Helmholtz Resonators with Large Aperture

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The lowest resonant frequency of a cavity resonator is usually approximated by the classical Helmholtz formula. However, if the opening is rather large and the front wall is narrow this formula is no longer valid. Here we present a correction which is of third order in the ratio of the diameters of aperture and cavity. In addition to the high accuracy it allows to estimate the damping due to radiation. The result is found by applying the method of matched asymptotic expansions. The correction contains form factors describing the shapes of opening and cavity. They are computed for a number of standard geometries. Results are compared with numerical computations.

Acknowledgements

The paper is based on ideas used first by L. Ronkel[1] deriving the classical Helmholtz formula in a rigorous way from the wave equation [12]. We thank S.W. Rienstra[2] for some valuable comments. This work was supported in the frame of the program *Wirtschaftsnahe Forschung* of the Ministry of Economy, Rheinland Pfalz, Germany.

1 Introduction

A Helmholtz–resonator consists of a large cavity connected with the environment via a small aperture. The lowest resonant frequency does not correspond to waves standing between opposite walls of the cavity. Rather, it is induced by an alternating stream through the aperture compressing and decompressing the air inside the cavity. It is called Helmholtz frequency. Helmholtz and Lord Rayleigh predicted this frequency already in the last century assuming a potential flow through the opening, a uniform pressure inside the cavity and neglecting radiation losses [13, 11]. They found the Helmholtz frequency to be proportional to the square-root of the quotient of aperture radius and cavity volume. The opening of the resonator may be a long duct, as in case of a bottle, or just a hole in a narrow wall, as in case of guitars. The Helmholtz frequency of a bottle–shaped resonator is already well predictable taking into account only the kinetic energy of the air inside the duct, which shows a uniform velocity profile [3].

Things become more involved with guitar–like resonators, the subject of this paper. In particular, if the radius of the aperture is not so much smaller than the diameter of the cavity the

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classical Helmholtz formula yields wrong estimates of the lowest resonant frequency. Even for cavities with moderate aspect ratio there may occur errors of more than 10%. The essential parameter ε the smallness of which decides on the validity of the Helmholtz formula is the square-root of the ratio of the diameters of aperture and cavity. The classical formula is of first order in this parameter. A more recent approximation that applies still to arbitrarily shaped resonators is of third order [1]. For spherical resonators the Helmholtz frequency has been expanded to sixth order [2]. Here we present an approximation which is also of sixth order, but applies to quite a general class of resonators. In addition to the high accuracy it provides the damping constant associated with the resonant frequency. This is necessary to estimate the power radiated or absorbed by a cavity resonator.

The result is found by applying the method of matched asymptotic expansions to the Helmholtz equation. Solutions of this equation are represented in a different way and on different scales close to the opening, inside and outside the cavity. The partial solutions are matched by expanding them into the small parameter ε and comparing terms of equal powers of ε on the same scale. In some aspects the method is similar to the one used in [10] solving elliptic eigenvalue problems on singularly perturbed regions.

We stress that the new procedure does not require any physically motivated assumptions, e.g. a potential flow through the aperture or an essentially constant acoustic pressure inside the cavity. They arise automatically from the method at the correct order of accuracy.

The coefficients of the expansion consist of form factors describing the shape of the aperture and, respectively, of the cavity. They are independent of the size. This enables us to treat a large variety of resonators by only a few parameters. In order to compute these factors there has to be solved an integral equation on the aperture and, respectively, simple Poisson equations on the closed cavity. Explicit expressions of the factors are given for elliptic and rectangular openings and for hemispherical and rectangular cavities. Moreover, the influence of the position of the opening is investigated for cubes. We find the following qualitative result.

The classical Helmholtz formula underestimates the resonant frequency of cavities close to a hemisphere with an opening in the centre of the front wall, i.e. resonators where the other walls have a similar distance to the aperture (e.g. a half–cube). It overestimates the frequency of cavities with large aspect ratios such as pipes. Finally, we demonstrate that the approximation is in fact valid also for large apertures by comparing with numerical results.

2 Problem

We investigate Helmholtz resonators which are bounded by a narrow plane wall in the neighbourhood of the aperture. We are interested in the lowest resonant frequency and the radiation losses at this frequency. Losses due to viscosity or absorption at the walls are neglected. Since we focus on rather large apertures assuming narrow walls and an inviscid medium seems to be justified. The validity of the rigid wall assumption depends on the material of the cavity.

The Helmholtz frequency and the damping constant are closely related to the eigenvalue $\lambda = k^2$ of the Helmholtz equation subject to homogeneous Neumann conditions on the walls and the Sommerfeld radiation condition far away from the opening.

$$\Delta u + k^2 u = 0 \quad \text{on } \Omega \tag{1}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{2}$$

$$\frac{\partial u}{\partial r} - iku = o\left(r^{-1}\right) \ , \ r = |x| \to \infty \quad \text{on } \Omega_+ . \tag{3}$$

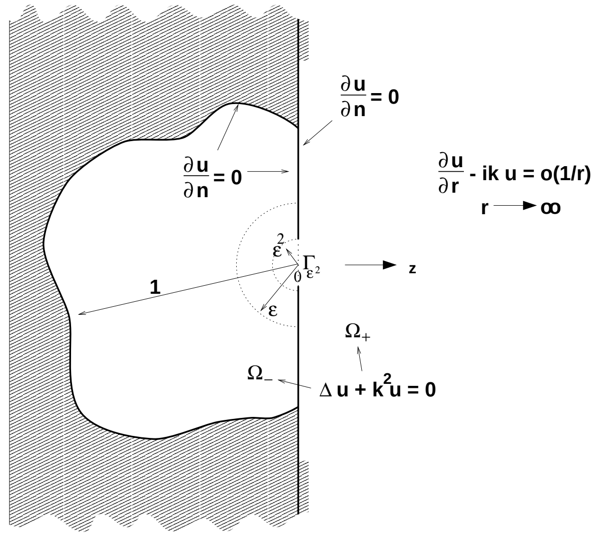


Figure 1: geometry

$u(x)\, e^{-(2\pi i f+\delta)t}$ is the acoustic pressure at position x and time t. The Helmholtz frequency f and the damping constant δ are related to the wavenumber k via

$$f = \frac{c}{2\pi} \operatorname{Re}(k), \quad \delta = -c \operatorname{Im}(k), \tag{4}$$

where c denotes the velocity of sound. The interior of the cavity $\Omega_- \subset I\!R^3$ is a connected bounded domain with smooth boundary, i.e. the Gauß Theorem shall be applicable. In a neighbourhood of the origin the boundary coincides with the x-y–plane. $\Omega_+ = I\!R^2 \times I\!R_+$ is the right half-space. Ω_+ and Ω_- are connected via the aperture

$$\Gamma_r = \left\{ (x, y, 0) \in I\!R^3 \middle| \, r^{-1}(x, y) \in \Gamma \right\}, \tag{5}$$

where $\Gamma \subset I\!R^2$ is a connected bounded domain with area π and smooth boundary which contains the origin. For instance, if Γ is a circle with unit radius then Γ_r is a circle of radius r. If the aperture is no circle we call $r = \sqrt{\frac{1}{\pi}\text{area}(\Gamma_r)}$ the radius of Γ_r. Finally, the domain Ω consists of the cavity, the aperture, and the exterior:

$$\Omega = \Omega_- \cup \Gamma_r \cup \Omega_+. \tag{6}$$

Assuming Ω_+ to be the right half-space means that the resonator is mounted in an infinite wall. However, results will remain valid as long as the aperture is considerably smaller than the penetrated wall.

3 Result

The present approximation of Helmholtz frequency and damping constant requires five parameters describing the geometry of the resonator:

- the *radius* of the actual aperture $r = \sqrt{\frac{1}{\pi}\text{area}(\Gamma_r)}$
- the *diameter* of the cavity $l = \text{vol}(\Omega_-)^{\frac{1}{3}}$
- the *capacitance* κ of the normalised aperture Γ with area π
- the *compactness* α and another form factor β describing the shape of the cavity.

The form factors κ, α, and β remain unchanged, if the cavity is blown up. Hence, we have two independent kinds of parameters for both, the aperture and the cavity, one describing the size and one describing the shape.The form factors κ, α, and β are defined as follows:

κ: Let Γ be the aperture with area π which has the same shape as Γ_r and let $\varphi : \Gamma \to I\!R$ be the solution of the integral equation

$$\int\limits_{\Gamma} \frac{\varphi(\eta)}{4\pi|\xi-\eta|} dS(\eta) = 1 \quad \text{for all } \xi \in \Gamma. \tag{7}$$

Then the capacitance of Γ reads

$$\kappa = \frac{1}{4\pi} \int\limits_{\Gamma} \varphi(\eta)\, dS(\eta)\,. \tag{8}$$

Note that κ does not change, if the aperture is blown up, as we consider always a normalised opening. The expression *capacitance* originates from electrostatics, where (7) is a standard problem [8]. There the charge density φ on the isolated conducting plate Γ is sought producing a constant potential. For instance, if Γ is a disc of radius 1 we have:

$$\varphi(\eta) = \frac{4}{\pi\sqrt{1-|\eta|^2}}, \quad \kappa = \frac{2}{\pi}. \tag{9}$$

Capacitances of other apertures are listed below.

