hep–th/9512138
NSF–ITP–95–158
BONN–TH–95–22

Heterotic Gauge Structure of Type II K3 Fibrations

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Abstract

We show that certain classes of K3 fibered Calabi-Yau manifolds derive from orbifolds of global products of K3 surfaces and particular types of curves. This observation explains why the gauge groups of the heterotic duals are determined by the structure of a single K3 surface and provides the dual heterotic picture of conifold transitions between K3 fibrations. Abstracting our construction from the special case of K3 hypersurfaces to general K3 manifolds with an appropriate automorphism, we show how to construct Calabi-Yau threefold duals for heterotic theories with arbitrary gauge groups. This generalization reveals that the previous limit on the Euler number of Calabi-Yau manifolds is an artifact of the restriction to the framework of hypersurfaces.

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1. Introduction

It has been recognized recently that the agreement found in [1][2][3] between the perturbative structure of the prepotentials for a number of heterotic string K3$\times$T^2 vacua and certain type II Calabi–Yau backgrounds can be traced back to the K3-fiber structure of the models considered [4][5][6]. Evidence based on the analysis of the weak coupling form of the prepotential however is not convincing in the light of recent discussions [7, 8, 9] from which one learns that moduli spaces of different Calabi–Yau manifolds intersect in certain submanifolds. Thus weak coupling arguments would appear insufficient to identify heterotic duals[1]. This makes it particularly important to develop different tools for identifying heterotic and type II vacua which do not rely on a comparison of the perturbative couplings.

In the present paper we wish to describe a way to identify heterotic duals directly from the structure of the K3 fibrations and vice versa. Instead of analyzing the prepotentials we focus on the detailed geometry of K3 fibered Calabi–Yau manifolds which turns out to contain sufficient information to derive the heterotic gauge structure. The basic observation is that the manifolds which have been encountered so far in the context of heterotic/type II duality can in fact be described as orbifolds of product manifolds defined by a K3 surface and an appropriately defined curve. This shows that the essential information of the fibration is determined by a single K3 surface and thus provides an explanation of the fact that the gauge groups of the heterotic duals of K3 fibered Calabi-Yau spaces are determined by the singularity structure of K3 manifolds[2]. Our construction then identifies the heterotic gauge group of these theories as the invariant part of the Picard lattice of the K3 fiber with respect to the group action which gives rise to the fibration. We will also see that the combination of this result with the conifold transitions between K3 fibered Calabi-Yau manifolds introduced in [12], and the analysis of the origin of the gauge group in D=6 theories presented in [11], provides complete control of the dual heterotic picture of the conifold transition on the Calabi-Yau manifold.

We can then turn around this observation and start from abstractly defined orbifolds in which the fibers are not described by some weighted hypersurface, or complete intersection, as has been assumed in the most general class of fibrations presented so far. This will allow us to construct Calabi–Yau manifolds for arbitrary gauge groups. It turns out that the theory is most easily understood in terms of elliptic fibrations of K3 surfaces in which the generic fiber is a torus. There are finitely many singular fibers, which were classified by Kodaira in the sixties. This classification is related to the classification of the simple rational doublepoints, and through this to the classification of the simple Lie algebras. From our present standpoint we can see another reason for this coincidence – we will find K3 surfaces with elliptic fibrations (elliptic K3's) which can be used to construct Calabi-Yau threefolds with K3 fibrations, which correspond dually to heterotic strings. Since the unbroken gauge group of the heterotic string is $E_8 \times E_8$, the broken gauge group is a subgroup of this. The correspondence between the singular fibers of the K3 and the lattice of the gauge group of the heterotic string then dictates that the possible singular

[1] A detailed investigation of this problem will appear in ref. [10].

[2] The relation in the context of N=4 theories between the ADE singularities of K3 and ADE gauge groups of the heterotic dual on the torus has been explicated in [11]. Using the adiabatic limit it was argued in [5] that this relation carries over to the N=2 framework.

fibers must also mirror the classification of the gauge group.

A further result we find along the way concerns the possible limits on the Hodge numbers of Calabi–Yau manifolds, i.e. the number of vector multiplets and hypermultiplets of the heterotic theory. Applying our construction to other K3's than the weighted hypersurfaces we find that the limit found in the context of weighted hypersurfaces is not in fact a characteristic of Calabi–Yau manifolds per se. In [13] weighted hypersurfaces $\mathbb{P}_{(1,1,12,28,84)}[84]$ and its mirror[3] $\mathbb{P}_{(11,41,42,498,1162,1743)}[3486]$, have been constructed which define the 'boundaries' of the mirror plot of [13] with the largest absolute value of the Euler number, namely 960. This value has proved to be rather robust over the years, turning up in different constructions, such as various classes of exactly solvable string vacua based on minimal N=2 [15] and Kazama-Suzuki models [16], the class of all Landau–Ginzburg theories [17], their abelian orbifolds [18], abelian orbifolds with torsion [19], as well as toric constructions [20, 21]. The fact that the space $\mathbb{P}_{(1,1,12,28,84)}[84]$ has the maximal Euler number in the class of Calabi–Yau hypersurfaces can be traced back to the fact that the smooth fiber of the K3 fibration has an automorphism of order 42. Results of Nikulin [22] show that this is not the highest possible value. There are in fact two higher values for the order of a K3 surface and we will discuss both examples in this paper. Applying our construction to them we will find Calabi-Yau threefolds with Euler number smaller than –960. This shows that the structure of the mirror plot of [13] is an artifact of the construction after all.

2. A Class of K3 fibered Calabi-Yau Manifolds

Before describing the geometry of general K3 fibrations more abstractly we explain the essential ingredients in the more familiar framework of weighted complete intersection manifolds. This will allow us to be fairly concrete. Thus consider the class of fibrations

$$\mathbb{P}_{(2k_1-1,2k_1-1,2k_2,2k_3,2k_4)}[2k] \sim \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{((2k_1-1),(2k_1-1),k_2,k_3,k_4)} \end{matrix} \begin{bmatrix} 2 & 0 \\ (2k_1-1) & k \end{bmatrix}, \qquad (1)$$

with $k = (2k_1 + k_2 + k_3 + k_4 - 1)$ and $2k/(2k_1 - 1) \in 2\mathbb{N}$, described in [12][4]. The relations (1) which can be established via Landau–Ginzburg considerations utilizing fractional transformations [14, 12], are useful for the analysis of the Yukawa structure of heterotic/type II theories.

