TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Tensor Products of Hilbert Spaces

Master's Thesis

TU Kaiserslautern
Fachbereich Mathematik

Robin Rüßmann

Supervisor:
Prof. Dr. Klaus Ritter

Kaiserslautern, January 2020

Contents

Introduction

Given a nonempty set N and a family of Hilbert spaces $(H_j)_{j\in N}$ over the same scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we study the *Hilbert space tensor product* $\bigotimes_{j\in N} H_j$. Both the case of N being finite and infinite have been covered by John von Neumann, see [10] and [11]. We convey the main results of both papers in a more detailed way and put them in context of each other. In the infinite case, we restrict ourselves to the case $N = \mathbb{N}$.

We also consider the special case when all spaces H_j are given as reproducing kernel Hilbert spaces $H(K_j)$. For given reproducing kernels K_j, if we restrict the domain appropriately, the tensor kernel $\bigotimes_{j\in N} K_j$ is given as the pointwise product of the kernels K_j. We are interested in results of the form

$$H\left(\bigotimes_{j\in N} K_j\right) \text{ is canonically isomorphic to } \bigotimes_{j\in N} H(K_j).$$

In the case of N being finite, this is a well-established result. We give similar results for the case of infinite N. Tensor products of reproducing kernel Hilbert spaces are interesting tools in dealing with certain approximation problems, see for example [12].

The Finite Case

To understand the problem at hand, consider at first the case that N is finite, which we study in Section 1. We wish to define a Hilbert space $\bigotimes_{j\in N} H_j$ that fulfills the three properties

1. $\bigotimes_{j\in N} H_j$ contains an element $\otimes_{j\in N} f_j$ for each $f = (f_j)_{j\in N} \in \bigtimes_{j\in N} H_j$,
2. $\langle \otimes_{j\in N} f_j, \otimes_{j\in N} g_j \rangle_{\bigotimes_{j\in N} H_j} = \prod_{j\in N} \langle f_j, g_j\rangle_{H_j}$ holds for any two such elements $\otimes_{j\in N} f_j, \otimes_{j\in N} g_j$ and
3. $H_0 = \operatorname{span}\left\{ \otimes_{j\in N} f_j \;\middle|\; (f_j)_{j\in N} \in \bigtimes_{j\in N} H_j \right\}$ is dense in $\bigotimes_{j\in N} H_j$.

The elements whose existence is required by property 1 are called *elementary tensors*. Property 2, along with the sesquilinear nature of the scalar product, assures that these tensors "behave in a multiplicative way", see Remark 1.1. Property 3 ensures the minimality of $\bigotimes_{j\in N} H_j$. In Subsection 1.1, we show that such a space always exists and give a general construction: Given any $f \in \times_{j\in N} H_j$, we define the elementary tensor

$$\otimes_{j\in N} f_j : \bigtimes_{j\in N} H_j \to \mathbb{K} : (g_j)_{j\in N} \mapsto \prod_{j\in N} \langle f_j, g_j\rangle_{H_j}.$$

By taking the linear span of these mappings in $\mathbb{K}^{(\times_{j\in N} H_j)}$, we obtain a vector space, and, thanks to property 2, there is only one way to define a scalar product on this space, which is indeed well-defined. Thus, we obtain a pre-Hilbert space, which can of course be completed in the usual way, but in this case also by using the pointwise limit of Cauchy sequences. This is useful, since it allows us to define the tensor product as a subspace of $\mathbb{K}^{(\times_{j\in N} H_j)}$.

Subsection 1.2 covers some interesting properties of the tensor product. Most importantly, the space $\bigotimes_{j\in N} H_j$ is uniquely determined up to a unique canonical isometric isomorphism, which allows us to view any space fulfilling the properties 1, 2 and 3 as the Hilbert space tensor product. For example, if each Hilbert space H_j is given as the L_2-space with respect to some σ-finite measure μ_j on some set Ω_j, the tensor product is canonically isomorphic to the L_2-space with respect to the product measure μ on the space $\times_{j\in\mathbb{N}}\Omega_j$.

We also briefly touch upon another way to define the Hilbert space tensor product which was given by Kadison in [6]. Further, we remark the connection to the tensor product of generic vector spaces, see Subsection 1.3.

Challenges in the Infinite Case

In the case $N = \mathbb{N}$, which we study in Section 2, we immediately run into problems if we want to proceed exactly as in the finite case. Most obviously, property 2 implies that

$$\|\otimes_{j\in\mathbb{N}} f_j\|_{\bigotimes_{j\in\mathbb{N}} H_j} = \prod_{j\in\mathbb{N}} \|f_j\|_{H_j}$$

must always hold, but this product might not even be convergent, so property 1 cannot be fulfilled for every sequence f. Worse still, even if two sequences f, g are given such that both $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j}$ and $\prod_{j\in\mathbb{N}} \|g_j\|_{H_j}$ do converge, this does not imply the convergence of $\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}$. This prevents us from defining elementary tensors as in the finite case and further puts properties 1 and 2 at odds with each other. We study two ways to handle this.

Firstly, we try to restrict property 1 as little as possible. To this end, we require the existence of elementary tensors $\otimes_{j\in\mathbb{N}} f_j$ only if f belongs to the set $\mathcal{C}$ of all sequences for which $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j}$ converges. In this case, we also need the notion of quasi-convergence: Put slightly simplified, whenever a product of the form $\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}$ does not converge but property 2 requires it to have a value, we set its value to 0. We then define

$$\otimes_{j\in\mathbb{N}} f_j : \mathcal{C} \to \mathbb{K} : (g_j)_{j\in\mathbb{N}} \mapsto \prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}$$

and then proceed as in the finite case. This way, we obtain the *complete tensor product*, denoted by $\bigotimes_{j\in\mathbb{N}} H_j$. This is not straight-forward however: We first need to establish several results on the set $\mathcal{C}$; in particular, we define an important equivalence relation on a subset of $\mathcal{C}$. This equivalence relation is of considerable importance and is studied in Subsection 2.2. This then allows us to fall back to the finite case when constructing the complete tensor product in Subsection 2.3.

The second approach to defining the tensor product is to fix a sequence e that fulfills $\|e_j\|_{H_j} = 1$ for all $j \in \mathbb{N}$. We then only require the existence of elementary tensors $\otimes_{j\in\mathbb{N}} f_j$ only if $f_j \neq e_j$ holds only finitely often. Then, the product given by property 2 always converges, allowing us to define elementary tensors in analogy to the finite case and then to proceed accordingly. This gives rise to the *incomplete tensor product* with respect to e, which is denoted by $\bigotimes_{j\in\mathbb{N}}^{e} H_j$.

In Subsection 2.4, we will define the incomplete tensor product as a certain subspace of the complete tensor product. We establish that it can be obtained as described above, even without having the complete tensor product as a bigger space; we also study its connection to the equivalence from Subsection 2.2.

Both the complete and the incomplete tensor product with respect to e are again uniquely determined up to a unique isometric isomorphism. This allows us to view the finite tensor product as a special incomplete tensor product. We also give an example where all spaces H_j are given by L_2-spaces with respect to some probability measure μ_j; the result is similar to the finite case.

The Special Case of Reproducing Kernel Hilbert Spaces

A case that is of special interest to us is the case where each Hilbert space H_j is given as a *reproducing kernel Hilbert space* (RKHS). This means the elements are $\mathbb{K}$-valued mappings with some domain D_j such that each evaluation functional

$$\delta_t : H_j \to \mathbb{K} : h \mapsto h(t)$$

is bounded. In this case, there exists a reproducing kernel, that is to say a nonnegative definite mapping $K_j : D_j^2 \to \mathbb{K}$, fulfilling both

$$K_j(\cdot, t) \in H_j$$

and

$$h(t) = \langle h, K_j(\cdot, t) \rangle_{H_j}$$

for every choice of $t \in D_j$ and $h \in H_j$. The second property is called the *reproducing property*. It implies that $\operatorname{span}\{\, K_j(\cdot, t) \mid t \in D_j \,\}$ is dense in H_j. A kernel uniquely determines a RKHS. We also write $H_j = H(K_j)$. We cover some basic results in Section 3.

In Section 4, we study in which cases the tensor product of RKHS can be viewed as a RKHS itself. In the finite case, there is a result that the mapping

$$\bigotimes_{j\in N} K_j : (\times_{j\in N} D_j)^2 \to \mathbb{K} : (x, y) \mapsto \prod_{j\in N} K_j(x_j, y_j)$$

is also a nonnegative definite kernel and that the corresponding RKHS is canonically isomorphic to the finite tensor product. We give a similar result in the countably infinite case: If a sequence e is given such that $\|e_j\|_{H_j} = 1$ holds for all $j \in \mathbb{N}$, we can define the set

$$X_e = \left\{ x \in \bigtimes_{j\in\mathbb{N}} D_j \;\middle|\; \prod_{j\in\mathbb{N}} e_j(x_j) \text{ converges in the stricter sense} \right\}.$$

If X_e is nonempty, the mapping

$$\bigotimes_{j\in\mathbb{N}} K_j : X_e^2 \to \mathbb{K} : (x,y) \mapsto \prod_{j\in\mathbb{N}} K_j(x_j, y_j)$$

is well-defined, a reproducing kernel, and there is a canonical isometric isomorphism

$$\Lambda : \bigotimes_{j\in\mathbb{N}}{}^{e} H(K_j) \to H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)$$

that fulfills

$$\Lambda\left(\otimes_{j\in\mathbb{N}} h_j\right) = \prod_{j\in\mathbb{N}} h_j$$

for each elementary tensor $\otimes_{j\in\mathbb{N}} h_j$ in $\bigotimes^{e}_{j\in\mathbb{N}} H(K_j)$.

1 The Finite Tensor Product

For $m \in \mathbb{N}$ and $N = \{1, \ldots, m\}$, let a family of Hilbert spaces $(H_j)_{j \in N}$ be given. We assume all Hilbert spaces are given over the same $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Our goal is to define the *tensor product* $\bigotimes_{j \in N} H_j$. This space should be a Hilbert space again and be the product of the spaces H_j in the following sense.

For each $(f_j)_{j \in N} \in \bigtimes_{j \in N} H_j$, the space $\bigotimes_{j \in N} H_j$ should contain a special element $\otimes_{j \in N} f_j$, which we call an *elementary tensor*. Further, two tensors $\otimes_{j \in N} f_j$ and $\otimes_{j \in N} g_j$ should always fulfill

$$\langle \otimes_{j \in N} f_j, \otimes_{j \in N} g_j \rangle_{\bigotimes_{j \in N} H_j} = \prod_{j \in N} \langle f_j, g_j \rangle_{H_j} . \tag{1}$$

This implies $\| \otimes_{j \in N} f_j \|_{\bigotimes_{j \in N} H_j} = \prod_{j \in N} \| f_j \|_{H_j}$, which is called the cross-norm property.

In Subsection 1.1, we will construct such a space. For this, we use the construction method found in [10]. Later on, in Theorem 1.14, we will see that any space that contains elementary tensors that fulfill Equation (1), the closure of the linear span of the elementary tensors is isomorphic in a canonical way to the space we construct.

Remark 1.1. A motivation why we require the cross-norm property is that it, along with the sesquilinearity of each scalar product, endows tensors with some linear properties you would expect from a product. For example, in the case $m = 2$, for $f_1, g_1 \in H_1$ and $f_2 \in H_2$ we have

$$(f_1 + g_1) \otimes f_2 = (f_1 \otimes f_2) + (g_1 \otimes f_2)$$

and for $a \in \mathbb{K}$, we have

$$a(f_1 \otimes f_2) = (a f_1) \otimes f_2 = f_1 \otimes (a f_2).$$

1.1 Construction

First, we must define what we mean by $\otimes_{j \in N} f_j$.

Definition 1.2. Given an element $f_j \in H_j$ for each $j \in N$, we define the mapping

$$\otimes_{j \in N} f_j : \bigtimes_{j \in N} H_j \to \mathbb{K} : (g_j)_{j \in N} \mapsto \prod_{j \in N} \langle f_j, g_j \rangle_{H_j} .$$

We also call these mappings *elementary tensors*.

We view elementary tensors as elements of $\mathbb{K}^{(\times_{j \in N} H_j)}$, allowing the usual vector space operations. Whenever it is convenient, we also write

$$f_1 \otimes \cdots \otimes f_m := \otimes_{j \in N} f_j .$$

Our next goal is to define a scalar product on the smallest space containing all these mappings, namely their linear span.

Definition 1.3. Define the space

$$H_1 \otimes' \cdots \otimes' H_m := \otimes'_{j\in N} H_j := \operatorname{span}\left\{ \otimes_{j\in N} f_j \;\middle|\; (f_j)_{j\in N} \in \bigtimes_{j\in N} H_j \right\}.$$

Remark 1.4. For any $a \in \mathbb{K}$ and any elementary tensor $\otimes_{j\in N} f_j$, the mapping $a\cdot\otimes_{j\in N} f_j$ is an elementary tensor too, since $a\cdot(f_1 \otimes \cdots \otimes f_m) = (af_1)\otimes\cdots\otimes f_m$ holds. Thus, any element Φ of $\otimes'_{j\in N} H_j$ is of the form $\Phi = \sum_{k=1}^{m_1} \otimes_{j\in N} f_{j,k}$ for elementary tensors $\otimes_{j\in N} f_{j,k}$.

We wish to equip $\otimes'_{j\in N} H_j$ with a scalar product, and we do in Lemma 1.7. Before that, we give two lemmas which will be of use here, and also when defining other scalar products in Sections 2 and 3.

Lemma 1.5. The pointwise product of two nonnegative definite matrices is again a nonnegative definite matrix.

Proof. This was first proven by Issai Schur, see page 14 in [9]. The statement and proof in English language can be found as Theorem 7.5.3 in [4]. □

Lemma 1.6. Let H be a $\mathbb{K}$-vector space and $\langle\cdot,\cdot\rangle_H : H\times H \to \mathbb{K}$ a sesquilinear, hermitian, nonnegative definite mapping. Then, for any $g, h \in H$ we have

$$|\langle g,h\rangle_H| \le \langle g,g\rangle_H \langle h,h\rangle_H .$$

Proof. This is a slight modification of the standard proof of the Cauchy-Schwarz inequality, see for example Theorem V.1.2 in [13]. We have $\langle h,h\rangle_H \ge 0$, so for any $\varepsilon > 0$, we can define $\lambda_\varepsilon = \frac{\langle g,h\rangle_H}{\langle h,h\rangle_H+\varepsilon}$. We obtain

$$\begin{aligned} 0 \le\ & \langle g-\lambda_\varepsilon h, g-\lambda_\varepsilon h\rangle_H \\ =\ & \langle g,g\rangle_H - \lambda_\varepsilon \langle h,g\rangle_H - \overline{\lambda_\varepsilon} \langle g,h\rangle_H + |\lambda_\varepsilon|^2 \langle h,h\rangle_H \\ =\ & \langle g,g\rangle_H - 2\frac{|\langle g,h\rangle_H|^2}{\langle h,h\rangle_H+\varepsilon} + \frac{|\langle g,h\rangle_H|^2 \langle h,h\rangle_H}{(\langle h,h\rangle_H+\varepsilon)^2}. \end{aligned}$$

By multiplying with $\langle h,h\rangle_H + \varepsilon$ and rearranging we get

$$2\,|\langle g,h\rangle_H|^2 - |\langle g,h\rangle_H|^2 \frac{\langle h,h\rangle_H}{\langle h,h\rangle_H+\varepsilon} \le \langle g,g\rangle_H (\langle h,h\rangle_H + \varepsilon).$$

Now, if $\langle h,h\rangle_H > 0$, we can let ε tend to 0 and obtain the desired result.

If $\langle h,h\rangle_H = 0$, we obtain

$$2\,|\langle g,h\rangle_H|^2 \le \langle g,g\rangle_H (\varepsilon),$$

and if we let ε tend to 0, we obtain $\langle g,h\rangle_H = 0$. □

Lemma 1.7. For $\Phi, \Psi \in \otimes'_{j\in N} H_j$ with representations $\Phi = \sum_{k=1}^{m_1} \otimes_{j\in N} f_{j,k}$ and $\Psi = \sum_{\ell=1}^{m_2} \otimes_{j\in N} g_{j,\ell}$, the mapping given by

$$\langle \Phi, \Psi \rangle_{\otimes'_{j\in N} H_j} = \sum_{k=1}^{m_1} \sum_{\ell=1}^{m_2} \prod_{j\in N} \langle f_{j,k}, g_{j,\ell} \rangle_{H_j} .$$

is well-defined and a scalar product on $\otimes'_{j\in N} H_j$.

Proof. By the definition of $\otimes_{j\in N} f_{j,k}$, we have

$$\begin{aligned} &\sum_{k=1}^{m_1} \sum_{\ell=1}^{m_2} \prod_{j\in N} \langle f_{j,k}, g_{j,\ell} \rangle_{H_j} \\ &= \sum_{k=1}^{m_1} \sum_{\ell=1}^{m_2} \otimes_{j'\in N} f_{j',k} \left((g_{j,\ell})_{j\in N} \right) \\ &= \sum_{\ell=1}^{m_2} \Phi \left((g_{j,\ell})_{j\in N} \right) . \end{aligned}$$

This shows that $\langle \Phi, \Psi \rangle_{\otimes'_{j\in \mathbb{N}} H_j}$ does not depend on the representation of Φ given by the $f_{j,k}$. Because each scalar product $\langle \cdot, \cdot \rangle_{H_j}$ is hermitian, we also obtain that $\langle \Phi, \Psi \rangle_{\otimes'_{j\in \mathbb{N}} H_j}$ does not depend on the representation of Ψ given by the $g_{j,\ell}$. Thus, $\langle \cdot, \cdot \rangle_{\otimes'_{j\in N} H_j}$ is well-defined.

Take a third element $\Xi \in \otimes'_{j\in N} H_j$ with representation $\Xi = \sum_{i=1}^{m_3} \otimes_{j\in N} h_{j,i}$ as well as $a, b \in \mathbb{K}$. We have

$$a\Phi + b\Psi = \sum_{k=1}^{m_1} (af_{1,k}) \otimes \cdots \otimes f_{n_k} + \sum_{\ell=1}^{m_2} (bg_{1,\ell}) \otimes \cdots \otimes g_{n_\ell}$$

and thus

$$\begin{aligned} &\langle a\Phi + b\Psi, \Xi \rangle_{\otimes'_{j\in N} H_j} \\ &= \sum_{k=1}^{m_1} \sum_{i=1}^{m_3} \langle af_{1,k}, h_{1,i} \rangle_{H_1} \prod_{j=2}^{n} \langle f_{j,k}, h_{j,i} \rangle_{H_j} + \sum_{\ell=1}^{m_2} \sum_{i=1}^{m_3} \langle bg_{1,\ell}, h_{1,i} \rangle_{H_1} \prod_{j=2}^{n} \langle g_{j,\ell}, h_{j,i} \rangle_{H_j} \\ &= a \left(\sum_{k=1}^{m_1} \sum_{i=1}^{m_3} \prod_{j\in N} \langle f_{j,k}, h_{j,i} \rangle_{H_j} \right) + b \left(\sum_{\ell=1}^{m_2} \sum_{i=1}^{m_3} \prod_{j\in N} \langle g_{j,\ell}, h_{j,i} \rangle_{H_j} \right) \\ &= a \langle \Phi, \Xi \rangle_{\otimes'_{j\in N} H_j} + b \langle \Psi, \Xi \rangle_{\otimes'_{j\in N} H_j} , \end{aligned}$$

so $\langle\cdot,\cdot\rangle_{\otimes'_{j\in N}H_j}$ is linear in its first component. It is also hermitian, since

$$\begin{aligned}\langle\Phi,\Psi\rangle_{\otimes'_{j\in N}H_j} &= \sum_{k=1}^{m_1}\sum_{\ell=1}^{m_2}\prod_{j\in N}\langle f_{j,k},g_{j,\ell}\rangle_{H_j}\\ &= \sum_{k=1}^{m_1}\sum_{\ell=1}^{m_2}\prod_{j\in N}\overline{\langle g_{j,\ell},f_{j,k}\rangle_{H_j}}\\ &= \overline{\sum_{k=1}^{m_1}\sum_{\ell=1}^{m_2}\prod_{j\in N}\langle g_{j,\ell},f_{j,k}\rangle_{H_j}}\\ &= \overline{\langle\Psi,\Phi\rangle_{\otimes'_{j\in N}H_j}}\end{aligned}$$

holds.

For a fixed $j_0\in N$, and for any choice of $x_1,\dots,x_n\in\mathbb{K}$, we have

$$\sum_{k=1}^{m_1}\sum_{i=1}^{m_1}x_k\overline{x_i}\,\langle f_{j_0,i},f_{j_0,k}\rangle_{H_{j_0}} = \left\langle\sum_{k=1}^{m_1}x_kf_{j_0,k},\sum_{i=1}^{m_1}x_if_{j_0,i}\right\rangle_{H_{j_0}} \geq 0,$$

which means the matrix $\left(\langle f_{j_0,k},f_{j_0,i}\rangle_{H_{j_0}}\right)_{k,i=1}^{m_1}$ is nonnegative definite. The pointwise product of nonnegative definite matrices is again a nonnegative definite matrix according to Lemma 1.5. So, the matrix $\left(\prod_{j\in N}\langle f_{j,k},f_{j,i}\rangle_{H_j}\right)_{k,i=1}^{m_1}$ nonnegative definite, which implies

$$\langle\Phi,\Phi\rangle_{\otimes'_{j\in N}H_j} = \sum_{k=1}^{m_1}\sum_{i=1}^{m_1}\prod_{j\in N}\langle f_{j,k},f_{j,i}\rangle_{H_j}\geq 0$$

and thus $\langle\cdot,\cdot\rangle_{\otimes'_{j\in N}H_j}$ is nonnegative definite.

All that is left to show is that $\langle\cdot,\cdot\rangle_{\otimes'_{j\in N}H_j}$ is positive definite. Consider $\Phi\in\otimes'_{j\in N}H_j$ with $\langle\Phi,\Phi\rangle_{\otimes_{j\in N}H_j}=0$. Lemma 1.6 then implies $\langle\Phi,\Psi\rangle_{\otimes'_{j\in N}H_j}=0$ for all $\Psi\in\otimes'_{j\in N}H_j$. For any $f\in\bigtimes_{j\in N}H_j$, we thus obtain

$$\Phi\left((f_j)_{j\in N}\right)=\langle\Phi,\otimes_{j\in N}f_j\rangle_{\otimes'_{j\in N}H_j}=0.$$

This means that $\Phi=0$. □

Corollary 1.8. For any $\Phi\in\otimes'_{j\in N}H_j$ and any elementary tensor $\otimes_{j\in N}f_j$,

$$\langle\Phi,\otimes_{j\in N}f_j\rangle_{\otimes'_{j\in N}H_j}=\Phi\left((f_j)_{j\in N}\right)$$

holds.

Proof. This follows immediately from the definition of $\langle\cdot,\cdot\rangle_{\otimes'_{j\in N}H_j}$. □

In general, $\otimes'_{j\in N} H_j$ is not a complete space, as we will later see in Remark 1.20. However, any Cauchy sequence converges pointwise:

Lemma 1.9. For any Cauchy sequence $(\Phi_n)_{n\in\mathbb{N}}$ in $\otimes'_{j\in N} H_j$ and given any $(f_j)_{j\in N} \in \bigtimes_{j\in N} H_j$, the pointwise limit $\lim_{n\to\infty} \Phi_n \left((f_j)_{j\in N}\right)$ exists.

Proof. For $n_1, n_2 \in \mathbb{N}$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} &\left|\Phi_{n_1}\left((f_j)_{j\in N}\right) - \Phi_{n_2}\left((f_j)_{j\in N}\right)\right| \\ =& \left|\langle \Phi_{n_1} - \Phi_{n_2}, \otimes_{j\in N} f_j\rangle_{\otimes_{j\in N} H_j}\right| \\ \leq& \left\|\Phi_{n_1} - \Phi_{n_2}\right\|_{\otimes'_{j\in N} H_j} \left\|\otimes_{j\in N} f_j\right\|_{\otimes'_{j\in N} H_j}. \end{aligned}$$

Thus, $\left(\Phi_n\left((f_j)_{j\in N}\right)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ and, as such, must converge. □

The idea now is to complete $\otimes'_{j\in N} H_j$ using these pointwise limits. Before we can do so, we must prove one more lemma, which ensures that our definition will be well-defined.

Lemma 1.10. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\otimes'_{j\in N} H_j$ and define Φ as the pointwise limit

$$\Phi : \bigtimes_{j\in N} H_j \to \mathbb{K} : (g_j)_{j\in N} \mapsto \lim_{n\to\infty} \Phi_n((g_j)_{j\in N}).$$

Then, for any Cauchy sequence $(\Psi_n)_{n\in\mathbb{N}}$ in $\otimes'_{j\in N} H_j$, we have

$$\lim_{n\to\infty} \left\|\Phi_n - \Psi_n\right\|_{\otimes'_{j\in N} H_j} = 0$$

if and only if the pointwise limit of $(\Psi_n)_{n\in\mathbb{N}}$ is Φ.

Proof. By linearity, we may assume that $\Phi = 0$ and $\Phi_n = 0$ for all $n \in \mathbb{N}$. If not, just replace Ψ with $\Phi - \Psi$ and Ψ_n with $\Phi_n - \Psi_n$.

First, let $\lim_{n\to\infty} \left\|\Psi_n\right\|_{\otimes'_{j\in N} H_j} = 0$. As in the proof of Lemma 1.9, for any $(f_j)_{j\in N} \in \bigtimes_{j\in N} H_j$ we obtain

$$\left|\Psi_n\left((f_j)_{j\in N}\right)\right| \leq \left\|\Psi_n\right\|_{\otimes'_{j\in N} H_j} \left\|\otimes_{j\in N} f_j\right\|_{\otimes'_{j\in N} H_j},$$

which implies $\lim_{n\to\infty} \Psi_n \left((f_j)_{j\in N}\right) = 0 = \Phi\left((f_j)_{j\in N}\right)$.

Now, let $\lim_{n\to\infty} \Psi_n \left((f_j)_{j\in N}\right) = 0$. Assume that $\lim_{n\to\infty} \left\|\Psi_n\right\|_{\otimes'_{j\in N} H_j} \neq 0$. In this case, there exists an $a > 0$ and a subsequence $(\Psi_{n_k})_{k\in\mathbb{N}}$ such that

$\|\Psi_{n_k}\|_{\otimes'_{j\in N} H_j} \geq a$ for all $k \in \mathbb{N}$. Choose $k_0 \in \mathbb{N}$ so that $\|\Psi_{n_k} - \Psi_{n_\ell}\|_{\otimes'_{j\in N} H_j} \leq \frac{a}{2}$ holds for all $k, \ell \geq k_0$. For $k \geq k_0$ we obtain

$$\begin{aligned}
&\left|\left\langle \Psi_{n_k}, \Psi_{n_{k_0}} \right\rangle_{\otimes'_{j\in N} H_j}\right| \\
\geq\ & \left|\left\langle \Psi_{n_{k_0}}, \Psi_{n_{k_0}} \right\rangle_{\otimes'_{j\in N} H_j}\right| - \left|\left\langle \Psi_{n_{k_0}} - \Psi_{n_k}, \Psi_{n_{k_0}} \right\rangle_{\otimes'_{j\in N} H_j}\right| \\
\geq\ & \left\|\Psi_{n_{k_0}}\right\|^2_{\otimes'_{j\in N} H_j} - \left\|\Psi_{n_{k_0}} - \Psi_{n_k}\right\|_{\otimes'_{j\in N} H_j} \left\|\Psi_{n_{k_0}}\right\|_{\otimes'_{j\in N} H_j} \\
\geq\ & \left(\left\|\Psi_{n_{k_0}}\right\|_{\otimes'_{j\in N} H_j} - \frac{a}{2}\right) \left\|\Psi_{n_{k_0}}\right\|_{\otimes'_{j\in N} H_j} \\
\geq\ & \frac{a^2}{2}.
\end{aligned}$$

This is a contradiction: Since

$$\lim_{n\to\infty} \langle \Psi_n, \otimes_{j\in N} f_j \rangle_{\otimes'_{j\in N} H_j} = \lim_{n\to\infty} \Psi_n \left((f_j)_{j\in N} \right) = 0$$

holds for any choice of $\otimes_{j\in N} f_j$, and $\Psi_{n_{k_0}}$ can be the written as the finite sum of such, we must necessarily have $\lim_{k\to\infty} \left\langle \Psi_{n_k}, \Psi_{n_{k_0}} \right\rangle_{\otimes'_{j\in N} H_j} = 0$. □

Now, as the main result of this subsection, we are able to construct the tensor product.

Theorem 1.11. Consider the subspace of $\mathbb{K}^{(\times_{j\in N} H_j)}$ defined by

$$\bigotimes_{j\in N} H_j := \left\{ \Phi \in \mathbb{K}^{(\times_{j\in N} H_j)} \;\middle|\; \Phi \text{ pointw. limit of a Cauchy seq. in } \otimes'_{j\in N} H_j \right\}.$$

For $\Phi, \Psi \in \bigotimes_{j\in N} H_j$ with Cauchy sequences $(\Phi_n)_{n\in\mathbb{N}}, (\Psi_n)_{n\in\mathbb{N}}$ in $\otimes'_{j\in N} H_j$ that converge pointwise to Φ and Ψ respectively, the mapping given by

$$\langle \Phi, \Psi \rangle_{\bigotimes_{j\in N} H_j} = \lim_{n\to\infty} \langle \Phi_n, \Psi_n \rangle_{\otimes'_{j\in N} H_j}$$

is well-defined and a scalar product. Further, $\bigotimes_{j\in N} H_j$ equipped with this scalar product is a Hilbert space.

Proof. We first have to show that $\lim_{n\to\infty} \langle \Phi_n, \Psi_n \rangle_{\otimes'_{j\in N} H_j}$ exists and does only depend on Φ and Ψ, not on the respective Cauchy sequences.

