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On the algebraic dimension of twistor spaces over the connected sum of four complex projective planes

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Abstract. We study the algebraic dimension of twistor spaces of positive type over $4\mathbb{CP}^2$. We show that such a twistor space is Moishezon if and only if its anticanonical class is not nef. More precisely, we show the equivalence of being Moishezon with the existence of a smooth rational curve having negative intersection number with the anticanonical class. Furthermore, we give precise information on the dimension and base locus of the fundamental linear system $|-\frac{1}{2}K|$. This implies, for example, $\dim|-\frac{1}{2}K| \leq a(Z)$. We characterize those twistor spaces over $4\mathbb{CP}^2$, which contain a pencil of divisors of degree one by the property $\dim|-\frac{1}{2}K| = 3$.

1. Introduction

Twistor spaces usually arise in four–dimensional conformal geometry. Their construction reflects the impossibility to equip in general a four–dimensional conformal manifold M with a compatible complex structure. It was shown in [**AHS**] that the conformal metric on M is self–dual if and only if the twistor space Z associated to M carries, in a natural way, the structure of a complex manifold. Therefore, the conformal geometry of M is closely related to the holomorphic geometry of Z. Since we shall only work with methods of complex geometry, we can use the following definition:
A twistor space Z is a complex three–manifold with the following additional structure:

- a proper differentiable submersion $\pi : Z \to M$ onto a real differentiable four–manifold M. The fibres of π are holomorphic curves in Z being isomorphic to $\mathbb{CP}^1$ and having normal bundle in Z isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$;
- an anti–holomorphic fixed point free involution $\sigma : Z \to Z$ with $\pi\sigma = \pi$.

The fibres of π are called "real twistor lines" and the involution σ is called the "real structure". A geometric object will be called "real" if it is σ–invariant. For example, a line bundle $\mathcal{L}$ on Z is real if $\sigma^*\bar{\mathcal{L}} \cong \mathcal{L}$, and a complex subvariety $D \subset Z$ is real if $\sigma(D) = D$. Instead of $\sigma(D)$ we shall often write $\bar{D}$.
We only consider compact and simply connected twistor spaces.
At the beginning of the 80's, the first classification result emerged in [**FK**], [**H2**]:
There exist exactly two compact Kählerian twistor spaces. They are automatically projective algebraic. The corresponding Riemannian four–manifolds are the 4–sphere S^4 and the complex projective plane $\mathbb{CP}^2$ (with Fubini–Study metric). This was generalized in

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[**C2**] to the result that a twistor space which is bimeromorphic to a compact Kähler manifold must be Moishezon and simply connected. This implies (see [**Don**], [**F**]) that M is homeomorphic to the connected sum $n\mathbb{CP}^2$ for some $n \geq 0$.
New examples of Moishezon twistor spaces were constructed by Y.S. Poon [**Po1**] (case $n = 2$) and C. LeBrun [**LeB1**], H. Kurke [**Ku**] (case $n \geq 3$).
Nowadays the situation is well understood for $n \leq 3$ (cf. [**H2**], [**FK**], [**Po1**], [**KK**], [**Po3**]). To become more precise, we have to introduce the notion of the "type" of a twistor space. By a result of R. Schoen [**Sch**], every conformal class of a compact Riemannian four–manifold contains a metric of constant scalar curvature. Its sign will be called the *type* of the twistor space. This is an invariant of the conformal class, hence of the twistor space. It was shown in [**Po2**] that a Moishezon twistor space is always of positive type. If $n \leq 3$, the converse is also true.
In this paper we focus on the positive type case, for two reasons. One reason is that we can then apply Hitchin's vanishing theorem (2.1). The other reason is the following: a result of P. Gauduchon [**Gau**] implies that any twistor space of negative type has algebraic dimension zero. From the results of M. Pontecorvo [**Pon**] we easily derive that a twistor space of type zero over $n\mathbb{CP}^2$ must also have algebraic dimension zero. It is not clear whether there exist twistor spaces of non–positive type over $n\mathbb{CP}^2$.
Computation of algebraic dimension is, therefore, interesting only in the case of positive type. A very important tool to compute the algebraic dimension of twistor spaces is the result of Y.S. Poon [**Po2**] (see also [**Pon**]) stating that the algebraic dimension is equal to the Iitaka dimension of the anticanonical bundle (cf. Section 2).
From [**DonF**] and [**C1**], [**LeBP**] it is known that the generic twistor space over $n\mathbb{CP}^2$ has algebraic dimension one (if $n = 4$), respectively zero (if $n \geq 5$).
For the case $n = 4$, the characterizing property $c_1^3 = 0$ is of central importance.
In this paper we study the following
Problem: *Compute the algebraic dimension $a(Z)$ of a twistor space Z over $4\mathbb{CP}^2$ in terms of geometric or numeric properties of certain divisors on Z.*
A first attempt to tackle this problem was made by Y.S. Poon [**Po3**, Section 7]. He assumes, additionally, the existence of a divisor D of degree one on Z. He studies a birational map $D \to \mathbb{P}^2$, which is the blow–up of four points. He seems to assume that these four points are actually in $\mathbb{P}^2$ (no infinitesimally near blown–up points). If these four points are in a special position he obtains $a(Z) = 3$. In the case of general position he can only show: $a(Z) \leq 2$.
We shall, in general, not assume the existence of divisors of degree one. Because in case $n = 4$ there exists at least a pencil of so–called fundamental divisors, we shall study their geometry to obtain our results. If $S \subset Z$ is a real fundamental divisor, we have a birational map $S \to \mathbb{P}^1 \times \mathbb{P}^1$ which is the blow–up of eight points. We shall study in detail the possible positions for these points. We take into account that some of these points can be infinitesimally near each other. We are able to derive the algebraic dimension $a(Z)$ from the knowledge of the positions of these eight points.
Similar considerations were made for general $n \geq 4$ in the paper [**PP2**]. But the authors of that paper are intersted in a study of small deformations of well–known Moishezon twistor spaces, and so they investigate only the case without infinitely near blown–up points.
As a consequence of our results, we give a new characterization of the twistor spaces over $4\mathbb{CP}^2$ which are first described by C. LeBrun [**LeB1**] (with methods from differential geometry). From the point of view of complex geometry the twistor space structure on