α: Let $p \in H^1(\Omega_-)$ be the solution of the Poisson equation

$$\begin{aligned} -\Delta p &= \frac{2\pi}{V} \quad \text{on } \Omega_- \\ \frac{\partial p}{\partial n} &= \frac{\partial}{\partial n}\frac{1}{|x|} \quad \text{on } \delta\Omega_- \setminus \{0\} \\ p(x) &= O(|x|)\,, \quad |x| \to 0, \end{aligned} \tag{10}$$

where V is the volume of Ω_-. Then the *compactness* is defined as

$$\alpha = \frac{1}{l^2} \int\limits_{\Omega_-} \frac{1}{|x|} - p(x)\, dx. \tag{11}$$

We call α compactness as it takes on high positive values for hemispheres or half–cubes and tends to $-\infty$ for cavities with very large aspect ratios, e.g. pipes.

β : Let $q \in H^1(\Omega_-)$ be the solution of the Poisson equation

$$\begin{aligned} -\Delta q &= \frac{2\pi}{V}\left(1 - \frac{l}{\alpha}\left(\frac{1}{|x|} - p(x)\right)\right) \\ \frac{\partial q}{\partial n} &= 0 \quad \text{on } \partial\Omega_- \\ q(x) &= O(|x|)\,, \quad |x| \to 0 \end{aligned} \tag{12}$$

where p is the solution of problem (10). Then the second form factor is defined as

$$\beta = -\frac{1}{l^2} \int\limits_{\Omega_-} q\,(x)\ \ dx. \tag{13}$$

Note that α and β remain invariant, if x is rescaled, i.e. the form factors depend only on the shape, but not on the actual size of the cavity. Values of the form factors will be given below for several shapes.

Now we formulate the main result of this paper, an approximation of the wave number k associated with the Helmholtz frequency. It is of order $(r/l)^3$. This turns out to be the highest order of accuracy which can be achieved solely with the five parameters r, l, κ, α, and β.

$$\boxed{\begin{gathered} lk = \sigma \left\{1 + \frac{\alpha}{4\pi}\sigma^2 \left(1 + \frac{3\alpha + 4\beta}{8\pi}\sigma^2\right)\right\} \ - \ \frac{i}{4\pi}\sigma^4 \left\{1 + \frac{\alpha}{\pi}\sigma^2\right\} \ + \ O(\sigma^7) \\ \sigma = \sqrt{\frac{\pi\kappa r}{l}}, \quad f = \frac{c}{2\pi}\,\mathrm{Re}\,(k)\,, \quad \delta = -c\,\mathrm{Im}\,(k)\,. \qquad (14) \end{gathered}}$$

The defining equation of k has been multiplied by l to emphasis its structure. (14) relates two non–dimensional quantities, on the one hand the ratio of cavity diameter and wave length $\frac{l}{1/k}$, and on the other hand the quotient of aperture radius and cavity diameter $\frac{r}{l}$. The right–hand side is a power series in $\varepsilon = \sqrt{r/l}$. The coefficients of the terms ε^n consist of the non–dimensional form factors of aperture and cavity. They have the form $\sqrt{\kappa^n\,\gamma_1\cdots\gamma_n}$, where the factors γ_i describing the cavity are composed of α and β and change with i, while the factor κ of the aperture is always the same. Therefore, we have included it into the quantity σ to make the representation more concise. The first non–trivial term is of order ε for the real part and of order ε^4 for the imaginary part. Taking into account only these terms we find the classical formula of the Helmholtz frequency and a nice expression of the damping constant:

$$f = \frac{c}{2\pi}\sqrt{\frac{\pi\kappa r}{V}}, \quad \delta = \frac{c}{4\pi}\frac{(\pi\kappa r)^2}{V}, \tag{15}$$

where c is the velocity of sound and V is the volume of the cavity. A reasonable relative accuracy is obtained for both, frequency and damping constant, if we restrict ourselves to a basic term and a correction term of the next higher order. In this case the σ^5 term may be skipped and the second form factor β is not needed. Hence the crucial term deciding on the validity of the low order expressions (15) is

$$\mu = \frac{\kappa\alpha r}{l}. \tag{16}$$

Al least for small apertures, i.e. if higher order components are negligible, we find: if μ is positive, Helmholtz frequency and damping constant are underestimated, otherwise they are overestimated.

4 Matched asymptotic expansions

We are looking for the lowest eigenvalue $\lambda = k^2$ of the Helmholtz equation (1) subject to the boundary conditions (2) and (3). This will be done by matching three representations of the pressure, one being valid outside the cavity, one inside the cavity close to the aperture and one inside the cavity away from the opening. The latter representation is developed on a macroscopic scale where the volume and the diameter of the cavity is one. In contrast to the original problem the opening is treated as a point source. Boundary conditions describing the behaviour close to this point source have to be found by matching.

Outside the cavity and close to the aperture inside the cavity, there the pressure is represented by two single layer potentials. The weight functions are restricted to the aperture. Hence, the Helmholtz equation, the radiation condition, and the Neumann conditions on the the plane front wall are automatically satisfied. The two single layer potentials are matched claiming continuity of pressure and velocity on the aperture. The scale is chosen such that the radius of the aperture is one. Boundary conditions are left undefined for points inside the cavity far away from the opening.

Finally, the two representations inside the cavity are matched. For that purpose the solution defined in the vicinity of the aperture is lifted to the macroscopic scale. Then the two representations are expanded into ε, the square-root of the ratio of aperture radius and cavity diameter. Finally, terms of equal power are fitted by balancing the constant and weakly singular components.

We stress that the subsequent derivation is only formal, i.e. we do not prove that our approximation is well defined and tends to the desired eigenvalue. However, the numerical tests at the end of the paper are reassuring.

4.1 Pressure in the vicinity of the aperture

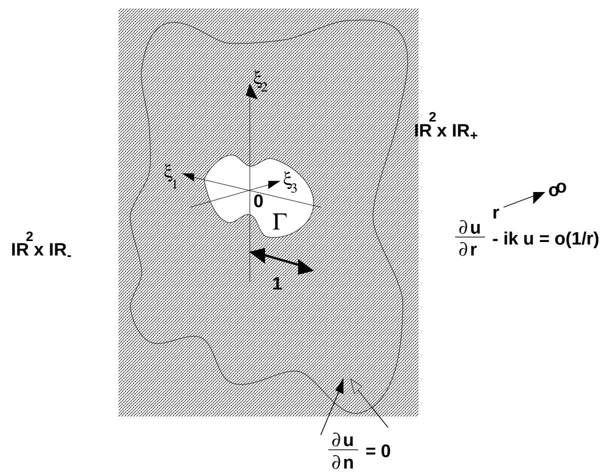


Figure 2: vicinity of aperture

First we introduce the two scales used in the vicinity of and away from the opening. Let x',k',l',V',r' the position of a point, the wave number, the cavity diameter etc. on an arbitrary scale. Scaling by l' yields the corresponding non–dimensional quantities on the macroscopic scale:

$$x = \frac{x'}{l'}, \quad r = \frac{r'}{l'} = \varepsilon^2, \quad l = 1, \quad V = l^3 = 1, \quad k = l'k'. \tag{17}$$

Scaling by r' yields position and aperture radius on the microscopic scale:

$$\xi = \frac{x'}{r'} = \frac{x}{\varepsilon^2}, \quad \rho_0 = 1. \tag{18}$$

Let us assume that the aperture is small with respect to the diameter of the cavity. Hence, if we focus on the solution close to the opening, it will resemble the solution of the simpler problem that is illustrated in fig. 2. On the microscopic scale , where the aperture has radius one, this problem reads:

$$\left(\Delta_\xi + \varepsilon^4 k^2\right) v(\varepsilon,\xi) = 0 \qquad , \xi \in \omega \tag{19}$$

$$\tfrac{\partial}{\partial n} v(\varepsilon,\xi) = 0 \qquad , \xi \in \partial\omega \tag{20}$$

$$\left(\tfrac{\partial}{\partial \rho} - i\varepsilon^2 k\right) v(\varepsilon,\xi) = o(\rho^{-1}) \qquad , \xi \in \omega_+,\ \rho = |\xi| \to \infty \tag{21}$$

$$\omega_\pm = I\!R^2 \times I\!R_\pm\ ,\ \omega = \omega_- \cup \omega_+ \cup \Gamma.$$

Note that this problem is incomplete as the behaviour of v is undefined for $\rho \to \infty$ in ω_-, which corresponds to the interior of the cavity in the original problem. This leads to free coefficients that have to be fitted later on by matching. Lifting v to the macroscopic scale it will provide an approximation of u, the solution of the original problem (1)–(3), i.e. $u(\varepsilon,x) \approx v(\varepsilon, x/\varepsilon^2)$ close to the opening.