For $r, s \in \mathbb{N}$ we obtain

$$\begin{aligned}
&\left| \langle \Phi_r, \Psi_r \rangle_{\otimes'_{j\in N} H_j} - \langle \Phi_s, \Psi_s \rangle_{\otimes'_{j\in N} H_j} \right| \\
\leq\ & \left| \langle \Phi_r - \Phi_s, \Psi_r \rangle_{\otimes'_{j\in N} H_j} \right| + \left| \langle \Phi_s, \Psi_s - \Psi_r \rangle_{\otimes'_{j\in N} H_j} \right| \\
\leq\ & \|\Phi_r - \Phi_s\|_{\otimes'_{j\in N} H_j} \|\Psi_r\|_{\otimes'_{j\in N} H_j} + \|\Phi_s\|_{\otimes'_{j\in N} H_j} \|\Psi_s - \Psi_r\|_{\otimes'_{j\in N} H_j}.
\end{aligned}$$

This shows that $\left(\langle \Phi_n, \Psi_n \rangle_{\otimes'_{j\in N} H_j}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ and as such converges.

Now consider Cauchy sequences $(\Phi'_n)_{n\in\mathbb{N}}$ and $(\Psi'_n)_{n\in\mathbb{N}}$ in $\otimes'_{j\in N} H_j$ that also converge pointwise to Φ and Ψ respectively. By the same calculation as before we obtain

$$\begin{aligned}&\left|\langle \Phi_r, \Psi_r \rangle_{\otimes'_{j\in N} H_j} - \langle \Phi'_r, \Psi'_r \rangle_{\otimes'_{j\in N} H_j}\right| \\ &\leq \|\Phi_r - \Phi'_r\|_{\otimes'_{j\in N} H_j} \|\Psi_r\|_{\otimes'_{j\in N} H_j} + \|\Phi'_r\|_{\otimes'_{j\in N} H_j} \|\Psi'_r - \Psi_r\|_{\otimes'_{j\in N} H_j}\end{aligned}$$

for any $r \in \mathbb{N}$. The right-hand side converges to 0 by Lemma 1.10, so the left-hand side converges to 0, too. Thus, $\langle \Phi, \Psi \rangle_{\bigotimes_{j\in N} H_j}$ is well-defined.

Since we already know that $\langle \cdot, \cdot \rangle_{\otimes'_{j\in N} H_j}$ is a scalar product, by taking the limit we directly obtain that $\langle \cdot, \cdot \rangle_{\bigotimes_{j\in N} H_j}$ is sesquilinear, conjugate symmetric and nonnegative definite.

Now let $\Phi \in \bigotimes_{j\in N} H_j$ such that $\langle \Phi, \Phi \rangle_{\bigotimes_{j\in N} H_j} = 0$ and let $(\Phi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\otimes'_{j\in N} H_j$ with pointwise limit Φ. Then, we have

$$0 = \lim_{n\to\infty} \langle \Phi_n, \Phi_n \rangle_{\otimes'_{j\in N} H_j} = \lim_{n\to\infty} \|\Phi_n\|_{\otimes'_{j\in N} H_j}.$$

By applying Lemma 1.10 with $\Psi_n = \Psi = 0$ for all $n \in \mathbb{N}$, we get $\Phi = 0$. Thus, $\langle \cdot, \cdot \rangle_{\bigotimes_{j\in N} H_j}$ is positive definite and a scalar product.

Now, consider a Cauchy sequence (Φ_n) in $\bigotimes_{j\in N} H_j$. For each $n \in \mathbb{N}$, let a Cauchy sequence $\left(\Phi_n^k\right)_{k\in\mathbb{N}}$ in $\otimes'_{j\in N} H_j$ that converges pointwise to Φ_n be given. Then,

$$\left\|\Phi_n - \Phi_n^k\right\|_{\bigotimes_{j\in N} H_j} = \lim_{\ell\to\infty} \left\|\Phi_n^\ell - \Phi_n^k\right\|_{\bigotimes_{j\in N} H_j}$$

converges to 0 as k tends to infinity. Thus, $\left(\Phi_n^k\right)_{k\in\mathbb{N}}$ converges to Φ_n in $\bigotimes_{j\in N} H_j$ and we can choose a $k(n) \in \mathbb{N}$ such that $\left\|\Phi_n - \Phi_n^{k(n)}\right\|_{\bigotimes_{j\in N} H_j} < \frac{1}{n}$. This means that $\lim_{n\to\infty} \left\|\Phi_n - \Phi_n^{k(n)}\right\|_{\bigotimes_{j\in N} H_j} = 0$. Because for any $n, \ell \in \mathbb{N}$

$$\begin{aligned}&\left\|\Phi_n^{k(n)} - \Phi_m^{k(\ell)}\right\|_{\bigotimes_{j\in N} H_j} \\ &\leq \left\|\Phi_n^{k(n)} - \Phi_n\right\|_{\bigotimes_{j\in N} H_j} + \|\Phi_n - \Phi_\ell\|_{\bigotimes_{j\in N} H_j} + \left\|\Phi_\ell - \Phi_\ell^{k(\ell)}\right\|_{\bigotimes_{j\in N} H_j}\end{aligned}$$

holds, $\left(\Phi_n^{k(n)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\otimes'_{j\in N} H_j$ and thus converges pointwise to some $\Phi \in \bigotimes_{j\in N} H_j$. As before, this implies that $\left(\Phi_n^{k(n)}\right)_{n\in\mathbb{N}}$ converges to Φ in $\bigotimes_{j\in N} H_j$. Finally, by

$$\|\Phi - \Phi_n\|_{\bigotimes_{j\in N} H_j} \leq \left\|\Phi - \Phi_n^{k(n)}\right\|_{\bigotimes_{j\in N} H_j} + \left\|\Phi_n^{k(n)} - \Phi_n\right\|_{\bigotimes_{j\in N} H_j},$$

we obtain that Φ_n converges to Φ in $\bigotimes_{j\in N} H_j$. Since $(\Phi_n)_{n\in\mathbb{N}}$ is an arbitrary Cauchy sequence, $\bigotimes_{j\in N} H_j$ equipped with $\langle \cdot, \cdot \rangle_{\bigotimes_{j\in N} H_j}$ is a Hilbert space. □

We summarize the construction of the tensor product with the following definition.

Definition 1.12. We call the space $\bigotimes_{j\in N} H_j$ equipped with $\langle\cdot,\cdot\rangle_{\bigotimes_{j\in N} H_j}$ as established in Theorem 1.11 the *tensor product* of the spaces $H_1,\ldots,H_m$.

Again, if it is convenient, we write $H_1 \bigotimes \cdots \bigotimes H_m := \bigotimes_{j\in N} H_j$.

Examining the proof of Theorem 1.11 gives us the following useful fact.

Corollary 1.13. $\otimes'_{j\in N} H_j$ is dense in $\bigotimes_{j\in N} H_j$.

Proof. In the proof of Theorem 1.11, we showed that any Cauchy sequence in $\bigotimes_{j\in N} H_j$ converges. In that part of the proof, we used sequences $\left(\Phi_n^k\right)_{k\in\mathbb{N}}$, which are sequences in $\otimes'_{j\in N} H_j$ converging to an arbitrary $\Phi_n \in \bigotimes_{j\in N} H_j$. This shows our claim. □

This shows that $\bigotimes_{j\in N} H_j$ is the smallest possible Hilbert space containing all elementary tensors, in that it is the completion of their linear span.

1.2 Properties and Examples

The way $\bigotimes_{j\in N} H_j$ is defined – as a space of certain mappings from $\bigtimes_{j\in N} H_j$ to $\mathbb{K}$ – is often not that easy to work with. We often work with other Hilbert spaces that contain elements $\tilde{\otimes}_{j\in N} f_j$ which fulfill the cross-norm property. The following theorem allows us to easily identify these with $\bigotimes_{j\in N} H_j$.

Theorem 1.14. If a Hilbert space H fulfills the three properties

1. H contains an element $\tilde{\otimes}_{j\in N} f_j$ for each $(f_j)_{j\in N} \in \bigtimes_{j\in N} H_j$,
2. $\left\langle \tilde{\otimes}_{j\in N} f_j, \tilde{\otimes}_{j\in N} g_j \right\rangle_H = \prod_{j\in N} \langle f_j, g_j\rangle_{H_j}$ holds for any two such elements $\tilde{\otimes}_{j\in N} f_j, \tilde{\otimes}_{j\in N} g_j$ and
3. $H_0 = \operatorname{span}\left\{ \tilde{\otimes}_{j\in N} f_j \;\middle|\; (f_j)_{j\in N} \in \bigtimes_{j\in N} H_j \right\}$ is dense in H,

there exists a unique isometric isomorphism $\Lambda : H \to \bigotimes_{j\in N} H_j$ that fulfills

$$\Lambda\left(\tilde{\otimes}_{j\in N} f_j\right) = \otimes_{j\in N} f_j$$

for every $\tilde{\otimes}_{j\in N} f_j \in H$.

Conversely if H is a Hilbert space and an isometric isomorphism

$$\Lambda : H \to \bigotimes_{j\in N} H_j$$

exists, H fulfills the three properties given above.

Proof. We show the last claim first. For this, we remark that $\bigotimes_{j\in N} H_j$ fulfills properties 1 and 2 by construction and property 3 by Corollary 1.13. Now, let $\Lambda : H \to \bigotimes_{j\in N} H_j$ be an isometric isomorphism. In this case, we can define $\tilde{\otimes}_{j\in N} f_j = \Lambda^{-1}\left(\otimes_{j\in N} f_j\right)$. Note that any isometric isomorphism is unitary and thus preserves the scalar product. Since the three properties hold for $\bigotimes_{j\in N} H_j$, by applying Λ^{-1}, they also hold for H.

Now consider any space H for which the three properties hold. On H_0, define $\bar{\Lambda} : H_0 \to \otimes'_{j\in N} H_j$ by setting $\bar{\Lambda}\left(\tilde{\otimes}_{j\in N} f_j\right) = \otimes_{j\in N} f_j$ and extending linearly. Then, $\bar{\Lambda}$ is well-defined, since for any $\Phi \in H_0$ with two representations

$$\Phi = \sum_{k=1}^{m_1} a_k \cdot \tilde{\otimes}_{j\in N} f_{j,k} = \sum_{\ell=1}^{m_2} b_\ell \cdot \tilde{\otimes}_{j\in N} g_{j,\ell},$$

we have

$$\left\| \sum_{k=1}^{m_1} a_k \cdot \tilde{\otimes}_{j\in N} f_{j,k} - \sum_{\ell=1}^{m_2} b_\ell \cdot \tilde{\otimes}_{j\in N} g_{j,\ell} \right\|_H = 0.$$

Since both $\|\cdot\|_H$ and $\|\cdot\|_{\bigotimes_{j\in N} H_j}$ are given by scalar products fulfilling the second property, this implies

$$\left\| \sum_{k=1}^{m_1} a_k \cdot \otimes_{j\in N} f_{j,k} - \sum_{\ell=1}^{m_2} b_\ell \cdot \otimes_{j\in N} g_{j,\ell} \right\|_{\bigotimes_{j\in N} H_j} = 0$$

and therefore $\sum_{k=1}^{m_1} a_k \cdot \otimes_{j\in N} f_{j,k} = \sum_{\ell=1}^{m_2} b_\ell \cdot \otimes_{j\in N} g_{j,\ell}$.

Clearly, $\bar{\Lambda}$ is an isometric isomorphism. Next, we define $\Lambda : H \to \bigotimes_{j\in N} H_j$. By the third property, for any $\Phi \in H$, there exists a sequence $(\Phi_n)_{n\in\mathbb{N}}$ in H_0 such that $\lim_{n\to\infty} \Phi_n = \Phi$. Set

$$\Lambda(\Phi) = \lim_{n\to\infty} \bar{\Lambda}(\Phi_n).$$

For two sequences $(\Phi_n)_{n\in\mathbb{N}}, (\Psi_n)_{n\in\mathbb{N}}$ in H_0 with $\lim_{n\to\infty} \Phi_n = \lim_{n\to\infty} \Psi_n$, we have $\lim_{n\to\infty} \|\Phi_n - \Psi_n\|_H = 0$. Thus, $\lim_{n\to\infty} \left\|\bar{\Lambda}(\Phi_n) - \bar{\Lambda}(\Psi_n)\right\|_{\bigotimes_{j\in N} H_j} = 0$ and therefore $\lim_{n\to\infty} \bar{\Lambda}(\Phi_n) = \lim_{n\to\infty} \bar{\Lambda}(\Psi_n)$. This means that Λ is well-defined.

Clearly, Λ is isometric. It is also surjective, because for any $\Phi \in \bigotimes_{j\in N} H_j$, there is an sequence $(\Phi_n)_{n\in\mathbb{N}}$ converging towards Φ. Thus, we have

$$\Phi = \Lambda\left(\lim_{n\to\infty} \bar{\Lambda}^{-1}(\Phi_n)\right).$$

Any two isometric isomorphisms $\Lambda_1, \Lambda_2 : H \to \bigotimes_{j\in N} H_j$ fulfilling

$$\Lambda_1\left(\tilde{\otimes}_{j\in N} f_j\right) = \Lambda_2\left(\tilde{\otimes}_{j\in N} f_j\right) = \otimes_{j\in N} f_j$$

for every $\tilde{\otimes}_{j\in N} f_j \in H$ must coincide on H_0 by linearity and on H because H_0 is dense in H and by continuity of the norm. □

If the three properties from Theorem 1.14 hold for a Hilbert space H, we say H and $\bigotimes_{j\in N} H_j$ are canonically isomorphic.

Remark 1.15. Another way to look at what we have done so far is the following: One might define the tensor product $\bigotimes_{j\in N} H_j$ as any Hilbert space satisfying the three properties given in Theorem 1.14. The existence of such a space would then be proven by Theorem 1.11 and everything leading up to it, while Theorem 1.14 would ensure that this definition is well-defined up to a unique isometric isomorphism given by Λ.

Remark 1.16. For $m = 1$, the spaces H_1 and $\bigotimes_{j\in N} H_j$ are canonically isomorphic.

For $m \geq 2$, the spaces $\left(\bigotimes_{j=1}^{m-1} H_j\right) \otimes H_m$ and $\bigotimes_{j\in N} H_j$ are canonically isomorphic.

Now, we can look at an interesting example, namely the tensor product of certain L_2-spaces. First, we need the following lemma.

Lemma 1.17. In the setting of Example 1.18,

$$\mathcal{E} = \left\{ \bigtimes_{j\in N} A_j \;\middle|\; A_j \in \mathcal{A}_j \right\}$$

is a semiring of sets, that is, it fulfills

1. $\emptyset \in \mathcal{E}$,
2. for $A, B \in \mathcal{A}$, we have $A \cap B \in \mathcal{A}$ and
3. for $A, B \in \mathcal{A}$, there is a finite number of pairwise disjoint sets $C_1, \ldots, C_k$ such that $A \setminus B = \bigcup_{j=1}^{k} C_j$ holds.

Proof. Obviously, property 1 holds.

For property 2, let $\bigtimes_{j\in N} A_j$ and $\bigtimes_{j\in N} B_j$ be two sets in $\mathcal{E}$. Then, we have

$$\left(\bigtimes_{j\in N} A_j\right) \cap \left(\bigtimes_{j\in N} B_j\right) = \bigtimes_{j\in N} (A_j \cap B_j) \in \mathcal{E}.$$

To show property 3, for any set $I \subseteq N$ and any $j \in N$, define

$$D_{I,j} = \begin{cases} A_j \setminus B_j, & \text{if } j \in I \\ B_j, & \text{if } j \notin I \end{cases}$$

and $C_I = \bigtimes_{j\in N} D_{I,j}$. Clearly, $C_I \in \mathcal{E}$ and for two subsets $I \neq I'$ of N, we have $C_I \cap C_{I'} = \emptyset$. Since N is finite, it only has finitely many subsets, and we obtain

$$\left(\bigtimes_{j\in N} A_j\right) \setminus \left(\bigtimes_{j\in N} B_j\right) = \bigcup_{\substack{I\subseteq N\\ I\neq\emptyset}} C_I.$$

This finishes the proof. $\square$

Now, we can give the desired example. The statement and the general idea of the proof are taken from Example 2.6.11 in [6].

Example 1.18. For $j \in \mathbb{N}$, consider a σ-finite measure space $(\Omega_j, \mathcal{A}_j, \mu_j)$ and the corresponding space of equivalence classes of real-valued, square-integrable functions $H_j = L_2(\Omega_j, \mathcal{A}_j, \mu_j)$. For the product-$\sigma$-algebra $\otimes_{j\in N}\mathcal{A}_j$ and the product measure $\bigtimes_{j\in N}\mu_j$, the space $H = L_2\left(\bigtimes_{j\in N}\Omega_j, \otimes_{j\in N}\mathcal{A}_j, \bigtimes_{j\in N}\mu_j\right)$ fulfills the properties in Theorem 1.14 and thus is canonically isomorphic to the tensor product $\bigotimes_{j\in N} H_j$.

Proof. For any square-integrable function f, we denote its corresponding equivalence class by $[f]$.

For property 1, for any $([f_j])_{j\in N} \in \bigtimes_{j\in N} H_j$, we put $\tilde{\otimes}_{j\in N}[f_j] = [\prod_{j\in N} f_j]$. This is well-defined: On the one hand, we have

$$\begin{aligned}
&\int_{\bigtimes_{j\in N}\Omega_j} \left|\prod_{j\in N} f_j\right|^2 \,\mathrm{d}\left(\bigtimes_{j\in N}\mu_j\right) \\
&= \int_{\Omega_1} \dots \int_{\Omega_n} \left|\prod_{j\in N} f_j\right|^2 \,\mathrm{d}\mu_n \dots \,\mathrm{d}\mu_1 \\
&= \int_{\Omega_1} |f_1|^2 \dots \int_{\Omega_n} |f_n|^2 \,\mathrm{d}\mu_n \dots \,\mathrm{d}\mu_1 \\
&= \prod_{j\in N} \int_{\Omega_j} |f_j|^2 \,\mathrm{d}\mu_j < \infty,
\end{aligned}$$

so $\prod_{j\in N} f_j$ is indeed square-integrable and we have $\tilde{\otimes}_{j\in N}[f_j] = [\prod_{j\in N} f_j] \in H$. On the other hand, for a sequence of square-integrable functions $(g_j)_{j\in N}$ where $g_j \in [f_j]$ holds for every $j \in N$, we have

$$\begin{aligned}
&\left\{ \omega \in \bigtimes_{j\in N}\Omega_j \;\middle|\; \prod_{j\in N} f_j(\omega_j) \neq \prod_{j\in N} g_j(\omega_j) \right\} \\
&\subseteq \bigcup_{j\in N} \left\{ \omega \in \bigtimes_{j\in N}\Omega_j \;\middle|\; f_j(\omega_j) \neq g_j(\omega_j) \right\},
\end{aligned}$$

and the latter is the finite union of sets of measure zero. This implies

$$[\prod_{j\in N} f_j] = [\prod_{j\in N} g_j],$$

so $\tilde{\otimes}_{j\in N}[f_j]$ depends on the equivalence-classes $[f_j]$ only, not on their representations.

Now, let $([f_j])_{j\in N}$ and $([g_j])_{j\in N}$ be two elements of $\bigtimes_{j\in N} H_j$. Property 2 follows by replacing all instances $\left|\prod_{j\in N} f_j\right|^2$ and $|f_j|^2$ in the calculation above with $\prod_{j\in N} f_j \prod_{j\in N} g_j$ and $f_j g_j$ respectively. This implies

$$\int_{\bigtimes_{j\in N}\Omega_j} \prod_{j\in N} f_j \prod_{j\in N} g_j \ \mathrm{d}\left(\bigtimes_{j\in N} \mu_j\right) = \prod_{j\in N} \int_{\Omega_j} f_j g_j \ \mathrm{d}\mu_j,$$

which in turn implies $\left\langle \tilde{\otimes}_{j\in N}[f_j], \tilde{\otimes}_{j\in N}[g_j]\right\rangle_H = \prod_{j\in N} \langle [f_j],[g_j]\rangle_{H_j}$.

For Property 3, consider the set $\mathcal{E}$ from Lemma 1.17, which is a semiring of sets. It is well-known that $\sigma(\mathcal{E}) = \otimes_{j\in N}\mathcal{A}_j$. Further, the product measure $\mu = \bigtimes_{j\in N}\mu_j$ is σ-finite. This implies that

$$S = \operatorname{span}\{\, [\mathbb{1}_A] \mid A \in \mathcal{E}, \mu(A) < \infty \,\}$$

is dense in H, as seen in Theorem VI 2.28 in [3]. Now, each $A \in \mathcal{E}$ is of the form $A = \bigtimes_{j\in N} A_j$ for some $A_j \in \mathcal{A}_j$. By this, we obtain

$$[\mathbb{1}_A] = \left[\prod_{j\in N} \mathbb{1}_{A_j}\right] = \tilde{\otimes}_{j\in N}[\mathbb{1}_{A_j}]$$

and thus

$$S \subseteq H_0 := \operatorname{span}\left\{\, \tilde{\otimes}_{j\in N}[f_j] \;\middle|\; [f_j] \in H_j \,\right\} \subseteq H.$$

This implies that H_0 is dense in H. □

Remark 1.19. Applying Example 1.18 gives us

1. $\mathbb{R}^n \bigotimes \mathbb{R}^m$ and $\mathbb{R}^{n\cdot m}$ are canonically isomorphic for any $n, m \in \mathbb{N}$ and
2. $H \bigotimes \mathbb{R}$ and H are canonically isomorphic for any real Hilbert space H.

Remark 1.20. Example 1.18 shows that each part of our construction of $\bigotimes_{j\in N} H_j$ was necessary.

1. Consider $H_1 = H_2 = L_2(\{1,2\}, 2^{\{1,2\}}, \Sigma)$, where Σ is the counting measure. In this case, we do not need to distinguish functions from their equivalence classes. Consider the function

$$h : \{1,2\}^2 \to \mathbb{R} : (x,y) \mapsto x + y,$$

which is clearly square-integrable in the product space. Obviously, we have $h \in \operatorname{span}\{\, f\cdot g \mid f,g \in \mathbb{R}^{\{1,2\}}\,\}$. However, there exist no functions $f, g \in \mathbb{R}^{\{1,2\}}$ such that $h = f\cdot g$. Indeed, if we have $f(1)g(1) = 2$ and $f(2)g(1) = f(1)g(2) = 3$, we obtain $g(2) = \frac{3}{2}g(1)$ and $f(2) = \frac{3}{2}f(1)$, which implies $f(2)g(2) = \frac{9}{4}f(1)g(1) = \frac{9}{2} \neq 4$.

This shows that $\otimes'_{j\in N} H_j$ may contain elements not of the form $\otimes_{j\in N} f_j$.

2. Consider $H_1 = H_2 = L_2\left([0,1], \mathcal{B}([0,1]), \lambda_1\right)$. Then, the product space $L_2\left([0,1]^2, \mathcal{B}([0,1]^2), \lambda_2\right)$ is canonically isomorphic to $H_1 \bigotimes H_2$. For notational convenience, we here use square-integrable functions instead of their equivalence classes. All functions used are continuous. Since each equivalence class contains at most one continuous function, the results given here also apply to the L_2-case.

   The function $h : [0,1] \to \mathbb{R} : (x,y) \mapsto \exp(xy)$ is clearly square-integrable. However, $h \notin \operatorname{span}\{\, f \cdot g \mid f, g \in \mathbb{R}^{[0,1]} \,\}$. To see this, we first show that for each $n \in \mathbb{N}$, the family $(h_k : [0,1] \to \mathbb{R} : x \mapsto \exp(kx))_{k \in \{1,\dots,n\}}$ is linearly independent in $\mathbb{R}^{[0,1]}$: Let $\lambda_k \in \mathbb{R}$ such that $f(x) := \sum_{k=1}^{n} \lambda_k h_k(x) = 0$ for all $x \in [0,1]$. Consider the function $g(x) = \sum_{k=1}^{n} \lambda_k x^k$ with domain $\mathbb{R}$. We then have $f(x) = g(\exp(x))$ for all $x \in [0,1]$ Since $f(x) = 0$ for all $x \in [0,1]$, we have $g(x) = 0$ for all $x \in [1, \exp(1)]$. Because g is a polynomial function, this implies $\lambda_k = 0$ for all $k \in \{1,\dots,n\}$.

   Now assume $h \in \operatorname{span}\{\, f \cdot g \mid f, g \in \mathbb{R}^{[0,1]} \,\}$ holds, which implies there is a representation $h(x,y) = \sum_{k=1}^{n} \lambda_k f_k(x) g_k(y)$. Then, for any fixed $j \in \mathbb{N}$, we define $h_j(x) := \exp(jx) = \sum_{k=1}^{n} \lambda_k f_k(x) g_k(j)$. This implies that

   $$h_j \in \operatorname{span}\{\, f_k \,\}_{k \in \{1,\dots,n\}}.$$

   Since this space has dimension at most n, the family $(h_j)_{j \in \{1,\dots,n+1\}}$ cannot be linearly independent. This is a contradiction.

   This shows that $\otimes'_{j \in N} H_j$ is not, in general, a complete space.

Given orthonormal bases of the spaces H_j, the following shows a canonical construction of an orthonormal basis of $\bigotimes_{j \in N} H_j$,

Proposition 1.21. Given orthonormal bases $B_j = (f_{j,i_j})_{i_j \in I_j}$ of H_j,

$$B = \left(\otimes_{j \in N} f_{j,i_j}\right)_{(i_j)_{j \in N} \in \times_{j \in N} I_j}$$

defines an orthonormal basis of $\bigotimes_{j \in N} H_j$.

Proof. For two members $\otimes_{j \in N} f_{j,i_j}, \otimes_{j \in N} f_{j,k_j}$ of B, their scalar product is

$$\left\langle \otimes_{j \in N} f_{j,i_j}, \otimes_{j \in N} f_{j,k_j} \right\rangle_{\bigotimes_{j \in N} H_j} = \prod_{j \in N} \left\langle f_{j,i_j}, f_{k,i_j} \right\rangle_{H_j}.$$

Because each B_j is an orthonormal system, each factor is equal to 1 if and only if $f_{j,i_j} = f_{j,k_j}$ and equal to 0 otherwise. This shows that B is an orthonormal system.

Now, let any $h \in \times_{j \in N} H_j$ be given. Because $\operatorname{span}(B_j)$ is dense in H_j, for any h_j we can find a sequence $\left(h_j^{(n)}\right)_{n \in \mathbb{N}}$ in $\operatorname{span}(B_j)$ that converges to h_j. Because $\|\otimes_{j \in N} h_j\|_{\bigotimes_{j \in N} H_j} = \prod_{j \in N} \|h_j\|_{H_j}$ holds, the sequence $\left(\otimes_{j \in N} h_j^{(n)}\right)_{n \in \mathbb{N}}$, which lies in $\operatorname{span}(B)$, converges to $\otimes_{j \in N} h_j$. This implies that $\operatorname{span}(B)$ is dense in $\otimes'_{j \in N} H_j$, which itself is dense in $\bigotimes_{j \in N} H_j$. □

In Remark 1.20, we saw that $\bigotimes_{j\in N} H_j$ and $\otimes'_{j\in N} H_j$ are in general not the same space. However, we can give a special case in which both spaces are equal.

Proposition 1.22. The space $\otimes'_{j\in N} H_j$ is complete if

$$\dim\left(\otimes'_{j\in N} H_j\right) < \infty.$$

Proof. If $\bigotimes_{j\in N} H_j$ is finite-dimensional, so is any subspace and in particular $\otimes'_{j\in N} H_j$. It is well known that any finite-dimensional $\mathbb{K}$-vector space equipped with a norm given by a scalar product is complete. □

The reverse need not be true, as the following example shows.

Example 1.23. Let H_1 be any infinite-dimensional space and $H_2 = \mathbb{K}$. Then for any $f_1 \in H_1$ and any $a \in H_2$ we obtain $f_1 \otimes a = (af_1 \otimes 1)$. Further, for any $g_1 \in H_1$ we obtain $(f_1 \otimes 1) + (g_1 \otimes 1) = (f_1 + g_1) \otimes 1$. This implies $H_1 \otimes' H_2 = \{ f_1 \otimes 1 \mid f_1 \in H_1 \}$. Clearly, this is a complete space.

1.3 Other Notions of Tensor Products

We briefly look into another way to define the tensor product of Hilbert spaces, as done in [6]. We also shortly compare the tensor product of Hilbert spaces with the tensor product of generic vector spaces.

Another Approach to the Tensor Product of Hilbert Spaces

There is another way to define the Hilbert space tensor product, which is laid out in [6]. It relies on orthonormal bases and Hilbert-Schmidt-mappings. We will not look at this in detail; rather, our goal is to see that both notions are essentially the same. For this, we give a quick rundown of how the tensor product of Hilbert spaces is defined in [6].

First, we need to understand what a Hilbert-Schmidt-mapping is.

Proposition 1.24. Let $H_1, \ldots, H_m$ be Hilbert spaces over $\mathbb{K}$ and

$$\varphi : \bigtimes_{j=1}^{m} H_j \to \mathbb{K}$$

a bounded multilinear functional, that is to say, φ is multilinear and there is a constant $c \in \mathbb{R}$ such that

$$|\varphi(h_1, \ldots, h_m)| \le c \cdot \prod_{j=1}^{m} \|h_j\|_{H_j}$$

holds for each choice of $h_j \in H_j$. If there are orthonormal bases B_j of H_j such that

$$\sum_{b_1\in B_1} \cdots \sum_{b_m\in B_m} |\varphi(b_1, \ldots, b_m)|^2 < \infty,$$

then this sum is finite for every choice of orthonormal bases $B_1, \ldots, B_m$ and its value is the same regardless of this choice.

Proof. See Proposition 2.6.1 in [6]. □

Definition 1.25.

