1}{\varepsilon}\left(f_2^2-f_1^2\right),\quad \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)f_2=\frac{1}{\varepsilon}\left(f_1^2-f_2^2\right),\qquad (1)$$

where $f_1(x,t)$ and $f_2(x,t)$ are the densities of particles moving with velocities (+1) and (-1) respectively along the x-axis, and ε denotes the Knudsen number. Let

$$\rho = f_1+f_2,\quad u = f_1-f_2$$

then the equations

$$\rho_t+u_x=0,\qquad u_t+\rho_x+\frac{2}{\varepsilon}\rho u=0,\qquad (2)$$

where lower indexes t and x mean partial derivatives, describe the evolution of the *hydrodynamic* variable $\rho(x,t)$ and the *kinetic* variable $u(x,t)$. We notice here that "the Navier-Stokes equation" for our system (1) can be written as the nonlinear diffusion equation

$$\frac{\partial\rho}{\partial t}=\frac{\varepsilon}{2}\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial\rho}{\partial x}.\qquad (3)$$

Let us consider small perturbations of (2) near equilibrium

$$\rho = 1 + \bar\rho, \quad u = \bar u ,$$

then we obtain from (2)

$$\bar\rho_t + \bar u_x = 0, \qquad \bar u_t + \bar\rho_x + \frac{2}{\varepsilon} \bar\rho\, \bar u = - \frac{2}{\varepsilon} \bar u , \tag{4}$$

or in the linear approximation

$$\rho_t + u_x = 0, \qquad u_t + \rho_x + \frac{2}{\varepsilon} u = 0, \tag{5}$$

where the sign "-" is omitted.

Characteristic solutions of this system can be written as

$$\rho = \rho_k \exp\,[ikx - \lambda(k)t], \qquad u = u_k \exp\,[ikx - \lambda(k)t], \tag{6}$$

where for any real k the eigenvalue $\lambda(k)$ satisfies the quadratic equation

$$\lambda^2 - \frac{2}{\varepsilon}\lambda + k^2 = 0 \tag{7}$$

Thus we obtain two eigenvalues

$$\lambda_1(k) = \frac{1 - [1 - (\varepsilon k)^2]^{1/2}}{\varepsilon}, \quad \lambda_2(k) = \frac{1 + [1 - (\varepsilon k)^2]^{1/2}}{\varepsilon}$$

which correspond in the limiting case $\varepsilon \to 0$

$$\lambda_1 \simeq \varepsilon k^2/2, \qquad \lambda_2 \simeq 2/\varepsilon \tag{8}$$

to the two branches (hydrodynamic and kinetic) of roots of the equation (7) on the complex plane λ.

We notice that the solutions (6) can berepresentedin real form as

$$\rho = 2\theta\, e^{-\lambda t} \cos kx\,, \quad u = 2\theta \frac{\lambda}{k} e^{-\lambda t} \sin kx, \tag{9}$$

where $0<\varepsilon k<1$, $\lambda = \lambda_{1,2}(k)$ and a parameter $0<\theta<1$ characterizes a deviation from the equilibrium state.

Then we use an assumption based on the Poincaré normal form theorem [11,12]. Let us suppose that there exists an analytical "change of variables", characterized by a nonlinear operator $\hat{B}$,

$$\begin{pmatrix} \bar\rho \\ \bar u \end{pmatrix} = \hat{B}\left\{\begin{pmatrix} \rho \\ u \end{pmatrix}\right\} = \begin{pmatrix} \rho \\ u \end{pmatrix} + \hat{B}_2\left\{\begin{pmatrix} \rho \\ u \end{pmatrix},\begin{pmatrix} \rho \\ u \end{pmatrix}\right\} + \hat{B}_3\left\{...\right\} + ... \tag{10}$$

where $B_n\{...\}$, $n = 2,3,...$,mean n-linear operators, acting on the x-variable and such that $\bar\rho(x,t)$ and $\bar u(x,t)$ satisfy the nonlinear equations (4) for a non-empty class of solutions $\rho(x,t)$ and $u(x,t)$ of the linear system (5). Then, substituting (9) into (10), we can conclude that the solution corresponding to (6) has the following form:

$$\bar\rho = \sum_1^\infty \bar\rho_n(x)\theta^n exp(-n\lambda t), \quad \bar u = \sum_1^\infty \bar u_n(x)\theta^n exp(-n\lambda t), \tag{11}$$

where

$$\bar\rho_1(x) = 2\cos kx, \quad \bar u_2(x) = 2(\lambda/k)\sin kx, \quad \lambda = \lambda_{1,2}(k) \qquad (12).$$

Let $\bar\rho_n(x)$ and $\bar u_n(x)$ be for all n = 2,3,... periodic functions with period $2\pi/k$, then we obtain from (4) the following equations

$$\frac{d^2\bar u_n}{d\,x^2} + n[k^2 - (n-1)\lambda^2]\bar u_n + n\,\frac{\lambda^2 + k^2}{\lambda}\sum_1^{n-1}\frac{\bar u_{n-j}}{j}\,\frac{d\bar u_j}{dx} = 0, \quad n = 2,3,...$$

from which it is easy to find the final result

$$\bar\rho_n(x) = 2\cos nkx, \quad \bar u_n(x) = 2(\lambda/k)\sin nkx, \quad n = 1,2,...$$

We note that this result is valid for both roots $\lambda_{1,2}(k)$ since the equation (7) only was used in its derivation.

So, we constructed the solution

$$\rho(x,t) = 1 + 2\sum_1^{\infty}\theta^n e^{-n\lambda t}\cos nkx, \quad u(x,t) = \frac{2\lambda}{k}\sum_1^{\infty}\theta^n e^{-n\lambda t}\sin nkx,$$

$$\lambda^2 - (2/\varepsilon)\lambda + k^2 = 0, \quad 0 < \varepsilon k < 1 \qquad (13)$$

of the nonlinear equations (2).The summation in (13) is trivial because of the identity

$$\rho + i(k/\lambda)u = 1 + 2\sum_1^{\infty}[\theta e^{ikx-\lambda t}]^n = \frac{1 + \exp(ikx-\lambda t)}{1 - \exp(ikx-\lambda t)} \qquad (14)$$

Hence,the formulas

$$\rho(x,t) = \frac{1 - [s(t)]^2}{1 - 2s(t)\ cos\ kx + [s(t)]^2},$$

$$u(x,t) = 2(\lambda/k)\frac{s(t)\ sin\ kx}{1 - 2s(t)\ cos\ kx + [s(t)]^2}; \qquad (15)$$

$$s(t) = \theta \exp[-\lambda(k)t],\ \lambda(k) = \lambda_{\pm}(k) = \{1 \pm [1 - (\varepsilon k)^2]^{1/2}\}/\varepsilon$$

describe the explicit solution for both eigenvalues $\lambda_+(k)$ and $\lambda_-(k)$.