The restrictions of v to ω_+ and ω_-, respectively, may be represented as single layer potentials:

$$v^+(\varepsilon,\xi) = \int_\Gamma a(\varepsilon,\eta)\, \frac{e^{i\varepsilon^2 k(\varepsilon)|\xi-\eta|}}{4\pi|\xi-\eta|} dS(\eta) \tag{22}$$

$$v^-(\varepsilon,\xi) = \int_\Gamma b(\varepsilon,\eta)\, \frac{e^{i\varepsilon^2 k(\varepsilon)|\xi-\eta|}}{4\pi|\xi-\eta|} + c(\varepsilon,\eta)\, \frac{e^{-i\varepsilon^2 k(\varepsilon)|\xi-\eta|}}{4\pi|\xi-\eta|} dS(\eta) \tag{23}$$

Incoming waves are excluded in the representation of v^+. This is necessary and sufficient to satisfy the Sommerfeld condition (21). Note that v^+ solves automatically the Helmholtz equation (19) and the Neumann condition (20) on ω^+ for any weight function a keeping v^+ well defined. The same is true for v^- on ω_- with b and c.

Solutions v which are valid on the whole of ω are found by matching v^+ and v^- in such a way that the transition over Γ is continuous and continuously differentiable. Differentiability is considered first. From the theory of single layer potentials [6] we find by symmetry arguments:

$$\tfrac{\partial}{\partial \xi_3} v^+(\varepsilon,\xi) = -\tfrac{1}{2} a(\varepsilon,\xi) \qquad , \xi \in \Gamma \tag{24}$$

$$\tfrac{\partial}{\partial \xi_3} v^-(\varepsilon,\xi) = \tfrac{1}{2}\left(b(\varepsilon,\xi) + c(\varepsilon,\xi)\right) \quad , \xi \in \Gamma. \tag{25}$$

Hence, $c = -(a+b)$ and v^- may be rewritten as

$$v^-(\varepsilon,\xi) = \int_\Gamma \frac{2ib(\varepsilon,\eta)\sin\left(\varepsilon^2 k(\varepsilon)|\xi-\eta|\right) - a(\varepsilon,\eta)\, e^{-i\varepsilon^2 k(\varepsilon)|\xi-\eta|}}{4\pi|\xi-\eta|} dS(\eta)\,. \tag{26}$$

Claiming continuity of v then yields for any $\xi \in \Gamma$:

$$\begin{aligned} 0 &= \frac{v^+(\varepsilon,\xi) - v^-(\varepsilon,\xi)}{2} \\ &= \int_\Gamma \frac{a(\varepsilon,\xi)\cos\left(\varepsilon^2 k(\varepsilon)|\xi-\eta|\right) - ib(\varepsilon,\xi)\sin\left(\varepsilon^2 k(\varepsilon)|\xi-\eta|\right)}{4\pi|\xi-\eta|} dS(\eta)\,. \end{aligned} \tag{27}$$

We assume that a, b, and k permit the following expansions:

$$k(\varepsilon) = \varepsilon k_1 + \varepsilon^2 k_2 + \varepsilon^3 k_3 + \cdots \tag{28}$$

$$a(\varepsilon,\eta) = a_0(\eta) + \varepsilon a_1(\eta) + \varepsilon^2 a_2(\eta) \cdots \tag{29}$$

$$b(\varepsilon,\eta) = b_{-3}(\eta)\,\varepsilon^{-3} + b_{-2}(\eta)\,\varepsilon^{-2} + b_{-1}(\eta)\,\varepsilon^{-1} + \cdots\,. \tag{30}$$

k missing a constant reflects our aim to find the lowest eigenvalue of the Helmholtz equation $\lambda = k^2$, which tends to zero with the radius of the opening $r \sim \varepsilon^2$. The expansion of b starts with ε^{-3} since the lowest power contributed by sine is 3, whereas the lowest power contributed by cosine is 0.

If we plug these expansions into the continuity condition (27) and collect for equal powers of ε we end up with a series of conditions on the coefficients of a, b, and k. The first six conditions may be written in a concise form, if we introduce the constants

$$A_n = \frac{i}{2\pi} \int\limits_{\Gamma} \sum_{j=1}^{n+1} k_j b_{n-2-j}(\eta)\, dS(\eta) \quad , n \in I\!N. \tag{31}$$

They read:

$$\varepsilon^n : \quad \frac{1}{4\pi} \int\limits_{\Gamma} \frac{a_n(\eta)}{|\xi - \eta|} dS(\eta) = \frac{1}{2} A_n, \quad n = 0, \dots, 5. \tag{32}$$

From ε^6 on the right–hand sides also contain integrals including positive powers of $|\xi - \eta|$ owing to the expansions of sine and cosine:

$$\begin{aligned} \varepsilon^6 : \quad \frac{1}{4\pi} \int\limits_{\Gamma} \frac{a_6(\eta)}{|\xi - \eta|} dS(\eta) \quad &= \quad \frac{1}{2} A_6 + \frac{k_1^2}{8\pi} \int\limits_{\Gamma} a_0(\eta) \, |\xi - \eta| \, dS(\eta) \\ &\quad + \frac{k_1^3}{24\pi} \int\limits_{\Gamma} i b_{-3}(\eta) \, |\xi - \eta|^2 \, dS(\eta) \qquad (33) \\ \varepsilon^7 : \quad & \quad \dots \end{aligned}$$

Equations (32), (33), ... are integral equations defining the functions a_n provided the right–hand sides are known. Here, for the first time, the three problems enter which we introduced to define the form factors of aperture and cavity. (32) can be solved for the a_n, if we know the solution φ of problem (7), which appears in the definition of the capacitance κ. Provided the weight function φ is known the first six a_n read

$$a_n(\eta) = \frac{1}{2} A_n \varphi(\eta), \quad n = 0, \dots, 5 \tag{34}$$

Higher a_n require more complicated integral equations to be solved. However, in this paper we restrict ourselves on approximations that require only the capacitance of the aperture. Thus, the expansion will be continued only so far as solving for a_6 can be avoided.

The expansion of k is based on matching the solutions of two simplified problems, one characterising the situation in the vicinity of the aperture and one treating the opening as a point source. v^- is the solution of the first problem restricted to the left half-space, which corresponds to the interior of the cavity. To prepare the matching we will now expand v^- into powers of ε, which is proportional to the square-root of the radius of the aperture. Using (31), (8), and (34) we find after some lengthy but straight forward manipulations:

$$\begin{aligned} v^-(\varepsilon, \xi) = & \\ & \left(1 - \frac{1}{2} \int\limits_{\Gamma} \frac{\varphi(\eta)}{4\pi \, |\xi - \eta|} dS(\eta)\right) \left\{A_0 + \varepsilon A_1 + \dots + \varepsilon^5 A_5\right\} \\ + \tfrac{i\kappa}{2} \quad & \left\{\varepsilon^3 k_1 A_0 + \varepsilon^4 (k_1 A_1 + k_2 A_0) + \varepsilon^5 (k_1 A_2 + \dots + k_3 A_0)\right\} \end{aligned}$$

$$+\varepsilon^6 \quad \left\{ A_6 - \int_\Gamma \frac{a_6(\eta)}{4\pi|\xi-\eta|} dS(\eta) + \frac{i\kappa}{2}\ (k_1 A_3 + \cdots + k_4 A_0) \right.$$
$$\left. + \frac{k_1^2 A_0}{4} \frac{1}{4\pi} \int_\Gamma \varphi(\eta)|\xi-\eta|\, dS(\eta) - \frac{k_1^2}{6} \frac{i}{2\pi} \int_\Gamma b_{-3}(\eta)|\xi-\eta|^2\, dS(\eta) \right\}$$
$$+ \quad O\left(\varepsilon^7\right) \tag{35}$$

Let us summarise this section. Up to now we have only claimed that v solves the Helmholtz equation in the vicinity of the aperture. This has already been sufficient to reduce the freedom in v to a few constants and a remainder, which is less structured, of order $O(\varepsilon^6)$. In the next section these constants will be fitted to the special geometry of the cavity, which was ignored up to now.

