

Symmetric subgroups of rational groups of hermitian type

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Introduction

A rational group of hermitian type is a \mathbb{Q} -simple algebraic group G such that the symmetric space \mathcal{D} of maximal compact subgroups of the real Lie group $G(\mathbb{R})$ is a hermitian symmetric space of the non-compact type. There are two major classes of subgroups of G of importance to the geometry of \mathcal{D} and to arithmetic quotients $X_\Gamma = \Gamma \backslash \mathcal{D}$ of \mathcal{D} : *parabolic* subgroups and *reductive* subgroups. The former are connected with the boundary of the domain \mathcal{D} , while the latter are connected with submanifolds \mathcal{D}' of the *interior* of \mathcal{D} , *a priori* just symmetric spaces. Under certain assumptions on the reductive subgroup, $\mathcal{D}' \subset \mathcal{D}$ is a *holomorphic symmetric embedding*, displaying \mathcal{D}' as a sub-hermitian symmetric space. It is these latter groups we study in this paper, and we call them quite generally “symmetric subgroups”; these are the subgroups occurring in the title.

The purpose of this paper is to prove an existence result of the following kind: given a maximal \mathbb{Q} -parabolic $P \subset G$, which is the stabilizer of a boundary component F , meaning that $P(\mathbb{R}) = N(F)$, there exists a \mathbb{Q} -subgroup $N \subset G$, which is a symmetric subgroup defining a subdomain $\mathcal{D}_N \subset \mathcal{D}$ such that: F is a boundary component of \mathcal{D}_N . To formulate this precisely, we make the following assumptions on G which are to be in effect throughout the paper: G is isotropic, and $G(\mathbb{R})$ is not a product of groups of type $SL_2(\mathbb{R})$. The basis of our formulation is the notion of *incident* parabolic and symmetric subgroups. We first define this for the real groups. Fix a maximal \mathbb{R} -parabolic $P \subset G(\mathbb{R})$, and let as above $F \subset \mathcal{D}^*$ denote the corresponding boundary component stabilized by P . First assume that F is positive-dimensional. Let $N \subset G(\mathbb{R})$ be a symmetric subgroup; we shall say N and P are *incident*, if the following conditions are satisfied.

- 1) N has maximal \mathbb{R} -rank, that is, $\text{rank}_{\mathbb{R}} N = \text{rank}_{\mathbb{R}} G$.
- 2) N is a maximal symmetric subgroup.
- 3) $N = N_1 \times N_2$, where $N_1 \subset P$ is a hermitian Levi factor of P for some Levi decomposition (these terms are explained in more detail in the text).

From work of Satake and Ihara one knows, up to conjugation, all subgroups N fulfilling the above; these results are collected in Table 1, and such a group is uniquely determined up to conjugation (by an element of $K \subset G(\mathbb{R})$, a maximal compact subgroup). It is trickier to define something similar for the case $\dim(F) = 0$. This is easiest to see by considering the geometric equivalent of the conditions above. They state that the subdomain \mathcal{D}_N defined by N has the same \mathbb{R} -rank as \mathcal{D} , is a maximal subdomain, and finally, that \mathcal{D}_N contains the given boundary component F in *its* boundary, $F \subset \overline{\mathcal{D}_N}$, and furthermore that $\mathcal{D}_N = \mathcal{D}_1 \times \mathcal{D}_2$, where \mathcal{D}_1 is (as an abstract hermitian symmetric

space) isomorphic to F . The condition 1) is still a natural one to assume, but 2) is in general too strong, while 3), in the case of $\dim(F) = 0$, seems to imply \mathcal{D}_N should be irreducible and contain F as a boundary component. We then replace, for $\dim(F) = 0$, the conditions 2) and 3) by 2') and 3') below:

- 2') N is a maximal subgroup of tube type, i.e., such that \mathcal{D}_N is a tube domain.
- 3') \mathcal{D}_N is maximal irreducible and contains F as a boundary component.

The subgroups N fulfilling 1), 2), 3') or 1), 2'), 3') are also known and are listed in Table 2. But there simply do not always exist such subgroups; more precisely, for the domains

$$(\mathcal{ED}) \quad \mathbf{I}_{q,q}, \mathbf{II}_n, \mathbf{n \text{ even}}, \mathbf{III}_n,$$

there are no subgroups N fulfilling the above. So in these cases, we introduce the conditions

- 2'') N is *minimal*, subject to 1).
- 3'') \mathcal{D}_N contains F as a boundary component.

It is clear that condition 1) and 2'') together imply that the domain \mathcal{D}_N is a polydisc, $\mathcal{D}_N \cong (\mathcal{D}_{1,1})^t$, where $t = \text{rank}_{\mathbb{R}} G$, and $\mathcal{D}_{1,1} = SL_2(\mathbb{R})/K$ is the one-dimensional disc. Altogether these conditions insure that for any maximal \mathbb{R} -parabolic $P \subset G(\mathbb{R})$, there is a finite set of incident symmetric subgroups $N \subset G(\mathbb{R})$, such that any subgroup incident to P is isomorphic to one of them. Moreover, if F is positive-dimensional, N is unique up to isomorphism.

Recall now that given $G(\mathbb{R})$ of hermitian type, $\text{rank}_{\mathbb{R}} G(\mathbb{R}) = t$, one can find t strongly orthogonal roots μ_1, \dots, μ_t which determine a unique maximal \mathbb{R} -split torus $A \subset G(\mathbb{R})$, the corresponding root system $\Phi(A, G)$, as well as a set of simple \mathbb{R} -roots. The closed, symmetric set of roots $\Psi = \{\pm\{\mu_1\}, \dots, \pm\{\mu_t\}\}$ then determines a (unique) \mathbb{R} -subgroup $N_{\Psi} \subset G(\mathbb{R})$, which is isomorphic to $(SL_2(\mathbb{R}))^t$. Conversely, the maximal \mathbb{R} -split torus A defines the root system $\Phi(A, G)$, and in this root system there is a good unique choice for the strongly orthogonal roots μ_1, \dots, μ_t (see section 1.1), so A determines Ψ and Ψ determines A . For each of the symmetric groups above, it is natural to consider also the condition

- 4) Assume the parabolic P is standard with respect to A . Then $N_{\Psi} \subset N$, where Ψ and A determine one another in the manner just described.

We then pose the following problem. Given the rational group of hermitian type G , and given a maximal \mathbb{Q} -parabolic $P \subset G$, find a symmetric \mathbb{Q} -subgroup $N \subset G$, such that P and N are *incident*, meaning that $N(\mathbb{R})$ and $P(\mathbb{R})$ are incident in $G(\mathbb{R})$ in the above sense, i.e., $N(\mathbb{R})$ fulfills 1), 2) and 3) (if $P(\mathbb{R}) = N(F)$, $\dim(F) > 0$), 1), 2') and 3') or 1), 2'') and 3'') in the corresponding cases $\dim(F) = 0, \mathcal{D} \notin (\mathcal{ED})$ respectively $\dim(F) = 0, \mathcal{D} \in (\mathcal{ED})$.

The main result of the paper is that the problem just posed can be solved in almost all cases, specified in the following theorem.

Main Theorem *Let G be \mathbb{Q} -simple of hermitian type subject to the restrictions above ($G(\mathbb{R})$ not a product of $SL_2(\mathbb{R})$'s and G isotropic), $P \subset G$ a maximal \mathbb{Q} -parabolic. Then there exists a reductive \mathbb{Q} -subgroup $N \subset G$ such that (P, N) are incident, in the sense just defined, with the following exceptions:
In the case of zero-dimensional boundary components,*

- Index $C_{2n,n}^{(2)}$, that is, G is isogenous to the group $\text{Res}_{k|\mathbb{Q}} G'$, $G' = SU(V, h)$, the special unitary group for a hermitian form $h : V \times V \rightarrow D$, where V is a $2n$ -dimensional right D -vector space, D a totally indefinite quaternion division algebra which is central simple over k (k a totally real number field), and the Witt index of h is n .

In this case the k -subgroups fulfilling 1), 2'') (over k) and 3'') and the corresponding domains are

- $n > 1$: $N = \text{Res}_{k|\mathbb{Q}} N'$, $N' \cong SU(V_1, h_{|V_1}) \times \cdots \times SU(V_n, h_{|V_n})$, where V_i is a hyperbolic plane, $V \cong V_1 \oplus \cdots \oplus V_n$, and then $N'(\mathbb{R}) \cong Sp(4, \mathbb{R}) \times \cdots \times Sp(4, \mathbb{R})$, and the domain $\mathcal{D}_{N'}$ is of type $\mathbf{III}_2 \times \cdots \times \mathbf{III}_2$.
- $n = 1$: $N = \text{Res}_{k|\mathbb{Q}} N'$, $N' \cong SU(V_L, h_{|V_L})$, where $V_L \subset V$ is a plane defined by the inclusion of an imaginary quadratic extension L of k , $L \subset D$ (viewing V as a k -vector space), and then $N'(\mathbb{R}) \cong SL_2(\mathbb{R})$ and the domain $\mathcal{D}_{N'}$ is of type \mathbf{III}_1 , a disc (in particular not fulfilling 1)).

Moreover, there is a \mathbb{Q} -subgroup N fulfilling also condition 4) with the exception of the index $C_{2,1}^{(2)}$.

Hence, for indices $C_{2n,n}^{(2)}$ with $n > 1$ we can find N fulfilling 1) and 3''), but neither 2') nor 2''), but rather only 2'') over k , that is, requiring N to be minimal over k . In the case $n = 1$, it would seem that there are no subgroups satisfying condition 1) at all; at any rate we could find none. This case was considered in detail in [Hyp]. As a final remark, we could eliminate the exceptional status of the cases $n > 1$ by changing 2'') accordingly, but we feel these groups are indeed exceptions.

The main application of the above theorem is to arithmetic quotients of bounded symmetric domains. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup, $X_\Gamma = \Gamma \backslash \mathcal{D}$ the arithmetic quotient. Since the subgroups $N \subset G$ of the theorem are defined over \mathbb{Q} , $\Gamma_N := \Gamma \cap N(\mathbb{Q})$ is an arithmetic subgroup in N , and $X_{\Gamma_N} := \Gamma_N \backslash \mathcal{D}_N$ is an arithmetic quotient. Clearly $X_{\Gamma_N} \subset X_\Gamma$, and, viewing X_Γ as an algebraic variety, X_{Γ_N} is a subvariety, which we call a *modular* subvariety. More precisely, one has the Baily-Borel embedding of the Satake compactification X_Γ^* , which makes X_Γ^* an algebraic variety and X_Γ a Zariski open subset of a projective variety. By a theorem of Satake [S2], the natural inclusion $X_{\Gamma_N} \subset X_\Gamma$ extends to an inclusion of the Baily-Borel compactifications, displaying $X_{\Gamma_N}^*$ as a (singular) algebraic subvariety of the algebraic variety X_Γ^* . The notion of incidence then manifests itself in the following way. A boundary component $V_i \subset X_\Gamma^*$ (here V_i is itself an arithmetic quotient) and modular subvariety $X_{\Gamma_N}^*$ are incident, if V_i is a boundary component of $X_{\Gamma_N}^*$. These matters will be taken up elsewhere.

Let us now sketch the idea of proof of the Main Theorem. We consider the different possibilities for the \mathbb{Q} -rank of G . By assumption, G is isotropic, so $\text{rank}_{\mathbb{Q}} G \geq 1$. We split the possible cases into the following three items, corresponding roughly to increasing order of difficulty:

- 1) G is split over \mathbb{R} (explained below).
- 2) G is not split over \mathbb{R} , but $\text{rank}_{\mathbb{Q}} G \geq 2$.
- 3) $\text{rank}_{\mathbb{Q}} G = 1$.

The first case occurs as follows. Since G is \mathbb{Q} -simple, there is a totally real number field k and an *absolutely* simple k -group G' such that $\text{Res}_{k|\mathbb{Q}} G' = G$. Let S be a maximal k -split torus of G' , $A \subset G'(\mathbb{R})$ a maximal \mathbb{R} -split torus, which one may assume contains S , $S \subset A$. Then G' is *split over* \mathbb{R} , if $S = A$, i.e., a maximal k -split torus is also a maximal \mathbb{R} -split torus. In this case the proof of the main theorem is more or less trivial, and is carried out in §4. Case 2) is already more subtle; we combine constructions based on the correspondence between semisimple Lie algebras (of the classical type) and central simple algebras with involution, as presented in [W], with the known classification of k -indices of absolutely simple groups, as presented in [T]. This is carried out in §5, and in this as well as the next section most of the work is necessary for the case of zero-dimensional boundary components. The most interesting case, done in the last paragraph, is the case of rank one. Here it turns out that similar arguments as above apply in the cases that the boundary component F is positive-dimensional; but in the case of zero-dimensional boundary components, new arguments are

required. In particular, the case of *hyperbolic planes* is fundamental, and this was studied in detail in [Hyp]. Drawing on the results of that paper we can complete the proof of the theorem.

1 Real parabolics of hermitian type

1.1 Notations

In this paper we will basically adhere to the notations of [BB]. In the first two paragraphs G will denote a real Lie group; later G will be a \mathbb{Q} -group of hermitian type. We assume G is reductive, connected and with compact center; $K \subset G$ will denote a maximal compact subgroup, $\mathcal{D} = G/K$ the corresponding symmetric space. Throughout this paper we will assume G is of *hermitian type*, meaning that \mathcal{D} is a hermitian symmetric space, hence a product $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_d$ of irreducible factors, each of which we assume is *non-compact*. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ denote a Cartan decomposition of the Lie algebra of G , $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^+ + \mathfrak{p}^-$ the decomposition of the complexified Lie algebra, with \mathfrak{p}^{\pm} abelian subalgebras (and $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$). Choosing a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the set of roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ is denoted $\Phi = \Phi(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$. As usual, we choose root vectors $E_{\alpha} \in \mathfrak{g}^{\alpha}$ such that the relations

$$[E_{\alpha}, E_{-\alpha}] = H_{\alpha} \in \mathfrak{h}_{\mathbb{C}}, \quad \alpha(H_{\beta}) = 2 \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}, \quad \alpha, \beta \in \Phi,$$

hold. Complex conjugation maps \mathfrak{p}^+ to \mathfrak{p}^- , in fact permuting E_{α} and $E_{-\alpha}$ for $E_{\alpha} \in \mathfrak{p}^{\pm}$. Moreover, if $\Sigma^{\pm} := \{\alpha | E_{\alpha} \in \mathfrak{p}^{\pm}\}$, then

$$\mathfrak{p}^{\pm} = \text{span}_{\mathbb{C}}(E_{\alpha}), \quad \alpha \in \Sigma^{\pm}; \quad \mathfrak{p} = \text{span}_{\mathbb{R}}(X_{\alpha}, Y_{\alpha}), \quad \alpha \in \Sigma^+,$$

where $X_{\alpha} = E_{\alpha} + E_{-\alpha}$, $Y_{\alpha} = i(E_{\alpha} - E_{-\alpha})$ (twice the real and the (negative of the) imaginary parts, respectively). Let μ_1, \dots, μ_t denote a maximal set of strongly orthogonal roots, determined as in [H]: μ_1 is the smallest root in Σ^+ , and μ_j is the smallest root in Σ^+ which is strongly orthogonal to μ_1, \dots, μ_{j-1} . This set will be fixed once and for all.

Once this set of strongly orthogonal roots has been chosen, a maximal \mathbb{R} -split torus A is uniquely determined by $\text{Lie}(A) = \mathfrak{a} = \text{span}_{\mathbb{R}}(X_{\mu_1}, \dots, X_{\mu_t})$. Then $\Phi_{\mathbb{R}} = \Phi(\mathfrak{a}, \mathfrak{g})$ will denote the set of \mathbb{R} -roots, and \mathfrak{g} has a decomposition

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{a}) \oplus \sum_{\eta \in \Phi_{\mathbb{R}}} \mathfrak{g}^{\eta},$$

where $\mathfrak{g}^{\eta} = \{x \in \mathfrak{g} | \text{ad}(s)x = \eta(s)x, \forall s \in A\}$. For each irreducible component of G , the set of \mathbb{R} -roots is either of type \mathbf{C}_t or \mathbf{BC}_t , and of type $\mathbf{C}_t \iff$ the corresponding domain is a tube domain. If ξ_i denote coordinates on \mathfrak{a} dual to X_{μ_i} , assuming for the moment \mathcal{D} to be irreducible, the \mathbb{R} -roots are explicitly

$$\begin{aligned} \Phi_{\mathbb{R}} : \quad & \pm(\xi_i \pm \xi_j), \pm 2\xi_i \quad (1 \leq i \leq t, i < j) && \text{(Type } \mathbf{C}_t) \\ & \pm(\xi_i \pm \xi_j), \pm 2\xi_i, \pm \xi_i \quad (1 \leq i \leq t, i < j) && \text{(Type } \mathbf{BC}_t) \\ \Delta_{\mathbb{R}} : \quad & \eta_i = \xi_i - \xi_{i+1}, i = 1, \dots, t-1, \text{ and } \eta_t = 2\xi_t \text{ (Type } \mathbf{C}_t), \eta_t = \xi_t \text{ (Type } \mathbf{BC}_t). \end{aligned} \quad (1)$$

Here the simple roots $\Delta_{\mathbb{R}}$ are with respect to the lexicographical order on the ξ_i . A general \mathbb{R} -root system is a disjoint union of simple root systems. The choice of maximal set of strongly orthogonal roots determines an order on \mathfrak{a} (the lexicographical order), which determines, on each simple \mathbb{R} -root system, an order as above; this is called the *canonical order*.

