

# Modular subvarieties of arithmetic quotients of bounded symmetric domains

Bruce Hunt  
FB Mathematik, Universität  
Postfach 3049  
67653 Kaiserslautern

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## Introduction

A reductive  $\mathbb{Q}$ -simple algebraic group  $G$  is of *hermitian type*, if the symmetric space  $\mathcal{D}$  defined by  $G(\mathbb{R})$  is a hermitian symmetric space. A discrete subgroup  $\Gamma \subset G(\mathbb{Q})$  is *arithmetic*, if for some faithful rational representation  $\rho : G \rightarrow GL(V)$ , and for some lattice  $V_{\mathbb{Z}} \subset V_{\mathbb{Q}}$ ,  $\Gamma$  is commensurable with  $\rho^{-1}(GL(V_{\mathbb{Z}}))$ . A non-compact hermitian symmetric space is holomorphically equivalent to a bounded symmetric domain. If this is the case,  $\Gamma$  acts on  $\mathcal{D}$  preserving the natural Bergmann metric, and  $X_{\Gamma} = \Gamma \backslash \mathcal{D}$  is, if  $\Gamma$  is torsion-free, a complex manifold, in general not compact (it is compact exactly when  $G$  is anisotropic). We call spaces  $X_{\Gamma}$  arithmetic quotients of bounded symmetric domains even when  $\Gamma$  has torsion; it is known that in this case  $X_{\Gamma}$  is a  $V$ -manifold (in the sense of Satake), locally the quotient of a smooth space by a finite group action. There is a natural compactification of  $X_{\Gamma}$ , the Satake compactification  $X_{\Gamma}^*$ , which has the property: the complement  $X_{\Gamma}^* - X_{\Gamma}$  is a finite disjoint union of arithmetic quotients of bounded symmetric domains of lower dimension. A natural problem in this respect is to consider, in addition to the data above, a symmetric subdomain  $\mathcal{D}' \subset \mathcal{D}$  which has the property that the restriction of the action of  $\Gamma$  to  $\mathcal{D}'$  is discrete, say by a discrete subgroup  $\Gamma'$ , resulting in a commutative square

$$\begin{array}{ccc} \mathcal{D}' & \hookrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ X_{\Gamma'} = \Gamma' \backslash \mathcal{D}' & \hookrightarrow & \Gamma \backslash \mathcal{D} = X_{\Gamma}. \end{array}$$

It is spaces arising as  $X_{\Gamma'}$  that we will refer to as *modular subvarieties*. This situation ensues in particular if  $N \subset G$  is a reductive subgroup of hermitian type; then the symmetric space  $\mathcal{D}_N$  of  $N$  has a holomorphic symmetric embedding  $\mathcal{D}_N \subset \mathcal{D}$ . As a matter of notation, we refer to such subgroups as *symmetric* subgroups. This explains the title and describes the topic of this paper.

We are in fact concerned with a very special set of modular subvarieties which have very special behavior at the cusps. Upstairs in the universal covers (in the domains  $\mathcal{D}'$  and  $\mathcal{D}$ ), this behavior defines a notion of incidence, implying a relation between (real) parabolic and symmetric subgroups. Conversely, one can define a notion of incidence, group theoretically, between parabolic and symmetric subgroups  $P$  and  $N$ , which implies the desired behavior of the subdomain  $\mathcal{D}_N$  (the  $\mathcal{D}'$  in the notations above) near the cusp  $F$  which corresponds to  $P$ . It is the geometric point of view we will adapt in this paper. Starting with a rational boundary component  $F$  (respectively the corresponding parabolic  $P$ ), we define what it means for  $F$  and a symmetric subdomain  $\mathcal{D}_N$  (respectively the parabolic  $P$  and a symmetric subgroup  $N$ ) to be incident. For  $F$  and  $\mathcal{D}_N$  this is easy to formulate:  $\mathcal{D}_N$  and  $F$  are incident

if  $F$  is a rational boundary component of  $\mathcal{D}_N$  and maximal with this property. For the subgroups  $N$  and  $P$  the condition is more complicated to formulate, but consists essentially in a maximality condition plus the condition  $N = N_1 \times N_2$ , and  $N_1$  is a hermitian Levi factor of  $P$  (if  $\dim(F) > 0$ ) or just that  $N$  is irreducible such that  $F$  is a boundary component of  $\mathcal{D}_N$  (if  $\dim(F) = 0$ ), see below for details.

This gives rise to a notion of incidence for  $\mathbb{Q}$ -subgroups  $P$  and  $N$ , by insisting that incidence holds as above for the groups of real points  $P(\mathbb{R})$  and  $N(\mathbb{R})$ . Everything being defined over  $\mathbb{Q}$ , one can proceed to form an arithmetic quotient by an arithmetic subgroup  $\Gamma \subset G(\mathbb{Q})$ , and  $\Gamma$  will act naturally both on  $F$  and on  $\mathcal{D}_N$ . The quotients will be a boundary variety  $W$  and a modular subvariety  $X_\Gamma$  as above, and the notion of incidence now becomes  $W \subset X_{\Gamma'}^*$ , where  $X_{\Gamma'}^* \subset X_\Gamma^*$  is the embedding of Baily-Borel embeddings described in [S2]. Thus the problem has two aspects

- a) The question of the *existence* of  $\mathbb{Q}$ -subgroups  $N$  which are incident with a given  $P$ , and
- b) the action of  $\Gamma$  on the set of such subgroups.

The first point a) was treated in detail in [H1] and [H2]; this paper contains a preliminary study of b).

Similar questions have been asked and answered for parabolic subgroups, but the situation for symmetric subgroups is totally different. The basic cause for this is the following elementary fact: if two  $\mathbb{Q}$ -parabolics are conjugate (over the algebraic closure), then they are in fact  $\mathbb{Q}$ -conjugate, i.e., conjugate by an element of  $G(\mathbb{Q})$ , whereas the corresponding statement for symmetric subgroups is totally wrong. The reason behind this is the very fundamental fact that the homogenous spaces  $G/P$  are *projective* for parabolic subgroups  $P$ , while  $G/N$  is *affine* for a reductive subgroup  $N$ , a property which in fact characterizes reductive subgroups  $N$ , as was proved in [BHC]. A consequence of this is that it holds for the parabolic subgroup  $P$  that  $G(\mathbb{Q})/P(\mathbb{Q}) = (G/P)(\mathbb{Q})$ , while for symmetric subgroups  $N$  only the inclusion  $G(\mathbb{Q})/N(\mathbb{Q}) \subset (G/N)(\mathbb{Q})$  holds. The elements of the left-hand side are those subgroups  $N'(\mathbb{Q}) \subset G(\mathbb{Q})$  which are  $G(\mathbb{Q})$ -conjugate to  $N(\mathbb{Q})$ , while on the right-hand side we have those  $G$ -conjugates of  $N$  which are *defined over*  $\mathbb{Q}$ . Considering only the former leads to the definition of *rational* symmetric subgroups (Definition 4.1). From this point on we restrict our attention to symmetric subgroups  $N$  which are incident with a given parabolic  $P$  (or the other way around, parabolic subgroups  $P$  which are incident with a given symmetric subgroup  $N$ ). This means we consider for each  $b \in \{1, \dots, s\}$ ,  $s = \text{rank}_{\mathbb{Q}} G$ , a fixed isomorphism class (with the exception  $b = s$  for the two exceptional domains, for which there are three, resp. two such isomorphism classes) of the groups of real points  $N(\mathbb{R})$  for subgroups  $N$  (conjugate to a fixed symmetric subgroup  $N_b$  incident with  $P_b$ , a standard parabolic), and consider pairs  $(N, P)$  of incident symmetric and parabolic subgroups. We consider rational symmetric subgroups  $N$  representing points of  $G(\mathbb{Q})/N_b(\mathbb{Q})$ , i.e.,  $N$  is  $G(\mathbb{Q})$ -conjugate to  $N_b$ . However, these turn out to still be too many such subgroups in the sense that there are in general infinitely many  $\Gamma$ -equivalence classes for any  $\Gamma$ . We are led to introduce the notion of  $\Gamma$ -*integral* symmetric subgroups (Definition 4.3), of which there are finitely many  $\Gamma$ -orbits. To formulate this we assume  $\dim(F) > 0$  and let  $N_b = N_{b,1} \times N_{b,2}$  (resp.  $N = N_1 \times N_2$ ) be the decomposition of  $N_b$  (resp. of  $N$ ) mentioned above. Then the definition in this case is:

$$N = gN_bg^{-1} \text{ is integral} \iff gN_1g^{-1} \cap \Gamma = g(N_{b,1} \cap \Gamma)g^{-1},$$

and it depends in fact on the choice of  $N_b$  (that is, on the choice of maximal  $\mathbb{Q}$ -split torus and order on it) as well as on  $\Gamma$ . (For  $\dim(F) = 0$ , the condition is simply  $gNg^{-1} \cap \Gamma = g(N_b \cap \Gamma)g^{-1}$ .) Fixing this data  $N_b$  and  $\Gamma$  leads to a finite number of  $\Gamma'$ -equivalence classes of  $\Gamma$ -integral symmetric subgroups conjugate to  $N_b$  for any arithmetic group  $\Gamma' \subset G(\mathbb{Q})$ , which we show by utilizing the basic finiteness result of [BHC].

We now consider  $\Gamma$ -integral symmetric subgroups  $N$  and arbitrary arithmetic subgroups  $\Gamma' \subset G(\mathbb{Q})$ , let  $\Gamma'_N = N \cap \Gamma'$  and consider the corresponding modular subvarieties they define,  $X_{\Gamma'_N} \subset X_{\Gamma'}$ ; we call these *integral* modular subvarieties (Definition 6.2). As mentioned above, the inclusion extends to the Baily-Borel embeddings  $X_{\Gamma'_N}^* \subset X_{\Gamma'}^*$ . We now take  $\Gamma$  to be  $G_{\mathbb{Z}}$  for some rational representation  $\rho : G \rightarrow GL(V)$ , that is  $\Gamma = \rho^{-1}(GL(V_{\mathbb{Z}}))$  for some  $\mathbb{Z}$ -structure  $V_{\mathbb{Z}}$  on  $V$ . Putting all the pieces together yields our main result, which we now formulate. For this we refer to the notations  $\nu_b(\Gamma')$ ,  $b = 1, \dots, s$  and  $\mu_b(\Gamma, \Gamma')$ ,  $b = 1, \dots, s$  of Definition 4.12 and 5.3, respectively, for the number of  $b^{th}$  boundary varieties and the number of  $b^{th}$  integral modular subvarieties, respectively. We let  $W_{b,i}$ ,  $b = 1, \dots, s$ ,  $i = 1, \dots, \nu_b(\Gamma')$  be the corresponding boundary varieties on the Satake compactification,  $Y_{b,j}$ ,  $b = 1, \dots, s$ ,  $j = 1, \dots, \mu_b(\Gamma, \Gamma')$  the corresponding  $\Gamma$ -integral modular varieties, everything on the arithmetic quotient  $X_{\Gamma'}$ . Then the main result of the paper is the following.

**Theorem 0.1** *Let  $\Gamma$  be as above,  $\Gamma' \subset G(\mathbb{Q})$  arithmetic, and  $X_{\Gamma'} \subset X_{\Gamma'}^*$  the Satake compactification,  $X_{\Gamma'}^* - X_{\Gamma'} = \sum_{b,i} W_{b,i}$ . Then  $\Xi := \sum_{b,j} Y_{b,j}$  is a complete (finite, non-empty) set of  $\Gamma'$ -equivalence classes of  $\Gamma$ -integral modular subvarieties, such that for each  $W_{b,i}$ , there is at least one  $Y_{b,j}$  incident to  $W_{b,i}$ .*

This gives us a *well-defined, non-empty, finite* set of subvarieties of the Baily-Borel embedding  $X_{\Gamma'}^* \subset \mathbb{P}^N$  for any subgroup  $\Gamma' \subset \Gamma$  of finite index. Furthermore these have a prescribed behavior near the cusps. For example, if  $f : \mathcal{D} \rightarrow \mathbb{C}$  is a modular form whose zero divisor  $D_f$  on  $X_{\Gamma'}^*$  contains the union of the integral modular subvarieties, then  $f$  is a cusp form for  $\Gamma'$ .

In the case that the arithmetic quotient  $X_{\Gamma}$  is a moduli space of abelian varieties with some PEL structure, it is not difficult to see the moduli interpretation of the modular subvarieties  $X_{\Gamma_N}$  determined by *rational* symmetric subgroups. See Example 4.2, where an interesting case is discussed in more detail. That is the case  $G = Sp(4, \mathbb{Q})$ , and it turns out that in this case many modular subvarieties which come from  $\mathbb{Q}$ -groups  $N$  conjugate to the standard one have a nice moduli interpretation: they parameterize abelian varieties (surfaces in this case) with real multiplication by a real quadratic field  $k$ . The group  $N$  is  $G(\mathbb{Q})$ -conjugate by the standard one  $N_1$  precisely when the field  $k$  splits into a product  $\mathbb{Q} \times \mathbb{Q}$ , and in this case the real multiplication “degenerates” into two copies of multiplication by  $\mathbb{Q}$ , in other words the abelian surface is no longer simple but *isogenous* to a *product*. In this particular case the set of  $\Gamma$ -integral subgroups ( $\Gamma = Sp(4, \mathbb{Z})$ ) corresponds to the abelian surfaces which are actually isomorphic to a product. We prove more generally the following:

**Theorem 0.2** *Let  $G$ ,  $S$ ,  $P_b$ ,  $N_b$  and  $\Gamma$  be as above ( $b < t$ ),  $\Gamma' \in G(\mathbb{Q})$  arithmetic, and let  $X_{\Gamma'_N}$  be a modular subvariety of  $X_{\Gamma'}$  for  $N$  rational symmetric, conjugate to  $N_b$ . Then  $X_{\Gamma'_N}$  is a finite quotient of a product, and the set of  $\Gamma'$ -equivalence classes of such modular subvarieties forms a locus in  $X_{\Gamma'}$  where the corresponding abelian varieties are isogenous to products, i.e., are not simple. If  $N$  is  $\Gamma$ -integral, then  $X_{\Gamma'_N}$  is a product, and the set of  $\Gamma'$ -equivalence classes of such modular subvarieties forms a locus in  $X_{\Gamma'}$  where the corresponding abelian varieties split while preserving the endomorphisms (but not necessarily the polarizations).*

An important application of the main result is to define a simplicial complex which is an analog for reductive groups of what the Tits building is for parabolic groups. Recall this complex  $\mathcal{T}(G)$  is constructed by forming the simplicial complex of the set of all rational parabolics, partially ordered by the inverse of the inclusion. Taking  $\Gamma$ -equivalence classes of the parabolics gives rise to a *finite* quotient complex  $\mathcal{T}(G)/\Gamma$ , whose vertices are in one to one correspondence with the boundary varieties of  $X_{\Gamma}^*$ . However, the Tits building  $\mathcal{T}(G)$  is a *rational* invariant of  $G$ , not depending in any way on the arithmetic group  $\Gamma$ . As an analog we can define a complex  $\mathcal{S}(\Gamma)$  by replacing the maximal  $\mathbb{Q}$ -parabolics by the  $\Gamma$ -integral symmetric subgroups incident with the maximal  $\mathbb{Q}$ -parabolics. The arithmetic group  $\Gamma$  acts on  $\mathcal{S}(\Gamma)$ , and the quotient complex  $\mathcal{S}(\Gamma)/\Gamma$  is again finite, by our main result above, and the

vertices are in one to one correspondence with the  $\Gamma$ -integral modular subvarieties which are incident with a rational boundary variety. In this case the complex  $\mathcal{S}(\Gamma)$  itself is an *integral*, not a rational, object. This will be dealt with elsewhere.

We now sketch the contents of the paper. The first paragraph is preliminary and recalls some mostly known facts on the classification of the rational groups of hermitian type for which we could find no reference. In the second we recall some notions and results from [H2] on which the theory is based. The third paragraph essentially describes all arithmetic subgroups of the rational groups of hermitian type, in terms of maximal orders and ideals therein in division algebras  $D$  for the classical cases and in terms of maximal orders of exceptional Jordan algebras for the exceptional cases. Theoretically this section could have been dispensed with, but it does help one get the feeling for the modular subvarieties later on. In the fourth paragraph we introduce the notion of  $\Gamma$ -integral symmetric subgroups and derive the basic finiteness result. In the fifth paragraph we discuss the compactifications of the arithmetic quotients, and finally in the sixth paragraph we define precisely modular subvarieties and derive the main results above.

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**Notations:** For an algebraic  $k$ -group  $H$ , the group of  $K$ -valued points for a field extension  $K|k$  will be denoted  $H_K$  or  $H(K)$ ; similarly, for a vector space  $V$  defined over  $k$ ,  $V_k$  will denote the set of  $k$ -points, and for the ring of integers  $\mathcal{O}_k$ ,  $V_{\mathcal{O}_k}$  will denote a  $\mathcal{O}_k$ -lattice in  $V_k$ . Throughout,  $s$  will denote the  $\mathbb{Q}$ -rank of a  $\mathbb{Q}$ -group  $G$ ,  $t$  will denote the  $\mathbb{R}$ -rank of  $G(\mathbb{R})$ , and  $f$  will denote the degree of  $k$  over  $\mathbb{Q}$ , when  $k$  is a totally real number field fixed in a discussion. Usually  $d$  will denote the degree of a division algebra  $D$ , and  $n$  will denote the dimension of a  $D$ -vector space  $V$ . For a group  $G$  and a subset  $\Xi \subset G$ , the normalizer (resp. centralizer) of  $\Xi$  in  $G$  will be denoted  $\mathcal{N}_G(\Xi)$  (resp.  $\mathcal{Z}_G(\Xi)$ ).

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# 1 Rational groups of hermitian type

## 1.1 Notations

We now fix some notations to be in effect for the rest of the paper. We will be dealing with algebraic groups defined over  $\mathbb{Q}$ , which give rise to hermitian symmetric spaces, groups of *hermitian type*, as we will say. As we are interested in the automorphism groups of domains, we may, without restricting generality, assume the group is *centerless*, and *simple* over  $\mathbb{Q}$ . We will also assume  $G$  is Zariski connected. Henceforth, if not indicated otherwise (occasionally  $G$  will denote a reductive group; in sections 2.1 and 2.2  $G$  will be a real Lie group)  $G$  will denote such an algebraic group. To avoid complications, we exclude in this paper the following case:

**Exclude:** All non-compact real factors of  $G(\mathbb{R})$  are of type  $SL_2(\mathbb{R})$ .

Finally, we shall only consider *isotropic* groups. This implies the hermitian symmetric space  $\mathcal{D}$  has no compact factors. By our assumptions, then, we have

- (i)  $G = Res_{k|\mathbb{Q}}G'$ ,  $k$  a totally real number field,  $G'$  absolutely simple over  $k$ .
- (ii)  $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_f$ , each  $\mathcal{D}_i$  a non-compact irreducible hermitian symmetric space,  $f = [k : \mathbb{Q}]$ .

### 1.1.1 Real parabolics

This material is presented in detail in [BB] and [H2], 1.2, so we just mention enough to fix notations. We work in this section in the category of real Lie groups.  $G$  will denote a connected reductive real Lie group of hermitian type, such that the symmetric space  $\mathcal{D} = G/K$  is irreducible. In a well-known manner one fixes a maximal set of strongly orthogonal (absolute) roots, defining a subalgebra  $a \subset g$ , such that  $A = \exp(a)$  is a maximal  $\mathbb{R}$ -split torus which will be fixed throughout this discussion. The set of strongly orthogonal roots is ordered, defining an order on  $A$ , which determines a set of simple  $\mathbb{R}$ -roots  $\Delta_{\mathbb{R}} = \{\eta_1, \dots, \eta_t\}$ ,  $t = \text{rank}_{\mathbb{R}} G = \dim(A)$ , in the  $\mathbb{R}$ -root system  $\Phi_{\mathbb{R}} := \Phi(a, g)$ . For each  $b \in \{1, \dots, t\}$ , the one-dimensional subtorus  $A_b$  is defined:  $a_b = \bigcap_{i \neq b} \text{Ker}(\eta_i)$ ,  $A_b = \exp(a_b)$ . We also set  $n = \sum_{\eta \in \Phi_{\mathbb{R}}^+} g^\eta$ ,  $N = \exp(n)$ . The *standard maximal  $\mathbb{R}$ -parabolic*,  $P_b$ ,  $b = 1, \dots, t$ , is the group generated by  $\mathcal{Z}_G(A_b)$  and  $N$ ; equivalently it is the semidirect product (Levi decomposition)

$$P_b = \mathcal{Z}_G(A_b) \rtimes U_b, \quad (1)$$

where  $U_b$  denotes the unipotent radical. For real parabolics of hermitian type one has a very useful refinement of (1). This is explained in detail in [SC] and especially in [S], §III.3-4. First we have the

decomposition of  $\mathcal{Z}_G(A_b)$

$$\mathcal{Z}_G(A_b) = M_b \cdot L_b \cdot \mathcal{R}_b, \quad (2)$$

where  $M_b$  is compact,  $L_b$  is the *hermitian Levi factor*,  $\mathcal{R}_b$  is reductive (of type  $\mathbf{A}_{b-1}$ ), and the product is almost direct (i.e., the factors have finite intersection). Secondly, the unipotent radical decomposes,

$$U_b = \mathcal{Z}_b \cdot V_b, \quad (3)$$

which is a direct product,  $\mathcal{Z}_b$  being the center of  $U_b$ . For this decomposition the groups are both Zariski connected and connected in the real Lie groups. The action of  $\mathcal{Z}_G(A_b)$  on  $U_b$  can be explicitly described, and is the basis for the compactification theory of [SC]. The main results can be found in [S], III §3-4, and can be summed up as follows.

**Theorem 1.1** *In the decomposition of the standard parabolic  $P_b$  (see (2) and (3))*

$$P_b = (M_b \cdot L_b \cdot \mathcal{R}_b) \rtimes \mathcal{Z}_b \cdot V_b,$$

*the following statements hold.*

- (i) *The action of  $M_b \cdot L_b$  is trivial on  $\mathcal{Z}_b$ , while on  $V_b$  it is by means of a symplectic representation  $\rho : M_b \cdot L_b \rightarrow Sp(V_b, J_b)$ , for a symplectic form  $J_b$  on  $V_b$ .*
- (ii)  *$\mathcal{R}_b$  acts transitively on  $\mathcal{Z}_b$  and defines a homogenous self-dual (with respect to a bilinear form) cone  $C_b \subset \mathcal{Z}_b$ , while on  $V_b$  it acts by means of a representation  $\sigma : \mathcal{R}_b \rightarrow GL(V_b, I_b)$  for some complex structure  $I_b$  on  $V_b$ .*

*Furthermore the representations  $\rho$  and  $\sigma$  are compatible in a natural sense. The decomposition and the representations in fact are valid for the corresponding real algebraic group  $G$  and its algebraic subgroups.*

Finally, there is a one to one correspondence between the maximal real parabolics  $P$  (each of which is conjugate to a unique  $P_b$ ) and the boundary components  $F$  (each of which is the image of a unique standard boundary component  $F_b$ ), given by  $P \longleftrightarrow F$ , where  $P = \mathcal{N}_G(F)$ . In particular,  $P_b = \mathcal{N}_G(F_b)$ .

### 1.1.2 Roots

We now return to the notation used above,  $G = Res_{k|\mathbb{Q}} G'$  the  $\mathbb{Q}$ -simple group of hermitian type. and introduce a few notations concerning the root systems involved. Let  $\Sigma_\infty$  denote the set of embeddings  $\sigma : k \rightarrow \mathbb{R}$ ; this set is in bijective correspondence with the set of infinite places of  $k$ . We denote these places by  $\nu$ , and if necessary we denote the corresponding embedding by  $\sigma_\nu$ . For each  $\sigma \in \Sigma_\infty$ , the group  ${}^\sigma G'$  is the algebraic group defined over  $\sigma(k)$  by taking the set of elements  $g^\sigma$ ,  $g \in G'$ . For each infinite prime  $\nu$  we have  $G_{k_\nu} \cong ({}^{\sigma_\nu} G')_{\mathbb{R}}$ , and the decomposition of  $\mathcal{D}$  above can be written

$$\mathcal{D} = \prod_{\sigma \in \Sigma_\infty} \mathcal{D}_\sigma, \quad \mathcal{D}_\sigma := ({}^\sigma G')_{\mathbb{R}} / K_{(\sigma)} = ({}^\sigma G')_{\mathbb{R}}^0 / K_{(\sigma)}^0.$$

We set  $G_\sigma = ({}^\sigma G')_{\mathbb{R}}^0$  and note that the discussion of the last section applies to  $G_\sigma$  for each  $\sigma$ . For convenience we now index the components  $\mathcal{D}_\sigma$  by  $i \in \{1, \dots, f\}$ . For each  $\mathcal{D}_i$  we have  $\mathbb{R}$ -roots  $\Phi_{i, \mathbb{R}}$ , of  $\mathbb{R}$ -ranks  $t_i$  and simple  $\mathbb{R}$ -roots  $\{\eta_{i,1}, \dots, \eta_{i,t_i}\}$ ,  $i = 1, \dots, d$ . For each factor we have standard parabolics  $P_{i,b_i}$  ( $1 \leq b_i \leq t_i$ ) and standard boundary components  $F_{i,b_i}$ . The standard parabolics of  $G_{\mathbb{R}}^0$  and boundary components of  $\mathcal{D}$  are then products

$$P_{\mathbf{b}}(\mathbb{R})^0 = P_{1,b_1} \times \cdots \times P_{d,b_d}, \quad F_{\mathbf{b}} = F_{1,b_1} \times \cdots \times F_{d,b_d}, \quad (\mathbf{b} = (b_1, \dots, b_d)), \quad (4)$$

where  $P_{i,b_i} \subset G_{\sigma_i}$ ,  $P_b \subset G$  is a maximal  $\mathbb{Q}$ -parabolic, and as above  $P_b(\mathbb{R})^0 = \mathcal{N}_{G_{\mathbb{R}}}(F_b)^0$ . Furthermore, there is a  $\mathbb{Q}$ -subgroup  $L_b \subset G$  such that

$$\text{Aut}(F_b)^0 = L_b(\mathbb{R})^0, \quad L_b(\mathbb{R})^0 = L_{1,b_1} \times \cdots \times L_{d,b_d}, \quad (5)$$

where  $L_{i,b_i} \subset P_{i,b_i}$  is the hermitian Levi component as above. As far as the domains are concerned, any of the boundary components  $F_{i,b_i}$  may be the *improper* boundary component  $\mathcal{D}_i$ , which is indicated by setting  $b_i = 0$ . Consequently,  $P_{i,0} = L_{i,0} = G_i$  and in (4) and (5) any  $\mathbf{b} = (b_1, \dots, b_d)$ ,  $0 \leq b_i \leq t_i$  is admissible.

Since  $G'$  is isotropic, there is a positive-dimensional  $k$ -split torus  $S' \subset G'$ , which we fix. Then  ${}^\sigma S'$  is a maximal  $\sigma(k)$ -split torus of  ${}^\sigma G'$  and there is a canonical isomorphism  $S' \rightarrow {}^\sigma S'$  inducing an isomorphism  $\Phi_k = \Phi(S', G') \longrightarrow \Phi_{\sigma(k)}({}^\sigma S', {}^\sigma G') =: \Phi_{k,\sigma}$ . The torus  $\text{Res}_{k|\mathbb{Q}} S'$  is defined over  $\mathbb{Q}$  and contains  $S$  as maximal  $\mathbb{Q}$ -split torus; in fact  $S \cong S'$ , diagonally embedded in  $\text{Res}_{k|\mathbb{Q}} S'$ . This yields an isomorphism  $\Phi(S, G) \cong \Phi_k$ , and the root systems  $\Phi_{\mathbb{Q}} = \Phi(S, G)$ ,  $\Phi_k$  and  $\Phi_{k,\sigma}$  (for all  $\sigma \in \Sigma_\infty$ ) are identified by means of the isomorphisms.