1. A bounded multilinear mapping satisfying the inequalities in Proposition 1.24 is called a Hilbert-Schmidt-functional.

2. For a Hilbert space H, a bounded multilinear mapping $L : \bigtimes_{j=1}^m H_j \to H$ is called weak Hilbert-Schmidt-mapping, if and only if for each $u \in H$, the mapping L_u defined by $L_u(h_1, \ldots, h_m) = \langle L(h_1, \ldots, h_m), u \rangle_H$ is a Hilbert-Schmidt-functional.

Remark 1.26. In [6], the notion of conjugate Hilbert spaces is defined to simplify some things. We do not do that here; instead, we remark that Proposition 1.24 also holds if φ is conjugate multilinear instead of multilinear, and we call bounded conjugate multilinear mappings satisfying the inequalities in Proposition 1.24 conjugate Hilbert-Schmidt functionals.

Having established this, a notion of the tensor product is given in the following way.

Definition 1.27. The tensor product of $H_1, \ldots, H_m$ is given by a Hilbert space H for which a weak Hilbert-Schmidt mapping $p : \bigtimes_{j=1}^m H_j \to H$ with the following property exists: For any (other) Hilbert space K and any (other) weak Hilbert-Schmidt mapping $L : \bigtimes_{j=1}^m H_j \to K$, there is a unique bounded linear mapping $T : H \to K$ such that $L = T \circ p$.

This is proven to exist and be unique in the following sense.

Theorem 1.28.

1. A space H as given in Definition 1.27 always exists.

2. For any other space $\widetilde{H}$ with a weak Hilbert-Schmidt mapping $\widetilde{p}$ fulfilling Definition 1.27, there exists an isometric isomorphism $U : H \to \widetilde{H}$ such that $\widetilde{p} = U \circ p$.

Proof. See Theorem 2.6.4 in [6]. □

We do not replicate the full proof here. We only briefly examine it to see that the notion of tensor product given in [6] agrees with the one given in [10] and in this thesis.

To prove the existence of such a space, the space H of all conjugate Hilbert-Schmidt functionals $\varphi : \bigtimes_{j=1}^m H_j \to \mathbb{K}$ is established to be a Hilbert space (see Proposition 2.6.2 in [6]) fulfilling Definition 1.27. This space is also shown to contain the functionals we know by Definition 1.2. In fact, for any finite sequence $(f_j)_{j \in N} \in \bigtimes_{j \in N} H_j$, we have

$$p\left((f_j)_{j \in N}\right) = \otimes_{j \in N} f_j,$$

where p is the weak Hilbert-Schmidt mapping from Definition 1.27. It is also shown that the scalar product on H fulfills the multiplicative property for these elements. This shows that $\otimes'_{j\in N} H_j$ is a subspace of H. On page 135 in [6], it is remarked that $\otimes'_{j\in N} H_j$ as the space of all finite linear combinations of elements $\otimes_{j\in N} f_j$ is dense in H. This implies that

$$H = \bigotimes_{j\in N} H_j.$$

We now know that $\bigotimes_{j\in N} H_j$ satisfies the conditions of Theorem 1.14 and fulfills Definition 1.25, with the weak Hilbert-Schmidt mapping given by

$$p\left((f_j)_{j\in N}\right) = \otimes_{j\in N} f_j.$$

For any other Hilbert space with a weak Hilbert-Schmidt mapping K fulfilling Definition 1.25 with a weak Hilbert-Schmidt mapping $\widetilde{p}$, there is an isometric isomorphism $U : \bigotimes_{j\in N} H_j \to K$ fulfilling $\widetilde{p} = U \circ p$. This, however, is the same as Λ^{-1} given by Theorem 1.14. Thus, K fulfills Theorem 1.14. Conversely, if any Hilbert space K satisfies Theorem 1.14, the isometric isomorphism Λ can be used to define $\widetilde{p} = \Lambda^{-1} \circ p$, which then has to be a weak Hilbert-Schmidt mapping too. Then, Definition 1.27 holds by this isomorphism.

Thus, the two notions of tensor product are equivalent.

The Tensor Product of Vector Spaces

We now look to define the tensor product of vector spaces with no additional structure, such as a norm or a scalar product, given. This is also known as the *algebraic tensor product* as opposed to the Hilbert space tensor product we considered before. As one might expect, this is more abstract than in the Hilbert space case. We will not delve deep into why these definitions make sense; we are only interested in how they relate to the Hilbert space case. Therefore, we will only state the basic facts needed without giving proofs. We take our information from Chapter 7.2 in [2]; it can also be found in many intermediate-level books on Algebra.

Let K be any field and V, W two K-vector spaces. We will not explicitly construct the algebraic tensor product here, but will take an approach similar to the one lined out in Remark 1.15 or Definition 1.27. Because of this, the definition will again not give us a truly unique space.

Definition 1.29. An algebraic tensor product $V \bigotimes W$ of V and W is a K-vector space for which a K-bilinear mapping $\tau : V \times W \to V \bigotimes W$ with the following property exists:

For any K-vector space Z and any K-bilinear mapping $\tau' : V \times W \to Z$ there exists a unique linear mapping $\Phi : V \bigotimes W \to Z$ such that $\tau' = \Phi \circ \tau$.

Before we give results on existence and uniqueness, let us form some intuition. For any $v \in V$ and $w \in W$, we require that $V \bigotimes W$ contains some element

$v \otimes w$ given by $\tau(v,w)$. We require that τ is bilinear so that $v \otimes w$ behaves like "you would expect" from a product, for example $(v+v') \otimes w = v \otimes w + v' \otimes w$ for $v, v' \in V, w \in W$. We do not require τ to be injective or surjective, so not every element of $V \bigotimes W$ needs to have a representation of the form $v \otimes w$ and such a representation need not be unique.

Now if another such space with special elements $v \otimes w$, given by Z and τ' here, exists, we require that the special elements of both spaces must be identified with each other in a unique way, given by Φ.

Now we state why this makes sense.

Proposition 1.30.

1. For any two K-vector spaces V and W, a space $V \bigotimes W$ as given in Definition 1.29 exists.
2. $V \bigotimes W$ is uniquely determined in the following way: If there is another space $(\widetilde{V \bigotimes W})$ along with a bilinear mapping $\tilde{\tau} : V \times W \to (\widetilde{V \otimes W})$ satisfying Definition 1.29, there exists a uniquely determined isomorphism $\tilde{\Phi} : V \bigotimes W \to (\widetilde{V \bigotimes W})$ such that $\tilde{\tau} = \tilde{\Phi} \circ \tau$.

Proof. See page 299 in [2]. □

Remark 1.31. The tensor product of more than two spaces can be defined inductively by $\bigotimes_{j=1}^{m} V_j := \left(\bigotimes_{j=1}^{m-1} V_j \right) \bigotimes V_m$. Since we could do the same in the Hilbert space case, see Remark 1.16, it suffices to consider $m = 2$.

Now we compare the notions of Hilbert space tensor product and algebraic tensor product and see if they agree.

Proposition 1.32. Let H_1, H_2 be two $\mathbb{K}$-Hilbert-spaces. Then, their algebraic tensor product is given by $H_1 \otimes' H_2$ as defined in Definition 1.3.

Proof. Consider the mapping

$$\tau : H_1 \times H_2 \to H_1 \otimes' H_2 : (f_1, f_2) \mapsto f_1 \otimes f_2,$$

where $f_1 \otimes f_2$ is defined as in Definition 1.2. This is bilinear, since the scalar products of H_1 and H_2 are linear in their first components.

Now, consider any $\mathbb{K}$-vector space Z with a bilinear mapping

$$\tau' : H_1 \times H_2 \to Z.$$

Define $\Phi : H_1 \otimes' H_2 \to Z$ by setting $\Phi(f_1 \otimes f_2) = \tau'(f_1, f_2)$ and extending linearly. This is well defined, as seen in Proposition 2.6.6 in [6].

By definition, Φ is linear and $\tau' = \Phi \circ \tau$ holds. Further, Φ is uniquely determined: Any mapping $\Phi' : H_1 \otimes' H_2 \to Z$ satisfying $\tau' = \Phi' \circ \tau$ must satisfy $\Phi'(f_1 \otimes f_2) = \tau'(f_1, f_2) = \Phi(f_1 \otimes f_2)$. If we require Φ' to be linear, we get $\Phi = \Phi'$ on $\operatorname{span}\{ f_1 \otimes f_2 \mid f_1 \in H_1, f_2 \in H_2 \} = H_1 \otimes' H_2$. □

This proof does not work if we consider $H_1 \bigotimes H_2$ instead of $H_1 \otimes' H_2$, since we cannot guarantee the uniqueness of Φ anymore. This might seem unsatisfying at first, because the algebraic tensor product and the Hilbert space tensor product are not the same vector space in general. However, we have already seen that there is only one way to equip the algebraic tensor product $H_1 \otimes' H_2$ with a scalar product satisfying the cross-norm property. This means we can not have it both ways: Either the notions of algebraic tensor product and Hilbert space tensor product do not coincide or the Hilbert space tensor product is not, in general, a Hilbert space. However, Theorem 1.14 shows that the algebraic tensor product lies everywhere dense in the Hilbert space tensor product.

2 The Infinite Tensor Product

The goal of this section is to generalize the results of Section 1 to countably infinite families of Hilbert spaces. This is rarely studied in literature, and we rely mostly on results from Sections 3 and 4 of [11].

We assume that a sequence of Hilbert spaces $(H_j)_{j\in\mathbb{N}}$ is given such that each Hilbert space H_j is given over the same $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Additionally, to avoid considering many special cases, we assume that

$$H_j \neq \{0\}$$

holds for all $j \in \mathbb{N}$, which is justified in Remark 2.6. As in Section 1, the tensor product $\bigotimes_{j\in\mathbb{N}} H_j$ should be a Hilbert space. We will have to restrict the existence of elementary tensors, but for suitable sequences $(f_j)_{j\in\mathbb{N}} \in \times_{j\in\mathbb{N}} H_j$, we still want elementary tensors $\otimes_{j\in\mathbb{N}} f_j$ to exist that fulfill

$$\|\otimes_{j\in\mathbb{N}} f_j\|_{\bigotimes_{j\in\mathbb{N}} H_j} = \prod_{j\in\mathbb{N}} \|f_j\|_{H_j}. \tag{2}$$

2.1 Additional Challenges that Arise in the Infinite Case

Notions of Convergence for Infinite Products of Complex Numbers

It is not immediately clear how Equation (2) is defined. The intuitive way to define the convergence of infinite products is as follows.

Definition 2.1. Let $(z_j)_{j\in\mathbb{N}}$ be a sequence of complex numbers. If the sequence of partial products $\left(\prod_{j=1}^{n} z_j\right)_{n\in\mathbb{N}}$ converges, the infinite product $\prod_{j\in\mathbb{N}} z_j$ is called *convergent* and its *value* is defined by

$$\prod_{j\in\mathbb{N}} z_j := \lim_{n\to\infty} \prod_{j=1}^{n} z_j.$$

Convergence criteria that are needed here can be found in Appendix A.

Remark 2.2. In literature, see for example Chapter 7 in [5], the convergence of infinite products is often defined in a stricter sense than in Definition 2.1: The infinite product $\prod_{j\in\mathbb{N}} z_j$ *converges in the stricter sense* if and only if there is a $j_0 \in \mathbb{N}$ such that $z_j \neq 0$ holds for all $j > j_0$ and the infinite product $\prod_{j=j_0+1}^{\infty} z_j$ converges (in the sense of Definition 2.1) to a value other than 0. The *value* is then just the same as the value according to Definition 2.1. Convergence in the stricter sense has some practical benefits. For example, it is not affected by replacing finitely many z_j. Further, the value of an infinite product that converges in the stricter sense is 0 if and only if one of the factors z_j is 0.

Since this notion of convergence is not explicitly used in [11], we will not use it here either. We will, however, remark if it arises naturally at some point.

In particular, the way $\mathcal{C}_0$-sequences and their equivalence are defined, see Definition 2.10 and Definition 2.12, is related to this. Later, when we apply the notion of tensor products to reproducing kernel Hilbert spaces in Section 4, convergence in the stricter sense is more important.

The notion of convergence given in Definition 2.1 (or that given in Remark 2.2) is not sufficient in our case. In Definition 1.2, we defined elementary tensors via (finite) products of scalar products. In the infinite case, this becomes a problem, even if we restrict the sequences that give rise to elementary tensors, as Example 2.3 shows.

Example 2.3. For each $j \in \mathbb{N}$, set $H_j = \mathbb{C}$ equipped with the standard scalar product and define $f_j = i, g_j = -1$. Clearly, $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j}$ and $\prod_{j\in\mathbb{N}} \|g_j\|_{H_j}$ are convergent and their value is 1. However, we have $\langle f_j, g_j\rangle_{H_j} = (-i)(-1) = i$ for each $j \in \mathbb{N}$. Thus, we have $\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j} = \prod_{j\in\mathbb{N}} i$, which is clearly not convergent, as it has the four limit points $1, -1, i, -i$.

To circumvent this, we define the notion of quasi-convergence in the following way.

Definition 2.4. Let $(z_j)_{j\in\mathbb{N}}$ be a sequence of complex numbers. The infinite product $\prod_{j\in\mathbb{N}} z_j$ is called *quasi-convergent* if and only if $\prod_{j\in\mathbb{N}} |z_j|$ is convergent. The *value* of a quasi-convergent product $\prod_{j\in\mathbb{N}} z_j$ is defined as in Definition 2.1 if it is convergent and 0 otherwise.

By Lemma A.4, convergence implies quasi-convergence. Example 2.3 shows that the reverse is not true. However, we do have the following important connection.

Lemma 2.5. Let $(f_j)_{j\in\mathbb{N}}, (g_j)_{j\in\mathbb{N}} \in \times_{j\in\mathbb{N}} H_j$. If $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j}$ and $\prod_{j\in\mathbb{N}} \|g_j\|_{H_j}$ are convergent, $\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}$ is quasi-convergent.

Proof. First, we observe that $\prod_{j\in\mathbb{N}} \|f_j\|^2_{H_j}$ and $\prod_{j\in\mathbb{N}} \|g_j\|^2_{H_j}$ also converge.

Next, if there is a $j \in \mathbb{N}$ such that $\|f_j\|_{H_j} = 0$ or $\|g_j\|_{H_j} = 0$ holds, we have $\langle f_j, g_j\rangle_{H_j} = 0$ and thus, $\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}$ even converges towards 0.

If $\|f_j\|_{H_j} \neq 0$ and $\|g_j\|_{H_j} \neq 0$ hold for all $j \in \mathbb{N}$, Lemma A.1 implies the convergence of

$$\sum_{j\in\mathbb{N}} \max\left(\|f_j\|^2_{H_j} - 1, 0\right) \text{ and } \sum_{j\in\mathbb{N}} \max\left(\|g_j\|^2_{H_j} - 1, 0\right).$$

Now, take any $j \in \mathbb{N}$. By the Cauchy-Schwarz inequality and the binomial theorem, we get

$$\left|\langle f_j, g_j\rangle_{H_j}\right| \leq \|f_j\|_{H_j} \|g_j\|_{H_j} \leq \frac{1}{2}\|f_j\|^2_{H_j} + \frac{1}{2}\|g_j\|^2_{H_j}.$$

This implies

$$\begin{aligned}
&\left|\langle f_j, g_j\rangle_{H_j}\right| - 1 \leq \frac{1}{2}\left\|f_j\right\|_{H_j}^2 + \frac{1}{2}\left\|g_j\right\|_{H_j}^2 \\
&= \frac{1}{2}\left(\left\|f_j\right\|_{H_j}^2 - 1\right) + \frac{1}{2}\left(\left\|g_j\right\|_{H_j}^2 - 1\right) \\
&\leq \frac{1}{2}\max\left(\left\|f_j\right\|_{H_j}^2 - 1, 0\right) + \frac{1}{2}\max\left(\left\|g_j\right\|_{H_j}^2 - 1, 0\right),
\end{aligned}$$

and, since the right-hand side is trivially nonnegative, this in turn implies

$$\max\left(\left|\langle f_j, g_j\rangle_{H_j}\right| - 1, 0\right) \leq \frac{1}{2}\max\left(\left\|f_j\right\|_{H_j}^2 - 1, 0\right) + \frac{1}{2}\max\left(\left\|g_j\right\|_{H_j}^2 - 1, 0\right).$$

So, the series $\sum_{j\in\mathbb{N}} \max\left(\left|\langle f_j, g_j\rangle_{H_j}\right| - 1, 0\right)$ converges, which, by Lemma A.1 implies the convergence of $\prod_{j\in\mathbb{N}} \left|\langle f_j, g_j\rangle_{H_j}\right|$. □

Notions of Infinite Tensor Products

We will study two ways to define the tensor product of infinitely many spaces.

1. A starting point to define the *complete tensor product* is to define the set

$$\mathcal{C} := \left\{ \left(f_j\right)_{j\in\mathbb{N}} \in \times_{j\in\mathbb{N}} H_j \;\middle|\; \prod_{j\in\mathbb{N}} \left\|f_j\right\|_{H_j} \text{ converges} \right\}$$

   and to then define elementary tensors for each element of $\left(f_j\right)_{j\in\mathbb{N}} \in \mathcal{C}$ as in the finite case, so by

$$\otimes_{j\in\mathbb{N}} f_j : \mathcal{C} \to \mathbb{K} : \left(g_j\right)_{j\in\mathbb{N}} \mapsto \prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}.$$

   This is the intuitive way of dealing with our now infinite products, but might not be well-defined, see Example 2.3. To circumvent this, we introduce the notion of quasi-convergence, see Definition 2.4. We will first study the structure of $\mathcal{C}$ in Subsection 2.2, which will allow us to construct the complete tensor product $\bigotimes_{j\in\mathbb{N}} H_j$ analogously to the finite tensor product in Subsection 2.3.

2. *Incomplete tensor products* are obtained as subspaces of the complete tensor product. However, a more concrete characterization is given in the following way, see also Remark 2.42. We fix, for each $j \in \mathbb{N}$, an element $e_j \in H_j$ with $\left\|e_j\right\|_{H_j} = 1$. Then, we can form the set

$$\mathcal{C}_e := \left\{ \left(f_j\right)_{j\in\mathbb{N}} \in \times_{j\in\mathbb{N}} H_j \;\middle|\; f_j \neq e_j \text{ only for finitely many j} \right\}$$

and define, for any $(f_j)_{j\in\mathbb{N}} \in \mathcal{C}_e$, an elementary tensor

$$\otimes_{j\in\mathbb{N}} f_j : \mathcal{C}_e \to \mathbb{K} : (g_j)_{j\in\mathbb{N}} \mapsto \prod_{j\in\mathbb{N}} \langle f_j, g_j \rangle_{H_j} .$$

This approach has the benefit that all infinite products only have finitely many factors that are not equal to 1, and thus quasi-convergence or even convergence do not need to be considered. It does, however, require the additional information about the e_j and is not uniquely determined. We will see the connection between the structure of $\mathcal{C}$ and different incomplete tensor products in Subsection 2.4.

Remark 2.6. As mentioned above, we assume $H_j \neq \{ 0 \}$ holds for all $j \in \mathbb{N}$. This is justified in the following way: If we had $H_j = \{ 0 \}$ for any $j \in \mathbb{N}$, any elementary tensor fulfilling the cross-norm property necessarily has to be 0. So, if any elementary tensor is 0, and if we proceed in any way resembling the finite case, we will obtain

$$\bigotimes_{j\in\mathbb{N}} H_j = \{ 0 \} ,$$

which is also what happens in the finite case. So, to avoid many special cases, we just define the complete *and* any incomplete tensor product to be the trivial Hilbert space $\{ 0 \}$, if any $H_j = \{ 0 \}$. This allows us to assume $H_j \neq \{ 0 \}$ for the remainder of this section.

2.2 The Set $\mathcal{C}$ and its Structure

First, we define the set of sequences that will give rise to elementary tensors.

Definition 2.7. We define

$$\mathcal{C} := \left\{ (f_j)_{j\in\mathbb{N}} \in \times_{j\in\mathbb{N}} H_j \;\middle|\; \prod_{j\in\mathbb{N}} \|f_j\|_{H_j} \text{ converges} \right\} .$$

Elements of $\mathcal{C}$ are also called $\mathcal{C}$-sequences.

Any $\mathcal{C}$-sequence gives rise to an elementary tensor in the following way, which is well defined by Lemma 2.5.

Definition 2.8. For any $\mathcal{C}$-sequence $(f_j)_{j\in\mathbb{N}}$, we define the mapping

$$\otimes_{j\in\mathbb{N}} f_j : \mathcal{C} \to \mathbb{K} : (g_j)_{j\in\mathbb{N}} \mapsto \prod_{j\in\mathbb{N}} \langle f_j, g_j \rangle_{H_j} .$$

These mappings are called *elementary tensors*.

For now, we only need the following property of elementary tensors.

Lemma 2.9. If $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j} = 0$ holds, $\otimes_{j\in\mathbb{N}} f_j = 0$ holds as well.

Proof. If $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j} = 0$ holds, for any $\mathcal{C}$-sequence $(g_j)_{j\in\mathbb{N}}$, we have

$$\prod_{j\in\mathbb{N}} \|f_j\|_{H_j} \|g_j\|_{H_j} = \left(\prod_{j\in\mathbb{N}} \|f_j\|_{H_j}\right)\left(\prod_{j\in\mathbb{N}} \|g_j\|_{H_j}\right) = 0,$$

and since $\left|\langle f_j, g_j\rangle_{H_j}\right| \leq \|f_j\|_{H_j} \|g_j\|_{H_j}$ holds for every $j \in \mathbb{N}$, this implies $\prod_{j\in\mathbb{N}} \left|\langle f_j, g_j\rangle_{H_j}\right| = 0$. Now, employing Lemma A.4, we get

$$\otimes_{j\in\mathbb{N}} f_j \left((g_j)_{j\in\mathbb{N}}\right) = \prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j} = 0.$$

This is what we wanted to show. □

As in the finite case, our goal is to define a scalar product on the linear span of the set of all elementary tensors and, as in the finite case, there is only one possibility to do this because we require the cross-norm property. However, this is not straightforward, because quasi-convergent products do not necessarily fulfill even basic properties of limits. So, before we actually construct the tensor product in Subsection 2.3, we will examine the set $\mathcal{C}$ to circumvent this. First, we drop down to a subset, eliminating some uninteresting sequences.

Definition 2.10. We define

$$\mathcal{C}_0 := \left\{ (f_j)_{j\in\mathbb{N}} \in \times_{j\in\mathbb{N}} H_j \;\middle|\; \sum_{j\in\mathbb{N}} \left| \|f_j\|_{H_j} - 1 \right| < \infty \right\}.$$

Elements of $\mathcal{C}_0$ are also called $\mathcal{C}_0$-sequences.

Remark A.3 implies that a sequence f is a $\mathcal{C}_0$-sequence if and only if the product $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j}$ converges in the stricter sense, as defined in Remark 2.2.

The sets $\mathcal{C}$ and $\mathcal{C}_0$ relate in the following way.

Lemma 2.11.

1. We have $\mathcal{C}_0 \subseteq \mathcal{C}$.
2. For every $(f_j)_{j\in\mathbb{N}} \in \mathcal{C} \setminus \mathcal{C}_0$, we have $\otimes_{j\in\mathbb{N}} f_j = 0$.

Proof.

1. Let $(f_j)_{j\in\mathbb{N}}$ be a $\mathcal{C}_0$-sequence. If $\|f_j\|_{H_j} = 0$ for some $j \in \mathbb{N}$, we have $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j} = 0$ in the sense of convergence. If $\|f_j\|_{H_j} \neq 0$ holds for all $j \in \mathbb{N}$, the convergence of $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j}$ follows by Lemma A.2. Either way, $(f_j)_{j\in\mathbb{N}}$ is a $\mathcal{C}$-sequence.

2. If $(f_j)_{j\in\mathbb{N}}$ is a $\mathcal{C}$-sequence, $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j}$ converges. Now, if $\otimes_{j\in\mathbb{N}} f_j \neq 0$, Lemma 2.9 implies $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j} \neq 0$, and in that case Lemma A.4 implies $\sum_{j\in\mathbb{N}} \left| \|f_j\|_{H_j} - 1 \right| < \infty$, which means that $(f_j)_{j\in\mathbb{N}}$ is a $\mathcal{C}_0$-sequence.

□

Lemma 2.11 shows that all nontrivial elementary tensors arise from $\mathcal{C}_0$-sequences.

Our next goal is to establish an equivalence relation on $\mathcal{C}_0$. Although the Definition will seem technical at first, the equivalence relation itself will be an important tool to construct the complete tensor product as well as the incomplete tensor product. The key to understanding this equivalence lies primarily in Lemma 2.17. When we eventually define the scalar product, we will see that two tensors given by $\mathcal{C}_0$-sequences from different equivalence classes are always orthogonal towards each other. If they are instead given by $\mathcal{C}_0$-sequences from the same equivalence class, it is often easy to retreat to the finite case via simple limit laws. In Subsection 2.4, these equivalence classes will also give rise to the different incomplete tensor products.

Definition 2.12. We define the relation

$$R = \left\{ (f,g) \in \mathcal{C}_0 \times \mathcal{C}_0 \;\middle|\; \sum_{j\in\mathbb{N}} \left| \langle f_j, g_j \rangle_{H_j} - 1 \right| \text{converges} \right\}$$

and call two $\mathcal{C}_0$-sequences f, g equivalent if and only if $(f,g) \in R$ holds.

By Remark A.3, we have $(f,g) \in R$ if and only if the product $\prod_{j\in\mathbb{N}} \langle f_j, g_j \rangle_{H_j}$ converges in the stricter sense, as defined in Remark 2.2.

Before we can work with this, we have to show that the relation established in Definition 2.12 is in fact an equivalence relation. First, we show two technical lemmas.

Lemma 2.13. Let f be a $\mathcal{C}_0$-sequence. Then, $\sum_{j\in\mathbb{N}} \left| \|f_j\|_{H_j}^2 - 1 \right|$ converges.

Proof. Since f is a $\mathcal{C}_0$-sequence, there exists a constant $B > 0$ such that $\left| \|f_j\|_{H_j} - 1 \right| \leq B$ holds for all $j \in \mathbb{N}$. This implies $\left| \|f_j\|_{H_j} + 1 \right| \leq B + 2$. With this, we obtain

$$\begin{aligned}
\sum_{j\in\mathbb{N}} \left| \|f_j\|_{H_j}^2 - 1 \right| &= \sum_{j\in\mathbb{N}} \left| \|f_j\|_{H_j} + 1 \right| \left| \|f_j\|_{H_j} - 1 \right| \\
&\leq (B+2) \sum_{j\in\mathbb{N}} \left| \|f_j\|_{H_j} - 1 \right| \\
&< \infty.
\end{aligned}$$

□

Lemma 2.14. Let $x \geq \frac{1}{2}$. Then,

$$\left|\frac{1}{x} - 1\right| \leq 2\left|x - 1\right|.$$

holds.

Proof. We distinguish two cases.

If $x \geq 1$ holds, we can easily see that $x + \frac{1}{x} \geq 2$ also holds. This implies $1 - \frac{1}{x} \leq x - 1$ which, in this case, is equivalent to

$$\left|\frac{1}{x} - 1\right| \leq \left|x - 1\right|.$$

If $\frac{1}{2} \leq x < 1$ holds, we can easily see that $2x + \frac{1}{x} \leq 3$ also holds. This implies $\frac{1}{x} - 1 \leq 2(1 - x)$ which, in this case, is equivalent to

$$\left|\frac{1}{x} - 1\right| \leq 2\left|x - 1\right|.$$

This finishes the proof. □

Proposition 2.15. The relation established in Definition 2.12 is an equivalence relation.

Proof. Lemma 2.13 directly implies the reflexivity of the relation.

That the relation is symmetric is obvious.

The only thing left to show is that the relation is transitive. So, let $\mathcal{C}_0$-sequences $f = (f_j)_{j\in\mathbb{N}}$ as well as $g = (g_j)_{j\in\mathbb{N}}$ and $h = (h_j)_{j\in\mathbb{N}}$ be given in such a way that f and g are equivalent and that g and h are equivalent. We need to show that f is equivalent to h.

First, we observe that there is some constant $C > 0$ such that for all $j \in \mathbb{N}$, C is an upper bound of $\|f_j\|_{H_j}$, of $\|h_j\|_{H_j}$, of $\left|\langle f_j, g_j\rangle_{H_j}\right|$ and of $\left|\langle g_j, h_j\rangle_{H_j}\right|$. This is true since any convergent sequence is bounded.

To show the convergence of $\sum_{j\in\mathbb{N}} \left|\langle f_j, h_j\rangle_{H_j} - 1\right|$, we show the existence of a sequence $(b_j)_{j\in\mathbb{N}}$ such that $\sum_{j\in\mathbb{N}} b_j$ converges and such that $\left|\langle f_j, h_j\rangle_{H_j} - 1\right| \leq b_j$ holds always except for finitely many $j \in \mathbb{N}$.

Since $\left|\|g_j\|_{H_j} - 1\right| > \frac{1}{4}$ and $\|g_j\|^2_{H_j} < \frac{1}{4}$ hold only for finitely many $j \in \mathbb{N}$, we may assume that

$$\left|\|g_j\|_{H_j} - 1\right| \leq \frac{1}{4} \leq \|g_j\|^2_{H_j}$$

holds for all $j \in \mathbb{N}$. This also implies $\frac{1}{2} \leq \|g_j\|_{H_j}$.