1.2 Real parabolics

The maximal \mathbb{R} -split abelian subalgebra \mathfrak{a} , together with the order on it (induced by the choice of strongly orthogonal roots), determines a unique nilpotent Lie algebra of \mathfrak{g} , $\mathfrak{n} = \sum_{\eta \in \Phi_{\mathbb{R}}^+} \mathfrak{g}^{\eta}$. Set $A = \exp(\mathfrak{a})$, $N = \exp(\mathfrak{n})$, and

$$B := \mathcal{Z}(A) \rtimes N; \quad (2)$$

this is a minimal \mathbb{R} -parabolic, the *standard* one, uniquely determined by the choice of strongly orthogonal roots. Every minimal \mathbb{R} -parabolic of G is conjugate to B . Note that, setting $M = \mathcal{Z}(A) \cap K$, we have $\mathcal{Z}(A) = M \times A$, and the group M is the *semisimple anisotropic kernel* of G .

Assume again for the moment that \mathcal{D} is irreducible, and let η_i , $i = 1, \dots, t$ denote the simple \mathbb{R} -roots. Set $\mathfrak{a}_b := \bigcap_{j \neq b} \text{Ker} \eta_j$, $b = 1, \dots, t$, a one-dimensional subspace of \mathfrak{a} , and $A_b := \exp(\mathfrak{a}_b)$, a one-dimensional \mathbb{R} -split subtorus of A . Equivalently, $A_b = (\bigcap_{j \neq b} \text{Ker} \eta_j)^0$, where η_j is viewed as a character of A . The *standard maximal \mathbb{R} -parabolic*, P_b , $b = 1, \dots, t$, is the group generated by $\mathcal{Z}(A_b)$ and N ; equivalently it is the semidirect product (Levi decomposition)

$$P_b = \mathcal{Z}(A_b) \rtimes U_b, \quad (3)$$

where U_b denotes the unipotent radical. The Lie algebra \mathfrak{u}_b of U_b is the direct sum of the \mathfrak{g}^{η} , $\eta \in \Phi_{\mathbb{R}}^+$, $\eta|_{\mathfrak{a}_b} \neq 0$. The Lie algebra $\mathfrak{z}(\mathfrak{a}_b)$ of $\mathcal{Z}(A_b)$ has a decomposition:

$$\mathfrak{z}(\mathfrak{a}_b) = \mathfrak{m}_b \oplus \mathfrak{l}_b \oplus \mathfrak{l}'_b \oplus \mathfrak{a}_b, \quad \mathfrak{l}_b = \sum_{\eta \in [\eta_{b+1}, \dots, \eta_t]} \mathfrak{g}^{\eta} + [\mathfrak{g}^{\eta}, \mathfrak{g}^{-\eta}], \quad \mathfrak{l}'_b = \sum_{\eta \in [\eta_1, \dots, \eta_{b-1}]} \mathfrak{g}^{\eta} + [\mathfrak{g}^{\eta}, \mathfrak{g}^{-\eta}], \quad (4)$$

and \mathfrak{m}_b is an ideal of \mathfrak{m} , the Lie algebra of the (semisimple) anisotropic kernel M . Both \mathfrak{l}_b and \mathfrak{l}'_b are simple, and the root system $[\eta_{b+1}, \dots, \eta_t]$ is of type \mathbf{C}_{t-b} or \mathbf{BC}_{t-b} , while the root system $[\eta_1, \dots, \eta_{b-1}]$ is of type \mathbf{A}_{b-1} . Let L_b , L'_b denote the analytic groups with Lie algebras \mathfrak{l}_b and \mathfrak{l}'_b , respectively, and let $\mathcal{R}_b = L'_b A_b$, a reductive group (of type \mathbf{A}_{b-1}). We call L_b the *hermitian factor* of the Levi component and \mathcal{R}_b the *reductive factor*. It is well known that L_b defines the hermitian symmetric space which is the b^{th} (standard) boundary component of \mathcal{D} . Indeed, letting $K_b \subset L_b$ denote a maximal compact subgroup, $\mathcal{D}_b = L_b/K_b$ is hermitian symmetric, and naturally contained in \mathcal{D} as a subdomain, $\mathcal{D}_b \subset \mathcal{D}$. Let $\zeta : \mathcal{D} \rightarrow \mathfrak{p}^+$ be the Harish-Chandra embedding, and let $\mathbf{D} = \zeta(\mathcal{D})$, $\mathbf{D}_b = \zeta(\mathcal{D}_b)$ denote the images; \mathbf{D}_b is a bounded symmetric domain contained in a linear subspace (which can be identified with $\mathfrak{p}_b^+ = \mathfrak{l}_{b, \mathbb{C}} \cap \mathfrak{p}^+$). Let $o_b = -(E_{\mu_1} + \dots + E_{\mu_b})$, $1 \leq b \leq t$; as the elements o_b are in \mathfrak{p}^+ , one can consider the orbits $o_b \cdot G$ and $o_b \cdot L_b$. Since for $g \in L_b$ the action is described by $o_b g = o_b + \zeta(g)$, one has $o_b \cdot L_b = o_b + \zeta(\mathcal{D}_b) = o_b + \mathbf{D}_b$, and this is the domain \mathbf{D}_b , translated into an affine subspace ($o_b + \mathfrak{p}_b^+$) of \mathfrak{p}^+ . One denotes this domain by $F_b := o_b \cdot L_b$, and this is the b^{th} *standard boundary component* of \mathbf{D} . G acts by translations on the various F_b , and the images are the *boundary components* of \mathbf{D} ; one has

$$\overline{\mathbf{D}} = \mathbf{D} \cup \{ \text{boundary components} \} = \mathbf{D} \cup \left(\bigcup_{b=1}^t o_b \cdot G \right),$$

and $\overline{\mathbf{D}} \subset \mathfrak{p}^+$ is the compactification of \mathbf{D} in the Euclidean topology. For any boundary component F one denotes by $N(F)$, $Z(F)$ and $G(F)$ the normalizer, centralizer and automorphism group $G(F) = N(F)/Z(F)$, respectively. Then, letting $U(F)$ denote the unipotent radical of $N(F)$,

$$N(F_b) = P_b, \quad U(F_b) = U_b, \quad Z(F_b) = Z_b, \quad G(F_b) = L_b, \quad (5)$$

where Z_b is a closed normal subgroup of P_b containing every normal subgroup of P_b with Lie algebra $\mathfrak{z}_b = \mathfrak{m}_b \oplus \mathfrak{l}'_b \oplus \mathfrak{a}_b \oplus \mathfrak{u}_b$, which is an ideal in \mathfrak{p}_b .

Now consider the general case, $\mathcal{D} = \mathcal{D}_1 \times \dots \times \mathcal{D}_d$, \mathcal{D}_i irreducible. For each \mathcal{D}_i we have \mathbb{R} -roots $\Phi_{i, \mathbb{R}}$, of \mathbb{R} -ranks t_i and simple \mathbb{R} -roots $\{\eta_{i,1}, \dots, \eta_{i,t_i}\}$, $i = 1, \dots, d$. For each factor we have standard

parabolics P_{i,b_i} ($1 \leq b_i \leq t_i$) and standard boundary components F_{i,b_i} . The standard parabolics of G and boundary components of \mathcal{D} are then products

$$P_{\mathbf{b}} = P_{1,b_1} \times \cdots \times P_{d,b_d}, \quad F_{\mathbf{b}} = F_{1,b_1} \times \cdots \times F_{d,b_d}, \quad (\mathbf{b} = (b_1, \dots, b_d)), \quad (6)$$

and as above $P_{\mathbf{b}} = N(F_{\mathbf{b}})$. Furthermore,

$$G(F_{\mathbf{b}}) =: L_{\mathbf{b}} = L_{1,b_1} \times \cdots \times L_{d,b_d}. \quad (7)$$

As far as the domains are concerned, any of the boundary components F_{i,b_i} may be the *improper* boundary component \mathcal{D}_i , which is indicated by setting $b_i = 0$. Consequently, $P_{i,0} = L_{i,0} = G_i$ and in (6) and (7) any $\mathbf{b} = (b_1, \dots, b_d)$, $0 \leq b_i \leq t_i$ are admissible.

1.3 Fine structure of parabolics

For real parabolics of hermitian type one has a very useful refinement of (3). This is explained in detail in [SC] and especially in [S], §III.3-4. First we have the decomposition of $\mathcal{Z}(A_b)$ as described above,

$$\mathcal{Z}(A_b) = M_b \cdot L_b \cdot \mathcal{R}_b, \quad (8)$$

where M_b is compact, L_b is the hermitian Levi factor, \mathcal{R}_b is reductive (of type \mathbf{A}_{b-1}), and the product is almost direct (i.e., the factors have finite intersection). Secondly, the unipotent radical decomposes,

$$U_b = \mathcal{Z}_b \cdot V_b, \quad (9)$$

which is a direct product, \mathcal{Z}_b being the center of U_b . The action of $\mathcal{Z}(A_b)$ on U_b can be explicitly described, and is the basis for the compactification theory of [SC]. Before we recall this, let us note the notations used in [SC] and [S] for the decomposition. In [SC], we find

$$P(F) = (M(F)G_h(F)G_l(F)) \rtimes U(F) \cdot V(F), \quad (10)$$

and in [S], where the author uses Hermann homomorphisms $\kappa : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathfrak{g}$ to index the boundary component,

$$B_{\kappa} = (G_{\kappa}^{(1)} \cdot G_{\kappa}^{(2)}) \rtimes U_{\kappa} V_{\kappa}. \quad (11)$$

In (10), $M = M_b$, $G_h = L_b$, $G_l = \mathcal{R}_b$, while in (11), $G_{\kappa}^{(1)} = M_b \cdot L_b$, $G_{\kappa}^{(2)} = \mathcal{R}_b$ in our notations. The action can be described as follows ([S], III §3-4).

Theorem 1.1 *In the decomposition of the standard parabolic P_b (see (8) and (9))*

$$P_b = (M_b \cdot L_b \cdot \mathcal{R}_b) \rtimes \mathcal{Z}_b \cdot V_b,$$

the following statements hold.

- (i) *The action of $M_b \cdot L_b$ is trivial on \mathcal{Z}_b , while on V_b it is by means of a symplectic representation $\rho : M_b \cdot L_b \rightarrow Sp(V_b, J_b)$, for a symplectic form J_b on V_b .*
- (ii) *\mathcal{R}_b acts transitively on \mathcal{Z}_b and defines a homogenous self-dual (with respect to a bilinear form) cone $C_b \subset \mathcal{Z}_b$, while on V_b it acts by means of a representation $\sigma : \mathcal{R}_b \rightarrow GL(V_b, I_b)$ for some complex structure I_b on V_b .*

Furthermore the representations ρ and σ are compatible in a natural sense.

2 Holomorphic symmetric embeddings of symmetric domains

2.1 Symmetric subdomains

We continue with the notations of the previous paragraph. Hence G is a real Lie group of hermitian type (reductive), \mathcal{D} is the corresponding domain. We wish to consider reductive subgroups $N \subset G$, also of hermitian type, defining domains \mathcal{D}_N , such that the inclusion $N \subset G$ induces a holomorphic injection of the domains $i : \mathcal{D}_N \subset \mathcal{D}$, and the $i(\mathcal{D}_N)$ are totally geodesic. First of all we may assume that K_N , a maximal compact subgroup of N , is the intersection $K_N = K \cap N$; equivalently, letting $o \in \mathcal{D}$ and $o_N \in \mathcal{D}_N$ denote the base points, $i(o_N) = o$. Note that conjugating N by an element of K yields an isomorphic group N' and subdomain $i' : \mathcal{D}_{N'} \subset \mathcal{D}$ such that $i'(o_{N'}) = o$, and this defines an equivalence relation on the set of reductive subgroups $N \subset G$ as described. For the irreducible hermitian symmetric domains, the equivalence classes of all such N have been determined by Satake and Ihara ([S1] for the cases of \mathcal{D} of type $\mathbf{I}_{p,q}$, \mathbf{II}_n , \mathbf{III}_n ; [I] for the other cases).

Before quoting the results we will need, let us briefly remark on the mathematical formulation of the conditions. For this, let \mathcal{D} , \mathcal{D}' be hermitian symmetric domains, G , G' the automorphism groups, \mathfrak{g} , \mathfrak{g}' the Lie algebras, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ the Cartan decompositions and θ , θ' the Cartan involutions on \mathfrak{g} and \mathfrak{g}' , respectively. To say that for an injection $i_{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{D}'$ of symmetric spaces, $i_{\mathcal{D}}(\mathcal{D})$ is totally geodesic in \mathcal{D}' is to say that $i_{\mathcal{D}}$ is induced by an injection $i : \mathfrak{g} \hookrightarrow \mathfrak{g}'$ of the Lie algebras. If this holds, $i_{\mathcal{D}}$ is said to be *strongly equivariant*. Then, $\theta = \theta'_{|_{\mathfrak{g}}}$, or $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{k}'$, $\mathfrak{p} = \mathfrak{g} \cap \mathfrak{p}'$. Since both \mathcal{D} and \mathcal{D}' are hermitian symmetric, there is an element ξ in the center of \mathfrak{k} (resp. ξ' in the center of \mathfrak{k}'), such that $J = \text{ad}(\xi)$ (resp. $J' = \text{ad}(\xi')$) gives the complex structure. To say that the injection $i_{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{D}'$ is holomorphic is the same as saying $i \circ J = J' \circ i$, or equivalently,

$$(H_1) \quad i \circ \text{ad}(\xi) = \text{ad}(\xi') \circ i.$$

This is the condition utilized by Satake and Ihara in their classifications. The condition (H_1) is clearly implied by

$$(H_2) \quad i(\xi) = \xi',$$

which however, if fulfilled, gives additional information. For example ([S2], Proposition 4) if \mathcal{D} is a tube domain and i satisfies (H_2) , then \mathcal{D}' is also a tube domain. Furthermore, ([S], Proposition II 8.1), if $i_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}'$ is a holomorphic map which is strongly equivariant, the corresponding homomorphism i fulfills (H_1) , and, moreover, if \mathcal{D} and \mathcal{D}' are viewed as bounded symmetric domains \mathbf{D} , \mathbf{D}' via the Harish-Chandra embeddings, then $i_{\mathbf{D}} : \mathbf{D} \rightarrow \mathbf{D}'$ is the restriction of a \mathbb{C} -linear map $i^+ : \mathfrak{p}^+ \rightarrow (\mathfrak{p}')^+$. If $i_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{g}'_{\mathbb{C}}$ denotes the \mathbb{C} -linear extension of i , and $\sigma : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{g}_{\mathbb{C}}$, $\sigma' : \mathfrak{g}'_{\mathbb{C}} \rightarrow \mathfrak{g}'_{\mathbb{C}}$ denote the conjugations over \mathfrak{g} and \mathfrak{g}' , respectively, then the condition $\theta = \theta'_{|_{\mathfrak{g}}}$ is equivalent to the condition $i_{\mathbb{C}} \circ \sigma = \sigma' \circ i_{\mathbb{C}}$. This implies that $i : (\mathfrak{g}, \xi) \rightarrow (\mathfrak{g}', \xi')$ gives rise to a symmetric Lie algebra homomorphism $(\mathfrak{g}_{\mathbb{C}}, \sigma) \rightarrow (\mathfrak{g}'_{\mathbb{C}}, \sigma')$, and therefore, by [S], Proposition I 9.1, to a homomorphism of Jordan triple systems $i^+ : \mathfrak{p}^+ \rightarrow (\mathfrak{p}')^+$. It follows ([S], p. 85) that the following three categories are equivalent:

- (SD) Category whose objects are symmetric domains (\mathcal{D}, o) with base point o , whose morphisms $\rho_{\mathcal{D}} : (\mathcal{D}, o) \rightarrow (\mathcal{D}', o')$ are strongly equivariant holomorphic maps $\rho_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}'$ with $\rho_{\mathcal{D}}(o) = o'$.
- (HL) Category whose objects are semisimple Lie algebras (\mathfrak{g}, ξ) of hermitian type (without compact factors), whose morphisms $\rho : (\mathfrak{g}, \xi) \rightarrow (\mathfrak{g}', \xi')$ are homomorphisms satisfying (H_1) .
- (HJ) Category whose objects are positive definite hermitian Jordan triple systems \mathfrak{p}^+ , whose morphisms $\rho_+ : \mathfrak{p}^+ \rightarrow (\mathfrak{p}')^+$ are \mathbb{C} -linear homomorphisms of Jordan triple systems.