In each group  ${}^\sigma G'$  one chooses a maximal  $\mathbb{R}$ -split torus  $A_\sigma \supset {}^\sigma S'$ , contained in a maximal torus defined over  $\sigma(k)$ . Fixing an order on  $X(S')$  induces one also on  $X({}^\sigma S')$  and  $X(S)$ . Then, for each  $\sigma$ , one chooses an order on  $X(A_\sigma)$  which is compatible with that on  $X({}^\sigma S')$ , and  $r : X(A_\sigma) \longrightarrow X({}^\sigma S') \cong X(S)$  denotes the restriction homomorphism. The canonical numbering on  $\Delta_{\mathbb{R},\sigma}$  of simple  $\mathbb{R}$ -roots of  $G$  with respect to  $A_\sigma$  is compatible by restriction with the canonical numbering of  $\Delta_{\mathbb{Q}}$  ([BB], 2.8). Recall also that each  $k$ -root in  $\Phi_k$  is the restriction of at most one simple  $\mathbb{R}$ -root of  $G'(\mathbb{R})$  (which is a simple Lie group). Let  $\Delta_k = \{\beta_1, \dots, \beta_s\}$ ; for  $1 \leq i \leq s$  set  $c(i, \sigma) :=$  index of the simple  $\mathbb{R}$ -root of  ${}^\sigma G'$  restricting on  $\beta_i$ . Then  $i < j$  implies  $c(i, \sigma) < c(j, \sigma)$  for all  $\sigma \in \Sigma$ .

Each simple  $k$ -root defines a unique standard boundary component: for  $b \in \{1, \dots, s\}$ ,

$$P_b := \prod_{\sigma \in \Sigma_\infty} P_{c(b,\sigma)} \quad (\text{resp. } F_b := \prod_{\sigma \in \Sigma_\infty} F_{c(b,\sigma)}), \quad (6)$$

which is the product of standard (with respect to  $A_\sigma$  and  $\Delta_{\mathbb{R},\sigma}$ ) parabolics  $P_{c(b,\sigma)} \subset G_\sigma$  (resp. boundary components  $F_{c(b,\sigma)}$  of  $\mathcal{D}_\sigma$ ). It follows that  $\overline{F_j} \subset \overline{F_i}$  for  $1 \leq i \leq j \leq s$ . Furthermore, setting  $o_b := \prod o_{c(b,\sigma)}$ , then ([BB], p. 472)

$$F_b = o_b \cdot L_b(\mathbb{R})^0, \quad (7)$$

where  $L_b(\mathbb{R})^0$  denotes the hermitian Levi component (5). As these are the only boundary components of interest to us, we will henceforth refer to any conjugates of the  $F_b$  of (6) by elements of  $G$  as *rational boundary components* (these should more precisely be called rational with respect to  $G$ ), and to the conjugates of the parabolics  $P_b$  as the *rational parabolics*.

## 1.2 Classification

For the convenience of the reader we sketch the classification of rational groups of hermitian type. As  $G = \text{Res}_{k|\mathbb{Q}} G'$  for an absolutely simple  $G'$  over  $k$  we need only classify these.

### 1.2.1 Classical cases

By means of the correspondence given by Weil in [W] if  $G$  is of classical type, classifying the (semi)simple  $k$ -groups of interest to us is equivalent to classifying the central (semi)simple  $k$ -algebras with involution such that  $\text{Aut}(A, *)$  is of hermitian type. We now just list the cases, the possible  $k$ -groups  $G'$ , the set of  $\mathbb{R}$ -points of  $G'$  as well as of the  $\mathbb{Q}$ -group  $G$ , and the corresponding domains. Let  $k$  be a totally real number field of degree  $f$  over  $\mathbb{Q}$ . For the bounded symmetric domains we shall use the notations **I<sub>p,q</sub>**, **II<sub>n</sub>**, **III<sub>n</sub>**, **IV<sub>n</sub>**, **V**, **VI**. In what follows  $G'$  will be simple but not necessarily centerless.

## O Orthogonal type

**O.1** split case:  $G' = SO(V, h)$ ,  $V$  a  $k$ -vector space of dimension  $n + 2$ ,  $h$  a symmetric bilinear form such that, at all real primes  $\nu$ ,  $h_\nu$  has signature  $(n, 2)$ .

$$G'(\mathbb{R}) \cong SO(n, 2), \quad G(\mathbb{R}) \cong \prod_{i=1}^f SO(n, 2)_i, \quad \mathcal{D} \cong \mathbf{IV_n} \times \cdots \times \mathbf{IV_n}, \quad f \text{ factors.}$$

**O.2** non-split case:  $G' = SU(V, h)$ ,  $V$  a right  $D$ -vector space of dimension  $n$ ,  $h$  is a skew-hermitian form; here  $D$  is a quaternion division algebra, central simple over  $k$ , and for all real primes, either

- \*  $D_\nu \cong \mathbb{H}$ ,  $G'_\nu \cong SU(\mathbb{H}^n, h)$ ,  $h$  a skew-hermitian form on  $\mathbb{H}^n$ ,
- \*  $D_\nu \cong M_2(\mathbb{R})$ ,  $G'_\nu \cong SO(2n - 2, 2)$ .

and in the first case  $h$  has Witt index  $[\frac{n}{2}]$ . Number the primes such that for  $\nu_1, \dots, \nu_{f_1}$  the first case occurs and for  $\nu_{f_1+1}, \dots, \nu_f$  the second occurs. Then  $G'(\mathbb{R}) \cong SU(\mathbb{H}^n, h)$ , and

$$G(\mathbb{R}) \cong \prod_{i=1}^{f_1} SU(\mathbb{H}^n, h)_i \times \prod_{i=f_1+1}^f SO(2n-2, 2)_i, \quad \mathcal{D} \cong \underbrace{\mathbf{II_n} \times \cdots \times \mathbf{II_n}}_{f_1 \text{ factors}} \times \underbrace{\mathbf{IV_{2n-2}} \times \cdots \times \mathbf{IV_{2n-2}}}_{f-f_1 \text{ factors}}.$$

## S Symplectic type

**S.1** split case:  $G' = Sp(2n, k)$ ,  $G'(\mathbb{R}) \cong Sp(2n, \mathbb{R})$ ,  $G(\mathbb{R}) \cong (Sp(2n, \mathbb{R}))^f$ ,  $\mathcal{D} \cong (\mathbf{III_n})^f$ .

**S.2** non-split case:  $G' = SU(V, h)$ , where  $V$  is an  $n$ -dimensional right vector space over a quaternion division algebra  $D$ , central over  $k$ , which is however now required to be totally indefinite, and  $h$  is a hermitian form on  $V$ . Then  $G'(\mathbb{R}) = Sp(2n, \mathbb{R})$ , and

$$G(\mathbb{R}) \cong (Sp(2n, \mathbb{R}))^f, \quad \mathcal{D} = (\mathbf{III_n})^f.$$

## U Unitary type

**U.1** split case:  $G' = SU(V, h)$ , where  $V$  is an  $n$ -dimensional  $K$ -vector space,  $K|k$  an imaginary quadratic extension, and  $h$  is a hermitian form. Let for each real prime  $\nu$   $(p_\nu, q_\nu)$  denote the signature of  $h_\nu$ . Then

$$G(\mathbb{R}) \cong \prod_\nu SU(p_\nu, q_\nu), \quad \mathcal{D} = \mathbf{I_{p_{\nu_1},q_{\nu_1}}} \times \cdots \times \mathbf{I_{p_{\nu_f},q_{\nu_f}}}.$$

**U.2** non-split case:  $G' = SU(V, h)$ , where  $D$  is a division algebra of degree  $d$ , central simple over  $K$  ( $K$  as in **U.1**) with a  $K|k$ -involution and  $V$  is an  $n$ -dimensional right  $D$ -vector space with hermitian form  $h$ . If  $d = 1$  this reduces to **U.1**, so we may assume  $d \geq 2$ . Again letting  $(p_\nu, q_\nu)$  denote the local signatures, we have

$$G(\mathbb{R}) \cong \prod_\nu SU(p_\nu, q_\nu), \quad \mathcal{D} \cong \mathbf{I_{p_{\nu_1},q_{\nu_1}}} \times \cdots \times \mathbf{I_{p_{\nu_f},q_{\nu_f}}}.$$

### 1.2.2 Exceptional groups

The exceptional groups can be classified by results of Ferrar as we now describe. A general reference to non-associative algebra used here is [Sch]. See also [FF] for an excellent survey and further references.

**Definition 1.2** For an alternative algebra with involution  $(A, -)$  let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be a diagonal matrix with coefficients in  $k$ , and set  $L(A^n, \gamma) := \{g \in M_n(A) \mid \gamma g^* \gamma^{-1} = g\}$ . One defines the Jordan algebra  $J(A, \gamma)$  by taking  $n = 3$ ,

$$J(A, \gamma) := L(A^3, \gamma).$$

If  $A$  is an octonion algebra we call  $J(A, \gamma)$  the *exceptional simple Jordan algebra* defined by  $A$  and  $\gamma$ .

In particular, the following cases for exceptional simple Jordan algebras can occur over  $\mathbb{R}$ :

- (i)  $J^c = J(C, (1, 1, 1))$  (the compact form)
  - (ii)  $J^b = \eta J(C, (1, -1, 1))\eta^{-1}$ ,  $\eta = \text{diag}(1, i, 1)$ ,
  - (iii)  $J^s = J(\mathbb{O}, (1, 1, 1))$  (the split form).
- (8)

There is only one  $\mathbb{R}$ -form for the split octonion algebra  $\mathbb{O}$ . Furthermore, for an algebraic number field  $k$ , there are  $3^t$  isomorphism classes of Jordan algebras, where  $t$  denotes the number of real primes of  $k$ . The Jordan algebras  $J^c$ ,  $J^b$ , and  $J^s$  have the following explicit matrix realisations (see [Dr], p. 33)

$$J^c \text{ (respectively } J^s) \cong \left\{ g = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mid \xi_i \in \mathbb{R}, x_i \in C \text{ (respectively } x_i \in \mathbb{O}) \right\} \quad (9)$$

$$J^b \cong \left\{ g = \begin{pmatrix} \xi_1 & ix_3 & \bar{x}_2 \\ i\bar{x}_3 & \xi_2 & ix_1 \\ x_2 & i\bar{x}_1 & \xi_3 \end{pmatrix} \mid \xi_i \in \mathbb{R}, x_i \in C \right\}, \quad (10)$$

and the algebra of Definition 1.2 is given explicitly as a matrix algebra as follows:

$$J(A, (\gamma_1, \gamma_2, \gamma_3)) = \left\{ x = \begin{pmatrix} \xi_1 & \gamma_2 x_3 & \gamma_3 \bar{x}_2 \\ \gamma_1 \bar{x}_3 & \xi_2 & \gamma_3 x_1 \\ \gamma_1 x_2 & \gamma_2 \bar{x}_1 & \xi_3 \end{pmatrix} \mid \xi_i \in k, x_i \in A \right\}. \quad (11)$$

Utilizing composition algebras and Jordan algebras one can construct, with the following construction of *Tits algebras*, exceptional Lie algebras.

**Definition 1.3** Let  $A$  be a composition algebra over  $k$ , and  $J = J(B, \gamma)$  a Jordan algebra over another composition algebra  $B$  as in Definition 1.2. Set:

$$\mathcal{L}(A, J) = \text{Der}(A) \oplus (A_0 \otimes J_0) \oplus \text{Der}(J).$$

One defines a multiplication  $[\cdot, \cdot]$  on  $\mathcal{L}(A, J)$ , which extends the  $[\cdot, \cdot]$  products on  $\text{Der}(A)$  and  $\text{Der}(J)$ , by the rules:

- (a)  $[\cdot, \cdot]$  is bilinear and  $[x, x] = 0$  for all  $x \in \mathcal{L}(A, J)$ ;
- (b)  $[\cdot, \cdot]$  restricts to the usual commutator on  $\text{Der}(A)$  and  $\text{Der}(J)$ , and these are orthogonal with respect to  $[\cdot, \cdot]$ , i.e.,  $[D, E] = 0$  for all  $D \in \text{Der}(A)$ ,  $E \in \text{Der}(J)$ ;

(c) For  $D \in \text{Der}(A), E \in \text{Der}(J)$ ,  $D + E$  acts on  $A_0 \otimes J_0$  by:

$$[D + E, a \otimes x] = D(a) \otimes x + a \otimes E(x);$$

(d)  $[\cdot, \cdot]$  is defined on  $A_0 \otimes J_0$  by the formula:

$$[a \otimes x, b \otimes y] = \frac{1}{3}T(x \circ y) \langle a, b \rangle + (a * b) \otimes (x * y) + \frac{1}{2}T(a \cdot b) \langle x, y \rangle,$$

where the  $*$  and  $\langle \cdot, \cdot \rangle$  products are defined as in [Dr] (in the cases which we require the definition simplifies somewhat and will be described below).

This makes  $\mathcal{L}(A, J)$  a Lie algebra.

For later use we mention that (11) allows us to write elements in  $J(A, \gamma)$  in the following way:

$$x = \sum_{i=1}^3 \xi_i e_{ii} + \sum_{i=1}^3 x_i[j, k], \quad x_i[j, k] := \gamma_k x e_{jk} + \gamma_j \bar{x} e_{kj}, \quad (12)$$

and the second sum is over cyclic permutations  $(i, j, k)$  of  $(1, 2, 3)$ . In these terms the norm and trace forms are given by (see [FF], 4.11)

$$N(x) = \xi_1 \xi_2 \xi_3 + \xi_1 \gamma_2 \gamma_3 n(x_1) + \gamma_1 \xi_2 \gamma_3 n(x_2) + \gamma_1 \gamma_2 \xi_3 n(x_3), \quad (13)$$

$$T(x) = \xi_1 + \xi_2 + \xi_3. \quad (14)$$

In the first formula  $n(a) = a \cdot \bar{a}$  is the norm in  $A$ . The norm above is of course analogous to the determinant in a usual matrix algebra. In particular,  $N(x) \neq 0$  is a necessary and sufficient condition for  $x$  to be *invertible* in  $J$ , i.e.,  $N(x) \neq 0 \iff \exists_{y \in J}$  with  $x \cdot y = 1, x^2 \cdot y = x$ , and the inverse of  $x$  is given by:

$$x^{-1} = \frac{x^\#}{N(x)}, \quad (15)$$

where  $x^\#$  satisfies  $x \cdot x^\# = N(x) \cdot 1$ , or explicitly

$$x^\# = \sum (\xi_j \xi_k - \gamma_j \gamma_k n(x_i)) e_{ii} + \sum (\gamma_i (\bar{x}_j \bar{x}_k) - \xi_i x_i) [j, k]. \quad (16)$$

**E<sub>6</sub>** There are two constructions leading to the real Lie algebra of hermitian type  $e_{6(-14)}$ . On the one hand there is the algebra  $\mathcal{L}(\mathbb{C}, J^b)$  (see Definition 1.3), where  $J^b$  is isomorphic to the Jordan algebra  $J(C, (1, -1, 1))$  of Definition 1.2 and is given explicitly as a matrix algebra in (10). Note that in this case the general definition of the algebra  $\mathcal{L}(\mathbb{C}, J^b)$  simplifies to

$$\mathcal{L}(\mathbb{C}, J^b) \cong i \cdot J_0^b \oplus \text{Der}(J^b), \quad (17)$$

which, identifying  $J_0^b$  with the right translations by traceless algebra elements  $\mathcal{R}_{J_0^b}$ , is nothing but Albert's twisted  $\mathcal{L}(J)_\lambda = \sqrt{\lambda} \mathcal{R}_{J_0} \oplus \text{Der}(J)$ ,  $\mathcal{L}(J) = \mathcal{R}_{J_0} \oplus \text{Der}(J)$ , as mentioned in [F1], p. 62. In our case  $\lambda = -1$  and  $J = J^b$ , and this implies the  $e_6$ -form is of *outer* type (see [F1], §4 and Theorem 5 b), p. 70). The Lie multiplication with respect to the decomposition in (17) is given as follows. Writing an element of  $\mathcal{L}(\mathbb{C}, J^b)$  as  $x = i \otimes A + D$ ,  $A \in J_0^b$  and  $D \in \text{Der}(J^b)$  and identifying  $i \otimes J_0^b$  and  $J_0^b$  so that  $x = A + D$ , the Lie multiplication is given by

$$[A + D, A' + D'] = (D(A') - D'(A)) + ([D, D'] - [L(A), L(A')]), \quad (18)$$

where  $L(A)$  is left multiplication in  $J^b$  by  $A$  (cf. [Dr], 3.2, p. 46). It turns out that this construction is insufficient to describe all  $k$ -forms for number fields  $k$ .

The other description of  $e_{6(-14)}$  is as  $\mathcal{L}(C, J_1^b)$ , where  $J_1^b$  is isomorphic to the Jordan algebra  $J(A_1^c, (1, -1, 1)) = J(\mathbb{C}, (1, -1, 1))$  which can be explicitly described in matrix terms as

$$\begin{aligned} J_1^b &\cong \mathcal{H}_3(\mathbb{C}, (1, -1, 1)) \\ &= \left\{ \begin{pmatrix} r_1 & \alpha_3 & \bar{\alpha}_2 \\ -\bar{\alpha}_3 & r_2 & \alpha_1 \\ \alpha_2 & -\bar{\alpha}_1 & r_3 \end{pmatrix} \mid r_i \in \mathbb{R}, \alpha_i \in \mathbb{C} \right\}. \end{aligned} \quad (19)$$

It is then clear that  $J_1^b \cong D^+$  for an associative algebra  $D$  whose traceless elements form a Lie algebra of type  $su(2, 1)$ . With this information we can exhibit an explicit isomorphism:

$$\mathcal{L}(\mathbb{C}, J^b) \xrightarrow{\sim} \mathcal{L}(C, J_1^b).$$

By means of the isomorphism (17) we may represent an element as a  $k$ -linear transformation of  $J^b$ , i.e., as an element of  $C \otimes M_3(k)$ . Write an element in  $\mathcal{L}(C, J_1^b)$  as follows:  $D + c \otimes a + \text{ady}$ , where  $D \in \text{Der}(C)$ ,  $c \in C_0$ ,  $a \in (J_1^b)_0$ ,  $y \in M_3(k)$ . The isomorphism is given by ([F2], 2.1)

$$\begin{aligned} \psi : \mathcal{L}(C, J_1^b) &\longrightarrow \mathcal{L}(\mathbb{C}, J^b) \\ D + c \otimes a + \text{ady} &\mapsto D \otimes 1 + (c \otimes a)_r + (\bar{c} \otimes {}^t a)_l + I \otimes (y_r + {}^t y_l). \end{aligned} \quad (20)$$

Now we have the following result of Ferrar concerning  $k$ -forms of  $e_6$ :

**Theorem 1.4** ([F2], p. 201) *If  $L$  is a Lie algebra of type  $e_6$  over an algebraic number field  $k$ , then*

$$L \cong \mathcal{L}(A_k, J(B, \gamma))$$

as in Definition 1.3 for some octonion algebra  $A$  and Jordan algebra  $J(B, \gamma)$  as in Definition 1.2, with  $B$  an alternative  $k$ -algebra of dimension two, and  $\gamma = \text{diag}(\gamma_1, \gamma_2, \gamma_3)$  is a diagonal  $k$ -matrix.

For our situation of  $k$ -forms of the  $\mathbb{R}$ -algebra  $e_{6(-14)}$  this means:

**Corollary 1.5** *Any  $k$ -form of  $e_{6(-14)}$  (with  $k$  totally real) is of the form*

$$\mathcal{L}(C_k, (J_1^b)_k),$$

where  $C_k$  is an anisotropic octonion algebra over  $k$  and  $(J_1^b)_k$  is a  $k$ -form of the algebra (19).

As a corollary of this we get a classification of  $k$ -groups of hermitian  $\mathbf{E}_6$  type:

**Corollary 1.6** *Let  $G'$  be an absolutely almost simple  $k$ -group of hermitian type, in the class of structures of type  $\mathbf{E}_6$ . Then  $(G')^0 \sim \text{Aut}(\mathcal{L}(C_k, (J_1^b)_k))^0$  with the notations of the preceding corollary, where “ $\sim$ ” means isogenous.*

Since an octonion algebra  $A$  over  $k$  is uniquely determined up to isomorphism by the set of real primes at which it ramifies, the totally definite (Cayley) algebra  $C_k$  is unique, and we need only apply the classification of  $k$ -forms of the Lie algebra  $su(2, 1)$  to get a complete classification of  $k$ -forms of  $J_1^b$ ,

and hence a classification of the  $k$ -forms of  $e_{6(-14)}$ . There are essentially three cases which can occur (let  $D$  denote the associative algebra with involution and  $D^-$  the  $k$ -form of the Lie algebra  $u(2,1)$ ):

- (i)  $(V, h)$  is a  $k$ -vector space with hermitian form  $h$  of Witt index 1, represented by a matrix  $H$ , and  $D^- = \{g \in \text{End}(V) | gH - Hg = 0\}$ .
- (ii)  $(V, h)$  is a  $k$ -vector space with *anisotropic* hermitian form  $h$ , represented by a matrix  $H$ , and  $D^- = \{g \in \text{End}(V) | gH - Hg = 0\}$ . (21)
- (iii)  $D$  is a central simple division algebra of degree three over an imaginary quadratic extension  $K$  of  $k$  with a  $K|k$ -involution, and  $D = D^-$ .

Considering the Tits index of these  $\mathbb{Q}$ -groups, note that since a  $\mathbb{Q}$ -split torus is all the more  $\mathbb{R}$ -split, it follows that the set of split roots of the index of  $G$  (usually drawn white in the Tits index) are a subset of the split roots of  $G(\mathbb{R})$ . This gives a simple criterion for deciding which indices may give rise to the given  $\mathbb{R}$ -form. Looking now at the list of  $\mathbf{E}_6$  indices (of outer type) in [T1], the following four possibilities arise for  $\mathbb{Q}$ -forms of  $E_{6(-14)}$ :  ${}^2E_{6,0}^{78}$ ,  ${}^2E_{6,1}^{35}$ ,  ${}^2E_{6,1}^{29}$ ,  ${}^2E_{6,2}^{16'}$ . However, as shown in [K], the index  ${}^2E_{6,1}^{29}$  does not give rise to a bounded symmetric domain, but rather has symmetric space **EIV** in the notation of [H]. The argument is roughly as follows. If  $H \subset G$  is the anisotropic kernel, of type **D<sub>4</sub>**, then, since  $\dim[U, U] = 8$  for a maximal unipotent subgroup (in the maximal  $\mathbb{Q}$ -parabolic  $P_{\alpha_1} \cap P_{\alpha_6}$ ), it follows that  $H \subset \text{End}_{\mathbb{Q}}([U, U])$ , a relation preserved upon tensoring with  $\mathbb{R}$ , so that  $P_{\alpha_2}$  is still not defined over  $\mathbb{R}$ ; thus the index of  $G(\mathbb{R})$  is  ${}^1E_{6,2}^{28}$ , giving rise to the symmetric space denoted **EIV** in [H]. Hence there are only three possible Tits indices, namely  ${}^2E_{6,2}^{16'}$ ,  ${}^2E_{6,1}^{35}$  and  ${}^2E_{6,0}^{78}$  for  $k$ -forms of  $e_{6(-14)}$ , and it may hold that the three possibilities in (21) coincide with the three possible indices.

**E<sub>7</sub>** There are two constructions utilizing the Tits algebra leading to the real form of type  $e_{7(-25)}$ . On the one hand there is the algebra  $\mathcal{L}(A, J^b) \cong \mathcal{L}(A, J^c)$  (see Definition 1.3), where  $A \cong M_2(\mathbb{R})$  and  $J^b$  (respectively  $J^c$ ) is isomorphic to the Jordan algebra  $J(C, (1, -1, 1))$  (respectively is the Jordan algebra  $J(C, (1, 1, 1))$ ) in Definition 1.2 and is given explicitly as a matrix algebra in (10) (respectively in (9)). In this case the direct sum decomposition analogous to (17) is ([Dr], 4.6, p. 50)

$$\mathcal{L}(A, J^b) \cong (A_0 \otimes J^b) \oplus \text{Der}(J^b). \quad (22)$$

The multiplication is given by the rules

- (i)  $[a \otimes A, b \otimes B] = \frac{1}{2}[a, b] \otimes A \circ B + \frac{1}{2}\text{Tr}(ab)[L(A), L(B)]$ , for  $a, b \in A_0$ ,  $A, B \in J^b$ ;
- (ii)  $[D, b \otimes B] = b \otimes D(B)$ ,  $D \in \text{Der}(J^b)$ ,  $b \in A_0$ ,  $B \in J^b$ ; (23)
- (iii)  $[D, D'] = \text{usual commutator of } D, D' \in \text{Der}(J^b)$ .

The other description of  $e_{7(-25)}$  is as the algebra  $\mathcal{L}(C, \mathcal{JO}_6(\mathbb{R}))$ , where  $\mathcal{JO}_6(\mathbb{R})$  is the Jordan algebra  $\mathcal{H}_3(M_2(\mathbb{R}), (1, 1, 1))$ , and is given as a matrix algebra by (11). Of course we could derive an explicit isomorphism as in (20) between the two. But in this case it turns out that the first description is sufficient to get all  $k$ -forms. Namely, we have the following result of Ferrar:

**Theorem 1.7 ([F3], Theorem 4.3)** *Let  $k$  be an algebraic number field  $k$  and let  $L$  be a  $k$ -form of the Lie algebra  $e_7$ . Then*

$$L \cong \mathcal{L}(A, J)$$

*as in Definition 1.3 for some quaternion algebra  $A$  over  $k$ , and exceptional simple Jordan algebra  $J$  over  $k$ .*

For the case at hand here, namely  $k$ -forms of the  $\mathbb{R}$ -algebra  $e_{7(-25)}$ , this implies

**Corollary 1.8** *Let  $G'$  be an almost absolutely simple  $k$ -group of hermitian type, of type  $E_7$ , and let  $g'$  be the Lie algebra. Then*

$$g' \cong \mathcal{L}(A_k, J_k),$$

where  $J_k$  is exceptional simple such that for each real prime of  $k$ ,  $(J_k)_\nu \cong J^b$  or  $J^c$ , and  $A_k$  is a quaternion algebra over  $k$  which splits at all infinite primes  $\nu$ .

There are the following possibilities over  $\mathbb{Q}$ :

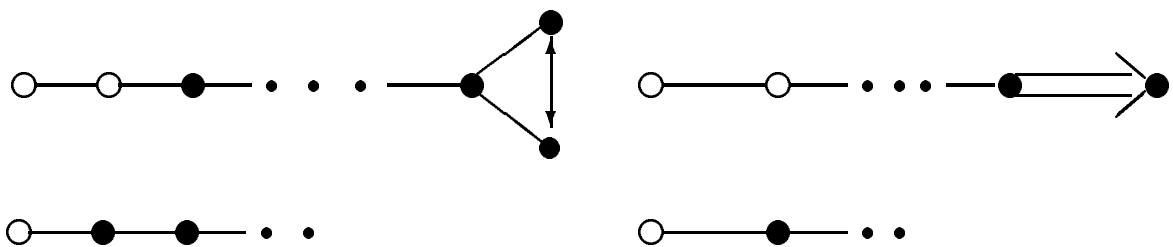
- (i)  $A$  is split,  $J_\mathbb{Q}$  is a  $\mathbb{Q}$ -form of  $J^b$ ;
  - (ii)  $A$  is split,  $J_\mathbb{Q}$  is a  $\mathbb{Q}$ -form of  $J^c$ ;
  - (iii)  $A$  is division,  $J_\mathbb{Q}$  is a  $\mathbb{Q}$ -form of  $J^b$ ;
  - (iv)  $A$  is division,  $J_\mathbb{Q}$  is a  $\mathbb{Q}$ -form of  $J^c$ .
- (24)

There are three possible Tits indices, namely  $E_{7,3}^{28}$ ,  $E_{7,2}^{31}$  and  $E_{7,0}^{133}$ . It is rather clear that the first (respectively the last) case above gives rise to  $E_{7,3}^{28}$  (respectively to  $E_{7,0}^{133}$ ), and it seems natural to expect the other two cases to give rise to  $E_{7,2}^{31}$ .

### 1.3 Boundary components

We briefly discuss the rational boundary components occurring in each of the cases. Again we tabulate this, giving the Tits index in each case and describing the boundary components. We also describe, in the classical cases, the corresponding isotropic subspaces of the vector space  $V$ . Throughout,  $\mathcal{D}^*$  denotes the union of  $\mathcal{D}$  and the rational boundary components.