these complex manifolds was found by H. Kurke [**Ku**]. Following the literature, we call them *LeBrun twistor spaces.* These twistor spaces are characterized in [**Ku**] and [**Po3**] by the property to contain a pencil of divisors of degree one. In the case $n = 4$ we show (Theorem (6.5)) that they can also be characterized by the property $h^0(K^{-\frac{1}{2}}) = 4$ or by the structure of the base locus of $|-\frac{1}{2}K|$.
Besides this, our main results are a precise description of the set of irreducible curves intersecting $K^{-\frac{1}{2}}$ negatively (Theorem 6.1) and the following theorems, where Z denotes always a simply connected compact twistor space of positive type over $4\mathbb{CP}^2$:

THEOREM 6.2. *$a(Z) = 3 \iff K^{-\frac{1}{2}}$ is not nef;*
$a(Z) = 2 \iff K^{-\frac{1}{2}}$ is nef and $\exists m \geq 1 : h^1(K^{-\frac{m}{2}}) \neq 0$;
$a(Z) = 1 \iff \forall m \geq 1 : h^1(K^{-\frac{m}{2}}) = 0$.

THEOREM 6.3. *The following conditions are equivalent:*
(i) *$a(Z) = 3$;*
(ii) *$K^{-\frac{1}{2}}$ is not nef;*
(iii) *there exists a smooth rational curve $C \subset Z$ with $C.(-\frac{1}{2}K) < 0$.*

THEOREM 6.6. *$a(Z) \geq \dim|-\frac{1}{2}K|$.*

THEOREM 6.7. *If $\dim|-\frac{1}{2}K| \geq 2$, then:*
$a(Z) = 2 \iff K^{-\frac{1}{2}}$ is nef $\iff |-\frac{1}{2}K|$ does not have base points.

This paper is organized as follows:
In Section 2 well–known but necessary facts about simply connected compact twistor spaces of positive type are collected.
Also Section 3 has preparatory character. We study there the structure of fundamental divisors for general n, using results and techniques contained in [**PP2**]. Technically important for the following sections will be Proposition 3.6 where the structure of effective anticanonical curves on real fundamental divisors is described in detail.
In the remaining three sections we assume $n = 4$.
In Section 4 we study the case where the anti–canonical bundle K_Z^{-1} is nef (in the sense of Mori theory). We shall prove that the algebraic dimension is, in this case, at most two. We also see how to distinguish between algebraic dimension one and two. This generalizes results of [**CK**].
In Section 5 we assume K_Z^{-1} to be not nef. We collect detailed information on the fundamental linear system $|-\frac{1}{2}K|$ and on the set of curves which intersect $K^{-\frac{1}{2}}$ negatively. In this cases the algebraic dimension is three.
The final Section 6 combines the results of the previous part to prove the main theorems stated above.
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2. Preliminaries

We briefly collect well–known facts which will be frequently used later. We refer the reader to [**AHS**], [**ES**], [**H2**], [**Kr**], [**Ku**] and [**Po1**]. For brevity, we assume, throughout this section, Z to be a simply connected compact twistor space of positive type. As mentioned in Section 1 the corresponding Riemannian four–manifold M is homeomorphic to $n\mathbb{CP}^2$.