Table 1: Symmetric subdomains incident with positive-dimensional boundary components

\mathcal{D}	$F_{\mathbf{b}}, (b < t)$	\mathcal{D}_N	(H_2)
$\mathbf{I}_{p,q}$	$\mathbf{I}_{p-b,q-b}$	$\mathbf{I}_{p-b,q-b} \times \mathbf{I}_{b,b}$	$p = q$
\mathbf{II}_n	\mathbf{II}_{n-2b}	$\mathbf{II}_{n-2b} \times \mathbf{II}_{2b}$	<i>yes</i>
\mathbf{III}_n	\mathbf{III}_{n-b}	$\mathbf{III}_{n-b} \times \mathbf{III}_b$	<i>yes</i>
\mathbf{IV}_n	\mathbf{IV}_1	$\mathbf{IV}_1 \times \mathbf{IV}_1$	<i>yes</i>
\mathbf{V}	$\mathbf{I}_{5,1}$	$\mathbf{I}_{5,1} \times \mathbf{I}_{1,1}$	<i>yes</i>
\mathbf{VI}	\mathbf{IV}_{10}	$\mathbf{IV}_{10} \times \mathbf{IV}_1$	<i>yes</i>
	\mathbf{IV}_1	$\mathbf{IV}_1 \times \mathbf{IV}_{10}$	<i>yes</i>

2.2 Positive-dimensional boundary components

We now quote some results which we will be using. First, assume we have fixed $A \subset G$ as above, and let $F_{\mathbf{b}} \subset \overline{\mathcal{D}}$ be a standard boundary component of positive dimension, i.e., if \mathcal{D} is irreducible, of rank t , then $b < t$; if $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_d$, then in the notations of (6), $\mathbf{b} = (b_1, \dots, b_d)$, we have $b_i < t_i$ for *at least one* i . If \mathcal{D} is irreducible, we list in Table 1 a positive-dimensional boundary component and a symmetric subdomain $\mathcal{D}_M \subset \mathcal{D}$ with the property that $\mathcal{D}_N = \mathcal{D}_F \times \mathcal{D}'$, where \mathcal{D}_F is, as a hermitian symmetric space, isomorphic to the given boundary component. If \mathcal{D} is reducible, $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_d$, and $F_{\mathbf{b}} \subset \overline{\mathcal{D}}$ is a standard boundary component, we get a subdomain $\mathcal{D}_N = \mathcal{D}_{N_1} \times \cdots \times \mathcal{D}_{N_d}$ such that $\mathcal{D}_{N_i} \subset \mathcal{D}_i$ is of the type just mentioned with respect to the boundary component $F_{b_i} \subset \overline{\mathcal{D}_i}$.

Next, choose $N \subset G$ with \mathcal{D}_N as in Table 1, such that $A \subset N$ is a maximal \mathbb{R} -split torus in N , so that we can speak of standard boundary components of \mathcal{D}_N . Then the subdomains \mathcal{D}_N listed in Table 1 have the following property. For simplicity we will assume from now on that G is semisimple.

Proposition 2.1 *Given G , simple of hermitian type with maximal \mathbb{R} -split torus A and simple \mathbb{R} -roots η_i ($1 \leq i \leq t = \text{rank}_{\mathbb{R}} G$), let $P_{\mathbf{b}}$ and $F_{\mathbf{b}}$ denote the standard maximal \mathbb{R} -parabolic and standard boundary component determined by $\eta_{\mathbf{b}}$ ($b < t$). Let $N \subset G$ be a symmetric subgroup with $A \subset N$, defining a subdomain \mathcal{D}_N as in Table 1, such that $N = N_1 \times N_2$ and N_1 is a hermitian Levi factor of $P_{\mathbf{b}}$. Let $P_0 \times P_{t_2}$ be the standard maximal parabolic defined by the last simple \mathbb{R} -root η_{t_2} of the second factor in the decomposition $N = N_1 \times N_2$. Then if $F := F_0 \times F_{t_2}$ ($\cong \mathcal{D}_{N_1} \times \{\text{pt.}\}$, $t_2 = \text{rank}_{\mathbb{R}} N_2$) denotes the corresponding standard boundary component, the equality $i_N(F) = F_{\mathbf{b}}$ holds, where $i_N : \mathcal{D}_N \rightarrow \mathcal{D}$ denotes the injection.*

Proof: From construction, $F \cong \mathcal{D}_{N_1} \times \{\text{pt.}\} \cong F_{\mathbf{b}}$ as a hermitian symmetric space; to see that they coincide under i_N , recall from (4) the root space decomposition of the hermitian Levi component of $P_{\mathbf{b}}$. Since $A \subset N$ is also a maximal \mathbb{R} -split torus of N , in the root system $\Phi(A, N)$ we have the subsystem $[\eta_{b+1}, \dots, \eta_t]$ giving rise, on the one hand to the hermitian Levi factor $\mathfrak{l}_{\mathbf{b}}$ in $\mathfrak{p}_{\mathbf{b}}$, on the other hand to the Lie algebra of the first factor \mathfrak{n}_1 of N . From this it follows that $P_{\mathbf{b}}$ stabilizes $i_N(F)$, hence $i_N(F) = F_{\mathbf{b}}$. \square

Before proceeding to the case of zero-dimensional boundary components, we briefly explain how the subgroups $N \subset G$ (which are not unique, of course) arise in terms of \pm -symmetric/hermitian forms, at least for the classical cases. For this, we note that G can be described as follows (we describe here

certain reductive groups; the simple groups are just the derived groups):

- I $\mathbf{I}_{p,q}$: G is the unitary group of a hermitian form on \mathbb{C}^{p+q} of signature (p, q) ($p \geq q$).
 - II \mathbf{II}_n : G is the unitary group of a skew-hermitian form on \mathbb{H}^n .
 - III \mathbf{III}_n : G is the unitary group of a skew-symmetric form on \mathbb{R}^{2n} .
 - IV \mathbf{IV}_n : G is the unitary group of a symmetric bilinear form on \mathbb{R}^{n+2} of signature $(n, 2)$.
- (12)

Each of the \pm -symmetric/hermitian forms is isotropic, and if $t = \text{rank}_{\mathbb{R}} G$, the maximal dimension of a totally isotropic subspace is $t = q$, $\lfloor \frac{n}{2} \rfloor$, n , 2 in the cases I, II, III, and IV, respectively. Each maximal real parabolic is the stabilizer of a totally isotropic subspace, and using the canonical order on the \mathbb{R} -roots as above, P_b stabilizes a totally isotropic subspace of dimension b . Choosing a maximal torus T (resp. a maximal \mathbb{R} -split torus $A \subset T$) amounts to choosing a basis of V (resp. choosing a subset of this basis which spans a maximal totally isotropic subspace), and the standard parabolic is the stabilizer of a totally isotropic subspace spanned by some part of this basis. Now let $H \subset V$ be a totally isotropic subspace with basis h_1, \dots, h_b . Then there are elements h'_1, \dots, h'_b of V such that $h(h_i, h'_j) = \delta_{ij}$, $h(h_i, h_i) = h(h'_i, h'_i) = 0$, and $h_1, \dots, h_b, h'_1, \dots, h'_b$ span (over D) a vector subspace $W \subset V$ on which h restricts to a non-degenerate form. Let W^\perp denote the orthogonal complement of W in V , $W \oplus W^\perp = V$. Then

$$N := U(W, W^\perp; h) := \{g \in U(V, h) \mid g(W) \subset W, g(W^\perp) \subset W^\perp\} \cong U(W, h|_W) \times U(W^\perp, h|_{W^\perp}). \quad (13)$$

N is a reductive subgroup of G , and as one easily sees, its symmetric space is just the domain denoted \mathcal{D}_N in Table 1 above. The relation “boundary component \subset symmetric subdomain” translates into “totally isotropic subspace \subset non-degenerate subspace”, $H \subset W$, and *because* $h|_W$ is non-degenerate, any $g \in U(V, h)$ which stabilizes W automatically stabilizes its orthogonal complement in V as in (13).

2.3 Zero-dimensional boundary components

We now would like to consider the zero-dimensional boundary components, which correspond in the above picture to maximal totally isotropic subspaces. The construction above (13) doesn't necessarily work in this case, as W^\perp may be $\{0\}$, and $N = G$. However, in terms of domains, given *any* subdomain $\mathcal{D}' \subset \mathcal{D}$, it can be translated so as to contain a given zero-dimensional boundary component. We therefore place the following three conditions on such a subdomain:

- 1) The subdomain \mathcal{D}' has maximal rank ($\text{rank}_{\mathbb{R}} G' = \text{rank}_{\mathbb{R}} G$).
- 2) The subdomain \mathcal{D}' is maximal and G' is a maximal subgroup, or
- 2') The subdomain \mathcal{D}' is maximal of tube type and G' is maximal with this property.
- 3') The subdomain \mathcal{D}' is maximal irreducible, and F is a boundary component of \mathcal{D}' .

In Table 2 we list the subdomains (after [I]) \mathcal{D}' fulfilling 1), 2) and 3') in the column titled “ \mathcal{D}_N ”. We have listed also those subgroups fulfilling 1), 2') and 3') in the column titled “maximal tube”.

In Table 2, if there is no entry in the column “ \mathcal{D}_N ”, no such subgroups exist. In these cases it is natural to take the polydisc $\mathcal{D}_{N_\#}$ defined by the maximal set of strongly orthogonal roots

Table 2: Symmetric subdomains incident with zero-dimensional boundary components

\mathcal{D}	\mathcal{D}_N	(H_2)	maximal tube
$\mathbf{I}_{p,q}, p > q$	$\mathbf{I}_{p-1,q}$	<i>no</i>	$\mathbf{I}_{q,q}$
$\mathbf{I}_{q,q}$	–	–	–
$\mathbf{II}_n, n \text{ even}$	–	–	–
$\mathbf{II}_n, n \text{ odd}$	\mathbf{II}_{n-1}	<i>yes</i>	\mathbf{II}_{n-1}
\mathbf{III}_n	–	–	–
\mathbf{IV}_n	\mathbf{IV}_{n-1}	<i>yes</i>	\mathbf{IV}_{n-1}
\mathbf{V}	$\mathbf{I}_{2,4}, \mathbf{II}_5, \mathbf{IV}_8$	<i>yes, no, no</i>	$\mathbf{I}_{2,2}, \mathbf{II}_4, \mathbf{IV}_8$
\mathbf{VI}	$\mathbf{I}_{3,3}, \mathbf{II}_6$	<i>yes</i>	$\mathbf{I}_{3,3}, \mathbf{II}_6$

In the column “ \mathcal{D}_N ” the subgroups fulfilling 1), 2) and 3’) are listed, and in the column “maximal tube” the subgroups fulfilling 1), 2’) and 3’) (i.e., not necessarily 2)) are listed.

$\Psi = \{\pm\{\mu_1\}, \dots, \pm\{\mu_t\}\}$ as the subdomain \mathcal{D}_N , as there is no irreducible subdomain, and other products already occur in Table 1. Hence for these cases we require the conditions 2’’) and 3’’) of the introduction. To sum up these facts we make the following definition.

Definition 2.2 Let G be a simple real Lie group of hermitian type, A a fixed maximal \mathbb{R} -split torus defined as above by a maximal set of strongly orthogonal roots, $\eta_i, i = 1, \dots, t$ the simple \mathbb{R} -roots, F_b a standard boundary component and P_b the corresponding standard maximal \mathbb{R} -parabolic. A reductive subgroup $N \subset G$ (respectively the subdomain $\mathcal{D}_N \subset \mathcal{D}$) will be called *incident* to P_b (respectively to F_b), if \mathcal{D}_N is isomorphic to the corresponding domain of Table 1 ($b < t$) or Table 2 ($b = t$), and if N fulfills:

- $b < t$, then N satisfies 1), 2), 3).
- $b = t, \mathcal{D} \notin (\mathcal{ED})$, then N satisfies 1), 2’), 3’).
- $b = t, \mathcal{D} \in (\mathcal{ED})$, then N satisfies 1), 2’), 3’).

For reducible $\mathcal{D} = \mathcal{D}_1 \times \dots \times \mathcal{D}_d$, we have the product subgroups $N_{b_1,1} \times \dots \times N_{b_d,d}$, where $\mathcal{D}_{N_{b_i,i}}$ is incident to the standard boundary component F_{b_i} of \mathcal{D}_i (and $N_{0,i} = G_i$).

Next we briefly discuss uniqueness. We consider first the case of positive-dimensional boundary components. Let $P_b, 1 \leq b < t$ be a standard parabolic and let L_b be the “standard” hermitian Levi factor, i.e., such that $\text{Lie}(L_b) = \mathfrak{l}_b$; then

$$N_b := L_b \times \mathcal{Z}_G(L_b) \tag{14}$$

is a subgroup having the properties of Proposition 2.1, unique since L_b is unique. We shall refer to this unique subgroup as the *standard* incident subgroup. The different Levi factors \mathcal{L} in Levi decompositions $P_b = \mathcal{L} \rtimes \mathcal{R}_u(P_b)$ are conjugate by elements of $\mathcal{R}_u(P_b)$, as is well known. This implies for the hermitian factors $L = \mathcal{L}^{\text{herm}} \subset \mathcal{L}$ (which are uniquely determined by \mathcal{L}) by Theorem 1.1 the following.

Lemma 2.3 *Two hermitian Levi factors $L, L' \subset P_b$ are conjugate by an element of $V_b \subset P_b$.*

It follows, since $g(L_b \times \mathcal{Z}_G(L_b))g^{-1} = gL_b g^{-1} \times \mathcal{Z}_G(gL_b g^{-1})$, that two subgroups N, N' , both incident with P_b , are conjugate by an element of V_b :

$$N, N' \text{ incident to } P_b \iff N, N' \text{ conjugate (in } G) \text{ by } g \in V_b.$$

Proposition 2.4 *If (N, P_b) are incident, there is $g \in V_b$ such that N is conjugate by g to the standard N_b of (14).*

Proof: Since N is incident, $N \cong N_1 \times N_2$, where N_1 is a hermitian Levi factor of N . By Lemma 2.3, N_1 is conjugate by $g \in V_b$ to L_b , the hermitian Levi factor with Lie algebra \mathfrak{l}_b in the notations of the last section. Hence $gNg^{-1} = g(N_1 \times N_2)g^{-1} = gN_1g^{-1} \times gN_2g^{-1} = L_b \times N_{2,b} = N_b$, with $N_{2,b} = \mathcal{Z}(L_b)$ (this follows from the maximality of N_b). Consequently, N is conjugate by $g \in V_b$ to N_b . \square

The situation for zero-dimensional boundary components is more complicated, so we just observe the following. Suppose $\mathcal{D} \notin (\mathcal{E}\mathcal{D})$, and that $\mathcal{D}_N \subset \mathcal{D}$ is incident to F_t , $F_t = \text{point}$. For any $g \in N(F_t) = P_t$, $g\mathcal{D}_N = \mathcal{D}' \subset \mathcal{D}$ is another subdomain, again incident to F_t . If $g \in P_t \cap N_t$, then $g\mathcal{D}' = \mathcal{D}_N$. In this sense, letting $Q_t = P_t \cap N_t$, the coset space P_t/Q_t is a parameter space of subdomains incident with F_t .