Now, we fix an arbitrary $j \in \mathbb{N}$. In H_j, there is an orthonormal system $(\varphi_1^{(j)}, \varphi_2^{(j)}, \varphi_3^{(j)})$ such that

$$\begin{aligned}
g_j &= a_{1,1}^{(j)}\varphi_1^{(j)} \\
f_j &= a_{2,1}^{(j)}\varphi_1^{(j)} + a_{2,2}^{(j)}\varphi_2^{(j)} \\
h_j &= a_{3,1}^{(j)}\varphi_1^{(j)} + a_{3,2}^{(j)}\varphi_2^{(j)} + a_{3,3}^{(j)}\varphi_3^{(j)}
\end{aligned}$$

hold for suitable constants $a_{i,k}^{(j)}$. Note that $a_{1,1} \neq 0$, since $g_j \neq 0$ by assumption. Using these representations, we obtain

$$\begin{aligned}
\|g_j\|_{H_j}^2 &= \left|a_{1,1}^{(j)}\right|^2 \\
\|f_j\|_{H_j}^2 &= \left|a_{2,1}^{(j)}\right|^2 + \left|a_{2,2}^{(j)}\right|^2 \\
\|h_j\|_{H_j}^2 &= \left|a_{3,1}^{(j)}\right|^2 + \left|a_{3,2}^{(j)}\right|^2 + \left|a_{3,3}^{(j)}\right|^2 \\
\langle f_j, g_j\rangle_{H_j} &= a_{2,1}^{(j)}\overline{a_{1,1}^{(j)}} \\
\langle g_j, h_j\rangle_{H_j} &= a_{1,1}^{(j)}\overline{a_{3,1}^{(j)}} \text{ and} \\
\langle f_j, h_j\rangle_{H_j} &= a_{2,1}^{(j)}\overline{a_{3,1}^{(j)}} + a_{2,2}^{(j)}\overline{a_{3,2}^{(j)}}.
\end{aligned}$$

Now, we can begin estimating. We have

$$\begin{aligned}
&\left|\langle f_j, h_j\rangle_{H_j} - 1\right| = \left|a_{2,1}^{(j)}\overline{a_{3,1}^{(j)}} + a_{2,2}^{(j)}\overline{a_{3,2}^{(j)}} - 1\right| \\
\leq &\left|a_{2,1}^{(j)}\overline{a_{3,1}^{(j)}} - 1\right| + \left|a_{2,2}^{(j)}\overline{a_{3,2}^{(j)}}\right| \\
= &\left|a_{2,1}^{(j)}\overline{a_{1,1}^{(j)}}a_{1,1}^{(j)}\overline{a_{3,1}^{(j)}}\left|a_{1,1}^{(j)}\right|^{-2} - 1\right| + \left|a_{2,2}^{(j)}\overline{a_{3,2}^{(j)}}\right| \\
= &\left|\langle f_j, g_j\rangle_{H_j}\langle g_j, h_j\rangle_{H_j}\|g_j\|_{H_j}^{-2} - 1\right| + \left|a_{2,2}^{(j)}\overline{a_{3,2}^{(j)}}\right|.
\end{aligned}$$

We now consider both summands separately.

1. With the help of what we already established and Lemma 2.14, we obtain

$$
\begin{aligned}
&\left|\langle f_j, g_j\rangle_{H_j} \langle g_j, h_j\rangle_{H_j} \|g_j\|_{H_j}^{-2} - 1\right| \\
&= |\, \langle f_j, g_j\rangle_{H_j} - 1 + \langle f_j, g_j\rangle_{H_j} \langle g_j, h_j\rangle_{H_j} - \langle f_j, g_j\rangle_{H_j} \\
&\quad + \langle f_j, g_j\rangle_{H_j} \langle g_j, h_j\rangle_{H_j} \|g_j\|_{H_j}^{-2} - \langle f_j, g_j\rangle_{H_j} \langle g_j, h_j\rangle_{H_j} \,| \\
&\leq \left|\langle f_j, g_j\rangle_{H_j} - 1\right| + \left|\langle f_j, g_j\rangle_{H_j}\right| \left|\langle g_j, h_j\rangle_{H_j} - 1\right| \\
&\quad + \left|\langle f_j, g_j\rangle_{H_j} \langle g_j, h_j\rangle_{H_j}\right| \left|\|g_j\|_{H_j}^{-2} - 1\right| \\
&\leq \left|\langle f_j, g_j\rangle_{H_j} - 1\right| + C\left|\langle g_j, h_j\rangle_{H_j} - 1\right| + 2C^2 \left|\|g_j\|_{H_j}^{2} - 1\right| \\
&=: c_j.
\end{aligned}
$$

Since f and g as well as g and h are equivalent and by Lemma 2.13, the series $\sum_{j\in\mathbb{N}} c_j$ converges.

2. First, we estimate

$$
\left|a_{2,2}^{(j)}\overline{a_{3,2}^{(j)}}\right| \leq \max\left(\left|a_{2,2}^{(j)}\right|^2, \left|a_{3,2}^{(j)}\right|^2\right) \leq \left|a_{2,2}^{(j)}\right|^2 + \left|a_{3,2}^{(j)}\right|^2
$$

and again consider both summands separately.

(a) We have

$$
\begin{aligned}
&\left|a_{2,2}^{(j)}\right|^2 = \left|a_{2,1}^{(j)}\right|^2 + \left|a_{2,2}^{(j)}\right|^2 - \frac{\left|a_{2,1}^{(j)}\overline{a_{1,1}^{(j)}}\right|^2}{\left|a_{1,1}^{(j)}\right|^2} \\
&= \|f_j\|_{H_j}^2 - \left|\langle f_j, g_j\rangle_{H_j}\right|^2 \|g_j\|_{H_j}^{-2} \\
&= \left(\|f_j\|_{H_j} + \left|\langle f_j, g_j\rangle_{H_j}\right| \|g_j\|_{H_j}^{-1}\right)\left(\|f_j\|_{H_j} - \left|\langle f_j, g_j\rangle_{H_j}\right| \|g_j\|_{H_j}^{-1}\right) \\
&\leq (C + 2C)\left(\|f_j\|_{H_j} - \left|\langle f_j, g_j\rangle_{H_j}\right| \|g_j\|_{H_j}^{-1}\right) \\
&\leq 3C\left(\left|\|f_j\|_{H_j} - 1\right| + \left|1 - \left|\langle f_j, g_j\rangle_{H_j}\right|\right| + \left|\left|\langle f_j, g_j\rangle_{H_j}\right| - \left|\langle f_j, g_j\rangle_{H_j}\right| \|g_j\|_{H_j}^{-1}\right|\right) \\
&\leq 3C\left(\left|\|f_j\|_{H_j} - 1\right| + \left|1 - \langle f_j, g_j\rangle_{H_j}\right| + 2C\left|1 - \|g_j\|_{H_j}\right|\right) \\
&=: c_j'
\end{aligned}
$$

The series $\sum_{j\in\mathbb{N}} c_j'$ then converges.

(b) We have

$$\left|a_{3,2}^{(j)}\right|^2 \leq \left|a_{3,1}^{(j)}\right|^2 + \left|a_{3,2}^{(j)}\right|^2 + \left|a_{3,3}^{(j)}\right|^2 - \frac{\left|a_{1,1}^{(j)}\overline{a_{3,1}^{(j)}}\right|^2}{\left|a_{1,1}^{(j)}\right|^2}$$

$$= \|h_j\|_{H_j}^2 - \frac{\left|\langle g_j, h_j\rangle_{H_j}\right|^2}{\|g_j\|_{H_j}^2}$$

and by proceeding as in (a) we get

$$\leq 3C\left(\left|\|h_j\|_{H_j} - 1\right| + \left|1 - \langle h_j, g_j\rangle_{H_j}\right| + 2C\left|1 - \|g_j\|_{H_j}\right|\right)$$
$$=: c_j''$$

The series $\sum_{j\in\mathbb{N}} c_j''$ then converges.

By setting $b_j = c_j + c_j' + c_j''$, we obtain a convergent series $\sum_{j\in\mathbb{N}} b_j$. Further, $\left|\langle f_j, h_j\rangle_{H_j} - 1\right| \leq b_j$ holds for all $j \in \mathbb{N}$. Thus, we are done. □

Remark 2.16. We only use the equivalence relation from Definition 2.12 to divide $\mathcal{C}_0$ into disjoint sets in a convenient way. We do not use it to identify two elements of the same equivalence class with each other in any way.

The following lemma shows two important aspects of the equivalence relation established in Definition 2.12. These are actually more important than the definition itself.

Lemma 2.17. Let $f = (f_j)_{j\in\mathbb{N}}$ and $g = (g_j)_{j\in\mathbb{N}}$ be two $\mathcal{C}_0$-sequences.

1. If f and g belong to different equivalence classes, we have

$$\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j} = 0$$

either in the sense of convergence or in the sense of quasi-convergence.

2. If f and g belong to the same equivalence class,

$$\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j} = 0$$

holds if and only if $\langle f_j, g_j\rangle_{H_j} = 0$ for some $j \in \mathbb{N}$. In particular, the product $\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}$ is convergent and not just quasi-convergent in this case.

Proof.

1. In this case, $\sum_{j\in\mathbb{N}} \left|\langle f_j, g_h\rangle_{H_j} - 1\right|$ does not converge. Lemma A.2 then implies that, even if $\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}$ converges, its value must be 0.

2. In this case, $\sum_{j\in\mathbb{N}} \left|\langle f_j, g_h\rangle_{H_j} - 1\right|$ converges. If $\langle f_j, g_j\rangle_{H_j} \neq 0$ holds for all $j \in \mathbb{N}$, Lemma A.2 implies the convergence of $\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}$ to a value other than 0. On the other hand, if $\langle f_j, g_j\rangle_{H_j} = 0$ for some $j \in \mathbb{N}$, we obviously have $\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j} = 0$ in the sense of convergence.

□

With this, we can construct the complete tensor product, which we will do in Subsection 2.3.

The remainder of this subsection is devoted to giving some technical lemmas regarding the equivalence classes. These are needed to construct and study the incomplete tensor product and its relation to the complete tensor product.

Lemma 2.19 gives a useful criterion for the equivalence of $\mathcal{C}_0$-sequences. To prove it, we prove Lemma 2.18 first.

Lemma 2.18. Let $f = (f_j)_{j\in\mathbb{N}}$ and $g = (g_j)_{j\in\mathbb{N}}$ be two $\mathcal{C}_0$-sequences. Then, f and g are equivalent if and only if the two series $\sum_{j\in\mathbb{N}} \|f_j - g_j\|^2_{H_j}$ and $\sum_{j\in\mathbb{N}} \left|\Im(\langle f_j, g_j\rangle_{H_j})\right|$ converge.

Proof. First, we show that $\sum_{j\in\mathbb{N}} \left|\langle f_j, g_j\rangle_{H_j} - 1\right|$ is convergent if and only if

$$\sum_{j\in\mathbb{N}} \left|\langle f_j, g_j\rangle_{H_j} - \frac{1}{2}\|f_j\|^2_{H_j} - \frac{1}{2}\|g_j\|^2_{H_j}\right|$$

is convergent. This is true because

$$\begin{aligned}
&\left|\langle f_j, g_j\rangle_{H_j} - \frac{1}{2}\|f_j\|^2_{H_j} - \frac{1}{2}\|g_j\|^2_{H_j}\right| \\
=& \left|\langle f_j, g_j\rangle_{H_j} - 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\|f_j\|^2_{H_j} - \frac{1}{2}\|g_j\|^2_{H_j}\right| \\
\leq& \left|\langle f_j, g_j\rangle_{H_j} - 1\right| + \left|\frac{1}{2}\|f_j\|^2_{H_j} - \frac{1}{2}\right| + \left|\frac{1}{2}\|g_j\|^2_{H_j} - \frac{1}{2}\right|
\end{aligned}$$

and

$$\begin{aligned}
&\left|\langle f_j, g_j\rangle_{H_j} - 1\right| \\
=& \left|\langle f_j, g_j\rangle_{H_j} - \frac{1}{2}\|f_j\|^2_{H_j} - \frac{1}{2}\|g_j\|^2_{H_j} + \left(\frac{1}{2}\|f_j\|^2_{H_j} - \frac{1}{2}\right) + \left(\frac{1}{2}\|g_j\|^2_{H_j} - \frac{1}{2}\right)\right| \\
\leq& \left|\langle f_j, g_j\rangle_{H_j} - \frac{1}{2}\|f_j\|^2_{H_j} - \frac{1}{2}\|g_j\|^2_{H_j}\right| + \frac{1}{2}\left|\|f_j\|^2_{H_j} - 1\right| + \frac{1}{2}\left|\|g_j\|^2_{H_j} - 1\right|
\end{aligned}$$

hold. Note that $\sum_{j\in\mathbb{N}} \left| \|f_j\|_{H_j}^2 - 1 \right|$ and $\sum_{j\in\mathbb{N}} \left| \|g_j\|_{H_j}^2 - 1 \right|$ always converge due to reflexivity of the equivalence relation.

Now, for any sequence of complex numbers $(z_j)_{j\in\mathbb{N}}$, the series $\sum_{j\in\mathbb{N}} |z_j|$ converges if and only if both $\sum_{j\in\mathbb{N}} |\Re(z_j)|$ and $\sum_{j\in\mathbb{N}} |\Im(z_j)|$ converge. So, by

$$\begin{aligned} &\Re\left(\langle f_j, g_j \rangle_{H_j} - \frac{1}{2} \|f_j\|_{H_j}^2 - \frac{1}{2} \|g_j\|_{H_j}^2 \right) \\ = &-\frac{1}{2} \left(\|f_j\|_{H_j}^2 - 2\Re\left(\langle f_j, g_j \rangle_{H_j} \right) + \|g_j\|_{H_j}^2 \right) \\ = &-\frac{1}{2} \|f_j - g_j\|_{H_j}^2 \end{aligned}$$

and

$$\begin{aligned} &\Im\left(\langle f_j, g_j \rangle_{H_j} - \frac{1}{2} \|f_j\|_{H_j}^2 - \frac{1}{2} \|g_j\|_{H_j}^2 \right) \\ = &\Im\left(\langle f_j, g_j \rangle_{H_j} \right), \end{aligned}$$

our claim holds. □

Lemma 2.19 gives us an important special case of equivalent $\mathcal{C}_0$-sequences.

Lemma 2.19. Let $f = (f_j)_{j\in\mathbb{N}}$ and $g = (g_j)_{j\in\mathbb{N}}$ be two $\mathcal{C}_0$-sequences. If $f_j \neq g_j$ holds for only finitely many $j \in \mathbb{N}$, f and g are equivalent.

Proof. If $f_j = g_j$, we have $\|f_j - g_j\|_{H_j}^2 = 0$ and $\Im\left(\langle f_j, g_j \rangle_{H_j} \right) = 0$. Thus, if $f_j \neq g_j$ holds only finitely many times, the series $\sum_{j\in\mathbb{N}} \|f_j - g_j\|_{H_j}^2$ and $\sum_{j\in\mathbb{N}} \Im\left(\langle f_j, g_j \rangle_{H_j} \right)$ both only have finitely many non-zero summands and thus converge. By Lemma 2.18, f and g are then equivalent. □

We can now prove Lemma 2.20, which provides us with an important tool when considering incomplete tensor products.

Lemma 2.20. Each equivalence class contains an element $f^0 = \left(f_j^0\right)_{j\in\mathbb{N}}$ such that $\left\|f_j^0\right\|_{H_j} = 1$ holds for all $j \in \mathbb{N}$.

Proof. Let A be an equivalence class. Then, A is nonempty, so there is a $\mathcal{C}_0$-sequence $(g_j)_{j\in\mathbb{N}} \in A$. The definition of a $\mathcal{C}_0$-sequence implies that we have $g_j = 0$ only finitely often. By replacing these with nonzero elements, we obtain a $\mathcal{C}_0$-sequence $(f_j)_{j\in\mathbb{N}}$ that is in A by Lemma 2.19. We also have $f_j \neq 0$ for all $j \in \mathbb{N}$.

By Lemma A.2, we have $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j} \neq 0$, so $\prod_{j\in\mathbb{N}} \frac{1}{\|f_j\|_{H_j}}$ converges. Define $f_j^0 = \frac{1}{\|f_j\|_{H_j}} f_j$. Then, $\left(f_j^0\right)_{j\in\mathbb{N}}$ is clearly a $\mathcal{C}_0$-sequence. Further, we have

$$\sum_{j\in\mathbb{N}} \left| \left\langle f_j^0, f_j \right\rangle_{H_j} - 1 \right| = \sum_{j\in\mathbb{N}} \left| \|f_j\|_{H_j} - 1 \right| < \infty,$$

since $(f_j)_{j\in\mathbb{N}}$ is a $\mathcal{C}_0$-sequence. This implies the equivalence of $(f_j)_{j\in\mathbb{N}}$ and $(f_j^0)_{j\in\mathbb{N}}$ □

Lastly, we give a lemma concerning the multiplicative properties of elementary tensors.

Lemma 2.21. Let $(z_j)_{j\in\mathbb{N}}$ be a sequence of complex numbers such that $\prod_{j\in\mathbb{N}} z_j$ is quasi-convergent, and let $(f_j)_{j\in\mathbb{N}}$ be a $\mathcal{C}$-sequence. Then, whenever

$$\otimes_{j\in\mathbb{N}} (z_j f_j) = \prod_{j\in\mathbb{N}} z_j \cdot \otimes_{j\in\mathbb{N}} f_j$$

does *not* hold, $\prod_{j\in\mathbb{N}} z_j$ is not convergent and $\otimes_{j\in\mathbb{N}} f_j \neq 0$.

Proof. If $\otimes_{j\in\mathbb{N}} (z_j f_j) \neq \prod_{j\in\mathbb{N}} z_j \cdot \otimes_{j\in\mathbb{N}} f_j$, by Definition 2.8, there is a $\mathcal{C}$-sequence $(g_j)_{j\in\mathbb{N}}$ such that

$$\prod_{j\in\mathbb{N}} \langle z_j f_j, g_j\rangle_{H_j} \neq \prod_{j\in\mathbb{N}} z_j \prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}$$

holds. By Lemma A.5, this implies that both $\prod_{j\in\mathbb{N}} z_j$ and $\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}$ do not converge. So, what remains to show is $\otimes_{j\in\mathbb{N}} f_j \neq 0$. Lemma A.4 implies $\prod_{j\in\mathbb{N}} \left|\langle f_j, g_j\rangle_{H_j}\right| \neq 0$. The Cauchy-Schwarz inequality implies $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j} \neq 0$, which proves our claim by Lemma 2.9. □

The Cardinality of the Set of All Equivalence Classes

Finally, one might ask how many equivalence classes there are. This will later be useful to compare the dimension of the complete and the incomplete tensor product. We will just paraphrase the results given in [11] here, as these questions are only of minor interest to us.

In [11], another notion of equivalence is introduced, namely *weak equivalence*.

Definition 2.22. We call two $\mathcal{C}_0$-sequences $(f_j)_{j\in\mathbb{N}}$ and $(g_j)_{j\in\mathbb{N}}$ weakly equivalent if and only if there is a sequence of complex numbers $(z_j)_{j\in\mathbb{N}}$ such that $(z_j f_j)_{j\in\mathbb{N}}$ is a $\mathcal{C}_0$-sequence and $(z_j f_j)_{j\in\mathbb{N}}$ and $(g_j)_{j\in\mathbb{N}}$ are equivalent.

Remark 2.23.

1. That Definition 2.22 actually defines an equivalence relation is proven in Lemma 6.1.2 in [11].

2. By choosing $z_j = 1$ for all $j \in \mathbb{N}$, we see that equivalence implies weak equivalence. Thus, each weak equivalence class is partitioned by certain equivalence classes.

With this, we can give some results on the cardinality of the set of equivalence classes and the set of weak equivalence classes.

Lemma 2.24. Let Γ be the set of all equivalence classes with respect to the equivalence defined in Definition 2.12. Let Γ_w be the set of all equivalence classes with respect to weak equivalence, defined in Definition 2.22.

1. For each $G_w \in \Gamma_w$, there is a bijection between the set

   $$\{\, G \in \Gamma \mid G \subseteq G_w \,\}$$

   and the set $2^{\mathbb{N}}$.

2. If $\dim(H_j) \geq 2$ holds only finitely often, Γ_w contains only one element.

3. If $\dim(H_j) \geq 2$ holds infinitely often, Γ_w is uncountable.

In any case, Γ is uncountable.

Proof. This is Lemma 6.4.1 in [11], reformulated for this special case. □

2.3 The Complete Tensor Product

We are now able to construct the complete tensor product. This will mostly work analogously to the finite tensor product constructed in Section 1.

We have already defined elementary tensors in Definition 2.8. We view these as elements of $\mathbb{K}^{\mathcal{C}}$, which gives us a notion of addition and scalar multiplication.

Definition 2.25. Define the space

$$\otimes'_{j\in\mathbb{N}} H_j := \operatorname{span}\left\{ \otimes_{j\in\mathbb{N}} f_j \;\middle|\; (f_j)_{j\in\mathbb{N}} \in \mathcal{C} \right\}.$$

Remark 2.26.

1. As in the finite case, any element of $\otimes'_{j\in\mathbb{N}} H_j$ is of the form

   $$\Phi = \sum_{k=1}^{m} \otimes_{j\in\mathbb{N}} f_{j,k}$$

   for elementary tensors $\otimes_{j\in\mathbb{N}} f_{j,k}$. See Remark 1.4.

2. Lemma 2.11 implies

   $$\otimes'_{j\in\mathbb{N}} H_j = \operatorname{span}\left\{ \otimes_{j\in\mathbb{N}} f_j \;\middle|\; (f_j)_{j\in\mathbb{N}} \in \mathcal{C}_0 \right\},$$

   so we can in some cases assume that elementary tensors are given by $\mathcal{C}_0$-sequences.

In analogy to the finite case, we define a scalar product on $\otimes'_{j\in\mathbb{N}} H_j$.

Lemma 2.27. For $\Phi, \Psi \in \otimes'_{j\in\mathbb{N}} H_j$ with representations $\Phi = \sum_{k=1}^{m_1} \otimes_{j\in\mathbb{N}} f_{j,k}$ and $\Psi = \sum_{\ell=1}^{m_2} \otimes_{j\in\mathbb{N}} g_{j,\ell}$, the mapping given by

$$\langle \Phi, \Psi \rangle_{\otimes'_{j\in\mathbb{N}} H_j} = \sum_{k=1}^{m_1} \sum_{\ell=1}^{m_2} \prod_{j\in\mathbb{N}} \langle f_{j,k}, g_{j,\ell} \rangle_{H_j} .$$

is well-defined and a scalar product on $\otimes'_{j\in\mathbb{N}} H_j$.

Proof. By Lemma 2.5, each elementary tensor is a well-defined mapping, at least in the sense of quasi-convergence. The proof that $\langle \Phi, \Psi \rangle_{\otimes'_{j\in\mathbb{N}} H_j}$ does not depend on the representations of Φ and Ψ works exactly as in the proof of Lemma 1.7.

Next, take $\Phi, \Psi, \Xi \in \otimes'_{j\in\mathbb{N}} H_j$ with representations $\Phi = \sum_{k=1}^{m_1} \otimes_{j\in\mathbb{N}} f_{j,k}$, $\Psi = \sum_{\ell=1}^{m_2} \otimes_{j\in\mathbb{N}} g_{j,\ell}$, $\Xi = \sum_{i=1}^{m_3} \otimes_{j\in\mathbb{N}} h_{j,i}$ and $a, b \in \mathbb{K}$. As in the finite case, we have

$$a\Phi + b\Psi = \sum_{k=1}^{m_1} (af_{1,k}) \otimes \otimes_{j\in\mathbb{N}_{>1}} f_{j,k} + \sum_{\ell=1}^{m_2} (bg_{1,\ell}) \otimes \otimes_{j\in\mathbb{N}_{>1}} g_{j,\ell}.$$

Now, linearity in the first component follows as in the finite case, since

$$\begin{aligned}
&\langle a\Phi + b\Psi, \Xi \rangle_{\otimes'_{j\in\mathbb{N}} H_j} \\
&= \sum_{k=1}^{m_1} \sum_{i=1}^{m_3} \langle af_{1,k}, h_{1,i} \rangle_{H_1} \prod_{j=2}^{\infty} \langle f_{j,k}, h_{j,i} \rangle_{H_j} + \sum_{\ell=1}^{m_2} \sum_{i=1}^{m_3} \langle bg_{1,\ell}, h_{1,i} \rangle_{H_1} \prod_{j=2}^{\infty} \langle g_{j,\ell}, h_{j,i} \rangle_{H_j} \\
&= a \left(\sum_{k=1}^{m_1} \sum_{i=1}^{m_3} \prod_{j\in\mathbb{N}} \langle f_{j,k}, h_{j,i} \rangle_{H_j} \right) + b \left(\sum_{\ell=1}^{m_2} \sum_{i=1}^{m_3} \prod_{j\in\mathbb{N}} \langle g_{j,\ell}, h_{j,i} \rangle_{H_j} \right) \\
&= a \langle \Phi, \Xi \rangle_{\otimes'_{j\in\mathbb{N}} H_j} + b \langle \Psi, \Xi \rangle_{\otimes'_{j\in\mathbb{N}} H_j} ,
\end{aligned}$$

still holds.

That $\langle \cdot, \cdot \rangle_{\otimes'_{j\in\mathbb{N}} H_j}$ is hermitian is obtained as follows. For any sequence of complex numbers $(z_j)_{j\in\mathbb{N}}$, the convergence of $\prod_{j\in\mathbb{N}} z_j$ is equivalent to the convergence of $\prod_{j\in\mathbb{N}} \overline{z_j}$ by Lemma A.2. The same is obviously true for quasi-convergence. Because of this, $\prod_{j\in\mathbb{N}} \overline{z_j} = \overline{\prod_{j\in\mathbb{N}} z_j}$ always holds; in the case of convergence it holds by basic limit laws and in the case of quasi-convergence without convergence it holds because both sides are equal to 0. So, as in the finite case, we have

$$\begin{aligned}\langle \Phi, \Psi \rangle_{\otimes'_{j\in N} H_j} &= \sum_{k=1}^{m_1}\sum_{\ell=1}^{m_2}\prod_{j\in\mathbb{N}} \langle f_{j,k}, g_{j,\ell}\rangle_{H_j} \\ &= \sum_{k=1}^{m_1}\sum_{\ell=1}^{m_2}\prod_{j\in\mathbb{N}} \overline{\langle g_{j,\ell}, f_{j,k}\rangle_{H_j}} \\ &= \overline{\sum_{k=1}^{m_1}\sum_{\ell=1}^{m_2}\prod_{j\in\mathbb{N}} \langle g_{j,\ell}, f_{j,k}\rangle_{H_j}} \\ &= \overline{\langle \Psi, \Phi\rangle_{\otimes'_{j\in N} H_j}}.\end{aligned}$$

Next, we show that $\langle \Phi, \Phi \rangle_{\otimes'_{j\in\mathbb{N}} H_j} \geq 0$ always holds. We use the representation $\Phi = \sum_{k=1}^{m_1} \otimes_{j\in\mathbb{N}} f_{j,k}$. Because we have already established well-definedness, we can assume that $\otimes_{j\in\mathbb{N}} f_{j,k} \neq 0$ always holds, which implies that each sequence $(f_{j,k})_{j\in\mathbb{N}}$ is a $\mathcal{C}_0$-sequence. First, we assume that all these sequences belong to the same equivalence class. This means that, by Lemma 2.17, for any choice of $k_1, k_2 \in \{ 1, \dots, m_1 \}$, the product $\prod_{j\in\mathbb{N}} \langle f_{j,k_1}, f_{j,k_2}\rangle_{H_j}$ is convergent and not just quasi-convergent. Thus, we can employ simple limit laws and obtain

$$\begin{aligned}\langle \Phi, \Phi \rangle_{\otimes'_{j\in\mathbb{N}} H_j} &= \sum_{k=1}^{m_1}\sum_{\ell=1}^{m_1}\prod_{j\in\mathbb{N}} \langle f_{j,k}, f_{j,\ell}\rangle_{H_j} \\ &= \sum_{k=1}^{m_1}\sum_{\ell=1}^{m_1}\lim_{n\to\infty}\prod_{j=1}^{n} \langle f_{j,k}, f_{j,\ell}\rangle_{H_j} \\ &= \lim_{n\to\infty}\sum_{k=1}^{m_1}\sum_{\ell=1}^{m_1}\prod_{j=1}^{n} \langle f_{j,k}, f_{j,\ell}\rangle_{H_j} \\ &\geq 0.\end{aligned}$$

The last inequality holds because $\sum_{k=1}^{m_1}\sum_{\ell=1}^{m_1}\prod_{j=1}^{n} \langle f_{j,k}, f_{j,\ell}\rangle_{H_j} \geq 0$ always holds, which is true by Lemma 1.7.

In the general case, the sequences $(f_{j,k})_{j\in\mathbb{N}}$ belong to $p \in \mathbb{N}$ different equivalence classes. By reordering and renaming the elementary tensors, we obtain $\Phi = \sum_{i=1}^{p} \Phi_i$, with $\Phi_i = \sum_{k_i=1}^{p_i} \otimes_{j\in\mathbb{N}} f_{j,k_i,i}$ being the sum of elementary tensors given by $\mathcal{C}_0$-sequences of the same equivalence class. By what we already established, we have

$$\langle \Phi_i, \Phi_i \rangle_{\otimes'_{j\in\mathbb{N}} H_j} \geq 0$$

for all $i \in \{ 1, \dots, p \}$. Further, for $i \neq i' \in \{ 1, \dots, p \}$, we have

$$\langle \Phi_i, \Phi_{i'} \rangle_{\otimes'_{j\in\mathbb{N}} H_j} = \sum_{k_i=1}^{p_i}\sum_{k_{i'}=1}^{p_{i'}}\prod_{j\in\mathbb{N}} \langle f_{j,k_i,i}, f_{j,k_{i'},i'}\rangle_{H_j} = 0,$$

because each summand is 0 by Lemma 2.17. Thus, we obtain

$$\langle \Phi, \Phi \rangle_{\otimes'_{j\in\mathbb{N}} H_j} = \sum_{i=1}^{p} \sum_{i'=1}^{p} \langle \Phi_i, \Phi_{i'} \rangle_{\otimes'_{j\in\mathbb{N}} H_j} = \sum_{i=1}^{p} \langle \Phi_i, \Phi_i \rangle_{\otimes'_{j\in\mathbb{N}} H_j} \geq 0,$$

which means that $\langle \cdot, \cdot \rangle_{\otimes'_{j\in\mathbb{N}} H_j}$ is nonnegative definite.