In the following we will also be needing information about the the structure of the second cohomology group[5]. The Kähler sector of the theory receives contributions from two different sources. First there is the (1,1)–form which comes from the restriction of the Kähler form of the ambient space. Next there is a fixed $\mathbb{Z}_2$–curve

$$\mathbb{Z}_2: \; C = \mathbb{P}_{(k_2,k_3,k_4)}[k] = \{p(z_2, z_3, z_4) = 0\}, \qquad (2)$$

[3]It can be shown using the mirror transform of [14, 12] that these manifolds are indeed mirrors.

[4]As emphasized in [12], the assumption $2k/(2k_1 - 1) \in 2\mathbb{N}$ is merely a matter of convenience and dropping this condition merely complicates the discussion, as explicated with a concrete example. Recently a heterotic/type II dual pair based on such a more general space with $k_1 = 2$, $k_2 = 4$, $k_3 = 10$ and $k_4 = 7$ has been discussed in detail in [23].

[5]The necessary ingredients for the following remarks can be found for example in [24].

the resolution of which adds one further contribution to the Kähler sector. This curve lives in a weighted projective plane, whose resolution introduces a further number N_C of (1,1)–forms, depending on the relative divisibility properties of the weights of the curve as well as the type of its defining polynomial. Finally there is the $\mathbb{Z}_{(2k_1-1)}$ fixed point set. The precise structure of this set of singularities depends on the divisibility properties of $(2k_1-1)$ relative to the rest of the weights. To be concrete we will present our discussion assuming $\gcd(2k_1-1, 2k_i) = 1$, $i = 2, 3, 4$ (other situations being described by modifications which can be easily derived using the information contained in [24])[6]. In such a situation the singular set is

$$\mathbb{Z}_{2k_1-1}: \quad \mathbb{P}_1[2k/(2k_1-1)] = \frac{2k}{2k_1-1} \text{ pts}, \tag{3}$$

the resolution of which leads to an additional $(2k(k_1-1)/(2k_1-1))$ (1,1)–forms. Thus we find a total of

$$h^{(1,1)} = 1 + \frac{(k_1-1)2k}{2k_1-1} + 1 + N_C \tag{4}$$

(1,1)–forms.

The generic fiber is an element in the K3 configuration

$$K_\lambda = \mathbb{P}_{(2k_1-1,k_2,k_3,k_4)}[k] \ni \{(1+\lambda^{2k/(2k_1-1)})z_1^{k/(2k_1-1)} + p(z_2,z_3,z_4) = 0\}, \tag{5}$$

which can be chosen to describe a quasismooth surface everywhere on the base $\mathbb{P}_1$ except at the $2k/(2k_1-1)$ points λ_i which solve $(1+\lambda_i^{2k/(2k_1-1)}) = 0$. Over these points the fibers degenerate. Important for the following will be detailed structure of these degenerations. One of essential features of the class of manifolds (1) is that the structure of the fibers does not change as one moves in the fiber, they are of constant modulus. They do change rather drastically however when one hits one of the $2k/(2k_1-1)$ base points λ_i on $\mathbb{P}_1$ at which the fibers degenerate. At these points the coordinate z_1 is completely unrestricted and the degenerate fibers are cones over the curve C embedded in the generic fiber K.

The discussion so far suffices to derive the Euler number of the spaces (1) purely in terms of the fiber structure, a result which we will have use for later on. The necessary ingredients of this computation are the Euler number of the base, $\chi(\mathbb{P}_1) = 2$, the Euler number of generic fiber, $\chi(K3) = 24$, the number $(2k/(2k_1-1))$ of singular base points, and the Euler number of the degenerate fiber. The structure of the singular fibers depends crucially on whether k_1 is equal or larger than unity. If $k_1 > 1$ there is the additional complication that each vertex of the cone over C is a $\mathbb{Z}_{2k_1-1}$–singular point on the Calabi-Yau manifold whose resolution introduces (k_1-1) spheres $\mathbb{P}_1$. Thus the Euler number of the degenerate fibers $F_{\rm deg}$ is given by

$$\chi(F_{\rm deg}) = \chi(C) + 2(k_1-1) + 1 \tag{6}$$

and therefore the Euler number of the fibered threefold

$$\chi(M) = (2-N_s)\cdot 24 + N_s(\chi(C) + 2k_1 - 1). \tag{7}$$

[6]In the example $\mathbb{P}_{(3,3,8,20,34)}[68]$ the $\mathbb{Z}_3$–singular set e.g. is just a curve $\mathbb{P}_1$).

The second crucial property of the manifolds (1) is that the monodromy transformation $\mathbf{m}$, generated by

$$\mathbb{Z}_{k/(2k_1-1)} \ni \mathrm{m}: \quad (z_1, ..., z_4) \quad \longrightarrow \quad (\alpha z_1, z_2, z_3, z_4), \tag{8}$$

is nilpotent of degree $k/(2k_1-1)$, i.e. $\mathbf{m}^{k/(2k_1-1)} = 1$.

The structure of the fibrations (1) explicated thus far allows us to draw on some general results of birational geometry[7] in order to get further insight into their structure. Namely, since the monodromy is nilpotent and the modulus is constant, it follows that these manifolds can in fact be described (birationally) as orbifolds of products[8] of the form $\mathcal{C}_{k/(2k_1-1)} \times K$, where $p : \mathcal{C}_{k/(2k_1-1)} \longrightarrow \mathbb{P}_1$ is the projection of a $k/(2k_1-1)$–fold cover of the base space of the fibration. In order to see this consider the action of the cyclic group $\mathbb{Z}_{k/(2k_1-1)}$ on the product

$$\mathbb{Z}_{k/(2k_1-1)}: \ \mathcal{C}_{k/(2k_1-1)} \times K \ \longrightarrow \mathcal{C}_{k/(2k_1-1)} \times K \tag{9}$$