This first derivation of the exact solution [2,3] to the Carleman model was published only in 1988 in Russian [13]. J.Wick knew only the final result (15) and suggested in 1984 his original way to obtain this result [3].

2. The Wick approach based on the transition to complex variables (Carleman model)

We consider equations (2) with $\varepsilon = 1$.Let us introduce complex independent

$$z = t + ix,\ \bar{z} = t - ix, \qquad (16)$$

and dependent variables

$$F = \rho - iu,\ \bar{F} = \rho + iu. \qquad (17)$$

Then we look for a partial solution as an analytic function F(z). We obtain from (2) the equation

$$\mathrm{Im}\ [F'(z) + \frac{1}{2}F^2(z)] = const \qquad (18)$$

For any analytical function it means that

$$F'(z) + \frac{1}{2}[F^2(z) - F_\infty^2] = 0 \tag{19}$$

where F_∞ is arbitrary complex number. If we consider solutions, which have a physical meaning, then this constant is real and positive. The general solution of (19) has the following form:

$$F(z) = F_\infty \, cth \frac{F_\infty z}{2} = F_\infty \frac{1 + \exp(-F_\infty z)}{1 - \exp(-F_\infty z)} . \tag{20}$$

Hence, we obtain (14)-(15) in the case $F_\infty=1$, $\varepsilon = k = 1$. The erlier [1] Wick solution

$$F(z) = \frac{2}{z} = 2\frac{t - ix}{t + ix} = \rho + iu \tag{21}$$

can be easy obtained from (20) in the limiting case $F_\infty = 0$.

Using this approach we loose the difference between the two solutions for $\lambda_\pm(k)$ (15), but it is not difficult to improve this approach and to construct in such a way both solutions (15).

3. Some properties of the solutions

We consider in this Section the formulas (15) in the case $\theta=0$ and k=1,i.e. 2π-periodic solutions. Then one can rewrite (15) in the simpler form

$$\rho(x,t) = \frac{sh\ \lambda t}{ch\ \lambda t - cos\ x}, \qquad u = \frac{\lambda\ sin\ x}{ch\ \lambda t - cos\ \lambda t}; \tag{22}$$

$$\lambda^2 - 2\lambda/\varepsilon + 1 = 0, \quad \lambda = \lambda_\pm = \{1 \pm [1-\varepsilon^2]^{1/2}\}/\varepsilon, \; 0<\varepsilon<1.$$

The positivity condition for densities $f_{1,2}(1)$ is equivalent to the inequality

$$sh\ \lambda t \leq \lambda \;\Rightarrow\; \lambda t \geq Arsh\ \lambda = ln\ [\lambda + (1+\lambda^2)^{1/2}]$$

In the most interesting case $\varepsilon \to 0$ the behaviour of $\lambda_\pm$ is described by formulas (8) with k =1. Therefore we obtain for λ_- the restriction t >1 and for λ_+ the more complex restriction

$$t \geq t_+ = \frac{ln\ \lambda}{\lambda} = \frac{\varepsilon}{2}\, ln\, \frac{1}{\varepsilon}$$

So in the last case $\lambda_+ t \to \infty$ with $\varepsilon \to 0$. Hence, the positive solution of (1) with $\lambda=\lambda_+$ is close to equilibrium, i.e.$u(x,t)=O(\varepsilon)$ for all $t \geq t_+$. It explains the exceptionally short time $\tau = O(\varepsilon)$ of relaxation to equilibrium in this case. Our example shows that for non-positive initial data the relaxation time can be very small even in the hydrodynamical limit (see also [13]).

Some other interesting properties of our solutions can also be noticed. We restrict ourselves for simplicity to the case $\varepsilon = 1$ in (22), then $\lambda_+ = \lambda_- = 1$ and the two solutions are equivalent. Let us introduce the mean velocity $v = u/\rho$ and rewrite (22) in the form

$$\rho(x,t) = \frac{sh\ t}{sh\ t - cos\ x}, \qquad v(x,t) = \frac{sin\ x}{sh\ t}. \qquad (23)$$

The density $\rho(x,t)$ and the velocity $v(x,t)$ satisfy the usual equation of continuity

$$\rho_t + (\rho v)_x = 0 \qquad (24)$$

and also the second equation

$$\rho(v_t + vv_x) + [\rho(1-v^2)]_x = 0, \qquad (25)$$

as it follows from equations (2).

At the same time our solution (23) can be considered as the solution of two other nonlinear systems having a clear physical meaning:

(1) the Navier-Stokes equations for *the cold gas* with zero pressure

$$\rho_t + (\rho v)_x = 0, \quad \rho(v_t + vv_x) = v_{xx}; \qquad (26)$$

(2) the nonlinear diffusion equation (compare with (3))

$$\rho_t = (\rho^{-1}\rho_x)_x, \qquad (27)$$

that can be written also as a system of two equations: equation of continuity (24) and equation for the diffusive flux

$$\rho v = - \rho^{-1}\rho_x$$

Hence, our solution is typical not only for the discrete velocity models of the Boltzmann equation but also for some other nonlinear equations of mathematical physics.

The deep and non-trivial generalization of this simple solution of the Carleman model to a class of solutions of some more general and more physical discrete velocity models was made by H.Cornille in 1986-87.

4. The Cornille approach and its generalizations (Broadwell and other models)

In the early 80's H.Cornill e published some results [14] related to so-called *bisoliton solutions* of nonlinear partial differential equations. Roughly speaking, his idea was the following. He noticed that for a wide class of evolution equations

$$u_t = F(u, u_x, u_{xx}, \ldots)$$

one can construct exact solutions in the form

$$u(x,t) = \Phi(\omega_1, \omega_2),\ \omega_i = \exp(\alpha_i x + \beta_i t),$$

$$\alpha_i, \beta_i = const,\ i = 1,2 \qquad (28)$$

where $\Phi(\omega_1, \omega_2)$ is a rational function of the two variables ω_i, $i = 1,2$. The generalization to the case of ***n*-soliton solutions** for $n>2$ is obvious. A similar approach was suggested in [12] on the base of the Poincaré theorem, we called these solutions ***n*-modal** ones. The main difference between H.Cornille's and our approaches in the general case is that we usually consider analyticfunctions, as a consequence of the Poincaré transformation, while H.Cornille restricts himself to rational functions and does not consider the connection with the linearized equation.