That $\langle \cdot, \cdot \rangle_{\otimes'_{j\in\mathbb{N}} H_j}$ is positive definite follows as in the finite case, see again Lemma 1.7. □

Corollary 2.28. For any $\Phi \in \otimes'_{j\in\mathbb{N}} H_j$ and any elementary tensor $\otimes_{j\in\mathbb{N}} f_j$,

$$\langle \Phi, \otimes_{j\in\mathbb{N}} f_j \rangle_{\otimes'_{j\in\mathbb{N}} H_j} = \Phi\left((f_j)_{j\in\mathbb{N}} \right)$$

holds.

Proof. This follows immediately from the definition of $\langle \cdot, \cdot \rangle_{\otimes'_{j\in\mathbb{N}} H_j}$. □

As in the finite case, $\otimes'_{j\in\mathbb{N}} H_j$ is not necessarily a complete space. Constructing a bigger, complete space works exactly as in the finite case. The proofs of the corresponding Lemmas 1.9 and 1.10 as well as Theorem 1.11 still hold. These proofs do not use the fact that elementary tensors are defined using products, finite or infinite. All we need when we use the mapping-nature of elements of $\otimes'_{j\in\mathbb{N}} H_j$ is given by Corollary 2.28, which is the same as Corollary 1.8 in the finite case. So, without further difficulties, we obtain the following results.

Lemma 2.29. For any Cauchy sequence $(\Phi_n)_{n\in\mathbb{N}}$ in $\otimes'_{j\in\mathbb{N}} H_j$ and given any $\mathcal{C}$-sequence $(f_j)_{j\in\mathbb{N}}$, the pointwise limit $\lim_{n\to\infty} \Phi_n (f_j)_{j\in\mathbb{N}}$ exists.

Proof. See the proof of Lemma 1.9. □

Lemma 2.30. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\otimes'_{j\in\mathbb{N}} H_j$ and define Φ as the pointwise limit

$$\Phi : \bigtimes_{j\in\mathbb{N}} H_j \to \mathbb{K} : (g_j)_{j\in\mathbb{N}} \mapsto \lim_{n\to\infty} \Phi_n \left((g_j)_{j\in\mathbb{N}} \right).$$

Then, for any Cauchy sequence $(\Psi_n)_{n\in\mathbb{N}}$ in $\otimes'_{j\in\mathbb{N}} H_j$, we have

$$\lim_{n\to\infty} \|\Phi_n - \Psi_n\|_{\otimes'_{j\in\mathbb{N}} H_j} = 0$$

if and only if the pointwise limit of $(\Psi_n)_{n\in\mathbb{N}}$ is Φ.

Proof. See the proof of Lemma 1.10. □

Theorem 2.31. Consider the subspace of $\mathbb{K}^{(\times_{j\in\mathbb{N}} H_j)}$ defined by

$$\bigotimes_{j\in\mathbb{N}} H_j := \left\{ \Phi \in \mathbb{K}^{(\times_{j\in\mathbb{N}} H_j)} \;\middle|\; \Phi \text{ pointw. limit of a Cauchy seq. in } \otimes'_{j\in\mathbb{N}} H_j \right\}.$$

For $\Phi, \Psi \in \bigotimes_{j\in\mathbb{N}} H_j$ with approximating sequences $(\Phi_n)_{n\in\mathbb{N}}$ and $(\Psi_n)_{n\in\mathbb{N}}$ in $\otimes'_{j\in\mathbb{N}} H_j$, the mapping given by

$$\langle \Phi, \Psi \rangle_{\bigotimes_{j\in\mathbb{N}} H_j} = \lim_{n\to\infty} \langle \Phi_n, \Psi_n \rangle_{\otimes'_{j\in\mathbb{N}} H_j}$$

is well-defined and a scalar product. Further, $\bigotimes_{j\in\mathbb{N}} H_j$ equipped with this scalar product is a Hilbert space.

Proof. See the proof of Theorem 1.11. □

Definition 2.32. We call the space $\bigotimes_{j\in\mathbb{N}} H_j$ equipped with $\langle \cdot, \cdot \rangle_{\bigotimes_{j\in\mathbb{N}} H_j}$ as established in Theorem 2.31 the *complete tensor product* of the spaces H_j.

Similarly to the finite case, we obtain the following result.

Corollary 2.33. $\otimes'_{j\in\mathbb{N}} H_j$ is dense in $\bigotimes_{j\in\mathbb{N}} H_j$.

Proof. See the proof of Corollary 1.13. □

Similarly to Theorem 1.14 in the finite case, the following theorem allows us to identify $\bigotimes_{j\in\mathbb{N}} H_j$ with certain other spaces.

Theorem 2.34. If a Hilbert space H fulfills the three properties

1. H contains an element $\tilde{\otimes}_{j\in\mathbb{N}} f_j$ for each $(f_j)_{j\in\mathbb{N}} \in \mathcal{C}$,
2. $\left\langle \tilde{\otimes}_{j\in\mathbb{N}} f_j, \tilde{\otimes}_{j\in\mathbb{N}} g_j \right\rangle_H = \prod_{j\in\mathbb{N}} \langle f_j, g_j \rangle_{H_j}$ holds for any two such elements $\tilde{\otimes}_{j\in\mathbb{N}} f_j, \tilde{\otimes}_{j\in\mathbb{N}} g_j$ and
3. $H_0 = \operatorname{span}\left\{ \tilde{\otimes}_{j\in\mathbb{N}} f_j \,\middle|\, (f_j)_{j\in\mathbb{N}} \in \mathcal{C} \right\}$ is dense in H,

there exists a unique isometric isomorphism $\Lambda : H \to \bigotimes_{j\in\mathbb{N}} H_j$ that fulfills

$$\Lambda\left(\tilde{\otimes}_{j\in\mathbb{N}} f_j\right) = \otimes_{j\in\mathbb{N}} f_j$$

for every $\tilde{\otimes}_{j\in\mathbb{N}} f_j \in H$.

Conversely if H is a Hilbert space and an isometric isomorphism

$$\Lambda : H \to \bigotimes_{j\in\mathbb{N}} H_j$$

exists, H fulfills the three properties given above.

Proof. Observe that $\bigotimes_{j\in\mathbb{N}} H_j$ fulfills the properties 1 and 2 by construction and property 3 by Corollary 2.33. Then, the rest of the proof is exactly as in Theorem 1.14. □

2.4 The Incomplete Tensor Product

With the complete tensor product already defined and employing the equivalence relation established in Definition 2.12, the incomplete tensor product simply arises as a special subspace. Note that for this, the complete tensor product can be given as any space fulfilling the three properties from Theorem 2.34.

Definition 2.35. Let A be any equivalence class with respect to the equivalence relation established in Definition 2.12. Then, the *incomplete tensor product* with respect to A is defined as

$$\bigotimes_{j\in\mathbb{N}}{}^{A} H_j := \overline{\operatorname{span}}\left\{ \otimes_{j\in\mathbb{N}} f_j \,\middle|\, (f_j)_{j\in\mathbb{N}} \in A \right\},$$

where the closure is with respect to the metric given by $\|\cdot\|_{\bigotimes_{j\in\mathbb{N}} H_j}$.

It is immediately clear that $\bigotimes_{j\in\mathbb{N}}^{A} H_j$ is a Hilbert space, since it is defined as a closed subspace of a Hilbert space.

Even though we defined $\bigotimes_{j\in\mathbb{N}}^{A} H_j$ independently of any explicit construction of $\bigotimes_{j\in\mathbb{N}} H_j$, we still establish a result akin to Theorem 2.34. This will be useful in examples, particularly in Section 4, as it allows us to construct $\bigotimes_{j\in\mathbb{N}}^{A} H_j$ without first constructing the bigger space $\bigotimes_{j\in\mathbb{N}} H_j$.

Theorem 2.36. Let A be any equivalence class with respect to the equivalence relation established in Definition 2.12. Let $\bigotimes_{j\in\mathbb{N}}^{A} H_j$ be given according to Definition 2.35, so as a subspace of $\bigotimes_{j\in\mathbb{N}} H_j$. Further, let H be any Hilbert space fulfilling the three properties

1. H contains an element $\tilde{\otimes}_{j\in\mathbb{N}} f_j$ for each $\mathcal{C}_0$-sequence $(f_j)_{j\in\mathbb{N}} \in A$,
2. $\left\langle \tilde{\otimes}_{j\in\mathbb{N}} f_j, \tilde{\otimes}_{j\in\mathbb{N}} g_j \right\rangle_H = \prod_{j\in\mathbb{N}} \langle f_j, g_j \rangle_{H_j}$ holds for any two such elements $\tilde{\otimes}_{j\in\mathbb{N}} f_j, \tilde{\otimes}_{j\in\mathbb{N}} g_j$ and
3. $H_0 = \operatorname{span}\left\{ \tilde{\otimes}_{j\in\mathbb{N}} f_j \,\middle|\, (f_j)_{j\in\mathbb{N}} \in A \right\}$ is dense in H.

In this case, there exists a unique isometric isomorphism $\Lambda : H \to \bigotimes_{j\in\mathbb{N}}^{A} H_j$ that fulfills

$$\Lambda\left(\tilde{\otimes}_{j\in\mathbb{N}} f_j\right) = \otimes_{j\in\mathbb{N}} f_j$$

for every $\tilde{\otimes}_{j\in\mathbb{N}} f_j$ according to property 1.

Conversely if H is a Hilbert space and an isometric isomorphism

$$\Lambda : H \to \bigotimes_{j\in\mathbb{N}}{}^{A} H_j$$

exists, H fulfills the three properties given above.

Proof. Note that $\bigotimes_{j\in\mathbb{N}}^{A} H_j$ fulfills properties 1, 2 and 3 by Definition 2.35. The rest of the proof, again, works as in Theorem 1.14. □

Theorem 2.36 permits us to view any space fulfilling the properties 1 to 3 as the incomplete tensor product with respect to A. In particular, it does not need to be a subspace of some complete tensor product given by Theorem 2.34.

The relation between the complete and the incomplete tensor product is as follows.

Proposition 2.37. Let Γ be the set of all equivalence classes with respect to the equivalence relation established in Definition 2.12. If the incomplete tensor product $\bigotimes_{j\in\mathbb{N}}^{A} H_j$ is given as a subspace of $\bigotimes_{j\in\mathbb{N}} H_j$ as in Definition 2.35 for every $A \in \Gamma$, the incomplete tensor products are mutually orthogonal and the closure of their direct sum is $\bigotimes_{j\in\mathbb{N}} H_j$.

Proof. Let $A, B \in \Gamma$ with $A \neq B$. Then, $\bigotimes_{j\in\mathbb{N}}^{A} H_j$ and $\bigotimes_{j\in\mathbb{N}}^{B} H_j$ are orthogonal. Indeed, for any two $\mathcal{C}_0$-sequences $(f_j)_{j\in\mathbb{N}} \in A$ and $(g_j)_{j\in\mathbb{N}} \in B$, we have

$$\langle \otimes_{j\in\mathbb{N}} f_j, \otimes_{j\in\mathbb{N}} g_j \rangle_{\bigotimes_{j\in\mathbb{N}} H_j} = 0$$

by Lemma 2.17. By the definition of the incomplete tensor product as the closure of the span of such tensors, this implies the orthogonality of $\bigotimes_{j\in\mathbb{N}}^{A} H_j$ and $\bigotimes_{j\in\mathbb{N}}^{B} H_j$.

Now, bearing in mind Remark 2.26, we know that $\otimes'_{j\in\mathbb{N}} H_j$ is the linear span of the set of all $\mathcal{C}_0$-sequences. Since every $\mathcal{C}_0$-sequence belongs to an equivalence class, we are done. □

Our next goal is to provide an alternate definition, using elemental tensors $\otimes_{j\in\mathbb{N}} e_j$ that fulfill $\|e_j\|_{H_j} = 1$ for each $j \in \mathbb{N}$. To this end, recall Lemma 2.20, by which $\bigotimes_{j\in\mathbb{N}}^{A} H_j$ always contains such a tensor.

First, we need a technical lemma that will help us with an estimation in Proposition 2.39.

Lemma 2.38. For $n \in \mathbb{N}$ and complex numbers $z_1, \ldots, z_n$, we have

$$\left| \prod_{j=1}^{n} z_j - 1 \right| \leq \exp\left(\sum_{j=1}^{n} |z_j - 1| \right) - 1.$$

Proof. It is well-known that for any $x \in \mathbb{R}$, we have $x + 1 \leq \exp(x)$. For positive real numbers $x_1, \ldots, x_n$ this implies $\prod_{j=1}^{n} \exp(x_j) \geq \prod_{j=1}^{n} (x_j + 1)$. In particular,

$$\prod_{j=1}^{n} (|z_j - 1| + 1) \leq \prod_{j=1}^{n} \exp(|z_j - 1|) = \exp\left(\sum_{j=1}^{n} |z_j - 1| \right).$$

Thus, we are done if we show $\left| \prod_{j=1}^{n} z_j - 1 \right| + 1 \leq \prod_{j=1}^{n} (|z_j - 1| + 1)$. For $n = 1$, this is trivial. For $n = 2$, we calculate

$$\begin{aligned} &|z_1 z_2 - 1| + 1 = |z_1 z_2 - z_1 - z_2 + 1 + z_1 - 1 + z_2 - 1| + 1 \\ &\leq |z_1 z_2 - z_1 - z_2 + 1| + |z_1 - 1| + |z_2 - 1| + 1 \\ &= (|z_1 - 1| + 1)(|z_2 - 1| + 1), \end{aligned}$$

and for $n > 2$, we inductively obtain

$$\prod_{j=1}^{n} |z_j - 1| + 1 \leq \left(\left| \prod_{j=1}^{n-1} z_j - 1 \right| + 1 \right) (|z_n - 1| + 1) \leq \prod_{j=1}^{n} (|z_j - 1| + 1) \,.$$

This finishes the proof. □

Proposition 2.39. Let $e = (e_j)_{j\in\mathbb{N}} \in A$ such that $\|e_j\|_{H_j} = 1$ holds for all $j \in \mathbb{N}$. Define

$$H := \overline{\operatorname{span}} \{ \otimes_{j\in\mathbb{N}} f_j \mid f_j \neq e_j \text{ for only finitely many } j \in \mathbb{N} \} \,.$$

Here, the linear hull and the closure may be taken either in $\bigotimes_{j\in\mathbb{N}} H_j$ or in $\bigotimes_{j\in\mathbb{N}}^A H_j$.

In this case,

$$H = \bigotimes_{j\in\mathbb{N}}{}^{A} H_j$$

holds.

Proof. By Lemma 2.19, any $\mathcal{C}_0$-sequence $f = (f_j)_{j\in\mathbb{N}}$ is equivalent to e if $f_j \neq e_j$ holds only for finitely many $j \in \mathbb{N}$. This implies $H \subseteq \bigotimes_{j\in\mathbb{N}}^A H_j$.

Now, let $f = (f_j)_{j\in\mathbb{N}}$ be any $\mathcal{C}_0$-sequence such that $\otimes_{j\in\mathbb{N}} f_j \in \bigotimes_{j\in\mathbb{N}}^A H_j$ holds. To show $H \supseteq \bigotimes_{j\in\mathbb{N}}^A H_j$, it suffices to show that $\otimes_{j\in\mathbb{N}} f_j$ is also an element of H. This is certainly true, if $\otimes_{j\in\mathbb{N}} f_j = 0$, since H contains 0 by definition. We thus assume $\otimes_{j\in\mathbb{N}} f_j \neq 0$, which implies $\prod_{j\in\mathbb{N}} \|f_j\|_{H_j} \neq 0$ and thus $\|f_j\|_{H_j} \neq 0$ for all $j \in \mathbb{N}$. In this case, we may consider the sequence $\tilde{f} = \left(\|f_j\|_{H_j}^{-1} f_j \right)_{j\in\mathbb{N}}$. This sequence is equivalent to $(f_j)_{j\in\mathbb{N}}$ and thus to e, since

$$\sum_{j\in\mathbb{N}} \left| \left\langle \frac{1}{\|f_j\|_{H_j}} f_j, f_j \right\rangle_{H_j} - 1 \right| = \sum_{j\in\mathbb{N}} \left| \|f_j\|_{H_j} - 1 \right| < \infty$$

holds. If we can show that $\tilde{f} \in H$, we also have shown $f \in H$, since

$$\otimes_{j\in\mathbb{N}} f_j = \left(\prod_{j\in\mathbb{N}} \|f_j\|_{H_j} \right) \cdot \left(\otimes_{j\in\mathbb{N}} \tilde{f}_j \right)$$

holds by Lemma 2.21, and H is a vector space. So, it suffices to show that any tensor $\otimes_{j\in\mathbb{N}} f_j \in \bigotimes_{j\in\mathbb{N}}^A H_j$ with $\|f_j\|_{H_j} = 1$ for all $j \in \mathbb{N}$ is also an element of H.

Let $\otimes_{j\in\mathbb{N}} f_j$ be such a tensor. Our goal now is to find elements of H arbitrarily close to $\otimes_{j\in\mathbb{N}} f_j$. So, let $0 < \varepsilon < 1$ be given. Since f and e are equivalent, the series $\sum_{j\in\mathbb{N}} \left|\langle f_j, e_j\rangle_{H_j} - 1\right|$ converges, and thus, there exists some $n \in \mathbb{N}$ such that

$$\sum_{j=n+1}^{\infty} \left|\langle f_j, e_j\rangle_{H_j} - 1\right| < \varepsilon$$

holds. Now, we define a sequence $g = (g_j)_{j\in\mathbb{N}}$ by

$$g_j = \begin{cases} f_j, & j \leq n \\ e_j, & j > n. \end{cases}$$

Obviously, $\otimes_{j\in\mathbb{N}} g_j \in H$ holds. Further, we have

$$\begin{aligned}
&\|\otimes_{j\in\mathbb{N}} f_j - \otimes_{j\in\mathbb{N}} g_j\|^2_{\bigotimes_{j\in\mathbb{N}} H_j} \\
=\ & \|\otimes_{j\in\mathbb{N}} f_j\|^2_{\bigotimes_{j\in\mathbb{N}} H_j} + \|\otimes_{j\in\mathbb{N}} g_j\|^2_{\bigotimes_{j\in\mathbb{N}} H_j} - 2\Re \langle \otimes_{j\in\mathbb{N}} f_j, \otimes_{j\in\mathbb{N}} g_j\rangle_{\bigotimes_{j\in\mathbb{N}} H_j} \\
=\ & \prod_{j\in\mathbb{N}} \|f_j\|^2_{H_j} + \prod_{j\in\mathbb{N}} \|g_j\|^2_{H_j} - 2\Re\left(\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle_{H_j}\right) \\
=\ & 2 - 2\Re\left(\prod_{j=1}^{n} \langle f_j, f_j\rangle_{H_j} \cdot \prod_{j=n+1}^{\infty} \langle f_j, e_j\rangle_{H_j}\right) \\
=\ & 2 - 2\Re\left(\prod_{j=n+1}^{\infty} \langle f_j, e_j\rangle_{H_j}\right) \\
=\ & 2\Re\left(1 - \prod_{j=n+1}^{\infty} \langle f_j, e_j\rangle_{H_j}\right) \\
\leq\ & 2\left|1 - \prod_{j=n+1}^{\infty} \langle f_j, e_j\rangle_{H_j}\right|.
\end{aligned}$$

To further estimate this, take any $m \in \mathbb{N}$. For this, we use Lemma 2.38 to obtain

$$\begin{aligned}
\left|\prod_{j=n+1}^{n+m} \langle f_j, e_j\rangle_{H_j} - 1\right| &\leq \exp\left(\sum_{j=n+1}^{n+m} \left|\langle f_j, e_j\rangle_{H_j} - 1\right|\right) - 1 \\
&\leq \exp\left(\sum_{j=n+1}^{\infty} \left|\langle f_j, e_j\rangle_{H_j} - 1\right|\right) - 1 \\
&\leq \exp(\varepsilon) - 1.
\end{aligned}$$

Now, the product $\prod_{j\in\mathbb{N}} \langle f_j, e_j\rangle_{H_j}$ is convergent by Lemma 2.17 and thus we have

$$\left|\prod_{n+1}^{\infty} \langle f_j, e_j\rangle_{H_j} - 1\right| = \lim_{m\to\infty} \left|\prod_{n+1}^{n+m} \langle f_j, e_j\rangle_{H_j} - 1\right| \leq \exp(\varepsilon) - 1.$$

All in all, we have shown that for any $0 < \varepsilon < 1$ there is a tensor $\otimes_{j\in\mathbb{N}} g_j \in H$ such that

$$\|\otimes_{j\in\mathbb{N}} f_j - \otimes_{j\in\mathbb{N}} g_j\|^2_{\bigotimes_{j\in\mathbb{N}} H_j} \leq \exp(\varepsilon) - 1$$

holds. Further, as ε tends to 0, so does $\exp(\varepsilon) - 1$. So, there are sequences in H that converge towards $\otimes_{j\in\mathbb{N}} f_j$. Since H is a closed space, this implies $\otimes_{j\in\mathbb{N}} f_j \in H$. □

Remark 2.40. Proposition 2.39 provides an alternative way to define the incomplete tensor product. Any sequence $(e_j)_{j\in\mathbb{N}}$ fulfilling $\|e_j\|_{H_j} = 1$ for all $j \in \mathbb{N}$ is clearly a $\mathcal{C}_0$-sequence and thus belongs to exactly one equivalence class A. We write

$$\bigotimes_{j\in\mathbb{N}}{}^{e} H_j = \overline{\operatorname{span}} \left\{ \otimes_{j\in\mathbb{N}} f_j \mid f_j \neq e_j \text{ for only finitely many } j \in \mathbb{N} \right\},$$

where linear hull and closure are taken with respect to either $\bigotimes^{A}_{j\in\mathbb{N}} H_j$ or $\bigotimes_{j\in\mathbb{N}} H_j$.

By Proposition 2.39, we have

$$\bigotimes_{j\in\mathbb{N}}{}^{e} H_j = \bigotimes_{j\in\mathbb{N}}{}^{A} H_j,$$

and thus it is justified to call $\bigotimes^{e}_{j\in\mathbb{N}} H_j$ the incomplete tensor product with respect to e.

Proposition 2.39 and Remark 2.40 clearly give rise to the following alternative characterization of the incomplete tensor product:

Corollary 2.41. Let A be any equivalence class and let e be any $\mathcal{C}_0$-sequence in A such that $\|e_j\|_{H_j} = 1$ holds for each $j \in \mathbb{N}$. Define

$$\mathcal{C}_e = \left\{ f \in \mathcal{C}_0 \mid f_j \neq e_j \text{ for only finitely many } j \in \mathbb{N} \right\}.$$

Let the incomplete tensor product $\bigotimes^{e}_{j\in\mathbb{N}} H_j = \bigotimes^{A}_{j\in\mathbb{N}} H_j$ be given either according to Definition 2.35 or as any space fulfilling the three properties given in Theorem 2.36. Further, let H be any Hilbert space fulfilling the three properties

1. H contains an element $\tilde{\otimes}_{j\in\mathbb{N}} f_j$ for each sequence $(f_j)_{j\in\mathbb{N}} \in \mathcal{C}_e$,
2. $\left\langle \tilde{\otimes}_{j\in\mathbb{N}} f_j, \tilde{\otimes}_{j\in\mathbb{N}} g_j \right\rangle_H = \prod_{j\in\mathbb{N}} \langle f_j, g_j \rangle_{H_j}$ holds for any two such elements $\tilde{\otimes}_{j\in\mathbb{N}} f_j, \tilde{\otimes}_{j\in\mathbb{N}} g_j$ and
3. $H_0 = \operatorname{span} \left\{ \tilde{\otimes}_{j\in\mathbb{N}} f_j \;\middle|\; (f_j)_{j\in\mathbb{N}} \in \mathcal{C}_e \right\}$ is dense in H.

In this case,there exists a unique isometric isomorphism $\Lambda : H \to \bigotimes^{e}_{j\in\mathbb{N}} H_j$ that fulfills

$$\Lambda\left(\tilde{\otimes}_{j\in\mathbb{N}} f_j\right) = \otimes_{j\in\mathbb{N}} f_j$$

for every $\tilde{\otimes}_{j\in\mathbb{N}} f_j$ according to property 1.

Conversely if H is a Hilbert space and an isometric isomorphism

$$\Lambda : H \to \bigotimes_{j\in\mathbb{N}}^{e} H_j$$

exists, H fulfills the three properties given above.

Proof. By Proposition 2.39 and Remark 2.40, the properties 1, 2 and 3 are fulfilled by $\bigotimes_{j\in\mathbb{N}}^{e} H_j$. The rest of the proof then works as in Theorem 1.14. $\square$

Remark 2.42. Corollary 2.41 allows us to define the incomplete tensor product without ever worrying about the convergence of infinite products or equivalence classes. We give an outline here.

Given any sequence $e \in \times_{j\in\mathbb{N}} H_j$ that fulfills $\|e_j\|_{H_j} = 1$ for each $j \in \mathbb{N}$, we define the set

$$\mathcal{C}_e = \{\, f \in \times_{j\in\mathbb{N}} H_j \mid f_j \neq e_j \text{ for only finitely many } j \in \mathbb{N} \,\}.$$

Note that this coincides with the set $\mathcal{C}_e$ from Corollary 2.41. For any $f \in \mathcal{C}_e$, we define the elementary tensor

$$\otimes_{j\in\mathbb{N}} f_j : \mathcal{C}_e \to \mathbb{K} : g \mapsto \prod_{j\in\mathbb{N}} \langle f_j, g_j \rangle_{H_j}.$$

The product used in this definition only has finitely many factors that are not equal to one, so it always converges and can even be treated as a finite product. From here on, we can proceed analogously to Section 1 and Subsection 2.3, and all proofs work as in the finite case: Using the vector space structure of $\mathbb{K}^{\mathcal{C}_e}$, we can consider

$$\operatorname{span} \{\, \otimes_{j\in\mathbb{N}} f_j \mid f \in \mathcal{C}_e \,\}$$

and define a scalar product in this space analogously to Lemma 1.7. In the resulting unitary space, Cauchy sequences converge pointwise, which allows us to define the incomplete tensor product withe respect to e analogously to Theorem 1.11. The resulting space then clearly fulfills the properties given in Corollary 2.41.

We now aim to construct an orthonormal basis of $\bigotimes_{j\in\mathbb{N}}^{e} H_j$ out of orthonormal bases of the H_j and thus study the link between the dimensions of these spaces. This will lead us to the relation of the finite tensor product and the (infinite) incomplete tensor product.

First, we need notation to replace a single "factor" of an elementary tensor.

Notation 2.43. Let $f = (f_j)_{j\in\mathbb{N}}$ be any $\mathcal{C}$-sequence and let $g \in H_{j_0}$ for some $j_0 \in \mathbb{N}$ that fulfills $f_{j_0} \neq 0$. We can then consider

$$\tilde{f}_j = \begin{cases} g, & j = j_0 \\ f_j, & \text{else,} \end{cases}$$

which defines a $\mathcal{C}$-sequence $\tilde{f}$ where only one member differs from f. For the corresponding elementary tensors, we write

$$g \otimes \left(\otimes_{\substack{j\in\mathbb{N} \\ j\neq j_0}} f_j \right) := \otimes_{j\in\mathbb{N}} \tilde{f}_j.$$

Lemma 2.44. Let $(f_j)_{j\in\mathbb{N}}$ be any $\mathcal{C}$-sequence, and let $j_0 \in \mathbb{N}$ be given such that $f_{j_0} \neq 0$. Then, the mapping

$$\Phi : H_{j_0} \to \bigotimes_{j\in\mathbb{N}} H_j : f \mapsto f \otimes \left(\otimes_{\substack{j\in\mathbb{N} \\ j\neq j_0}} f_j \right)$$

is linear and continuous.

Proof. Linearity follows by the definition of an elementary tensor and because the scalar product $\langle \cdot, \cdot \rangle_{H_{j_0}}$ is linear in its first component.

For continuity, take $f, g \in H_{j_0}$. We have

$$\begin{aligned} &\left\| f \otimes \left(\otimes_{\substack{j\in\mathbb{N} \\ j\neq j_0}} f_j \right) - g \otimes \left(\otimes_{\substack{j\in\mathbb{N} \\ j\neq j_0}} f_j \right) \right\|_{\bigotimes_{j\in\mathbb{N}} H_j} \\ = &\left\| (f-g) \otimes \left(\otimes_{\substack{j\in\mathbb{N} \\ j\neq j_0}} f_j \right) \right\|_{\bigotimes_{j\in\mathbb{N}} H_j} \\ = &\left\| f - g \right\|_{H_{j_0}} \prod_{\substack{j\in\mathbb{N} \\ j\neq j_0}} \left\| f_j \right\|_{H_j}, \end{aligned}$$

which implies continuity. □

Now, if given orthonormal bases of the spaces H_j, we can construct an orthonormal basis of $\bigotimes^e_{j\in\mathbb{N}} H_j$ in the following canonical way.

Proposition 2.45. Let $e = (e_j)_{j\in\mathbb{N}}$ be any $\mathcal{C}_0$-sequence such that $\|e_j\|_{H_j} = 1$ holds for all $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, let an orthonormal basis

$$B_j = \left(f_{j,i_j}\right)_{i_j\in I_j}$$

be given in such a way that $0 \in I_j$ and $e_j = f_{j,0}$. Then, the elements of

$$B = \left\{ \otimes_{j\in\mathbb{N}} f_{j,i_j} \;\middle|\; \left(f_{j,i_j}\right)_{j\in\mathbb{N}} \in \times_{j\in\mathbb{N}} B_j \text{ and } i_j \neq 0 \text{ for only finitely many j} \right\}$$

form an orthonormal basis of $\bigotimes^e_{j\in\mathbb{N}} H_j$.