defined by the projection p on the first factor and the monodromy action $\mathbf{m}$ (8) on the second factor. The action $\mathbf{m}$ leaves invariant the curve C and therefore the orbifold $\mathbb{Z}_{k/(2k_1-1)}\backslash\mathcal{C}_{k/(2k_1-1)}\times K$ will have $2k/(2k_1-1)$ singular fibers which are obtained by blowing up the curve C in the fiber K_{λ_i} (λ_i being any of the $2k/(2k_1-1)$ branch points of $\mathcal{C}_{k/(2k_1-1)}$), resulting in a ruled surface[9] E_i. The surface K_{λ_i} has as quotient the weighted projective plane $\mathbb{P}_{(k_2,k_3,k_4)}$ while each E_i descends to the orbit space (being fixed under the action of $\mathbb{Z}_{k/(2k_1-1)}$), and is in the branch locus there. Thus on the resolved orbit space each singular fiber over a $k/(2k_1-1))^{st}$ root of unity consists of two components, a plane $\mathbb{P}_{(k_2,k_3,k_4)}$ and a ruled surface E_i, the two intersecting in the curve C. Because C is just a hyperplane section of the original fiber K_{λ_i}, it follows that the weighted projective plane can be blown down to a point. In this process the intersection curve C is blown down to a point as well and the surface E_i becomes a cone over the curve C. This is precisely the structure we have previously found for the manifolds (1) and thus we have uncovered that the essential structure of the weighted hypersurfaces (1) is that of orbifolds of a global product involving K3 surfaces.

This may be described more explicitly as follows. Let $\pi : \mathcal{C}_{k/(2k_1-1)} \longrightarrow \mathbb{P}_1$ denote the projection. We may set $z_1 = y_1^{1/2}$ here because all weights are divisible by 2. Then define (with $\ell = k/(2k_1-1)$)

$$\begin{aligned} \phi : \mathbb{Z}_\ell\backslash\mathcal{C} \times K \quad &\hookrightarrow \quad \mathbb{P}_{(2k_1-1,2k_1-1,2k_2,2k_3,2k_4)} \\ (x_0, (x_1, z_2 \ldots, z_4)) \bmod \mathbb{Z}_\ell \quad &\mapsto \quad \left(\frac{\pi(x_0)}{(1+\pi(x_0)^{2\ell})^{1/2\ell}} x_1^{1/2}, \frac{1}{(1+\pi(x_0)^{2\ell})^{1/2\ell}} x_1^{1/2}, z_2, z_3, z_4\right). \end{aligned} \tag{10}$$

[7]For a single nilpotent fiber, by definition, the monodromy satisfies $\mathbf{m}^s = 1$, hence a ramified cover of the base, branched to order s at the base point of the singular fiber, has trivial monodromy. For a global fiber space with nilpotent monodromy, the same holds for a suitably chosen ramified cover of the base.

[8]On the ramified cover of the last footnote, the monodromy is trivial, and the modulus is constant; this means we have a product.

[9]A ruled surface is a fibration over a curve (here C) with fiber $\mathbb{P}_1$.

Here the map is defined for $\pi(x_0)^{2\ell} \neq -1$; we then take the closure in the projective space. Then clearly $\phi(x_0, K)$ is the hyperplane section $z_1 = \pi(x_0)z_2$ of

$$\{f(z_1, \ldots, z_5) = z_1^{2k/(2k_1-1)} + z_2^{2k/(2k_1-1)} + p(z_2, z_3, z_4) = 0\} \subset \mathbb{P}_{(2k_1-1,2k_1-1,2k_2,2k_3,2k_4)}. \qquad (11)$$

It is well defined on the quotient because only $\pi(x_0)$ occurs, and for $\pi(x_0)^{2\ell} = -1$ the image is the cone

$$C = \{p(z_2, z_3, z_4) = 0\} \subset \{z_1 = \pi(x_0)z_2\} \subset \mathbb{P}_{(2k_1-1,2k_1-1,2k_2,2k_3,2k_4)}. \qquad (12)$$

In this way the birational map described above is immediately performed and in particular one can determine whether the vertex of the cone $H_{\pi(x_0)}$ for $\pi(x_0)^{2m_1} = -1$, namely the point $(\pi(x_0), 1, 0, 0, 0)$ is a singular point of the threefold. Looking at the equation, it is clear that the vertex is a quasi-smooth point of the threefold, and in particular, if $k_1 = 1$, it is a smooth point, while if $k_1 > 1$, it is a singular point of the ambient projective space and must be resolved.

Combining the results of the previous paragraphs shows that the cohomology of the generic fiber of the fibration

$$\begin{array}{ccc} \mathbb{P}_{(2k_1-1,k_2,k_3,k_4)}[k] & \longrightarrow & \mathbb{P}_{(2k_1-1,2k_1-1,2k_2,2k_3,2k_4)}[2k] \\ & & \downarrow \\ & & \mathbb{P}_1 \end{array} \qquad (13)$$

receives contributions from both, the resolution of the orbifold singularities as well as those forms spanned by the polynomial ring. More precisely, the subgroup $\mathrm{H}^{(1,1)}(K) \subset \mathrm{H}^2(K)$, which is 20-dimensional for a K3 manifold, is spanned by the Kähler form of the ambient space of the fiber configuration $\mathbb{P}_{(2k_1-1,k_2,k_3,k_4)}[k]$, the contribution N_C of the resolution of the singular points lying on the curve C, and the complex deformations. The group $\mathrm{H}^2(K)$ is endowed with a natural inner product given by $< \omega, \eta >= \int \omega \wedge \eta$ for $\omega, \eta \in \mathrm{H}^2(K)$, the signature of which is (3,19), with 3–dimensional positive definite subspace.

We now see that the structure of the second cohomology group of any of the spaces of type (1) is determined by a single K3 hypersurface and the action of the automorphism. We thus have reduced the problem of deriving the heterotic gauge structure to the problem of deriving it from the K3. This is easily achieved by considering the invariant part of the Picard lattice with respect to the action which defines the fibrations. We will illustrate this with some examples in the next Section.