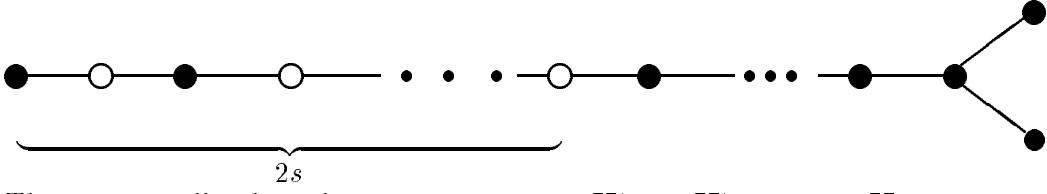
**O.1** The Tits index is  $D_{n,s}$  (for  $n \equiv 2(4)$ ),  ${}^2D_{n,s}$  (for  $n \equiv 0(4)$ ) or  $B_{n,s}$  (for  $n$  odd), where  $s$  is the Witt index of  $h$ . The corresponding diagrams are (the top diagrams are for the case  $s = 2$ , the lower ones giving the left ends for  $s = 1$ ):



where the Galois action in the left-hand diagram is present only for  $n \equiv 0(4)$ . The boundary components of  $\mathcal{D}' = G'(\mathbb{R})/K'$  are:

- $\{pt\} \subset \{1\text{-disc}\}^*$ , ( $s = 2$ )
- $\{pt\}$ , ( $s = 1$ ).

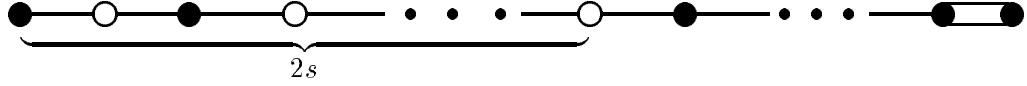
**O.2** The index in this case is  $D_{\frac{n}{2},s}^{(2)}$  ( $n$  even) or  ${}^2D_{\frac{n-1}{2},s}^{(2)}$  ( $n$  odd), where  $s$  is the Witt index of  $h$ . The corresponding diagrams are (with non-trivial Galois action identifying the two right most vertices for  $n$  odd):



The corresponding boundary components are  $\text{II}_{n-2}^* \supset \text{II}_{n-4}^* \supset \cdots \supset \text{II}_{n-2s}$ .

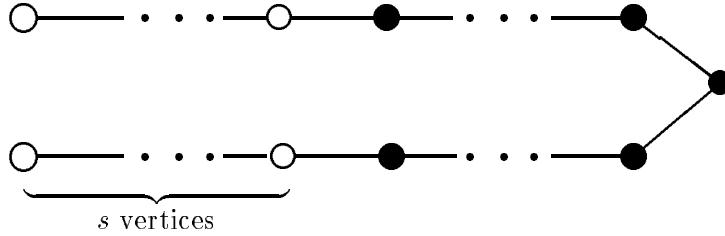
**S.1** The index is  $C_{n,n}$ , with the usual diagram and the following boundary components:  $\{pt\} \subset \text{III}_1^* \subset \cdots \subset \text{III}_{n-1}^*$ .

**S.2** The index is  $C_{n,s}^{(2)}$ , with diagram



The boundary components are then the following:  $\text{III}_{n-2}^* \supset \cdots \supset \text{III}_{n-2s}$ .

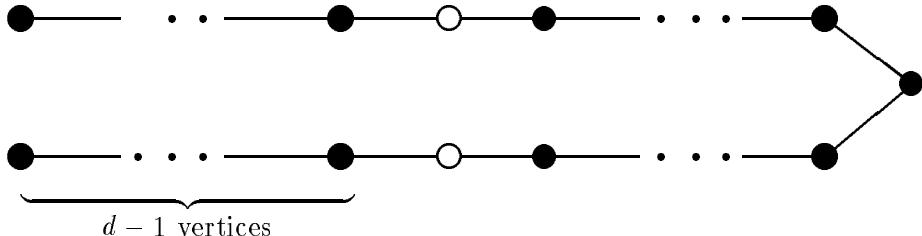
**U.1** The index is  ${}^2A_{n-1,s}$ , with the diagram



As above, let  $(p_\nu, q_\nu)$  denote the signature of  $h_\nu$ , then in the factor  $\mathcal{D}_\nu$  of  $\mathcal{D}$  we have the boundary components of the type  $I_{p_\nu-b, q_\nu-b}$  for  $1 \leq b \leq s$ . Hence a flag of boundary components will be

$$\prod I_{p_\nu-1, q_\nu-1}^* \supset \prod I_{p_\nu-2, q_\nu-2}^* \supset \cdots \supset \prod I_{p_\nu-s, q_\nu-s}^*.$$

**U.2** The index is in this case  ${}^2A_{nd-1,s}^{(d)}$ , with diagram



where there are  $2s$  white vertices altogether. Letting the notations be as for the case **U.1**, we have the following boundary components:

$$\prod I_{p_\nu-d, q_\nu-d}^* \supset \prod I_{p_\nu-2d, q_\nu-2d}^* \supset \cdots \supset \prod I_{p_\nu-sd, q_\nu-sd}^*.$$

We now describe briefly the parabolics in terms of the geometry of  $(V, h)$  for all the cases above. Fixing a maximal  $k$ -split torus and an order on it amounts to fixing a maximal totally isotropic ( $s$ -dimensional) subspace  $H_1 \subset V$  and a basis  $v_1, \dots, v_s$  of  $H_1$ . There are then  $k$ -vectors  $v'_1, \dots, v'_s$

spanning a complementary totally isotropic subspace  $H_2$  such that  $h(v_i, v'_j) = \delta_{ij}$ . Then each pair  $(v_i, v'_i)$  spans a hyperbolic plane  $V_i$  (over  $D$ ), and  $V$  decomposes:

$$V = V_1 \oplus \cdots \oplus V_s \oplus V', \quad V' \text{ anisotropic for } h. \quad (25)$$

Furthermore,  $V_1 \oplus \cdots \oplus V_s = H_1 \oplus H_2$ . With these notations, for  $1 \leq b \leq s$  the standard  $k$ -parabolic  $P'_b \subset G'$  is given as follows:

$$P'_b = \mathcal{N}_{G'}(\langle v_1, \dots, v_b \rangle), \quad (26)$$

where  $\langle v_1, \dots, v_b \rangle$  denotes the span, a  $b$ -dimensional totally isotropic subspace. The hermitian Levi factor of  $P'_b$  is

$$L'_b = \mathcal{N}_{G'}(V_{b+1} \oplus \cdots \oplus V_s \oplus V') / \mathcal{Z}_{G'}(V_{b+1} \oplus \cdots \oplus V_s \oplus V'). \quad (27)$$

It reduces to the  $k$ -anisotropic kernel for  $b = s$ .

For the exceptional cases we have the following possibilities:

- $E_6$ : Index:  ${}^2E_{6,2}^{16'}$ , boundary components:  $\{pt\} \subset \mathbb{B}_5^*$ .  
Index:  ${}^2E_{6,1}^{35}$ , boundary components:  $\mathbb{B}_5$ .
- $E_7$ : Index:  $E_{7,3}^{28}$ , boundary components  $\{pt\} \subset \mathbf{IV}_1^* \subset \mathbf{IV}_{10}^*$ .  
Index:  $E_{7,2}^{31}$ , boundary components  $\mathbf{IV}_1 \subset \mathbf{IV}_{10}^*$ .

## 2 Rational symmetric subgroups and incidence

### 2.1 Holomorphic symmetric embeddings

Recall that an injection  $i_{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{D}'$  of symmetric spaces is said to be *strongly equivariant* if  $i_{\mathcal{D}}$  is induced by an injection  $i : g \hookrightarrow g'$  of the Lie algebras  $g$  (resp.  $g'$ ) of the real Lie group  $G = \text{Aut}(\mathcal{D})$  (resp.  $G' = \text{Aut}(\mathcal{D}')$ ). This is equivalent to the condition that  $i_{\mathcal{D}}(\mathcal{D})$  is totally geodesic in  $\mathcal{D}'$  with respect to the  $G'$ -invariant metric on  $\mathcal{D}'$ . Assuming both  $\mathcal{D}$  and  $\mathcal{D}'$  are hermitian symmetric, there exist elements  $\xi$  (resp.  $\xi'$ ) in the center of the maximal compact subgroup  $K$  (resp.  $K'$ ) such that  $J = \text{ad}(\xi)$  (resp.  $J' = \text{ad}(\xi')$ ) gives the complex structure, and the condition that  $i_{\mathcal{D}}$  be holomorphic is

$$(H_1) \quad i \circ \text{ad}(\xi) = \text{ad}(\xi') \circ i.$$

For any given hermitian symmetric space  $\mathcal{D}'$ , the possible hermitian symmetric subdomains  $i_{\mathcal{D}}(\mathcal{D})$  have been classified by Satake and Ihara (see [I] and [S1]). Note in particular that the above applies to  $\mathcal{D}_N$ , where  $N$  is a reductive subgroup of hermitian type and  $\mathcal{D}_N$  is the associated hermitian symmetric space. We will refer to subgroups  $N \subset G$ , where  $G$  is the connected component of the automorphism group of  $\mathcal{D}$ , for which  $\mathcal{D}_N \subset \mathcal{D}$  is a hermitian symmetric subdomain, as *symmetric subgroups*  $N \subset G^1$ . For this notion it is irrelevant whether  $N$  is reductive, semisimple or even centerless.

### 2.2 Incidence over $\mathbb{R}$

In this section let  $G$  be a reductive Lie group of hermitian type such that the symmetric space  $\mathcal{D}$  is irreducible, and let  $A \subset G$  be the maximal  $\mathbb{R}$ -split torus (with order) defined by the maximal set of strongly orthogonal roots of  $G$  as in 1.1.1. Then we can speak of the standard parabolics  $P_b$ ,  $b = 1, \dots, t$ ,  $t = \text{rank}_{\mathbb{R}} G$ . We introduce the set of domains  $(\mathcal{ED})$  as follows.

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<sup>1</sup>The term “symmetric” arises from the fact that in most cases,  $N$  can be defined in terms of closed symmetric sets of roots.

$(\mathcal{ED})$

**I<sub>q,q</sub>, II<sub>n</sub>, n even, III<sub>n</sub>.**

With respect to a fixed  $P_b$  we consider the following conditions on a symmetric subgroup  $N \subset G$  as in section 2.1.

- 1)  $N$  has maximal  $\mathbb{R}$ -rank, that is,  $\text{rank}_{\mathbb{R}} N = \text{rank}_{\mathbb{R}} G$ .
- 2)  $N$  is a maximal symmetric subgroup.
- 2')  $N$  is a maximal subgroup of tube type, i.e., such that  $\mathcal{D}_N$  is a tube domain.
- 2'')  $N$  is *minimal*, subject to 1).
- 3)  $N = N_1 \times N_2$ , where  $N_1 \subset P_b$  is a hermitian Levi factor of  $P_b$  for some Levi decomposition.
- 3')  $\mathcal{D}_N^*$  contains  $F$  as a boundary component.

**Definition 2.1** Let  $G$  be a simple real Lie group of hermitian type,  $A$  a fixed maximal  $\mathbb{R}$ -split torus defined as above by a maximal set of strongly orthogonal roots,  $\eta_i$ ,  $i = 1, \dots, t$  the simple  $\mathbb{R}$ -roots,  $F_b$  a standard boundary component and  $P_b$  the corresponding standard maximal  $\mathbb{R}$ -parabolic,  $b \in \{1, \dots, t\}$ . A symmetric subgroup  $N \subset G$  (respectively the subdomain  $\mathcal{D}_N \subset \mathcal{D}$ ) will be called *incident* to  $P_b$  (respectively to  $F_b$ ), if  $N$  fulfills:

- $b < t$ , then  $N$  satisfies 1), 2), 3).
- $b = t$ ,  $\mathcal{D} \notin (\mathcal{ED})$ , then  $N$  satisfies 1), 2) or 2'), 3').
- $b = t$ ,  $\mathcal{D} \in (\mathcal{ED})$ , then  $N$  satisfies 1), 2''), 3').

For reducible  $\mathcal{D} = \mathcal{D}_1 \times \dots \times \mathcal{D}_d$ , we have the product subgroups  $N_{b_1,1} \times \dots \times N_{b_d,d}$ , where  $\mathcal{D}_{N_{b_i,i}}$  is incident to the standard boundary component  $F_{b_i}$  of  $\mathcal{D}_i$  (and  $N_{0,i} = G_i$ ).

This defines the notion of symmetric subgroups incident with a standard parabolic. Any maximal  $\mathbb{R}$ -parabolic is conjugate to one and only one standard maximal parabolic,  $P = gP_bg^{-1}$  for some  $b$ . Let  $N_b$  be any symmetric subgroup incident with  $P_b$ . Then just as one has the pair  $(P_b, N_b)$  one has the pair  $(P, N)$ ,

$$P = gP_bg^{-1}, \quad N = gN_bg^{-1}. \quad (28)$$

**Definition 2.2** A pair  $(P, N)$  consisting of a maximal  $\mathbb{R}$ -parabolic  $P$  and a symmetric subgroup  $N$  is called *incident*, if the groups are conjugate by a common element  $g$  as in (28) to a pair  $(P_b, N_b)$  which is incident as in Definition 2.1.

The existence of the symmetric subgroups  $N_b$  was proved in the above mentioned work of Ihara and Satake. Let  $P_b$ ,  $1 \leq b < t$  (this means  $\dim(F_b) > 0$ ) be a standard parabolic and let  $L_b$  be the “standard” hermitian Levi factor, i.e., such that  $\text{Lie}(L_b) = l_b$ ; then

$$N_b := L_b \times \mathcal{Z}_G(L_b) \quad (29)$$

is a subgroup having the properties given in the definition, unique since  $L_b$  is unique. We shall refer to this unique subgroup as the *standard* incident subgroup. As to uniqueness, the following was shown in [H2], Prop. 2.4.

**Proposition 2.3** *If  $(N, P_b)$  are incident, there is  $g \in V_b$  such that  $N$  is conjugate by  $g$  to the standard  $N_b$  of (29), where  $V_b$  is the factor of  $P_b$  of Theorem 1.1.*

The situation for zero-dimensional boundary components was not considered in [H2] in detail, so we take this up now.

Consider first the case where  $\mathcal{D} \notin (\mathcal{ED})$ , so  $\mathcal{D}$  is a product of factors of types:

$$\mathbf{I}_{p,q} \quad (p > q), \mathbf{II}_n \quad (n \text{ even}), \mathbf{IV}_n, \mathbf{V} \text{ or } \mathbf{VI}.$$

The corresponding subgroups  $N_t$  are:  $\mathbf{I}_{p-1,q} \subset \mathbf{I}_{p,q}$ ,  $\mathbf{II}_{n-1} \subset \mathbf{II}_n$ ,  $\mathbf{IV}_{n-1} \subset \mathbf{IV}_n$ ,  $\mathbf{I}_{2,4}$ ,  $\mathbf{II}_5$  or  $\mathbf{IV}_8 \subset \mathbf{V}$ ,  $\mathbf{I}_{3,3}$  or  $\mathbf{II}_6 \subset \mathbf{VI}$ . Next note that if  $N$  is incident to  $P_t$ , so  $\mathcal{D}_N$  is incident to  $F_t$  (=pt.), then any other domain  $\mathcal{D}_{N'}$  incident to  $F_t$  (isomorphic to the given  $\mathcal{D}_N$ ) will be the conjugate by some element of  $G$  fixing  $F_t$ , that is by  $g \in P_t$ . If furthermore  $g \in N$ , then  $g$  leaves  $\mathcal{D}_N$  invariant. It follows that  $N$  is unique (in its isomorphism class for type  $\mathbf{V}$  and  $\mathbf{VI}$ ) up to elements in  $P_t$  modulo those in  $N_t$ . Hence we must find the intersection  $N_t \cap P_t$ . This can be done in the Lie algebras, i.e., we must find  $n_t \cap p_t$ .

Ihara has shown that all the subalgebras  $(n_t)_{\mathbb{C}}$  (with the exception of  $\mathbf{IV}_{n-1} \subset \mathbf{IV}_n$ ,  $n$  even) are *regular* subalgebras, i.e., are generated by the Cartan subalgebra  $t$  and the root spaces  $g^\alpha$  for  $\alpha \in \Psi_{sym}$ , where  $\Psi_{sym}$  is a closed, *symmetric* set of roots. Similarly,  $(p_t)_{\mathbb{C}}$  is the subalgebra generated by  $t$  and the root spaces  $g^\alpha$  for  $\alpha \in \Psi_{par}$ , where  $\Psi_{par}$  is a closed, *parabolic* set of roots,  $\Psi_{par} = \Phi^+ \cup [\Delta - \theta]$ , where  $\theta \subset \Delta$  is some subset of the set of simple roots, and for any subset  $\Xi \subset \Delta$ ,  $[\Xi]$  denotes the set of roots which are integral linear combinations of the elements of  $\Xi$ . Then the intersection of  $(p_t)_{\mathbb{C}}$  and  $(n_t)_{\mathbb{C}}$  is given by

$$(p_t)_{\mathbb{C}} \cap (n_t)_{\mathbb{C}} = t + \left( \sum_{\alpha \in \Psi_{sym}} g^\alpha \cap \sum_{\alpha \in \Psi_{par}} g^\alpha \right) = t + \sum_{\alpha \in \Psi_{sym} \cap \Psi_{par}} g^\alpha.$$

From this it follows that the complement of  $(p_t)_{\mathbb{C}} \cap (n_t)_{\mathbb{C}}$  in  $(p_t)_{\mathbb{C}}$  is given by

$$c = \sum_{\alpha \in \Psi_{par} - (\Psi_{sym} \cap \Psi_{par})} g^\alpha.$$

This is of course not a subalgebra, but we can determine the dimension of the parameter space of non-trivial conjugates of  $N_t$  incident with  $P_t$ . In other words, the homogenous space  $P_t/(P_t \cap N_t)$  can be identified with the set of symmetric subgroups  $N$  incident with  $P_t$ ; its dimension is the cardinality of the set of roots  $\Psi_{par} - \Psi_{sym} \cap \Psi_{par}$ . To demonstrate this consider  $SU(4, 1)$ . Let  $\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_4 = \varepsilon_4 - \varepsilon_5$  denote the simple roots for  $g_{\mathbb{C}}$ , we have :

$$\Psi_{sym} = \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_2 - \varepsilon_4), \pm(\varepsilon_2 - \varepsilon_5), \pm(\varepsilon_3 - \varepsilon_4), \pm(\varepsilon_3 - \varepsilon_5), \pm(\varepsilon_4 - \varepsilon_5),$$

$$\Psi_{par} = +(\varepsilon_i - \varepsilon_j), \quad (10 \text{ of these}), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_2 - \varepsilon_4), \pm(\varepsilon_3 - \varepsilon_4),$$

so that  $\Psi_{par} - (\Psi_{sym} \cap \Psi_{par}) = +(\varepsilon_1 - \varepsilon_j)$ ,  $j = 2, \dots, 5$ . Hence, taking the relation  $\sum \varepsilon_i = 0$  into account, there are three effective parameters. Geometrically this can be seen as follows. The bounded symmetric domain is a four-dimensional ball, the boundary component is a point, and the symmetric subdomain  $\mathcal{D}_N$  is an embedded three-ball passing through the point. Now think of the four-ball as embedded in  $\mathbb{P}^4$  via the Borel embedding; the three-ball is the intersection of  $\mathbb{B}_4 \subset \mathbb{P}^4$  with a hyperplane  $\mathbb{P}^3$  passing through the given point. There is an infinitesimal  $\mathbb{P}^3$  of hyperplanes through the point, so we see three effective parameters.

Now we turn to the embedding  $\mathbf{IV}_{n-1} \subset \mathbf{IV}_n$ ,  $n$  even. If  $G = SO(V, h)$ ,  $h$  symmetric of signature  $(n, 2)$ , let  $v \in V$  be an anisotropic vector. Then  $v^\perp$  is of codimension one,  $h|_{v^\perp}$  has signature  $(n-1, 2)$  and  $N_t = N_G(v^\perp)$ . On the other hand the parabolic  $P_t$  is the stabilizer of a (maximal) two-dimensional totally isotropic subspace  $I \subset V$ . Then  $V$  splits off two hyperbolic planes  $H_1, H_2$ , and  $v$  is in the

orthogonal complement of  $H := H_1 \oplus H_2$ . So the intersection  $N_t \cap P_t$  is just the stabilizer of  $v$  in  $P_t$ , i.e.,

$$N_t \cap P_t = \{g \in G | g(I) \subseteq I, g(v) \in \langle v \rangle\}.$$

Finally we mention the case  $\mathcal{D} \in (\mathcal{ED})$ . Then  $\mathcal{D}_N$  is a polydisc and it is easy to see that the intersection  $N_t \cap P_t$  is just the parabolic in  $N_t$  corresponding to the given point. Since  $N_t \cong (SL_2)^t$ ,  $t = \text{rank}_{\mathbb{R}} \mathcal{D}$ , the parabolic is  $(P_1)^t$ , where  $P_1 \subset SL_2$  is the standard one-dimensional parabolic. So the number of parameters in this case is the dimension of  $P_t$  minus  $t$ .

We now list the sets  $\Psi_{sym}$ , following Ihara, but we will use the notations of the root systems as in [Bo1].

- **I<sub>p,q</sub>**:  $\Delta = \{\alpha_1, \dots, \alpha_{p+q-1}\}$ ,  $\Psi_{sym} = [\alpha_2, \dots, \alpha_{p+q-1}]$ .
- **II<sub>n</sub>**: ( $n$  even).  $\Delta = \{\alpha_1, \dots, \alpha_{[\frac{n}{2}]}\}$ ,  $\Psi_{sym} = [\alpha_2, \dots, \alpha_{[\frac{n}{2}]}]$ .
- **IV<sub>n</sub>**: ( $n = 2\ell + 1$ ),  $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ ,  $\beta := \alpha_{\ell-1} + 2\alpha_\ell$ . The following set of roots forms a diagram of type  $D_\ell$  as indicated:

$$\begin{array}{ccccccccc} \alpha_1 & \alpha_2 & \cdots & \alpha_{\ell-2} & \alpha_\ell \\ & & & & \beta \\ & & & & & & & & \end{array}$$

- **V**:  $\Delta = \{\alpha_1, \dots, \alpha_6\}$ ,  $\beta_1 := \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$ ,  $\beta_2 := \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$ . Then the subalgebras are determined by the following sets of roots:

$$\begin{array}{ccc} \mathbf{I_{2,4}} \times \mathbf{SU(2)} & \mathbf{II_5} & \mathbf{IV_8} \\ \beta_1 \ \alpha_1 \ \alpha_3 \ \alpha_4 \ \alpha_2 \cup \ \alpha_6 & \alpha_1 \ \alpha_3 \ \alpha_4 \ \alpha_5 & \alpha_1 \ \alpha_3 \ \alpha_4 \ \alpha_5 \\ & \beta_2 & \alpha_2 \end{array}$$

- **VI**:  $\Delta = \{\alpha_1, \dots, \alpha_7\}$ ,  $\beta_1 := \alpha_6 + 2\alpha_5 + 3\alpha_4 + 2\alpha_3 + \alpha_1 + 2\alpha_2$ ,  $\beta_2 := \alpha_5 + 2\alpha_4 + 2\alpha_3 + \alpha_1 + \alpha_2$ . Then the subalgebras are determined by the following sets of roots:

$$\begin{array}{ccc} \mathbf{I_{3,3}} & & \mathbf{II_6} \\ (-\alpha_2) \ \beta_1 \ \alpha_7 \ \alpha_6 \ \alpha_5 & & \alpha_7 \ \alpha_6 \ \alpha_5 \ \alpha_4 \ \alpha_2 \\ & & \beta_2 \end{array}$$

We can also consider the converse question, i.e., given a symmetric subgroup, what is the set of parabolics to which it is incident? The answer to this is easier: if  $\dim(F) > 0$ , then for any other boundary component  $F'$  of  $\mathcal{D}_N$ , conjugate to  $F$ , the parabolic  $P_{F'} = N(F')$  is also incident to  $N$ . These boundary components are in 1-1 correspondence with the zero-dimensional boundary components of the second factors  $\mathcal{D}_2$  of  $\mathcal{D}_N = \mathcal{D}_1 \times \mathcal{D}_2$ . If  $\dim(F) = 0$ , then, assuming  $\mathcal{D}_N$  is irreducible (i.e.,  $\mathcal{D} \notin (\mathcal{ED})$ ), then for any other zero-dimensional boundary component  $F'$ , the corresponding stabilizer  $P_{F'}$  is incident with  $N$ . If  $\mathcal{D} \in (\mathcal{ED})$ , then we have the set of zero-dimensional boundary components of the polydisc.

### 2.3 Incidence over $\mathbb{Q}$

We now return to the notations of section 1.1.2;  $G$  is a simple  $\mathbb{Q}$ -group of hermitian type. The following definition gives a  $\mathbb{Q}$ -form of Definition 2.2.

**Definition 2.4** Let  $P \subset G$  be a maximal  $\mathbb{Q}$ -parabolic,  $N \subset G$  a reductive  $\mathbb{Q}$ -subgroup. Then we shall say that  $(P, N)$  are *incident* (over  $\mathbb{Q}$ ), if  $(P(\mathbb{R}), N(\mathbb{R}))$  are incident in the sense of Definition 2.2.

Note that in particular  $N$  must itself be of hermitian type, and such that the Cartan involution of  $G(\mathbb{R})$  restricts to the Cartan involution of  $N(\mathbb{R})$ . The main result of [H2] is the following existence result.

**Theorem 2.5** *Let  $G$  be  $\mathbb{Q}$ -simple of hermitian type subject to the restrictions above ( $G$  is isotropic and  $G(\mathbb{R})$  is not a product of  $SL_2(\mathbb{R})$ 's),  $P \subset G$  a  $\mathbb{Q}$ -parabolic. Then there exists a reductive  $\mathbb{Q}$ -subgroup  $N \subset G$  such that  $(P, N)$  are incident over  $\mathbb{Q}$ , with the exception of the indices  $C_{2n,n}^{(2)}$  for the zero-dimensional boundary components.*

We now describe the standard symmetric subgroups  $N'_b$  incident to  $P'_b$  for the classical cases. For details, see [H2]. We consider the vector space  $V$  with the  $\pm$ symmetric/hermitian form  $h$ . In the notation of (27), if the standard hermitian Levi factor  $L'_b$  is  $L'_b = \mathcal{N}_G(W)/\mathcal{Z}_G(W)$ ,  $W = V_{b+1} \oplus \cdots \oplus V_s \oplus V'$  in the notations used there, then for  $b < s$  or  $c(s, \sigma_i) < t_i$  for some  $i = 1, \dots, f$ ,

$$N'_b = \mathcal{N}_G(W). \quad (30)$$

If  $b = s$  and  $c(s, \sigma_i) = t_i$  for all  $i = 1, \dots, f$ , the boundary component is a point, and  $L'_s$  is the anisotropic kernel, and  $N'_s$  as in (30) is not the standard symmetric subgroup incident to  $P'_s$  as we have defined it. Rather, these subgroups correspond to the following constructions. We consider first the case where  $\mathcal{D} \notin (\mathcal{ED})$ . Pick an anisotropic vector  $v \in V$  which is defined over  $k$ , and consider the subspace  $W = v^\perp$ , the space of vectors orthogonal to  $v$ . We describe the subgroup  $N'_s = \mathcal{N}_{G'}(W)$ , which depends on the choice of  $v$ .

**O.1:**  $V$  is a  $k$ -vector space; the subgroup  $N'_s$  gives rise to a subdomain  $\mathcal{D}_{N'_s}$  of type  $\mathbf{IV}_{n-1} \subset \mathbf{IV}_n$ .

**O.2:** Here  $V$  is an  $n$ -dimensional  $D$ -vector space, and we have  $n$  odd;  $W \subset V$  is of codimension one over  $D$ , giving rise to a subdomain of type  $\mathbf{II}_{n-1} \subset \mathbf{II}_n$ .

**U.1:** In this case we get subdomains  $\mathbf{I}_{p-1,q} \subset \mathbf{I}_{p,q}$ .

**U.2:**  $V$  is  $n$ -dimensional over  $D$ , where  $D$  has degree  $d$  over  $K$ ; the subspace  $W$  gives rise to a subdomain of type  $\mathbf{I}_{p-d,q} \subset \mathbf{I}_{p,q}$ . Iteration of this gives subdomains of types  $\mathbf{I}_{p-jd,q} \subset \mathbf{I}_{p,q}$ , and for  $j = s$  the boundary component will be a point  $\iff sd = q$ , in which case  $\mathbf{I}_{q,q} \subset \mathbf{I}_{p,q}$  is a maximal tube domain and fulfills 2').