Proof. First, we have $B \subseteq \bigotimes^e_{j\in\mathbb{N}} H_j$, since $i_j \neq 0$ in the definition of B is another way of saying $f_{j,i_j} \neq e_j$. Now, take $\otimes_{j\in\mathbb{N}} f_{j,i_j}, \otimes_{j\in\mathbb{N}} g_{j,k_j} \in B$. Since B_j is an orthonormal system for each $j \in \mathbb{N}$, we have

$$\left\langle f_{j,i_j}, g_{j,k_j} \right\rangle_{H_j} = \begin{cases} 1, & f_{j,i_j} = g_{j,k_j} \\ 0, & \text{else.} \end{cases}$$

This implies

$$\left\langle \otimes_{j\in\mathbb{N}} f_{j,i_j}, \otimes_{j\in\mathbb{N}} g_{j,k_j} \right\rangle_{\bigotimes_{j\in\mathbb{N}} H_j} = \prod_{j\in\mathbb{N}} \left\langle f_{j,i_j}, g_{j,k_j} \right\rangle_{H_j} = \begin{cases} 1, & \otimes_{j\in\mathbb{N}} f_{j,i_j} = \otimes_{j\in\mathbb{N}} g_{j,k_j} \\ 0, & \text{else,} \end{cases}$$

which implies that B is an orthonormal system. Note that the scalar product does not depend on the specific representation of the elementary tensors.

Now, let $H = \overline{\operatorname{span}}(B)$. We have already remarked that $B \subseteq \bigotimes_{j\in\mathbb{N}}^{e} H_j$ holds, and so $H \subseteq \bigotimes_{j\in\mathbb{N}}^{e} H_j$ must hold as well. If we show $\bigotimes_{j\in\mathbb{N}}^{e} H_j \subseteq H$, we are done. To this end, it suffices to show that for any $f = (f_j)_{j\in\mathbb{N}}$, for which $f_j \neq e_j$ holds only finitely many times, we have $\otimes_{j\in\mathbb{N}} f_j \in H$.

This is proven via induction. For any such f, we have $f_j \notin B_j$ for only finitely many $j \in \mathbb{N}$, since $e_j \in B_j$. So, at first we assume that $f_j \in B_j$ always holds. In this case, we have $\otimes_{j\in\mathbb{N}} f_j \in B \subseteq H$ by definition of B.

Now, assume that for some $n \in \mathbb{N}_0$, we already have that any $f = (f_j)_{j\in\mathbb{N}}$ fulfilling both

$$f_j \neq e_j \text{ for only finitely many } j$$

and

$$f_j \notin B_j \text{ exactly } n \text{ times}$$

also fulfills $\otimes_{j\in\mathbb{N}} f_j \in H$. Take then any $f = (f_j)_{j\in\mathbb{N}}$ that fulfills

$$f_j \neq e_j \text{ for only finitely many } j$$

and

$$f_j \notin B_j \text{ exactly } n+1 \text{ times.}$$

Additionally, take any $k \in \mathbb{N}$ with $f_k \notin B_k$. By assumption, for any $f_{k,i_k} \in B_k$, we have

$$f_{k,i_k} \otimes \left(\otimes_{\substack{j\in\mathbb{N} \\ j\neq k}} f_j \right) \in H.$$

Now, consider the mapping $\Phi : H_k \to \bigotimes_{j\in\mathbb{N}} H_j$ as given in Lemma 2.44. Since it is linear and continuous, we have, for any $g_k \in \overline{\operatorname{span}}(B_k)$,

$$g_k \otimes \left(\otimes_{\substack{j\in\mathbb{N} \\ j\neq k}} f_j \right) \in \overline{\operatorname{span}}(H) = H.$$

Since B_k is an orthonormal basis of H_k, we have

$$g_k \otimes \left(\otimes_{\substack{j\in\mathbb{N} \\ j\neq k}} f_j \right) \in H$$

for any $g_k \in H$, and, in particular, $\otimes_{j\in\mathbb{N}} f_j \in H$. □

Remark 2.46. Proposition 2.45, along with Proposition 2.37 and Lemma 2.24, can be used to determine the dimension of the incomplete and, with some restrictions, the complete tensor product.

Now, we form a connection to the finite case.

Remark 2.47. Suppose that there is some $m \in \mathbb{N}$, such that H_j is one-dimensional for each $j > m$. If we apply Proposition 2.45, choosing an e accordingly, we see that, for $j > m$, the basis B_j has only one element, namely e_j. Thus, we have

$$B = \times_{j \in \mathbb{N}} B_j$$

and

$$|B| = \left| \times_{j=1}^{m} B_j \right|.$$

Comparing this to Proposition 1.21, it becomes clear that in this case, the finite tensor product $\bigotimes_{j \in N} H_j$ has the same dimensionality as the incomplete tensor product $\bigotimes_{j \in \mathbb{N}}^{e} H_j$. The same need not be true for the complete tensor product: If all spaces H_j are in addition of finite dimension, so is $\bigotimes_{j \in N} H_j$. However, by Proposition 2.37 and Lemma 2.24, $\bigotimes_{j \in \mathbb{N}} H_j$ is always of infinite dimension.

We will now see that the connection between the finite tensor product and the incomplete tensor product is even stronger.

Proposition 2.48. Let the H_j be given in such a way that, for some $m \in \mathbb{N}$, the space H_j is one-dimensional for each $j > m$. Set $N = \{ 1, \ldots, m \}$, and consider the finite tensor product $\bigotimes_{j \in N} H_j$. Further, let $e = (e_j)_{j \in \mathbb{N}}$ be any $\mathcal{C}_0$-sequence such that $\|e_j\|_{H_j} = 1$ holds for all $j \in \mathbb{N}$.

Then, $\bigotimes_{j \in \mathbb{N}}^{e} H_j$ fulfills the properties of Theorem 1.14 and can thus be identified with $\bigotimes_{j \in N} H_j$ via the canonical isometric isomorphism given by that theorem.

Proof. To show property 1, let any $(f_j)_{j \in N} \in \times_{j \in N} H_j$ be given. We define

$$\tilde{f}_j = \begin{cases} f_j, & j \leq m \\ e_j, & j > m \end{cases}$$

and

$$\tilde{\otimes}_{j \in N} f_j = \otimes_{j \in \mathbb{N}} \tilde{f}_j. \tag{3}$$

This is clearly an element of $\bigotimes_{j \in \mathbb{N}}^{e} H_j$.

For property 2, let $(f_j)_{j \in N}, (g_j)_{j \in N} \in \times_{j \in N} H_j$ be given. With $\tilde{\otimes}_{j \in N} f_j$ and $\tilde{\otimes}_{j \in N} g_j$ defined according to Equation (3), we obtain

$$\begin{aligned} &\left\langle \tilde{\otimes}_{j \in N} f_j, \tilde{\otimes}_{j \in N} g_j \right\rangle_{\bigotimes_{j \in N} H_j} = \left\langle \otimes_{j \in \mathbb{N}} \tilde{f}_j, \otimes_{j \in \mathbb{N}} \tilde{g}_j \right\rangle_{\bigotimes_{j \in \mathbb{N}} H_j} \\ &= \prod_{j \in \mathbb{N}} \left\langle \tilde{f}_j, \tilde{g}_j \right\rangle_{H_j} \\ &= \prod_{j=1}^{m} \langle f_j, g_j \rangle_{H_j} \cdot \prod_{j=m+1}^{\infty} \langle e_j, e_j \rangle_{H_j} \\ &= \prod_{j=1}^{m} \langle f_j, g_j \rangle_{H_j}. \end{aligned}$$

To show property 3, for each $j \in \mathbb{N}$, we choose an orthonormal basis B_j fulfilling the conditions of Proposition 2.45. In particular, in the case of $j > m$, the basis B_j has only one element, namely e_j. Proposition 2.45 then gives us an orthonormal basis B of $\bigotimes_{j\in\mathbb{N}}^{e} H_j$. As discussed in Remark 2.47, in this particular case, we have $B = \times_{j\in\mathbb{N}} B_j$. So, each basis vector in B is an elementary tensor of the form given in Equation (3). Since B is an orthonormal basis, we obtain

$$\begin{aligned}
&\bigotimes_{j\in\mathbb{N}}{}^{e} H_j = \overline{\operatorname{span}} \{ \otimes_{j\in\mathbb{N}} f_j \mid \otimes_{j\in\mathbb{N}} f_j \in B \} \\
&\subseteq \overline{\operatorname{span}} \left\{ \tilde{\otimes}_{j\in N} f_j \;\middle|\; (f_j)_{j\in\mathbb{N}} \in \times_{j\in N} H_j \right\} \\
&\subseteq \bigotimes_{j\in\mathbb{N}}{}^{e} H_j .
\end{aligned}$$

This gives us property 3. □

We give an example similar to Example 1.18, examining the incomplete tensor product of infinitely many L_2-spaces. We restrict ourselves to the case of each measure μ_j being a probability measure, in which case we can use a theorem from Andersen and Jessen. The statement and the proof can be found in Section 10.6 in [8].

Lemma 2.49. Let I be any nonempty set, and let $(\Omega_i, \mathcal{A}_i, \mu_i)_{i\in I}$ be a family of probability spaces. For any $J \subseteq I$, define the projections

$$\pi_J : \bigtimes_{i\in I} \Omega_i \to \bigtimes_{i\in J} \Omega_i : \omega \mapsto (\omega_i)_{i\in J} .$$

Then, there exists exactly one probability measure $\mu = \bigtimes_{i\in I} \mu_i$ on the measurable space $\left(\bigtimes_{i\in I} \Omega_i, \bigotimes_{i\in I} \mathcal{A}_i\right)$ that fulfills

$$\mu \circ (\pi_J)^{-1} = \bigtimes_{i\in J} \mu_j$$

for each finite, nonempty set $J \subseteq I$.

Proof. See Section 10.6 in [8]. □

Before we can give the example, we need to show the following lemma, similar to the finite case.

Lemma 2.50. In the setting of Example 2.51,

$$\mathcal{E} = \left\{ \bigtimes_{j\in\mathbb{N}} A_j \;\middle|\; A_j \neq \Omega_j \text{ only finitely often and } A_j \in \mathcal{A}_j \right\}$$

is a semiring of sets.

Proof. Clearly, $\emptyset \in \mathcal{E}$ holds.

Now, let $\bigtimes_{j\in\mathbb{N}} A_j$ and $\bigtimes_{j\in\mathbb{N}} B_j$ be two sets in $\mathcal{E}$. Then, we have

$$\left(\bigtimes_{j\in\mathbb{N}} A_j\right) \cap \left(\bigtimes_{j\in\mathbb{N}} B_j\right) = \bigtimes_{j\in\mathbb{N}} (A_j \cap B_j) \in \mathcal{E}.$$

Lastly, there is some $j_0 \in \mathbb{N}$ such that $A_j = B_j = \Omega_j$ for holds for all $j > j_0$. Define $N = \{ 1, \ldots, j_0 \}$. For any set $I \subseteq N$ and any $j \in N$, define

$$D_{I,j} = \begin{cases} A_j \setminus B_j, & \text{if } j \in I \\ B_j, & \text{if } j \notin I. \end{cases}$$

For $j \in \mathbb{N} \setminus N$, define $D_{I,j} = \Omega_j$. Finally, define and $C_I = \bigtimes_{j\in\mathbb{N}} D_{I,j}$. Clearly, $C_I \in \mathcal{E}$ and for two subsets $I \neq I'$ of N, we have $C_I \cap C_{I'} = \emptyset$. Since N is finite, it only has finitely many subsets. Now, as in the finite case, we obtain

$$\left(\bigtimes_{j\in\mathbb{N}} A_j\right) \setminus \left(\bigtimes_{j\in\mathbb{N}} B_j\right) = \bigcup_{\substack{I\subseteq N \\ I\neq\emptyset}} C_I.$$

This finishes the proof. □

Now, we are able to give the desired example.

Example 2.51. For every $j \in \mathbb{N}$, let a probability space $(\Omega_j, \mathcal{A}_j, \mu_j)$ be given. Define $H_j = L_2(\Omega_j, \mathcal{A}_j, \mu_j)$ and $e_j = [\mathbb{1}_{\Omega_j}] \in H_j$. Then, the space

$$H = L_2\left(\bigtimes_{j\in\mathbb{N}} \Omega_j, \bigotimes_{j\in\mathbb{N}} \mathcal{A}_j, \bigtimes_{j\in\mathbb{N}} \mu_j\right)$$

is canonically isomorphic to $\bigotimes^e_{j\in\mathbb{N}} H_j$.

Proof. First, $\bigtimes_{j\in\mathbb{N}} \mu_j$ exists by Lemma 2.49, and thereby H also exists. We clearly have $\|e_j\|_{H_j} = 1$ for all $j \in \mathbb{N}$, so $\bigotimes^e_{j\in\mathbb{N}} H_j$ is defined. We now aim to employ Corollary 2.41, and we proceed similarly to Example 1.18.

We define, as in Corollary 2.41,

$$\mathcal{C}_e = \{ f \in \mathcal{C}_0 \mid f_j \neq e_j \text{ for only finitely many } j \in \mathbb{N} \}.$$

Take any $([f_j])_{j\in\mathbb{N}} \in \mathcal{C}_e$. Then, there is some $j_0 \in \mathbb{N}$ such that $[f_j] = [e_j]$ holds for all $j > j_0$. We set $\tilde{\otimes}_{j\in\mathbb{N}}[f_j] = [\prod_{j\in\mathbb{N}} f_j]$.

We use the theorem of Tonelli to obtain

$$\begin{aligned}
&\int_{\times_{j\in\mathbb{N}}\Omega_j} \left|\prod_{j\in\mathbb{N}} f_j\right|^2 \,\mathrm{d}\left(\bigtimes_{j\in\mathbb{N}} \mu_j\right)\\
&= \int_{\times_{j\in\mathbb{N}}\Omega_j} \left|\prod_{j\in\mathbb{N}} f_j\right|^2 \,\mathrm{d}\left(\left(\bigtimes_{j=1}^{j_0} \mu_j\right) \times \left(\bigtimes_{j\in\mathbb{N}_{>j_0}} \mu_j\right)\right)\\
&= \int_{\times_{j=1}^{j_0}\Omega_j} \int_{\times_{j\in\mathbb{N}_{>j_0}}\Omega_j} \left|\prod_{j\in\mathbb{N}} f_j\right|^2 \,\mathrm{d}\left(\bigtimes_{j\in\mathbb{N}_{>j_0}} \mu_j\right) \,\mathrm{d}\left(\bigtimes_{j=1}^{j_0} \mu_j\right)\\
&= \left(\int_{\times_{j=1}^{j_0}\Omega_j} \left|\prod_{j=1}^{j_0} f_j\right|^2 \,\mathrm{d}\left(\bigtimes_{j=1}^{j_0} \mu_j\right)\right) \cdot \left(\int_{\times_{j\in\mathbb{N}_{>j_0}}\Omega_j} \left|\prod_{j\in\mathbb{N}_{>j_0}} e_j\right|^2 \,\mathrm{d}\left(\bigtimes_{j\in\mathbb{N}_{>j_0}} \mu_j\right)\right)\\
&= \left(\int_{\times_{j=1}^{j_0}\Omega_j} \left|\prod_{j=1}^{j_0} f_j\right|^2 \,\mathrm{d}\left(\bigtimes_{j=1}^{j_0} \mu_j\right)\right) \cdot 1\\
&= \int_{\Omega_1} \dots \int_{\Omega_{j_0}} \left|\prod_{j\in N} f_j\right|^2 \,\mathrm{d}\mu_{j_0} \dots \,\mathrm{d}\mu_1\\
&= \int_{\Omega_1} |f_1|^2 \dots \int_{\Omega_{j_0}} |f_n|^2 \,\mathrm{d}\mu_{j_0} \dots \,\mathrm{d}\mu_1\\
&= \prod_{j=1}^{j_0} \int_{\Omega_j} |f_j|^2 \,\mathrm{d}\mu_j < \infty.
\end{aligned}$$

This shows that $\prod_{j\in\mathbb{N}} f_j$ is indeed square-integrable and that we have

$$\tilde{\otimes}_{j\in\mathbb{N}}[f_j] = \left[\prod_{j\in\mathbb{N}} f_j\right] \in H.$$

On the other hand, for a sequence of square-integrable functions $(g_j)_{j\in\mathbb{N}}$ where $g_j \in [f_j]$ holds for every $j \in \mathbb{N}$, we have

$$\begin{aligned}
&\left\{ \omega \in \bigtimes_{j\in\mathbb{N}} \Omega_j \;\middle|\; \prod_{j\in\mathbb{N}} f_j(\omega_j) \neq \prod_{j\in\mathbb{N}} g_j(\omega_j) \right\}\\
&\subseteq \bigcup_{j\in\mathbb{N}} \left\{ \omega \in \bigtimes_{j\in\mathbb{N}} \Omega_j \;\middle|\; f_j(\omega_j) \neq g_j(\omega_j) \right\},
\end{aligned}$$

(this includes cases where $\prod_{j\in\mathbb{N}} g_j(\omega_j)$ does not converge) and the latter is the countable union of sets of measure zero and so has measure zero too. This implies $[\prod_{j\in\mathbb{N}} f_j] = [\prod_{j\in\mathbb{N}} g_j]$, so $\tilde{\otimes}_{j\in\mathbb{N}}[f_j]$ depends on the equivalence-classes $[f_j]$ only, not on their representations. This shows property 1.

Now, let $([f_j])_{j\in\mathbb{N}}$ and $([g_j])_{j\in\mathbb{N}}$ be two sequences in $\mathcal{C}_e$. Then, there is some $j_0 \in \mathbb{N}$ such that $[f_j] = [g_j] = [e_j]$ holds for all $j > j_0$. Then, as in the finite case, see Example 1.18, we obtain property 2 by a calculation analogously to the one we used to establish property 1.

For property 3, consider the set $\mathcal{E}$ from Lemma 2.50, which is a semiring of sets. Define

$$\widetilde{\mathcal{E}} := \bigcup_{k\in\mathbb{N}} \left\{ A_j \times \bigtimes_{\substack{j\in\mathbb{N}\\ j\neq k}} \Omega_j \;\middle|\; A_j \in \mathcal{A}_j \right\}.$$

Clearly, we have $\widetilde{\mathcal{E}} \subseteq \mathcal{E} \subseteq \bigotimes_{j\in\mathbb{N}} \mathcal{A}_j$, and by Example III.5.3 in [3], we have $\sigma(\tilde{\mathcal{E}}) = \bigotimes_{j\in\mathbb{N}} \mathcal{A}_j$. This implies $\sigma(\mathcal{E}) = \bigotimes_{j\in\mathbb{N}} \mathcal{A}_j$. Since all probability measures are trivially σ-finite, we can employ Theorem V.2.28 from [3] as in the finite case, and obtain that

$$S = \operatorname{span}\{\, [\mathbb{1}_A] \mid A \in \mathcal{E}, \mu(A) < \infty \,\}$$

is dense in H. Now, each $A \in \mathcal{E}$ is of the form $A = \bigtimes_{j\in\mathbb{N}} A_j$ for some $A_j \in \mathcal{A}_j$, and $A_j \neq \Omega_j$ only holds finitely often. By this, we obtain $\big([\mathbb{1}_{A_j}]\big)_{j\in\mathbb{N}} \in \mathcal{C}_e$ and

$$[\mathbb{1}_A] = [\prod_{j\in\mathbb{N}} \mathbb{1}_{A_j}] = \tilde{\otimes}_{j\in\mathbb{N}}[\mathbb{1}_{A_j}]$$

and thus

$$S \subseteq H_0 := \operatorname{span}\left\{ \tilde{\otimes}_{j\in\mathbb{N}}[f_j] \;\middle|\; ([f_j])_{j\in\mathbb{N}} \in \mathcal{C}_e \right\} \subseteq H.$$

This implies that H_0 is dense in H. □

3 Hilbert Spaces with Reproducing Kernels

Let D be any nonempty set. In this section, we focus on special Hilbert spaces that are connected to *nonnegative definite functions* on $D \times D$. These spaces are called *reproducing kernel Hilbert spaces*, or RKHS. This topic was first thoroughly studied by Aronszajn in [1]; we also employ ideas and reformulations found in [7].

3.1 Construction, Existence and Uniqueness

Definition 3.1. A function $K : D \times D \to \mathbb{K}$ is called nonnegative definite, if it is conjugate symmetric and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c_i} c_j K(x_i, x_j) \geq 0$$

holds for any choice of $c_1, \dots, c_n \in \mathbb{K}$ and $x_1, \dots, x_n \in D$. Equivalently, the matrix $M = (K(x_i, x_j))_{i,j=1}^{n}$ is nonnegative definite regardless of the choice of $x_1, \dots, x_n \in D$.

Let $K : D \times D \to \mathbb{K}$ be a nonnegative definite function. Our first goal in this section is to show that there exists a uniquely determined Hilbert space $H(K)$ of $\mathbb{K}$-valued functions on D that fulfills

$$K(\cdot, t) \in H(K) \tag{4}$$

and

$$h(t) = \langle h, K(\cdot, t) \rangle_{H(K)} \tag{5}$$

for every choice of $t \in D$ and $h \in H(K)$. Equation (5) is also called the *reproducing property* and Hilbert spaces given in this way are called *reproducing kernel Hilbert spaces.*

The construction of $H(K)$ is somewhat similar to that of the Hilbert space tensor product we considered in section 1. We first consider the smallest space fulfilling (4).

Lemma 3.2. Consider the subspace of $\mathbb{K}^D$

$$H_0 := \operatorname{span} \left\{ K(\cdot, t) \in \mathbb{K}^D \;\middle|\; t \in D \right\}.$$

For $h, g \in H_0$ with representations $h = \sum_{i=1}^{n} a_i K(\cdot, s_i)$ and $g = \sum_{j=1}^{\ell} b_j K(\cdot, t_j)$,

$$\langle g, h \rangle_{H_0} = \sum_{i=1}^{n} \sum_{j_1}^{\ell} \overline{a_i} b_j K(s_i, t_j).$$

is well-defined. Further, $\langle \cdot, \cdot \rangle_{H_0}$ is a scalar product on H_0 and the properties (4) and (5) are fulfilled.

Proof. Since we have

$$\begin{aligned}\sum_{i=1}^{n} \overline{a_i} g(s_i) &= \sum_{i=1}^{n}\sum_{j=1}^{\ell} \overline{a_i} b_j K(s_i, t_j)\\ &= \langle g, h\rangle_{H_0}\\ &= \sum_{j=1}^{\ell} b_j \sum_{i=1}^{n} \overline{a_i} K(s_i, t_j)\\ &= \sum_{j=1}^{\ell} b_j \overline{\sum_{i=1}^{n} a_i K(t_j, s_i)}\\ &= \sum_{j=1}^{\ell} b_j h(t_j),\end{aligned}$$

$\langle g, h\rangle_{H_0}$ depends only on g and h and not their specific representations. Thus, it is well-defined. Sesquilinearity of $\langle \cdot, \cdot\rangle_{H_0}$ follows by definition. That $\langle \cdot, \cdot\rangle_{H_0}$ is hermitian follows directly because K is conjugate symmetric. Because K is nonnegative definite, we have

$$\langle h, h\rangle_{H_0} = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{a_i} a_j K(s_i, s_j) \geq 0,$$

and because h is an arbitrary element of H_0, $\langle \cdot, \cdot\rangle_{H_0}$ is nonnegative definite too.

Clearly, $K(\cdot, t) \in H_0$ for any $t \in D$, so (4) is fulfilled. Now we show that the mapping $\langle \cdot, \cdot\rangle_{H_0}$ fulfills (5) for any choice of $t \in D$ and $h \in H_0$ with the representation given above. We have

$$\langle h, K(\cdot, t)\rangle_{H_0} = \sum_{i=1}^{n} a_i K(t, s_i) = h(t)$$

by definition.

Now let $h \in H_0$ with $\langle h, h\rangle_{H_0} = 0$. For any $t \in D$, Lemma 1.6 along with the reproducing property yield

$$|h(t)|^2 = \left|\langle h, K(\cdot, t)\rangle_{H_0}\right|^2 \leq \langle h, h\rangle_{H_0} \langle K(\cdot, t), K(\cdot, t)\rangle_{H_0} = 0,$$

which implies $h = 0$. Thus, $\langle \cdot, \cdot\rangle_{H_0}$ is even positive definite and thus a scalar product on H_0. □

In general, H_0 equipped with $\langle \cdot, \cdot\rangle_{H_0}$ is not a complete space. However, as in the construction of the Hilbert space tensor product, we will show that any Cauchy sequence converges at least pointwise and use the pointwise limits to construct the desired Hilbert space.

Lemma 3.3. For any Cauchy sequence $(h_n)_{n\in\mathbb{N}}$ in H_0 and any $t \in D$, the pointwise limit

$$h(t) = \lim_{n\to\infty} h_n(t)$$

exists. If $h(t) = 0$ for all $t \in D$, we have

$$\lim_{n\to\infty} \langle h_n, h_n \rangle_{H_0} = 0.$$

Proof. Let $(h_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. For $n_1, n_2 \in \mathbb{N}$ and $t \in D$, the reproducing property and the Cauchy-Schwarz inequality yield

$$|h_{n_1}(t) - h_{n_2}(t)| = \left|\langle h_{n_1} - h_{n_2}, K(\cdot, t)\rangle_{H_0}\right| \leq \|h_{n_1} - h_{n_2}\|_{H_0} \|K(\cdot, t)\|_{H_0},$$

which shows the existence of the pointwise limit $h(t)$. Now, if $h(t) = 0$ for all $t \in D$ holds, for any $n, \ell \in \mathbb{N}$, we have

$$\begin{aligned} \langle h_n, h_n \rangle_{H_0} &\leq \left|\langle h_n, h_\ell\rangle_{H_0}\right| + \left|\langle h_n, h_n - h_\ell\rangle_{H_0}\right| \\ &\leq \left|\langle h_n, h_\ell\rangle_{H_0}\right| + \|h_n\|_{H_0} \|h_n - h_\ell\|_{H_0}. \end{aligned}$$

In the last term, because $(h_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and thus bounded, the second summand tends to 0 as n tends to infinity while ℓ remains fixed. The same is true for the first summand, as we see in the following: If h_ℓ has the representation $h_\ell = \sum_{i=1}^{m_\ell} a_{i,\ell} K(\cdot, t_{i,\ell})$, we obtain

$$\begin{aligned} \langle h_n, h_\ell\rangle_{H_0} &= \left\langle h_n, \sum_{i=1}^{m_\ell} a_{i,\ell} K(\cdot, t_{i,\ell})\right\rangle_{H_0} \\ &= \sum_{i=1}^{m_\ell} a_{i,\ell} \langle h_n, K(\cdot, t_{i,\ell})\rangle_{H_0} \\ &= \sum_{i=1}^{m_\ell} a_{i,\ell} h_n(t_{i,\ell}), \end{aligned}$$

which tends to 0 as n tends to infinity. This shows that $\lim_{n\to\infty} \langle h_n, h_n\rangle_{H_0} = 0$ holds. □

Theorem 3.4. Consider the subspace of $\mathbb{K}^D$

$$H(K) := \left\{ \Phi \in \mathbb{K}^D \;\middle|\; \Phi \text{ is the pointwise limit of a Cauchy sequence in } H_0 \right\}.$$

For $h, g \in H(K)$ with approximating sequences $(h_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ in H_0, the mapping given by

$$\langle h, g\rangle_{H(K)} = \lim_{n\to\infty} \langle h_n, g_n\rangle_{H_0}$$

is well-defined and a scalar product. Further, $H(K)$ equipped with this scalar product is the only Hilbert space fulfilling

$$K(\cdot, t) \in H(K)$$

and

$$h(t) = \langle h, K(\cdot, t) \rangle_{H(K)}$$

for every choice of $t \in D$ and $h \in H(K)$.

Proof. That $\lim_{n\to\infty} \langle h_n, g_n \rangle_{H_0}$ always exists is shown exactly as in the proof of Theorem 1.11. Now, let $(h_n)_{n\in\mathbb{N}}$ and $(h'_n)_{n\in\mathbb{N}}$ be two Cauchy sequences in H_0 with pointwise limit h as well as $(g_n)_{n\in\mathbb{N}}$ and $(g'_n)_{n\in\mathbb{N}}$ two Cauchy sequences in H_0 with pointwise limit g. It follows that $(h_n - h'_n)_{n\in\mathbb{N}}$ and $(g_n - g'_n)_{n\in\mathbb{N}}$ are Cauchy sequences with pointwise limit 0. We have

$$\begin{aligned}
&\left|\langle h_n, g_n \rangle_{H_0} - \langle h'_n, g'_n \rangle_{H_0}\right| \\
=\ &\left|\langle h_n - h'_n, g_n \rangle_{H_0} + \langle h'_n, g_n \rangle_{H_0} - \langle h'_n, g'_n - g_n \rangle_{H_0} - \langle h'_n, g_n \rangle_{H_0}\right| \\
\le\ &\left|\langle h_n - h'_n, g_n \rangle_{H_0}\right| + \left|\langle h'_n, g'_n - g_n \rangle_{H_0}\right| \\
\le\ &\|h_n - h'_n\|_{H_0} \|g_n\|_{H_0} + \|h'_n\|_{H_0} \|g'_n - g_n\|_{H_0},
\end{aligned}$$

which tends to 0 as n tends to infinity by Lemma 3.3. Thus, $\langle h, g \rangle_{H(K)}$ is well-defined.

As in Theorem 1.11, because we already know that $\langle \cdot, \cdot \rangle_{H_0}$ is a scalar product on H_0, by taking the limit, we obtain that $\langle \cdot, \cdot \rangle_{H(K)}$ is sesquilinear, hermitian and nonnegative definite. The reproducing property also follows by taking the limit: For any $t \in D$, we have

$$\langle h, K(\cdot, t) \rangle_{H(K)} = \lim_{n\to\mathbb{N}} \langle h_n, K(\cdot, t) \rangle_{H_0} = \lim_{n\to\mathbb{N}} h_n(t) = h(t).$$

As in Lemma 3.2, this already suffices to show that $\langle \cdot, \cdot \rangle_{H(K)}$ is positive definite and thus a scalar product. The proof that $H(K)$ is a Hilbert space works exactly as in Theorem 1.11. That proof also gives us that any Cauchy-sequence in H_0 with pointwise limit h also tends to h with respect to $\|\cdot\|_{H(K)}$. With this, it is clear that H_0 is dense in $H(K)$.