Above we have defined the notion of symmetric subgroups incident with a standard parabolic. Any maximal \mathbb{R} -parabolic is conjugate to one and only one standard maximal parabolic, $P = gP_b g^{-1}$ for some b . Let N_b be any symmetric subgroup incident with P_b . Then just as one has the pair (P_b, N_b) one has the pair (P, N) ,

$$P = gP_b g^{-1}, \quad N = gN_b g^{-1}. \quad (15)$$

Definition 2.5 A pair (P, N) consisting of a maximal \mathbb{R} -parabolic P and a symmetric subgroup N is called *incident*, if the groups are conjugate by a common element g as in (15) to a pair (P_b, N_b) which is incident as in Definition 2.2.

3 Rational parabolic and rational symmetric subgroups

3.1 Notations

We now fix some notations to be in effect for the rest of the paper. We will be dealing with algebraic groups defined over \mathbb{Q} , which give rise to hermitian symmetric spaces, groups of *hermitian type*, as we will say. As we are interested in the automorphism groups of domains, we may, without restricting generality, assume the group is *centerless*, and *simple* over \mathbb{Q} . Henceforth G will denote such an algebraic group. To avoid complications, we exclude in this paper the following case:

Exclude: All non-compact real factors of $G(\mathbb{R})$ are of type $SL_2(\mathbb{R})$.

Finally, we shall only consider *isotropic* groups. This implies the hermitian symmetric space \mathcal{D} has no compact factors. By our assumptions, then, we have

- (i) $G = \text{Res}_{k|\mathbb{Q}} G'$, k a totally real number field, G' absolutely simple over k .
- (ii) $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_d$, each \mathcal{D}_i a non-compact irreducible hermitian symmetric space, $d = [k : \mathbb{Q}]$.

We now introduce a few notations concerning the root systems involved. Let Σ_∞ denote the set of embeddings $\sigma : k \rightarrow \mathbb{R}$; this set is in bijective correspondence with the set of infinite places of k . We denote the latter by ν , and if necessary we denote the corresponding embedding by σ_ν . For each $\sigma \in \Sigma_\infty$, the group ${}^\sigma G'$ is the algebraic group defined over $\sigma(k)$ by taking the set of elements g^σ , $g \in G'$. For each infinite prime ν we have $G_{k_\nu} \cong ({}^\nu G')_{\mathbb{R}}$, and the decomposition of \mathcal{D} above can be written

$$\mathcal{D} = \prod_{\sigma \in \Sigma_\infty} \mathcal{D}_\sigma, \quad \mathcal{D}_\sigma := ({}^\sigma G')_{\mathbb{R}} / K_{(\sigma)} = ({}^\sigma G')_{\mathbb{R}}^0 / K_{(\sigma)}^0.$$

Since G' is isotropic, there is a positive-dimensional k -split torus $S' \subset G'$, which we fix. Then ${}^\sigma S'$ is a maximal $\sigma(k)$ -split torus of ${}^\sigma G'$ and there is a canonical isomorphism $S' \rightarrow {}^\sigma S'$ inducing an isomorphism $\Phi_k = \Phi(S', G') \rightarrow \Phi_{\sigma(k)}({}^\sigma S', {}^\sigma G') =: \Phi_{k,\sigma}$. The torus $\text{Res}_{k|\mathbb{Q}} S'$ is defined over \mathbb{Q} and contains S as maximal \mathbb{Q} -split torus; in fact $S \cong S'$, diagonally embedded in $\text{Res}_{k|\mathbb{Q}} S'$. This yields an isomorphism $\Phi(S, G) \cong \Phi_k$, and the root systems $\Phi_{\mathbb{Q}} = \Phi(S, G)$, Φ_k and $\Phi_{k,\sigma}$ (for all $\sigma \in \Sigma_\infty$) are identified by means of the isomorphisms.

In each group ${}^\sigma G'$ one chooses a maximal \mathbb{R} -split torus $A_\sigma \supset {}^\sigma S'$, contained in a maximal torus defined over $\sigma(k)$. Fixing an order on $X(S')$ induces one also on $X({}^\sigma S')$ and $X(S)$. Then, for each σ , one chooses an order on $X(A_\sigma)$ which is compatible with that on $X({}^\sigma S')$, and $r : X(A_\sigma) \rightarrow X({}^\sigma S') \cong X(S)$ denotes the restriction homomorphism. The canonical numbering on $\Delta_{\mathbb{R},\sigma}$ of simple \mathbb{R} -roots of G with respect to A_σ is compatible by restriction with the canonical numbering of $\Delta_{\mathbb{Q}}$ ([BB], 2.8). Recall also that each k -root in Φ_k is the restriction of at most one simple \mathbb{R} -root of $G'(\mathbb{R})$ (which is a simple Lie group). Let $\Delta_k = \{\beta_1, \dots, \beta_s\}$; for $1 \leq i \leq s$ set $c(i, \sigma) :=$ index of the simple \mathbb{R} -root of ${}^\sigma G'$ restricting on β_i . Then $i < j$ implies $c(i, \sigma) < c(j, \sigma)$ for all $\sigma \in \Sigma$.

Each simple k -root defines a unique standard boundary component: for $b \in \{1, \dots, s\}$,

$$F_{\mathbf{b}} := \prod_{\sigma \in \Sigma_\infty} F_{c(b,\sigma)}, \quad (16)$$

which is the product of standard (with respect to A_σ and $\Delta_{\mathbb{R},\sigma}$) boundary components $F_{c(b,\sigma)}$ of \mathcal{D}_σ . It follows that $\overline{F}_j \subset \overline{F}_i$ for $1 \leq i \leq j \leq s$. Furthermore, setting $o_{\mathbf{b}} := \prod o_{c(b,\sigma)}$, then ([BB], p. 472)

$$F_{\mathbf{b}} = o_{\mathbf{b}} \cdot L_{\mathbf{b}}, \quad (17)$$

where $L_{\mathbf{b}}$ denotes the hermitian Levi component (7) of the parabolic $P_{\mathbf{b}}(\mathbb{R}) = N(F_{\mathbf{b}})$. As these are the only boundary components of interest to us, we will henceforth refer to any conjugates of the $F_{\mathbf{b}}$ of (16) by elements of G as *rational boundary components* (these should more precisely be called rational with respect to G), and to the conjugates of the parabolics $P_{\mathbf{b}}$ as the *rational parabolics*.

3.2 Rational parabolics

Let G' be as above, $\Delta_k = \{\beta_1, \dots, \beta_s\}$ the set of simple k -roots (having fixed a maximal k -split torus S' and an order on $X(S')$). For $b \in \{1, \dots, s\}$ we have the standard maximal k -parabolic P'_b of G' , whose group of \mathbb{R} -points is the normalizer of the standard rational boundary component $F_{c(b)}$ of the domain $\mathcal{D}' = G'_{\mathbb{R}}/K'$, where $c(b)$ denotes the index of the simple \mathbb{R} -root restricting to β_b ; since G' is absolutely simple, $G'_{\mathbb{R}}$ is simple and \mathcal{D}' is irreducible. Hence Theorem 1.1 applies to $P'_b(\mathbb{R})$. Of the factors given there, the following are defined over k : the product $M'_b L'_b$ as well as L'_b (but M'_b is *not* defined over k , so the k -subgroups are (instead of L'_b and M'_b) L'_b and $G_b^{(1)} := M'_b L'_b$), $\mathcal{R}'_b, \mathcal{Z}'_b$ and V'_b . As is well known, any maximal k -parabolic of G' is conjugate to one and only one of the P'_b , and two parabolics are conjugate \iff they are conjugate over k . There is a 1-1 correspondence between the set of k -parabolics of G' and the set of \mathbb{Q} -parabolics of G , given by $P' \mapsto \text{Res}_{k|\mathbb{Q}} P' =: P$. The standard maximal \mathbb{Q} -parabolic $P_{\mathbf{b}}$ of G gives a \mathbb{Q} -structure on the real parabolic $P_{\mathbf{b}}(\mathbb{R})$, which is the normalizer in \mathcal{D} of the standard boundary component $F_{\mathbf{b}}$ as in (16) (see also (6) and (7)), where $\mathbf{b} = (c(b, \sigma_1), \dots, c(b, \sigma_d))$. In the decomposition of Theorem 1.1, the factors $G_{\mathbf{b}}^{(1)} = M_{\mathbf{b}} L_{\mathbf{b}}$, $\mathcal{R}_{\mathbf{b}}, \mathcal{Z}_{\mathbf{b}}$ and $V_{\mathbf{b}}$ are all defined over \mathbb{Q} . In particular, for the factor $G_{\mathbf{b}}^{(1)}$, which we will call the \mathbb{Q} -hermitian Levi factor (and similarly, we will call $G_b^{(1)}$ the k -hermitian Levi factor of P'_b), we have

$$G_{\mathbf{b}}^{(1)}(\mathbb{Q}) \cong \prod_{\sigma} ({}^\sigma G_b^{(1)})_{\sigma(k)}, \quad \mathcal{Z}_G(G_{\mathbf{b}}^{(1)})(\mathbb{Q}) \cong \prod_{\sigma} (\mathcal{Z}_{({}^\sigma G'_{\sigma(k)})} (({}^\sigma G_b^{(1)})_{\sigma(k)}). \quad (18)$$

Furthermore, the hermitian Levi factor $L_{\mathbf{b}}$ is defined over \mathbb{Q} , and

$$L_{\mathbf{b}}(\mathbb{Q}) = \prod_{\sigma} (\sigma L'_{\mathbf{b}})_{\sigma(k)}.$$

We now make a few remarks about the factors of $G(\mathbb{R})$ and of $L_{\mathbf{b}}(\mathbb{R})$. Since the map $G' \rightarrow \sigma G'$ is an isomorphism of a k -group onto a $\sigma(k)$ -group, the algebraic groups (over \mathbb{C}) are isomorphic, hence the various $\sigma G'_{\mathbb{R}}$ are all \mathbb{R} -forms of some fixed algebraic group. Similarly, the factors of $L_{\mathbf{b}}(\mathbb{R})$ are all \mathbb{R} -forms of a single \mathbb{C} -group. However, they need not be isomorphic, unless the given \mathbb{C} -group has a unique \mathbb{R} -form of hermitian type (like $Sp(2n, \mathbb{C})$). Next we note the following.

Lemma 3.1 $L_{\mathbf{b}}$ is anisotropic $\iff b = s$.

Proof: The group $L_{\mathbf{b}}$ is anisotropic precisely when the boundary component $F_{\mathbf{b}}$ defined by it contains no other boundary components $F_c^* \subset F_{\mathbf{b}}^*$, which means $b \geq c$ for all c , or $b = s$. \square

In this case the group $L_{\mathbf{b}}$ does not fulfill the assumptions we have placed on G , and our results up to this point are not directly applicable to $L_{\mathbf{b}}$. Let us see how the phenomenon of compact factors of $L_{\mathbf{b}}(\mathbb{R})$ manifests itself in $F_{\mathbf{b}} = \prod_{\sigma} F_{c(b, \sigma)}$. Suppose some factor of $L_{\mathbf{b}}(\mathbb{R})$ is compact, say $L_{1, b}$. Then the symmetric space $\mathcal{D}_{b, \sigma_1}$ of $L_{1, b}$ is compact, so it is not true that $\mathcal{D}_{b, \sigma_1} \cong F_{c(b, \sigma_1)}$, hence it is also not true that $\mathcal{D}_{\mathbf{b}} \cong F_{\mathbf{b}}$, where $\mathcal{D}_{\mathbf{b}} = \prod_{\sigma} \mathcal{D}_{b, \sigma}$ is the symmetric space of $L_{\mathbf{b}}(\mathbb{R})$. However, letting $\mathcal{D}'_{\mathbf{b}}$ be the product of all compact factors, $\mathcal{D}_{\mathbf{b}}/\mathcal{D}'_{\mathbf{b}}$ is a symmetric space which is isomorphic to $F_{\mathbf{b}}$. What happens is that in the product $F_{\mathbf{b}} = \prod F_{c(b, \sigma)}$, all factors $F_{c(b, \sigma)}$ are *points* for which $\mathcal{D}_{b, \sigma}$ is *compact*. Hence whether this occurs depends on whether any factors \mathcal{D}_{σ} have zero-dimensional (rational) boundary components or not.

3.3 Incidence

We keep the notations used above; G is a simple \mathbb{Q} -group of hermitian type. Our main definition gives a \mathbb{Q} -form of Definition 2.5, and is the following.

Definition 3.2 Let $P \subset G$ be a maximal \mathbb{Q} -parabolic, $N \subset G$ a reductive \mathbb{Q} -subgroup. Then we shall say that (P, N) are *incident* (over \mathbb{Q}), if $(P(\mathbb{R}), N(\mathbb{R}))$ are incident in the sense of Definition 2.5.

Note that in particular N must itself be of hermitian type, and such that the Cartan involution of $G(\mathbb{R})$ restricts to the Cartan involution of $N(\mathbb{R})$. Furthermore, N must be a \mathbb{Q} -form of a product of groups, defining domains each of which is as in either Table 1 or Table 2.

The main result of this paper is the following existence result.

Theorem 3.3 Let G be \mathbb{Q} -simple of hermitian type subject to the restrictions above (G is isotropic and $G(\mathbb{R})$ is not a product of $SL_2(\mathbb{R})$'s), $P \subset G$ a \mathbb{Q} -parabolic. Then there exists a reductive \mathbb{Q} -subgroup $N \subset G$ such that (P, N) are incident over \mathbb{Q} , with the exception of the indices $C_{2n, n}^{(2)}$ for the zero-dimensional boundary components.

We will give the proof in the following sections, where we consider separately different cases (of the \mathbb{Q} -rank, the dimension of a maximal \mathbb{Q} -split torus). But before we start, we note here that by definition, if the theorem holds for *standard* parabolics, then it holds for all parabolics, so it will suffice to consider only standard parabolics. The case that G' has index $C_{2, 1}^{(2)}$ was considered in [Hyp]; in that case there is a unique standard parabolic P_1 , with zero-dimensional boundary component; the associated N_1 described in [Hyp] has domain \mathcal{D}_{N_1} which is not a two-disc, but only a one-dimensional disc.

4 Split over \mathbb{R} case

In this paragraph we consider the easiest case. This could loosely be described as an \mathbb{R} -Chevalley form.

Definition 4.1 Let G' be as in the last paragraph, absolutely simple over k , and let Φ_k be a root system (irreducible) for G' with respect to a maximal k -split torus $S' \subset G'$. Let $\Phi_{\mathbb{R}}$ be the root system of $G'(\mathbb{R})$ with respect to a maximal \mathbb{R} -split torus A' of the real (simple) group $G'(\mathbb{R})$. We call G' *split over \mathbb{R}* , if $\Phi_k \cong \Phi_{\mathbb{R}}$ as root systems, and if the indices of G' and $G'(\mathbb{R})$ coincide.

Note that the indices are independent of the split tori used to form the root system, so there is no need to assume $S' \subset A'$ in the above definition (the notion of isomorphism of indices is obvious). However, one can always find split tori S', A' such that $S' \subset A'$. From $\Phi_k \cong \Phi_{\mathbb{R}}$ it follows then that $S' = A'$, as both tori have the same dimension.

Lemma 4.2 Let G be simple over \mathbb{Q} ($= \text{Res}_{k|\mathbb{Q}} G'$), $\mathcal{D} = \prod_{\sigma \in \Sigma_{\infty}} \mathcal{D}_{\sigma}$ the domain defined by the real Lie group $G(\mathbb{R}) \cong \prod_{\sigma \in \Sigma_{\infty}} G_{\sigma} =: \prod_{\sigma} {}^{\sigma}G'_{\mathbb{R}}$. If G' is split over \mathbb{R} , then $G_{\sigma} = G_{\tau}$ for all $\sigma, \tau \in \Sigma_{\infty}$.