Let us consider now our solutions (14),(15) from this point of view.Then

$$\rho(x,t) = -1 + \frac{1}{1 - \theta\exp(ikx - \lambda t)} + \frac{1}{1 - \theta\exp(-ikx - \lambda t)},$$

$$u(x,t) = \frac{1}{1 - \theta\exp(ikx - \lambda t)} - \frac{1}{1 - \theta\exp(-ikx - \lambda t)}, \tag{29}$$

i.e. we obtained *the bisoliton solution* in the Cornille sense The combination of this specific exact solution with the general idea of *bisolitons* was apparently the main starting point for the Cornille approach to the construction of exact solutions for Broadwell and other models [4-6]0. We describe here briefly this approach in the generalized form for the general discrete velocity model taking into account the paper of H.Cabannes and T.H.Tiem [9] and more recent papers of H.Cornille. Let us consider the general d-dimensional (d = 1,2,3) discrete model of the Boltzmann equation in the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{v}}\right) f(\mathbf{x},\mathbf{v},t) = I\{f,f\}, \quad \mathbf{x},\mathbf{v} \in \mathbf{R}^d \tag{30}$$

completely similar to the Boltzmann equation, where $I\{f,f\}$ means bilinear collision term. The main difference is that the variable v is here a discrete one and can accept the finite set of values $v_1, v_2, \dots, v_N$ only. The function $f(x,v,t)$ can also be considered as N-vector function with components

$$f_1(x,t)= f(x,v_1,t), \qquad f_2(x,t)= f(x,v_2,t),\dots,$$
$$f_N(x,t)= f(x,v_n,t) \qquad (31)$$

The collision term $I\{f_1,f_2\}$ denotes here the matrix bilinear operator acting on the pair of N-vectors $f_1(v)$ and $f_2(v)$.

Let us try now to search for the solution of (30) in the following form :

$$f(x,v,t) = f_o(v) + \sum_1^J f_j(v)\Psi(k_jx+\lambda_jt+\gamma_j), \qquad (32)$$

where $J\geq 2$ and the function $\Psi(z)$ is the simplest solution of the Riccati equation

$$\Psi'(z) = \Psi^2(z) - \Psi(z), \qquad \Psi(z) = \frac{1}{1+\exp(z)}. \qquad (33)$$

Substituting (32) in (30) we obtain

$$\sum_1^J f_j(v)(\lambda_j+k_jv)[\Psi^2(z_j)-\Psi(z_j)] = I\{f_0,f_0\} + \sum_1^J \Psi^2(z_j)I\{f_j,f_j\} +$$
$$\sum_1^J \Psi(z_j)[I\{f_0,f_j\}+I\{f_j,f_0\}] + \sum\sum_{1\leq i<j\leq J} \Psi(z_i)\Psi(z_j)[I\{f_i,f_j\}+I\{f_j,f_i\}].$$

Then we obtain the final equations for $f_j(v)$, considering

z_j, j=1,...J as independent variables

$$I\{f_0,f_0\} = I\{f_0+ f_j,f_0+ f_j\} = I\{f_i,f_j\} + I\{f_j,f_i\} = 0, \tag{34}$$

$$I\{f_0,f_j\} + I\{f_j,f_o\} = - (\lambda_j+ \mathbf{k}_j \mathbf{v})f_j(v);\ i,j = 1,\dots,J;\ i\neq j.$$

Here f(v) and I{f.f} denote respectively a constant N-vector and N×N matrix bilinear operator. Therefore we obtained finally the system of pure algebraic equations. Every non-trivial solution of (34) corresponds by formula (32) to the solution of the discrete Boltzmann equation (30). We need only to formulate the condition of the existence of nontrivial solutions to the algebraic system (34).

This system is written for (J+1)N unknown *densities* $f_j(\mathbf{v}_n)$ ($0\le j\le J$, $1\le n\le N$) and formally J(d+1) undefined parameters λ_j and $\mathbf{k}_j$.We can calculate now the number of independent equations in (34) and consider in detail the case, when this number is less than thetotal number of unknown *densities* and parameters. It is clear that we briefly described here only a rough scheme for the construction of solutions having a form (32).

The fact is that such solutions do exist and even satisfy the positivity conditions for the Broadwell model as well as for some more complex models. The algebraic construction of these solutions was the main result of the Cornille approach and we have now a class of solutions (32) to different discrete models [6].

The next problems are: (1) constructing the exact solutions in more general form and (2) using these solutions for a deeper understanding of some physical and mathematical problems. We try to do something in both directions for the Broadwell model in the second part of the paper.

II. Stationary problems for the plane Broadwell model

5. The connection with the Carleman model

We consider the four velocity Broadwell model equations with velocities rotated of 45°, that is

$$v_1 = \frac{c}{\sqrt{2}}(1,1), \qquad v_2 = \frac{c}{\sqrt{2}}(-1,1), \quad v_3 = -v_1, \qquad v_4 = -v_2 \qquad (35)$$

Then we obtain for the densities $f_i(x,y,t)$, $i=1,\dots,4$, the equations

$$\frac{\partial f_1}{\partial t} + \frac{c}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) f_1 = \frac{\partial f_3}{\partial t} - \frac{c}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) f_3 =$$

$$= 2c\sigma\,(f_2 f_4 - f_1 f_3) \qquad (36)$$

$$\frac{\partial f_2}{\partial t} - \frac{c}{\sqrt{2}}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) f_2 = \frac{\partial f_4}{\partial t} + \frac{c}{\sqrt{2}}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) f_4 =$$

$$=2c\sigma\ (f_1 f_3 - f_2 f_4)$$

where c is the velocity modulus, σ is the *cross section.*

Let us consider these equations in the stationary case for new dependent variables

$$p_1=(f_1+f_3)/\sqrt{2}, \qquad p_2=(f_2+f_4)/\sqrt{2},$$

$$u_1=(f_1-f_3)/\sqrt{2}, \qquad u_2=(f_4-f_2)/\sqrt{2}, \tag{37}$$

then

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) p_1 = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) p_2 = 0,$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u_1 = -\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u_2 = 2\sigma I, \tag{38}$$

$$I = u_1^2 - u_2^2 - p_1^2 + p_2^2$$

The general solution of the first two equations has the form

$$p_1 = P_1(x-y), \quad p_2 = P_2(x+y), \tag{39}$$

where $P_1(z)$ and $P_2(z)$ are arbitrary functions. Suppose that we have a boundary value problem with some horizontal and vertical boundaries,and also specular reflection conditions on these boundaries ,i.e. $p_1=p_2$. In particular these conditions are satisfied if $p_1 = p_2 = p = const$ (for any positive number p). Then we obtain from (38) the equations

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial y} = \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_2}{\partial y} \right) = 2\sigma\,(u_1^2 - u_2^2), \qquad (40)$$

which are obviously equivalent to the Carleman equations (1). We note that the space density

$$\rho = f_1 + f_2 + f_3 + f_4 = 2\sqrt{2}\,p = const, \qquad (41)$$

that is our *Broadwell gas* is incompressible. The components of the dimensionless mean velocity

$$u_x = \frac{u_1 + u_2}{\rho} \qquad u_y = \frac{u_1 - u_2}{\rho} \qquad (41')$$

satisfy the equations

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0\,, \quad \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = 4\rho\sigma u_x u_y \qquad (40')$$

If we pass now to dimensionless variables x/L and y/L where L is the macroscopic lenght, then we obtain finally the system

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \; \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = \frac{2}{\varepsilon} u_x u_y, \; \varepsilon = Kn = (2\rho\sigma L)^{-1}, \qquad (42)$$

that can be reduced to the Carleman system (2). At the same time the physical meaning of the variables in (2) and (42) is quite different.