Now, let H be any Hilbert space of $\mathbb{K}$-valued functions on D fulfilling both

$$K(\cdot, t) \in H$$

and

$$h(t) = \langle h, K(\cdot, t) \rangle_H$$

for every choice of $t \in D$ and $h \in H$. Because H is a linear space, $K(\cdot, t) \in H$ implies $H_0 \subseteq H$. Since the reproducing property holds and any scalar product is bilinear, $\langle \cdot, \cdot \rangle_H$ must coincide with $\langle \cdot, \cdot \rangle_{H_0}$ on $H_0 \times H_0$. This means that H_0 is a subspace of H. Since H is a Hilbert space, it must also contain the closure of H_0. We already established that this must be $H(K)$. By continuity of the scalar product, the scalar products $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_{H(K)}$ must coincide on $H(K)$. Thus, $H(K)$ is a closed subspace of H. Any $h \in H$ can thus be represented as $h = f + g$ with $f \in H(K)$ and $g \in H(K)^\perp$, where $H(K)^\perp$ denotes the orthogonal complement of $H(K)$. For any $t \in D$, we have $\langle g, K(\cdot,t)\rangle_H = 0$, since $K(\cdot,t) \in H(K)$. The reproducing property then yields

$$h(t) = \langle h, K(\cdot,t)\rangle_H = \langle f, K(\cdot,t)\rangle_H + \langle g, K(\cdot,t)\rangle_H = \langle f, K(\cdot,t)\rangle_H = f(t).$$

This implies $g = 0$, and thus $h \in H(K)$ and finally $H = H(K)$ holds. □

Definition 3.5. Any Hilbert space H of $\mathbb{K}$-valued functions on D for which a nonnegative definite kernel K exists such that $H = H(K)$ is called a *reproducing kernel Hilbert space* (RKHS).

The structure of a RKHS can always be described in the following way.

Corollary 3.6. H(K) is the closure of

$$\operatorname{span}\{\, K(\cdot,t) \mid t \in D \,\}\,.$$

Proof. This follows immediately from the way we constructed $H(K)$. □

We have seen that, given a kernel K, the corresponding RKHS is unique. The converse is also true.

Proposition 3.7. Let H be a RKHS. Then, the nonnegative kernel such that $H = H(K)$ holds is unique.

Proof. Let $K, K' : D^2 \to \mathbb{K}$ be two nonnegative kernels given in such a way that $H(K) = H(K')$ holds. Then, for any $y \in D$, we have

$$\begin{aligned}
&\|K(\cdot,y) - K'(\cdot,y)\|_H^2 = \langle K(\cdot,y) - K'(\cdot,y), K(\cdot,y) - K'(\cdot,y)\rangle_H \\
&= \langle K(\cdot,y) - K'(\cdot,y), K(\cdot,y)\rangle_H - \langle K(\cdot,y) - K'(\cdot,y), K'(\cdot,y)\rangle_H \\
&= K(y,y) - K'(y,y) - K(y,y) + K'(y,y) \\
&= 0,
\end{aligned}$$

since both K and K' fulfill the reproducing property. This shows $K = K'$. □

3.2 Examples and Characterization

Example 3.8.

1. $\mathbb{K}^n$ can be viewed as the space of all $\mathbb{K}$-valued function with domain $\{1,\ldots,n\}$. As commonly known, this, equipped with the euclidean scalar product, is a Hilbert space. Now, consider the kernel

$$K : \{1,\ldots,n\}^2 \to \mathbb{K} : (i,j) \mapsto \begin{cases} 1, & i = j \\ 0, & \text{else.} \end{cases}$$

This is clearly a nonnegative kernel. Further, for any $j \in \{1,\ldots,n\}$, the function $K(\cdot,j)$ is just the j-th canonical unit vector e_j and thus is contained in $\mathbb{K}^n$. The reproducing property also holds: For any $v \in \mathbb{K}^n$ and $j \in \{ 1, \dots, n \}$, we have

$$\langle v, K(\cdot, j) \rangle = \langle v, e_j \rangle = v_j.$$

We have thus shown that $\mathbb{K}^n$ is a RKHS.

2. Similarly, if we define

$$K : \mathbb{N}^2 \to \mathbb{R} : (i, j) \mapsto \begin{cases} 1, & i = j \\ 0, & \text{else,} \end{cases}$$

this is still clearly a nonnegative kernel. The corresponding RKHS is the space of all square-summable sequences $l_2(\mathbb{R})$. Indeed, as in the case of $\mathbb{R}^n$, for any $j \in \mathbb{N}$, the function $K(\cdot, j)$ is just the j-th canonical unit vector and thus

$$\langle f, K(\cdot, j) \rangle_{l_2(\mathbb{R})} = v_j$$

holds for any $v \in l_2(\mathbb{R})$.

Note that general L_2-spaces are *not* reproducing kernel Hilbert spaces, since their elements generally are not functions, but rather equivalence classes of functions, making point-evaluations impossible.

3. Moving away from well-known spaces, a class of well-known *kernels* is that of scalar products. Indeed, for any real Hilbert-space H, a nonnegative kernel is given by its scalar product:

$$K = \langle \cdot, \cdot \rangle_H$$

is a nonnegative kernel by definition. The corresponding RKHS is the dual space

$$H' := \{ T : H \to \mathbb{K} \mid T \text{ is linear and bounded} \}$$

equipped with the operator norm. First, H' can indeed be seen as a Hilbert space: Via the Riesz isomorphism $J : H \to H'$, a scalar product can be defined on H' by setting

$$\langle J(h), J(g) \rangle_{H'} := \langle h, g \rangle_H$$

for all $h, g \in H$, and since the Riesz isomorphism is isometric, the norm given by this scalar product coincides with the operator norm. So, H' is a Hilbert space of $\mathbb{K}$-valued functions on H.

Now, for any $h \in H$, the function $\langle \cdot, h \rangle_H$ is linear by sesquilinearity and bounded by the Cauchy-Schwarz inequality. Further, for any $T \in H'$, via the Riesz isomorphism, we get the element $t := J^{-1}(T) \in H$ and for this we have $T = \langle \cdot, t \rangle_H$. We obtain the reproducing property via

$$T(h) = \langle h, t \rangle_H = \langle \langle \cdot, h \rangle_H, T \rangle_{H'} = \langle T, K(\cdot, h) \rangle_{H'} .$$

An alternative way to obtain the notion of a reproducing kernel Hilbert space is given by the following characterization.

Proposition 3.9. Let H be a Hilbert space of real-valued functions on D. Then, H is a RKHS if and only if for every $t \in D$, the linear functional

$$\delta_t : H \to \mathbb{K} : h \mapsto h(t)$$

is bounded.

Proof. First, let H be a RKHS, that is to say there is a nonnegative definite kernel K such that $H = H(K)$. Then, we have

$$|\delta_t(h)| = |h(t)| = |\langle h, K(\cdot,t)\rangle_H| \leq \|h\|_H \|K(\cdot,t)\|_H$$

for every $t \in D$ and $h \in H$, so δ_t is bounded.

Conversely, let H be a Hilbert space of $\mathbb{K}$-valued functions on D such that for every $s \in D$, δ_s is bounded. This means δ_s is an element of the dual space H' of H. Then, by the Riesz representation theorem, for each δ_s there exists a unique Element $K_s \in H$ for which $\delta_s = \langle \cdot, K_s\rangle_H$ holds. For any $t \in D$, we obtain

$$K_s(t) = \delta_t(K_s) = \langle K_s, K_t\rangle_H .$$

We thus define $K : D \times D \to \mathbb{K} : (s,t) \mapsto \langle K_s, K_t\rangle_H$, which is conjugate symmetric due to the corresponding property of the scalar product. We also have, for any $n \in \mathbb{N}$, $c_1, \dots, c_n \in \mathbb{K}$ and $t_1, \dots, t_n \in D$ that

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} c_j K(t_i, t_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} c_j \left\langle K_{t_i}, K_{t_j}\right\rangle_H \\
&= \left\langle \sum_{i=1}^{n} c_i K_{t_i}, \sum_{i=1}^{n} c_i K_{t_i} \right\rangle_H \\
&= \left\| \sum_{i=1}^{n} c_i K_{t_i} \right\|_H^2 \geq 0
\end{aligned}$$

holds, so K is a nonnegative definite function.

We now check that $H = H(K)$ by using the uniqueness given in Theorem 3.4. For any $s,t \in D$ we have

$$K(s,t) = \langle K_s, K_t\rangle_H = \langle K_t, K_s\rangle_H = K_t(s),$$

which shows $K(\cdot,t) = K_t \in H$. Furthermore, for any $h \in H$, we get

$$h(t) = \delta_t(h) = \langle h, K_t\rangle_H = \langle h, K(\cdot,t)\rangle_H .$$

Thus, H must be the unique Hilbert space given by Theorem 3.4. □

This allows us to give an example of a space of functions that is *not* a RKHS. This example is found as Remark 2.4 in [12].

Example 3.10. Let $D \neq \emptyset$ be any set and $z \notin D$. Let H be any infinite-dimensional RKHS of functions with domain D. We construct a Hilbert space of functions with domain $D_z = D \cup \{ z \}$ that is not a RKHS.

Let $\Phi : H \to \mathbb{K}$ be any linear, discontinuous mapping. Define

$$H_z : \left\{ f \in \mathbb{K}^{D_z} \;\middle|\; f_{|D} \in H \text{ and } f(z) = \Phi(f_{|D}) \right\}.$$

This is a linear subspace of $\mathbb{K}^{D_z}$, since H is a linear space and Φ is linear. For $f, g \in H_z$ we define

$$\langle f, g \rangle_{H_z} = \left\langle f_{|D}, g_{|D} \right\rangle_H .$$

This is clearly well-defined and a scalar product; further H_z equipped with this is a Hilbert space. However, the mapping

$$\delta_z : H_z \to \mathbb{K} : f \mapsto f(z) = \Phi(f)$$

is not continuous, since even $\delta_{z|D} = \Phi$ is not continuous. By Proposition 3.9, H_z is not a RKHS.

4 Tensor Products of Reproducing Kernel Hilbert Spaces

In Sections 1 and 2, we have seen how, for an at most countably big set N and a family of Hilbert spaces $(H_j)_{j\in N}$, the tensor product $\bigotimes_{j\in N} H_j$ can generally be constructed. In this section, our goal is to find more convenient constructions in the special case that all spaces are reproducing kernel Hilbert spaces. More specifically, suppose that for each $j \in N$, there is a set $D_j \neq \emptyset$ and a nonnegative definite kernel $K_j : D_j^2 \to \mathbb{K}$ such that $H_j = H(K_j)$ holds. In this case, we seek to find a suitable subset $X \subseteq (\times_{j\in N} D_j)$ and some nonnegative definite kernel

$$\bigotimes_{j\in N} K_j : X^2 \to \mathbb{K},$$

such that $H\left(\bigotimes_{j\in N} K_j\right)$ fulfills, depending on the notion of tensor product, the properties of Theorem 1.14, Theorem 2.36 or Theorem 2.41 and can thus be identified with the tensor product in a canonical way.

4.1 The Finite Case

In the finite case, the construction of $\bigotimes_{j=1}^n K_j$ is straightforward: It arises as the pointwise product of the Kernels K_j.

Lemma 4.1. For each $j \in \{1, \ldots, n\}$, let a nonempty set D_j and a nonnegative kernel $K_j : D_j \to \mathbb{K}$ be given. Then, the function

$$\bigotimes_{j=1}^n K_j : \left(\bigtimes_{j=1}^n D_j\right)^2 \to \mathbb{K} : (x, y) \mapsto \prod_{j=1}^n K_j(x_j, y_j)$$

is also a nonnegative definite kernel.

Proof. For notational convenience, we only consider the case $n = 2$, the general case follows by induction. We need to show that for any $m \in \mathbb{N}$ and any $(x_1, y_1), \ldots, (x_m, y_m) \in D_1 \times D_2$ the matrix

$$M = (K_1(x_i, x_j)K_2(y_i, y_j))_{i,j=1}^m$$

is nonnegative definite. We know that the matrices $M_1 = (K_1(x_i, x_j))_{i,j=1}^m$ and $M_2 = (K_2(y_i, y_j))_{i,j=1}^m$ are nonnegative definite. Our claim follows by Lemma 1.5. □

The following result can already be found in [1]. We give an alternative proof that fits the work of von Neumann in [10] better.

Theorem 4.2. In the setting of Lemma 4.1,

$$H\left(\bigotimes_{j=1}^n K_j\right) \text{ is canonically isomorphic to } \bigotimes_{j=1}^n H(K_j).$$

Proof. We want to show that the three properties from Theorem 1.14 hold. The first two properties will be shown by structural induction, while the third follows almost immediately.

First, we assume that for each $j \in \{ 1, \dots, n \}$, we have $h_j = K_j(\cdot, x_j)$ and $g_j = K_j(\cdot, y_j)$ for some $x_j, y_j \in D_j$. In this case, we have

$$\prod_{j=1}^{n} h_j = \prod_{j=1}^{n} K_j(\cdot, x_j) = \bigotimes_{j=1}^{n} K_j(\cdot, x) \in H\left(\bigotimes_{j=1}^{n} K_j\right)$$

by the definition of a RKHS, so the first property holds. The second property follows by

$$\begin{aligned}
\left\langle \prod_{j=1}^{n} h_j, \prod_{j=1}^{n} g_j \right\rangle_{H\left(\bigotimes_{j=1}^{n} K_j\right)} &= \left\langle \bigotimes_{j=1}^{n} K_j(\cdot, x), \bigotimes_{j=1}^{n} K_j(\cdot, y) \right\rangle_{H\left(\bigotimes_{j=1}^{n} K_j\right)} \\
&= \bigotimes_{j=1}^{n} K_j(y, x) \\
&= \prod_{j=1}^{n} K_j(y_j, x_j) \\
&= \prod_{j=1}^{n} \langle K_j(\cdot, x_j), K_j(\cdot, y_j) \rangle_{H(K_j)} \\
&= \prod_{j=1}^{n} \langle h_j, g_j \rangle_{H(K_j)},
\end{aligned}$$

using the reproducing property of the kernels K_j and of the product kernel $\bigotimes_{j=1}^{n} K_j$.

Our next goal is to show that the properties 1 and 2 still hold if each h_j and each g_j is an element of $\operatorname{span}\{ K_j(\cdot, x_j) \mid x_j \in D_j \}$. Therefore, for an arbitrary but constant $1 \leq j_0 \leq n$, suppose the properties 1 and 2 hold for all functions $\prod_{j=1}^{n} h_j$ and $\prod_{j=1}^{n} g_j$ of the following form: For each $j \geq j_0$, we have the elementary case $h_j = K_j(\cdot, x_j)$ and $g_j = K_j(\cdot, y_j)$ with $x_j, y_j \in D_j$; and for each $j < j_0$, the functions h_j and g_j are elements of $\operatorname{span}\{ K_j(\cdot, x_j) \mid x_j \in D_j \}$. Given functions $\prod_{j=1}^{n} h_j, \prod_{j=1}^{n} g_j$ of this form, we replace h_{j_0} and g_{j_0} by defining

$$\widetilde{h_{j_0}} = \sum_{i=1}^{m_1} a_i K_{j_0}(\cdot, x_i) \text{ and } \widetilde{g_{j_0}} = \sum_{k=1}^{m_2} b_k K_{j_0}(\cdot, y_k)$$

for any $x_i, y_k \in D_j$, as well as $\widetilde{h_j} = h_j$ and $\widetilde{g_j} = g_j$ for each $j \neq j_0$. If we show that properties 1 and 2 hold for $\prod_{j=1}^{n} \widetilde{h_j}, \prod_{j=1}^{n} \widetilde{g_j}$, by induction over j_0, we have reached our intermediate goal. We have

$$\prod_{j=1}^{n} \widetilde{h_j} = \left(\sum_{i=1}^{m_1} a_i K_{j_0}(\cdot, x_i) \right) \left(\prod_{\substack{j \in \{1, \cdots, n\} \\ j \neq j_0}} \widetilde{h_j} \right)$$

$$= \sum_{i=1}^{m_1} a_i \left(K_{j_0}(\cdot, x_i) \prod_{\substack{j \in \{1, \cdots, n\} \\ j \neq j_0}} \widetilde{h_j} \right).$$

For each summand of the last sum, the expression in brackets is an element of $H\left(\bigotimes_{j=1}^{n} K_j\right)$ by assumption, since we replaced $\widetilde{h_{j_0}}$ with a function of the form $K_{j_0}(\cdot, x_{j_0})$. This means the entire sum is a linear combination of elements of $H\left(\bigotimes_{j=1}^{n} K_j\right)$, and thus we have $\prod_{j=1}^{n} \widetilde{h_j} \in H\left(\bigotimes_{j=1}^{n} K_j\right)$, which is property 1.

For property 2, we compute

$$\left\langle \prod_{j=1}^{n} \widetilde{h_j}, \prod_{j=1}^{n} \widetilde{g_j} \right\rangle_{H\left(\bigotimes_{j=1}^{n} K_j\right)}$$

$$= \left\langle \left(\sum_{i=1}^{m_1} a_i K_{j_0}(\cdot, x_i) \right) \prod_{\substack{j \in \{1, \cdots, n\} \\ j \neq j_0}} \widetilde{h_j}, \left(\sum_{k=1}^{m_2} b_k K_{j_0}(\cdot, y_k) \right) \prod_{\substack{j \in \{1, \cdots, n\} \\ j \neq j_0}} \widetilde{g_j} \right\rangle_{H\left(\bigotimes_{j=1}^{n} K_j\right)}$$

$$= \sum_{i=1}^{m_1} \sum_{k=1}^{m_2} a_i \overline{b_k} \left\langle K_{j_0}(\cdot, x_i) \cdot \prod_{\substack{j \in \{1, \cdots, n\} \\ j \neq j_0}} \widetilde{h_j}, K_{j_0}(\cdot, y_k) \cdot \prod_{\substack{j \in \{1, \cdots, n\} \\ j \neq j_0}} \widetilde{g_j} \right\rangle_{H\left(\bigotimes_{j=1}^{n} K_j\right)}$$

and by assumption we get

$$= \sum_{i=1}^{m_1} \sum_{k=1}^{m_2} a_i \overline{b_k} \left\langle K_{j_0}(\cdot, x_i), K_{j_0}(\cdot, y_k) \right\rangle_{H(K_{j_0})} \prod_{\substack{j \in \{1, \cdots, n\} \\ j \neq j_0}} \left\langle \widetilde{h_j}, \widetilde{g_j} \right\rangle_{H(K_j)}$$

$$= \left(\sum_{i=1}^{m_1} \sum_{k=1}^{m_2} a_i \overline{b_k} \left\langle K_{j_0}(\cdot, x_i), K_{j_0}(\cdot, y_k) \right\rangle_{H(K_{j_0})} \right) \prod_{\substack{j \in \{1, \cdots, n\} \\ j \neq j_0}} \left\langle \widetilde{h_j}, \widetilde{g_j} \right\rangle_{H(K_j)}$$

$$= \left\langle \widetilde{h_{j_0}}, \widetilde{g_{j_0}} \right\rangle_{H(K_{j_0})} \prod_{\substack{j \in \{1, \cdots, n\} \\ j \neq j_0}} \left\langle \widetilde{h_j}, \widetilde{g_j} \right\rangle_{H(K_j)}$$

$$= \prod_{j=1}^{n} \left\langle \widetilde{h_j}, \widetilde{g_j} \right\rangle_{H(K_j)}.$$

Now, we show that the properties 1 and 2 hold for generic h_j and g_j. We proceed as before: For an arbitrary but constant $1 \leq j_0 \leq n$, suppose the properties 1 and 2 hold for all functions $\prod_{j=1}^{n} h_j$ and $\prod_{j=1}^{n} g_j$ of the following form: For each $j \geq j_0$, the functions h_j and g_j are elements of $\operatorname{span}\{\, K_j(\cdot, x_j) \mid x_j \in D_j \,\}$ and for $j < j_0$, we have $h_j, g_j \in H(K_j)$. Given such functions, we consider $\prod_{j=1}^{n} \widetilde{h_j}$ and $\prod_{j=1}^{n} \widetilde{g_j}$, which are obtained by setting $\widetilde{h_j} = h_j$ and $\widetilde{g_j} = g_j$ for each $j \neq j_0$ and choosing $h_{j_0}, g_{j_0} \in H(K_{j_0})$ arbitrarily.

Now there exists a sequence $\left(\widetilde{h_{j_0,m}}\right)_{m\in\mathbb{N}}$, such that each member of the sequence is an element $\operatorname{span}\{\, K_j(\cdot, x_j) \mid x_j \in D_j \,\}$ and such that $\left(\widetilde{h_{j_0,m}}\right)_{m\in\mathbb{N}}$ converges to $\widetilde{h_{j_0}}$ pointwise as well as in $H(K_{j_0})$. A sequence $\left(\widetilde{g_{j_0,m}}\right)_{m\in\mathbb{N}}$ with the same properties exists for $\widetilde{g_{j_0}}$. Now, for each $m \in \mathbb{N}$, the function

$$\widetilde{h_{j_0,m}} \prod_{\substack{j\in\{\,1,\cdots,n\,\}\\ j\neq j_0}} \widetilde{h_j}$$

is an element of $H\left(\bigotimes_{j=1}^{n} K_j\right)$ by assumption. Since

$$\left|\prod_{j=1}^{n} \widetilde{h_j}(x_j) - \widetilde{h_{j_0,m}}(x_{j_0}) \prod_{\substack{j\in\{\,1,\cdots,n\,\}\\ j\neq j_0}} \widetilde{h_j}(x_j)\right|$$

$$= \left|\widetilde{h_{j_0}}(x_{j_0}) - \widetilde{h_{j_0,m}}(x_{j_0})\right| \left|\prod_{\substack{j\in\{\,1,\cdots,n\,\}\\ j\neq j_0}} \widetilde{h_j}(x_j)\right|$$

holds for any $x \in \times_{j=1}^{n} D_j$, the function $\prod_{j=1}^{n} \widetilde{h_j}$ is the pointwise limit of the functions

$$\widetilde{h_{j_0,n}} \prod_{\substack{j\in\{\,1,\cdots,n\,\}\\ j\neq j_0}} \widetilde{h_j}$$

and is thus an element of $H(\bigotimes_{j=1}^{n} K_j)$. By induction, we have shown property 1.

For property 2, we calculate

$$\left\langle \prod_{j=1}^{n} \widetilde{h_j}, \prod_{j=1}^{n} \widetilde{g_j} \right\rangle_{H(\bigotimes_{j=1}^{n} K_j)} = \left\langle \widetilde{h_{j_0}} \prod_{\substack{j\in\{\,1,\cdots,n\,\}\\ j\neq j_0}} \widetilde{h_j}, \widetilde{g_{j_0}} \prod_{\substack{j\in\{\,1,\cdots,n\,\}\\ j\neq j_0}} \widetilde{g_j} \right\rangle_{H(\bigotimes_{j=1}^{n} K_j)} .$$

By the general definition of the scalar product in a RKHS, see Theorem 3.4, we get

$$= \lim_{m\to\infty} \left\langle \widetilde{h_{j_0,m}} \prod_{\substack{j\in\{\,1,\cdots,n\,\}\\ j\neq j_0}} \widetilde{h_j}, \widetilde{g_{j_0,m}} \prod_{\substack{j\in\{\,1,\cdots,n\,\}\\ j\neq j_0}} \widetilde{g_j} \right\rangle_{H(\bigotimes_{j=1}^{n} K_j)}$$

and by assumption we get

$$= \lim_{m\to\infty} \left\langle \widetilde{h_{j_0,m}}, \widetilde{g_{j_0,m}} \right\rangle_{H(K_{j_0})} \prod_{\substack{j\in\{1,\cdots,n\}\\ j\neq j_0}} \left\langle \widetilde{h_j}, \widetilde{g_j} \right\rangle_{H(K_j)}$$

$$= \left\langle \widetilde{h_{j_0}}, \widetilde{g_{j_0}} \right\rangle_{H(K_{j_0})} \prod_{\substack{j\in\{1,\cdots,n\}\\ j\neq j_0}} \left\langle \widetilde{h_j}, \widetilde{g_j} \right\rangle_{H(K_j)}$$

$$= \prod_{j=1}^{n} \left\langle \widetilde{h_j}, \widetilde{g_j} \right\rangle_{H(K_j)} .$$

To show property 3 we recall that by Corollary 3.6, the linear subspace

$$H_0 := \operatorname{span} \left\{ \bigotimes_{j=1}^{n} K_j(\cdot, x) \;\middle|\; x \in \bigtimes_{j\in N} D_j \right\}$$

is dense in $H\left(\bigotimes_{j=1}^{n} K_j\right)$. Clearly,

$$H_0 \subseteq H_1 := \operatorname{span} \left\{ \prod_{j=1}^{n} h_j \;\middle|\; h_j \in H(K_j) \right\} \subseteq H\left(\bigotimes_{j=1}^{n} K_j\right)$$

holds, so H_1 must necessarily be dense in $H\left(\bigotimes_{j=1}^{n} K_j\right)$ too, which is exactly property 3. □

4.2 The Infinite Case

We now aim to generalize the result for finitely many spaces to countably infinitely many spaces $H(K_j)$. As in the finite case, we define $\bigotimes_{j\in\mathbb{N}} K_j$ as the pointwise product of the kernels K_j, though we restrict the domain if necessary. We will, in some cases, establish the existence of a canonical isomorphism between the RKHS $H(\bigotimes_{j\in\mathbb{N}} K_j)$ and the incomplete tensor product $\bigotimes_{j\in\mathbb{N}}^{e} H_j$ for an appropriately chosen $\mathcal{C}_0$-sequence e. There seem to be no interesting results involving the complete tensor product.

Remark 4.3. In this section, convergence of products *in the stricter sense* is more important than before, as it is often a requirement for results given. The definition of convergence in the stricter sense can be found in Remark 2.2. However, we also make use of convergence as defined in Definition 2.1, and we only mean convergence in the stricter sense when we specifically say so. We will make no use at all of quasi-convergence.

The Incomplete Tensor Product of RKHS

Let a set $D_j \neq \emptyset$ and a nonnegative definite kernel $K_j : D_j^2 \to \mathbb{K}$ be given for each $j \in \mathbb{N}$. We assume that for each kernel K_j there is some $x_j \in D_j$ such that $K_j(x_j, x_j) \neq 0$, which assures $H(K_j) \neq \{0\}$.

Define $D = \times_{j\in\mathbb{N}} D_j$. The following lemma gives us nonnegative definite kernels where the domain is a suitable subset of D^2.

Lemma 4.4. Let $X \subseteq D$ be a nonempty set such that $\prod_{j\in\mathbb{N}} K_j(x_j, y_j)$ converges for all $x, y \in D$. Then, the mapping

$$\bigotimes_{j\in\mathbb{N}} K_j : X^2 \to \mathbb{K} : (x, y) \mapsto \prod_{j\in\mathbb{N}} K_j(x_j, y_j)$$

is a nonnegative definite kernel.

Proof. By Lemma 4.1, for any $m, n \in \mathbb{N}$, any choice of $c_1, \ldots, c_n \in \mathbb{K}$ and any choice of $x^{(1)}, \ldots, x^{(n)} \in X$,

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \overline{c_i} c_k \prod_{j=1}^{m} K_j \left(x_j^{(i)}, x_j^{(k)} \right) \geq 0$$

holds. So, we also have

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \overline{c_i} c_k \prod_{j\in\mathbb{N}} K_j \left(x_j^{(i)}, x_j^{(k)} \right) = \lim_{m\to\infty} \sum_{i=1}^{n} \sum_{k=1}^{n} \overline{c_i} c_k \prod_{j=1}^{m} K_j \left(x_j^{(i)}, x_j^{(k)} \right) \geq 0.$$

Clearly, $\bigotimes_{j\in\mathbb{N}} K_j$ is also symmetric and thus a nonnegative definite kernel. □

Remark 4.5. Given a kernel $\bigotimes_{j\in\mathbb{N}} K_j$ as in Lemma 4.4 and $x \in X$, we have to carefully distinguish between $\bigotimes_{j\in\mathbb{N}} K_j(\cdot, x) \in H(\bigotimes_{j\in\mathbb{N}} K_j)$ and the potentially existing elementary tensor $\otimes_{j\in\mathbb{N}} K_j(\cdot, x_j) \in \bigotimes_{j\in\mathbb{N}} H(K_j)$.

Now, we want to find out for which sequences $h \in \times_{j\in\mathbb{N}} H(K_j)$ and which $x \in X$, the product $\prod_{j\in\mathbb{N}} h_j(x_j)$ is convergent. In particular, we establish a connection to $\mathcal{C}$-sequences, $\mathcal{C}_0$-sequences and their equivalence.

First, we obtain two results for sequences given directly by the kernels K_j.

Lemma 4.6. For any $x \in D$, the product $\prod_{j\in\mathbb{N}} K(x_j, x_j)$ converges if and only if $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ is a $\mathcal{C}$-sequence.

Proof. By the reproducing property, we get

$$\|K_j(\cdot, x_j)\|_{H_j}^2 = \langle K_j(\cdot, x_j), K_j(\cdot, x_j) \rangle_{H_j} = K_j(x_j, x_j)$$

for each $j \in \mathbb{N}$. Thus, the convergence of the products $\prod_{j\in\mathbb{N}} K_j(x_j, x_j)$ and $\prod_{j\in\mathbb{N}} \|K_j(\cdot, x_j)\|_{H_j}^2$ is equivalent.

Further, the convergence of $\prod_{j\in\mathbb{N}} \|K_j(\cdot, x_j)\|_{H_j}^2$ implies the convergence of $\prod_{j\in\mathbb{N}} \|K_j(\cdot, x_j)\|_{H_j}$ by continuity of the square root and the convergence of $\prod_{j\in\mathbb{N}} \|K_j(\cdot, x_j)\|_{H_j}$ clearly implies the convergence of $\prod_{j\in\mathbb{N}} \|K_j(\cdot, x_j)\|_{H_j}^2$ by simple limit laws. □

Lemma 4.7. For any $x \in D$, the sequence $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ is a $\mathcal{C}_0$-sequence if and only if $\prod_{j\in\mathbb{N}} K_j(x_j, x_j)$ converges in the stricter sense.