Cohomology ring of Z. $H^i(Z,\mathbb{Z})$ is a free $\mathbb{Z}$–module.
$H^1(Z,\mathbb{Z}) = H^3(Z,\mathbb{Z}) = H^5(Z,\mathbb{Z}) = 0$ and $H^0(Z,\mathbb{Z}) \cong H^6(Z,\mathbb{Z}) \cong \mathbb{Z}$.
$H^2(Z,\mathbb{Z})$ and $H^4(Z,\mathbb{Z})$ are free modules of rank $n+1$. There exists a basis $x_1,\ldots,x_n,w$ of $H^2(Z,\mathbb{Z})$ such that the pull–back $H^2(Z,\mathbb{Z}) \to H^2(F,\mathbb{Z}) \cong \mathbb{Z}$ (for any real twistor line $F \subset Z$) sends x_i to 0 and w to the positive generator.
The cohomology ring $H^*(Z,\mathbb{Z})$ is isomorphic to the graded ring $\mathbb{Z}[x_1,\ldots,x_n,w]/R$ where R is the ideal generated by

$$x_i^2 - x_j^2, \quad x_ix_j \ (i \neq j), \quad w^2 + w\sum_{i=1}^n x_i + x_1^2.$$

The grading is given by $\deg x_i = \deg w = 2$.
$H^4(Z,\mathbb{Z})$ is a free $\mathbb{Z}$–module with generators $wx_1,\ldots,wx_n,w^2$. The dual class of a real twistor fibre $F \subset Z$ is $-x_i^2 \in H^4(Z,\mathbb{Z})$.
$c_1(Z) = 4w + 2\sum_{i=1}^n x_i, \quad c_2(Z) = -6x_1^2 = 6F, \quad c_3(Z) = 2(n+2)$. This yields the following Chern numbers: $c_1^3 = 16(4-n), \quad c_1c_2 = 24, \quad c_3 = 2(n+2)$.

Cohomology of sheaves. The main reason to assume Z to be of positive type is Hitchin's vanishing theorem. We shall only use the following special case:

THEOREM 2.1 (Hitchin [**H1**]). *If Z is of positive type then we have for any $\mathcal{L} \in \mathrm{Pic}\,(Z)$*

$$\deg(\mathcal{L}) \leq -2 \quad \Rightarrow \quad H^1(Z,\mathcal{L}) = 0.$$

On the other hand, since the twistor lines cover Z, we obtain:

$$\deg(\mathcal{L}) \leq -1 \quad \Rightarrow \quad H^0(Z,\mathcal{L}) = 0.$$

By Serre duality this gives the following important vanishing results:

$$\deg(\mathcal{L}) \geq -2 \quad \Rightarrow \quad H^2(Z,\mathcal{L}) = 0, \tag{1}$$

$$\deg(\mathcal{L}) \geq -3 \quad \Rightarrow \quad H^3(Z,\mathcal{L}) = 0. \tag{2}$$

In particular, we obtain $h^2(\mathcal{O}_Z) = h^3(\mathcal{O}_Z) = 0$. Because Z is simply connected, we also have $h^1(\mathcal{O}_Z) = 0$. Hence, we obtain an isomorphism of abelian groups, given by the first Chern class:

$$\mathrm{Pic}\,(Z) \xrightarrow{\sim} H^2(Z,\mathbb{Z}).$$

There exists a unique line bundle whose first Chern class is $\frac{1}{2}c_1$. We shall denote it by $K^{-\frac{1}{2}}$. Following Poon, we call it the *fundamental* line bundle. The divisors in the linear system $|-\frac{1}{2}K|$ will be called *fundamental divisors*. The description of the cohomology ring gives $(-\frac{1}{2}K)^3 = 2(4-n)$. If $S \in |-\frac{1}{2}K|$ is a smooth fundamental divisor, we obtain by the adjunction formula $K_S^{-1} \cong K^{-\frac{1}{2}} \otimes \mathcal{O}_S$. If $n \leq 4$, there exist smooth real fundamental divisors (cf. [**CK**, Lemma 3.1]).
The degree of a line bundle $\mathcal{L} \in \mathrm{Pic}\,(Z)$ will be by definition the degree of its restriction to a real twistor line. For example, $\deg(K^{-\frac{1}{2}}) = 2$. We obtain in this way a *surjective* degree map