6. The Broadwell *incompressible fluid* and its flows in channels

Let us consider the stationary Broadwell equations (38) and restrict ourselves by the case of constant density

$$\rho = \sqrt{2}\,(p_1 + p_2) = const$$

It is possible if and only if $p_{1,2} = const_{1,2}$ because of the equality (39). Therefore the most general *incompressible* stationary equations (38) can be reduced to the system of two equations of Carleman type

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u_1 = -\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u_2 = 2\sigma(u_1^2 - u_2^2 + \hat{\theta}),$$

$$\hat{\theta} = p^2 - p^2 = const,$$

or to the dimensionless system for the components u_x and u_y of the mean velocity (41)

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \qquad \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = \frac{2}{\varepsilon}(u_x u_y + \theta),$$

$$\theta = \frac{p_1 - p_2}{2(p_1 + p_2)} = const, \quad \varepsilon = (2\rho\sigma L)^{-1} \tag{43}$$

The parameter θ characterizes the relative difference between partial densities p_1 and p_2 of the particles, moving along the positive $(v_x = v_y)$ and the negative $(v_x = v_y)$ diagonals on (x,y)-plane. This difference disappears if we have at least one vertical or horizontal boundary with the specular reflection condition because $p_1 = p_2$ on this boundary, i.e. $\theta=0$ in the equations (43). So we obtain the Carleman-like system (42) $u_x=0$ or $u_y=0$ on the vertical or horizontal specular reflecting boundaries respectively.

Let us consider a flow of the Broadwell *incompressible fluid* in the rectangular domain $0<x<L$, $-\pi<y<\pi$. Let $y=+\pi$ and $y=-\pi$ be specular reflecting boundaries (see Fig. 1) and the densities of incoming particles are given on vertical boundaries, i.e.

$$f_{1,4}(0,y) = f^{(0)}_{1,4}(0,y), \qquad f_{2,3}(L,y) = f^{(0)}_{2,3}(L,y), \quad -\pi \le y \le \pi \tag{44}$$

For the incompressible case $\rho = const$, we can immediately define the solution on this boundaries

$$f_{2,3}(0,y) = \frac{\rho}{2} - f^{(0)}_{4,1}(y), \qquad f_{1,4}(L,y) = \frac{\rho}{2} - f_{3,2}(y)$$

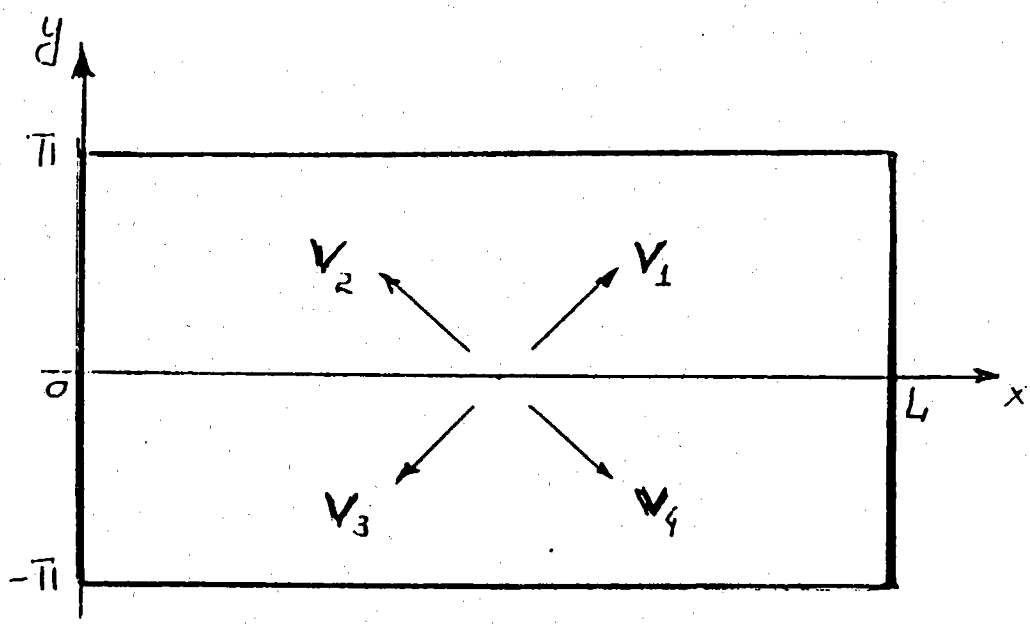

Fig. 1

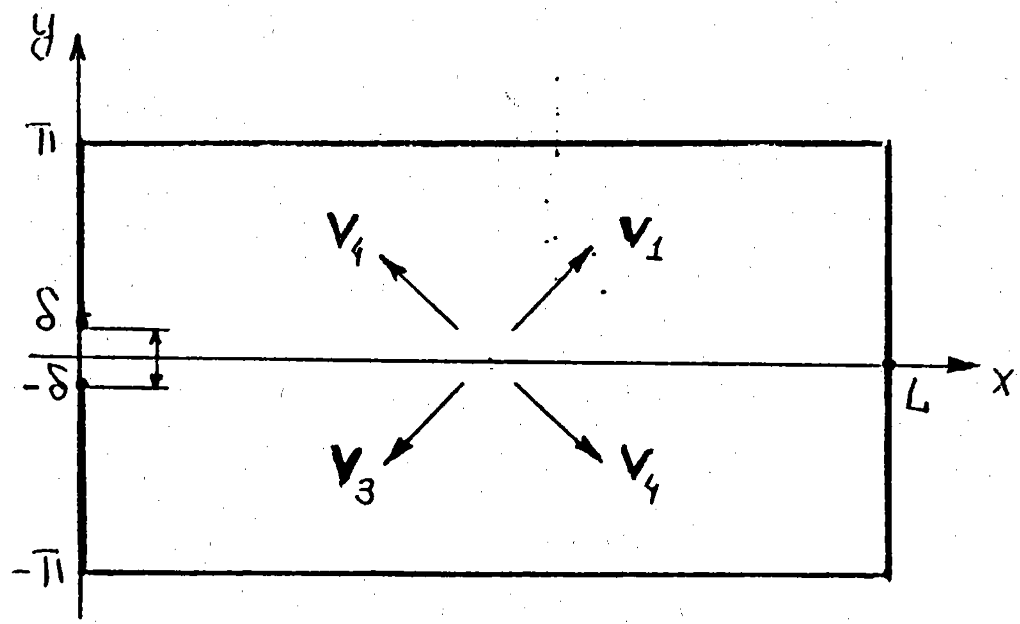

Fig. 2

Let us consider a flow of the Broadwell *incompressible fluid* in the rectangular domain $0<x<L$, $-\pi<y<\pi$. Let $y= +\pi$ and $y = -\pi$ be specular reflecting boundaries (see Fig.1) and the densities of incoming particles are given on vertical boundaries, i.e.