Finally we consider  $\mathcal{D} \in (\mathcal{ED})$ . In these cases, if the zero-dimensional boundary component is rational, then  $V$  splits into a direct sum of hyperbolic planes (no anisotropic kernel). We can define a unique polydisc by the prescription: letting  $V = \bigoplus_{i=1}^s V_i$  be the decomposition into hyperbolic planes as above, set:

$$N'_s := \{g \in G' \mid g(V_i) \subseteq V_i, i = 1, \dots, s\}.$$

For the individual cases this gives rise to the following subdomains:

**I<sub>q,q</sub>:**  $\mathcal{D}_{N'_s} \cong \mathbf{I}_{d,d} \times \cdots \times \mathbf{I}_{d,d}$ . In each of the factors  $\mathbf{I}_{d,d}$  we can apply the results of [H1] to get a uniquely determined polydisc.

**II<sub>n</sub>, n even:** In this case we get a subdomain  $\mathbf{II}_2 \times \cdots \times \mathbf{II}_2$ , which is a polydisc, as  $\mathbf{II}_2$  is a disc.

**III<sub>n</sub>:** In case **S.1**, the result is well known, giving just a polydisc. In case **S.2**, we get as a subdomain  $\mathbf{III}_2 \times \cdots \times \mathbf{III}_2$ , and this case represents the exception in Theorem 2.5; in general no polydisc (defined over  $k$ ) can be found in each factor.

### 3 Arithmetic groups

By definition, an arithmetic subgroup  $\Gamma \subset G(\mathbb{Q})$  is one which is commensurable with  $\varrho^{-1}(GL(V_{\mathbb{Z}})) \cap G(\mathbb{Q})$ , for some ( $\iff$  for any) faithful rational representation  $\varrho : G \rightarrow GL(V)$ , where  $V$  is a finite-dimensional  $\mathbb{Q}$ -vector space, and  $V_{\mathbb{Z}}$  is a  $\mathbb{Z}$ -structure, i.e., a  $\mathbb{Z}$ -lattice such that  $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = V_{\mathbb{Q}}$ . In the classical cases, it is natural to take the fundamental representation as  $\varrho$  (more precisely the fundamental representation  $\varrho' : G' \rightarrow GL_D(V)$  determines  $\varrho : G \rightarrow Res_{k|\mathbb{Q}} GL_D(V)$ ), and for the exceptional structures, one has either representations in exceptional Jordan algebras and related algebras, or simply the adjoint representation.

Consider first the classical groups. For these,  $D$  is a central simple division algebra over  $K$ , where  $K$  is either the totally real number field  $k$  or an imaginary quadratic extension of  $k$ , and  $D$  has a  $K|k$ -involution. The rational vector space  $V$  is an  $n$ -dimensional right  $D$ -vector space,  $A = M_n(D)$  is a central simple algebra over  $K$  with a  $K|k$ -involution extending the involution on  $D$  by (33). We have a  $\pm$ symmetric/hermitian form  $h : V \times V \rightarrow D$  such that

$$G' = \{g \in GL_D(V) | \forall_{x,y \in V}, h(x,y) = h(gx,gy)\} \quad (31)$$

is the unitary group of the situation. We take the natural inclusion given by (31),  $\varrho' : G' \rightarrow GL_D(V)$  and let the representation  $\varrho : G \rightarrow Res_{k|\mathbb{Q}} GL_D(V)$  determined by  $\varrho'$  be our rational representation. We now consider  $\mathbb{Z}$ -structures on  $V$ , for which we require an *order*  $\Delta \subset D$ , i.e., a lattice that is a subring of  $D$ , and consider  $\Delta$ -lattices  $\mathcal{L} \subset V$ . The analog of (31), after fixing the  $\mathbb{Z}$ -structure on  $V$ , is

$$\Gamma_{\mathcal{L}} = \{g \in G | g\mathcal{L} \subseteq \mathcal{L}\}. \quad (32)$$

Then  $\Gamma_{\mathcal{L}} \subset G(\mathbb{Q})$  is an arithmetic subgroup, as it is the set of elements which preserve the  $\Delta$ -structure on  $V$  defined by  $\mathcal{L}$ , which itself is a  $\mathbb{Z}$ -lattice in the rational vector space  $V$  (viewing  $V$  as a  $\mathbb{Q}$ -vector space). If, for example,  $\Gamma_{\mathcal{L}'} \subset \Gamma_{\mathcal{L}}$  is a normal subgroup of finite index, we get an induced representation of  $\Gamma_{\mathcal{L}}/\Gamma_{\mathcal{L}'}$  in  $\mathcal{L}/\mathcal{L}'$ , where  $\mathcal{L}'$  is the sublattice of  $\mathcal{L}$  preserved by  $\Gamma_{\mathcal{L}'}$ . This is the general formulation of an occurrence which is well-known in specific cases. For example, if  $\Gamma_{\mathcal{L}'} = \Gamma(N) \subset Sp(2n, \mathbb{Z}) = \Gamma_{\mathcal{L}}$  is the principal congruence subgroup of level  $N$ , there is a representation of  $\Gamma/\Gamma(N) \cong Sp(2n, \mathbb{Z}/N\mathbb{Z})$  in  $(\mathbb{Z}/N\mathbb{Z})^{2n}$  ( $= \mathcal{L}/\mathcal{L}'$ ).

Now consider the exceptional groups. In the case of  $E_6$  we have the 27-dimensional representation in the exceptional Jordan algebra  $J$ , while in the case of  $E_7$  we have the 56-dimensional representation in the exceptional algebra of  $2 \times 2$  matrices over  $J^2$ . In both cases we can also use the adjoint representation, so we require a  $\mathbb{Z}$ -structure on the Lie algebra itself. Such can be readily constructed, utilizing the Tits algebras, from lattices in the constituents, composition algebras and (exceptional) Jordan algebras.

After these introductory remarks we proceed to give a few details, which in particular allow us to give some relevant references in each case. We start by discussing orders, then describe the arithmetic groups these give rise to.

#### 3.1 Orders in associative algebras

A general reference for this section is [R]. We first fix some notations.  $k$  is a totally real Galois extension of degree  $f$  over  $\mathbb{Q}$ , and  $\mathcal{O}_k$  will denote the ring of integers in  $k$ .  $D$  will denote a division algebra (skew field), central simple of degree  $d$  over  $K$ , with a  $K|k$  involution ( $K = k$  for involutions of the first kind, and  $K$  is an imaginary quadratic extension of  $k$  for involutions of the second kind).

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<sup>2</sup>This is what W. Baily utilized in his beautiful paper [Ba].

$V$  denotes an  $n$ -dimensional right  $D$ -vector space, so that  $\text{Hom}_D(V, V) \cong M_n(D)$ .  $A = M_n(D)$  is a central simple algebra over  $K$  with involution extending the involution on  $D$  by

$$M \mapsto M^*, \text{ where } (M^*)_{ij} = \overline{m}_{ji}, \text{ for } M = (m_{ij}), \quad (33)$$

where “ $-$ ” denotes the involution in  $D$ .  $(V, h)$  is a  $\pm$ -hermitian space with  $\pm$ -hermitian form  $h$  (with respect to the involution on  $D$ ). Hence  $[D : K] = d^2$ ,  $[A : K] = (nd)^2 = t^2$ ,  $t = nd$ .

Let  $F$  be a number field, for example  $F = K, k$  as above, and let  $W$  be an  $F$ -vector space. A full  $\mathcal{O}_F$ -lattice  $\mathcal{L}$  in  $W$  is an  $\mathcal{O}_F$ -module, finitely generated, such that  $F \cdot \mathcal{L} = W$ . Usually we work with full lattices and delete the word full. If  $W$  is an  $F$ -algebra, then an  $\mathcal{O}_F$ -lattice  $\mathcal{L}$  is an  $\mathcal{O}_F$ -order, if  $\mathcal{L}$  is a subring of  $W$ . In particular in  $W = D$ , an  $\mathcal{O}_F$ -order is a (full) lattice which is a subring. Let  $\Delta \subset D$  denote an order in  $D$ , and let  $V$  be an  $n$ -dimensional vector space over  $D$ . Then a (full)  $\Delta$ -lattice in  $V$  is a  $\Delta$ -module  $\mathcal{M}$  with  $\mathcal{M} \cdot D = V$ ; if again  $A$  is the algebra  $M_n(D)$ , then a  $\Delta$ -lattice in  $A$  is a  $\Delta$ -order, if it is a subring of  $A$ .

Let an  $\mathcal{O}_F$ -lattice  $\mathcal{L} \subset A$  be given.  $\mathcal{L}$  determines a right (respectively left)  $\mathcal{O}_F$ -order:

$$\mathcal{O}_r(\mathcal{L}) = \{x \in A \mid \mathcal{L} \cdot x \subset \mathcal{L}\}, \text{ (respectively } \mathcal{O}_l(\mathcal{L}) = \{x \in A \mid x \cdot \mathcal{L} \subset \mathcal{L}\}). \quad (34)$$

If  $\mathcal{L}$  is a  $\Delta$ -lattice, then  $\mathcal{O}_r(\mathcal{L})$  and  $\mathcal{O}_l(\mathcal{L})$  are  $\Delta$ -orders. If an  $\mathcal{O}_F$ -order  $\mathcal{O} \subset A$  is given, and  $\mathcal{L} \subset A$  is a lattice with  $\mathcal{O} = \mathcal{O}_r(\mathcal{L})$  (respectively  $\mathcal{O}_l(\mathcal{L})$ ), then one also calls  $\mathcal{L}$  an  $\mathcal{O}$ -lattice, and says that  $\mathcal{L}$  and  $\mathcal{O}$  are associated. An element  $a \in A$  is called integral, if its characteristic polynomial has integer coefficients,  $\chi_a \in \mathcal{O}_F[X]$ . It is a basic result that every element  $a \in \mathcal{O}$  is integral for any  $\mathcal{O}_F$ -order  $\mathcal{O}$  in  $A$ . An order  $\mathcal{O}$  is maximal, if it is not properly contained in any other order. It is a basic fact that maximal orders exist in  $D$  and in  $A$ , and that any order is contained in a maximal one ([R], 10.4). One has the following description of maximal orders in  $A$ :

**Theorem 3.1 ([R], 21.6)** *Notations as above, let  $\Delta \subset D$  be a fixed maximal  $\mathcal{O}_F$ -order in  $D$ , and let  $\mathcal{M}$  be any (full) right  $\Delta$ -lattice in  $V$ . Then  $\text{Hom}_\Delta(\mathcal{M}, \mathcal{M})$  is a maximal  $\mathcal{O}_F$ -order in  $A$ , and for any maximal  $\mathcal{O}_F$ -order  $\mathcal{O}$  in  $A$ , there exists a (full) right  $\Delta$ -lattice  $\mathcal{N} \subset V$  with  $\mathcal{O} = \text{Hom}_\Delta(\mathcal{N}, \mathcal{N})$ .*

The following result of Chevalley describes maximal orders in associative algebras.

**Theorem 3.2 ([R], 27.6)** *Let  $\Delta \subset D$  be a maximal  $\mathcal{O}_K$ -order in  $D$ ; for each right ideal  $J \subset \Delta$ , set  $\Delta' = \mathcal{O}_l(J)$ . Then every maximal order of  $A = M_n(D)$  is of the form*

$$\mathcal{O}_J = \begin{pmatrix} \Delta & \cdots & \Delta & J^{-1} \\ \vdots & \ddots & \vdots & \vdots \\ \Delta & \cdots & \Delta & J^{-1} \\ J & \cdots & J & \Delta' \end{pmatrix},$$

for some right ideal  $J$ , and for each  $J$ , the lattice  $\mathcal{O}_J$  above is a maximal order.

In other words, to give a maximal order in  $A$  is the same as giving a maximal order  $\Delta \subset D$ , together with a right ideal  $J \subset \Delta$ , i.e., the same as giving a pair  $(\Delta, J)$ . In particular if the class number  $h(\Delta) = 1$  (note that  $h(\Delta)$ , which is defined as the number of left  $\Delta$ -ideal classes, is also equal to the number of right  $\Delta$ -ideal classes, see [R], Ex. 7 iii), p. 232), then up to  $\Delta$ -isomorphism there is a 1-1 correspondence between isomorphism classes of maximal orders in  $D$  and  $A$ .

### 3.2 Orders in Jordan algebras

First recall the result on orders in the (definite) Cayley algebra from [BS]. Let  $e_0, \dots, e_7$  be the base of  $C_{\mathbb{Q}}$  given as follows

$$C_{\mathbb{Q}} = e_0\mathbb{Q} + e_1\mathbb{Q} + \dots + e_7\mathbb{Q}, \text{ with center } e_0\mathbb{Q} \text{ and relations:}$$

$$e_i \cdot e_{i+1} = e_{i+3}, e_{i+1} \cdot e_{i+3} = e_i, e_{i+3} \cdot e_i = e_{i+1}, e_i^2 = -e_0, i \in \mathbb{Z}/7\mathbb{Z}.$$

Define

$$\mathcal{M} := \{x = \sum \xi_i e_i \mid 2\xi_i \in \mathbb{Z}, \xi_i - \xi_j \in \mathbb{Z}, \sum \xi_i \in 2\mathbb{Z}\}. \quad (35)$$

Then

**Lemma 3.3** ([BS], 4.6)  $\mathcal{M}$  is a maximal order in  $C_{\mathbb{Q}}$ , and any other maximal order is isomorphic to  $\mathcal{M}$ .

(In [BS] the authors call  $\mathcal{M}$  an octave-ring: a subring of  $C_{\mathbb{Q}}$  containing 1, on which the norm form is integral, and maximal with these properties; we just call  $\mathcal{M}$  a maximal order.)

A general reference for the remainder of this section is [Ra]. Let  $R$  be a commutative ring. A *Jordan algebra* over  $R$  is an  $R$ -module which is commutative and satisfies the relation

$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x), \forall_{x,y}.$$

**Definition 3.4** Let  $J$  be a Jordan algebra over a number field  $K$ , and let  $\mathcal{O}_K$  denote the ring of integers in  $K$ . A full  $\mathcal{O}_K$ -lattice  $\mathcal{L} \subset J$  is an *order*, if  $\mathcal{L}$  is a Jordan algebra over  $\mathcal{O}_K$ .

An element  $x \in J$  is *integral*, if the characteristic polynomial is integral, i.e., if  $N(x), Q(x)$  and  $T(x)$  are integral (see [Ja], pp. 91, also [Jc], Chapter VI, for details). Let  $\mathcal{L}$  be an order in  $J$ , and  $x \in \mathcal{L}$ ; then  $\mathcal{O}_K[x] \subset \mathcal{L}$  is an associative subalgebra, hence finitely generated, so  $x$  is integral ([Ra] Prop. 1, p. 19). Conversely, any integral element of  $J$  is contained in an order (*loc. cit.* Prop. 2).

Once again it is a basic fact that maximal orders exist (*loc. cit.* Thm. 2) and that an order is maximal if and only if it is maximal locally everywhere (*loc. cit.* Lemma 1). A maximal order  $\mathcal{L} \subset J$  is said to be *distinguished*, if  $\mathcal{L}$  is a maximal lattice of integral elements. For example, if  $\mathcal{O} \subset \mathbb{O}$  is a maximal order in an octonion algebra, then  $J(\mathcal{O}, \gamma) \subset J(\mathbb{O}, \gamma)$  (notations as in 1.2) is a distinguished maximal order. Conversely, for  $\gamma = 1$ ,

**Proposition 3.5** ([Ra], Prop. 5, p. 115) *If  $J = J(\mathbb{O}_K, 1)$  is the exceptional Jordan algebra over the totally indefinite octonion algebra  $\mathbb{O}_K$ , then any distinguished order  $\mathcal{P} \subset J$  is of the form  $J(\mathcal{O}, 1) \subset J$ , with  $\mathcal{O}$  a maximal order in  $\mathbb{O}_K$ .*

This may be considered in some sense as an analogue of Theorem 3.1 for orders in exceptional Jordan algebras.

### 3.3 Lattices in Tits algebras

Let  $A$  be a composition algebra over  $K$ , and  $J = J(A', 1)$  a Jordan algebra as in 1.2 over a second composition algebra  $A'$ . For a totally indefinite octonion algebra over  $K$ ,  $A'$ , and a maximal order  $\Delta' \subset A'$ , then, as we have seen (Proposition 3.5),  $\mathcal{L} = J(\Delta', 1)$  is a distinguished order in  $J$  and conversely. More generally it is easy to see:

**Lemma 3.6** *Let  $\Delta \subset A$  be a maximal order in the composition algebra  $A$ . Then  $J(\Delta, \gamma)$  is a maximal order in the Jordan algebra  $J(A, \gamma)$  of Definition 1.2.*

**Proof:**  $\mathcal{L} := J(\Delta, \gamma)$  is clearly a Jordan algebra over  $\mathcal{O}_K$ , hence it is an order in  $J$ . To see it is maximal, the method of [Ra] can be used. Let  $L_1 = \{a \in \mathcal{O}_K \mid a[j, k] \in \mathcal{L}\}$  (notations as in (12)); this is a lattice in  $\mathcal{O}_K$ , and, as can be checked, is the lattice  $\Delta$  which we started with. If  $\mathcal{L}$  is not maximal, then  $\mathcal{L} \subsetneq \mathcal{L}'$ , and the corresponding  $L'_1$  will be an  $\mathcal{O}_K$ -lattice in  $A'$  with  $L_1 \subsetneq L'_1$ , contradicting the maximality of  $\Delta$ .  $\square$

Let  $\Delta \subset A$  be a maximal order and  $\mathcal{L} \subset J$  a maximal order. Consider the Tits algebra  $\mathcal{L}(A, J)$  of Definition 1.3. Recall that the construction of Tits algebras requires, in addition to the algebras  $A$  and  $J$ , also the Lie algebras  $Der(A)$  and  $Der(J)$ . If we have maximal orders  $\Delta \subset A$ ,  $\mathcal{L} \subset J$ , then we define:

$$Der(\Delta) := \{D \in Der(A) \mid D(\Delta) \subseteq \Delta\}, \quad Der(\mathcal{L}) := \{D \in Der(J) \mid D(\mathcal{L}) \subseteq \mathcal{L}\}.$$

Since we know that  $Der(A)$  is a Lie algebra of type  $G_2$  and  $Der(J)$  is a Lie algebra of type  $F_4$ , we are asking for  $\mathbb{Z}$ -structures on these Lie algebras. Clearly  $Der(\Delta)$  and  $Der(\mathcal{L})$  are lattices in the corresponding Lie algebras, which are furthermore closed under the Lie bracket. It then is natural to consider the following lattice in the Tits algebra:

$$\Lambda_{\Delta, \mathcal{L}} := Der(\Delta) \oplus \Delta_0 \otimes \mathcal{L}_0 \oplus Der(\mathcal{L}), \quad (36)$$

and the corresponding arithmetic group it defines (for  $G = \text{Aut}(\mathcal{L}(A, J))^0$ )

$$\Gamma_{\delta, \mathcal{L}} := \{g \in G \mid \text{ad}(g)(\Lambda_{\Delta, \mathcal{L}}) \subseteq \Lambda_{\Delta, \mathcal{L}}\}, \quad (37)$$

where  $G$  is acting by means of the adjoint representation on  $\mathcal{L}(A, J)$ .

### 3.4 Arithmetic groups – classical cases

In this subsection  $G'$  will denote an absolutely (almost) simple  $k$ -group ( $k$  a totally real number field) which we assume is classical,  $G = \text{Res}_{k|\mathbb{Q}}G'$  the  $\mathbb{Q}$ -simple group it defines, which we assume is of hermitian type. We let  $\varrho' : G' \rightarrow GL_D(V)$  be the natural inclusion and  $\varrho : G \rightarrow \text{Res}_{k|\mathbb{Q}}GL_D(V)$  be the natural representation of  $G$  defined by  $\varrho'$ . Fix a maximal order  $\Delta \subset D$ , and let  $\mathcal{L} \subset V$  be a  $\Delta$ -lattice (which is in particular a  $\mathbb{Z}$ -lattice of the underlying  $\mathbb{Q}$ -vector space). As above,  $\mathcal{O}_r(\mathcal{L})$  (respectively  $\mathcal{O}_l(\mathcal{L})$ ) will denote the right (respectively left) order of  $\mathcal{L}$ , given by the equation (34). First of all, we have the arithmetic subgroup  $GL_\Delta(\mathcal{L}) \subset GL_D(V)$ , and we define the subgroup

$$\Gamma_{\mathcal{L}} := \{g \in G'(k) \mid \varrho(g)(\mathcal{L}) \subseteq \mathcal{L}\} = \varrho^{-1}(GL_\Delta(\mathcal{L})) \subset G'(k),$$

and similarly  $\Gamma_{\mathcal{L}} \subset G(\mathbb{Q})$ . By definition these are arithmetic subgroups of  $G'(k)$  and  $G(\mathbb{Q})$ , respectively. Let us see how this is related to the orders  $\mathcal{O}_r(\mathcal{L})$  and  $\mathcal{O}_l(\mathcal{L})$ . By Theorem 3.2  $\mathcal{O}_r(\mathcal{L})$  is of the form  $\mathcal{O}_J$  for some right ideal  $J \subset \mathcal{O}$ . Our central simple algebra is in this case  $A = M_n(D)$ , and  $\mathcal{O}_r(\mathcal{L})$  is a maximal order in  $A$ . Recall how the group  $G'$  and the algebra are related ([W], Thm. 2, p. 598). Let  $U = \{z \in A \mid zz^* = 1\}$ ,  $U_0$  the connected component of  $U$ ,  $((G')^0 =)G_0 := (\text{Aut}(A))^0$ , and let  $C \subset U_0$  be the center of  $U_0$ . Then we have an exact sequence

$$1 \longrightarrow C \longrightarrow U_0 \longrightarrow G_0 \longrightarrow 1.$$

As a lattice in  $A$  we consider  $\mathcal{O} := \mathcal{O}_r(\mathcal{L})$  and its intersection with  $U_0$ ,

$$\mathcal{O}_0 = \mathcal{O} \cap U_0.$$

Similarly,  $\mathcal{C} := C \cap \mathcal{O}_0$  is the center of  $\mathcal{O}_0$ , and we have the sequence

$$1 \longrightarrow \mathcal{C} \longrightarrow \mathcal{O}_0 \longrightarrow \Gamma' \longrightarrow 1,$$

where  $\Gamma' \cong \mathcal{O}_0/\mathcal{C}$  is the arithmetic subgroup  $\Gamma' \subset G'(k)$ , showing how the maximal orders are related to the arithmetic groups. In our situation here,  $(G')^0$  plays the role of  $G_0$ , while  $(U')^0 = (\{z \in A \mid z^*z = 1\})^0$  plays the role of  $U_0$ . Let further  $C' \subset (U')^0$  be the center. We have  $\mathcal{O} \cong \mathcal{O}_J$  for some right ideal  $J$ , and  $(\mathcal{O}_J)_0 = \mathcal{O}_J \cap U_0$  plays the role of  $\mathcal{O}_0$ . Then  $C' \cap (\mathcal{O}_J)_0 = \mathcal{C}$  is the center of  $(\mathcal{O}_J)_0$ , and we have sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & C' & \longrightarrow & (U')^0 & \longrightarrow & (G')^0 & \longrightarrow & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & \mathcal{C} & \longrightarrow & (\mathcal{O}_J)_0 & \longrightarrow & \Gamma'_L & \longrightarrow & 1. \end{array}$$

In this sense, maximal orders give rise to arithmetic subgroups. Viewing the  $\mathcal{O}_k$ -lattice  $\mathcal{L}$  as a  $\mathbb{Z}$ -lattice gives the corresponding diagram for the  $\mathbb{Q}$ -groups (with hopefully obvious notations)

$$\begin{array}{ccccccc} 1 & \longrightarrow & C & \longrightarrow & U^0 & \longrightarrow & G^0 & \longrightarrow & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & \mathcal{Z}(\mathcal{O}_0) & \longrightarrow & \mathcal{O}_0 & \longrightarrow & \Gamma_L & \longrightarrow & 1. \end{array}$$

We now describe this more precisely for the following special cases:

- a) Siegel modular groups.
- b) Picard modular groups.
- c) Hyperbolic plane modular groups.

These are examples of  $\mathbb{Q}$ -groups which are of both inner type (for a)) and outer type (for b) and c)), of split over  $\mathbb{R}$ -type, meaning the  $\mathbb{Q}$ -rank is equal to the  $\mathbb{R}$ -rank (for a) and b)) and more or less the *opposite* of split over  $\mathbb{R}$ -type ( $\mathbb{Q}$ -rank equal to one,  $\mathbb{R}$ -rank unbounded) (for c)). Case a) is well-known, b) is also to a certain extent, while c) was introduced in [H1].

- a) Siegel case:

- $A = M_{2n}(\mathbb{Q})$  with the involution  $* : X \mapsto JX^tJ$ ,  $J = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$ .
- $\text{Aut}(A, *) \cong PSp(2n, \mathbb{Q})$ ,  $V = \mathbb{Q}^{2n}$ .
- $D = \mathbb{Q}$ , a maximal order is  $\Delta = \mathbb{Z}$ ,  $V_{\mathbb{Z}} = \mathbb{Z}^{2n}$ .
- $\Gamma = PSp(2n, \mathbb{Z})$ .

The sequence above becomes:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}/(2) & \longrightarrow & Sp(2n, \mathbb{Q}) & \longrightarrow & PSp(2n, \mathbb{Q}) & \longrightarrow & 1 \\ & & \parallel & & \cup & & \cup & & \\ 1 & \longrightarrow & \mathbb{Z}/(2) & \longrightarrow & Sp(2n, \mathbb{Z}) & \longrightarrow & \Gamma & \longrightarrow & 1. \end{array}$$

- b) Picard case:

- $A = M_n(K)$  with involution  $* : X \mapsto HX^tH$ ,  $H$  hermitian, where  $K|\mathbb{Q}$  is imaginary quadratic.

- $\text{Aut}(A, *) \cong PSU(K^n, h)$ ,  $V = K^n$ ,  $h$  is a hermitian form represented by  $H$ .
- $D = K$ , a maximal order is  $\Delta = \mathcal{O}_K$ ,  $V_{\mathbb{Z}} = \mathcal{O}_K^n$ .
- $\Gamma = PSU(\mathcal{O}_K^n, h)$  (or  $PU(\mathcal{O}_K^n, h)$ , which is not simple, but is often considered anyway).

The sequence above becomes:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{C} & \longrightarrow & SU(K^n, h) & \longrightarrow & PSU(K^n, h) & \longrightarrow & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & \mathcal{C} & \longrightarrow & SU(\mathcal{O}_K^n, h) & \longrightarrow & PSU(\mathcal{O}_K^n, h) & \longrightarrow & 1. \end{array}$$

Note that  $\mathcal{C}$  is given essentially by  $\mathcal{O}_K \cap U(1)$ , which is  $\pm 1$  except for the two fields  $K = \mathbb{Q}(\sqrt{-1})$ ,  $K = \mathbb{Q}(\sqrt{-3})$  which contain fourth (respectively third) roots of unity.

c) Hyperbolic plane case:

- $D$  is a division algebra, central simple of degree  $d \geq 2$  over  $K$ , with a  $K|\mathbb{Q}$ -involution,  $\Delta \subset D$  is a maximal order.
- $A = M_2(D)$  with involution  $* : X \mapsto {}^t \bar{X}$ , where  ${}^t \bar{X} = (\bar{x}_{ji})$ , if  $X = (x_{ij})$ , and  $\bar{x}$  denotes the involution in  $D$ .
- $\text{Aut}(A, *)$  is a  $\mathbb{Q}$ -form of  $PSU(d, d)$ , and  $V = D^2$ , with a hermitian form  $h : V \times V \rightarrow D$  which is isotropic,  $V_{\mathbb{Z}} = \Delta^2$ .
- $\Gamma = PSU(\Delta^2, h)$ .