Proof: For each σ we have $A_{\sigma} \supset {}^{\sigma}S'$, so by assumption $A_{\sigma} \cong {}^{\sigma}S'$, and for each σ the map $\phi : \Phi_k \rightarrow \Phi_{\sigma(k)}({}^{\sigma}G')$ is an isomorphism, and since $\Phi_k \cong \Phi_{\mathbb{R}}$,

$$\Phi_{\sigma(k)}({}^{\sigma}G') \cong \Phi_{\mathbb{R}}({}^{\sigma}G'_{\mathbb{R}}).$$

It follows that $\Phi_{\mathbb{R}} \cong \Phi_k \xrightarrow{\phi} \Phi_{\sigma(k)}({}^{\sigma}G') \cong \Phi_{\mathbb{R}}({}^{\sigma}G'_{\mathbb{R}}) \cong \Phi_{\mathbb{R}}$. Similarly, since the index of G' is isomorphic to the index of $G'(\mathbb{R})$ (which determines the isomorphism class of $G'(\mathbb{R})$), the index of ${}^{\sigma}G'$ is isomorphic to that of ${}^{\sigma}G'(\mathbb{R})$. But the index of $G'(\mathbb{R})$ is the same as ${}^{\sigma}G'(\mathbb{R})$, as an easy case by case check verifies. For example, for type (I), all factors have the same \mathbb{R} -rank q , hence are all isomorphic to $SU(p, q)$. See Examples 4.3 below for the other cases. Hence ${}^{\sigma}G'(\mathbb{R}) \cong {}^{\tau}G'(\mathbb{R})$ for all σ, τ , as claimed. \square

From this it follows in particular that the (standard) boundary components are determined by $c(b, \sigma) = b$, $\forall \sigma$, $\mathbf{b} = (b, \dots, b)$, $1 \leq b \leq t = \text{rank}_{\mathbb{Q}} G = \text{rank}_k G' = \text{rank}_{\mathbb{R}} G'(\mathbb{R})$. Hence they are of the form

$$F_{\mathbf{b}} = \prod_{\sigma \in \Sigma_{\infty}} F_{b, \sigma}, \quad (19)$$

and $F_{b, \sigma}$ is the standard rational boundary component of \mathcal{D}_{σ} .

Examples 4.3: We now give examples of split over \mathbb{R} groups in each of the cases, and any such will be of one of the listed types. Let k be a totally real number field.

- I. Let $K|k$ be imaginary quadratic, V a K -vector space of dimension $n = p + q$, and h a hermitian form on V defined over K . Then the unitary group $U(V, h)$ is split over $\mathbb{R} \iff$ the hermitian form h has Witt index q and for all infinite primes, h_{ν} has signature (p, q) .
- II. Let $D|k$ be a totally definite quaternion algebra over k (with the canonical involution), V an n -dimensional right vector space over D , h a skew-hermitian form on V defined over k . Then the unitary group $U(V, h)$ is split over $\mathbb{R} \iff$ the skew-hermitian form has Witt index $\lfloor \frac{n}{2} \rfloor$ ($n > 4$).
- III. Take $G = Sp(2n, k)$.
- IV. Let V be a $(n + 2)$ -dimensional k -vector space, h a symmetric bilinear form defined over k of Witt index 2. Then if $U(V, h)$ is of hermitian type, it is split over \mathbb{R} .

- V. The Lie algebra in this case is of the form $\mathcal{L}(\mathfrak{C}_k, (J_1^b)_k)$, the Tits algebra, where \mathfrak{C}_k is an anisotropic octonion algebra and $(J_1^b)_k$ is the Jordan algebra \mathfrak{B}^+ for an associative algebra \mathfrak{B} whose traceless elements with the Lie product form a Lie algebra of type $\mathfrak{su}(2, 1)$; since G' is split over \mathbb{R} , the algebra \mathfrak{B}^- is the Lie algebra of a unitary group of a K -hermitian form ($K|k$ imaginary quadratic as in (I)) of Witt index 1.
- VI. The Lie algebra is isomorphic to $\mathcal{L}(\mathfrak{A}_k, \mathfrak{J}_k)$, the Tits algebra, where \mathfrak{A}_k is a totally indefinite quaternion algebra over k and \mathfrak{J}_k is a k -form of the exceptional Jordan algebra denoted J^b by Tits.

Lemma 4.4 *In the notations above, let $N'(\mathbb{R}) \subset G'(\mathbb{R})$ be a subgroup such that the Lie algebra $\mathfrak{n}' \subset \mathfrak{g}'$ is a regular subalgebra, i.e., defined by a closed symmetric set of roots Ψ of the (absolute) root system Φ of G' . Then N' is defined over k , $N' \subset G'$.*

Proof: From the isomorphism of the indices of G' and $G'(\mathbb{R})$, it follows that any subalgebra $\mathfrak{g}' \subset \mathfrak{g}$, such that for some subset $\Psi \subset \Phi$, the subalgebra \mathfrak{g}' is given by $\mathfrak{g}' = \mathfrak{t} + \sum_{\eta \in \Psi} \mathfrak{g}^\eta$, is defined over $\mathbb{R} \iff$ it is defined over k . The regular subalgebra \mathfrak{n}' is of this type, and it follows that N' is defined over k . \square

Corollary 4.5 *Let $N' \subset G'$ be as in Lemma 4.4, $N = \text{Res}_{k|\mathbb{Q}} N' \subset G$. Then N is defined over \mathbb{Q} .*

To apply Lemma 4.4 to (k -forms of) subgroups whose domains are listed in Tables 1 and 2, we need to know which of the subgroups are defined by regular subalgebras. Ihara in [I] considered this question, and the result is: all isomorphism classes of groups in Table 1 and all isomorphism classes of groups in Table 2, with the exception of $SO(n-1, 2) \subset SO(n, 2)$ for n even, have representatives which are defined by (maximal) regular subalgebras.

Corollary 4.6 *Let G' be split over \mathbb{R} . Then Theorem 3.3 holds for $G = \text{Res}_{k|\mathbb{Q}} G'$.*

Proof: By Lemma 4.2, $G(\mathbb{R})/K = \mathcal{D} = \prod \mathcal{D}_\sigma$, and all \mathcal{D}_σ are isomorphic to $\mathcal{D}' = G'(\mathbb{R})/K'$; the rational boundary components are as in (19), products of copies of F'_b , the standard boundary component of \mathcal{D}' , and each \mathbb{Q} -parabolic of G is conjugate to one of $P_b = \text{Res}_{k|\mathbb{Q}} P'_b$, where $P'_b(\mathbb{R}) = N(F'_b)$ is the standard maximal real parabolic of $G'(\mathbb{R})$. Now locate F'_b in Table 1 or 2 as the case may be; the corresponding group N'_b is isomorphic to one defined by a regular subalgebra of $\mathfrak{g}'_{\mathbb{R}}$ with the one exception mentioned above. Then by Lemma 4.4, N'_b is defined over k , hence (Corollary) $N_b = \text{Res}_{k|\mathbb{Q}} N'_b$ is defined over \mathbb{Q} , and is incident with P_b . This takes care of all cases except the exception just mentioned, $\mathbf{IV}_{n-1} \subset \mathbf{IV}_n$, $n > 3$ even. So let V be a k -vector space of dimension $n+2$, h a symmetric bilinear form on V . By assumption, G' is split over \mathbb{R} , so the Witt index of h is 2. Let $H \subset V$ be a maximal totally isotropic subspace (two-dimensional) defined over k , and h_1, h_2 a k -basis. Then there are k -vectors h'_i such that $H_1 := \langle h_1, h'_1 \rangle$ and $H_2 := \langle h_2, h'_2 \rangle$ are hyperbolic planes; let $W = H_1 \oplus H_2$ denote their direct sum. From $n > 3$, W has codimension ≥ 1 in V . Let $v \in W^\perp$ be a k -vector, and set:

$$U := v^\perp = \{w \in V | h(v, w) = 0\}.$$

Then $W \subset U$, the dimension of U is $n+2-1 = n+1$, and $h|_U$ still has Witt index 2. Hence

$$N' := \{g \in U(V, h) | g(U) \subset U\}$$

is a k -subgroup, and $N'(\mathbb{R})^0 \cong SO(n-1, 2)$. This is a group which is incident to a parabolic whose group of real points is the stabilizer of the zero-dimensional boundary component F'_2 of the domain \mathcal{D}' of type \mathbf{IV}_n . \square

This completes the discussion of the split over \mathbb{R} case. We just mention that, at least in the classical cases, we could have argued case for case with \pm symmetric/hermitian forms as in the proof of the exception above. Using the root systems simplified the discussion, and, in particular, gives the desired results for the exceptional groups without knowing their explicit construction.

5 Rank ≥ 2

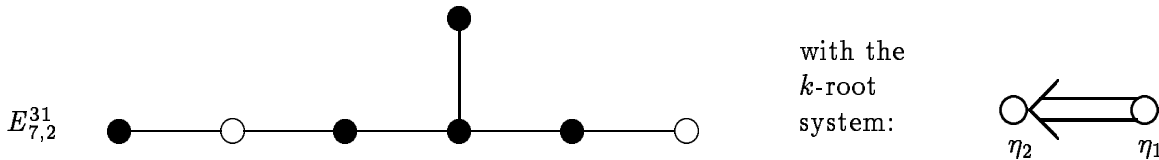
In this paragraph we assume G in *not* split over \mathbb{R} , but that $\text{rank}_k G' = \text{rank}_{\mathbb{Q}} G \geq 2$. Under these circumstances, it is known precisely which k -indices are possible for G' of hermitian type.

Proposition 5.1 *Assume $\text{rank}_{\mathbb{Q}} G \geq 2$ and that G' is not split over \mathbb{R} . Then the k -index of G' is one of the following:*

- (I) ${}^2A_{n,s}^{(d)}$; $s \geq 2$, $d|n+1$, $2sd \leq n+1$; if $d=1$, then $2s < n+1$.
- (II) ${}^1D_{n,s}^{(2)}$, $s \geq 2$, $s < \ell$ ($n=2\ell$); ${}^2D_{n,s}^{(2)}$, $s \geq 2$, $s < \ell$ ($n=2\ell+1$).
- (III) $C_{n,s}^{(2)}$, $s \geq 2$, $s < \lfloor \frac{n}{2} \rfloor$.
- (IV) none
- (V) none
- (VI) $E_{7,2}^{31}$.

Proof: All statements are self-evident from the description of the indices in [T]; in the case (V) there are three possible indices, only one of which has rank ≥ 2 ; this is the split over \mathbb{R} index. Similarly, in the case (IV), rank ≥ 2 implies split over \mathbb{R} . For type (III), the indices $C^{(1)}$ are also split over \mathbb{R} . \square

There is only one exceptional index to consider, so we start by dealing with this case. The index we must discuss is



There are two simple k -roots, η_1 and η_2 ; let P'_b be the corresponding standard maximal k -parabolics, F'_b the corresponding standard boundary components of the irreducible domain \mathcal{D}' . Then F'_2 is the one-dimensional boundary component, F'_1 is the ten-dimensional one. The k -root system is of type \mathbf{BC}_2 (since the highest simple \mathbb{R} -root is anisotropic, see [BB], 2.9). Consider the decomposition of Theorem 1.1 for $P'_b(\mathbb{R})$; in both cases L'_b is non-trivial, and, as mentioned above, $M'_b \cdot L'_b$ is defined over k . Here we have $b=1$ or 2 . But for E_7 , the compact factor M'_b is in fact *absent*¹, and as L'_b is defined over k , we can set

$$N'_b = L'_b \times \mathcal{Z}_{G'}(L'_b).$$

¹see [S], p. 117

This is a k -subgroup which is a k -form of the corresponding \mathbb{R} -subgroup whose domain is listed in Table 1. Now consider $G = \text{Res}_{k|\mathbb{Q}} G'$. It also has two standard maximal parabolics P_1 and P_2 , and in each we have a non-trivial hermitian Levi factor² $L_{\mathbf{b}} := \text{Res}_{k|\mathbb{Q}} L'_b$, such that

$$L_{\mathbf{b}}(\mathbb{R}) = \prod_{\sigma \in \Sigma_{\infty}} \sigma(L'_b)_{\mathbb{R}}.$$

Also the symmetric subgroup $N_{\mathbf{b}} := \text{Res}_{k|\mathbb{Q}} N'_b$ is defined over \mathbb{Q} and satisfies $N_{\mathbf{b}}(\mathbb{R}) = \prod_{\sigma \in \Sigma_{\infty}} \sigma(N'_b)_{\mathbb{R}}$. It follows that $(P_{\mathbf{b}}, N_{\mathbf{b}})$ are incident: conditions 1) and 2) follow from the corresponding facts for (P'_b, N'_b) ; we should check 3). But since it is obvious that $\sigma(L'_b)_{\mathbb{R}} \subset \sigma(P'_b)_{\mathbb{R}}$ is a hermitian Levi factor, the same holds for $L_{\mathbf{b}} \subset P_{\mathbf{b}}$; 3) is satisfied. This completes the proof of

Proposition 5.2 *Theorem 3.3 is true for the exceptional groups in the rank ≥ 2 , not split over \mathbb{R} case.*

We are left with the classical cases. Here we may use the interpretation of $G(\mathbb{R})$ as the unitary group of a \pm -symmetric/hermitian form as in (12), and G is a \mathbb{Q} -form of this. The precise realisation of this is the interpretation in terms of central simple algebras with involution; this is discussed in [W]. More precisely, the algebraic groups G' which represent the indices of Proposition 5.1 are (here we describe reductive groups; the corresponding derived groups are the simple groups G').

(I) D : degree d central simple division algebra over K , $K|k$ an imaginary quadratic extension, D has a $K|k$ -involution (involution of the second kind).

V : right D -vector space, of dimension m over D , $dm = n + 1$.

h : hermitian form $h : V \times V \rightarrow D$ of Witt index s , $2s \leq m$ ($2s < m$ if $d = 1$), given by a matrix H .

G' : unitary group $U(V, h) = \{g \in GL_D(V) | gHg^* = H\}$.

index: ${}^2A_{n,s}^{(d)}$.

(II) D : totally definite quaternion division algebra, central simple over k , with canonical involution.

V : right D -vector space of dimension m over D .

h : skew-hermitian form h of Witt index $s < [\frac{m}{2}]$, given by a matrix H .

G' : unitary group $U(V, h) = \{g \in GL_D(V) | gHg^* = H\}$.

index: $D_{m,s}^{(2)}$ (m even), ${}^2D_{m,s}^{(2)}$ (m odd).

(III) D : totally indefinite quaternion division algebra, central simple over k , with the canonical involution.

V : right D -vector space of dimension m .

h : hermitian form $h : V \times V \rightarrow D$ of Witt index s , $2s \leq m$, given by a matrix H .

G' : unitary group $U(V, h) = \{g \in GL_D(V) | gHg^* = H\}$.

index: $C_{m,s}^{(2)}$.

Finally, we must consider the following ‘‘mixed cases’’, which still can give rise to groups of hermitian type:

(II-IV): D : a quaternion division algebra over k , with D_{ν} definite for ν_1, \dots, ν_a , D_{ν} indefinite for ν_{a+1}, \dots, ν_f , where $f = [k : \mathbb{Q}]$.

²we note a change of notation here in that in (7), $L_{\mathbf{b}}$ denotes a real Lie group

V : same as for (II) above.

h : same as for (II) above, h of Witt index s .

G' : same as for (II) above.

$G(\mathbb{R})$ is then a product $(SU(n, \mathbb{H}))^a \times (SO(2n-2, 2))^{f-a}$, where we have taken into account that we are assuming G to be isotropic and of hermitian type. Note however, that since the factors $SO(2n-2, 2)$ corresponding to the primes ν_{a+1}, \dots, ν_f have \mathbb{R} -split torus of dimension two, the k -rank of G' must be ≤ 2 . Hence the only indices where this can occur are: ${}^i D_{n,1}^{(2)}$ and ${}^i D_{n,2}^{(2)}$, $i = 1, 2$.

In terms of the spaces (V, h) , the standard parabolics are stabilizers of totally isotropic subspaces $H_b \subset V$, where H_1 is one-dimensional (over D), while H_s is a maximal totally isotropic subspace. The latter case corresponds to zero-dimensional boundary components. We consider first the case H_b , $b < s$, of which at least H_1 exists, because of the assumption $\text{rank} \geq 2$. Fix a basis h_1, \dots, h_b of H_b of isotropic vectors $h(h_i, h_i) = 0$ for all $i = 1, \dots, b$. Then there exist, in V , elements h'_i , $i = 1, \dots, b$ with $h(h_i, h'_j) = \delta_{ij}$, and h'_1, \dots, h'_b span a complementary totally isotropic subspace; denote it by H'_b . Then $H := H_b \oplus H'_b$ is a *non-degenerate* space for h , $h|_H$ is a non-degenerate form. It follows that $\{g \in GL(V) | g(H) \subset H\} = \{g \in GL(V) | g(H^\perp) \subset H^\perp\}$. In the following we will work in the (reductive) unitary group $G' = U(V, h)$; for any subgroup $H \subset G'$ we can take the intersection $SL(V) \cap H \subset SL(V) \cap G'$ to give subgroups of the simple group. Furthermore, up to Corollary 5.5 below, we omit the primes in the notations for the subgroups of G' . Set

$$N = U(H, H^\perp; h) = \{g \in GL(V) | g(H) \subset H, g(H^\perp) \subset H^\perp\}; \quad (20)$$

then $N = U(H, h|_H) \times U(H^\perp, h|_{H^\perp})$, and $U(H, h|_H) \cong \mathcal{Z}_G(U(H^\perp, h|_{H^\perp}))$. So setting $L = U(H^\perp, h|_{H^\perp})$, we have

$$N \cong L \times \mathcal{Z}_G(L). \quad (21)$$

Next we note that the basis h_1, \dots, h_b of H_b determines a unique \mathbb{R} -split torus $A_b \subset A$, where A is the maximal \mathbb{R} -split torus defined by a basis h_1, \dots, h_s of a maximal totally isotropic subspace $H_s \supset H_b$, namely the scalars $\alpha = \alpha \cdot \mathbf{1} \in GL(H_b)$, extended to $GL(V)$ by unity. Taking the centralizer of the torus A_b gives a Levi factor of the parabolic $P_b = \mathcal{N}_G(H_b)$, $b < s$ (the normalizer in G of H_b).