$$f_{1,4}(0,y) = f^{(0)}_{1,4}(0,y), \quad f_{2,3}(L,y) = f^{(0)}_{2,3}(L,y), \; -\pi\le y \le\pi \tag{44}$$

For the incompressible case $\rho = const$ we can immediately define the solution on this boundaries

$$f_{2,3}(0,y) = \frac{\rho}{2} - f^{(0)}_{4,1}(y), \quad f_{1,4}(L,y) = \frac{\rho}{2} - f_{3,2}(y),$$

or in terms of components of the dimensionless mean velocity (41)

$$u_x(0,y)= u^{(0)}_x(y)= \frac{f_1+ f_4- f_2- f_3}{\sqrt{2}\,\rho} = \frac{1}{\sqrt{2}}\left(\frac{2(f^{(0)}_1+ f^{(0)}_4)}{\rho} - 1\right)$$

$$u_y(0,y)= u^{(0)}_y(y)= \frac{f_1+ f_2- f_3- f_4}{\sqrt{2}\,\rho} = \frac{2\,(f_1- f_4)}{\rho} \tag{45}$$

$$u(L,y)= u^{(L)}_x(y)= \frac{1}{\sqrt{2}}\left[1 - \frac{2}{\rho}\,(f^{(L)}_2 + f^{(L)}_3)\right]$$

$$u_y(L,y)= u^{(L)}_y(y)= \frac{1}{\sqrt{2}\,\rho}\,(f^{(L)}_2- f^{(L)}_3)$$

We assume here that the given functions $f_{1,4}^{(0)}(y)$ and $f_{2,3}^{(L)}(y)$ satisfy the conditions

$$f_1^{(0)}(\pm\pi)= f_4^{(0)}(\pm\pi), \quad f_2^{(L)}(\pm\pi)= f_3^{(L)}(\pm\pi).$$

At last we can find the stationary density ρ from *the conservation law*

$$\frac{d}{dx}\int_{-\pi}^{+\pi} u_x(x,y)dy =0, \tag{46}$$

so that

$$\rho= \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left(f_1^{(0)}(y) + f_2^{(L)}(y) + f_3^{(L)}(y) + f_4^{(0)}(y)\right) dy \tag{47}$$

Thus we have shown that the above described stationary incompressible flow for the Broadwell model equations is defined by the Carleman-like system (42) and the following boundary conditions:

$$u_y(x,\pm\pi)= 0, \ u_x(0,y)= u_x^{(0)}(y), \ u_y(0,y)= u_y^{(0)}(y)$$

$$u_x(L,y)= u_x^{(L)}(y), \ u_y(L,y)= u_y^{(L)}(y), \tag{48}$$

$$\int_{-\pi}^{+\pi} u_x^{(0)}(y)dy = \int_{-\pi}^{+\pi} u_x^{(L)}(y)dy, \quad u_y^{(0)}(\pm\pi) = u_y^{(L)}(\pm\pi) =0$$

Does the solution of the boundary value problem (42),(48) exist? The answer is unclear because the boundary conditions at $x=0$ and $y=\pm\pi$ have already defined the initial boundary value problem for the Carleman-like system (42) in the domain $(x>0,\ -\pi<y<\pi)$, so the conditions at the line $x=L$ can not be satisfied in general case. This question becomes more clear if we reduce (42) to the Carleman model (2) by the substitution

$$\bar{t}=x,\ \bar{x}=y,\ \bar{\rho}=-u_x,\ \bar{u}=-u_y,\ \bar{T}=L.$$

Then we obtain from (42),(48) the Carleman equations

$$\rho_t+u_x=0, \qquad u_t+\rho_x=-\frac{2}{\varepsilon}\rho u, \tag{49}$$

where the sign "-" is omitted. Boundary conditions at the points $x=\pm\pi$ also correspond to the specular reflection

$$u(\pm\pi,t)=u(\pm\pi,t)=0 \tag{50}$$

but in this case we have also *initial conditions*

$$\rho(x,0)=-u_x^{(0)}(x), \qquad u(x,0)=-u_y^{(0)}(x) \tag{51}$$

and *final conditions*

$$\rho(x,T)=-u_x^{(L)}(x), \qquad u(x,T)=-u_y^{(L)}(x). \tag{52}$$

We can always assume that

$$\int_{-\pi}^{+\pi} u_x^{(0)}(x)dx \le 0, \tag{53}$$

more than that we assume for a moment that

$$u_x^{(0)}(x)\le 0, \quad |u_y^{(0)}(x)| \le |u_x^{(0)}(x)|, \tag{54}$$

Then our problem (49)-(51) is completely equivalent to the initial-boundary value problem for the Carleman model (1) with positive initial data. This problem has a unique solution for a wide class of positive initial data, some results are obtained also for non-positive data [15]. Inequalities (54) are satisfied in the most physically interesting case of the strong flux. Therefore the initial-boundary value problem (49)-(51) has a unique solution that does not satisfy given *final conditions* (52) at the time t=T.

Hence, for the Broadwell equations (36) the stationary solution of the above described boundary value problem is in general case *compressible*, i.e. $\rho \neq const$, and can not be found from the Carleman-like system (42).