Proof. By the reproducing property, we get

$$\|K_j(\cdot, x_j)\|_{H_j}^2 = \langle K_j(\cdot, x_j), K_j(\cdot, x_j)\rangle_{H_j} = K_j(x_j, x_j)$$

for all $j \in \mathbb{N}$. As in Lemma 4.6, the convergence in the stricter sense of $\prod_{j\in\mathbb{N}} \|K_j(\cdot, x_j)\|_{H_j}^2$ is equivalent to the convergence in the stricter sense of $\prod_{j\in\mathbb{N}} \|K_j(\cdot, x_j)\|_{H_j}$. By Remark A.3, the product $\prod_{j\in\mathbb{N}} \|K_j(\cdot, x_j)\|_{H_j}$ converges in the stricter sense if and only if $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ is a $\mathcal{C}_0$-sequence. □

With this, we can employ $\mathcal{C}_0$-sequences and their equivalence to give criteria for the convergence and convergence in the stricter sense of products of the form $\prod_{j\in\mathbb{N}} h_j(x_j)$.

Lemma 4.8. Let $x \in D$ be given in such a way that $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ is a $\mathcal{C}$-sequence. Further, let h be any $\mathcal{C}$-sequence.

1. If either h or $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ is not a $\mathcal{C}_0$-sequence, we have

$$\prod_{j\in\mathbb{N}} h_j(x_j) = 0$$

   in the sense of convergence.

2. If both h and $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ are $\mathcal{C}_0$-sequences, they are equivalent if and only if the product $\prod_{j\in\mathbb{N}} h_j(x_j)$ converges in the stricter sense.

3. If both h and $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ are $\mathcal{C}_0$-sequences, they are *not* equivalent, but $\prod_{j\in\mathbb{N}} h_j(x_j)$ still converges,

$$\prod_{j\in\mathbb{N}} h_j(x_j) = 0$$

   holds.

Proof.

1. Without loss of generality, we assume that h is not a $\mathcal{C}_0$-sequence. Then, Lemma 2.11 implies $\prod_{j\in\mathbb{N}} \|h_j\|_{H_j} = 0$. By the reproducing property, we get

$$|h_j(x_j)| = \left|\langle h_j, K_j(\cdot, x_j)\rangle_{H(K_j)}\right| \leq \|h_j\|_{H(K_j)} \|K_j(\cdot, x_j)\|_{H_j}$$

   for all $j \in \mathbb{N}$. This implies

$$\prod_{j\in\mathbb{N}} |h_j(x_j)| \leq \prod_{j\in\mathbb{N}} \|h_j\|_{H(K_j)} \prod_{j\in\mathbb{N}} \|K_j(\cdot, x_j)\|_{H_j} = 0.$$

   By Lemma A.4, our claim then holds.

2. By the definition of equivalence, see Definition 2.12, and by Remark A.3, the sequences h and $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ are equivalent if and only if the product

$$\prod_{j\in\mathbb{N}} \langle h_j, K_j(\cdot, x_j)\rangle_{H(K_j)}$$

converges in the stricter sense. By the reproducing property, this is exactly the same product as $\prod_{j\in\mathbb{N}} h_j(x_j)$.

3. By Lemma 2.17, we have

$$\prod_{j\in\mathbb{N}} \langle h_j, K_j(\cdot, x_j)\rangle_{H(K_j)} = 0.$$

Again, by the reproducing property, our claim follows.

□

We now observe equivalence classes A that contain some $\mathcal{C}_0$-sequence of the form $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ for some $x \in D$. Later, in Theorem 4.10, we will see that any incomplete tensor product $\bigotimes_{j\in\mathbb{N}}^{A} H_j$ with respect to such an equivalence class can be represented as a RKHS in a canonical way. In preparation for that, we first show that the sequences of the mentioned form give rise to a dense subset of $\bigotimes_{j\in\mathbb{N}}^{A} H_j$.

Lemma 4.9. Let A be any equivalence class. If the set

$$X_A = \left\{ x \in D \;\middle|\; (K_j(\cdot, x_j))_{j\in\mathbb{N}} \text{ is a } \mathcal{C}_0\text{-sequence in A} \right\}$$

is nonempty, we have

$$\bigotimes_{j\in\mathbb{N}}{}^{A} H(K_j) = \overline{\operatorname{span}} \left\{ \otimes_{j\in\mathbb{N}} K_j(\cdot, x_j) \mid x \in X_A \right\}.$$

Proof. The inclusion

$$\bigotimes_{j\in\mathbb{N}}{}^{A} H(K_j) \supseteq \overline{\operatorname{span}} \left\{ \otimes_{j\in\mathbb{N}} K_j(\cdot, x_j) \mid x \in X_A \right\}$$

trivially holds.

Now, take any $y \in X_A$. By the definition of a $\mathcal{C}_0$-sequence, it is obvious that $K_j(\cdot, y_j) = 0$ can only hold for finitely many $j \in \mathbb{N}$. If we replace these by some $K_j(\cdot, x_j) \neq 0$, which is possible because we always assume $H(K_j) \neq \{0\}$, we obtain the existence of a $\mathcal{C}_0$-sequence of the form $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ in A, such that $K_j(\cdot, x_j) \neq 0$ always holds.

This allows us to define

$$e_j = \frac{K_j(\cdot, x_j)}{\|K_j(\cdot, x_j)\|_{H(K_j)}}$$

for each $j \in \mathbb{N}$. The tensor $\otimes_{j\in\mathbb{N}} e_j$ is then a $\mathcal{C}_0$-sequence in $\bigotimes_{j\in\mathbb{N}}^{A} H(K_j)$; this is shown as in the proof of Lemma 2.20.

Since $\left(K_j(\cdot, x_j)\right)_{j\in\mathbb{N}}$ is a $\mathcal{C}_0$-sequence, Lemma 4.8 implies the convergence in the stricter sense of $\prod_{j\in\mathbb{N}} \|K_j(\cdot, x_j)\|_{H(K_j)}$. Its value is not 0 by Lemma A.2. So, $\prod_{j\in\mathbb{N}} \|K_j(\cdot, x_j)\|_{H(K_j)}^{-1}$ also converges, and by this, we also have

$$\otimes_{j\in\mathbb{N}} e_j = \left(\prod_{j\in\mathbb{N}} \frac{1}{\|K_j(\cdot, x_j)\|_{H(K_j)}} \right) \left(\otimes_{j\in\mathbb{N}} K_j(\cdot, x_j)\right)$$

and thus

$$\otimes_{j\in\mathbb{N}} e_j \in \overline{\operatorname{span}}\left\{ \otimes_{j\in\mathbb{N}} K_j(\cdot, x_j) \mid x \in X_A \right\}.$$

Now, if we replace e_1 by any other element $h_1 \in H(K_1)$, the resulting tensor $h_1 \otimes \left(\otimes_{\substack{j\in\mathbb{N} \\ j\neq 1}} e_j \right)$ is still an element of $\overline{\operatorname{span}}\left\{ \otimes_{j\in\mathbb{N}} K_j(\cdot, x_j) \mid x \in X_A \right\}$. This is true because elementary tensors are linear and continuous in the sense of Lemma 2.44 and because

$$H(K_1) = \overline{\operatorname{span}}\left\{ K_1(\cdot, x_1) \mid x_1 \in D_1 \right\}$$

holds. If we now inductively replace finitely many e_j, we obtain

$$\begin{aligned} &\left\{ \otimes_{j\in\mathbb{N}} h_j \mid h_j \neq e_j \text{ for only finitely many } j \in \mathbb{N} \right\} \\ \subseteq\ &\overline{\operatorname{span}}\left\{ \otimes_{j\in\mathbb{N}} K_j(\cdot, x_j) \mid x \in X_A \right\}. \end{aligned}$$

Now, taking linear span and completion on both sides, we obtain

$$\bigotimes_{j\in\mathbb{N}}^{A} H_j \subseteq \overline{\operatorname{span}}\left\{ \otimes_{j\in\mathbb{N}} K_j(\cdot, x_j) \mid x \in X_A \right\}$$

by Proposition 2.39. This is what we wanted to show. □

Having established this, we can give the desired result.

Theorem 4.10. Under the assumptions of Lemma 4.9, the mapping

$$\bigotimes_{j\in\mathbb{N}} K_j : X_A^2 \to \mathbb{K} : (x, y) \mapsto \prod_{j\in\mathbb{N}} K_j(x_j, y_j)$$

is well-defined and a nonnegative definite kernel. Further, there exists a canonical isometric isomorphism

$$\Lambda : \bigotimes_{j\in\mathbb{N}}^{A} H(K_j) \to H\left(\bigotimes_{j\in\mathbb{N}} K_j \right)$$

that fulfills

$$\Lambda\left(\otimes_{j\in\mathbb{N}} h_j\right) = \prod_{j\in\mathbb{N}} h_j$$

for each $\mathcal{C}_0$-sequence $h \in A$. In other words, $H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)$ can be identified with $\bigotimes_{j\in\mathbb{N}}^{A} H(K_j)$ in the sense of Theorem 2.36.

Proof. Take $x, y \in X_A$. Since $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ and $(K_j(\cdot, y_j))_{j\in\mathbb{N}}$ are equivalent, Lemma 4.8 implies the convergence (even in the stricter sense) of the product $\prod_{j\in\mathbb{N}} K_j(x_j, y_j)$. Thus, $\bigotimes_{j\in\mathbb{N}} K_j$ is well-defined. By Lemma 4.4, it is a nonnegative definite kernel.

Now set

$$H_0 = \operatorname{span}\{\, \otimes_{j\in\mathbb{N}} K_j(\cdot, x_j) \mid x \in X_A \,\}$$

and define

$$\Lambda_0 : H_0 \to H\left(\bigotimes_{j\in\mathbb{N}} K_j\right) : \sum_{i=1}^{n} a_i \left(\otimes_{j\in\mathbb{N}} K_j(\cdot, x_j^{(i)})\right) \mapsto \sum_{i=1}^{n} a_i \left(\prod_{j\in\mathbb{N}} K_j(\cdot, x_j^{(i)})\right).$$

We show that this is well-defined by considering an element of H_0 with two representations

$$\sum_{i=1}^{n_1} a_i \left(\otimes_{j\in\mathbb{N}} K_j(\cdot, x_j^{(i)})\right) = \sum_{k=1}^{n_2} b_k \left(\otimes_{j\in\mathbb{N}} K_j(\cdot, y_j^{(k)})\right).$$

Then, for any $z \in X_A$, we have

$$\begin{aligned}
&\left\langle \otimes_{j\in\mathbb{N}} K_j(\cdot, z_j), \sum_{i=1}^{n_1} a_i \left(\otimes_{j\in\mathbb{N}} K_j(\cdot, x_j^{(i)})\right) \right\rangle_{\bigotimes^A_{j\in\mathbb{N}} H(K_j)} \\
&= \left\langle \otimes_{j\in\mathbb{N}} K_j(\cdot, z_j), \sum_{k=1}^{n_2} b_k \left(\otimes_{j\in\mathbb{N}} K_j(\cdot, y_j^{(k)})\right) \right\rangle_{\bigotimes^A_{j\in\mathbb{N}} H(K_j)},
\end{aligned}$$

which implies

$$\sum_{i=1}^{n_1} a_i \prod_{j\in\mathbb{N}} \left\langle K_j(\cdot, z_j), K_j(\cdot, x_j^{(i)})\right\rangle_{H(K_j)} = \sum_{k=1}^{n_2} b_k \prod_{j\in\mathbb{N}} \left\langle K_j(\cdot, z_j), K_j(\cdot, y_j^{(k)})\right\rangle_{H(K_j)}.$$

The reproducing property then implies

$$\sum_{i=1}^{n_1} a_i \prod_{j\in\mathbb{N}} K_j(z_j, x_j^{(i)}) = \sum_{k=1}^{n_2} b_k \prod_{j\in\mathbb{N}} K_j(z_j, y_j^{(k)}).$$

Since z was chosen arbitrarily, this implies that Λ_0 is well-defined.

Obviously, Λ_0 is linear. We show that Λ_0 is isometric. For $x, y \in X_A$, we have

$$\begin{aligned}
&\langle \otimes_{j\in\mathbb{N}} K_j(\cdot, x_j), \otimes_{j\in\mathbb{N}} K_j(\cdot, y_j)\rangle_{\bigotimes^A_{j\in\mathbb{N}} H_j} = \prod_{j\in\mathbb{N}} \langle K_j(\cdot, x_j), K_j(\cdot, y_j)\rangle_{H(K_j)} \\
&= \langle \Lambda_0\left(\otimes_{j\in\mathbb{N}} K_j(\cdot, x_j)\right), \Lambda_0\left(\otimes_{j\in\mathbb{N}} K_j(\cdot, y_j)\right)\rangle_{H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)}.
\end{aligned}$$

The last equality immediately follows from the general form of the scalar product of a RKHS, see Lemma 3.2. By linearity of the scalar product, Λ_0 leaves the scalar product invariant and thus is isometric. This also implies that Λ_0 is injective.

The isometric nature of Λ_0 now allows us to extend the domain of Λ_0 to $\bigotimes_{j\in\mathbb{N}}^{A} H(K_j)$. By Lemma 4.9, for any $h \in \bigotimes_{j\in\mathbb{N}}^{A} H(K_j)$, there is a sequence $(h_n)_{n\in\mathbb{N}}$ which has elements in H_0 and converges to h. We set

$$\Lambda(h) = \lim_{n\to\infty} \Lambda_0(h_n).$$

We show that this is well-defined. Since Λ_0 is isometric, the sequence given as $(\Lambda_0(h_n))_{n\in\mathbb{N}}$ is a Cauchy-sequence and thus converges. For a second sequence $(g_n)_{n\in\mathbb{N}}$ in H_0 converging to h, we obtain

$$\lim_{n\to\infty} \|\Lambda_0(h_n) - \Lambda_0(g_n)\|_{H(\otimes_{j\in\mathbb{N}} H(K_j))} = \lim_{n\to\infty} \|h_n - g_n\|_{\bigotimes_{j\in\mathbb{N}}^{A} H(K_j)} = 0.$$

Thus,

$$\Lambda : \bigotimes_{j\in\mathbb{N}}{}^{A} H(K_j) \to H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)$$

is well-defined. It is clearly linear and isometric. It is also surjective: By Corollary 3.6, we know that $\Lambda_0(H_0)$ is dense in $H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)$. Thus, for any $h \in H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)$, there is a sequence $(h_n)_{n\in\mathbb{N}}$ in H_0 that fulfills

$$\lim_{n\to\infty} \Lambda_0(h_n) = h.$$

This implies $\Lambda(\lim_{n\to\infty} h_n) = h$.

That $\Lambda(\otimes_{j\in\mathbb{N}} h_j) = \prod_{j\in\mathbb{N}} h_j$ holds for each $\mathcal{C}_0$-sequence $(h_j)_{j\in\mathbb{N}}$ is immediately obvious from the construction of Λ. □

If the incomplete tensor product is not given with respect to an equivalence class A but rather with respect to a sequence of unit vectors e as in Remark 2.40, we obtain the following result.

Corollary 4.11. Let $e \in \mathcal{C}_0$ be given in such a way that $\|e_j\|_{H_j} = 1$ holds for every $j \in \mathbb{N}$. Define the set

$$X_e = \left\{ x \in \bigtimes_{j\in\mathbb{N}} D_j \;\middle|\; \prod_{j\in\mathbb{N}} e_j(x_j) \text{ converges in the stricter sense} \right\}.$$

If X_e is nonempty, the mapping

$$\bigotimes_{j\in\mathbb{N}} K_j : X_e^2 \to \mathbb{K} : (x, y) \mapsto \prod_{j\in\mathbb{N}} K_j(x_j, y_j)$$

is a nonnegative definite kernel and there exists a canonical isometric isomorphism

$$\Lambda : \bigotimes_{j\in\mathbb{N}}{}^{e} H(K_j) \to H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)$$

that fulfills

$$\Lambda\left(\otimes_{j\in\mathbb{N}} h_j\right) = \prod_{j\in\mathbb{N}} h_j$$

for each $\mathcal{C}_0$-sequence h that is equivalent to e.

Proof. By Lemma 4.8, the product $\prod_{j\in\mathbb{N}} e_j(x_j)$ converges in the stricter sense if and only if the sequence $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ is equivalent to e. So, if A is the equivalence class e belongs to, and we define the set X_A as in Lemma 4.9, we have $X_A = X_e$. Apply Theorem 4.10. □

In the case that $\prod_{j\in\mathbb{N}} K_j(x_j, y_j)$ always converges in the stricter sense, we obtain the following.

Corollary 4.12. If $\prod_{j\in\mathbb{N}} K_j(x_j, y_j)$ converges in the stricter sense for every choice of $x, y \in D$,

$$\bigotimes_{j\in\mathbb{N}} K_j : D^2 \to \mathbb{K} : (x, y) \to \prod_{j\in\mathbb{N}} K_j(x_j, y_j)$$

is a nonnegative definite kernel. Further, there is an equivalence class A such that $H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)$ is canonically isomorphic to $\bigotimes_{j\in\mathbb{N}}^{A} H(K_j)$. If $e \in \mathcal{C}_0$ is given in such a way that $\|e_j\|_{H_j} = 1$ holds for every $j \in \mathbb{N}$, and the product $\prod_{j\in\mathbb{N}} e_j(x_j)$ converges in the stricter sense for any $x \in D$, it converges in the stricter sense regardless of the choice of x and we have

$$\bigotimes_{j\in\mathbb{N}}{}^{e} H(K_j) = \bigotimes_{j\in\mathbb{N}}{}^{A} H(K_j).$$

Proof. That $\bigotimes_{j\in\mathbb{N}} K_j$ is a nonnegative definite kernel follows by Lemma 4.4. By Lemma 4.8, all sequences $(K_j(\cdot, y_j))_{j\in\mathbb{N}}$ belong to the same equivalence class A, regardless of the choice of y. The canonical isomorphism is then given by Theorem 4.10.

Convergence in the stricter sense of $\prod_{j\in\mathbb{N}} e_j(x_j)$ implies that $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ and e are equivalent by Lemma 4.8. By the transitive property, $(K_j(\cdot, y_j))_{j\in\mathbb{N}}$ and e are equivalent and Lemma 4.8 then implies the convergence of $\prod_{j\in\mathbb{N}} e_j(y_j)$.

The last claim follows because $e \in A$. □

We can give a slight relaxation of the requirements of Corollary 4.12.

Proposition 4.13. Assume that $\prod_{j\in\mathbb{N}} K_j(x_j, y_j)$ converges for every choice of $x, y \in D$, but not necessarily in the stricter sense.

Assume convergence in the stricter sense of the products $\prod_{j\in\mathbb{N}} K_j(x_j, x_j)$ and $\prod_{j\in\mathbb{N}} K_j(y_j, y_j)$ implies convergence in the stricter sense of $\prod_{j\in\mathbb{N}} K_j(x_j, y_j)$ for every choice of $x, y \in D$. Further assume that $\prod_{j\in\mathbb{N}} K_j(x_j, x_j)$ converges in the stricter sense for at least one $x \in D$. Then,

$$\bigotimes_{j\in\mathbb{N}} K_j : D^2 \to \mathbb{K} : (x, y) \to \prod_{j\in\mathbb{N}} K_j(x_j, y_j)$$

is a nonnegative definite kernel. Further, there is an equivalence class A such that $H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)$ is canonically isomorphic to $\bigotimes_{j\in\mathbb{N}}^{A} H(K_j)$.

If there is a $\mathcal{C}_0$-sequence e such that $\|e_j\|_{H_j} = 1$ holds for all $j \in \mathbb{N}$, and additionally there is some $x \in D$ such that the products $\prod_{j\in\mathbb{N}} K_j(x_j, x_j)$ and $\prod_{j\in\mathbb{N}} e_j(x_j)$ converge in the stricter sense, we have

$$\bigotimes_{j\in\mathbb{N}}{}^{e} H(K_j) = \bigotimes_{j\in\mathbb{N}}{}^{A} H(K_j).$$

Proof. Under the requirements of this proposition, Lemma 4.8 implies that all $\mathcal{C}_0$-sequences of the form $(K_j(\cdot, x_j))_{j\in\mathbb{N}}$ are elements of the same equivalence class, say A. If we define

$$X = \left\{ x \in D \;\middle|\; \prod_{j\in\mathbb{N}} K_j(x_j, x_j) \text{ converges in the stricter sense} \right\},$$

which is nonempty by assumption, and

$$\widetilde{\bigotimes_{j\in\mathbb{N}}} K_j : X^2 \to \mathbb{K} : (x, y) \to \prod_{j\in\mathbb{N}} K_j(x_j, y_j),$$

we can apply Theorem 4.10 to obtain a canonical isomorphism

$$\Lambda : \bigotimes_{j\in\mathbb{N}}{}^{A} H(K_j) \to H\left(\widetilde{\bigotimes_{j\in\mathbb{N}}} K_j\right)$$

as described in that theorem.

Now, define

$$\Phi : H\left(\bigotimes_{j\in\mathbb{N}} K_j\right) \to H\left(\widetilde{\bigotimes_{j\in\mathbb{N}}} K_j\right) : h \mapsto h_{|X}.$$

This is well defined: For $x, y \in D$, if $x \in D \setminus X$ or $y \in D \setminus X$, Lemma 4.8 implies $\prod_{j\in\mathbb{N}} K_j(x_j, y_j) = 0$. In particular, for any $x \in D \setminus X$, we have

$$\bigotimes_{j\in\mathbb{N}} K_j(\cdot, x) = 0.$$

So, in this case, we can reword Corollary 3.6 in the following way: The space $H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)$ is the closure of

$$\operatorname{span}\left\{ \bigotimes_{j\in\mathbb{N}} K_j(\cdot, x) \;\middle|\; x \in X \right\},$$

so we replaced D with X.

Now, each $h \in H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)$ is the pointwise limit of some sequence of the form $\left(\sum_{k_n=1}^{m_n} a_{k_n} \cdot \bigotimes_{j\in\mathbb{N}} K(\cdot, x_{(k_n)})\right)_{n\in\mathbb{N}}$, where the a_{k_n} are complex numbers and the $x_{(k_n)}$ can be chosen from X according to the above. Clearly, $h_{|X}$ is the pointwise limit of the sequence $\left(\sum_{k_n=1}^{m_n} a_{k_n} \cdot \widetilde{\bigotimes_{j\in\mathbb{N}}} K(\cdot, x_{(k_n)})\right)_{n\in\mathbb{N}}$. Thus, we have shown that $h_{|X} \in H\left(\widetilde{\bigotimes_{j\in\mathbb{N}}} K_j\right)$ and thus that Φ is well-defined. It is isometric, which is clear by the definition of the scalar product in a RKHS and by the above consideration. Now, for $h \in H\left(\widetilde{\bigotimes_{j\in\mathbb{N}}} K_j\right)$, define

$$\hat{h} : D \to \mathbb{K} : x \mapsto \begin{cases} h(x), & \text{if } x \in X \\ 0, & \text{else.} \end{cases}$$

Clearly, $\hat{h} \in H\left(\bigotimes_{j\in\mathbb{N}} K_j\right)$ and $\Phi(\hat{h}) = h$, so Φ is surjective.

The canonical isomorphism is given by $\Phi^{-1} \circ \Lambda$.

Now, if e is given as required, the convergence in the stricter sense of $\prod_{j\in\mathbb{N}} K_j(x_j, x_j)$ implies that $(K_j(\cdot, x_j))$ is a $\mathcal{C}_0$-sequence and thus an element of A. The convergence in the stricter sense of $\prod_{j\in\mathbb{N}} e_j(x_j)$ then implies the equivalence of e and $(K_j(\cdot, x_j))$, and thus we obtain $e \in A$. This implies $\bigotimes_{j\in\mathbb{N}}^{e} H(K_j) = \bigotimes_{j\in\mathbb{N}}^{A} H(K_j)$. □

A Convergence of Infinite Products

The lemmas here are taken from Section 2 of [11] if not otherwise specified.

Lemma A.1. Let $(z_j)_{j\in\mathbb{N}}$ be a sequence of real nonnegative numbers. Then, $\prod_{j\in\mathbb{N}} z_j$ converges if and only if $\sum_{j\in\mathbb{N}} \max(z_j - 1, 0)$ converges or $z_j = 0$ for some $j \in \mathbb{N}$.

Lemma A.2. Let $(z_j)_{j\in\mathbb{N}}$ be a sequence of complex numbers. Then, $\prod_{j\in\mathbb{N}} z_j$ converges to a value other than 0 if and only if $\sum_{j\in\mathbb{N}} |z_j - 1|$ converges and $z_j \neq 0$ for all $j \in \mathbb{N}$.

Remark A.3. Changing finitely many summands does not change whether an infinite series converges or not, so Lemma A.2 implies that $\sum_{j\in\mathbb{N}} |z_j - 1|$ converges if and only if there is some $j_0 \in \mathbb{N}$ such that $z_j \neq 0$ holds for all $j \geq j_0$ and the product $\prod_{j=j_0}^{\infty} z_j$ converges to a value other than 0. In other words, $\sum_{j\in\mathbb{N}} |z_j - 1|$ converges if and only if $\prod_{j\in\mathbb{N}} z_j$ converges in the stricter sense, see Remark 2.2.

Lemma A.4. Let $(z_j)_{j\in\mathbb{N}}$ be a sequence of complex numbers. Let $(\varphi_j)_{j\in\mathbb{N}}$ be a sequence such that $\varphi_j \in [-\pi, \pi)$ and $z_j = |z_j| \exp(i\varphi_j)$ for all $j \in \mathbb{N}$ (polar coordinates).

1. $\prod_{j\in\mathbb{N}} z_j = 0$ if and only if $\prod_{j\in\mathbb{N}} |z_j| = 0$.
2. $\prod_{j\in\mathbb{N}} z_j$ converges to a value $a \neq 0$ if and only if $\prod_{j\in\mathbb{N}} |z_j|$ converges to a value $b \neq 0$ and $\sum_{j\in\mathbb{N}} |\varphi_j|$ converges.

In particular, the convergence of $\prod_{j\in\mathbb{N}} z_j$ implies the convergence of $\prod_{j\in\mathbb{N}} |z_j|$, but not vice versa.

Lemma A.5. Let $(z_j)_{j\in\mathbb{N}}$ and $\left(z'_j\right)_{j\in\mathbb{N}}$ be two sequences of complex numbers. Then, the equality

$$\prod_{j\in\mathbb{N}} z_j z'_j = \prod_{j\in\mathbb{N}} z_j \prod_{j\in\mathbb{N}} z'_j$$

holds, in the sense that the values of both sides are the same, if both $\prod_{j\in\mathbb{N}} z'_j$ and $\prod_{j\in\mathbb{N}} z'_j$ are quasi-convergent and at least one of the two is convergent.

Proof. If both $\prod_{j\in\mathbb{N}} z_j$ and $\prod_{j\in\mathbb{N}} z'_j$ are convergent, this is trivial. So, assume $\prod_{j\in\mathbb{N}} z_j$ is convergent and $\prod_{j\in\mathbb{N}} z'_j$ is not. In this case, we have

$$\prod_{j\in\mathbb{N}} z_j \prod_{j\in\mathbb{N}} z'_j = 0.$$

If $\prod_{j\in\mathbb{N}} z_j = 0$ holds, we obtain $\prod_{j\in\mathbb{N}} z_j z'_j = 0$ by simple limit laws, since $\prod_{j\in\mathbb{N}} \left|z'_j\right|$ is convergent.

Now, let $\prod_{j\in\mathbb{N}} z_j \neq 0$. In this case, $\prod_{j\in\mathbb{N}} 1/z_j$ also converges. If $\prod_{j\in\mathbb{N}} z_j z'_j$ is not convergent, we are done; if it is convergent, so is $\prod_{j\in\mathbb{N}} z_j z'_j 1/z_j = \prod_{j\in\mathbb{N}} z'_j$, which is a contradiction. $\square$

References

[1] Aronszajn. Theory of reproducing kernels. *Transactions of the American Mathematical Society*, 68:337–404, 1950.

[2] Bosch. *Algebra*. Springer, Berlin, seventh edition, 2009.

[3] Elstrod. *Maß- und Integrationstheorie*. Springer, Berlin, fourth edition, 2005.

[4] Horn; Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, second edition, 2013.

[5] Knopp. *Theory and application of infinite series*. Dover Publications Inc., New York, Dover edition, 1990.

[6] Kadison; Ringrose. *Fundamentals of the theory of operator algebras. Vol. I*. American Mathematical Society, Providence, 1997. Elementary theory, reprint of the 1983 original.

[7] Ritter. *Average-case analysis of numerical problems*. Springer, Berlin, 2000.

[8] Schmidt. *Maß und Wahrscheinlichkeit*. Springer, Berlin, second edition, 2011.

[9] Schur. Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen. *Journal für die reine und angewandte Mathematik*, 140:1–28, 1911.

[10] von Neumann. On rings of operators. *Annals of Mathematics*, 37:116–229, 1936.

[11] von Neumann. On infinite direct products. *Compositio Mathematica*, 6:1–77, 1939.

[12] Gnewuch; Hefter; Hinrichs; Ritter; Wasilkowski. Embeddings for infinite-dimensional integration and L_2-approximation with increasing smoothness. *Journal of Complexity*, 54:Article 101406, 2019.

[13] Werner. *Funktionalanalysis*. Springer, Berlin, eighth edition, 2018.

Schriftliche Erklärung

Ich erkläre hiermit, dass ich, Robin Rüßmann, die vorliegende Masterarbeit selbstständig verfasst, keine anderen Hilfsmittel als die angegebenen Quellen und Hilfsmittel verwendet, und Zitate kenntlich gemacht habe.

Kaiserslautern, 15. Januar 2020

Robin Rüßmann