Lemma 5.3 $L = U(H^\perp, h|_{H^\perp})$ is the k -hermitian factor $G_b^{(1)} = M_b \cdot L_b$ of P_b in the decomposition of P_b as in Theorem 1.1.

Proof: First observe that $L \subset P_b$, as H^\perp is orthogonal to the totally isotropic subspace, hence L normalizes H_b . Since L is reductive, there is a Levi decomposition of P_b for which L is contained in the Levi factor. It is clearly of hermitian type, and maximal with this property. We must explain why the Levi factor is the standard one $\mathcal{Z}(A_b)$. But this follows from the fact that H_b is constructed by means of a basis, which in turn was determined by the choice of \mathbb{R} -split torus A_b . It therefore suffices to explain the ‘‘compact’’ factor M_b . This factor occurs only in the cases $\mathbf{I}_{p,q}$ and \mathbf{IV}_n . We don't have to consider the latter case, as this is split over \mathbb{R} if $\text{rank} \geq 2$. So suppose $G \cong U(V, h)$, where (V, h) is as in (I) above. We first determine the anisotropic kernel. Let H_s be a maximal totally isotropic subspace, $S := H_s \oplus H'_s$ as above. Then $U(S^\perp, h|_{S^\perp})$ is the anisotropic kernel, $U(S^\perp, h|_{S^\perp})(\mathbb{R}) \cong U(md - 2sd)$. In particular, for $m = 2s$, there is no anisotropic kernel. Now consider the group $L = U(H^\perp, h|_{H^\perp})$. Clearly, for $b < s$, we have

$$U(S^\perp, h|_{S^\perp}) \subset U(H^\perp, h|_{H^\perp}) = L,$$

so that L contains the anisotropic kernel. Note that $SU(H^\perp, h|_{H^\perp})(\mathbb{R}) \cong L_b(\mathbb{R})$, while (if $H^\perp \neq \{0\}$)

$$U(H^\perp, h|_{H^\perp})(\mathbb{R}) / SU(H^\perp, h|_{H^\perp})(\mathbb{R}) \cong M_b(\mathbb{R}) \cong U(1).$$

Here we have used that $U(H^\perp, h|_{H^\perp}) \subset SU(V, h)$, as it is for the group $SU(V, h)$ (and not for $U(V, h)$) that $M_b(\mathbb{R}) \cong U(1)$ (see [S], p. 115). This verifies the Lemma for the groups of type I. \square

Now note that $L(\mathbb{R}) \cong (M_b \cdot L_b)(\mathbb{R}) = M_b(\mathbb{R})L_b(\mathbb{R})$, so for the domain defined by L we have $\mathcal{D}_L = M_b(\mathbb{R})L_b(\mathbb{R})/M_b(\mathbb{R})K_b = L_b(\mathbb{R})/K_b \cong F_b$, hence $\mathcal{D}_N \cong \mathcal{D}_{N_b}$ as in Table 1. Consider also $\mathcal{Z}_G(L_b)$ and $\mathcal{Z}_G(M_b L_b)$; both are defined over \mathbb{R} , and clearly $\mathcal{Z}_{G(\mathbb{R})}(L_b(\mathbb{R}))/M_b(\mathbb{R}) \cong \mathcal{Z}_{G(\mathbb{R})}(M_b(\mathbb{R})L_b(\mathbb{R}))$, so the group $L \times \mathcal{Z}_G(L)$ (both these factors being defined over k) is, over \mathbb{R} ,

$$\begin{aligned} L(\mathbb{R}) \times \mathcal{Z}_G(L)(\mathbb{R}) &\cong M_b(\mathbb{R}) \cdot L_b(\mathbb{R}) \times \mathcal{Z}_{G(\mathbb{R})}(M_b(\mathbb{R})L_b(\mathbb{R})) \\ &\cong M_b(\mathbb{R}) \cdot L_b(\mathbb{R}) \times \mathcal{Z}_{G(\mathbb{R})}(L_b(\mathbb{R}))/M_b(\mathbb{R}) \\ &\cong L_b(\mathbb{R}) \times \mathcal{Z}_{G(\mathbb{R})}(L_b(\mathbb{R})). \end{aligned} \tag{22}$$

This completes the proof of

Proposition 5.4 *The subgroup N of (20) satisfies $N(\mathbb{R}) \cong N_b(\mathbb{R})$, the latter group being the standard symmetric subgroup (14) standard incident to $P_b(\mathbb{R})$.*

Corollary 5.5 *The parabolic P_b and the symmetric subgroup N of (20) are incident over k , i.e., $(P_b(\mathbb{R}), N(\mathbb{R}))$ are incident in the sense of Definition 2.2.*

Up to this point we have been working with the absolutely simple k -group; we now denote this situation by G' as in section 3.1, and consider $G = Res_{k|\mathbb{Q}}G'$. Let again primes in the notations denote subgroups of G' , the unprimed notations for subgroups of G . As above we set $P_b := Res_{k|\mathbb{Q}}(P'_b)$, and we denote the subgroup N' of (20) henceforth by N'_b and set: $N_b := Res_{k|\mathbb{Q}}(N'_b)$. Then Corollary 5.5 tells us that $(P'_b(\mathbb{R}), N'(\mathbb{R}))$ are incident. We now claim

Lemma 5.6 *The \mathbb{Q} -groups (P_b, N_b) are incident.*

Proof: $P_b(\mathbb{R})$ is a product $P_b(\mathbb{R}) \cong P_{b,1}(\mathbb{R}) \times \cdots \times P_{b,d}(\mathbb{R})$ corresponding to (16); by assumption F_b is not zero-dimensional. Hence for at least one factor $P_{b,\sigma}(\mathbb{R})$ the incident group ${}^\sigma N'_b(\mathbb{R}) \cong {}^\sigma L'_b(\mathbb{R}) \times \mathcal{Z}_{{}^\sigma G'(\mathbb{R})}({}^\sigma L'_b(\mathbb{R}))$ is defined. Consequently N_b is not trivial, and it is clearly a \mathbb{Q} -form of $N_b(\mathbb{R}) = \prod_{\sigma} {}^\sigma N'_b(\mathbb{R})$. \square

With Corollary 5.5 and Lemma 5.6, we have just completed the proof of the following.

Proposition 5.7 *To each standard maximal \mathbb{Q} -parabolic P_b of G with $b < s$, there is a symmetric \mathbb{Q} -subgroup $N_b \subset G$ such that (P_b, N_b) are incident.*

Finally, we remark on what happens for the parabolic corresponding to the zero-dimensional boundary components. We have, in the notations above, $H = H_s \oplus H'_s$, and H^\perp is anisotropic for h . It follows that the group L of Lemma 5.3 is anisotropic; its semisimple part is the semisimple anisotropic kernel of G' . If $s = \frac{1}{2}\dim V$, then $H = V$ already, $H^\perp = \{0\}$. Hence the group N of (21) is the whole group ($L = 1 \Rightarrow \mathcal{Z}_G(L) = G$). Otherwise it is of the form $\{\text{anisotropic}\} \times \{k\text{-split}\}$. We list these in Table 3. Note that the domains occuring have \mathbb{R} -rank equal to the \mathbb{Q} -rank of G , suggesting this as a possible modification of the definition of incident:

1') N has \mathbb{R} -rank equal to the \mathbb{Q} -rank of G .

Viewing things this way, we see that again indices $C^{(1)}$ represent an exception; for these 1) and 1') are equivalent.

Let us now see which of the subgroups listed in Table 2 are defined over k . We use the notations D, V, h and G as described above in the cases (I)-(III).

Table 3: k -subgroups incident with zero-dimensional boundary components

Index	L	$\mathcal{Z}_G(L)$	subdomains	$\mathcal{Z}_G(L)(\mathbb{R})$
${}^2A_{n,s}^{(d)}$	${}^2A_{n-2ds,0}^{(d)}$	${}^2A_{2ds-1,s}^{(d)}$	$\mathbf{I}_{\mathbf{p-ds,q-ds}} \times \mathbf{I}_{\mathbf{ds,ds}}$	$SU(ds, ds)$
${}^1D_{n,s}^{(2)}$	${}^1D_{n-2s,0}^{(2)}$	${}^1D_{2s,s}^{(2)}$	$\mathbf{II}_{n-s} \times \mathbf{II}_s$	$SU(2s, \mathbb{H})$ (n even)
${}^2D_{n,s}^{(2)}$	${}^2D_{n-2s,0}^{(2)}$	${}^2D_{2s,s}^{(2)}$	$\mathbf{II}_{n-s} \times \mathbf{II}_s$	$SU(2s, \mathbb{H})$ (n odd)
$C^{(1)}$	–	G	–	–
$C_{n,s}^{(2)}$	$C_{n-s,0}^{(2)}$	$C_{s,s}^{(2)}$	$\mathbf{III}_{n-s} \times \mathbf{III}_s$	$Sp(2s, \mathbb{R})$

(I) Again d denotes the degree of D . In $U(V, h)$ we have the subgroup $U(V', h|_{V'})$ for any codimension one subspace $V' \subset V$. Let $W = (V')^\perp$ be the one-dimensional (over D) subspace orthogonal to V' . Then $U(W, h|_W)$ is again a unitary group whose set of \mathbb{R} -points is isomorphic to $U(p_W, q_W)$ for some p_W, q_W . Actually each h_ν for each infinite prime ν gives an \mathbb{R} -group $U(p_{W,\nu}, q_{W,\nu})$. Let (p_ν, q_ν) be the signature of h_ν on V_ν . Then $U(V'_\nu, h_\nu|_{V'_\nu}) \cong U(p_\nu - p_{W,\nu}, q_\nu - q_{W,\nu})$. This gives rise to a product $N = \prod U(p_{W,\nu}, q_{W,\nu}) \times U(p_\nu - p_{W,\nu}, q_\nu - q_{W,\nu})$, and the factors of the domain \mathcal{D}_N are of type $\mathbf{I}_{\mathbf{p}_{W,\nu}, \mathbf{q}_{W,\nu}} \times \mathbf{I}_{\mathbf{p}_\nu - \mathbf{p}_{W,\nu}, \mathbf{q}_\nu - \mathbf{q}_{W,\nu}}$. In particular, for $p_{W,\nu} = 0$, this is an irreducible group of type $\mathbf{I}_{\mathbf{p}_\nu, \mathbf{q}_\nu - \mathbf{q}_{W,\nu}}$ and for $q_{W,\nu} = 0$, of type $\mathbf{I}_{\mathbf{p}_\nu - \mathbf{p}_{W,\nu}, \mathbf{q}_\nu}$. Now since k is the degree of D , all of $p_\nu, q_\nu, p_{W,\nu}, q_{W,\nu}$ are divisible by d and the net subdomains these subgroups (possibly) define are

$$\mathbf{I}_{\mathbf{p-jd,q}}, \quad \mathbf{I}_{\mathbf{p,q-jd}}, \quad \mathbf{I}_{\mathbf{p-id,q-jd}} \subset \mathbf{I}_{\mathbf{p,q}}, \quad \mathbf{i, j} = 1, \dots, s. \quad (23)$$

(II) In $U(V, h)$ we have as above $U(V', h|_{V'})$; now if h is non-degenerate on V' , then $U(V', h|_{V'}) \cong U(n-1, D)$, giving subgroups of the real groups, defined over k , of type $U(n-1, \mathbb{H}) \subset U(n, \mathbb{H})$, with a corresponding subdomain of type $\mathbf{II}_{n-1} \subset \mathbf{II}_n$. This occurs at the primes for which D is definite; at the others $SU(V', h|_{V'}) \subset SU(V, h)$ is of the type $SO(2n-4, 2) \subset SO(2n-2, 2)$ (for $n = \text{dimension of } V \text{ over } D$). So we have maximal k -domains

$$\mathbf{II}_{n-1} \subset \mathbf{II}_n, \quad (\nu \text{ definite}), \quad \mathbf{IV}_{2n-4} \subset \mathbf{IV}_{2n-2}, \quad (\nu \text{ indefinite}).$$

(III) The index is $C_{n,s}^{(2)}$; this case is considered in more detail below.

From this, we deduce

Proposition 5.8 *Let G' have $\text{rank}_k G' = s \geq 2$, not split over \mathbb{R} , and let P'_s be a standard k -parabolic defining a zero-dimensional boundary component, $P'_s(\mathbb{R}) = N(F)$, and $\dim(F) = 0$. Then there is a k -subgroup N' incident with P'_s , with the following exception: Index $C_{2s,s}^{(2)}$.*

Proof: We first deduce for which of the indices listed in Proposition 5.1 zero-dimensional boundary components of \mathcal{D}' are rational (this is necessary for the zero-dimensional boundary components of \mathcal{D} to be rational). We need not consider exceptional cases or type \mathbf{IV}_n . We first consider the groups of type 2A .

Lemma 5.8.1 *For G' with the index ${}^2A_{n,s}^{(d)}$, let $G'(\mathbb{R}) \cong SU(p, q)$. Then the zero-dimensional boundary components are rational $\iff sd = q$.*

Proof: Let H_s be an s -dimensional (over D) totally isotropic subspace, with basis h_1, \dots, h_s . Let $h'_i \in V$ be vectors such that $h(h_i, h'_j) = \delta_{ij}$, and set $H'_s = \langle h'_1, \dots, h'_s \rangle$. Then h , restricted to $H :=$

$H_s \oplus H'_s$, is non-degenerate, and $SU(H^\perp, h|_{H^\perp})$ is the anisotropic kernel. The group $SU(H, h|_H)(\mathbb{R}) \cong SU(sd, sd)$, while $SU(H^\perp, h|_{H^\perp})(\mathbb{R}) \cong SU(p - ds, q - ds)$. This defines the subdomain of type $\mathbf{I}_{\mathbf{d}s, \mathbf{d}s} \times \mathbf{I}_{\mathbf{p}-\mathbf{d}s, \mathbf{q}-\mathbf{d}s}$ of Table 1, hence the boundary component, which is the second factor, is zero-dimensional $\iff q = ds$. \square

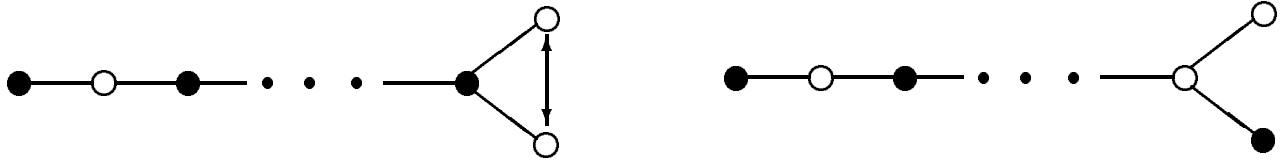
As to indices of type D we observe the following.