3. Hypersurface Examples

Example I: Consider the manifold $\mathbb{P}_{(1,1,12,28,42)}[84]$ which has been playing a somewhat distinguished role over the last five years because it, together with its mirror, features the to date largest known absolute value of Euler number, defining the extreme boundary of the mirror plot of [13]. The candidate heterotic dual has been discussed in [1, 25, 26, 6]. This manifold has the $\mathbb{Z}_2$–singular curve $C = \mathbb{P}_{(6,14,21)}[42]$, on top of which we have the orbifold ponts

$\mathbb{Z}_4 : \mathbb{P}_{(3,7)}[21] = 1\ pt$, $\mathbb{Z}_6 : \mathbb{P}_{(2,7)}[14] = 1\ pt$, and $\mathbb{Z}_{14} : \mathbb{P}_{(2,3)}[6] = 1\ pt$. The resolution of these points introduces one, two, and nine (1,1)–forms respectively and therefore we find, including the one (1,1)–form coming down from the ambient form, that the smooth resolved space has a total of $h^{(1,1)} = 11$ (1,1)–forms. Using $c_3 = -164978h^3$ one finds for the Euler number $\chi = -960$.

Considering the K3 fibration of this space then shows that the second cohomology of the generic fiber

$$\mathbb{P}_{(1,6,14,21)}[42] \ni K_\lambda = \{(1+\lambda^{84})y^{42} + z_3^7 + z_4^3 + z_5^2 = 0\} \tag{14}$$

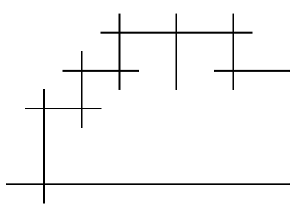
decomposes as $20 = 1+9+10$, the first form coming from the ambient Kähler form of $\mathbb{P}_{(1,6,14,21)}$, the nine forms decomposing into $9 = 1+2+6$ resolution modes, and finally the 10 monomials

$$z_1^{28-6n} z_2^n z_3, \ 0 \le n \le 4; \qquad z_1^{42-6m} z_2^m, \ 1 \le m \le 5. \tag{15}$$

We therefore see that the manifold $\mathbb{P}_{(1,1,12,28,42)}[84]$ arises by choosing $K = \mathbb{P}_{(1,6,14,21)}[42]$ and considering the action of $\mathbb{Z}_{42}$ on the product $\mathcal{C}_{42} \times K$. The quotient $\mathbb{Z}_{42} \backslash \mathcal{C}_{42} \times K$ has 84 singular fibers obtained by first blowing up the curve C in the fiber K_{λ_i} and then blowing down the projective plane to a point. The resulting degenerate fibers are cones over the curve C with $\chi_C = 11$ and plugging the values of $N_C = 84$ and χ_C into our fibration formula (7) reproduces the known result.

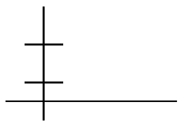
Thus we should look for the invariant part of the Picard lattice of the K3 hypersurface $K = K_0$ (14) with respect to the group $\mathbb{Z}_{42}$. It is generated by the Kähler form descending from the ambient space and the 9 modes coming from the resolution. Since the remaining part of $\mathrm{H}^2(\mathrm{K3})$ transforms nontrivially, as can easily be inferred from the transformation behavior of the monomials (15), the invariant sublattice of the K3 lattice $\Gamma^{(3,19)}$ is determined by the intersection form of the K3 $\Gamma^{(1,9)} \subset \Gamma^{(3,19)}$. This sublattice is a selfdual Lorentzian even lattice and decomposes into $\Gamma^{(1,1)} \oplus \Gamma^{(0,8)}$ in which the second term denotes the root lattice of the E_8.

The geometry of this situation is encoded by the resolution of the curve C. As described above resolving the orbifold singularities of the ambient space leads to a contribution of $N_C = 9$, coming from the resolution of the three singular points sitting on top of the curve. The resolution of each of these points introduces the Hirzebruch-Jung trees described by the diagrams A_1, A_2 and A_6 respectively, and the curve C glues these trees together, resulting in the graph



whose dual describes precisely the lattice $E_8 \oplus U$, where U denotes the hyperbolic plane. Thus we see that the heterotic dual should be determined by higgsing the first E_8 completely while retaining the second E_8. We also see that we should not fix the radii of the torus at some particular symmetric point but instead embed the full gauge bundle structure into the E_8. Doing precisely as instructed by the manifold we recover the heterotic model of [1].

Example II: Consider the manifold $\mathbb{P}_{(1,1,2,4,4)}[12]$ whose Hodge numbers were found [1] to match that of a particular heterotic model. The detailed understanding of this space is of particular interest because it is known to be connected via a conifold transition to a codimension two Calabi–Yau manifold [12]. We therefore wish to see whether we in fact can derive the heterotic theory from this Calabi–Yau manifold. For this we have to determine the invariant sector of the Picard lattice of $\mathrm{H}^2(K3)$ under the orbifolding. In order to do so we only need to observe that the orbifold singularities of the curve $\mathbb{P}_{(1,2,2)}[6]$ are three $\mathbb{Z}_2$–points whose resolution leads to a total of 3 (1,1)–forms which, together with the Kähler form of the ambient space, determines the sublattice $\Gamma^{(1,3)} \subset \Gamma^{(3,19)} = \mathrm{H}^2(\mathrm{K3}, \mathbb{Z})$. Taking into account the divisor coming from the curve C we find the resolution diagram



with $\chi(C) = 3$. We see from this that the intersection matrix is precisely given by the Cartan matrix of the group SO(8). We also see that in the heterotic dual we need to take the torus at the SU(3) point in the moduli space and break this SU(3) by embedding the K3 gauge bundle structure groups appropriately. In this way we recover the heterotic construction of [1].

4. The General Construction

In this Section we abstract from the framework of weighted spaces and describe our construction for general K3 fibers which do not necessarily admit a description as weighted hypersurfaces. For this we make the following assumptions. We are given a smooth K3 surface K with an automorphism group $\mathbb{Z}_k$, such that the fixed point set of the group is a curve on K, that is, there are no isolated fixed points. Then we use the curve $\mathcal{C}_k$ of Section 2, the k-fold cover of $\mathbb{P}_1$ branched at the N roots of unity on $\mathbb{P}_1$, where k divides N. A somewhat involved computation then shows that the quotient $\mathcal{C}_k \times K$ by $\mathbb{Z}_k$ will be (birationally) Calabi-Yau if and only if

$$N = 2k. \tag{16}$$

This is the value we found above, and it is true more generally. Now the quotient space $\mathbb{Z}_k \backslash \mathcal{C}_k \times K$ will be singular at the curve $C \subset K$ which is the fixed point curve of the action on K. This must be blown up, and the exceptional divisor is a ruled surface, i.e., a $\mathbb{P}_1$-bundle over C. After this blow up, the fiber over one of the roots of unity consist in two components: the quotient of the original fiber (that is a copy of K), and this ruled surface. This is depicted in the middle picture of Figure 1. After this blow up, however, the component which is the quotient of K becomes *exceptional*, and can be blown down. This is depicted by the second arrow of Figure 1.