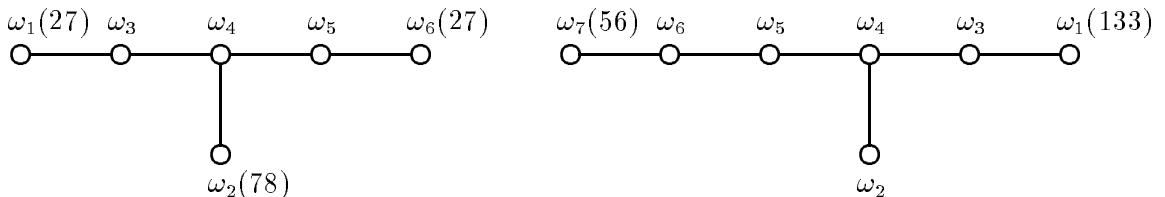
The above sequence becomes in this case

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{C} & \longrightarrow & SU(D^2, h) & \longrightarrow & PSU(D^2, h) & \longrightarrow & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & \mathcal{C} \cap \Delta & \longrightarrow & SU(\Delta^2, h) & \longrightarrow & PSU(\Delta^2, h) & \longrightarrow & 1. \end{array}$$

As  $D$  is central simple over  $K$ , the center is as in the last case,  $\mathcal{C} \cong \mathcal{O}_K \cap U(1)$ , hence it is  $\pm 1$  except for the case  $K = \mathbb{Q}(\sqrt{-1})$  and  $K = \mathbb{Q}(\sqrt{-3})$  as above.

### 3.5 Arithmetic groups – exceptional cases

We mentioned above that for the exceptional cases, there are (at least) two natural types of representations we can consider: representations in algebras derived from exceptional Jordan algebras (Tits algebras), and the adjoint representation. These representations correspond to the following fundamental weights:



In the case of  $E_6$ , the 27-dimensional (respectively the adjoint, 78-dimensional) representation corresponds to the weights  $\omega_1$  and  $\omega_6$  (respectively to  $\omega_2$ ), while in the case of  $E_7$ , the 56-dimensional (respectively the adjoint, 133-dimensional) representation corresponds to the weight  $\omega_7$  (respectively to  $\omega_1$ ). We briefly discuss the arithmetic groups arising in this way.

We first consider the 27-dimensional representation. For this we assume  $G'$  has index  ${}^2E_{6,2}^{16'}$  and we use the model

$$g' = \mathcal{L}(J)_\lambda = \sqrt{\lambda}R_{J_0} \oplus \text{Der}(J)$$

(Albert's twisted  $\mathcal{L}(J)$ ), where  $\lambda < 0, \lambda \in k$ . We then choose a maximal order  $\mathcal{M} \subset J_k$  and set

$$\text{Der}(\mathcal{M}) = \{a \in \text{Der}(J) \mid a(\mathcal{M}) \subseteq \mathcal{M}\}.$$

Then we may consider the lattice

$$\mathcal{L}(\mathcal{M})_\lambda := \sqrt{\lambda}R_{\mathcal{M}_0} \oplus \text{Der}(\mathcal{M}).$$

This defines an arithmetic group:

$$\Gamma_{\mathcal{M}} := \{g \in G' \mid \varrho(g)(\mathcal{L}(\mathcal{M})_\lambda) \subseteq \mathcal{L}(\mathcal{M})_\lambda\},$$

where  $\varrho$  is the 27-dimensional representation in  $\mathcal{L}(J)_\lambda$ .

Next we consider the adjoint representation. For this we utilize the lattice in the Tits algebra constructed in (36), and the corresponding arithmetic group (37). That lattice depends on the choice of a maximal order  $\Delta$  in the Cayley algebra, as well as on one in the algebra  $B$ . More explicitly,

**Theorem 3.7** *Let  $g'$  be a  $k$ -form of  $e_{6(-14)}$  as in Corollary 1.5,  $G'$  as in Corollary 1.6, i.e.,*

$$g' \cong \mathcal{L}(C_k, (J_1)_k^b), \quad (G')^0 \cong (\text{Aut}(g'))^0.$$

*Let  $\Delta \subset C_k$  be a maximal order in the Cayley algebra  $C_k$  as above, let  $\mathcal{L} \subset (J_1)_k^b$  be a maximal order in the Jordan algebra (Definition 3.4), and set*

$$g'_{(\Delta, \mathcal{L})} := \mathcal{L}(\Delta, \mathcal{L}) = \left\{ X \in \mathcal{L}(C_k, (J_1)_k^b) \mid \begin{array}{l} X = X_1 + x \otimes y + Y_1 : X_1 \in \text{Der}(\Delta), \\ Y_1 \in \text{Der}(\mathcal{L}), x \in \Delta_0, y \in \mathcal{L}_0 \end{array} \right\}.$$

*Then  $g'_{(\Delta, \mathcal{L})}$  is an  $\mathcal{O}_k$ -lattice in the  $k$ -vector space  $g'$ . Set*

$$\Gamma'_{(\Delta, \mathcal{L})} := \{g \in G'(k) \mid \text{ad}(g)(g'_{(\Delta, \mathcal{L})}) \subseteq g'_{(\Delta, \mathcal{L})}\}.$$

*Then  $\Gamma'_{(\Delta, \mathcal{L})} \subset G'(k)$  is an arithmetic subgroup.*

Now consider type  $E_7$ . We first consider the 56-dimensional representation. This is the situation considered by Baily in [Ba]. In this example  $k = \mathbb{Q}$ , and  $J$  is the exceptional Jordan algebra over  $\mathbb{Q}$ ,  $J_{\mathbb{Q}} = J^b = J(C_{\mathbb{Q}}, (1, -1, 1))$  in the notation of Definition 1.2, and  $A_{\mathbb{Q}} = M_2(\mathbb{Q})$ . Let  $\mathcal{M} \subset C_{\mathbb{Q}}$  be the maximal order (35). This determines, as in 3.6, a maximal order  $\mathcal{L}$  in  $J_{\mathbb{Q}}$ . Also  $\mathbb{Z} \subset \mathbb{Q}$  defines the maximal order  $\Delta = M_2(\mathbb{Z}) \subset M_2(\mathbb{Q})$ . This then gives rise to an arithmetic group  $G_{(\Delta, \mathcal{L})}$ , which Baily shows is maximal and has only one cusp.

Again in this case we can also consider the adjoint representation. For this we again utilize the lattice (36), and as above, this determines an arithmetic group as in (37). This time, we need a lattice in the totally indefinite quaternion algebra  $A$  as well as one in the Jordan algebra  $(J_1)_k^b$ . More explicitly,

**Theorem 3.8** *Let  $g'$  be a  $k$ -form of  $e_{7(-25)}$  as in Theorem 1.8, i.e.,*

$$g' \cong \mathcal{L}(A_k, J_k), \quad (G')^0 \cong (\text{Aut}(g'))^0.$$

Let  $\Delta \subset A_k$  be a maximal order in the indefinite quaternion algebra  $A_k$  as in section 3.1, and let  $\mathcal{L} \subset J_k$  be a maximal order in the Jordan algebra  $J_k$  as in 3.4, and set:

$$g'_{(\Delta, \mathcal{L})} := \mathcal{L}(\Delta, \mathcal{L}) \text{ as above .}$$

Then  $g'_{(\Delta, \mathcal{L})} \subset g'$  is an  $\mathcal{O}_k$ -lattice, and

$$G'_{(\Delta, \mathcal{L})} := \{g \in \text{Aut}(g') \mid \text{ad}(g)(g'_{(\Delta, \mathcal{L})}) \subset g'_{(\Delta, \mathcal{L})}\} \cap (G')^0$$

is an arithmetic subgroup in  $G'(k)$ .

A more detailed discussion of these matters can be found in [H3].

## 4 Integral symmetric subgroups

Let  $G$  be a  $\mathbb{Q}$ -simple algebraic group of hermitian type, and let  $A \subset G$  be a maximal  $\mathbb{R}$ -split torus defined by the set of strongly orthogonal roots as in section 1.1, given the canonical order. Let  $S \subset A$  be a maximal  $\mathbb{Q}$ -split torus with the canonical order, compatible with the given order on  $A$ . Let further  $\Delta_{\mathbb{Q}} = \{\eta_1, \dots, \eta_s\}$  be the set of simple  $\mathbb{Q}$ -roots, and let  $F_b, P_b$  be the standard boundary components and parabolics as explained above,  $\mathbf{b} = (c(b, \sigma_1), \dots, c(b, \sigma_f)), b = 1, \dots, s$ . Finally, let  $N_b$  be the standard incident symmetric subgroup (i.e., given by (29) if  $\dim(F_b) > 0$ , and in terms of root systems as explained in section 2.2 for  $\dim(F_b) = 0$ ). Since  $N_b$  is a reductive subgroup, it is not true that any  $G$ -conjugate  $N'$  of  $N_b$  is already  $G_{\mathbb{Q}}$ -conjugate. Therefore we make the following definition, yielding a proper subset of the set of  $G$ -conjugates of the given  $N_b$ .

**Definition 4.1** Let  $G, S, P_b, N_b$  be given as above. A symmetric subgroup  $N' \subset G$  which is conjugate to  $N_b$  by an element of  $G(\mathbb{Q})$  is called a *rational symmetric* subgroup of  $G$ .

The following well-known example illustrates the difference between rational and more general symmetric  $\mathbb{Q}$ -subgroups.

**Example 4.2** Let  $G'$  be the symplectic group  $G' = Sp(V, h)$ ,  $G = \text{Res}_{k|\mathbb{Q}} G'$ , where  $V$  is a  $k$ -vector space of dimension  $2n$  and  $h$  is skew-symmetric. If  $n = 2$ , the corresponding domain is a product of copies of the Siegel space of degree 2 (type **III**<sub>2</sub>). The boundary components corresponding to  $P_1$  (respectively  $P_2$ ) are products of one-dimensional (respectively zero-dimensional) boundary components. Then  $N_1$  is also a product of two factors,  $N_1 = N_{1,1} \times N_{1,2}$ , and each  $N_{1,i}$  is a polydisc  $(\mathbf{H})^f$ . If we consider the universal family of abelian varieties parameterized by the domain  $\mathcal{D}$ , say  $\mathcal{A} \rightarrow \mathcal{D}$ , we may consider the following conditions on the fibres  $A_t \in \mathcal{A}$  ( $t \in \mathcal{D}$ ):

- 1)  $A_t$  is isogenous to a product.
- 2)  $A_t$  is simple with real multiplication by some real quadratic extension  $k'|k$ .

We claim that the locus 1) is the locus of subdomains  $\mathcal{D}_{N'}$ , where  $N'$  is rational symmetric, while the locus 2) is the union of  $\mathcal{D}_{N'}$ , where  $N'$  is a  $\mathbb{Q}$ -subgroup conjugate to  $N_1$ , but not in  $G(\mathbb{Q})$ . To see this, let us suppose  $k = \mathbb{Q}$ ; we have the familiar description for the domains  $\mathcal{D}_{N'}$  of 2): in this case the standard symmetric subdomain is  $\mathbb{S}_1 \times \mathbb{S}_1 \subset \mathbb{S}_2$  (given by the diagonal  $2 \times 2$  matrices), and it is conjugated (in  $G_{\mathbb{R}}$ ) by the matrix

$$\mathcal{S} = \begin{pmatrix} S^{-1} & 0 \\ 0 & {}^t S \end{pmatrix}, \quad S = \begin{pmatrix} 1 & w \\ 1 & \overline{w} \end{pmatrix}, \quad w \in \mathcal{O}_{k'}$$

for a real quadratic extension  $k'|\mathbb{Q}$ . More precisely, the subdomains  $\mathcal{D}_{N'}$  are given by the equations

$$H_{(a,b,c,d,e)} := \left\{ \tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \mid a\tau_{11} + b\tau_{12} + c\tau_{22} + d(\tau_{12}^2 - \tau_{11}\tau_{22}) + e = 0 \right\}, \quad (38)$$

for some integral tuple  $(a, b, c, d, e)$  with  $c, d \equiv 0(p)$  for some prime  $p$ . Then the *discriminant* is  $\Delta = b^2 - 4ac - 4de$ , and the field  $k' = \mathbb{Q}(\sqrt{\Delta})$  is the field mentioned above; the element  $w \in \mathcal{O}_{k'}$  can be taken here, for example, as  $w = \frac{b+\sqrt{\Delta}}{2}$ , yielding a Humbert surface with  $a = 1$ ,  $d = e = 0$ . The standard symmetric subgroup  $\cong SL(2, \mathbb{Q}) \times SL(2, \mathbb{Q})$  gets conjugated onto groups  $\cong SL(2, k')$  by the elements  $S \in Sp(4, k') \subset Sp(4, \mathbb{R})$ . The rational boundary components of  $\mathcal{D}_{N'}$  which are  $SL_2(k')$ -rational, are zero-dimensional, and are also rational boundary components of  $\mathcal{D}$ . Note that the domain  $\mathcal{D}_{N'}$  defined by the subgroup  $SL(2, k')$  also contains one-dimensional cusps of the domain  $\mathcal{D}$ , the normalizers of which are defined over  $k'$ , but not over  $\mathbb{Q}$  and these boundary components are consequently not rational (for either  $G'$  or  $N'$ ).

It is clear that  $\mathcal{D}_\Delta$ , the union of the subdomains of given discriminant  $\Delta$ , is the union of conjugates of the standard one by elements of  $G(\mathbb{Q})$  if and only if  $\Delta$  is a square, giving 1). If  $\Delta$  is not a square, then  $k' = \mathbb{Q}(\sqrt{\Delta})$ ,  $N'$  is conjugate to  $N_1$  by an element in  $G(k')$ , and these are the cases occurring in 2).  $\square$

Next we note that the set of subgroups defined in Definition 4.1 is independent of the maximal  $\mathbb{Q}$ -split torus used to define  $N_1$ ; if  $S'$  is another it is conjugate in  $G(\mathbb{Q})$  to  $S$ , and  $N'$  will be rational with respect to  $S$  exactly when it is so with respect to  $S'$ . For a fixed  $N$  the set of rational symmetric subgroups conjugate to  $N$  is naturally identified with  $\mathcal{H} = G(\mathbb{Q})/N(\mathbb{Q})$ . Since  $G(\mathbb{Q})$  acts on  $\mathcal{H}$  so does any arithmetic subgroup  $\Gamma \subset G(\mathbb{Q})$  and one can consider the double coset space  $\Gamma \backslash \mathcal{H}$ . By definition,  $N_b^g$  and  $N_b^{g'}$  will be in the same  $\Gamma$ -orbit if  $g(g')^{-1} \in \Gamma$ , so the orbits are determined by the denominators occurring in  $g$  and in  $(g')^{-1}$ , respectively. In the example above, for each prime  $p$ , the group  $N_\Delta$ ,  $\Delta = p^2$  lies in a separate  $\Gamma$ -orbit. In particular there are in general infinitely many orbits. It turns out that the following definition gives a convenient notion. For  $b < t$  (by which we mean  $\dim(F_b) > 0$ ) let  $N_b = N_{b,1} \times N_{b,2}$  be the decomposition above, and for  $N = N_b^g$ , let  $N = N_1 \times N_2$  denote the corresponding decomposition. If  $b = t$  we set  $N_b = N_{b,1}$ ,  $N = N_1$ .

**Definition 4.3** Let  $G$ ,  $S$ ,  $P_b$ ,  $N_b$  be fixed as above,  $\Gamma \subset G(\mathbb{Q})$  arithmetic. A rational symmetric subgroup  $N \subset G$ , conjugate to  $N_b$  by  $g \in G(\mathbb{Q})$ ,  $N = N_b^g := gN_bg^{-1}$  will be called  $\Gamma$ -integral (respectively *strongly  $\Gamma$ -integral*), if

$$N_1 \cap \Gamma = g(N_{b,1} \cap \Gamma)g^{-1} \quad (\text{respectively } N \cap \Gamma = g(N_b \cap \Gamma)g^{-1})$$

for the element  $g$  above.

For  $b = t$ , both notions coincide, otherwise strongly  $\Gamma$ -integral implies  $\Gamma$ -integral. For our purposes, the weaker notion will be most important. Note that since  $N = gN_bg^{-1}$  the conditions are equivalent to

$$N_{b,1} \cap g^{-1}\Gamma g = N_{b,1} \cap \Gamma \quad (\text{respectively } N_b \cap g^{-1}\Gamma g = N_b \cap \Gamma). \quad (39)$$

This in turn means that  $g^{-1}\Gamma g$  is integral on  $N_{b,1}$  (respectively integral on  $N_b$ ), in other words, that for some rational representation  $\rho : G \longrightarrow GL(V)$  we have  $\rho|_{N_{b,1}}(N_{b,1} \cap g^{-1}\Gamma g) \subset GL(V_{\mathbb{Z}})$  (respectively  $\rho|_{N_b}(N_b \cap g^{-1}\Gamma g) \subset GL(V_{\mathbb{Z}})$ ). Note that this definition depends on the chosen maximal torus, as well as on  $\Gamma$ . If  $S' = xSx^{-1}$  is another maximal  $\mathbb{Q}$ -split torus, then  $N'_b = xN_bx^{-1}$  is the standard symmetric subgroup with respect to  $S'$ . If  $N \subset G$  is  $\Gamma$ -integral with respect to  $N_b$  (i.e., there is  $g \in G(\mathbb{Q})$  such that  $gN_bg^{-1} = N$  and  $N_{b,1} \cap g^{-1}\Gamma g = N_{b,1} \cap \Gamma$ ), then  $N'_{b,1} \cap (gx^{-1})^{-1}\Gamma(gx^{-1}) = N'_{b,1} \cap x\Gamma x^{-1}$ ; in other words when  $N$  is  $\Gamma$ -integral with respect to  $N_b$ , then  $N$  is  $x\Gamma x^{-1}$ -integral with respect to  $N'_b$ .

(with similar statements for strongly  $\Gamma$ -integral). At least in the classical cases, when  $G$  is a matrix group, there is a very canonical choice for  $N_b$ , namely as a subgroup consisting of block matrices, so this dependence is not unreasonable.

Let us now suppose  $G$  is classical,  $\rho' : G' \rightarrow GL_D(V)$  the fundamental representation,  $\rho : G \rightarrow Res_{k|\mathbb{Q}}(GL_D(V))$  the corresponding representation of  $G$ . We have  $P_b = Res_{k|\mathbb{Q}}P'_b$ ,  $N_b = Res_{k|\mathbb{Q}}N'_b$  and  $P'_b$  (resp.  $N'_b$ ) are given in terms of  $(V, h)$  by (26) (resp. by (30)). Clearly if  $N = N_b^g$  and  $N_b = \mathcal{N}_G(W)$ , then  $N = \mathcal{N}_G(g(W))$ .

**Lemma 4.4**  $N = \mathcal{N}_G(g(W))$  is  $\Gamma$ -integral  $\iff W_{\mathbb{Z}} = W \cap V_{\mathbb{Z}} = W \cap \rho(g^{-1})(V_{\mathbb{Z}})$ .

**Proof:** By definition  $N_1 \cap \Gamma = gN_{b,1}g^{-1} \cap \Gamma = g(N_{b,1} \cap \Gamma)g^{-1}$ , and this holds if and only if  $N_{b,1} \cap g^{-1}\Gamma g = N_{b,1} \cap \Gamma$ , i.e.,  $g^{-1}\Gamma g$  meets  $N_{b,1}$  in the arithmetic group  $\Gamma$ . But this holds precisely when  $g^{-1}\Gamma g$  maps  $W_{\mathbb{Z}} = W \cap V_{\mathbb{Z}}$  into itself, and this is equivalent to  $W_{\mathbb{Z}} = W \cap \rho(g^{-1})(V_{\mathbb{Z}})$ , as  $\rho(g^{-1}\Gamma g)$  maps  $\rho(g^{-1})(V_{\mathbb{Z}})$  into itself, and this is the statement of the lemma.  $\square$

Recall also

**Definition 4.5** A lattice  $V_{\mathbb{Z}} \subset V_{\mathbb{Q}}$  being given, a submodule  $W_{\mathbb{Z}}$  is *pure*, if  $n \cdot x \in W_{\mathbb{Z}}$ ,  $n \in \mathbb{Z} \Rightarrow x \in W_{\mathbb{Z}}$ .

**Lemma 4.6** There is a 1-1 correspondence between rational subspaces  $W_{\mathbb{Q}} \subset V_{\mathbb{Q}}$  and pure  $\mathbb{Z}$ -submodules  $W_{\mathbb{Z}} \subset V_{\mathbb{Z}}$ , given by

$$W_{\mathbb{Q}} \mapsto W_{\mathbb{Q}} \cap V_{\mathbb{Z}}, \quad W_{\mathbb{Z}} \mapsto W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

**Proof:** Clear.  $\square$

Note that the statement of Lemma 4.4 is also equivalent to  $g(W_{\mathbb{Z}}) = g(W) \cap V_{\mathbb{Z}}$ , and the latter is by Lemma 4.6, a pure submodule. This then yields:

**Corollary 4.7** There is a 1-1 correspondence between the set of  $\Gamma$ -integral symmetric subgroups and pure submodules of the form  $g(W_{\mathbb{Z}})$ , with  $g \in G(\mathbb{Q})$  and  $W_{\mathbb{Z}} \subset V_{\mathbb{Z}}$  the submodule above (cf. Lemma 4.4).

**Proof:** As we just remarked,  $N$  is  $\Gamma$ -integral  $\iff g(W_{\mathbb{Z}})$  is pure, and for any  $g(W_{\mathbb{Z}})$ ,  $g \in G(\mathbb{Q})$ , which is pure,  $N_b^g$  is clearly  $\Gamma$ -integral.  $\square$

Next we consider, for  $d = \dim(W)$ , the representation

$$R = \bigwedge^d \rho : G \rightarrow GL(\mathcal{V}), \quad \mathcal{V} = \bigwedge^d V.$$

Since  $\rho(N_b) = \mathcal{N}_G(W)$ , it follows that  $R(N_b) = \bigwedge^d \rho(N_b) = \mathcal{N}_G(\bigwedge^d W) = \mathcal{N}_G(W_b)$ , where  $W_b$  is one-dimensional in  $\mathcal{V}$ , defined over  $\mathbb{Q}$ , and we have slightly abused notation by denoting by  $\mathcal{N}_G$  the inverse image under  $R$  of the corresponding normalizer in  $GL(\mathcal{V})$ . Our lattice  $V_{\mathbb{Z}}$  produces a lattice in  $\mathcal{V}$ ,  $\mathcal{V}_{\mathbb{Z}} = \bigwedge^d V_{\mathbb{Z}}$ , and  $\Gamma$  is commensurable with  $R^{-1}(GL(\mathcal{V}_{\mathbb{Z}}))$ .

We now return to the general situation;  $G$  is  $\mathbb{Q}$ -simple of hermitian type,  $P_b$  is a standard parabolic and  $N_b$  is an incident symmetric subgroup, which we take to be the standard one. Since  $N_b$  is reductive, by [BHC], Theorem 3.8, there exists a rational representation  $\pi : G \rightarrow GL(\mathcal{V})$ , defined over  $\mathbb{Q}$ , and an element  $v \in \mathcal{V}_{\mathbb{Q}}$ , such that  $v \cdot \pi(G)$  is a closed orbit and  $N_b = \pi^{-1}(\mathcal{N}_{GL(\mathcal{V})}(v))$ . For example, in the classical cases, the representation  $R$  above is such a  $\pi$ . We now assume that  $\mathcal{V}$  is given a  $\mathbb{Z}$ -structure  $\mathcal{V}_{\mathbb{Z}}$  such that  $\Gamma$  is given by

$$\Gamma = \pi^{-1}(GL(\mathcal{V}_{\mathbb{Z}})). \tag{40}$$

Let  $W_b = \mathbb{Q}\langle v \rangle$  be the one-dimensional vector subspace spanned by  $v$ ; then we may choose a primitive integral vector  $w \in W_b$  such that  $N_b = \pi^{-1}(\mathcal{N}_{GL(V)}(w))$ . We consider the orbit  $w \cdot \pi(G)$ ; as is well-known there is a natural isomorphism  $w \cdot \pi(G) \cong G/N_b$  given by  $w \cdot \pi(g) \mapsto gN_b$ . We may consider the lattice  $\mathcal{V}_\mathbb{Z}$ , defining *integral points*  $\mathcal{V}_\mathbb{Z} \cap w \cdot \pi(G) \xrightarrow{\sim} \mathcal{V}_\mathbb{Z} \cap G/N_b$ .

**Lemma 4.8** *Assume  $\Gamma$  fulfills (40). A subgroup  $N$  given by  $N = N_b^g$  is  $\Gamma$ -integral  $\iff$  under the isomorphism  $w \cdot \pi(G) \xrightarrow{\sim} G/N_b$ ,  $N$  is given by an integral point  $gN_b$ , i.e.,  $w \cdot \pi(g) \in \mathcal{V}_\mathbb{Z}$ .*

**Proof:** We have the following equivalences:

$$\begin{aligned}
& N_1 \cap \Gamma = g(N_{b,1} \cap \Gamma)g^{-1} \\
\stackrel{(40)}{\iff} & N_1 \cap \pi^{-1}(GL(\mathcal{V}_\mathbb{Z})) = g(N_{b,1} \cap \pi^{-1}(GL(\mathcal{V}_\mathbb{Z})))g^{-1} \\
\stackrel{\text{apply } \pi}{\iff} & \pi(N_1) \cap GL(\mathcal{V}_\mathbb{Z}) = \pi(g)(\pi(N_{b,1}) \cap GL(\mathcal{V}_\mathbb{Z}))\pi(g^{-1}) \\
\iff & \pi(N_{b,1}) \cap \pi(g^{-1})GL(\mathcal{V}_\mathbb{Z})\pi(g) = \pi(N_{b,1}) \cap GL(\mathcal{V}_\mathbb{Z}) \\
\stackrel{\pi(N_{b,1}) =}{\iff} & \mathcal{N}_{\pi(G)}(W_b)/\mathcal{Z}_{\pi(G)}(W_b) \cap \pi(g^{-1})GL(\mathcal{V}_\mathbb{Z})\pi(g) \\
& = \mathcal{N}_{\pi(G)}(W_b)/\mathcal{Z}_{\pi(G)}(W_b) \cap GL(\mathcal{V}_\mathbb{Z}) \\
\iff & \pi(g)(W_b \cap \mathcal{V}_\mathbb{Z}) \subset \mathcal{V}_\mathbb{Z} \\
\iff & w \cdot \pi(g) \in \mathcal{V}_\mathbb{Z}
\end{aligned}$$

where the last equivalence follows from the fact that  $w$  is primitive. The Lemma follows.  $\square$

**Corollary 4.9** *The set of  $\Gamma$ -integral symmetric subgroups is the set of subgroups corresponding to the integral points,*

$$\left\{ \begin{array}{l} \text{$\Gamma$-integral symmetric subgroups} \\ \text{conjugate to $N_b$} \end{array} \right\} \cong G/N_b \cap \mathcal{V}_\mathbb{Z}.$$

**Proof:** This follows immediately from the proceeding Lemma.  $\square$

Utilizing Corollary 4.9, we can prove finiteness of the set of  $\Gamma'$ -equivalence classes of  $\Gamma$ -integral symmetric subgroups, for any arithmetic subgroup  $\Gamma' \subset G(\mathbb{Q})$ . Recall the basic finiteness result of [BHC].

**Theorem 4.10 ([BHC], 6.9)** *Let  $G$  be a reductive algebraic group defined over  $\mathbb{Q}$ ,  $\pi : G \longrightarrow GL(V)$  a rational representation defined over  $\mathbb{Q}$ ,  $\mathcal{L} \subset V$  a lattice in  $V_\mathbb{Q}$  invariant under  $G_\mathbb{Z}$ , and  $X$  a closed orbit of  $G$ . Then  $X \cap \mathcal{L}$  consists of a finite number of orbits of  $G_\mathbb{Z}$ .*

Here  $G \subset GL(n, \mathbb{C})$  and  $G_\mathbb{Z} = G \cap M_n(\mathbb{Z})$ .