Lemma 5.8.2 *$\dim(F) = 0$ does not occur for the indices of type (II) in Proposition 5.1.*

Proof: Recall that D is a quaternion division algebra, central simple over k , with the canonical involution, V is an n -dimensional right D -vector space, and $h : V \times V \rightarrow D$ is a skew-hermitian form. Let ν_1, \dots, ν_a denote the infinite primes for which D_ν is definite, ν_{a+1}, \dots, ν_d the primes at which D_ν is split. Then $G(\mathbb{R})$ is a product

$$(SU(n, \mathbb{H}))^a \times (SO(2n - 2, 2))^{d-a},$$

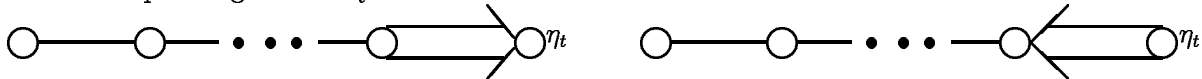
where we have taken into account that G is assumed to be of hermitian type. At each of the first factors we have the Satake diagram



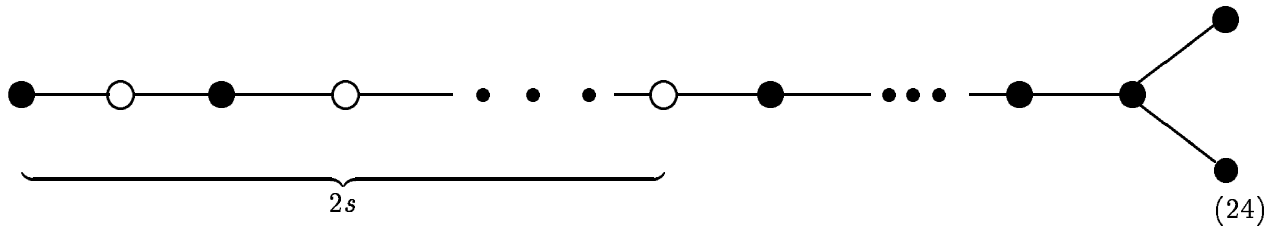
for n odd,

for n even.

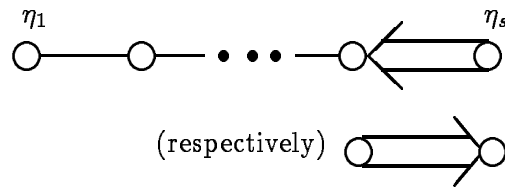
The corresponding \mathbb{R} -root systems are then:



In particular, the \mathbb{R} -root corresponding to the parabolic P_t with $\dim(F_t) = 0$ is the right-most one. On the other hand, the k -index is

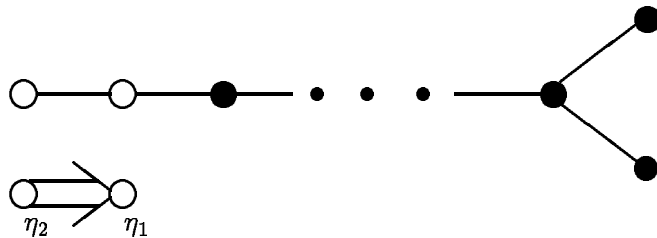


with the k -root system



from which it is evident that P_t is defined over $k \iff s = t (= \lfloor \frac{n}{2} \rfloor)$. But this is the split over \mathbb{R} case. Consequently, $a = 0$ and D is totally indefinite.

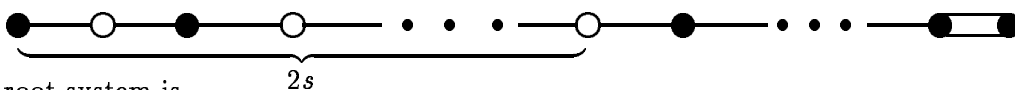
So we consider a prime ν where D_ν is split; the \mathbb{R} -index is



the \mathbb{R} -root η_2 corresponding to the two-dimensional totally isotropic subspace and zero-dimensional boundary component. The k -index is as in (24), so η_2 is always anisotropic; the boundary components are actually one-dimensional. This verifies the statements of the lemma. \square

Note that this proves Proposition 5.8 for the indices of type (II).

Now consider index $C_{n,s}^{(2)}$. The k -index is



and the k -root system is



The same reasoning as above shows that F_t is rational $\iff 2s = t$, but that is only possible if the index is $C_{2n,n}^{(2)}$. Hence:

Lemma 5.8.3 *The only indices of Proposition 5.1, case (III), for which zero-dimensional boundary components occur are $C_{2n,n}^{(2)}$.*

This index is that of the unitary group $U(V, h)$, where V is a $2n$ -dimensional vector space over D , and h has Witt index n . We can find n hyperbolic planes V_i such that

$$V = V_1 \oplus \cdots \oplus V_n.$$

This decomposition is defined over k , hence the subgroup

$$N = U(V_1, h|_{V_1}) \times \cdots \times U(V_n, h|_{V_n}),$$

which is a product of groups with index $C_{2,1}^{(2)}$, is also defined over k . We have

$$N(\mathbb{R}) \cong \underbrace{Sp(4, \mathbb{R}) \times \cdots \times Sp(4, \mathbb{R})}_{n \text{ times}} \tag{25}$$

and the domain \mathcal{D}_N is of type $(\mathbf{III}_2)^n$. This is the exception in the statement of the main theorem.

Proof of Proposition 5.8: We have already completed the proof for (II) and (III), and as we mentioned above, the exceptional cases and (IV) need not be considered. It remains to show the existence of groups of the stated types for indices 2A . We explained above how one can find k -subgroups N such that \mathcal{D}_N has irreducible components of types $\mathbf{I}_{p-jd,q}$ (see (23)). Here we take a maximal totally isotropic subspace H_s , and $H := H_s \oplus H'_s$ as described there. Let H^\perp denote the orthogonal complement, so that $SU(H^\perp, h|_{H^\perp})$ is the anisotropic kernel. Then, if $G'(\mathbb{R}) = SU(p, q)$, we have

$$SU(H, h|_H)(\mathbb{R}) \cong SU(sd, sd), \quad SU(H^\perp, h|_{H^\perp}) \cong SU(p - sd, q - sd).$$

Therefore we get a subdomain of type

$$\mathbf{I}_{sd, sd} \times \mathbf{I}_{p-sd, q-sd},$$

which is irreducible $\iff sd = q$; Then $N = \{g \in G \mid g(H) \subseteq H\}$ is a k -subgroup with $N(\mathbb{R}) \sim SU(q, q) \times \{\text{compact}\}$, and N then fulfills 1), 2') and 3'). By Lemma 5.8.1, this holds precisely when the boundary component F_s is a point. This completes the proof if $p > q$. It remains to consider the case where \mathcal{D}' is of type $\mathbf{I}_{q, q}$. In this case, $q = d \cdot j$ for some j , and the hermitian form $h : V \times V \rightarrow D$ has Witt index j . The vector space V is then $2j$ -dimensional, and it is the orthogonal direct sum of hyperbolic planes, $V = V_1 \oplus \dots \oplus V_j$, $\dim_D V_i = 2$. Consider the k -subgroup

$$N = \{g \in GL_D(V) \mid g(V_i) \subset V_i, i = 1, \dots, j\}.$$

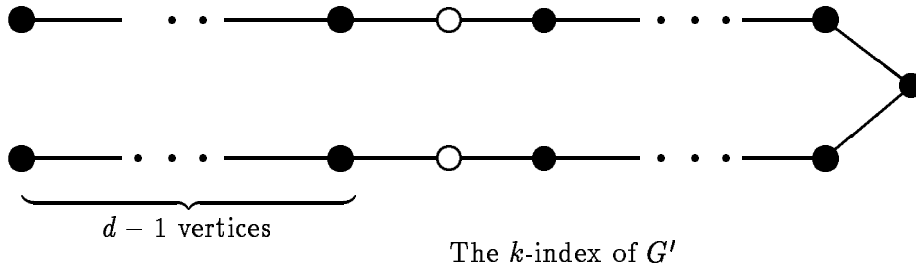
Clearly $N \cong N_1 \times \dots \times N_j$, and each N_i is a subgroup of rank one with index ${}^2A_{2d-1, 1}^{(d)}$. As was shown in [Hyp], in each N_i we have a k -subgroup $N'_i \subset N_i$, with $\mathcal{D}_{N'_i}$ of type $(\mathbf{I}_{1, 1})^d$. Then

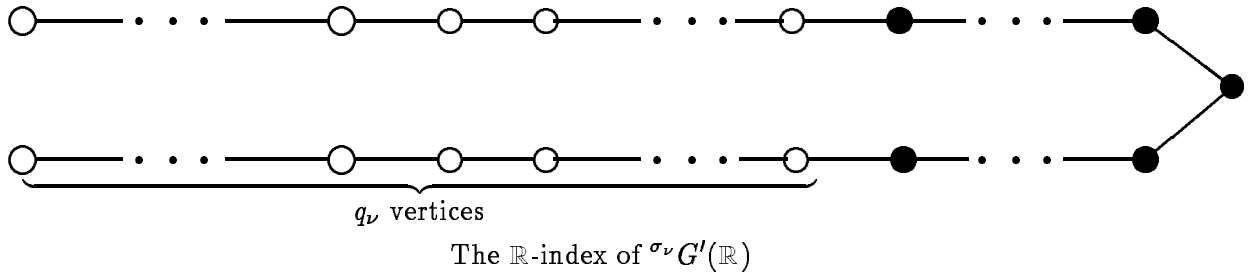
$$N' := N'_1 \times \dots \times N'_j$$

is a k -subgroup with $\mathcal{D}_{N'}$ of type $((\mathbf{I}_{1, 1})^d)^j = (\mathbf{I}_{1, 1})^{d \cdot j} = (\mathbf{I}_{1, 1})^q$, which is a maximal polydisc, i.e., satisfies 1), 2") and 3"). This completes the proof of Proposition 5.8 in this case also. \square

6 Rank one

We now come to the most interesting and challenging case. In this last paragraph G' will denote an absolutely simple k -group, G the corresponding \mathbb{Q} -simple group, both assumed to have rank one. There is only one standard maximal parabolic $P'_1 \subset G'$ in this case, so we may delete the subscript $_1$ in the notations. Let $P \subset G$ be the corresponding \mathbb{Q} -parabolic, so $P(\mathbb{R}) = P_1(\mathbb{R}) \times \dots \times P_d(\mathbb{R})$, where $P_\nu(\mathbb{R}) \subset {}^{\sigma_\nu}G'(\mathbb{R})$ is a standard maximal parabolic, say $P_\nu(\mathbb{R}) = N(F_{b_\nu})$, $F_{b_\nu} \subset \overline{D}_{\sigma_\nu}$. As we observed above, the F_{b_ν} are all hermitian spaces whose automorphism group is an \mathbb{R} -form of some fixed algebraic group. As we are now assuming the rank to be one, it follows from Lemma 3.1 that $L (=L_{\mathfrak{b}}$ in the notations above) is anisotropic. One way that this may occur was explained there, namely that if one of the factors F_{b_ν} is a *point*, in which case the symmetric space of $L(\mathbb{R})$ has a compact factor. Another possibility is that all F_{b_ν} are positive-dimensional, in which case L is a “genuine” anisotropic group. The type of F_{b_ν} can be determined from the k -index of G' and the \mathbb{R} -index of ${}^{\sigma_\nu}G'$. For example, for G' of type 2A , these indices are:





From this we see that the boundary component is of type $\mathbf{I}_{p-d, q-d}$.

There are basically two quite different cases at hand; the first is that the boundary components are positive-dimensional, the second occurs when the boundary components reduce to points. The former can be easily handled with the same methods as above, by splitting off orthogonal complements. The real interest is in the latter case, and here a basic role is played by the *hyperbolic planes*, which have been dealt with in detail in [Hyp]. We will essentially reduce the rank one case (at least for the classical groups) to the case of hyperbolic planes, then we explain how the results of [Hyp] apply to the situation here.

6.1 Positive-dimensional boundary components

Let G', P' be as above, and consider the hermitian Levi factor $G'^{(1)} = M'L'$, which is defined over k . Over \mathbb{R} the factors $M'(\mathbb{R})$ and $L'(\mathbb{R})$ are defined. In this section we consider the situation that the boundary component F' of \mathcal{D}' defined by P' (i.e., $P'(\mathbb{R}) = N(F')$) is positive-dimensional, or equivalently, that the hermitian Levi factor $L'(\mathbb{R})$ is non-trivial. As above, we get the following k -group

$$N' := G'^{(1)} \times \mathcal{Z}_{G'}(G'^{(1)}). \quad (26)$$

The same calculation as in (22) shows that the domain $\mathcal{D}_{N'}$ defined by N' is the same as that defined by $L'(\mathbb{R}) \times \mathcal{Z}_{G'(\mathbb{R})}(L'(\mathbb{R}))$. Taking the subgroup $N = \text{Res}_{k|\mathbb{Q}} N'$ defines a subdomain $\mathcal{D}_N \subset \mathcal{D}$, which is a product $\mathcal{D}_N = \mathcal{D}_{N, \sigma_1} \times \cdots \times \mathcal{D}_{N, \sigma_f}$. Each factor $\mathcal{D}_{N, \sigma}$ is determined by the corresponding factor of $\sigma L'(\mathbb{R})$. The \mathbb{R} -groups $N'(\mathbb{R})$ and $N(\mathbb{R})$ are determined in terms of the data D, V, h as follows.

- (I) If $F' \cong \mathbf{I}_{p-d, q-d}$, then $\mathcal{D}_{N'} \cong \mathbf{I}_{p-d, q-d} \times \mathbf{I}_{d, d}$. Note that in terms of the hermitian forms, this amounts to the following. Since h has Witt index 1, the maximal totally isotropic subspaces are one-dimensional. Let $H_1 = \langle v \rangle$ be such a space; we can find a vector $v' \in V$ such that $H = \langle v, v' \rangle$ is a hyperbolic plane, that is, $h|_H$ has Witt index 1. It follows that $h|_{H^\perp}$ is anisotropic. Consider the subgroup

$$N_k := \{g \in U(V, h) \mid g(H) \subset H\}. \quad (27)$$

It is clear that for $g \in N_k$, it automatically holds that $g(H^\perp) \subset H^\perp$, hence

$$N_k \cong U(H, h|_H) \times U(H^\perp, h|_{H^\perp}). \quad (28)$$

The first factor has \mathbb{R} -points $U(H, h|_H)(\mathbb{R}) \cong U(d, d)$, while the second fulfills $U(H^\perp, h|_{H^\perp})(\mathbb{R}) \cong U(p-d, q-d)$. Thus $N_k \cong N'$ as in (26). At any rate, this gives us subdomains of type

$$\mathbf{I}_{d, d} \times \mathbf{I}_{p-d, q-d} \subset \mathcal{D}_{N'},$$

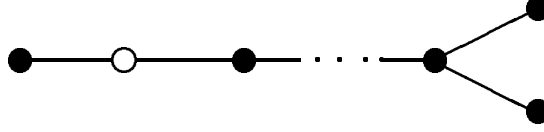
which, in case $d = p = q$ is the whole domain; in all other cases it is a genuine subdomain as listed in Table 1, defined over k , and (N', P') are incident. It follows from this that (N, P) are incident over \mathbb{Q} . The components $N_\sigma(\mathbb{R})$ of $N_{\mathbf{b}}(\mathbb{R})$ are determined as follows. Let (p_ν, q_ν) be the signature of h_ν (so that $p_\nu + q_\nu = dm$ for all ν). This implies

$$G(\mathbb{R}) \cong \prod_{\nu} SU(p_\nu, q_\nu).$$

For each factor, we have the boundary component $F_\sigma \cong SU(p_\nu - d, q_\nu - d)/K$, and for each factor for which $q_\nu > d$ this is positive-dimensional. As above, this leads to subdomains, in each factor, of type $\mathbf{I}_{d,d} \times \mathbf{I}_{p_\nu-d, q_\nu-d}$, so that in sum

$$\mathcal{D}_N \cong \prod_{\nu} \mathcal{D}_\nu, \quad \mathcal{D}_\nu \text{ of type } \mathbf{I}_{d,d} \times \mathbf{I}_{p_\nu-d, q_\nu-d}. \quad (29)$$

(II) Here rank 1 means we have the following k -index, $D_{n,1}^{(2)}$



In particular, the boundary component is of type \mathbf{II}_{n-2} if \mathcal{D}' is of type \mathbf{II}_n . This means also that the ‘‘mixed cases’’ only can occur if \mathcal{D}' is of type \mathbf{II}_4 , for then $\mathbf{II}_2 \cong$ one-dimensional disc. Of course $\mathbf{II}_4 \cong \mathbf{IV}_6$ anyway, so we can conclude from this that mixed cases do not occur in the hermitian symmetric setting (for \mathbb{Q} -simple G of rank 1). The domain $\mathcal{D}_{N'}$ defined by N' is of type $\mathbf{II}_{n-2} \times \mathbf{II}_2$. The components $N_\sigma(\mathbb{R})$ of $N_1(\mathbb{R})$ are all of type $U(n-2, \mathbb{H}) \times U(2, \mathbb{H}) \subset U(n, \mathbb{H})$, so the domain \mathcal{D}_N is of type

$$(\mathbf{II}_{n-2} \times \mathbf{II}_2)^f. \quad (30)$$

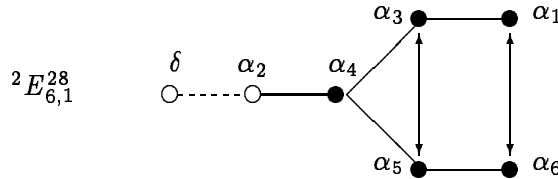
(III) Here rank 1 implies the index is one of $C_{1,1}^{(1)}$ (which we have excluded) or $C_{n,1}^{(2)}$. The corresponding boundary components in these cases are of type \mathbf{III}_{n-2} . The case $C_{2,1}^{(2)}$, for which the boundary component is a point, will be dealt with later, the others give rise to a subdomain of type $\mathbf{III}_2 \times \mathbf{III}_{n-2}$. Consequently, \mathcal{D}_N is of type $(\mathbf{III}_{n-2} \times \mathbf{III}_2)^f$, $f = [k : \mathbb{Q}]$.