$$\deg : \mathrm{Pic}\,(Z) \twoheadrightarrow \mathbb{Z}.$$

From the above equations on Chern numbers we obtain, by applying the Riemann–Roch theorem,

$$\chi(Z, K^{-\frac{m}{2}}) = m + 1 + 2(4-n)\binom{m+2}{3}. \tag{3}$$

Algebraic dimension. We denote by $a(Z)$ the algebraic dimension of Z, which is by definition the transcendence degree of the field of meromorphic functions of Z over $\mathbb{C}$. If $\dim Z = a(Z)$, then Z is called Moishezon. To compute the algebraic dimension of twistor spaces we shall frequently use, without further reference, the following theorem of Y.S. Poon:

THEOREM 2.2. [**Po2**], [**Pon**, Prop. 3.1]
$\kappa(Z,K^{-1}) \geq 0 \quad \Rightarrow \quad a(Z) = \kappa(Z,K^{-1})$
$\kappa(Z,K^{-1}) = -\infty \quad \Rightarrow \quad a(Z) = 0.$

The number $\kappa(Z,K^{-1})$ is usually called the Iitaka dimension (or L–dimension = line bundle dimension) of the line bundle K^{-1}. Its definition generalizes the well–known notion of Kodaira dimension. For details, including the following facts, we refer the reader to [**U**].
For any line bundle $\mathcal{L} \in \mathrm{Pic}\,(Z)$ there holds: $\dim Z \geq a(Z) \geq \kappa(Z,\mathcal{L})$.
If $f : Z \to Y$ is a dominant morphism, then $a(Z) \geq a(Y)$. Particularly, if $f : Z \to \mathbb{P}^N$ is a meromorphic map, then $a(Z) \geq \dim f(Z)$, because any projective variety is Moishezon.
If we define $g := \gcd\{m \in \mathbb{Z} \mid m > 0, h^0(Z,L^m) \neq 0\}$ and denote by $\Phi_{|L^m|}$ the meromorphic map given by the linear system $|L^m|$, then $\kappa(Z,L) = \max\{\dim \Phi_{|L^m|}(Z) \mid m \in g\mathbb{Z},\ m > 0\}$. If there exists a polynomial $P(X)$ such that for all large positive $m \in \mathbb{Z}$ we have $h^0(Z,L^{mg}) \leq P(m)$, then $\kappa(Z,L) \leq \deg P$.
We apply these basic facts to obtain our first result on the algebraic dimension in case $n = 4$. The following proposition is a generalization of a result contained in [**CK**]. For convenience we introduce the following
Definition: *If there exists an integer $m \geq 1$ with $h^1(K^{-\frac{m}{2}}) \neq 0$ then we define $\tau := \min\{m | m \geq 1, h^1(K^{-\frac{m}{2}}) \neq 0\}$. Otherwise we set $\tau := \infty$.*

PROPOSITION 2.3. *Let Z be a simply connected compact twistor space of positive type with $c_1^3 = 0$. Then:*
(i) $a(Z) \geq 1$
(ii) $a(Z) = 1 \iff \forall m \geq 1 \quad h^1(K^{-\frac{m}{2}}) = 0.$

PROOF: From Riemann–Roch and Hitchin's vanishing theorem we know: $h^0(K^{-\frac{m}{2}}) = m+1+h^1(K^{-\frac{m}{2}})$. Therefore, $a(Z) = \kappa(Z,K^{-\frac{1}{2}}) \geq 1$ and if $\tau = \infty$ we have $\kappa(Z,K^{-\frac{1}{2}}) = 1$. Assume $\tau < \infty$ and $a(Z) = 1$. Let $S \in |-\frac{1}{2}K|$ be smooth and real. (Such a divisor exists, because we assume $n = 4$, cf. [**CK**].) Since $h^1(K^{-\frac{\tau-1}{2}}) = 0$ the exact sequence

$$0 \to K^{-\frac{\tau-1}{2}} \to K^{-\frac{\tau}{2}} \to K_S^{-\tau} \to 0$$

gives an exact sequence

$$0 \to H^0(K^{-\frac{\tau-1}{2}}) \to H^0(K^{-\frac{\tau}{2}}) \to H^0(K_S^{-\tau}) \to 0.$$