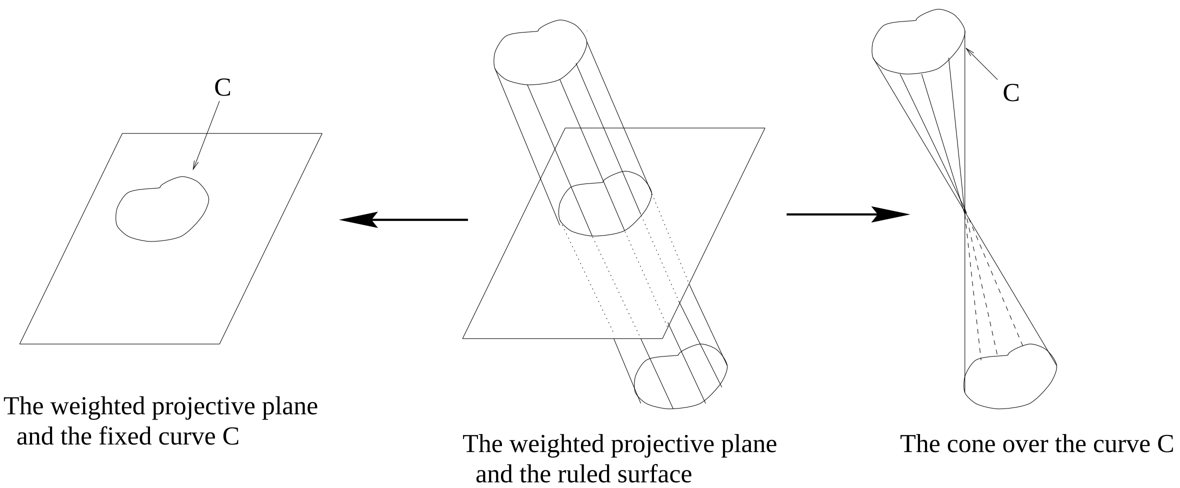


Figure 1: *The left hand arrow is the blow up along the fixed curve C. The right hand arrow is the blow down of the original quotient of the fiber K.*

The result is a cone over the curve C. Of course, the curve C need not be irreducible, and if there are several components, the resulting ruled surface consists of several components. One can still blow up the threefold along the curve C, and the result is a ruled surface over the (reducible) curve C. This is depicted in Figure 2.

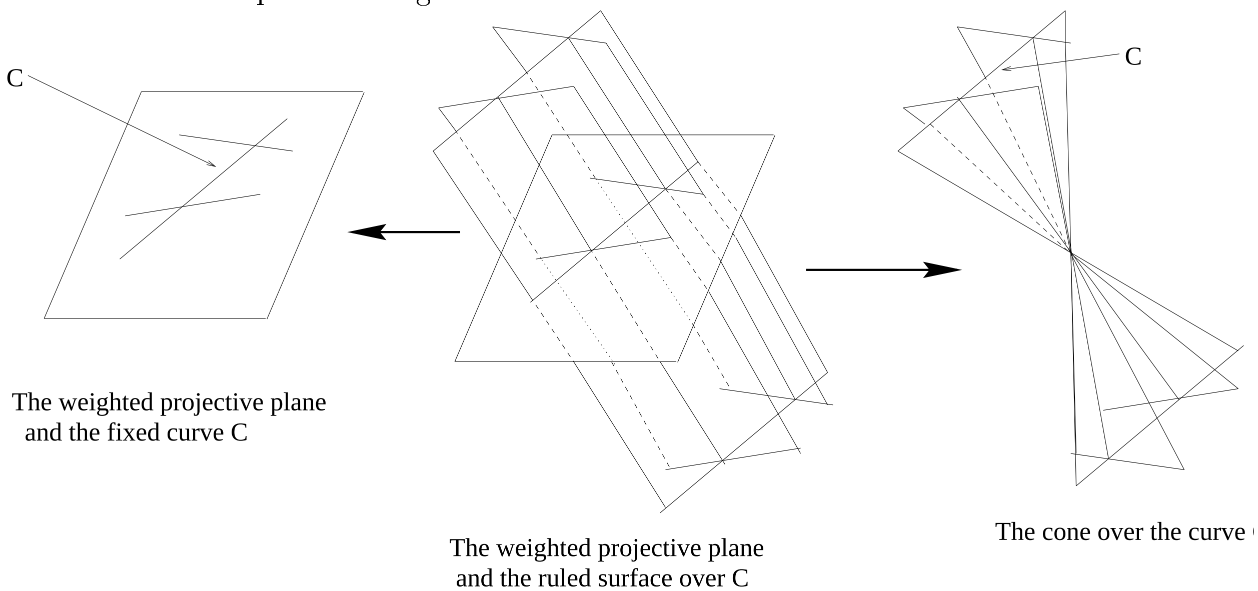


Figure 2: *Here the fixed curve C is reducible. In this case the situation is the same, except that the ruled surface has several components now.*

After this blow down, the vertex of the cone may or may not be a singular point of the threefold. We saw above that for the weighted hypersurfaces with $k_1 > 1$, this was indeed a singular point and needed to be resolved. The reason is that in those cases the ambient projective space is singular at the vertex. That is also the reason for the occurance of the degree k_1 in the χ-formula. For the elliptic surfaces considered below, this problem does not occur.