4.2 Pressure in the cavity away from the opening

We return to the original scaling $x = \varepsilon^2\xi$, where the diameter and the volume of the cavity is one. We look for an approximation w to the pressure inside the cavity away from the aperture. We assume w to allow an expansion of the form

$$w(\varepsilon, x) = w_0(x) + \varepsilon w_1(x) + \varepsilon^2 w(x) + \cdots \ . \tag{36}$$

For k we assume once more the expansion (28). Note that w is an approximate eigenfunction of the desired eigenvalue k^2, i.e. w is only determined up to a factor. To fix one family of eigenfunctions $w(\varepsilon,\cdot)$, $\varepsilon > 0$ we introduce the following normalisation:

$$\int_{\Omega_-} w_0\, dx = 1, \qquad \int_{\Omega_-} w_n\, dx = 0, \quad n \geq 1, \tag{37}$$

where 1 is the volume of Ω_-. w is determined as solution of the following problem:

$$\Delta w + k^2 w = 0 \quad \text{on } \Omega_-\backslash\{0\} \tag{38}$$
$$\frac{\partial w}{\partial n} = 0 \qquad \text{on } \partial\Omega_-\backslash\{0\} \tag{39}$$

This is again an incomplete problem as we allow a singular behaviour of w in the vicinity of the origin without prescribing boundary conditions there. Matching with the pressure v^- found in the last section will provide the missing information. Inserting the expansions of w and k into (38),(39) and collecting equal powers of ε yields a series of Poisson equations on $\Omega_-\backslash\{0\}$:

$$\begin{aligned} -\Delta w_0 &= 0 \\ -\Delta w_1 &= 0 \\ -\Delta w_2 &= k_1^2\, w_0 \\ &\cdots \\ -\Delta w_n &= \sum_{m=0}^{n-1} \left(\sum_{j=1}^{n-m-1} k_j k_{n-m-j} \right) w_n \end{aligned} \tag{40}$$

subject to the boundary conditions

$$\frac{\partial w_n}{\partial n} = 0 \text{ on } \partial\Omega_-\backslash\{0\}, \quad n = 0, 1, \ldots \ . \tag{41}$$

Solvability conditions on the components of w and k arise, if we integrate these equations over the punched domain $\Omega_\varepsilon^- = \Omega_- \backslash \left\{ x \in I\!R^3 \,\middle|\, |x| \le \varepsilon \right\}$, apply the Gauß Theorem, use the normalisation (37), and turn to the limit $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} \int\limits_{S_\varepsilon^-} \frac{\partial w_n}{\partial r}\, dS = \sum_{j=1}^{n-1} k_j k_{n-j}, \quad n = 0, 1, \ldots, \tag{42}$$

where $S_\varepsilon^- = \left\{ x \in I\!R^3 \,\middle|\, |x| = \varepsilon,\ x_3 < 0 \right\}$ is the left hemisphere with radius ε. Now w is matched with the pressure v^- describing the situation close to the opening. For this purpose we lift v^- to the macroscopic scaling, i.e. we substitute $\xi = \varepsilon^{-2}x$ in (35) and recollect for equal powers of ε. Then we claim that w and v^- coincide except for strongly singular components that appear in v^- by expanding the surface integrals and that vanish quickly inside the cavity.

Let us consider a general surface integral of the kind we find in (35). Using the representation

$$|\xi - \eta| = \frac{r}{\varepsilon^2}\left(1 - 2\frac{\varepsilon^2}{r}\langle \nu, \eta\rangle + \frac{\varepsilon^4}{r^2}\langle \eta, \eta\rangle\right)^{\frac{1}{2}}, \qquad r = |x|, \quad \nu = \frac{x}{r} \tag{43}$$

we find that the surface integrals in (35) may be expanded into Laurent series in $\frac{r}{\varepsilon^2}$ with coefficients that depend only on ν:

$$\int\limits_\Gamma f(\eta)\, |\xi - \eta|^n\, dS(\eta) = \sum_{j=-\infty}^{n} f_j^n(\nu) \left(\frac{r}{\varepsilon^2}\right)^j. \tag{44}$$

Now we redistribute the surface integrals in (35) according to

$$\begin{aligned} &\varepsilon^m \int\limits_\Gamma f(\eta)\, |\xi - \eta|^n\, dS(\eta) \\ &= \sum_{j=0}^{n} \varepsilon^{m-2j} \left\{ f_j^n(\nu)\, r^j \right\} + \varepsilon^{m+2} \left\{ \frac{f_{-1}^n(\nu)}{r} + \sum_{j=-\infty}^{-2} f_j^n(\nu) \frac{r^j}{\varepsilon^{2j+2}} \right\}, \end{aligned} \tag{45}$$

i.e. the non–negative powers of r are collected as usual, whereas all singular components are collected under the power of ε corresponding to r^{-1}. In particular, we write

$$\int\limits_\Gamma \frac{\varphi(\eta)}{4\pi\, |\varepsilon^{-2}x - \eta|} dS(\eta) = \kappa\varepsilon^2 \left(\frac{1}{r} + s(\varepsilon, x)\right). \tag{46}$$

Have a look at (8), the definition of the capacitance κ, to see that (46) is correct. From representation (45) we see that the strongly singular part s satisfies:

$$\int\limits_{S_\varepsilon^-} \frac{\partial s(\varepsilon, r\nu)}{\partial r}\, dS = O(\varepsilon), \tag{47}$$

i.e. s does not influence the solvability condition (42). Reordering the terms of v^- finally yields:

$$
\begin{aligned}
& v^-\left(\varepsilon, \varepsilon^{-2} x\right) \\
= \;& \varepsilon^0 \left\{A_0\right\} + \varepsilon^1 \left\{A_1\right\} + \varepsilon^2 \left\{A_2 - \frac{\kappa A_0}{2}\left(\frac{1}{|x|} - s(\varepsilon, x) - p_2(x)\right)\right\} \\
+ \;& \varepsilon^3 \left\{A_3 - \frac{\kappa A_1}{2}\left(\frac{1}{|x|} - s(\varepsilon, x) - p_3(x)\right) + \frac{i\kappa}{2} k_1 A_0\right\} \\
+ \;& \varepsilon^4 \left\{A_4 - \frac{\kappa A_2}{2}\left(\frac{1}{|x|} - s(\varepsilon, x) - p_4(x)\right) + \frac{i\kappa}{2}\left(k_1 A_1 + k_2 A_0\right)\right\} \\
+ \;& \varepsilon^5 \left\{A_5 - \frac{\kappa A_3}{2}\left(\frac{1}{|x|} - s(\varepsilon, x) - p_5(x)\right) + \frac{i\kappa}{2}\left(k_1 A_2 + \cdots + k_3 A_0\right)\right\} \\
+ \;& \varepsilon^6 \left\{A_6 - \frac{\kappa A_4}{2}\left(\frac{1}{|x|} - s(\varepsilon, x) - p_6(x)\right) + \frac{i\kappa}{2}\left(k_1 A_3 + \cdots + k_4 A_0\right) + C_6\right\} \\
+ \;& \varepsilon^7 \left\{A_7 - \frac{\kappa A_5}{2}\left(\frac{1}{|x|} - s(\varepsilon, x) - p_7(x)\right) + \frac{i\kappa}{2}\left(k_1 A_4 + \cdots + k_5 A_0\right) + C_7\right\} \\
+ \;& O(\varepsilon^8)
\end{aligned}
\tag{48}
$$

The p_n consist of the regular terms that arise expanding surface integrals with positive powers of $|\xi - \eta|$. These appear in (35) at level ε^6 and higher and have now, after changing the scale, contributions to lower powers of ε. All we need to know about the p_n is the fact that they contain merely positive powers of r, i.e.

$$
p_n(x) = O(|x|), \quad n = 2, \ldots, 7. \tag{49}
$$

Expanding the original representation (26) of v^- shows that positive powers of r appear first at level ε^2, thus there are no p_0 and p_1. The C_n consist of the constant parts of the surface integrals with positive powers of $|\xi - \eta|$. Note that C_6 includes the integral $\int_\Gamma b_{-3}(\eta)\,|\eta|^2\, dS(\eta)$, i.e. we need the second moment of b_{-3}, while we know merely the the 0^{th} moment by (31). Ignorance of C_6 will limit the order of our approximation. The strongly singular term s vanishes rapidly inside the cavity and does not even influence the solvability condition (42) by virtue of (47). Hence, w and v^- may be matched simply by assigning the ε^n term of (48) to w_n without s :

$$
w_n(x) = A_n - \frac{\kappa A_{n-2}}{2}\left(\frac{1}{|x|} - p_n(x)\right) + \frac{i\kappa}{2}\sum_{j=1}^{n-2} k_j A_{n-2-j} + C_n, \quad n = 0, \ldots, 7, \tag{50}
$$

where non–existing parameters have to be set to zero. The normalisation condition (37) yields

$$
\begin{aligned}
A_0 &= 1, \qquad A_1 = 0, \qquad \text{and for } n = 2, \ldots, 5 : \\
A_n &= \frac{\kappa A_{n-2}}{2} \int\limits_{\Omega_-} \frac{1}{|x|} - p_n(x)\, dx \;-\; \frac{i\kappa}{2}\sum_{j=1}^{n-2} k_j A_{n-2-j}.
\end{aligned}
\tag{51}
$$