Nevertheless there exists a class of boundary value problems problems for the Broadwell model, which is exactly equivalent to the initial boundary value problem (49)-(51). It is the limit case $L=\infty$ of the above considered problem (42),(48) provided that

$$u_x^{(L)}(y) \to u_\infty = const, \qquad u_y^{(L)}(y) \to 0 \tag{55}$$

if $L \to \infty$. It is easy to obtain from (47) the following relation between ρ, u_∞ and $\overset{(0)}{f}_{1,4}(y)$ ($\rho>0$, $-1/\sqrt{2} < u_\infty < 0$):

$$\rho(1+\sqrt{2}\, u_\infty) = \frac{1}{\pi} \int_{-\pi}^{+\pi} \left(\overset{(0)}{f}_1(y) + \overset{(0)}{f}_4(y) \right) dy \tag{56}$$

Thus for given functions $\overset{(0)}{f}_{1,4}(y)$ the only one from the two parameters ρ and u_∞ can be chosen independently. We assume that $\rho, f_1^{(0)}$ and $f_4^{(0)}$ are given. Then we can find u_∞ from (56) and define *the initial data* from (45). The conditions of thepositivity (54) can be written as

$$0< f_1^{(0)} + f_4^{(0)} < \frac{\rho}{2} - |f_1^{(0)} - f_4^{(0)}|, \tag{57}$$

If these conditions are satisfied then we can immediately reduce our boundary value problem for the Broadwell model in half-infinite domain ($x>0$, $-\pi<y<+\pi$) to the initial boundary value problem with positive data for the Carleman model. So the existence (not uniqueness!) theorems for such problems are also equivalent for these two models.

It is easy to include into consideration boundary conditions of different kinds. For example, let us consider a very clear physical situation (see Fig.2): a specular reflecting wall at the line $x=0$ with a small hole of a diameter 2δ in the center of the wall. Then we have the following boundary condition at the line $x=0$:

$$f_1(0,y)= f_4(0,y) \qquad \text{if} \qquad |y|<\delta$$

$$f_1(0,y)= f_2(0,y), f_4(0,y)= f_3(0,y) \quad \text{if} \quad \delta<|y|<\pi \tag{58}$$

Therefore we can obtain from (45) the boundary conditions for velocity components

$$u_x(0,y)=0 \quad \text{if} \quad \delta<|y|<\pi;$$

$$u_x(0,y)= -1/\sqrt{2} \qquad u_y(\ 0,y)=0 \quad \text{if } |y|<\delta \qquad (59)$$

In this case we can consider the boundary value problem (42), (59),(45) even for finite values of L because theabove mentioned contradiction does not appear in this case. The matter is that *the initial data* at the line x=0 is given in this problem only for u_x; $u_y(0,y)$ is unknown for $|y|>\delta$. Therefore we have to start from *the final data* at the line x=L in the corresponding initial boundary value problem, but this problem seems to be incorrect (the inversion of time in a nonlinear dissipative system). Thus the *incompressible* stationary solutions are apparently typical for this class of problems and it was verified also by numerical experiments [8].

At last we return to the problem for the equations (40')

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \qquad \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = 4\rho\sigma u_x u_y$$

in the half-infinite domain $(x>0,\ |y|<\pi)$ with the boundary conditions for the small hole on the line x=0 (see Fig.2 with $L\to\infty$)

$$u_x=-\sqrt{2} \qquad u_y=0 \quad \text{if } x=0,\ |y|<\sigma;$$

$$u_x=0 \quad \text{if} \quad x=0,\ \delta<|y|<\pi$$

and the equilibrium state at $x\to\infty$

$$u_x\to u_\infty<0, \quad u_y\to 0$$

Let us pass to new variables $u_x=\bar{u}_x|u_\infty|, u_y=\bar{u}_y|u_\infty|$ and omit the sign "-" in final results. Then we obtain the following problem

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \quad \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = \frac{2}{\varepsilon} u_x u_y, \qquad \varepsilon=(2\rho\sigma|u_\infty|)^{-1};$$

$$u_x(0,y)=-\pi/\delta \ \text{ if } \ |y|<\delta, \qquad u_x(0,y)=0 \ \text{ if } \ \delta<|y|<\pi, \tag{60}$$

$$u_y(0,y)=0; \qquad u_x(\infty,y)=-1, \quad u_y(\infty,y)=0.$$

It is easy to verify that our exact solution (22) to the Carleman model gives us the two exact solutions

$$u_x = -\frac{\text{sh}\ \lambda x}{\text{ch}\ \lambda x - \cos x}, \qquad u_y = -\frac{\lambda \sin y}{\text{ch}\ \lambda x - \cos x},$$

$$\lambda = \lambda_\pm(\varepsilon) = [1 \pm (1-\varepsilon^2)^{1/2}]/\varepsilon \tag{61}$$

of the boundary value problem (60) for the limit case $\delta \to 0$ (infinitely small hole).

We have here two solutions for the boundary value problem. The numerical calculations showed that only one of these solutions, with $\lambda = \lambda_-(\varepsilon)$, is the true limit in time of time-dependent solutions; the second solution, with $\lambda = \lambda_+(\varepsilon)$, seems to be unstable [8].

In the next Section we consider the solutions in detail.

7. Potential flows and exact solutions

We consider now the general case (43) for *incompressible* stationary solution of the Broadwell equations. Substituting

$u=u_x$, $v=u_y$ and $\varepsilon=1$ in (43) we obtain the following equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 2(uv + \theta), \tag{62}$$

where θ is a positive constant. We apply to this system the standard method of plane fluid dynamics [16] and consider the so-called potential flows. Let $\phi(x,y)$ be a potential function for our system, i.e.

$$u = \frac{\partial}{\partial x}\phi(x,y), \qquad v = \frac{\partial}{\partial x}\phi(x,y), \tag{63}$$

then it satisfies the Laplace equation

$$\Delta\phi = \phi_{xx} + \phi_{yy} = 0$$

because of the first equation in (62). We can introduce also the current function $\psi(x,y)$ by the equalities

$$u = \frac{\partial}{\partial y}\psi(x,y), \qquad v = -\frac{\partial}{\partial x}\psi(x,y), \tag{64}$$

We obtain the standard Cauchy-Riemann conditions $\phi_x=\psi_y$, $\phi_y=-\psi_x$ for the analytic function

$$f(z=x+iy) = \phi(x,y) + i\psi(x,y), \qquad \Delta\phi = \Delta\psi = 0. \tag{65}$$

Thus, the Wick complex variables (see Section 1) appear here very naturally. The velocity components are defined by standard formulas

$$u - iv = f'(z),\ u = \mathrm{Re}\ f'(z),\quad v=-\mathrm{Im}\ f'(z). \tag{66}$$

The second equation in (62) can be written as

$$\phi_{xy} = \phi_x \phi_y + \theta. \tag{67}$$

We note that

$$[f'(z)]^2 = \phi_x^2 - \phi_y^2 - i\phi_x\phi_y\ , \quad f''(z) = \phi_{xx} - i\phi_{yy}\ ,$$

so the equation (67) means that

$$\mathrm{Im}\left(f''(z) - \frac{1}{2}[f'(z)]^2 \right) = -\theta = const \tag{68}$$

This is possible for the analytic function f(z) if and only if

$$f''(z) - \frac{1}{2}[f'(z)]^2 = A - i\theta,$$

where A is a real constant.