Finally we mention that this construction, starting from just a smooth K3 with an automorphism, is applicable to *any* such surface, and in particular, can be applied to elliptic K3 surfaces (see below) as well as to weighted complete intersection K3's. This is especially important, as the latter appear in the weighted conifold transitions described in [12], generalizing the splitting conifold construction of [27]. Combining the results of [12] with what we have learnt so far allows us to gain a complete understanding of the dual heterotic gauge structure of the conifold transition. More precisely we need to collect the following ingredients: 1) The fibered Calabi-Yau threefold is completely determined by a single K3 surface. 2) The conifold transition connects a fibration with another fibration. For general conifold transitions this will not be the case, but as shown in [12] there exist conifold transitions for which this holds. It was furthermore shown in [12] that such transitions proceed via a degeneration of the fibers. 3) The singularity structure of the K3 surface determines the dual heterotic gauge group [11]. Combining these facts we see that in conifold transitions between K3-fibered threefolds the resolution graph of the K3 surface changes because of the vanishing and appearance of 2–cycles when the K3 fibers go through the degenerate configuration. Since it is this graph which determines the Dynkin diagram we thus gain an understanding of the heterotic dual of the transition.

5. Automorphisms of K3 Surfaces

Our discussion of the previous Sections shows that we need to understand the automorphisms of K3 surfaces, in particular when group actions by some $\mathbb{Z}_k$ exist. The first observation is that if Ω_K denotes the non-vanishing holomorphic two form on K, then any automorphism g acts via $g^*\Omega_K = \alpha_K(g)\Omega_K$, where $\alpha_K(g) \in \mathbb{C}^*$, yielding an exact sequence

$$1 \longrightarrow G_K \longrightarrow Aut(K) \xrightarrow{\alpha_K} \mathbb{Z}_k \longrightarrow 1, \tag{17}$$

where $\mathbb{Z}_k$ is the cyclic group of k^{th} roots of unity in $\mathbb{C}^*$ and G_K is the kernel, i.e., the set (a group actually) of automorphisms preserving the form Ω_K. This gives a representation of $\mathbb{Z}_k$ in $T_K \otimes \mathbb{Q}$, which is by results of Nikulin the direct sum of irreducibles, of maximal rank $\phi(k)$, where ϕ denotes the Euler function. More precisely, Nikulin's result [22] is that all eigenvalues of $\mathbb{Z}_k$ acting on $T_K \otimes \mathbb{Q}$ are primitive k^{th} roots of unity. Each irreducible component has the maximal possible rank, namely $\phi(k)$. Since $\phi(k) \leq rank(T_K)$, it follows that $k \leq 66$. Particularly interesting are the automorphisms that act trivially on the Picard group. This group, denoted by H_K, is in fact the $\mathbb{Z}_k$ as above (which shows the sequence splits). As a consequence of this we learn that for an element $g \in H_K$, the invariant lattice under g is precisely S_K, the Picard lattice. It is clear from this that our main interest will be in elements of H_K, so it is desirable to know more about what can possibly occur. A first result in this direction was given by Kondo. It was shown in [28] that for unimodular T_K k must be a divisor of any of the values in $S = \{66, 44, 42, 36, 28, 12\}$. Furthermore, if $\phi(k) = \mathrm{rk}(T_K)$ then k takes precisely the values of S, and in these cases there exists a unique (up to isomorphism) K3 surface with given k.

For the examples indicated by Kondo's list S the invariant lattice S_K and its complement T_K are as follows.

k	S_K	T_K
66	U	$U \oplus U \oplus E_8 \oplus E_8$
44	U	$U \oplus U \oplus E_8 \oplus E_8$
42	$U \oplus E_8$	$U \oplus U \oplus E_8$
36	$U \oplus E_8$	$U \oplus U \oplus E_8$
28	$U \oplus E_8$	$U \oplus U \oplus E_8$
12	$U \oplus U \oplus E_8$	$U \oplus U$

The situation for G_K is just the opposite of the one for H_K. The invariant sublattice is this time T_K, and the action on S_K was described for abelian groups G_K by Nikulin. The possible such G_K which can occur are the following:

$$(\mathbb{Z}_2)^m,\ 0 \leq m \leq 4;\quad \mathbb{Z}_4;\quad \mathbb{Z}_2 \times \mathbb{Z}_4;\quad (\mathbb{Z}_4)^2;\quad \mathbb{Z}_8;\quad \mathbb{Z}_3;\quad (\mathbb{Z}_3)^2;\quad \mathbb{Z}_5;\quad \mathbb{Z}_7;\quad \mathbb{Z}_6;\quad \mathbb{Z}_2 \times \mathbb{Z}_6.$$

Since the largest cyclic group occuring is $\mathbb{Z}_8$ it follows that if a K3 surface admits a cyclic automorphism of order $k \geq 9$, then this automorphism is in H_K. Therefore, depending on our aims, it may be more useful to consider H_K or G_K. Note that by mirror symmetry (which for K3 surfaces is a theorem), there is another K3 surface K^m for which T_K and S_K are exchanged. Consequently H_K and G_K are switched also.

We now apply these results to our construction. Let K be a K3 surface with an automorphism of order k in H_K, and let S_K be the Picard lattice. Then with our construction above, we have: a Calabi-Yau threefold Y with a K3 fibration with S_K is the invariant part of the lattice. Hence the data $(K, \mathbb{Z}_k = H_K, S_K)$ determines a Calabi-Yau, dual to a heterotic string with gauge group lattice which is isomorphic to S_K. In other words, to identify the invariant lattice, it is sufficient to determine the Picard lattice S_K of K. Note that, to get a K3 surface with a given S_K, it is sometimes sufficient to give a combination of singular fibers and an elliptic K3 surface with those singular fibers. Let us give a brief description of this class of K3 surfaces. An elliptic curve can always be realized as a cubic in $\mathbb{P}_2$. To get an elliptic surface, one lets this cubic curve in the plane vary. This is described by an affine equation of the type

$$y^2 = x^3 - g_2(t)x - g_3(t), \tag{18}$$

where $g_i(t)$ is a section of a line bundle $L^{\otimes 2i}$ on some curve C (the base curve of the projection of the surface $S \longrightarrow C$). Here x and y are affine coordinates in a $\mathbb{P}_2$, and the entire surface is contained in a $\mathbb{P}_2$-bundle over C. If S is a K3 surface, then necessarily $C = \mathbb{P}_1$, and the sections $g_i(t)$ are just homogeneous polynomials of degrees $2i \cdot deg(L)$, and for S to be K3 again we need $deg(L) = 2$. The fiber over a point $t \in \mathbb{P}^1$ will be singular precisely when the discriminant of the Weierstraß polynomial vanishes there, $\Delta(t) := g_2(t)^3 - 27g_3(t)^2 = 0$. The type of singular fiber is completely determined by the degrees of vanishing of g_2, g_3 and Δ at the point, according to

Table.