Since $h^1(K^{-\frac{\tau}{2}}) \geq 1$ we have, furthermore, $h^0(K^{-\frac{\tau}{2}}) = \tau + 1 + h^1(K^{-\frac{\tau}{2}}) \geq \tau + 2$. The linear system $|-\frac{\tau}{2}K|$ cannot have a fixed component since $\tau S \in |-\frac{\tau}{2}K|$ and $\dim|S| = \dim|-\frac{1}{2}K| \geq 1$. If necessary blow up Z to obtain a morphism $\Phi_\tau : \tilde{Z} \to \mathbb{P}^d$ defined by $|-\frac{\tau}{2}K|$. Here $d := \dim|-\frac{\tau}{2}K| \geq \tau + 1 \geq 2$. By assumption $\dim \Phi_\tau(\tilde{Z}) = 1$. Since the curve $\Phi_\tau(\tilde{Z})$ is not contained in a linear subspace of $\mathbb{P}^d$, its degree must be at least d. Hence, a generic member of the linear system $|-\frac{\tau}{2}K|$ is the sum of λ algebraically equivalent divisors and so it is linearly equivalent to λS_0 with $\lambda \geq d \geq \tau + 1$. This gives

$2\tau = \deg(-\frac{\tau}{2}K) = \lambda \deg(S_0)$, which is only possible if $\lambda = 2\tau$ and $\deg(S_0) = 1$. But then we have infinitely many divisors of degree one in Z. This implies $a(Z) = 3$ by the Theorem of Kurke–Poon (see [**Ku**], [**Po3**]). This contradiction proves the proposition. □

REMARK 2.4. If $|-K_S|$ contains a smooth curve C, then we computed in [**CK**] that τ is the order of $N := K_S^{-1} \otimes \mathcal{O}_C$ in the Picard group $\operatorname{Pic} C$ of the elliptic curve C. Under this additional assumption Proposition 2.3 was shown in [**CK**].

3. The structure of fundamental divisors

In this section Z always denotes a simply connected compact twistor space of positive type.

LEMMA 3.1. *Let $S \in |-\frac{1}{2}K|$ be a smooth surface. Then the restriction map $\operatorname{Pic} Z \to \operatorname{Pic} S$ is injective.*

PROOF: By assumption we have $h^1(\mathcal{O}_Z) = h^2(\mathcal{O}_Z) = 0$. Since S is a rational surface [**Po1**], we also have $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$. Therefore, taking the first Chern class defines isomorphisms $\operatorname{Pic} Z \xrightarrow{\sim} H^2(Z,\mathbb{Z})$ and $\operatorname{Pic} S \xrightarrow{\sim} H^2(S,\mathbb{Z})$. Let us denote the inclusion of S into Z by i. The above isomorphisms transform then the restriction morphism $\operatorname{Pic} Z \to \operatorname{Pic} S$ into the map i^* on cohomology groups.
We shall apply standard facts from algebraic topology to verify the injectivity of i^*. Let $\mathrm{o}_S \in H_4(Z,\mathbb{Z})$ and $\mathrm{o}_Z \in H_6(Z,\mathbb{Z})$ be the fundamental classes of S and Z respectively. By $d_Z(S) \in H^2(Z,\mathbb{Z})$ we denote the Poincaré dual of $i_*(\mathrm{o}_S) \in H_4(Z,\mathbb{Z})$, this means $i_*(\mathrm{o}_S) = d_Z(S) \frown \mathrm{o}_Z$ (cap–product).
For any cohomology class $\alpha \in H^2(Z,\mathbb{Z})$ we obtain by the associativity of cap–product $\alpha \frown i_*(\mathrm{o}_S) = \alpha \frown (d_Z(S) \frown \mathrm{o}_Z) = (\alpha \smile d_Z(S)) \frown \mathrm{o}_Z$. The naturalness of cap–product implies $\alpha \frown i_*(\mathrm{o}_S) = i_*(i^*(\alpha) \frown \mathrm{o}_S)$. Therefore, we obtain a commutative diagram:

$$\begin{array}{ccc}
H^2(Z,\mathbb{Z}) & \xrightarrow{i^*} & H^2(S,\mathbb{Z}) \\
 & & \downarrow \frown \mathrm{o}_S \\
\Big\downarrow \smile d_Z(S) & & H_2(S,\mathbb{Z}) \\
 & & \downarrow i_* \\
H^4(Z,\mathbb{Z}) & \xrightarrow{\frown \mathrm{o}_Z} & H_2(Z,\mathbb{Z})
\end{array}$$