**Corollary 4.11** *Given  $G$ ,  $S$ ,  $P_b$ ,  $N_b$  and  $\Gamma$  as above ( $\Gamma$  as in (40)), there are finitely many  $\Gamma'$ -equivalence classes of  $\Gamma$ -integral symmetric subgroups, for any arithmetic subgroup  $\Gamma' \subset G(\mathbb{Q})$ .*

**Proof:** Since  $\Gamma$  satisfies (40), Corollary 4.9 holds and Theorem 4.10 may be applied to  $\Gamma$ , hence finiteness holds for any  $\Gamma'$ .  $\square$

Note that under the action of  $\Gamma$  on  $\rho(G) \cdot v \cap V_\mathbb{Z}$ , all orbits are bijective to  $\Gamma / (\Gamma \cap N_b)$ . Let the orbit decomposition with respect to  $\Gamma'$  be

$$\Gamma' \backslash \rho(G) \cdot v \cap V_\mathbb{Z} = \mathcal{O}_1 \cup \dots \cup \mathcal{O}_q.$$

Choose, in each orbit  $\mathcal{O}_i$ , a representative  $x_i$ , and let  $N_{x_i}$  be the corresponding integral symmetric subgroup. The set  $\{N_{x_i}\}$  serves as a finite set of  $\Gamma$ -integral symmetric subgroups representing all  $\Gamma'$ -equivalence classes of such. The following is then well defined.

**Definition 4.12** Given  $G$ ,  $N_b$ ,  $\Gamma$  as above,  $\Gamma' \subset G(\mathbb{Q})$  arithmetic, the *class number* of  $\Gamma'$ -equivalence classes of  $\Gamma$ -integral symmetric subgroups is the cardinality

$$\mu(G, N_b, \Gamma, \Gamma') := |\Gamma' \backslash (G/N_b \cap \mathcal{V}_{\mathbb{Z}})|.$$

If in a discussion  $G$  and a maximal  $\mathbb{Q}$ -split torus  $S \subset G$  are fixed, then  $N_b$  depends only on the integer  $b \in \{1, \dots, s\}$  ( $s = \text{rank}_{\mathbb{Q}} G$ )<sup>3</sup>, and we will denote this class number by  $\mu_b(\Gamma, \Gamma')$ .

## 5 Arithmetic quotients

In this section we keep the above notations.  $G$  is  $\mathbb{Q}$ -simple of hermitian type,  $\mathcal{D} = G(\mathbb{R})/K = G(\mathbb{R})^0/K^0$  the hermitian symmetric space,  $\Gamma \subset G(\mathbb{Q})$  an arithmetic subgroup. The group  $\Gamma$  acts on  $\mathcal{D}$  by means of holomorphic isometries, preserving the natural Bergmann metric.

**Definition 5.1** The quotient  $X_\Gamma := \Gamma \backslash \mathcal{D}$ , where  $\Gamma \subset G(\mathbb{R})$  is arithmetic, is called an *arithmetic quotient*.

If  $\Gamma$  acts without fix points, then the quotient  $X_\Gamma$  is a smooth complex manifold, not compact in general. If  $\Gamma$  has fix points, then  $X_\Gamma$  has certain quotient singularities, which can be described as follows. Let  $\Gamma_1 \subset \Gamma$  be a normal subgroup of finite order without elements of finite order, so that  $\Gamma_1$  acts freely and hence  $X_{\Gamma_1}$  is smooth. We have a Galois cover,

$$X_{\Gamma_1} \longrightarrow X_\Gamma, \tag{41}$$

with  $X_{\Gamma_1}$  smooth, and Galois group acting, yielding the singularities of  $X_\Gamma$ . It is clear that the Galois group,  $\Gamma/\Gamma_1$ , creates the singularities, so they are controlled by certain properties of  $\Gamma/\Gamma_1$ , such as the orders of the elements, etc. In particular,  $X_\Gamma$  is *still* smooth if  $\Gamma/\Gamma_1$  is generated by reflections, as the quotient is then smooth by Chevalley's theorem. It is well-known that  $X_\Gamma$  is compact  $\iff G$  is anisotropic. Suppose this is the case, and that in addition  $\Gamma$  has no elements of finite order. Then, as Kodaira showed in 1954 as one application of his embedding theorem,  $X_\Gamma$  is a smooth projective variety, the canonical bundle  $K_{X_\Gamma}$  being ample. In this case one has Hirzebruch proportionality, which states that the ratios of the Chern numbers of  $X_\Gamma$  are equal to the ratios of the corresponding Chern numbers of the compact hermitian symmetric spaces  $\mathcal{D}$ , and the overall factor of proportionality is just the volume of  $X_\Gamma$ , which is the same as the volume in  $\mathcal{D}$  of a fundamental domain of  $\Gamma$ , where volume is taken with respect to the Bergmann metric.

### 5.1 Satake compactification and Baily-Borel embedding

In case  $G$  is not anisotropic,  $X_\Gamma$  is not compact. It has a topological compactification  $X_\Gamma^*$ , the so-called *Satake compactification*. This is constructed by putting an appropriate topology, the Satake topology, on  $\mathcal{D}^* := \mathcal{D} \cup \{\text{rational boundary components}\}$  ([BB], 4.8). With the Satake topology, the action of  $\Gamma$  on  $\mathcal{D}$  extends to one on  $\mathcal{D}^*$  ([BB], 4.9), and the quotient  $\Gamma \backslash \mathcal{D}^* = X_\Gamma^*$  is the sought for compactification. It has the following property:

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<sup>3</sup>here again with the two exceptions for  $b = s = t$  and the two exceptional domains where there are three, resp. two isomorphism classes of  $N_b$

**Proposition 5.2** ([BB], 4.11)  $X_\Gamma^*$  is a compact, Hausdorff space, and the complement  $X_\Gamma^* \setminus X_\Gamma$  is a finite disjoint union  $X_\Gamma^* \setminus X_\Gamma = V_1 \cup \dots \cup V_N$ , with each  $V_i$  an arithmetic quotient of dimension and  $\mathbb{Q}$ -rank less than that of  $X_\Gamma$ . The length  $k$  of a maximal chain  $V_{i_1} \subsetneq V_{i_2}^* \subsetneq \dots \subsetneq V_{i_k}^*$  is the  $\mathbb{Q}$ -rank of  $G$ .

In our discussion of the  $\mathbb{Q}$ -hermitian symmetric subgroups in section 1.2 we determined the rational boundary components for each  $G$  giving rise to quotients  $V_i$ . One case of particular interest are the hyperbolic planes, discussed in detail in [H1]. We know that in this case all rational boundary components are zero-dimensional, i.e., points. Hence the finite union of 5.2 is a disjoint union of points; the number of such is just the number of cusps, defined as follows. Suppose again we have the fixed  $\mathbb{Q}$ -split torus  $S$  and the standard subgroups  $P_b$  and  $N_b$  with respect to  $S$ .

### Definition 5.3

- (i) For  $b \in \{1, \dots, s\}$ , the number of boundary varieties, conjugate to the  $b^{th}$  standard one, is the cardinality

$$\nu_b(\Gamma) = |\Gamma \backslash G(\mathbb{Q}) / P_b(\mathbb{Q})|.$$

- (ii) The *number of  $\Gamma$ -cusps* is the cardinality (where  $B$  is a Borel subgroup)

$$h(\Gamma) = |\Gamma \backslash G(\mathbb{Q}) / B(\mathbb{Q})|.$$

Note that  $h(\Gamma)$  is also the number of *maximal flags* of boundary varieties, and it is often given by a class number, hence the notation. Since in the case of hyperbolic planes ( $s = 1$ ),  $\nu_1(\Gamma) = h(\Gamma)$ , both of these are given by the results of [H1] in terms of class numbers of certain fields. More generally, the number of components  $N$  occurring in Proposition 5.2 is a sum  $N = r_1 + \dots + r_s$ , where  $s = \mathbb{Q}$ -rank of  $G$ ,  $r_b = \#$  equivalence classes of boundary components conjugate to  $F_b$ . Then

**Proposition 5.4** For any  $X_\Gamma$ , the number  $N$  of Proposition 5.2 can be expressed:  $N = r_1 + \dots + r_s$ , and  $r_b = \nu_b(\Gamma)$  as in Definition 5.3.

The term Baily-Borel embedding of  $X_\Gamma^*$  refers to the following result.

**Theorem 5.5** ([BB], 10.11, 10.12)  $X_\Gamma^*$  can be embedded in projective space as a normal algebraic variety  $V$ . If  $G$  has no normal  $\mathbb{Q}$ -subgroups of dimension three, then the field of rational functions  $K(V)$  is canonically isomorphic with the field of automorphic functions for  $\Gamma$ .

It follows in particular that  $X_\Gamma$  is a normal, quasi-projective variety, which is even smooth if  $\Gamma$  is torsion free.

## 5.2 Toroidal embeddings

Recall the decomposition (of algebraic groups over  $\mathbb{R}$ )  $P_b = M_b L_b \mathcal{R}_b \rtimes \mathcal{U}_b$  for the real parabolic, with the exact sequence

$$1 \longrightarrow M_b L_b \mathcal{R}_b \longrightarrow P_b \longrightarrow \mathcal{U}_b \longrightarrow 1;$$

this gives rise to a similar sequence for  $\Gamma_b = \Gamma \cap P_b(\mathbb{R})$ ,

$$1 \longrightarrow \Gamma_b^\ell \longrightarrow \Gamma_b \longrightarrow \Gamma_b^r \longrightarrow 1,$$

with  $\Gamma_b^\ell$  being the intersection with the Levi factor and  $\Gamma_b^r$  the intersection with the radical of  $P_b$ . Recall further that  $\mathcal{U}_b = \mathcal{Z}_b V_b$ , where  $\mathcal{Z}_b$  is the center, and  $M_b L_b$  acts trivially on  $\mathcal{Z}_b$  and by means

of a symplectic representation on  $V_b$ , while  $\mathcal{R}_b$  acts transitively on  $\mathcal{Z}_b$  defining a homogenous self dual cone  $C_b \subset \mathcal{Z}_b$ , and  $\mathcal{R}_b$  acts on  $V_b$  by means of complex linear transformations, see Theorem 1.1. This then gives us the following results about the factors of  $\Gamma_b$  (and similar results hold for any  $\Gamma_F = N(F) \cap \Gamma$ ):

- i)  $M_b(\mathbb{R})$  is compact, hence  $\Gamma \cap M_b(\mathbb{R})$  is finite. In particular, if  $\Gamma$  has no torsion,  $\Gamma \cap M_b(\mathbb{R}) = e$ .
- ii)  $\Gamma \cap L_b(\mathbb{R})$  is an arithmetic subgroup of  $L_b(\mathbb{R})$ , and  $\Gamma \cap L_b(\mathbb{R})^0 =: \Gamma_L^0$  acts on the boundary component  $F_b$ , with the boundary variety  $W_b = \Gamma_L^0 \backslash F_b \subset X_\Gamma^*$ .
- iii) Let  $V_{\mathbb{Z}} = \Gamma \cap V_b(\mathbb{R})$ . Then the group  $\Gamma_L^0 \rtimes V_{\mathbb{Z}}$  acts on  $F_b \times V_b(\mathbb{R})$  (recall that  $V_b(\mathbb{R})$  has the structure of complex vector space), and if  $\Gamma$  is torsion free, the quotient is an analytic family of abelian varieties over the arithmetic quotient  $W_b = \Gamma_L^0 \backslash F_b$ .
- iv) ([SC], p. 248) There is an exact sequence

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma_b \longrightarrow \Gamma'' \longrightarrow 1,$$

- $\Gamma'$  = subgroup of elements in  $\Gamma_b$  acting trivially by conjugation on  $\text{Lie}(\mathcal{Z}_b)$ ,
- $\Gamma''$  = group of automorphisms of  $\text{Lie}(\mathcal{Z}_b)$  induced by  $\Gamma_b$ ; these map  $C_b$  into itself.

The fourth point is important for the compactification theory, as one lets first  $\Gamma''$  act, then  $\Gamma'$ . A sketch of the construction is as follows: fix a boundary component  $F$ , rational with respect to  $\Gamma$  (i.e.,  $\Gamma \cap N(F)$  is a lattice). Let  $\mathbf{E}_F \subset \mathcal{Z}(F)_\mathbb{C} \times V(F) \times F$  be the realisation of  $\mathcal{D}$  as a Siegel domain as in [SC], and let  $1 \longrightarrow \Gamma' \longrightarrow \Gamma_F \longrightarrow \Gamma'' \longrightarrow 1$  be the sequence above for  $\Gamma_F = N(F) \cap \Gamma$ . Furthermore, the objects denoted above by a subscript  $?_b$  will be denoted here by  $?(F)$ , for example  $\mathcal{Z}(F)$  instead of  $\mathcal{Z}_b$ ,  $C(F)$  instead of  $C_b$ , etc.

**Proposition 5.6** ([SC], p.249) *A partial compactification along  $F$  can be constructed as follows:*

- 1) *Let  $\mathcal{Z}(F)_\mathbb{Z}$  act on  $\mathcal{Z}(F)_\mathbb{C}$  defining the algebraic torus  $T_F$ ; do this in the fibration*

$$\mathbf{E}_{\{1\}} = \mathbf{E}_F / \mathcal{Z}(F)_\mathbb{Z} \subset \mathcal{Z}(F)_\mathbb{C} / \mathcal{Z}(F)_\mathbb{Z} \times V(F) \times F \longrightarrow F.$$

*More precisely, the map  $\mathcal{Z}(F)_\mathbb{C} \longrightarrow \mathcal{Z}(F)_\mathbb{C} / \mathcal{Z}(F)_\mathbb{Z}$  is given by  $\exp(2\pi i \lambda_1), \dots, \exp(2\pi i \lambda_k)$ , where one chooses a  $\mathbb{Z}$ -base  $\xi_1, \dots, \xi_k$  of  $\mathcal{Z}(F)_\mathbb{Z}$ , and  $\lambda_i : \mathcal{Z}(F)_\mathbb{C} \longrightarrow \mathbb{C}$  is the dual base.*

- 2) *Now compactify the algebraic torus  $T_F$  by  $T_F \subset T_{F\{\sigma_\alpha\}}$ ,  $\{\sigma_\alpha\}$  a  $\Gamma''$ -admissible polyhedral decomposition of  $C(F) \subset \mathcal{Z}(F)$  ( $\Gamma''$  as in iv) above).  $T_{F\{\sigma_\alpha\}}$  is locally of finite type, but will have infinitely many components corresponding to integral vectors  $v \in \mathcal{Z}(F)_\mathbb{Z} \cap C(F)$ . The cones  $\sigma_\alpha$  themselves correspond to orbits of highest codimension, i.e., to points. If  $\sigma_\alpha \cap \mathcal{Z}(F)_\mathbb{Z}$  is spanned by  $v_1, \dots, v_k$ , then  $\sigma_\alpha$  corresponds to  $\Delta_1 \cap \dots \cap \Delta_k$ , where  $\Delta_j$  is the divisor corresponding to  $v_j$ .*

- 3) *Glue these into  $\mathbf{E}_{\{1\}}$  by forming the fibre product*

$$(\mathbf{E}_{\{1\}}) \times^{T_F} (T_{F\{\sigma_\alpha\}})$$

*and setting  $(\mathbf{E}_{\{1\}})_{\{\sigma_\alpha\}} = \text{interior of the closure of } \mathbf{E}_{\{1\}} \text{ in } (\mathbf{E}_{\{1\}}) \times^{T_F} (T_{F\{\sigma_\alpha\}})$ . Hence  $(\mathbf{E}_{\{1\}})_{\{\sigma_\alpha\}}$  has a fibre structure over  $F \times V(F)$  with fibres  $T_{F\{\sigma_\alpha\}}$ . If  $\Delta_i$  is the divisor corresponding to  $\xi_i$  as in 1), then  $\Delta_i = \{z_i = 0\}$ , where  $(z_1, \dots, z_k)$  are local coordinates on  $T_F$ .*

- 4)  $\Gamma_F$  still acts on  $(\mathbf{E}_{\{1\}})_{\{\sigma_\alpha\}}$ , as follows.  $\Gamma''$  now acts freely on  $(\mathbf{E}_{\{1\}})_{\{\sigma_\alpha\}}$ , giving a fibre space over  $F \times V(F)$  with fibre a finite  $T_F$  compactification, i.e., modulo  $\Gamma''$  there are only finitely many integral vectors, hence components, in the fibre. Now  $\Gamma'$  acts on  $F \times V(F)$ ; as  $\mathcal{Z}_b$  acts trivially this amounts to an action of  $\Gamma_L^0 \rtimes V_{\mathbb{Z}}$  as in iii) above, and this action extends to  $\Gamma'' \setminus (\mathbf{E}_{\{1\}})_{\{\sigma_\alpha\}}$ , hence an open neighborhood of  $F$  will give an open neighborhood of the boundary variety  $W(F) = \Gamma_L^0 \setminus F$  in  $\overline{X_\Gamma}$ ; this is the sought for partial compactification.

Next one glues these partial compactifications together by means of  $\{\sigma_{\alpha,F}\}$ , a  $\Gamma$ -admissible collection of polyhedral cones, one such collection for each boundary component. The main result is:

**Theorem 5.7 ([SC], Main Theorem 1, p. 252)** *With  $\Gamma$ ,  $D$  as above, for every  $\Gamma$ -admissible collection of polyhedral cones  $\{\sigma_{\alpha,F}\}$ , there is a unique compactification  $\overline{X_\Gamma}$  ( $= (\overline{X_\Gamma})_{\{\sigma_\alpha\}}$ ) which is locally given at each  $F$  (more precisely at  $W(F)$ ) by the partial compactification above, corresponding to the given collection of cones.  $\overline{X_\Gamma}$  is a compact Hausdorff, analytic variety, which is an algebraic space. Furthermore, for properly chosen  $\Gamma$ -admissible collections of polyhedral cones, the compactification is 1) a projective resolution of the Satake compactification:  $\overline{X_\Gamma} \rightarrow X_\Gamma^*$ , hence a projective variety, and 2) smooth with  $\Delta_\Gamma := \overline{X_\Gamma} - X_\Gamma$  a normal crossings divisor.*

## 6 Modular subvarieties

In this paragraph, the data  $G$ ,  $S$ ,  $\mathcal{D}$  will be fixed as above, so that for each  $b = 1, \dots, s = \text{rank}_{\mathbb{Q}} G$  we have the standard boundary components  $F_b$ , the standard parabolic  $P_b$  and the standard incident symmetric subgroup  $N_b$ .

### 6.1 Baily-Borel compactification

Let  $N \subset G$  be a reductive subgroup of hermitian type (this implies in particular that  $N$  is defined over  $\mathbb{R}$ , and we assume the inclusion  $N \subset G$  is also),  $\mathcal{D}_N \subset \mathcal{D}$  the subdomain (holomorphic symmetric embedding) determined by  $N$ .

**Definition 6.1** The subdomain  $\mathcal{D}_N \subset \mathcal{D}$  will be said to be *defined over  $\mathbb{Q}$* , if  $N$  is a  $\mathbb{Q}$ -subgroup of  $G$ .

Suppose a subdomain  $\mathcal{D}_N \subset \mathcal{D}$  is defined over  $\mathbb{Q}$ , and consider an arithmetic subgroup  $\Gamma \subset G(\mathbb{Q})$  and the arithmetic quotient  $X_\Gamma = \Gamma \backslash \mathcal{D}$ . Note that for a reductive subgroup of hermitian type  $N \subset G$ , the intersection  $\Gamma_N := \Gamma \cap N$  will be an arithmetic subgroup if and only if  $N$  is defined over  $\mathbb{Q}$ , and this is the case if and only if  $\mathcal{D}_N$  is defined over  $\mathbb{Q}$ . Hence the arithmetic quotient  $X_{\Gamma_N} := \Gamma_N \backslash \mathcal{D}_N$  is defined, and clearly fits into a commutative square

$$\begin{array}{ccc} \mathcal{D}_N & \hookrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ X_{\Gamma_N} & \hookrightarrow & X_\Gamma. \end{array} \tag{42}$$

**Definition 6.2** A *modular subvariety* on  $X_\Gamma$  is a sub-arithmetic quotient  $X_{\Gamma_N}$  as in (42), where  $\mathcal{D}_N$  is defined over  $\mathbb{Q}$ . A modular subvariety  $X_{\Gamma_N} \subset X_\Gamma$  will be called *rational* (resp.  $\Gamma$ -integral), if  $N$  is a rational (resp.  $\Gamma$ -integral) symmetric subgroup as in Definition 4.1 (resp. 4.3).

The embedding of (42) turns out to extend to one of the Baily-Borel embeddings, legitimizing the terminology *subvariety*. This is given by the following result of Satake.

**Theorem 6.3** Let  $X_{\Gamma_N}^* \subset \mathbb{P}^N$ ,  $X_\Gamma^* \subset \mathbb{P}^{N'}$  be Baily-Borel embeddings. Then there is a linear injection  $\mathbb{P}^N \hookrightarrow \mathbb{P}^{N'}$  making the diagram

$$\begin{array}{ccc} X_{\Gamma_N}^* & \hookrightarrow & \mathbb{P}^N \\ \cap & & \cap \\ X_\Gamma^* & \hookrightarrow & \mathbb{P}^{N'} \end{array}$$

commute and making  $X_{\Gamma_N}^* \subset X_\Gamma^*$  an algebraic subvariety.

**Proof:** We have an injective holomorphic embedding  $\mathcal{D}_N \hookrightarrow \mathcal{D}$  which comes from a  $\mathbb{Q}$ -morphism  $\rho : (N)_\mathbb{C} \hookrightarrow (G)_\mathbb{C}$  such that  $\rho(\Gamma_N) \subset \Gamma$ . Hence we map apply [S2], Theorem 3, and the theorem follows from this.  $\square$

**Definition 6.4** We say that  $X_{\Gamma_N}$  and a boundary variety  $W_i$  are *incident*, if in  $\mathcal{D}^*$  there is rational boundary component  $F$  with parabolic  $P = N(F)$  covering  $W_i$ , such that  $N$  and the corresponding parabolic  $P$  are incident.

Note the following

**Lemma 6.5**  $X_{\Gamma_N}$  and  $W_i$  are incident, if and only if  $W_i^* \subset X_{\Gamma_N}^*$  is a maximal-dimensional boundary component of  $X_{\Gamma_N}^*$  (if  $\dim(W_i) > 0$ ), resp. if and only if  $W_i \subset X_{\Gamma_N}^*$  (if  $\dim(W_i) = 0$ ).

**Proof:** If  $\dim(W_i) > 0$ , then the groups  $P$  and  $N$  are incident if  $F \subset \mathcal{D}_N^*$  and  $F$  is maximal with this property, and if  $F \in \mathcal{D}_N^*$  is rational and maximal with this property, then  $P$  and  $N$  are incident. If  $\dim(W_i) = 0$  and  $N$  is an incident symmetric subgroup, then  $W_i \subset \mathcal{D}_N^*$  is a (point) rational boundary component, and conversely.  $\square$

We now consider  $\Gamma$ -integral symmetric subgroups  $N$  and arbitrary arithmetic subgroups  $\Gamma' \subset G(\mathbb{Q})$ , let  $\Gamma'_N = N \cap \Gamma'$  and consider the corresponding integral modular subvarieties they define,  $X_{\Gamma'_N} \subset X_{\Gamma'}$ . As described above, the inclusion extends to the Baily-Borel embeddings  $X_{\Gamma'_N}^* \subset X_{\Gamma'}^*$ . We now take  $\Gamma$  to be  $G_\mathbb{Z}$  for some rational representation  $\rho : G \longrightarrow GL(V_\mathbb{Z})$ , that is  $\Gamma = \rho^{-1}(GL(V_\mathbb{Z}))$  for some  $\mathbb{Z}$ -structure  $V_\mathbb{Z}$  on  $V$ . Recall the notations  $\nu_b(\Gamma')$ ,  $b = 1, \dots, s$  and  $\mu_b(\Gamma, \Gamma')$ ,  $b = 1, \dots, s$  of Definition 4.12 and 5.3, respectively, for the number of  $b^{th}$  boundary varieties and the number of  $b^{th}$  integral modular subvarieties, respectively. We let  $W_{b,i}$ ,  $b = 1, \dots, s$ ,  $i = 1, \dots, \nu_b(\Gamma')$  be the corresponding boundary varieties on the Satake compactification,  $Y_{b,j}$ ,  $b = 1, \dots, s$ ,  $j = 1, \dots, \mu_b(\Gamma, \Gamma')$  the corresponding  $\Gamma$ -integral modular varieties, everything on the arithmetic quotient  $X_{\Gamma'}$ . Then the main result of the paper is the following.

**Theorem 6.6** Let  $\Gamma$  be as above,  $\Gamma' \subset G(\mathbb{Q})$  arithmetic, and  $X_{\Gamma'} \subset X_{\Gamma'}^*$  the Satake compactification,  $X_{\Gamma'}^* - X_{\Gamma'} = \sum_{b,i} W_{b,i}$ . Then  $\Xi := \sum_{b,j} Y_{b,j}$  is a complete (finite, non-empty) set of  $\Gamma'$ -equivalence classes of  $\Gamma$ -integral modular subvarieties, such that for each  $W_{b,i}$ , there is at least one  $Y_{b,j}$  incident to  $W_{b,i}$ .

**Proof:** There is for each  $W_{b,i}$  an incident  $\Gamma$ -integral modular subvariety because for any representative parabolic there is a  $\Gamma$ -integral symmetric subgroup which is incident. The finiteness result Corollary 4.11 implies that for each  $N_b$  (of which there are finitely many) there are finitely many  $\Gamma'$ -equivalence classes of  $\Gamma$ -integral symmetric subgroups of  $G$  conjugate to  $N_b$ , so a complete set of  $\Gamma'$ -equivalence classes is finite.  $\square$

This gives us a *well-defined, non-empty, finite* set of subvarieties of the Baily-Borel embedding  $X_{\Gamma'}^* \subset \mathbb{P}^N$  for any subgroup  $\Gamma' \subset \Gamma$  of finite index. Furthermore these have a prescribed behavior near the cusps. For example, if  $f : \mathcal{D} \longrightarrow \mathbb{C}$  is a modular form whose zero divisor  $D_f$  on  $X_{\Gamma'}^*$  contains the union of the integral modular subvarieties, then  $f$  is a cusp form for  $\Gamma'$ .

## 6.2 Incidence

Consider a toroidal compactification  $\overline{X}_\Gamma$  which is smooth and projective; consider what incidence means here. Let  $W_i$  be a rational boundary variety, and let  $P$  be a parabolic  $P = \mathcal{N}_{G(\mathbb{R})}(F)$  for some rational boundary component  $F$  which covers  $W_i$ . We have the decomposition  $P = (MLR) \rtimes \mathcal{Z}V$  of the parabolic. Recall from the construction 5.6 that the inverse image  $\pi^{-1}(W_i)$  in  $\overline{X}_\Gamma$  of  $W_i \subset X_\Gamma^*$  is a divisor which is a torus embedding bundle over the family of abelian varieties “ $V/V_{\mathbb{Z}}$  over  $W_i$ ”. On the other hand, if  $X_{\Gamma_N}$  is an integral modular subvariety incident to  $W_i$ , and if  $\dim(W_i) > 0$ , then the proper transform of  $X_{\Gamma_N}^*$  on  $\overline{X}_\Gamma$  will meet  $\pi^{-1}(W_i)$  in a *section* of the family of abelian varieties over  $W_i$ . In a sense, the standard one will meet in the zero-section, the others meet in certain sections associated with level structures (e.g. sections of torsion points). For  $\dim(W_i) = 0$  the situation is slightly different. We now discuss this in more detail.