Fiber	$\mathbf{I}_0$	$\mathbf{I}_n, n>0$	**II**	**III**	**IV**	$\mathbf{I}_0^*$	$\mathbf{I}_n^*, n>0$	$\mathbf{II}^*$	$\mathbf{III}^*$	$\mathbf{IV}^*$
$\nu(g_2)$	0	0	≥ 1	1	≥ 2	$2\ />2\ /2$	2	≥ 4	3	≥ 3
$\nu(g_3)$	0	0	1	≥ 2	2	$>3\ /3\ /3$	3	5	≥ 5	4
$\nu(\Delta)$	0	n	2	3	4	6	$n+6$	10	9	8
$\mathcal{J}$	$\neq 0,1,\infty$	order n pole	0	1	0	$1\ /0\ /\neq 0,1,\infty$	order n pole	0	1	0

Table: *In the last row we have listed the value of the $\mathcal{J}-$function, which is defined by $\mathcal{J}=g_2^3/\Delta$. If $\mathcal{J}$ is constant, then the modulus of the elliptic curve is constant in the family. The fibers are as shown in Figure 3.*

Looking at the table, the following is clear. If we consider the dual graph of each fiber type (with the exception of **II**), then we get an extended Dynkin diagram of one of the simple Lie algebras. The correspondence is given as follows:

II	**III**	**IV**	$\mathbf{I}_n$	$\mathbf{I}_n^*$	$\mathbf{IV}^*$	$\mathbf{III}^*$	$\mathbf{II}^*$
$-$	A_1	A_2	A_{n-1}	D_{n+4}	E_6	E_7	E_8

(see [28]). In this way it is often possible to see what the lattice S_K of such an elliptic surface is. More precisely, if there is a unique section, then the Picard lattice S_K can be read off directly from the singular fibers. This is the situation with all the examples of Kondo.

We now describe the three examples of Kondo which we shall use later. These are all elliptic fibrations.

$k=66$: There are two ways to construct this surface: Kondo describes it as an elliptic K3, with 12 fibers of type **II** at $t=0$ and at the 11^{th} roots of unity. The affine equation is

$$y^2 = x^3 + t\prod_1^{11}(t-\alpha_{11}^i), \tag{19}$$

and the automorphism is given by $(x,y,t)\mapsto(\alpha_{66}^2 x, \alpha_{66}^3 y, \alpha_{66}^6 t)$. Alternatively, one can consider the (non-Gorenstein) weighted hypersurface

$$\{x^2+y^3+z^{11}+w^{66}=0\}\in\mathbb{P}_{(1,6,22,33)}[66], \tag{20}$$

which, upon resolution, yields a smooth K3 [29]. Here the automorphism is given by $(x,y,z,w)\mapsto(x,y,z,\alpha_{66}w)$. From this second description we see that the fixed point set is the (total transform of the) curve $\{x^2+y^3+z^{11}=0\}\subset\mathbb{P}_{(6,22,33)}$. In the above description that curve is given by at most the fibers $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$ as 0 and ∞ are the only fixed points of t, together with the zero section, the locus (in $\mathbb{P}_1\times\mathbb{P}_2$) given by setting $x=0$ and $y=0$. It should be noted that the fiber $\pi^{-1}(\infty)$ is smooth, hence the group acts on it, and does not fix it. Hence the fix point set is $\pi^{-1}(0)\cup\sigma_0$, where σ_0 denotes the zero section. This will be used below.

$k=44$: There is a Weierstraß equation

$$y^2 = x^3 + x + t^{11} \tag{21}$$

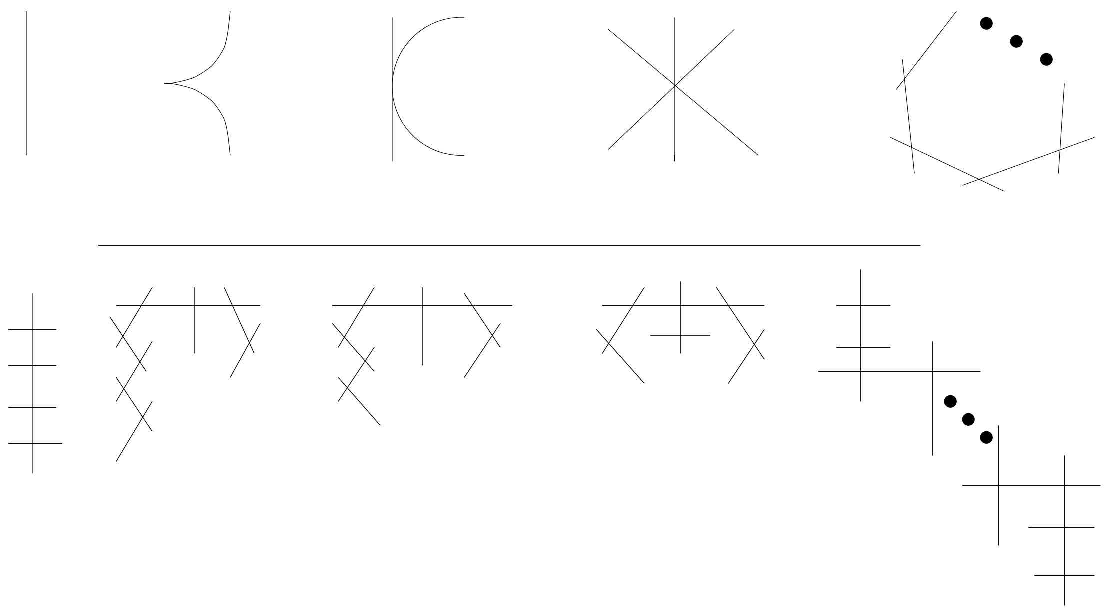


Figure 3: *In the first row are the fibers of types* $\mathbf{I}_0$, $\mathbf{II}$, $\mathbf{III}$, $\mathbf{IV}$ *and* $\mathbf{I}_n$*, respectively. In the second row are the types* $\mathbf{I}_0^*$, $\mathbf{II}^*$, $\mathbf{III}^*$, $\mathbf{IV}^*$ *and* $\mathbf{I}_n^*$*, respectively. The fibers of the second row are the minimal resolutions of the quotients of the fibers of the first row by the involution* $z \mapsto -z$ *of the elliptic curve.*

and the automorphism is given by $(x, y, t) \mapsto (\alpha_{44}^{22}x, \alpha_{44}^{11}y, \alpha_{44}^{2}t)$. This elliptic fibration has a singular fiber of type $\mathbf{II}$ over $t = \infty$ and 22 fibers of type I_1, over the roots of $t^{22} = -4/27$.