Since, by Poincaré duality, the cap–product with o_Z is an isomorphism, we obtain $\ker(i^*) \subset \ker(\smile d_Z(S))$.
The description of the cohomology ring given above allows us to compute the kernel of the cup–product with the dual class $d_Z(S)$ of S. With the notation of Section 2 the elements $x_1, \ldots, x_n, \omega$ form a basis of the free $\mathbb{Z}$–module $H^2(Z,\mathbb{Z})$. The dual class of S is $d_Z(S) = c_1(K^{-\frac{1}{2}}) = 2\omega + x_1 + \cdots + x_n$. If we use $\omega x_1, \ldots, \omega x_n, x_1^2$ as basis of $H^4(Z,\mathbb{Z})$ then the cup–product with $d_Z(S)$ is described by the $(n+1) \times (n+1)$–matrix:

$$\begin{pmatrix}
2 & 0 & \ldots & 0 & 1 \\
0 & 2 & & 0 & 1 \\
\vdots & & & \vdots & \vdots \\
0 & & 2 & 0 & 1 \\
0 & \ldots & 0 & 2 & 1 \\
-1 & \ldots & -1 & -1 & -2
\end{pmatrix}$$

whose determinant is equal to $2^{n-1}(n-4)$.
If $n \neq 4$ we obtain the injectivity of the map $\alpha \mapsto \alpha \smile d_Z(S)$ and thus of the restriction map $\operatorname{Pic} Z \hookrightarrow \operatorname{Pic} S$. If $n = 4$ it is easy to see that $\alpha \smile d_Z(S) = 0$ if and only if $\alpha \in \mathbb{Z} \cdot d_Z(S) \subset H^2(Z, \mathbb{Z})$. To prove the injectivity of $\operatorname{Pic} Z \to \operatorname{Pic} S$ it remains, therefore, to show that $K^{-\frac{m}{2}} \otimes \mathcal{O}_S \cong \mathcal{O}_S$ implies $m = 0$.
By adjunction we have $K^{-\frac{m}{2}} \otimes \mathcal{O}_S \cong K_S^{-m}$. But S is rational, hence $\operatorname{Pic} S$ is torsion free and $K_S \not\cong \mathcal{O}_S$. Thus, $K_S^{-m} \cong \mathcal{O}_S$ if and only if $m = 0$. This proves the Lemma. □

LEMMA 3.2. *Let Z be a simply connected compact twistor space of positive type and $D \subset Z$ a divisor of degree one. If $S \in |-\frac{1}{2}K|$ is a smooth surface, then $C := D \cap S$ is connected.*

PROOF: We shall show $h^0(\mathcal{O}_C) = 1$, which implies connectedness of C.
Consider first the exact sequence

$$0 \to \mathcal{O}_Z(-\bar{D}) \to \mathcal{O}_Z \to \mathcal{O}_{\bar{D}} \to 0. \tag{4}$$

From $h^1(\mathcal{O}_Z) = 0$ we obtain $h^1(\mathcal{O}(-\bar{D})) = h^0(\mathcal{O}_{\bar{D}}) - h^0(\mathcal{O}_Z) = 0$ since Z and $\bar{D}$ are connected.
As $D + \bar{D} \in |-\frac{1}{2}K|$ we obtain an exact sequence

$$0 \to K(\bar{D}) \to K^{\frac{1}{2}} \to \mathcal{O}_D(-C) \to 0.$$

But the degree of $K^{\frac{1}{2}}$ is -2 and, therefore, Hitchin's vanishing theorem gives $h^i(K^{\frac{1}{2}}) = 0$ for all i. Therefore, using Serre duality, $h^1(\mathcal{O}_D(-C)) = h^2(K(\bar{D})) = h^1(\mathcal{O}_Z(-\bar{D})) = 0$.
Consider finally the exact sequence

$$0 \to \mathcal{O}_D(-C) \to \mathcal{O}_D \to \mathcal{O}_C \to 0.$$

We have $h^0(\mathcal{O}_D) = 1$ since any divisor of degree one is connected [**Po1**]. Because C is effective, $h^0(\mathcal{O}_D(-C))$ must vanish. Hence, $h^0(\mathcal{O}_C) = h^0(\mathcal{O}_D) = 1$. □