We consider first the case that  $\dim(W_i) > 0$ . Then by (29),  $N_b$  (hence any  $G(\mathbb{Q})$ -conjugate) has the form  $N_1 \times N_2$ , where  $N_1 \subset P_b$  is a hermitian Levi factor. Considering the arithmetic group  $\Gamma_{N_b}$  acting on  $D_{N_1} \times D_{N_2}$ , since the product is defined over  $\mathbb{Q}$ , the quotient  $\Gamma_{N_b} \backslash D_N$  is at most a finite quotient of a product itself. We assume that in fact  $X_{\Gamma_N}$  is a product (we will show below in Lemma 6.21 that for  $N$   $\Gamma$ -integral this always holds); then  $X_{\Gamma_N} = X_1 \times X_2$ , where  $X_1$  is the arithmetic quotient  $\Gamma_1 \backslash D_{N_1}$  and this is isomorphic to the boundary variety  $W_i$ . It follows that  $X_2 = \Gamma_2 \backslash D_{N_2}$  has rational boundary components which are zero-dimensional, say  $w \in X_2^*$ , such that with respect to the natural inclusion  $i : X_{\Gamma_N}^* \subset X_\Gamma^*$  we have

$$i(X_1 \times \{w\}) = W_i. \quad (43)$$

Recall further that any two hermitian Levi factors are conjugate by an element  $g \in V$ , and that, modulo  $\Gamma$ , this means a point of the abelian variety “ $V/V_{\mathbb{Z}}$ ”. This is of course true for any point  $t \in W_i$ , so we get

**Lemma 6.7** *Given  $W_i$ ,  $X_{\Gamma_N}^*$  any integral modular subvariety incident to it. Then the proper transform of  $X_{\Gamma_N}^*$  in  $\overline{X}_\Gamma$  determines a section of the family of abelian varieties of  $\pi^{-1}(W_i)$  over  $W_i$ .*

**Proof:** Since  $X_{\Gamma_N}^*$  is integral, by Lemma 6.21 below,  $X_{\Gamma_N}^*$  is in fact a product  $X_{\Gamma_N}^* = X_{\Gamma_1}^* \times X_{\Gamma_2}^*$ . The boundary component  $W_i$  is by (43) given by a zero-dimensional boundary component  $w$  of  $X_{\Gamma_2}^*$ , which gets modified under  $\pi$ ,  $\pi^{-1}(X_{\Gamma_2}^*) = \overline{X}_{\Gamma_2}$ . We have fibre spaces (at least locally over  $W_i$ )

$$\pi^{-1}(W_i) \xrightarrow{\eta} A_i \xrightarrow{\zeta} W_i,$$

where  $A_i = W_i \times V_b(\mathbb{R})/\Gamma_b^0 \rtimes V_b(\mathbb{Z})$  is the natural family of abelian varieties parameterized by  $W_i$ . Note that the zero of  $V_b(\mathbb{R})$  determines a zero section  $\sigma_0 : W_i \rightarrow A_i$ ,  $t \mapsto$  the image of  $0 \in V_b(\mathbb{R})$  in  $(A_i)_t = V_b(\mathbb{R})/\Lambda_b(t)$ , where  $\Lambda_b(t)$  denotes the lattice at the point  $t$ , and any element  $x \in V_b(\mathbb{Q})$  determines locally a section  $\sigma_x = \sigma_0 + x$ . Recall that  $N (= N_b^g)$  is determined by an element  $g \in V_b(\mathbb{Q})$ , so the proper transform of  $X_{\Gamma_N}^*$  in the abelian variety part of the exceptional locus is  $[X_{\Gamma_1}^*] = (\sigma_0 + g)(W_i) \subset A_i$ , which is a global section.  $\square$

If  $\dim(W_i) = 0$ , two different situations occur, depending on whether  $\mathcal{D}$  is a tube domain or not. They are (assume for the moment that  $\mathcal{D}$  is irreducible)

- $\mathcal{D}$  is a tube domain, then  $V$  is trivial and there is no abelian variety;  $\pi^{-1}(W_i)$  is a torus embedding.
- $\mathcal{D}$  is not a tube domain,  $V$  is not trivial, and  $\pi^{-1}(W_i)$  has an abelian variety factor and a torus embedding factor.

In the first case there is not much more to say than that each irreducible component  $W_{ij}$  of  $\pi^{-1}(W_i)$  meets the proper transform of  $X_{\Gamma_N}^*$  in a divisor on  $X_{\Gamma_N}^*$  (which gets itself blown up at the point). In the second case, the dimension of the abelian variety factors and of the corresponding integral symmetric subvarieties are given as follows:

	$I_{p,q}$	$II_n$	$V$	
$\dim(V)$	$q(p-q)$	$n-1$	16	
$\dim(\mathcal{D}_N)$	$q(p-1)$	$\binom{n-2}{2}$	8, 10, 8	

(44)

At any rate, we have the following result:

**Theorem 6.8** *The proper transform of  $X_{\Gamma_N}^*$  on  $\overline{X}_{\Gamma}$  is  $\overline{X}_{\Gamma_N}$ , a partial compactification for some  $\Gamma_N$ -admissible collection of polyhedral cones.*

**Proof:** Let  $P_N$  be the parabolic in  $N$ ,  $P$  the corresponding parabolic in  $G$ . Consider the decompositions (we omit the subscript  $b = s$ )

$$P_N = (M_N L_N \mathcal{R}_N) \rtimes \mathcal{Z}_N V_N, \quad P = (ML\mathcal{R}) \rtimes \mathcal{Z} V.$$

Then  $L_N = L$  is trivial (as the boundary component is a point), and there is a natural inclusion  $\mathcal{Z}_N \subset \mathcal{Z}$ . Letting  $C_N, C$  denote the corresponding homogenous self dual cones, we have  $C_N \subset C$ , and both inclusions are defined over  $\mathbb{Q}$ . Finally we have  $\Gamma_N = \Gamma \cap N$  which implies  $\Gamma_N \cap C_N = C_N \cap (C \cap \Gamma)$ . We know by assumption that we have a  $\Gamma$ -admissible cone decomposition of  $C$ , and since  $C_N \subset C$  is defined over  $\mathbb{Q}$ , this gives one also for  $\Gamma_N$ , as follows from [O], Theorem 1.13. If  $\{\sigma\}$  is the cone decomposition of  $C$ , then  $\{\sigma_N\}$ ,  $\sigma_N := \sigma \cap C_N$  gives a corresponding cone decomposition of  $C_N$ , and the theorem just mentioned applies. This argument applies to each boundary component of  $X_{\Gamma_N}^*$ , and it is clear that a  $\Gamma$ -admissible collection restricts to a  $\Gamma_N$ -admissible collection.  $\square$

### 6.3 Intersection

First note the following:

**Lemma 6.9** *Given  $X_{\Gamma}$  and two modular subvarieties  $X_1, X_2 \subset X_{\Gamma}$ , the intersection, if of dimension  $\geq 1$ , is again a modular subvariety.*

**Proof:** We are given two injections defined over  $\mathbb{Q}$ ,  $i_1 : N_1 \hookrightarrow G$ ,  $i_2 : N_2 \hookrightarrow G$ , and commutative squares

$$\begin{array}{ccccc} \mathcal{D}_{N_1} & \longrightarrow & \mathcal{D} & \longleftarrow & \mathcal{D}_{N_2} \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_{\Gamma} & \longleftarrow & X_2; \end{array}$$

it follows that  $X_1 \cap X_2$  is covered by  $\mathcal{D}_{N_1} \cap \mathcal{D}_{N_2}$  with a corresponding injection  $i_{12} : N_1 \cap N_2 \hookrightarrow G$ , again defined over  $\mathbb{Q}$ . Since  $X_1$  and  $X_2$  are modular subvarieties,  $\mathcal{D}_{N_1}$  and  $\mathcal{D}_{N_2}$  are by definition defined over  $\mathbb{Q}$ , hence so is  $\mathcal{D}_{N_1} \cap \mathcal{D}_{N_2}$ . It is also a symmetric subspace since  $\mathcal{D}_{N_1} \cap \mathcal{D}_{N_2}$  is totally geodesic in  $\mathcal{D}$ . Consequently  $X_1 \cap X_2$  is a modular subvariety.  $\square$

This can be applied in particular to the integral modular subvarieties. Hence for any two integral modular subvarieties  $X_i$ , the intersection defines a (maybe empty) modular subvariety. As there are finitely many possible intersections, from the finite set of Corollary 4.11 we get a finite set of modular subvarieties. Note that if  $X_1$  and  $X_2$  are both integral, then also the intersection is, in the following sense: Let  $N_1 = N_{b_1}^{g_1}$ ,  $N_2 = N_{b_2}^{g_2}$ , then  $\Gamma$ -integral means:

$$N_{b_1,1} \cap \Gamma = N_{b_1,1} \cap g_1 \Gamma g_1^{-1}, \quad N_{b_2,2} \cap \Gamma = N_{b_2,2} \cap g_2 \Gamma g_2^{-1},$$

and  $N_1 \cap N_2 = (g_1 N_{\mathbf{b}_1} g_1^{-1}) \cap (g_2 N_{\mathbf{b}_2} g_2^{-1})$ . Hence

$$\begin{aligned} (N_{\mathbf{b}_1,1} \cap N_{\mathbf{b}_2,2}) \cap \Gamma &= (N_{\mathbf{b}_1,1} \cap \Gamma) \cap (N_{\mathbf{b}_2,2} \cap \Gamma) \\ &= (N_{\mathbf{b}_1,1} \cap g_1 \Gamma g_1^{-1}) \cap (N_{\mathbf{b}_2,2} \cap g_2 \Gamma g_2^{-1}) \\ &= (N_{\mathbf{b}_1,1} \cap N_{\mathbf{b}_2,2}) \cap (g_1 \Gamma g_1^{-1} \cap g_2 \Gamma g_2^{-1}). \end{aligned}$$

Note that adjoining these to the integral modular subvarieties implies that on an arithmetic quotient  $X_{\Gamma'}$  for  $\Gamma' \subset \Gamma$  of finite index, there is a well-defined, finite, non-empty set of subvarieties, all of which are either integral modular subvarieties incident to rational cusps or intersections of such.

Finally consider the boundary varieties  $W_1$  and  $W_2$  to which  $X_1$  and  $X_2$  are incident. Since  $X_1 \cap X_2$  is a modular subvariety, it is itself an arithmetic quotient (in general a product), and has a boundary variety  $W_{12}^* = W_1^* \cap W_2^*$ . In this sense, we make the

**Definition 6.10** Let  $X_1$  and  $X_2$  be integral modular subvarieties, incident with  $W_1$  and  $W_2$ , respectively. Then we say  $X_{12}^* := X_1^* \cap X_2^*$  is *incident to*  $W_{12}^* := W_1^* \cap W_2^*$ .

Next suppose that we are given the two parabolics, say  $P_1$  and  $P_2$ , which are the stabilizers of the boundary components  $F_1$  and  $F_2$ , of which  $W_1$  and  $W_2$  are the quotients,  $W_1 = \Gamma_1 \backslash F_1$ ,  $W_2 = \Gamma_2 \backslash F_2$ . Assume that  $F_1^* \cap F_2^* \subset F_i^*$ ,  $i = 1, 2$ , is a maximal boundary component in  $F_i^*$ . Under this assumption, the intersection  $P_1 \cap P_2$  is a parabolic, associated with  $F_1^* \cap F_2^*$ . Either of the inclusions  $F_1^* \cap F_2^* \subset F_i^*$  determines the parabolic which is the (non-maximal) parabolic stabilizing a flag of two terms. Similarly,  $X_1^* \cap X_2^*$  contains  $F_1^* \cap F_2^*$  as a rational boundary component, and either of the inclusions  $X_1^* \cap X_2^* \subset X_i^*$  determines a symmetric subgroup, also the stabilizer of a flag with two terms. This is of course just  $(N_1 \cap N_2) \times \mathcal{Z}_G(N_1 \cap N_2)$ , where  $N_i$  is the group giving rise to  $\mathcal{D}_{N_i}$ , covering  $X_i$ . So we have:  $N_i$  incident with  $P_i$ ,  $i = 1, 2$ ,  $P_{12} := P_1 \cap P_2$  a parabolic, then  $(N_1 \cap N_2) \times \mathcal{Z}_G(N_1 \cap N_2)$  is incident to  $P_{12}$ .

## 6.4 Moduli interpretation

In this section we suppose the algebraic group  $G$  comes from a moduli problem of PEL structures, and will discuss the moduli-theoretic description of the modular subvarieties  $X_{\Gamma_N}$  and the corresponding arithmetic quotients  $X_{\Gamma}^*$ . We then also briefly describe the notion of incidence from this point of view.

### 6.4.1 PEL structures

Let  $V$  be an abelian variety over  $\mathbb{C}$ ,  $End(V)$  the endomorphism ring and  $End_{\mathbb{Q}}(V) = End(V) \otimes_{\mathbb{Z}} \mathbb{Q}$  the endomorphism algebra. A polarization, i.e., a linear equivalence class of ample divisors giving a projective embedding of  $V$ , gives rise to a positive involution on  $End_{\mathbb{Q}}(V)$ , the so-called Rosatti involution:

$$\begin{aligned} \varrho : End_{\mathbb{Q}}(V) &\longrightarrow End_{\mathbb{Q}}(V) \\ \phi &\mapsto \phi^{\ell}. \end{aligned} \tag{45}$$

If  $A$  is a central simple algebra over  $\mathbb{Q}$ , an involution  $*$  on  $A$  is called *positive*, if  $tr_{A|\mathbb{Q}}(x \cdot x^*) > 0$  for all  $x \in A$ ,  $x \neq 0$ , where  $tr_{A|\mathbb{Q}}$  denotes the reduced trace. Assuming  $(A, *)$  to be simple with positive involution, the  $\mathbb{R}$ -algebra  $A(\mathbb{R})$  is isomorphic to one of the following (see [Sh2], Lemma 1)

- (i)  $M_r(\mathbb{R})$  with involution  $X^* = {}^t X$ ;
- (ii)  $M_r(\mathbb{C})$  with involution  $X^* = {}^t \overline{X}$ , where  $\overline{\phantom{x}}$  is complex conjugation;
- (iii)  $M_r(\mathbb{H})$  with involution  $X^* = {}^t \overline{X}$ , where  $\overline{\phantom{x}}$  is quaternionic conjugation.

The algebras  $A$  occurring in (i) and (iii) are central simple over  $\mathbb{R}$ , while those of (ii) are central simple over  $\mathbb{C}$ . The  $\mathbb{Q}$ -algebra  $A$  itself is a  $\mathbb{Q}$ -form of one of these. The central simple algebras  $A$  over  $\mathbb{Q}$  are known to be the  $M_n(D)$ , where  $D$  is a division algebra over  $\mathbb{Q}$ . If the algebra  $A$  has a positive involution, the same holds for  $D$ . The division algebras  $D$  which can occur are also known.

**Proposition 6.11** *Let  $D$  be a division algebra over  $\mathbb{Q}$  with a positive involution. Then  $D$  occurs in one of the following cases:*

- I.  *$A$  totally real algebraic number field  $k$ ;*
- II.  *$D$  a totally indefinite quaternion algebra over  $k$ ;*
- III.  *$D$  a totally definite quaternion algebra over  $k$ ;*
- IV.  *$D$  is central simple over  $K$  with a  $K|k$  involution of the second kind, where  $K$  is an imaginary quadratic extension of  $k$ .*

In case III the canonical involution on  $D$  is the unique positive involution, while in case II the positive involutions correspond to  $x \in D$  such that  $x^2$  is totally negative in  $k$ . If the algebra  $D$  has an involution of the second kind it is easy to see that it admits a positive one. It follows from the fact that  $\text{End}_{\mathbb{Q}}(V)$  is a semisimple algebra over  $\mathbb{Q}$  with a positive involution that each simple factor is a total matrix algebra  $M_n(D)$ , with  $D$  as in the proposition.

Let  $(A, *)$  be a semisimple algebra over  $\mathbb{Q}$  with positive involution, and let

$$\Phi : A \longrightarrow GL(n, \mathbb{C}) \quad (46)$$

be a faithful representation. Shimura considers data  $\mathcal{P} = (V, \mathcal{C}, \theta)$  and  $\{A, \Phi, *\}$  and defines the notion of *polarized abelian variety of type  $\{A, \Phi, *\}$*  by the conditions:

- (i)  $V$  is an abelian variety over  $\mathbb{C}$ ,  $\mathcal{C}$  is a polarization;
- (ii)  $\theta : A \xrightarrow{\sim} \text{End}_{\mathbb{Q}}(V)$  is an algebra isomorphism, and for  $\theta(x) : \tilde{V} \longrightarrow \tilde{V}$  (the  $\sim$  denoting the universal cover, i.e.,  $\tilde{V}$  is a complex vector space) one has  $\theta(x) = \Phi(x)$ ; (47)
- (iii) the involution  $\varrho$  determined by  $\mathcal{C}$  as in (45) coincides on  $\theta(A)$  with the involution coming from  $(A, *)$ , i.e.  $\theta(x)^{\varrho} = \theta(x^*)$ .

The condition (ii) is to be understood as follows. Fixing an isomorphism

$$\psi : V \cong \mathbb{C}^n / \Lambda, \quad (48)$$

each  $a \in \text{End}_{\mathbb{Q}}(V)$  is represented by a linear transformation of  $\mathbb{C}^n$  preserving  $\Lambda$ ; that is each  $a$  can be represented by a matrix, and  $\theta(x) = a$  is the matrix corresponding to  $x \in A$  via  $\theta$ . Recall also that a complex torus  $\mathbb{C}^n / \Lambda$  is an abelian variety if and only if there exists a *Riemann form*: each positive (1,1) form  $\omega$  gives rise to a skew symmetric matrix  $(q_{ij})$ :

$$\omega = \sum q_{ij} dx_i \wedge dx_j,$$

where the  $x_i$  are canonical coordinates on  $\mathbb{C}^n$ . Hence if we fix a positive divisor  $C \subset V$ , it determines an involution as in (45) and a Riemann form  $E_C(x, y)$  on  $\mathbb{C}^n / \Lambda$ , and these are related by

$$E_C(\psi(a)x, y) = E_C(x, \psi(a^{\varrho})y), \quad (49)$$

where for  $a \in \text{End}_{\mathbb{Q}}(V)$ ,  $\psi(a)$  denotes the matrix representation for  $a$  arising from the identification  $\psi : V \cong \mathbb{C}^n/\Lambda$  in (48).

Let  $(V, \mathcal{C}, \theta)$  be an abelian variety of type  $(D, \Phi, *)$  with  $D$  a division algebra, so that  $(D, *)$  is one of the algebras of Proposition 6.11. In the notations used there, put

$$[k : \mathbb{Q}] = f, \quad [D : K] = d^2, \text{ if } D \text{ is of type IV} \quad (50)$$

defining the numbers  $f$  and  $d$ . Let  $n = \dim(V)$ ; then, assuming  $D$  to be a division algebra,  $2n$  is a multiple of  $[D : \mathbb{Q}]$ , i.e.,  $2n = [D : \mathbb{Q}]m$ . Note that  $[D : \mathbb{Q}] = f$  for type I,  $[D : \mathbb{Q}] = 4f$  for types II and III, while  $[D : \mathbb{Q}] = 2d^2f$  if  $D$  is of type IV. For the existence of  $(V, \mathcal{C}, \theta)$  of type  $(D, \Phi, *)$ , certain restrictions are placed on  $\Phi$ ; we assume these are fulfilled. So under the isomorphism  $\theta$ , each  $x \in D$  is represented by the matrix  $\Phi(x)$ . This makes the lattice  $\Lambda$  with  $V \cong \mathbb{C}^n/\Lambda$ , tensored with  $\mathbb{Q}$ , a (left)  $D$ -module, i.e.,

$$Q := \mathbb{Q} \cdot \Lambda = \sum_1^m \Phi(D) \cdot x_i \quad (51)$$

for a suitable set of vectors  $x_i$ . But this is the same as saying there exists a  $\mathbb{Z}$ -lattice  $\mathcal{M} \subset D$ , such that

$$\Lambda = \left\{ \sum_1^m \Phi(a_i)x_i \mid (a_1, \dots, a_m) \in \mathcal{M} \right\}. \quad (52)$$

If  $D$  is central over  $K$ , then  $\mathcal{M}$  is clearly also an  $\mathcal{O}_K$ -lattice in  $D$ . The integrality of the Riemann form can be expressed in terms of  $\text{tr}_{D|K}$ :

$$E_C\left(\sum_1^m \Phi(a_i)x_i, \sum_1^m \Phi(b_j)x_j\right) = \text{tr}_{D|K}\left(\sum_{i,j}^m a_i t_{ij} b_j^*\right), \quad (53)$$

and  $T = (t_{ij}) \in M_m(D)$  is a skew-hermitian matrix:

$$T^* = -T, \quad (54)$$

where  $T^*$  denotes the matrix  $(t_{ji}^*)$ , where  $*$  is the involution on  $D$ . For the lattice  $\mathcal{M}$  one has

$$\text{tr}_{D|K}(\mathcal{M} T \mathcal{M}^*) \subset \mathbb{Z}. \quad (55)$$

Hence to each  $(V, \mathcal{C}, \theta)$  of type  $(D, \Phi, *)$  one gets a  $*$ -skew hermitian  $T \in M_m(D)$  and a lattice  $\mathcal{M} \subset D$ . To this situation there is a naturally associated  $\mathbb{Q}$ -group. On the vector space  $D^m$  we consider

$$G(D, T) := \{g \in GL(D^m) \mid g T g^* = T\}, \quad (56)$$

the symmetry group of the  $*$ -skew hermitian form determined by  $T$ . It is now easy to determine the  $\mathbb{R}$ -group:

$$G(D, T)(\mathbb{R}) = \begin{cases} \text{Type I: } Sp(m, \mathbb{R}) \times \cdots \times Sp(m, \mathbb{R}) & (m \text{ is even}) \\ \text{Type II: } Sp(2m, \mathbb{R}) \times \cdots \times Sp(2m, \mathbb{R}) \\ \text{Type III: } SO^*(2m) \times \cdots \times SO^*(2m) \\ \text{Type IV: } U(p_1, q_1) \times \cdots \times U(p_g, q_g), \end{cases} \quad (57)$$

where the number of factors is in each case  $f$ , and  $p_\nu + q_\nu = md$ , and  $(p_\nu, q_\nu)$  is the signature corresponding to the  $\nu^{th}$  real prime. For each  $\nu$ , there is a matrix  $W_\nu$  which transforms  $T_\nu$  into the

standard form, i.e.,

$$W_\nu T_\nu^{-1 t} W_\nu = \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix}, l = \frac{m}{2} \text{ for Type I, } l = m \text{ for Type II}; \quad (58)$$

$$W_\nu T_\nu^{-1} W_\nu^* = -i \begin{pmatrix} -1_m & 0 \\ 0 & 1_m \end{pmatrix}, \text{ Type III}; \quad (59)$$

$$W_\nu (iT_\nu^{-1}) W_\nu^* = \begin{pmatrix} 1_{p_\nu} & 0 \\ 0 & -1_{q_\nu} \end{pmatrix}, \text{ Type IV}. \quad (60)$$

Let  $\mathcal{D} = \mathcal{D}_{(D,T)}$  denote the domain determined by  $G(D,T)(\mathbb{R})$  (actually a particular unbounded realisation of this domain, see [Sh2], 2.6). Then  $\mathcal{D} = \prod \mathcal{D}_\nu$ , and  $z_\nu \in \mathcal{D}_\nu$  gives rise to a normalised period (i.e., one of the form  $(1, \Omega)$ ) for an abelian variety, by setting  $X_\nu = Y_\nu \overline{W}_\nu$ , where

$$Y_\nu = \begin{pmatrix} z_\nu & 1_l \\ \bar{z}_\nu & 1_l \end{pmatrix}, l = \frac{m}{2}, \text{ Type I, } l = m, \text{ Type II}; \quad (61)$$

$$Y_\nu = \begin{pmatrix} -z_\nu & 1_m \\ 1_m & \bar{z}_\nu \end{pmatrix}, \text{ Type III}; \quad (62)$$

$$Y_\nu = \begin{pmatrix} 1_{p_\nu} & z_\nu \\ t \bar{z}_\nu & 1_{q_\nu} \end{pmatrix}, \text{ Type IV}. \quad (63)$$

The matrix  $X_\nu$  determine  $m$  vectors  $x_1, \dots, x_m$  of  $\mathbb{C}^n$  (in a rather complicated fashion, see formulas (17)-(20) in [Sh2]), which determine a lattice  $\Lambda = \Lambda(z, T, \mathcal{M})$  by the formula in equations (51)-(52) above.

Note that the representation  $\Phi$  contains the representations  $\chi_\nu$ = projection on the  $\nu$ th real factor with multiplicities. For Type IV,  $p_\nu + q_\nu = md$ , and  $p_\nu$ =multiplicity of  $\chi_\nu$  while  $q_\nu$ =multiplicity of  $\bar{\chi}_\nu$ . For things to work out one must therefore assume, in case of Type IV, that  $iT_\nu^{-1}$  has the same signature  $(p_\nu, q_\nu)$  as occurs in  $\Phi$ . With this restriction, the following holds:

**Theorem 6.12 ([Sh2], Thm. 1)** *For every  $z \in \mathcal{D} = \mathcal{D}_{(D,T)}$ , and every lattice  $\mathcal{M} \subset D$ , we get a polarized abelian variety  $V_z = \mathbb{C}^n / \Lambda(z, T, \mathcal{M})$  of type  $(D, \Phi, *)$ , and conversely, every such  $V$  is of the form  $V = \mathbb{C}^n / \Lambda(z, T, \mathcal{M})$  for some  $z \in \mathcal{D}_{(D,T)}$ ,  $\mathcal{M} \subset D$  a lattice.*

The lattice  $\mathcal{M} \subset D$  gives rise to an arithmetic subgroup

$$\Gamma = \Gamma_{(D,T,\mathcal{M})} = \{g \in G(D,T) \mid g\mathcal{M} \subset \mathcal{M}\} \quad (64)$$

as discussed in section 3. If one defines an isomorphism  $\phi : V_z \rightarrow V_{z'}$  of two abelian varieties of type  $(D, \Phi, *)$  as an isomorphism of the underlying varieties, such that  $\phi^{-1}(\mathcal{C}') = \mathcal{C}$  and  $\phi\theta(a) = \theta'(a)\phi$ , for all  $a \in D$ , then one has

**Theorem 6.13 ([Sh2], Thm. 2)** *The arithmetic quotient  $X_\Gamma = \Gamma \backslash \mathcal{D}_{(D,T)}$  is the moduli space of isomorphism classes of abelian varieties  $V_z = \mathbb{C}^n / \Lambda(z, T, \mathcal{M})$  of type  $\{(D, \Phi, *), (T, \mathcal{M})\}$ , where  $\Gamma$  is the arithmetic group of (64).*

Moreover, one calls two such pairs  $(T_1, \mathcal{M}_1)$ ,  $(T_2, \mathcal{M}_2)$  equivalent, if  $\exists_{U \in M_m(D)}$ , such that  $UT_2U^* = \delta T_1$  for some positive  $\delta \in \mathbb{Q}$  and  $\mathcal{M}_1 U = \mathcal{M}_2$ . Equivalent pairs give rise to isomorphic families of abelian varieties ([Sh2], Prop. 4). Summing up,  $*$ -skew hermitian matrices  $T \in M_m(D)$  determine certain

$\mathbb{Q}$ -groups, lattices  $\mathcal{M} \subset D$  determine certain arithmetic groups, and the corresponding arithmetic quotients are moduli spaces for certain families of abelian varieties.

**Remark 6.14** The complex multiplication by  $\mathcal{M}$  describes the endomorphism ring. The *automorphisms* determined by  $\mathcal{M}$  are the invertible elements, i.e.,  $\text{Aut}(V) = \mathcal{M}^*$ , the group of units.

One can also accomodate level structures in this setup, introduced in [Sh3], cf. also [Sh4]. This is done by fixing  $s$  points  $y_1, \dots, y_s$  in the  $D$ -module  $Q$ , as in (51), and  $s$  points  $t_1, \dots, t_s$  of the abelian variety  $V$ . One requires that the map  $\psi$  of (48) maps the  $y_i$  onto the  $t_i$ . More precisely,

**Definition 6.15** Let  $Q$  be a  $D$ -vector space of dimension  $m$ , and  $\mathcal{M} \subset Q$  a  $\mathbb{Z}$ -lattice. Consider a conglomeration:

$$\mathcal{T} := \{(D, \Phi, *), (Q, T, \mathcal{M}); y_1, \dots, y_s\},$$

where  $(D, \Phi, *)$  is as above,  $(Q, T, \mathcal{M})$  is a  $D$ -vector space with lattice  $\mathcal{M}$  and  $*$ -skew hermitian ( $D$ -valued) form  $T$  on  $Q$ , and  $y_i$  points in  $Q$ . This is called a PEL-type. Consider a conglomeration:

$$\mathcal{Q} := \{(V, \mathcal{C}, \theta); t_1, \dots, t_s\},$$

where  $(V, \mathcal{C}, \theta)$  is a polarized abelian variety with analytic coordinate  $\theta$  as above and  $t_i$  are points of *finite order* on  $V$ . This is called a PEL-structure. Then  $\mathcal{Q}$  is of type  $\mathcal{T}$ , if there exists a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M} & \longrightarrow & Q(\mathbb{R}) & \longrightarrow & Q(\mathbb{R})/\mathcal{M} \longrightarrow 0 \\ & & \downarrow & & f \downarrow & & \downarrow \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}^n & \xrightarrow{\psi} & V \longrightarrow 0 \end{array}, \quad (65)$$

satisfying the conditions

- (i)  $\psi$  gives a holomorphic isomorphism (strictly speaking, this is the  $\psi^{-1}$  of above);
- (ii)  $f$  is an  $\mathbb{R}$ -linear isomorphism, and  $f(\mathcal{M}) = \Lambda$ ;
- (iii)  $f(ax) = \Phi(a)f(x)$ , and  $\Phi(a)$  defines  $\theta(a)$  for every  $a \in D$  as (47), (ii);
- (iv)  $C \in \mathcal{C}$  determines a Riemann form  $E_C$  as in (49).