$k = 42$: Here we again have a description as an elliptic surface,

$$y^2 = x^3 + t^5 \prod_1^7 (t - \alpha_7^i) \tag{22}$$

and the automorphism is given by $(x, y, t) \mapsto (\alpha_{42}^{2}x, \alpha_{42}^{3}y, \alpha_{42}^{18}t)$. It has a fiber of type $\mathbf{II}^*$ at $t = 0$, and fibers of type $\mathbf{II}$ at all seventh roots of unity. This example may also be described as a weighted hypersurface:

$$\{x^2 + y^3 + z^7 + w^{42} = 0\} \subset \mathbb{P}_{(1,6,14,21)}. \tag{23}$$

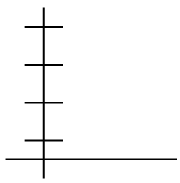
Here the automorphism is given by $(x, y, z, w) \mapsto (x, y, z, \alpha_{42}w)$.

The important point following from these remarks is that we can now pose the problem the other way around: given a gauge group of a heterotic string, we can find a Calabi-Yau with a K3 fibration such that the invariant lattice under the monodromy is the lattice of the given gauge group. More precisely, suppose we can find a K3 surface K, such that (i) it has S_K given by the lattice of the given gauge group, and (ii) it has a non-trivial automorphism group H_K. Then we can apply the construction above, and the result is a Calabi-Yau threefold, fibered in K3 surfaces, such that the invariants under the monodromy are exactly the lattice S_K.

Now let us try to find some interesting lattices which could play the role of gauge groups for hererotic strings. Suppose, for example we are looking for a type IIA string on a Calabi-Yau with gauge group $SO(8)$ and with $(n_v, n_h) = (8, 272)$. First we note that the following combination of singular fibers would do the job: $1\mathbf{I}_0^*$, $9\mathbf{II}$; as mentioned above, this would give a Picard lattice on a K3 $S_K = D_4 \oplus U$. We now construct such an elliptic surface with an automorphism of order 18, as follows. The Weierstraß equation will be

$$y^2 = x^3 + t^3 \prod_1^9 (t - \alpha_9^i), \tag{24}$$

and an automorphism is given by $(x, y, t) \mapsto (\alpha_{18}^2 x, \alpha_{18}^3 y, \alpha_{18}^2 t)$. From the general theory of elliptic surfaces, since $g_3 = t^3 \prod_1^9 (t - \alpha_9^i)$, which vanishes to order 3 at $t = 0$ and order 1 at $t = \alpha_9^i$, while $\Delta = g_2^3 - 27g_3^2 = -27g_3^2$ vanishes to order 6 at $t = 0$ and to order 2 at $t = \alpha_9^i$, we find that there are precisely the mentioned singular fibers, i.e., $1\mathbf{I}_0^*$, $9\mathbf{II}$. We may do our construction above, using the curve C_{18}, and the result is a Calabi-Yau with 36 singular fibers. Furthermore, the number $h^{1,1}$ is easy to find – it is just one more than the corresponding value for the K3 surface (as this automorphism is in H_K, the invariant lattice is just S_K which has rank 6), that is $h^{1,1} = 7$. We now calculate the Euler number of this fibration to find the other Hodge number. The singular fibers are in this case also, cones over a reducible curve. On the elliptic surface, this curve is given by the fibers over 0 and the zero section. Note that this corresponds to the invariance of the Picard lattice S_K, as S_K is spanned by the classes: the fiber $\pi^{-1}(0)$ and the pair (fiber,section), producing the lattice $D_4 \oplus U$. It is the curve on the elliptic surface



and has Euler number $\chi(C) = 7$. There are 36 singular fibers, and our formula (7) for the Euler number gives $\chi(X) = -528$. Thus we find $(h^{1,1}, h^{2,1}) = (7, 271)$, precisely as needed.

Returning briefly to the example for $k = 42$ recall the elliptic fibration had $1\mathbf{II}^*$ fiber and 7 of type $\mathbf{II}$, spanning a lattice $E_8 \oplus U$. These give the fixed curves under the action of $\mathbb{Z}_{42}$, namely the fiber $\pi^{-1}(0)$ and the zero section, which is the curve C with Euler number 11 described in Section 3. The Calabi-Yau constructed from this example has 84 singular fibers, and thus we recover our discussion of this example in Section 3.

We may also apply the above construction to the two Kondo examples with $k = 44, 66$. In a sense, the situation of these examples is considerably easier than with the case $k = 42$, simply because there are not so many rational curves. Indeed, as mentioned above, the fixed point set under the automorprhism consists of the fiber $\pi^{-1}(0)$ and the zero section for the example $k = 66$, and for the example $k = 44$ a similar argument shows that the fixed curve consists of $\pi^{-1}(\infty)$ and the zero section. In both cases, this is a singular fiber of type $\mathbf{II}$ and a smooth $\mathbb{P}_1$, so has Euler number 3. Applying our formula (7) for χ of the Calabi-Yau, we get

$$k = 44 \quad : \qquad \chi(X) = (2 - 88) \cdot 24 + 88 \cdot 4 = -1712,$$

$$k = 66 \quad : \qquad \chi(X) = (2-132)\cdot 24 + 132\cdot 4 = -2592. \tag{25}$$

These two examples hence have an Euler number far smaller than all examples known up till now, and also the highest number of singular fibers of any Calabi-Yau threefold with K3 fibration known to date.

Acknowledgement

The first author is indebted to Igor Dolgachev for discussions on the examples of Kondo. The second author is grateful to Per Berglund, Shyamoli Chaudhuri, Jens Erler, David Lowe and Andy Strominger for discussions. This work was supported in part by NSF grant PHY–94–07194.

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