LEMMA 3.3 (cf. [**PP2**], p. 693). *Let Z be as above and $S \in |-\frac{1}{2}K|$ an irreducible real divisor. Then S is smooth and contains a real twistor fibre $F \subset S$. The linear system $|F|$ is one–dimensional and its real elements are precisely the real twistor fibres contained in S.*

PROOF: The smoothness of S was shown in [**PP1**, Lemma 2.1]. If S does not contain a real twistor fibre, the restriction of the twistor fibration to S would give an unramified double cover over a simply connected manifold, since Z does not contain real points. But S is connected and must, therefore, contain a real twistor fibre F. From the adjunction formula we obtain $F^2 = 0$ on S. Hence, we have an exact sequence $0 \to \mathcal{O}_S \to \mathcal{O}_S(F) \to \mathcal{O}_F \to 0$. From $h^1(\mathcal{O}_S) = 0$ we infer, therefore, $\dim |F| = 1$.
Since the linear system $|F|$ defines a flat family of curves in S, its elements form a curve in the Douady space $\mathcal{D}$ of curves on Z (cf. [**Dou**]). Since $h^0(\mathbb{P}^1, \mathcal{O}(1) \oplus \mathcal{O}(1)) = 4$ and $h^1(\mathbb{P}^1, \mathcal{O}(1) \oplus \mathcal{O}(1)) = 0$, $\mathcal{D}$ is a four–dimensional complex manifold near points which correspond to smooth rational curves on Z with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. The real structure of Z induces one on $\mathcal{D}$. If the set of real points $\mathcal{D}(\mathbb{R})$ is non–empty, then it is a four–dimensional real manifold near points as before. Since the real twistor lines are smooth rational curves with the above normal bundle, the real manifold $M = 4\mathbb{CP}^2$ is a submanifold of $\mathcal{D}(\mathbb{R})$. Since M is compact and has the same dimension as $\mathcal{D}(\mathbb{R})$, it must be a connected component.

The set U of members of $|F|$ which are smooth rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is open and dense in $\mathbb{P}^1 \cong |F|$ with respect to the Zariski topology. Therefore, the set $U(\mathbb{R})$ of real points in U is open and dense in the one–sphere of real members of $|F|$. Since $\mathcal{D}(\mathbb{R})$ is smooth near M and M is a component of $\mathcal{D}(\mathbb{R})$, we have $U(\mathbb{R}) \subset M$. But M is compact and must, therefore, contain the closure of $U(\mathbb{R})$ in $\mathcal{D}(\mathbb{R})$, which is the set of all real members of $|F|$. Therefore, any real member of $|F|$ is a real twistor fibre and, in particular, smooth and ireducible. This proves the claim. □

To obtain more information on the structure of real irreducible fundamental divisors $S \in |-\frac{1}{2}K|$ one can study the morphism $S \to \mathbb{P}^1$ given by $|F|$ (cf. [**PP2**, p. 693]). Since the general fibre of this morphism is a smooth rational curve it factors through a rational ruled surface. Since $(-K_S)^2 = (-\frac{1}{2}K)^3 = 8 - 2n$, the surface S is a blow–up of a ruled surface at $2n$ points. The exceptional curves of these blow–ups are contained in fibres of the morphism $S \to \mathbb{P}^1$. By Lemma 3.3 none of the exceptional curves is real and none of the blown–up points lie on a real fibre of the ruled surface. Using this, in [**PP2**, Lemma 3.5] it has been shown that the ruled surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, we obtain a morphism $\sigma : S \to \mathbb{P}^1 \times \mathbb{P}^1$ which is a succession of blow–ups. Let us equip $\mathbb{P}^1 \times \mathbb{P}^1$ with the real structure given by the antipodal map on the first factor and the usual real structure on the second. Then σ is equivariant (or "real"). Since we can always contract a conjugate pair of disjoint (-1)–curves, σ is the succession of n blow–ups. At each step a conjugate pair of points is blown–up to give a surface without real points.

We should bear in mind that it is possible to have infinitesimally near blown–up points.

As in [**PP2**] we shall call curves of type $(1,0)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ "lines" and curves of type $(0,1)$ "fibres". Then there do not exist real lines. But the images of real twistor fibres in $|F|$ are exactly the real "fibres".