Note that A_6 cannot be determined by the normalisation condition, since C_6 is unknown. Next we consider the solvability condition (42). As the surface of the hemisphere S_ε^- is of order ε^2 the influence of the regular terms vanishes with $\varepsilon \to 0$. Using $\int_{S_\varepsilon^-} \frac{\partial}{\partial r}\frac{-1}{r} dS = 2\pi$ we find:

$$
\pi\kappa A_{n-2} = \sum_{j=1}^{n-1} k_j k_{n-j}, \quad n = 2, \ldots, 7, \quad \text{i.e.} \tag{52}
$$

$$k_1 = \sqrt{\pi\kappa}, \quad k_2 = 0, \quad k_n = \frac{1}{2k_1}\left(\pi\kappa A_{n-1} - \sum_{j=2}^{n-1} k_j k_{n+1-j}\right), \; n = 3, \ldots, 6. \tag{53}$$

The system of Poisson equations (40),(41) becomes

$$\begin{aligned} A_{n-2}\, \Delta P_n(x) &= -2\pi \sum_{j=0}^{n-2} A_{n-2-j}\, u_j(x) \quad \text{on } \Omega_- \backslash \{0\} \\ \frac{\partial}{\partial n} p_n(x) &= \frac{\partial}{\partial n}\frac{1}{|x|} \quad \text{on } \partial\Omega_- \backslash \{0\}, \quad n = 2, \ldots, 5. \end{aligned} \tag{54}$$

Here we have used (52) and the fact that constants and $|x|^{-1}$ are harmonic outside 0. For instance, if $n = 2$ we find $\Delta p_2 = -2\pi$, i.e. p_2 is the solution p of problem (10) defining the form factor α. (Keep in mind that $V = 1$ and $l = 1$ on our scale.) Hence, $A_2 = \frac{\kappa\alpha}{2}$, $k_3 = \frac{\pi\kappa}{2}\frac{A_2}{k_1}$, and $w_2(x) = \frac{\kappa}{2}\left\{\alpha - \left(\frac{1}{|x|} - p(x)\right)\right\}$. It is noteworthy that the A_n, $n \geq 2$ are directly related to the coefficients of the eigenvalue

$$\lambda = k^2 = \lambda_2 \varepsilon^2 + \lambda_3 \varepsilon^3 + \cdots. \tag{55}$$

As $\lambda_n = \sum_{j=1}^{n-1} k_j k_{n-j}$ relation (52) induces $\lambda_n = \pi\kappa A_{n-2}$, $n \geq 2$. Finally, going iteratively through (53),(54), and (51) yields the desired components of the eigenvalue $\lambda = k^2$ and of the approximate eigenmode w.

n	λ_{n+2}			w_n
0	$\pi\kappa$			1
1	0			0
2	$\pi\kappa$	$\frac{1}{2}\kappa\alpha$		$\frac{1}{2}\kappa\left(\alpha - \frac{1}{\vert x\vert} + p(x)\right)$
3	$-\frac{i}{2\pi}(\pi\kappa)^{\frac{5}{2}}$			0
4	$\pi\kappa$	$\frac{1}{2}\kappa\alpha$	$\frac{1}{2}\kappa(\alpha+\beta)$	$\frac{1}{4}\kappa^2\alpha\left(\alpha + \beta - \frac{1}{\vert x\vert} + p(x) + q(x)\right)$
5	$-\frac{i}{2\pi}(\pi\kappa)^{\frac{5}{2}}$	$\frac{5}{4}\kappa\alpha$		$-\frac{i}{4\pi^2}(\pi\kappa)^{\frac{5}{2}}\left(\alpha - \frac{1}{\vert x\vert} + p(x)\right)$

The capacitance κ, the form factors α and β, and the corresponding functions p and q are defined in section 3. The second form factor β and the corresponding function q enter with solving the Poisson equation (54) for p_4:

$$\begin{aligned} -\Delta p_4 &= 2\pi\left(A_2 + A_2^{-1} w_2(x)\right) \\ &= 2\pi\left(1 \; + \; 1 - \frac{1}{\alpha}\left(\frac{1}{|x|} - p(x)\right)\right). \end{aligned} \tag{56}$$

If we compare (56) with problem (12), which defines q and β, we see that p_4 may just be written as sum of p and q. This yields the representations of λ_6 and w_4. We stress once more that there is no simple way to continue the expansions of λ and w, since constants appear that cannot be determined.

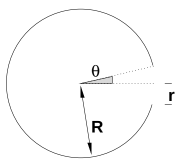
Finally we have to deduce the approximation (14) of the wave number k from the approximation of the eigenvalue $k^2 = \lambda = \lambda_2\varepsilon^2 + \cdots + \lambda_7\varepsilon^7$. This is done by expanding the square-root of λ with respect to ε and by returning to a general scale, i.e. ε becomes r/l and k becomes lk.

5 Comparison with earlier results

Helmholtz resonators have been discussed by many authors since the last century. Here we present only results that are close to ours. Of course, the first order component of (14) yields the classical formula by Helmholtz [13] and Rayleigh [11]:

$$f = \frac{c}{2\pi}\sqrt{\frac{\pi\kappa r}{V}} \tag{57}$$

The term $\pi\kappa r$ is called *conductivity* in [11]. Collins [2] considers spherical resonators with a small circular opening. Improving results by White [14] and Levine [9] he finds the following expression for the wavenumber:



$$\begin{aligned} k &= \frac{1}{R}\left(\frac{3\vartheta}{2\pi}\right)^{\frac{1}{2}}\left\{1+\frac{3}{10}\left(\frac{3\vartheta}{2\pi}\right)-\frac{i}{6}\left(\frac{3\vartheta}{2\pi}\right)^{\frac{3}{2}}\right. \\ &\quad \left. +\frac{9601}{113400}\left(\frac{3\vartheta}{2\pi}\right)^{2}-\frac{19i}{120}\left(\frac{3\vartheta}{2\pi}\right)^{\frac{5}{2}}+O\left(\vartheta^3\right)\right\}, \end{aligned} \tag{58}$$

where R denotes the radius of the sphere and ϑ the angle of the aperture. Note that for small angles we have $\vartheta \approx \frac{r}{R} \sim \frac{r}{l}$, i.e. the approximation has the same structure and order of accuracy as ours. However, as spherical resonators do not fit into the framework of this paper — the aperture is not surrounded by a piece of plane wall — the result cannot be reproduced by our method. In particular, problem (10) has no solution bounded in 0. On the other hand, the work by Collins is based on the fact that inner and outer solution may be represented explicitly in terms of special functions and within the same polar coordinate system. This prevents the method from being applicable to more general cavities.

Bigg and Tuck [1] consider resonators with both, apertures in narrow plane walls and axis symmetric apertures with a given profile in z–direction. In the former case their approximation of the Helmholtz wavenumber reads

$$\begin{gathered} k = \left(-\frac{v^2}{3}+\frac{1}{3}\sqrt{v^4+\eta v}\right)^{\frac{1}{2}} \\ v = \frac{2}{s}-\frac{1}{\Lambda}, \quad \eta = \frac{12\pi}{V}, \end{gathered} \tag{59}$$

where the parameters s and Λ are related to our parameters *capacitance* and *compactness* via $s = r\kappa$ and $\Lambda = l/\alpha$. Expanding k^2 in powers of r and comparing with (14) we see that (59) is a second order approximation. This means that the damping constant cannot be recovered from (59), whereas this is possible by our method without introducing any additional parameters. The idea underlying (59) is as follows. Inside and outside the cavity there are found approximate solutions of the Helmholtz equation by treating the aperture as a point source. About the opening the flow is assumed to be incompressible. This yields a relation between the pressure away from the opening and the flux through the opening, which may be used to match the three partial solutions. Finally, the Helmholtz wavenumber is found by differentiating the mean pressure inside the cavity with respect to k at a given flux. However, a systematic expansion of the partial solutions and the wavenumber into powers of a small parameter is missing, which limits the order of accuracy of the method.

The approach chosen in the present paper is close to that of Mazja [10] and Gadyl'schin [5], who applied it to elliptic eigenvalue problems in singularly perturbed regions and to finding higher eigenmodes of Helmholtz resonators, respectively. It is noteworthy that the formulae given in the latter paper cannot be applied to the fundamental frequency, as the wavenumber of the eigenmode of the unperturbed problem, which is zero, appears in the nominator. In contrast to [5] we expand the wavenumber into powers of $(r/l)^{\frac{1}{2}}$ rather than r/l. This is necessary to capture the imaginary parts of k. Missing this fact might have been the reason to exclude the Helmholtz frequency in [5].