Thus we obtain from the assumption (63) the modification of the equation (19)

$$f''(z) = \frac{1}{2}\left([f'(z)]^2 - [f'(\infty)]^2 \right). \tag{69}$$

Its general solution (20)-(21)

$$f'(z) = f'(\infty)\ \mathrm{cth}\left[\frac{f'(\infty)}{2}\right] \qquad \text{if}\ \ f'(\infty) \neq 0$$

$$f'(z) = 2/z \qquad \text{if}\ \ f(\infty) = 0 \tag{70}$$

defines completely the *incompressible* potential flows for the stationary Broadwell model.

Let us try to generalize this result to the case of the *compressible* potential flows. We consider the stationary Broadwell model equations (38) with $\sigma = 1/2$ for the functions

$$u = u_1 + u_2, \quad v = u_1 - u_2. \tag{71}$$

Then we obtain

$$u_x + v_y = 0, \quad v_x + u_y = 2\,[uv + \theta(x,y)], \tag{72}$$

where

$$\theta(x,y) = P_2^2(x+y) - P_1^2(x-y), \quad \theta_{xx} = \theta_{yy} \tag{73}$$

in accordance with (39).

If we repeat again all the steps from the introduction of the potential (63) to the final equation (68), then we obtain the direct generalization of this equation in the following form:

$$\mathrm{Im}\left(f''(z) - \frac{1}{2}\,[f'(z)]^2 \right) = -\,\theta(x,y). \tag{74}$$

Hence the function $\theta(x,y)$ satisfies the Laplace equation

$$\Delta\theta = \theta_{xx} + \theta_{yy} = 0$$

as imaginary part of an analytic function. It follows from the comparison with (73) that

$$\theta_{xx} = \theta_{yy} = 0, \; \theta(x,y) = \mathrm{Im}\,(\alpha z^2 + \beta z + \gamma),$$

$$\alpha,\beta,\gamma = const, \quad \mathrm{Im}\,\alpha = 0 \tag{75}$$

We can conclude that the most general class of potential flows for the stationary Broadwell model equations (38) can be described by the formulas (66),(71)-(75) and the equation

$$f''(z) = \frac{1}{2} [f'(z)]^2 - (\alpha z^2 + \beta z + \gamma) \qquad (76)$$

for two complex constants β,γ and a real constant α. The case $\alpha=\beta=0$ corresponds to the *incompressible* potential flows and was considered above. The last equation can be reduced to the linear equation

$$w''(z) = \frac{1}{2} (\alpha z^2 + \beta z + \gamma)\, w(z) \qquad (77)$$

by the standard substitution

$$f(z) = -2 \ln w(z)\ . \qquad (78)$$

The general solution of the equation (77) can be expressed in terms of the classical special functions [17]; we obtain in such a way a wide class of exact solutions. For brevity we restrict ourselves here to only one example.

Let partial densities $p_{1,2}$ be in (38),(73) the following functions

$$p_1(x,y)=x-y,\ p_2(x,y)=[(x+y)^2+\lambda]^{1/2},\quad x\geq y,\quad \lambda>0. \qquad (79)$$

Then

$$\theta(x,y) = p_2^2-p_1^2= 4xy+\lambda = \mathrm{Im}\ (2z^2+i\lambda),$$

that is $\alpha=2$, $\beta=0$, $\gamma=i\lambda$ and the equation (77) can be written as

$$w''(z) = (z^2+i\lambda/2)\ w(z) \qquad (80)$$

For any solution $w(z)$ of this equation the corresponding solution of the Broadwell equations (72) can be found by formulas (66),(78), i.e.

$$u(x,y) = -2\ \mathrm{Re}\ \frac{w'(z)}{w\ (z)}, \qquad v(x,y) = 2\ \mathrm{Im}\ \frac{w'(z)}{w\ (z)}.$$

After the substitution

$$w(z) = \psi(z^2)\exp(-z^2/2)$$

we obtain from (80) the equation for the function θ(t) in the form

$$t\psi'' + (1/2 - t)\psi' - [(1+\lambda/2)/4]\psi = 0. \qquad (81)$$

Two linearly independent solutions of this equation are expressed in terms of the degenerate hypergeometric function $\Phi(\alpha,\beta;t)$ ([17],p.1059)

$$\begin{aligned} &1.\ \psi = \Phi\ [(2+i\lambda)/8, 1/2; t]; \\ &2.\ \psi = \Phi\ [(6+i\lambda)/8, 3/2; t]\ t^{1/2}. \end{aligned} \qquad (82)$$

We do not consider here some interesting properties of these solutions, it will be done in a separate paper.

In the next Section 9 we discuss the case of small Knudsen numbers (hydrodynamics).

8. Hydrodynamics for the Broadwell model

We consider here time-dependent Broadwell equations (36) for dimensionless variables

$$\bar{x}=x/L,\ \bar{y}=y/L,\ \bar{t}=\frac{ct}{\sqrt{2}\,L},\ \bar{f}_i=\frac{\rho_0 f_i}{\sqrt{2}\,L},\quad i=1,\dots,4 \tag{83}$$

where L and ρ_0 are correspondingly typical macroscopic length and density for our problem. Omitting the sign "-" in final results we obtain

$$\frac{\partial f_1}{\partial t}+\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)f_1=\frac{\partial f_3}{\partial t}-\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)f_3=\frac{1}{\varepsilon}\,(f_2f_4-f_1f_3)$$

$$\varepsilon=(2\rho\ \sigma L)^{-1}, \tag{84}$$

$$\frac{\partial f_2}{\partial t}-\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)f_2=\frac{\partial f_4}{\partial t}+\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)f_4=\frac{1}{\varepsilon}\,(f_1f_3-f_2f_4).$$

We introduce hydrodynamic variables by formulas

$$\rho=f_1+f_2+f_3+f_4,\quad \rho u_x=f_1-f_2-f_3+f_4,$$
$$\rho u_y=f_1+f_2-f_3-f_4,\quad \pi=f_1-f_2+f_3-f_4, \tag{85}$$

where ρ, u_x, u_y and π denote correspondingly a density, components of mean velocity and non-diagonal component of the pressure tensor.