(IV) Here we just have a symmetric bilinear form of Witt index 1. The k -index in this case is necessarily of the form



The corresponding boundary component is a point, a case to be considered below. Splitting off an anisotropic vector (defined over k) in this case yields a codimension one subspace H^\perp on which h still has Witt index 1, hence the stabilizer N' defines a subdomain $\mathcal{D}_{N'}$ of type \mathbf{IV}_{n-1} . \mathcal{D}_N is then of type $(\mathbf{IV}_{n-1})^f$.

(V) The only index of rank 1 is



The vertex denoted α_2 gives rise to the five-dimensional boundary component. If δ denotes the lowest root, then, as is well known, δ is isotropic (does not map to zero in the k -root system), so the root δ defines a k -subalgebra $\mathfrak{n}_\delta := \mathfrak{g}^\delta + \mathfrak{g}^{-\delta} + [\mathfrak{g}^\delta, \mathfrak{g}^{-\delta}] \subset \mathfrak{g}'$ which is split over k . On the other hand the anisotropic kernel \mathcal{K} is of type 2A_5 , and $\mathcal{K}(\mathbb{R}) \cong U(5, 1)$. Clearly \mathcal{K} and the k -subgroup N_δ defined by \mathfrak{n}_δ are orthogonal, so we get a k -subgroup

$$N' = N_\delta \times \mathcal{K},$$

both factors being defined over k . The set of \mathbb{R} -points is then of type $N'(\mathbb{R}) \cong SL_2(\mathbb{R}) \times SU(5, 1)$, and the subdomain $\mathcal{D}_{N'}$ is

$$\mathcal{D}_{N'} \cong \mathbf{I}_{1,1} \times \mathbf{I}_{5,1}.$$

This is one of the domains listed in Table 1, incident to the five-dimensional boundary component. It follows that \mathcal{D}_N is a product of factors of this type.

(VI) There are no indices of hermitian type with rank one for E_7 .

We sum up these results in the following.

Proposition 6.1 *If the rational boundary components for G' are positive-dimensional, then Theorem 3.3 holds for G . The subdomains defined by the symmetric subgroups $N' \subset G'$ are:*

(I) $\mathbf{I}_{d,d} \times \mathbf{I}_{p-d,q-d}$.

(II) $\mathbf{II}_{n-2} \times \mathbf{II}_2$.

(III) $\mathbf{III}_{n-2} \times \mathbf{III}_2$.

(IV) \mathbf{IV}_{n-1} (here there are no positive-dimensional boundary components).

(V) $\mathbf{I}_{1,1} \times \mathbf{I}_{5,1}$.

Note here $\mathbf{I}_{1,1} \cong \mathbf{II}_2 \cong \mathbf{III}_1 \cong \mathbf{IV}_1$. The corresponding domains \mathcal{D}_N in \mathcal{D} defined by the subgroups N are products of domains of the types listed above.

6.2 Zero-dimensional boundary components

The restrictions rank equal to one and zero-dimensional boundary components are only possible for the domains of type $\mathbf{I}_{p,q}$, \mathbf{III}_2 and \mathbf{IV}_n (see Lemmas 5.8.2 and 5.8.3). Of these, the last case requires no further discussion: as above we find a codimension one k -subspace $V' \subset V$, on which h still is isotropic, and take its stabilizer as N' . This gives a k -subgroup $N' \subset G'$, and defines a subdomain $\mathcal{D}_{N'}$ of type \mathbf{IV}_{n-1} . In the ${}^2A^{(d)}$ case we may assume $d \geq 3$: the $d = 1$ case is again easily dealt with as above. We have a K -vector space V ($K|k$ imaginary quadratic) of dimension $p + q$ and a (K -valued) hermitian form h of Witt index 1 on V . By taking a K -subspace $V' \subset V$ of codimension one, such that $h|_{V'}$ still has Witt index 1, we get the k -subgroup N' as the stabilizer of V' . Then the domain $\mathcal{D}_{N'}$ is either of type $\mathbf{I}_{p-1,q}$ or $\mathbf{I}_{p,q-1}$, and by judicious choice of V' we can assume the first case, which is the domain listed in Table 2. The $d = 2$ case is “lifted” from the corresponding $d = 2$ case with involution of the first kind: if D is central simple of degree 2 over K with a $K|k$ -involution, then ([A], Thm. 10.21) $D = D' \otimes_k K$, where D' is central simple of degree 2 over k with the canonical involution. Consequently,

$$U(V, h) = U(V' \otimes_k K, h' \otimes_k K) = U(V', h')_K,$$

the group is just the group $U(V', h')$ lifted to K . Since $U(V', h')$ has index $C_{n,1}^{(2)}$, while $U(V', h')_K$ has index $A_{2n-1,1}^{(2)}$, it follows that the boundary component is a point only if $n \leq 2$. This implies that if $d = 2$, the index is $A_{3,1}^{(2)}$, the domain is $\mathbf{I}_{2,2} \cong \mathbf{IV}_4$, so $U(V', h')_K$ is isomorphic to an orthogonal group over k in six variables. As we just saw, in this case there is a subdomain defined over k of type $\mathbf{IV}_3 \subset \mathbf{IV}_4$. So we assume $d \geq 3$. Then, as we have seen, the boundary component $F' \cong \mathbf{I}_{p-d,q-d}$ will be zero-dimensional $\iff q = d$ (respectively $F' \cong \mathbf{I}_{p_1-d,q_1-d} \times \cdots \times \mathbf{I}_{p_f-d,q_f-d}$ will be zero-dimensional $\iff q_\nu = d, \forall \nu$). Here there are two possibilities:

1) $p = q = d$, the group N_k of (27) is $N_k \cong G'$. This is the case of *hyperbolic planes*.

2) $p > q = d$, the group N_k of (27) is over \mathbb{R} just $N_k(\mathbb{R}) = U(d, d) \times U(p - d) \subset U(p, d) \cong G'(\mathbb{R})$.

Note that in the second case the domain \mathcal{D}_{N_k} defined by N_k is of type $\mathbf{I}_{d,d}$, a maximal tube domain in $\mathbf{I}_{p,q}$. So we are also finished in this case. For completeness, let us quickly go through the details to make sure nothing unexpected happens.

Proposition 6.2 *Let G' have index ${}^2A_{n,1}^{(d)}$, $d = q$, $p > q$, $n + 1 = p + q$, and let P' denote the corresponding standard parabolic and $N' = N_k$, where $N_k \subset G'$ the symmetric subgroup defined in (27), where H is the hyperbolic plane spanned by the vector which is stabilized by P' and its ortho-complement ($v': h(v, v') = 1$). Then (P', N') are incident, in fact standard incident. Consequently, $P = \text{Res}_{k|\mathbb{Q}} P'$ and $N = \text{Res}_{k|\mathbb{Q}} N'$ are incident over \mathbb{Q} .*

Proof: We know that $N'(\mathbb{R}) \cong U(q, q) \times U(p - q)$ which gives rise to the maximal tube subdomain $\mathbf{I}_{q,q} \subset \mathbf{I}_{p,q}$ of Table 2. We need to check that the standard boundary component F' stabilized by $P'(\mathbb{R})$ is also a standard boundary component of $\mathcal{D}_{N'}$; in particular we need the common maximal \mathbb{R} -split torus in P' and N' . This is seen in (27), the \mathbb{R} -split torus being contained in the hermitian Levi factor of $P'(\mathbb{R})$, which is contained in $N'(\mathbb{R})$. Consider the group $P' \cap N'$; this is nothing but the stabilizer of v in H , which is a maximal standard parabolic in N' . Since ν determines the boundary component F , both in $G'(\mathbb{R})$ and in $N'(\mathbb{R})$, it is clear that F is a boundary component of $\mathcal{D}_{N'}$. It follows that (P', N') are incident, and this implies (see the discussion preceding Proposition 5.7) that $(P, N) \subset G$ are incident. \square

We are left with the following cases: \mathbf{III}_2 with index $C_{2,1}^{(2)}$ and $\mathbf{I}_{q,q}$ with index ${}^2A_{2d-1,1}^{(d)}$, $d \geq 3$. These indices are described in terms of hermitian forms as follows. Let D be a central simple division algebra over K ($K = k$ for $d = 2$ and $K|k$ is imaginary quadratic if $d \geq 3$) and assume further that D has a $K|k$ -involution, V is a two-dimensional right vector space over D and $h : V \times V \rightarrow D$ is a hermitian form which is isotropic. Then $d = 2$ gives groups with index $C_{2,1}^{(2)}$, and $d \geq 3$ gives groups with indices ${}^2A_{2d-1,1}^{(d)}$.

Lemma 6.3 *There exists a basis v_1, v_2 of V over D such that the form h is given by $h(x, y) = x_1 \bar{y}_2 + x_2 \bar{y}_1$, $x = (x_1, x_2)$, $y = (y_1, y_2)$.*

Proof: Let v be an isotropic vector, defined over k . Then there exists an isotropic vector v' , such that $h(v, v') = 1$, hence also $h(v', v) = 1$. Let $v' = (v'_1, v'_2)$, and set $\delta = v'_1 \bar{v}'_2$, so that $h(v', v') = \delta + \bar{\delta}$. Then the matrix of h with respect to the basis v, v' is $H' = \begin{pmatrix} 0 & 1 \\ 1 & \varepsilon \end{pmatrix}$, where $\varepsilon = \delta + \bar{\delta}$. Now setting

$$w = (w_1, w_2) = (-v_1 \bar{\delta} + v'_1, -v_2 \bar{\delta} + v'_2)$$

we can easily verify $h(w, w) = 0$, $h(v, w) = h(w, v) = 1$. Since the change of basis transformation is defined over k , the matrix of the hermitian form with respect to this k -basis v, w is $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. \square

So as far as the \mathbb{Q} -groups are concerned, we may take the standard hyperbolic form given by the matrix H as defining the hermitian form on V . We remark that the situation changes when one considers arithmetic groups, but that need not concern us here. At any rate, a two-dimensional right D -vector space V with a hermitian form as in Lemma 6.3 is what we call a *hyperbolic plane*, and this case was studied in detail in [Hyp]. There it was determined exactly what kind of symmetric subgroups exist. These derive from the existence of splitting subfields $L \subset D$, which may be taken to be cyclic of degree d over K , if D is central simple of degree d over K . In fact, we have subgroups ([Hyp], Proposition 2.4) $U(L^2, h) \subset U(D^2, h)$, which give rise to the following subdomains:

1) $d = 2$; $\mathcal{D}_L \cong \left(\begin{pmatrix} \tau_1 & 0 \\ 0 & b\zeta_1 \tau_1 \end{pmatrix} \right) \times \cdots \times \left(\begin{pmatrix} \tau_1 & 0 \\ 0 & b\zeta_f \tau_1 \end{pmatrix} \right)$, where $\zeta_i : k \rightarrow \mathbb{R}$ denote the distinct real embeddings of k .

$$2) \ d \geq 3; \mathcal{D}_L \cong \begin{pmatrix} \tau_1 & & 0 \\ & \ddots & \\ 0 & & \tau_d \end{pmatrix}^f.$$

In other words, for hyperbolic planes we find subdomains of the following kinds

$$\mathbf{III}_1 \subset \mathbf{III}_2, \quad (\mathbf{I}_{1,1})^d \subset \mathbf{I}_{d,d}. \quad (31)$$

The latter one is a polydisc, coming from a maximal set of strongly orthogonal roots, i.e., satisfying 1), 2") and 3"). The first case is the only exception to the rule that we have symmetric subgroups $N' \subset G'$ with $\text{rank}_{\mathbb{R}} N' = \text{rank}_{\mathbb{R}} G'$.

Proof of Theorem 3.3: We have split the set of cases up into the three considered in §4, 5 and 6. Corollary 4.5 proves 3.3 for the split over \mathbb{R} case and Proposition 5.7 for the rank ≥ 2 case and positive-dimensional boundary components. For rank ≥ 2 and zero-dimensional boundary components, Proposition 5.8 shows that with the exception given Theorem 3.3 holds in this case also. In the case of rank 1, Proposition 6.1 verifies 3.3 for the case that the boundary components are positive-dimensional, and Proposition 6.2 took care of the rest of the cases excepting hyperbolic planes. Then the results of [Hyp] verify 3.3 for hyperbolic planes, thus completing the proof. \square

Proof of the Main Theorem: The first statement is covered by Theorem 3.3. The statements on the domains for the exceptions follow from (25) and (31). It remains to consider the condition 4). This is fulfilled for the groups N utilized above by construction. For the exceptional cases this is immediate, as we took subgroups defined by symmetric closed sets of roots. Let us sketch this again for the classical cases, utilizing the description in terms of \pm symmetric/hermitian forms. The objects D , V , h and G' will have the meanings as above. Let $s = \text{rank}_k G'$, and let H_s be an s -dimensional (maximal) totally isotropic subspace in V , with basis h_1, \dots, h_s . Let $h'_i \in V$ be vectors of V with $h(h_i, h'_j) = \delta_{ij}$, $H'_s = \langle h'_1, \dots, h'_s \rangle$ and set $H = H_s \oplus H'_s$. Then $h|_H$ is non-degenerate of index s , and H splits into a direct sum of hyperbolic planes, $H = V_1 \oplus \dots \oplus V_s$. The form h restricted to H^\perp is anisotropic; the semisimple anisotropic kernel is $SU(H^\perp, h|_{H^\perp})$. Fixing the basis $h_1, \dots, h_s, h'_1, \dots, h'_s$ for H amounts to the choice of maximal k -split torus S' . For each real prime ν , (H_ν, h_ν) is a $2ds$ -dimensional \mathbb{R} -vector space with \pm symmetric/hermitian form. Choosing an \mathbb{R} -basis of H_ν amounts to choosing a maximal \mathbb{R} -split torus of $SU(H_\nu, h_\nu)$, and a choice of basis for a maximal set of hyperbolic planes (over \mathbb{R}) amounts to the choice of maximal \mathbb{R} -split torus. Similarly, $(H_\nu^\perp, h_\nu|_{H_\nu^\perp})$ is an \mathbb{R} -vector space, $h|_{H_\nu^\perp}$ has some index q_ν , and one can find a maximal set of hyperbolic planes W_1, \dots, W_τ , such that $H_\nu^\perp = (W_1)_\nu \oplus \dots \oplus (W_\tau)_\nu \oplus W'$, where $h_\nu|_{W'}$ is anisotropic over \mathbb{R} . A choice of basis of the $(W_i)_\nu$ amounts to the choice of maximal \mathbb{R} -split torus, and a choice of basis, over \mathbb{R} , of V_ν amounts to the choice of maximal torus defined over \mathbb{R} . From these descriptions we see that the polydisc group N_Ψ defined by the maximal set of strongly orthogonal roots Ψ splits into a component in $SU(H, h|_H)$ and a component in $SU(H^\perp, h|_{H^\perp})$, say $N_\Psi = N_{\Psi,1} \times N_{\Psi,2}$. Then $N_{\Psi,2} \subset SU(H^\perp, h|_{H^\perp})(\mathbb{R})$ and $N_{\Psi,1} \subset SU(H, h|_H)$. Since the subgroup $SU(H^\perp, h|_{H^\perp})$ is contained in all the groups N we have defined, we need only consider $N_{\Psi,1}$. H is a direct sum of hyperbolic planes V_i , and the question is whether the corresponding polydisc group is contained in $SU(V_i, h|_{V_i})$. But this is what was studied in [Hyp]; the answer is affirmative. It follows that with the one exception stated, $C_{2,1}^{(2)}, N_\Psi \subset N$. \square

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