6 Parameters of special geometries

6.1 Capacitance

The capacitance was defined in (7) and (8). Recall that the capacitance associated with an aperture Γ_r is in fact the capacitance of the scaled aperture Γ which has the same shape but area π. Hence, the capacitances of ellipses and rectangles are merely functions of the aspect ratio ρ. The data is plotted in figure 3. The capacitance of ellipses may be expressed in a closed form

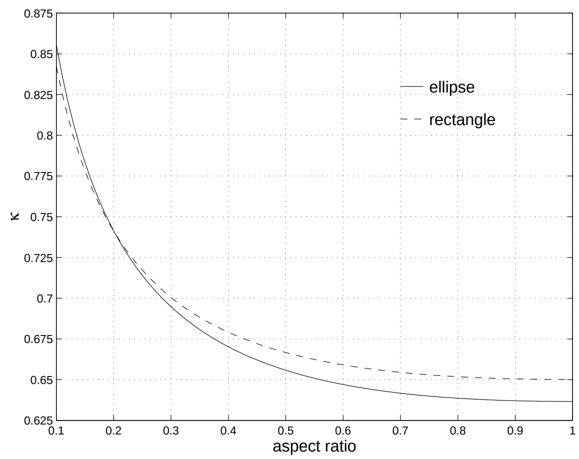


Figure 3: capacitances of ellipses and rectangles

via complete elliptic integrals [8, 15]:

$$\kappa = \left(\sqrt{\rho} \int_0^{\frac{\pi}{2}} \left(1 - \left(1 - \rho^2\right) \sin^2\theta \right)^{-\frac{1}{2}} d\theta \right)^{-1} , \tag{60}$$

A manageable approximation is found by expanding (60) in powers of $\ln^2 \rho$. This expansion is suggested by the fact that the capacitance κ is the same for ρ and ρ^{-1}.

$$\kappa = \frac{2}{\pi} \left(1 + \frac{1}{16} \ln^2 \rho + \frac{1}{3072} \ln^4 \rho + \frac{23}{737280} \ln^6 \rho - \frac{2519}{1321205760} \ln^8 \rho \right) + O\left(\ln^{10} \rho\right) \tag{61}$$

For ρ tending to 0 we find [4]:

$$\kappa \approx - \left(\sqrt{\rho} \ln \frac{\rho}{4} \right)^{-1} \tag{62}$$

If we apply (61) for $\rho > 0.073$ and (62) for $\rho < 0.073$ the relative error is always smaller than $1/1000$.

Rectangles are treated numerically. More precisely, equation (7) is approximated by the following system: find $\varphi(\xi) = \sum_{kl \in I} \varphi_{kl} \chi_{kl}(\xi)$ such that

$$\int_\Gamma \int_\Gamma \frac{\varphi(\eta)\, \chi_{ij}(\xi)}{4\pi\, |\xi - \eta|} dS(\eta)\, dS(\xi) = \int_\Gamma \chi_{ij}(\xi)\, dS(\xi) \quad \text{for all}\, ij \in I, \tag{63}$$

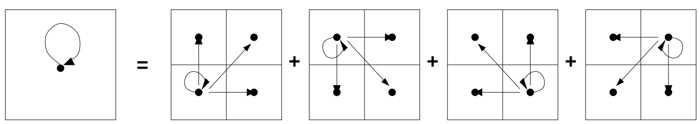
where χ_{ij} is the characteristic function of square ij and the double index ij runs through all squares of a uniform partitioning of the rectangle Γ. The matrix of the corresponding linear system has entries of the following kind:

$$a_{ij\,kl} = \int_\Gamma \int_\Gamma \frac{\chi_{ij}(\eta)\, \chi_{kl}(\xi)}{4\pi\, |\xi - \eta|} dS(\eta)\, dS(\xi)\,. \tag{64}$$

For distant squares $a_{ij\,kl}$ is approximated by expanding (64) in negative powers of the distance of the midpoints of the squares. If the squares are close to each other similarity arguments are used to express $a_{ij\,kl}$ in terms of matrix entries corresponding to more distant squares, e.g.:

$$a_{00\,00} = \frac{1}{2} a_{00\,00} + a_{00\,10} + \frac{1}{2} a_{00\,11}. \tag{65}$$

The easiest way to understand the principle is to have a look at the following figure which illustrates again the case $a_{00\,00}$. Reduced capacitances are computed for ten different aspect



ratios. Each capacitance is obtained by successively refining the grid and extrapolating the corresponding values at grid size zero. Errors are estimated by comparing extrapolations of different order. The grids contain up to 2500 squares. Interpolating the ten capacitances by powers of $\ln^2 \rho$ yields:

$$\begin{aligned} \kappa &= 6.5012 \times 10^{-1} + 3.4279 \times 10^{-2} \ln^2 \rho + 3.2154 \times 10^{-4} \ln^4 \rho \\ &+ 9.7460 \times 10^{-6} \ln^6 \rho + O\left(\ln^8 \rho\right) \end{aligned} \tag{66}$$

Both, the relative error in computing the ten capacitances and the relative error due to interpolation are of order 3×10^{-4} for $0.1 \leq \rho \leq 1$. Capacities of other apertures such as diamonds, rounded rectangles, or crosses are plotted in [4][3]. However, handy expansions in $\ln^2 \rho$ are not included.

The results of this section may be summarised as follows. The capacitance is minimal for a circle, does not change very much with the shape of the opening, if the aspect ratio is moderate, and it starts growing for large aspect ratios.

6.2 Form factors of the cavity

We consider cavities formed like hemispheres and cuboids. In [1][4] there is explained how to compute first form factors of cylinders. Hemispheres with central apertures may be treated analytically. Rewriting the Laplace operator in polar coordinates problems (10) and (12) prove to have easy solutions and we find:

[3]In [4] the quantity $L_0 = \pi\kappa$ is used in stead of κ.
[4]Mind the different notation: $\alpha = l/\Lambda$.

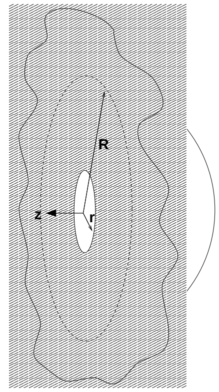


Figure 4: hemisphere with central aperture

Hemisphere with central aperture				
l	p	α	q	β
$R\left(\frac{2}{3}\pi\right)^{\frac{1}{3}}$	$-\frac{r^2}{2R^3}$	$\frac{9}{5}\left(\frac{2}{3}\pi\right)^{\frac{1}{3}}$	$\frac{r\left(20R^3-12R^2r+r^3\right)}{24R^5}$	$-\frac{12}{35}\left(\frac{2}{3}\pi\right)^{\frac{1}{3}}$

(66)

R is the radius of the hemisphere. The quality of the approximation is demonstrated in the next section where it is compared with numerical methods.

Next we have a look at cuboids. We restrict ourselves to computing the first form factor, the compactness α. This is enough to have a basic term and a correction term of two subsequent powers of r/l for both, the frequency and the damping constant, cf. (14). Two effects are investigated. First we change the aspect ratio of the cuboid, while the opening is fixed in the centre of the front wall. Then we consider a cube with the aperture moving along one middle axis of the front wall.

In both cases problem (10) is computed numerically for a number of constellations using the standard finite difference method on a regular grid. This is sufficient since problem (10) does not produce singularities. The integral (11) is computed using an adaptive algorithm that takes care of the singular term r^{-1}. As in case of the capacitance the grid is successively refined and the corresponding values of α extrapolated into $h = 0$. Grids of up to 27000 nodes are used. The error is estimated by comparing extrapolations of different order. Finally, the computed values are interpolated in such a way that the interpolation error is of the same order of magnitude. In case of a central opening α depends only on the distances X,Y, and Z of left, upper and back wall to the opening. Interchanging these distances has no influence. For instance, if we have a lengthy box it makes no difference placing the aperture in the centre of a short or a long wall. This may be seen by reflecting the solution of problem (10) at the front wall. This yields a solution of a problem which is symmetric with respect to the centre of the opening. Moreover, α stays the same if the cuboid is blown up without changing the aspect ratio. Hence, α may be parametrised by two arbitrary quotients of X, Y, and Z. To restrict the parameters to $[0, 1]$ we may choose the longest distance as nominator. In figure 5 this is assumed to be Z. The preceding considerations on how α depends on X, Y, and Z suggest the following interpolation of the numerical data:

$$
\begin{aligned}
\alpha \quad \approx \quad & 2.8899 - 1.0949\cdot 10^{-1}\left(s_1 + \cdots + s_6\right) + 5.5391\cdot 10^{-3}\left(s_1 s_6 + s_2 s_3 + s_4 s_5\right) \\
& + 1.2226\cdot 10^{-3}\left(s_1^2 + \cdots + s_6^2\right) - 7.9627\cdot 10^{-4}\left(s_1 s_2 + s_3 s_4 + s_5 s_6\right) \\
& - 1.4192\cdot 10^{-4}\left(s_1 s_6\left(s_1 + s_6\right) + s_2 s_3\left(s_2 + s_3\right) + s_4 s_5\left(s_4 + s_5\right)\right)
\end{aligned}
$$