The resulting equations can be written in the form

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\rho u_x + \frac{\partial}{\partial y}\rho u_y = 0, \qquad \frac{\partial}{\partial t}\rho u_x + \frac{\partial}{\partial x}\rho + \frac{\partial}{\partial y}\pi = 0$$

$$\frac{\partial}{\partial t}\rho u_y + \frac{\partial}{\partial x}\pi + \frac{\partial}{\partial y}\rho = 0, \tag{86}$$

$$\frac{\partial}{\partial t}\pi + \frac{\partial}{\partial x}\rho u_y + \frac{\partial}{\partial y}\rho u_x = -\frac{\rho}{\varepsilon}(\pi - \rho u_x u_y).$$

The first three equations can be expressed in the standard tensor form

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}\rho u_i = 0, \ \frac{\partial}{\partial t}u_i + u_j\frac{\partial}{\partial x_j}u_i + \frac{1}{\rho}\frac{\partial}{\partial x_j}p_{ij} = 0, \ i,j = 1,2; \tag{87}$$

$$p_{11} = \rho(1-u_1^2), \ \ p_{22} = \rho(1-u_2^2), \ \ p_{12} = p_{21} = \pi - \rho u_1 u_2,$$

where we used indexes 1 and 2 for x and y components of vectors and tensors. It is convenient to introduce the function

$$\omega = -p_{12} = \rho u_1 u_2 - \pi \tag{88}$$

as the fourth (kinetic) variable in equations (86).

Then we obtain finally three equations for hydrodynamical variables

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\rho u_x + \frac{\partial}{\partial y}\rho u_y = 0$$

$$\left(\frac{\partial}{\partial t} + u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y}\right)u_x + \frac{1}{\rho}\frac{\partial}{\partial x}\rho(1-u_x^2) = \frac{1}{\rho}\frac{\partial}{\partial y}\omega \tag{89}$$

$$\left(\frac{\partial}{\partial t} + u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y}\right)u_y + \frac{1}{\rho}\frac{\partial}{\partial y}\rho(1-u_y^2) = \frac{1}{\rho}\frac{\partial}{\partial x}\omega$$

and the fourth equation for the kinetic variable $\omega(x,y,t)$

$$\left(\frac{\partial}{\partial t} - u_x\frac{\partial}{\partial x} - u_y\frac{\partial}{\partial y}\right)\omega - \rho\left((1-u_x^2)\frac{\partial u_y}{\partial x} + (1-u_y^2)\frac{\partial u_x}{\partial y}\right) = -\frac{\rho\omega}{\varepsilon} \tag{90}$$

Let us consider now the first equations (Euler and Navier-Stokes) of the usual Chapman-Enskog expansion

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots$$

It follows from (90) that $\omega_0 = 0$ and in the first (Navier-Stokes) approximation we obtain

$$\omega \simeq \omega_{N\text{-}S} = \varepsilon\left((1-u_x^2)\frac{\partial u_y}{\partial x} + (1-u_y^2)\frac{\partial u_x}{\partial y}\right) \tag{91}$$

Thus the equations (89), where $\omega = \omega_{N\text{-}S}$(91), are the Navier-Stokes equations for the Broadwell system (84).

It is clear that our derivation of these equations is based on the time-dependent Broadwell model. As to the stationary case, the validity of these equations is not very clear, because it is difficult to understand how to derive the stationary Navier-Stokes equations directly from the stationary Broadwell model The same question arises for the Boltzman equation.

Let us try to clarify this question at least for the Broadwell model. Then the stationary Navier-Stokes equations have the following form:

$$\frac{\partial}{\partial x}\rho u_x + \frac{\partial}{\partial y}\rho u_y = 0, \left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y}\right)u_x + \frac{1}{\rho}\frac{\partial}{\partial x}\rho(1-u_x^2) = \frac{1}{\rho}\frac{\partial \omega}{\partial y}$$

$$\left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y}\right)u_y + \frac{1}{\rho}\frac{\partial}{\partial y}\rho(1-u_y^2) = \frac{1}{\rho}\frac{\partial \omega}{\partial x}, \qquad (92)$$

where ω is defined in (91). The exact stationary Broadwell equations are the same equations (92), where ω is defined by the stationary equation (90). This equation includes the space derivatives of $\omega(x,y)$ but they can be easy excluded by using the equations (92). We obtain in such a way the exact formula for ω

$$\omega = \frac{\varepsilon}{\rho}\left(\frac{\partial}{\partial x}\rho u_y + \frac{\partial}{\partial y}\rho u_x\right) \qquad (93)$$

and the explicit formula for the difference between exact and approximate expressions for ω

$$\omega - \omega_{N\text{-}S} = \varepsilon\Delta = \varepsilon\left(\left(u_x\frac{\partial}{\partial y} + u_y\frac{\partial}{\partial x}\right)\ln(\rho) + u_x^2\frac{\partial u_y}{\partial x} + u_y^2\frac{\partial u_x}{\partial y}\right) \qquad (94)$$

It is clear that $\Delta=0$ for any smooth solution of Euler equations, i.e. equations (92) with $\varepsilon=0$. Therefore there is no formal contradiction between exact and approximate systems. At the same time we notice that:

(1) for our Broadwell model the Navier-Stokes equations are more

complicated than exact equations, in particular in the case $\rho = const.$

(2) it is obvious from our consideration that *the Navier-Stokes approximation* for $\omega(x,y)$ in (92) is not unique; we can choose also other approximate formulas for ω with the same accuracy, the only restriction is that the corresponding difference Δ (94) is equal to zero on any smooth solution of the Euler equations.

The first remark is not valid for the Boltzmann equation but the second remark is also important in this case. Perhaps we can use more simple equations than Navier-Stokes equations with the same accuracy.

Conclusions

In conclusion we can present some remarks concerning mainly some open problems.

1. Almost all known exact solutions to discrete kinetic models have a very similar form. We showed in Sec. 3 that this form of solutions is typical also for some other equations. Hence,it is important to understand more deeply the properties of these solutions, their group nature, etc. (see also the discussion of related problems in [18]). At the same time there exist exact solutions of more complex form even for relatively simple (stationary Broadwell) models. We do not consider here some recent results [19] because they mainly concern travelling waves for more complex discrete models.

2. It is shown here and in [10] that boundary value problems

for the Broadwell model in two-dimensional unbounded domains can have more than one solution. The second solution is apparently unstable but it was "proved" only numerically [10]. The stability problem for stationary nonequilibrium solutions is also open.

3. The Carleman model is obviously the simplest one but it seemed to be "unphysical". It is proved in Sec.5-6 that this simple model describes also some classes of stationary solutions for the Broadwell model. Hence, we can in some cases apply directly known results related to the initial boundary value problem for the Carleman system, but in general we need new rigorous results for non-standard boundary value problems. It is interesting to use Cabannes results [15] concerning non-positive solutions of the Carleman model.

4. In the last Sec. 8 we considered the general problem of the validity of the stationary Navier-Stokes equations for the Broadwell model. In this case the fact that these equations are at least useless is almost self-evident. It is clear also that the.undefined fluxes (viscosity, etc.) in hydrodynamic equations can be choosen by different ways with the same accuracy as in the Navier-Stokes equations derived by the Chapman-Enskog method. The situation with the Boltzmann equation seems to be similar and it is necessary to clarify the role of the Navier-Stokes equation in the stationary case.

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