Note that the finite set of points  $y_i$  and  $t_i$  come both equipped with a form; on the former the form  $T$ , and the Riemann form  $E_C$  on the latter. These forms are preserved under the isomorphism.

There is a natural notion of isomorphism of abelian varieties with PEL structures. Given two PEL-structures  $\mathcal{Q}$  and  $\mathcal{Q}'$ , an isomorphism  $\phi : V \longrightarrow V'$  is an *isomorphism* from  $\mathcal{Q}$  to  $\mathcal{Q}'$ , if  $\phi\theta(a) = \theta'(a)\phi$  for all  $a \in D$ , and  $\phi(t_i) = t'_i$  for all  $i$ .

**Definition 6.16** A PEL-type  $\mathcal{T}$  is *equivalent* to a PEL-type  $\mathcal{T}'$ , if  $D = D'$ ,  $* = *'$ ,  $s = s'$ ,  $\Phi$  and  $\Phi'$  are equivalent as representations of  $D$ , and there is a  $D$ -linear automorphism  $\mu$  of  $Q$  such that  $T'(x\mu, y\mu) = T(x, y)$ ,  $\mathcal{M}\mu = \mathcal{M}'$ ,  $y_i\mu \equiv y'_i \pmod{\mathcal{M}'}$  for all  $i$ . If  $\mathcal{Q}$  is of type  $\mathcal{T}$ , then  $\mathcal{Q}$  is also of type  $\mathcal{T}'$  if and only if  $\mathcal{T}$  and  $\mathcal{T}'$  are equivalent. A PEL-type  $\mathcal{T}$  is called *admissible*, if there exists at least one PEL-structure of that type.

One has an analogue of Theorems 6.12 and 6.13 in this situation also.

**Theorem 6.17 ([Sh4], Thm. 3)** *For every admissible PEL-type  $\mathcal{T}$  there exists a bounded symmetric domain  $\mathcal{D}$  (this is the same domain as in Theorem 6.12) such that the statement of Theorem 6.12 holds in this situation, and every PEL-structure  $\mathcal{Q}$  of type  $\mathcal{T}$  occurs in this family.*

Tabelle 1: Rational groups for PEL-structures.

	Type II	Type III	Type IV
$D$	A totally indefinite quaternion algebra over $\mathbb{Q}$	a totally definite quaternion algebra over $\mathbb{Q}$	simple division algebra, central over $K$ , an imaginary quadratic extension of $\mathbb{Q}$ , with an involution of the second kind. One may assume $D$ to be a cyclic algebra.
$d$	2	2	$d$
$\dim(V)$	$2m$	$2m$	$d^2m$
Tits index	$C_{m,s}^{(2)}$	$i D_{m,s}^{(2)}, i = 1, 2$	${}^2 A_{dm-1,s}^{(d)}$

The types listed are absolutely  $\mathbb{Q}$ -simple, the  $\mathbb{Q}$ -rank is  $s$ , and this is the Witt index of the  $\pm$ hermitian form.

Now define a corresponding arithmetic group as follows:

$$\Gamma = \{g \in G(D, T) \mid \mathcal{M}g = \mathcal{M}, y_i g \equiv y_i \pmod{\mathcal{M}}, i = 1, \dots, s\} \quad (66)$$

Then the analogue of Theorem 6.13 is

**Theorem 6.18 ([Sh4], Thm. 4)** *Two members of the family of Theorem 6.17 corresponding to points  $z_1, z_2 \in \mathcal{D}$  are isomorphic if and only if  $z_1 = \gamma(z_2)$  for some  $\gamma \in \Gamma$ ,  $\Gamma$  as in (66).*

In Table 1 we list the data for each of the cases II, III and IV of (57).

#### 6.4.2 Modular subvarieties

We continue with the notations above,  $G$ ,  $S$ ,  $P_b$  and  $N_b$  being fixed,  $\Gamma$  an arithmetic group satisfying (40), and  $\Gamma' \subset G(\mathbb{Q})$  another arithmetic group. We consider the arithmetic quotient  $X_{\Gamma'}$ , its Baily-Borel embedding  $X_{\Gamma'}^*$ , and a smooth toroidal embedding  $\overline{X}_{\Gamma'}$ . Let  $N \subset G$  be a rational symmetric subgroup,  $N = N_b^g$  for some  $b = 1, \dots, s$  and some  $g \in G(\mathbb{Q})$ . Let us first suppose for the boundary point in question  $F_b$  that  $\dim(F_b) > 0$ . Under this assumption we know that  $N_b$  is a product

$$N_b = L_b \times \mathcal{Z}_G(L_b),$$

and  $L_b(\mathbb{R})^0 = (\text{Aut}(F_b))^0$ . This implies immediately that the domain  $\mathcal{D}_{N_b}$  is also a product,

$$\mathcal{D}_{N_b} \cong \mathcal{D}_1 \times \mathcal{D}_2.$$

Let  $\iota_i : \mathcal{D}_i \hookrightarrow \mathcal{D}_1 \times \mathcal{D}_2$  be the natural inclusion, and consider the inclusion  $\eta : \mathcal{D}_{N_b} \hookrightarrow \mathcal{D}$ . Then  $\eta(\iota_i(\mathcal{D}_i))$  is a symmetric subdomain, which itself has an interpretation in terms of PEL structures, which is a sub-PEL structure of that attached to  $\mathcal{D}$ . Let us now explain this for the individual cases. For PEL structures, only the domains of type **I<sub>p,q</sub>**, **II<sub>n</sub>**, **III<sub>n</sub>** occur, types **U.1**, **U.2**, **O.2**, **S.1**, **S.2**.

**U.1** : If  $b < t$ , then  $F_b$  is of type **I<sub>p-b,q-b</sub>**, and  $\mathcal{D}_{N_b}$  is of the type **I<sub>p-b,q-b</sub>**  $\times$  **I<sub>b,b</sub>**. The moduli interpretation is complex multiplication on abelian  $(p+q)$ -folds. In the locus **I<sub>p-b,q-b</sub>**  $\times$  **I<sub>b,b</sub>**, the variety  $A^{p+q}$  splits into  $A^{p+q-2b} \times A^{2b}$ , where the complex multiplication has signature  $(p-b, q-b)$  and  $(b, b)$ , respectively.

If  $b = s = t$ ,  $F_t$  is a point,  $\mathcal{D}_{N_b}$  is of type **I<sub>p-1,q</sub>**. Here the abelian variety  $A^{p+q}$  splits off an elliptic curve,  $A^{p+q} = A^{p+q-1} \times A^1$ . Since  $A^1$  has no moduli, only the moduli of  $A^{p+q-1}$  contributes.

**U.2** : If  $b < t$ , then  $F_b$  is of type  $\mathbf{I}_{\mathbf{p}\text{-}\mathbf{db},\mathbf{q}\text{-}\mathbf{db}}$  and  $\mathcal{D}_{N_b}$  is of type  $\mathbf{I}_{\mathbf{p}\text{-}\mathbf{db},\mathbf{q}\text{-}\mathbf{db}} \times \mathbf{I}_{\mathbf{db},\mathbf{db}}$ . The moduli involved here is a degree  $d$  division algebra  $D$ , central simple over  $K$  with  $K|k$ -involution, as endomorphism algebra. In the locus  $\mathbf{I}_{\mathbf{p}\text{-}\mathbf{db},\mathbf{q}\text{-}\mathbf{db}} \times \mathbf{I}_{\mathbf{db},\mathbf{db}}$  the abelian variety  $A^{p+q}$  splits  $A^{p+q} = A^{p+q-2db} \times A^{2db}$ , and each factor retains the endomorphisms by  $D$ .

If  $b = s = t$ , again  $F_t$  is a point,  $\mathcal{D}_{N_b}$  is of type  $\mathbf{I}_{\mathbf{q},\mathbf{q}} \subset \mathbf{I}_{\mathbf{p},\mathbf{q}}$ . Here the abelian variety splits as  $A^{2q} \times A^{p-q}$ , where the endomorphism ring on  $A^{p-q}$  is definite, and contributes no moduli. In this case the only moduli contributing is the modulus of  $A^{2q}$ .

**O.2** : If  $b < s$  or  $s < [\frac{n}{2}]$ , then  $F_b = \mathbf{II}_{\mathbf{n}-2\mathbf{b}}$ ,  $\mathcal{D}_{N_b} = \mathbf{II}_{\mathbf{n}-2\mathbf{b}} \times \mathbf{II}_{2\mathbf{b}}$ , and the splitting is evident. For  $b = t$ ,  $F_t = \mathbf{II}_0$  ( $n$  even) or  $\mathbf{II}_1$  ( $n$  odd), both of which are points. Then for  $n$  even,  $N_t$  is a polydisc by definition,  $\mathbf{II}_2 \times \cdots \times \mathbf{II}_2$ , and again the splitting is evident, this time as a product of abelian surfaces with multiplication by the quaternion division algebra  $D$ . For  $n$  odd,  $N_t$  is of type  $\mathbf{II}_{\mathbf{n}-1}$ , and the splitting of  $A^{2n}$  is as  $A^{2n} \cong A^{2n-s} \times A^s$ .

**S.1** : If  $b < t$ , then  $F_b$  is of type  $\mathbf{III}_{\mathbf{n}-\mathbf{b}}$ , the subdomain  $\mathcal{D}_{N_b}$  is of type  $\mathbf{III}_{\mathbf{n}-\mathbf{b}} \times \mathbf{III}_{\mathbf{b}}$ ,  $A^n$  splits  $A^n = A^{n-b} \times A^b$ . If  $b = s = t$ , then  $F_t$  is a point,  $\mathcal{D}_{N_t} \cong (\mathbf{III}_1)^n$ . Here the abelian variety splits into a product of elliptic curves.

**S.2** : If  $b < t$ ,  $F_b = \mathbf{III}_{\mathbf{n}-2\mathbf{b}}$ ,  $\mathcal{D}_{N_b} = \mathbf{III}_{\mathbf{n}-2\mathbf{b}} \times \mathbf{III}_{2\mathbf{b}}$ , where  $b < [\frac{n}{2}]$ . If  $b = s = t$ ,  $\mathcal{D}_{N_t} = (\mathbf{III}_2)^{\frac{n}{2}}$ , ( $n$  even follows from  $s = t$ ). Once again, in both cases the moduli-theoretic meaning is evident.

This explains the expression of sub-PEL structures, and we have established

**Proposition 6.19** *For any subdomain  $\mathcal{D}_{N_b} \subset \mathcal{D}$ , the corresponding abelian varieties split in the manner described above.*

**Proof:** We prove a typical case and leave the verification of the other cases to the reader. We will do case **U.2**. For this we consider the matrix  $Y_\nu$  of (63). We may assume that the realisation of the domain  $\mathcal{D}$  is such that for the subdomain  $\mathcal{D}_N$ , the corresponding  $z_\nu$  splits, i.e.,

$$z_\nu \in \mathcal{D}_N \Rightarrow z_\nu = \begin{pmatrix} z_{\nu,1} & 0 \\ 0 & z_{\nu,2} \end{pmatrix}, \quad (67)$$

where  $z_{\nu,1} \in \mathcal{D}_1$  and  $z_{\nu,2} \in \mathcal{D}_2$  for the decomposition  $\mathcal{D}_N = \mathcal{D}_1 \times \mathcal{D}_2$ . In this case,  $\mathcal{D}_1$  is of type  $\mathbf{I}_{\mathbf{p}\text{-}\mathbf{jd},\mathbf{q}\text{-}\mathbf{jd}}$ , while  $\mathcal{D}_2$  is of type  $\mathbf{I}_{\mathbf{jd},\mathbf{jd}}$ . We need the vectors  $x_1, \dots, x_m$  determined by  $X_\nu$ , given in this case by the formula (20) in [Sh2]

$$X_\nu = \begin{bmatrix} u_{11}^\nu & \cdots & u_{m1}^\nu & u_{12}^\nu & \cdots & u_{m2}^\nu & \cdots & u_{1d}^\nu & \cdots & u_{md}^\nu \\ \overline{v}_{11}^\nu & \cdots & \overline{v}_{m1}^\nu & \overline{v}_{12}^\nu & \cdots & \overline{v}_{m2}^\nu & \cdots & \overline{v}_{1d}^\nu & \cdots & \overline{v}_{md}^\nu \end{bmatrix}.$$

Here the vectors  $x_i$  are given by  ${}^t x_i^\nu = ({}^t u_{i1}^\nu \cdots {}^t u_{id}^\nu {}^t v_{i1}^\nu \cdots {}^t v_{id}^\nu)$  and  $u_{ik}^\nu \in \mathbb{C}^{p_\nu}$ ,  $v_{ik}^\nu \in \mathbb{C}^{q_\nu}$ . Now from the particular form of our  $z_\nu$ , we can conclude that also the vectors  $x_i$  have a special form. Indeed, comparing the above with (67), we see that for  $W_\nu = id$  we have

$$X_\nu = \left( \begin{array}{cc|cc} \mathbf{1}_{\mathbf{p}_\nu\text{-}\mathbf{jd}} & 0 & z_{\nu,1} & 0 \\ 0 & \mathbf{1}_{\mathbf{jd}} & 0 & z_{\nu,2} \\ \hline {}^t \overline{z}_{\nu,1} & 0 & \mathbf{1}_{\mathbf{q}_\nu\text{-}\mathbf{jd}} & 0 \\ 0 & {}^t \overline{z}_{\nu,2} & 0 & \mathbf{1}_{\mathbf{jd}} \end{array} \right),$$

so that the vector  ${}^t x_1$ , for example, has the form<sup>4</sup>

$${}^t x_1 = \left( \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^t \cdots \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}^t \begin{pmatrix} z_{\nu,1}^{(1)} \\ 0 \end{pmatrix}^t \cdots \begin{pmatrix} z_{\nu,1}^{(p-(j-1)d+1)} \\ 0 \end{pmatrix}^t \cdots \begin{pmatrix} 0 \\ z_{\nu,2}^{(1)} \end{pmatrix}^t \cdots \begin{pmatrix} 0 \\ z_{\nu,2}^{((j-1)d+1)} \end{pmatrix}^t \right),$$

where  $z_{\nu,j}^{(k)}$  denotes the  $k^{\text{th}}$  column of  $z_{\nu,j}$ , and similarly for the other  ${}^t x_i$ . From this it follows that the lattice  $\Lambda$  of (52) splits,  $\Lambda = \Lambda_1 \oplus \Lambda_2$ , where  $\Lambda_1$  and  $\Lambda_2$  are orthogonal to each other, each being itself a normalized period matrix

$$\Lambda_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mathbb{Z} \oplus \cdots \oplus \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \mathbb{Z} \oplus z_{\nu,1}^{(1)} \mathbb{Z} \oplus \cdots \oplus z_{\nu,1}^{(q-jd)} \mathbb{Z}, \quad \Lambda_2 = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mathbb{Z} \oplus \cdots \oplus \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \mathbb{Z} \oplus z_{\nu,2}^{(1)} \mathbb{Z} \oplus \cdots \oplus z_{\nu,2}^{(jd)} \mathbb{Z}.$$

The proposition follows from this for the case that  $W_\nu = \text{id}$ . Finally we note that if  $W_\nu \neq \text{id}$ , this does not influence the reasoning above, and the splitting remains (only the polarization is no longer principal). In particular, the case  $\dim(F_b) = 0$ , which occurs for  $b = s$ ,  $sd = q_\nu$ , is covered by the above,

$$z_\nu = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & z_{\nu,2} \end{pmatrix}$$

with  $z_{\nu,2} \in \mathbf{I}_{\mathbf{s}\mathbf{d},\mathbf{s}\mathbf{d}}$ . □

We now consider conjugates  $N = N_b^g$ . The following two lemmas apply to any  $G$  as considered in this paper so we assume for the moment only that  $G$  is  $\mathbb{Q}$ -simple of hermitian type,  $\Gamma$  fulfills (40) and  $\Gamma' \subset G(\mathbb{Q})$  is arithmetic.

**Lemma 6.20** *If  $g \in G(\mathbb{Q})$ ,  $N = N_b^g$ , then the modular subvariety  $X_{\Gamma'_N} \subset X_{\Gamma'}$  is a finite quotient of a product. Consequently, for all  $\mathcal{D}_N$ ,  $N$  rational symmetric, the arithmetic subvariety  $X_{\Gamma'_N}$  is in the locus of isomorphism classes of abelian varieties which are isogenous to products, i.e., are not simple.*

**Proof:** Since  $g \in G(\mathbb{Q})$ , we see that  $\mathcal{D}_N$  is  $\mathbb{Q}$ -equivalent to  $\mathcal{D}_{N_b}$ . For  $\mathcal{D}_{N_b}$  the statement follows from the fact that  $N_b = N_1 \times N_2$  is a product over  $\mathbb{Q}$ , so for  $g \in G(\mathbb{Q})$  it is likewise true for  $N = N_b^g$ . Consequently, the action of  $\Gamma$  on  $\mathcal{D}_N$  is up to a finite action a product action. The second statement follows from this, as on the finite cover which is a product, the splitting property follows as discussed above. □

Now suppose  $N$  is in fact  $\Gamma$ -integral, i.e.,  $N_1 \cap \Gamma = g(N_{b,\mathbf{1}} \cap \Gamma)g^{-1}$ .

**Lemma 6.21** *If  $N$  is  $\Gamma$ -integral, then the discrete subgroup  $N \cap \Gamma$  is in fact a product,  $\Gamma_N = N \cap \Gamma = \Gamma_1 \times \Gamma_2$ ,  $\Gamma_i \subset \text{Aut}(\mathcal{D}_i)$ ,  $i = 1, 2$ .*

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<sup>4</sup>for convenience the transposition  $t$  is placed to the right of the vector in this expression

**Proof:** We note that there is a natural inclusion  $N \cap \Gamma \subset g\Gamma_{b,1}g^{-1} \times g\Gamma_{b,2}g^{-1}$ , and since  $N_1 \cap \Gamma \subset N \cap \Gamma$  is equal to  $g\Gamma_{b,1}g^{-1}$  we get the exact diagram

$$\begin{array}{ccccccc}
& 1 & & 1 & & & \\
& \downarrow & & \downarrow & & & \\
1 & \longrightarrow & \Gamma_1 & \longrightarrow & g\Gamma_{b,1}g^{-1} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & N \cap \Gamma & \longrightarrow & g\Gamma_{b,1}g^{-1} \times g\Gamma_{b,2}g^{-1} & \longrightarrow & K_1 \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & Q & \longrightarrow & g\Gamma_{b,2}g^{-1} & \longrightarrow & K_2 \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
& 1 & \longrightarrow & 1 & \longrightarrow & 1, &
\end{array}$$

and the splitting  $N \cap \Gamma \cong \Gamma_1 \times Q$  follows from that of  $g\Gamma_{b,1}g^{-1} \times g\Gamma_{b,2}g^{-1}$ :  $Q$  is a subgroup of finite index in  $g\Gamma_{b,2}g^{-1}$ , and giving the injection  $\Gamma_1 \times Q \hookrightarrow g\Gamma_{b,1}g^{-1} \times g\Gamma_{b,2}g^{-1}$  is equivalent to giving the injection  $Q \hookrightarrow g\Gamma_{b,2}g^{-1}$ .  $\square$

It may well be that  $N$  is in fact  $\Gamma$ -integral if and only if  $X_{\Gamma_N}$  is a product, but I have no argument for this. At any rate, this can now be applied to derive the moduli interpretation of  $X_{\Gamma'_N}$  for  $N$   $\Gamma$ -integral.

Applying the two lemmas above again in the situation that  $G$  corresponds to a PEL-structure yields the following.

**Theorem 6.22** *Let  $G$ ,  $S$ ,  $P_b$ ,  $N_b$  and  $\Gamma$  be as above ( $b < t$ ),  $\Gamma' \in G(\mathbb{Q})$  arithmetic, and let  $X_{\Gamma'_N}$  be a modular subvariety of  $X_{\Gamma'}$  for  $N$  rational symmetric, conjugate to  $N_b$ . Then  $X_{\Gamma'_N}$  is a finite quotient of a product, and the set of  $\Gamma'$ -equivalence classes of such modular subvarieties forms a locus in  $X_{\Gamma'}$  where the corresponding abelian varieties are isogenous to products, i.e., are not simple. If  $N$  is  $\Gamma$ -integral, then  $X_{\Gamma'_N}$  is a product, and the set of  $\Gamma'$ -equivalence classes of such modular subvarieties forms a locus in  $X_{\Gamma'}$  where the corresponding abelian varieties split while preserving the endomorphisms.*

**Proof:** The first statement follows immediately from Lemma 6.20. By Lemma 6.21, if  $N$  is  $\Gamma$ -integral, the discrete subgroup  $\Gamma'_N$  is a product, hence so is the quotient  $X_{\Gamma'_N}$ , giving the second assertion. We know by the discussion above the moduli interpretation upstairs in  $\mathcal{D}$ , given in Proposition 6.19. Since  $X_{\Gamma'_N}$  itself is a product, it follows that the abelian varieties  $A^n$  also split as  $A^n \cong A^q \times A^{n-q}$ , where  $\tau_q$ , the modulus of  $A^q$ , defines a point in one of the factors of  $X_{\Gamma'_N}$ , while  $\tau_{n-q}$ , the modulus of  $A^{n-q}$ , defines a point in the second factor. That the endomorphisms are preserved was shown above in the proof of 6.19.  $\square$

We leave it to the reader to derive the correct result for  $b = t$ .

Finally we briefly mention the moduli interpretation of incidence. For this, recall that one has the Satake compactification and the (smooth projective) toroidal compactifications. The former relate to degenerations of the abelian varieties as follows. A quasi-abelian variety  $A'$  is an extension of an abelian variety by an algebraic torus

$$1 \longrightarrow (\mathbb{C}^*)^h \longrightarrow A' \longrightarrow B \longrightarrow 0. \quad (68)$$

Thus  $A'$  is still an abelian group. Let  $c$  denote the dimension of the abelian variety  $B$ ,  $n = h + c$  the dimension of  $A'$ . We now suppose that  $X_\Gamma$  is a moduli space of PEL structures, and assume the notations used above in this case. Let  $F_b$  be a standard boundary component of the domain  $\mathcal{D}$ , and  $W_i$  a boundary variety which is covered by  $F_b$ . Let  $n = \dim(A)$  for the abelian varieties parameterized by  $X_\Gamma$ ,  $m = \dim_D(V)$  so that  $n = mg$ ,  $g = f, 4f$  and  $2d^2f$  in the respective cases. Since  $F_b$  has rank

$b$ , it corresponds to a vector subspace  $W \subset V$  with  $\dim_D(W) = \dim_D(V) - b = m - b$  ( $b = 1, \dots, s$  = Witt index of the form,  $s \leq [\frac{m}{2}]$ ), and hence to abelian varieties  $B$  with  $\dim(B) = (m - b)g$  and  $g$  as above. We observe that the sequence (68) is relevant here, with  $h = bg$ ,  $c = (m - b)g$ . An extension as in (68) is far from being unique, and the precise degenerations have been constructed in many cases by utilizing methods from the theory of mixed variations of Hodge structures, and this can be brought into relation to the toroidal compactifications mentioned above, where the parameter spaces of the degenerations are divisors on  $X_\Gamma$ . For our purposes (68) is sufficient. We now consider an integral modular subvariety  $X_{\Gamma_N}$  incident with a boundary variety  $W_i$ . As we have seen above, the abelian varieties parameterized by  $X_{\Gamma_N}$  split, in this case as

$$0 \longrightarrow A^h \longrightarrow A' \longrightarrow A^c \longrightarrow 0, \quad (69)$$

and the relation to (68) is obvious; the boundary varieties are the loci in  $X_{\Gamma_N}$  where the  $A^h$  of (69) totally degenerate.

## Literatur

- [Ba] W. Baily, *An exceptional arithmetic group and its Eisenstein series*, Ann. of Math. **91** (1970), 512-549.
- [BB] W. Baily & A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. **84** (1966), 442-528.
- [BS] F. v. d. Blij, T. Springer, *Arithmetic of octaves and groups of type  $\mathbf{G}_2$* , Indag. Math. **21** (1959), 406-418.
- [BHC] A. Borel & S. Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. **75** (1962), 485-535.
- [Bo1] N. Bourbaki, “Éléments de mathématique, Groupes et algèbres de Lie”, Hermann: Paris 1968 & 1975.
- [Dr] W. Drucker, “Exceptional Lie algebras and the structure of hermitian symmetric spaces”, Mem. AMS, **208**, (1978).
- [FF] J. R. Faulkner & J. C. Ferrar, *Exceptional algebras and related algebraic and geometric structures*, Bull. Lond. Math. Soc. **9** (1977), 1-35.
- [F1] J. C. Ferrar, *Lie algebras of type  $E_6$* , J. of Alg. **13** (1969), 57-72.
- [F2] J. C. Ferrar, *Lie algebras of type  $E_6$ , II*, J. of Alg. **52** (1978), 201-209.
- [F3] J. C. Ferrar, *Algebras of type  $E_7$  over number fields*, J. of Alg. **39** (1976), 15-25.
- [H] S. Helgason, “Differential geometry, Lie groups and symmetric spaces,” Academic Press: New York 1978.
- [H1] B. Hunt, *Hyperbolic planes*, preprint, alggeom/9504001.
- [H2] B. Hunt, *Symmetric subgroups of rational groups of hermitian type*, preprint, alggeom / 9505033.

- [H3] B. Hunt, “The geometry of arithmetic quotients of bounded symmetric domains”, Habilitationsarbeit, Kaiserslautern, 1995.
- [I] S. Ihara, *Holomorphic embeddings of symmetric domains*, J. Math. Soc. Japan **19** (1967), 261-302.
- [Ja] N. Jacobson, “Exceptional Lie algebras”, Marcel Dekker, Inc.: New York 1971.
- [Jc] N. Jacobson, “Representations of Jordan algebras”, AMS Coll. Lectures, XXVII, AMS Rhode Island, 1974.
- [K] D. Kazdan, *On arithmetic varieties II*, Isreal J. Math. **44** (1983), 139-159.
- [O] Tadao Oda, “Convex bodies and algebraic geometry,” Ergebnisse der Mathematik und ihre Grenzgebiete, 3.Folge **15**, Springer: Berlin 1988.
- [R] I. Reiner, “Maximal orders”, Academic Press: New York 1969.
- [Ra] M. Racine, *Arithmetics of quadratic Jordan algebras*, Memoirs. AMS, **136** (1973).
- [S] I. Satake, “Arithmetic structures of symmetric domains,” Publ. of Math. Soc. Japan **14**, Iwanami Shoten Publ.: Tokyo 1980.
- [S1] I. Satake, *Holomorphic embeddings of symmetric domains into a Siegel space*, Amer. J. Math. **87** (1965), 425-461.
- [S2] I. Satake, *On some properties of holomorphic imbeddings of symmetric domains*, Amer. J. Math. **91** (1969), 289-305.
- [Sch] R. Schafer, “Introduction to nonassociative algebras,” Academic Press: New York 1966.
- [Sh2] G. Shimura, *On analytic families of polarized abelian varieties and automorphic functions*, Ann. of Math. **78** (1963), 149-192.
- [Sh3] G. Shimura, *Moduli and fibre systems of abelian varieties*, Ann. of Math. **83** (1966), 294-338.
- [Sh4] G. Shimura, *Moduli of abelian varieties and number theory*, in “Algebraic groups and discontinuous subgroups”, Proc. Symp. in Pure Math. **9**, AMS, Providence, Rhode Island 1966.
- [SC] A. Ash, D. Mumford, M. Rapoport, Y. Tai, “Smooth compactification of locally symmetric spaces,” Math. Sci. Press: Brookline 1975.
- [T1] J. Tits, *Classification of simple algebraic groups*, In: Algebraic groups and discontinuous subgroups, Proc. Symp. Pure Math. Vol. **9**, AMS: Providence 1966.
- [W] A. Weil, *Algebras with involution and semisimple groups*, J. Indian Math. Soc. **24** (1960), 589-623.