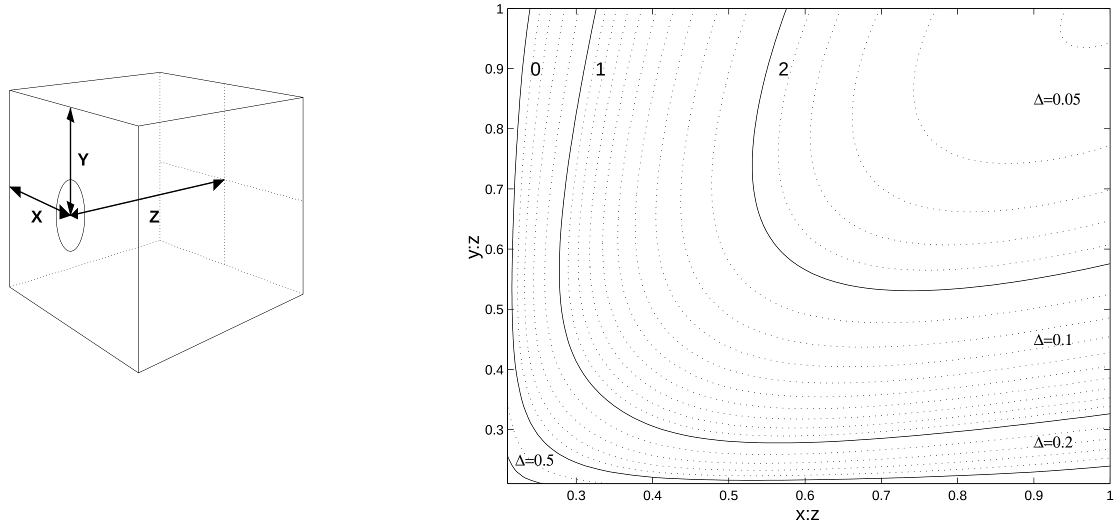


Figure 5: compactness of a cuboid with central aperture

$$
\begin{aligned}
&-6.6346\cdot 10^{-6}\left(s_1^3+\cdots+s_6^3\right)\\
&+1.5443\cdot 10^{-6}\left(s_1 s_6\left(s_1^2+s_6^2\right)+s_2 s_3\left(s_2^2+s_3^2\right)+s_4 s_5\left(s_4^2+s_5^2\right)\right)
\end{aligned}
\tag{68}
$$

where $s_1=\frac{X}{Y}$, $s_2=\frac{X}{Z}$, $s_3=\frac{Y}{Z}$, $s_4=\frac{Y}{X}$, $s_5=\frac{Z}{X}$, and $s_6=\frac{Z}{Y}$. The estimated absolute error is smaller than $2\cdot 10^{-3}$ for aspect ratios in $[0.25,1]^2$.

Note that α is maximal for a half–cube (X=Y=Z). The corresponding value is $\alpha\approx 2.2520$, which is only slightly smaller than that of the hemisphere ($\alpha\approx 2.3030$). For moderate aspect ratios α is smaller than this value and positive, cf. figure 5. There are cuboids with $\alpha=0$, e.g. a box with width = height = 1m, depth = 1.87m, i.e. $X:Y:Z=0.268:0.268:1$. Note that for such cuboids the classical Helmholtz formula describes already very well the actual resonant frequency. For very small aspect ratios α tends to $-\infty$. This is where our approximation fails and the theory of long ducts has to be applied (cf. any textbook on acoustics, e.g. [7]). Finally,

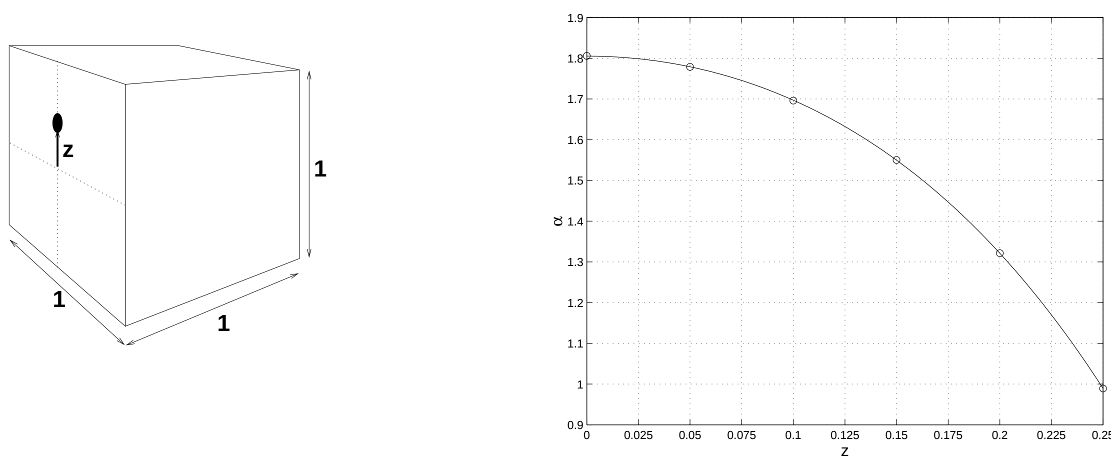


Figure 6: compactness form factor of cube with eccentric aperture

we investigate the influence of moving the opening from the centre of the front wall. For that purpose we restrict ourselves to cubes, i.e X=Y=Z/2. The interpolated numerical data is shown in figure 6 and the underlying approximation reads:

$$\alpha \approx 1.8053 - 10.402z^2 - 42.470z^4, \tag{69}$$

where z is the excursion of the opening along the middle axis relative to the width of the cube. The estimated relative accuracy of (69) is better than 0.003 for $z < 0.25$.

6.3 Other cavities

Although a complete proof is missing there are good reasons to believe that the compactness α is maximal for a hemisphere among all possible shapes satisfying the conditions listed in section 2. The value is $\alpha = \frac{9}{5}\left(\frac{2}{3}\pi\right)^{\frac{1}{3}} \approx 2.3030$. If our observation holds true plugging this value into (14) yields an upper bound of the Helmholtz frequency and the corresponding damping constant, provided the area and the capacitance of the aperture and the volume of the cavity are known. This result will be made more rigorous in a forthcoming paper.

7 Comparison with numerical results

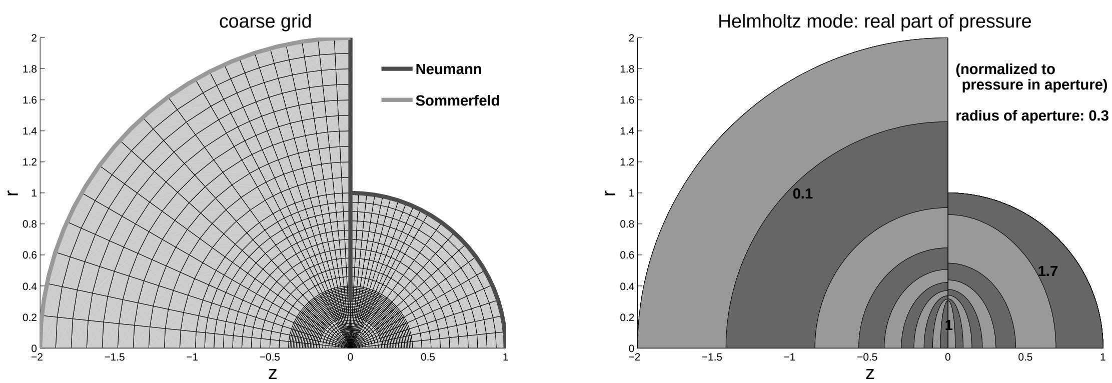


Figure 7: Helmholtz mode of hemisphere

The Helmholtz mode of a hemisphere is computed by switching to polar coordinates and applying the finite element method with elements bilinear in r and ϑ to equation (1). At the artificial boundary the pressure is assumed to behave like an outgoing spherical wave, i.e. $\frac{\partial u}{\partial r} = \left(ik - \frac{1}{r}\right)u$. Homogeneous Neumann conditions are prescribed on the walls. The final numerical value of $\lambda = k^2$, the lowest eigenvalue of the Helmholtz equation, is computed once more by extrapolation. For each radius of the aperture the corresponding eigenvalue problem is solved on three grids, the one shown in figure 7 and grids with four times and nine times as many elements. The finest grids consist of 16200 elements. In figure 8 there is presented a comparison of analytically and numerically computed values of the wave number k corresponding to the lowest eigenvalue $\lambda = k^2$ of the Helmholtz equation. The dotted line corresponds to the classical Helmholtz formula (57). The solid and dashed lines refer to the real and imaginary part, respectively, of the full analytical approximation (14). For large apertures the extrapolated numerical values should provide the best approximation and the relative error of the analytical values may be interpreted as errors with respect to the true solution. For very small apertures the analytical values are more precise than the numerical ones. This may be seen from the fact

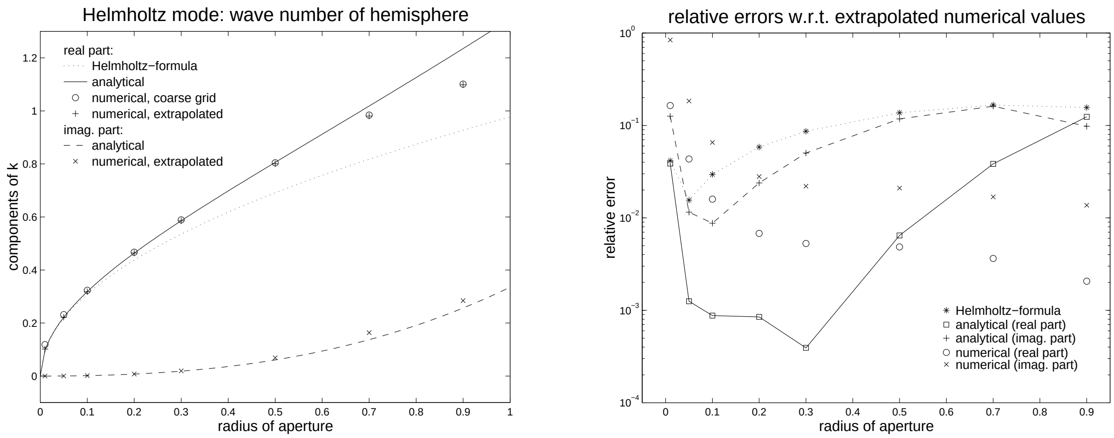


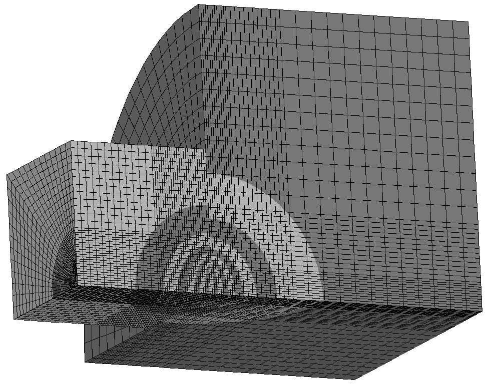
Figure 8: comparison of analytical and numerical results for hemisphere

that the errors relative to the extrapolated values increase again with the radius becoming very small. The actual errors of the analytical values will tend to zero for small apertures. Note that we may expect errors of less than 1% for the real part, if $r/R < 0.5$, and for the imaginary part, if $r/R < 0.1$. The error of the classical Helmholtz formula, which provides only the real part, is considerably higher: about 13% for $r/R = 0.5$.

Finally, we present some results for a cube and a half–cube. The width of the two cavities is 1 and the radius of the circular opening is 0.1. The numerical eigenvalue problems are solved with the commercial package *ANSYS*. On the artificial boundary the Sommerfeld condition is replaced by a homogeneous Dirichlet condition as a k–dependent mixed boundary condition is not implemented. This results in a systematic error and the full analytical solution will be closer to the actual one in figure 9. In particular, the numerical eigenvalues remain real. Therefore, we compare only the real part of k. The finest grids contain 60420 nodes and the computation takes more than one hour on a HP workstation. The coarser grids consist of about 18000 and 35000 nodes, respectively. This means that a considerable numerical effort is needed to achieve results that are comparable to the analytical one. In fact, also the analytical formula is based on numerically computed form factors. However, the corresponding problem is much simpler and independent of the aperture.

8 Conclusion

We have derived a formula for the Helmholtz frequency and the corresponding damping constant of a cavity resonator with narrow front wall (14). This approximation is of third order in the ratio of the aperture radius and the cavity diameter. It remains valid even for relatively large apertures. Comparing with numerical results shows that in case of a hemisphere the error is smaller than 1 % as long as the quotient of the radii of aperture and cavity is smaller than 0.5. The coefficients of the approximation depend on the *capacitance* and the position of the aperture and on form factors of the cavity, e.g. the *compactness*, but not on the size. Hence, it is possible to describe a large variety of combinations of apertures and cavities by a few parameters. Accurate expansions of these parameters are given for elliptic and rectangular apertures and hemispherical and rectangular cavities, c.f. (61) and (66)–(69). For general geometries there are given the integral and Poisson equations that have to be solved to find the form factors. On the validity of the classical Helmholtz formula (15) there is found the following qualitative result. Resonant



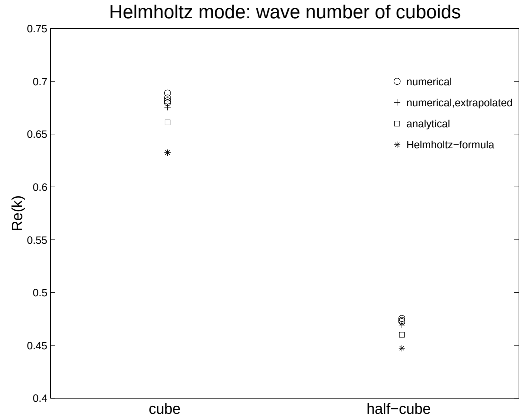


Figure 9: Helmholtz mode of half–cube ($\frac{1}{4}$ of the cavity and a cylindrical fraction of the exterior are shown) and comparison of analytical and numerical results for cuboids

frequency and damping constant are underestimated for compact cavities with central aperture, e.g. hemispheres or half–cubes. The underestimation is less, if the opening is not in the centre of the front wall. Overestimation occurs for cavities with large aspect ratio, e.g. pipes. The degree of over– or underestimation is minimal for circular openings and increases for apertures with large aspect ratio, if the area of the opening is fixed (16). The investigated geometries suggest that the hemisphere provides an upper bound of resonant frequency and damping constant among all cavities with a given volume. However, this assertion has to be proven, yet.

References

[1] G. BIGG, *The three dimensional cavity resonator*, Journal of Sound and Vibration, 85 (1982), pp. 85 – 103.

[2] W. D. COLLINS, *Some scalar diffraction problems for a spherical cap*, Archive for Rational Mechanics and Analysis, 10 (1962), pp. 249 – 266.

[3] A. CUMMINGS, *Acoustics of a wine bottle*, Journal of Sound and Vibration, 31 (1973), pp. 331 – 343.

[4] R. DE SMEDT, *Low frequency scattering through an aperture in a rigid screen – some numerical results*, Journal of Sound and Vibration, 75 (1981), pp. 371 – 386.

[5] R. R. GADYL'SHIN, *The method of matched asymptotic expansions in the problem of the Helmholtz acoustic resonator*, Journal of Applied Mathematics and Mechanics, 56 (1992), pp. 340 – 345.

[6] W. HACKBUSCH, *Integralgleichungen*, B. G. Teubner Stuttgart, 1989.

[7] L. E. KINSLER AND A. R. FREY, *Fundamentals of Acoustics*, Wiley, New York, 1962.

[8] L. D. LANDAU AND E. M. LIFSCHITZ, *Elektrodynamik der Kontinua*, Akademie–Verlag Berlin, 4 ed., 1985.

[9] H. LEVINE, *The wavelength of a spherical resonator with circular aperture*, Journal of the Acoustical Society of America, 23 (1951), pp. 307 – 311.

[10] W. MAZJA, S. NASAROW, AND B. PLAMENEWSKI, *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten I*, Akademie–Verlag Berlin, 1991.

[11] L. RAYLEIGH, *Theory of Sound II*, Macmillan, London, second ed., 1896.

[12] L. RONKEL, *Eigenmodes of cubic organ pipes*, Master's thesis, University of Kaiserslautern, Germany, 1997.

[13] H. V. HELMHOLTZ, *Lehre von den Tonempfindungen*, Vieweg Verlag, Braunschweig, 1877.

[14] F. P. WHITE, Proceedings of the Royal Society, A 92 (1916), pp. 265 – 275.

[15] W. E. WILLIAMS, *Diffraction by a disc*, Proceedings of the Royal Society, A 267 (1962), pp. 77 – 87.