Lemma 3.4. *Equip $\mathbb{P}^1 \times \mathbb{P}^1$ with the real structure $((a_0 : a_1), (b_0 : b_1)) \mapsto ((\bar{a}_1 : -\bar{a}_0), (\bar{b}_0 : \bar{b}_1))$ as described above. Then the reduced components of any <u>real</u> member of $|\mathcal{O}(2,2)| = |-K_{\mathbb{P}^1 \times \mathbb{P}^1}|$ are smooth. A non–reduced component of a real member of $|\mathcal{O}(2,2)|$ can only be of the form $2F$ with a real curve $F \in |\mathcal{O}(0,1)|$.*

Proof: As usual, $\mathcal{O}(k,l)$ denotes the locally free sheaf $p_1^*\mathcal{O}_{\mathbb{P}^1}(k) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(l)$ on the smooth rational surface $\mathbb{P}^1 \times \mathbb{P}^1$, where $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ $(i = 1,2)$ are the projections and k, l are integers. The Picard group Pic $(\mathbb{P}^1 \times \mathbb{P}^1)$ is free abelian of rank two with generators $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. In the proof we shall use the well–known fact that, if $k < 0$ or $l < 0$, then the linear system $|\mathcal{O}(k,l)|$ is empty.

Let $C \in |\mathcal{O}(2,2)|$ be a real curve and $C_0 \in |\mathcal{O}(a,b)|$ an *irreducible* component (with *reduced* scheme structure) of C. Let $\lambda \geq 1$ be the multiplicity of C_0 in C, that is the largest integer with $\lambda C_0 \subset C$. Then we must have $0 \leq \lambda a \leq 2$ and $0 \leq \lambda b \leq 2$. The case $\lambda a = \lambda b = 2$ can only occur if $\lambda C_0 = C$.

Assume first $\lambda C_0 \neq C$, hence $\lambda a \leq 1$ or $\lambda b \leq 1$. If $\lambda \geq 2$ or C_0 singular, there exists a point $y \in C_0$ such that any curve not contained in C_0 but containing y has intersection number at least two with λC_0. If $F \in |\mathcal{O}(0,1)|$ and $G \in |\mathcal{O}(1,0)|$ are the unique curves in these linear systems containing the point y, we obtain (as C_0 is irreducible) $F.(\lambda C_0) = \lambda a \geq 2$ or $G.(\lambda C_0) = \lambda b \geq 2$. By the above inequalities, this means $\lambda a = 2$ and $0 \leq \lambda b \leq 1$ or $0 \leq \lambda a \leq 1$ and $\lambda b = 2$. If $\lambda = 1$, the curve C_0 is, by assumption, irreducible, reduced, singular and a member of $|\mathcal{O}(2,0)|$, $|\mathcal{O}(2,1)|$, $|\mathcal{O}(0,2)|$, or $|\mathcal{O}(1,2)|$. But these linear systems do not contain such a curve. Hence, we must have $\lambda = 2$ and, therefore, $C_0 \in |\mathcal{O}(0,1)|$ or $C_0 \in |\mathcal{O}(1,0)|$. In particular, C_0 is smooth.

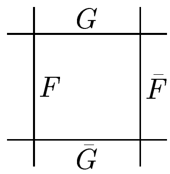
If $C_0 \in |\mathcal{O}(1,0)|$, this curve is not real, since by definition of the real structure on $\mathbb{P}^1 \times \mathbb{P}^1$ this linear system does not contain real members. Hence, the component $2C_0$ of C is not real, which implies $2C_0 + 2\bar{C}_0 \subset C$, since C is real. But $\bar{C}_0 \in |\mathcal{O}(1,0)|$ and so $2C_0 + 2\bar{C}_0 \in |\mathcal{O}(4,0)|$. Such a curve can never be contained in $C \in |\mathcal{O}(2,2)|$. So we obtain $\lambda C_0 = 2F$ with some $F \in |\mathcal{O}(0,1)|$. Again, since a curve of type $(0,4)$ can never be a component of C, the fibre F is necessarily real. This proves the lemma in the case $\lambda C_0 \neq C$.
Assume now $\lambda C_0 = C$. Then $\lambda = 1$ or $\lambda = 2$. If $\lambda = 2$, we have, by reality of $C = 2C_0$, that $C_0 \in |\mathcal{O}(1,1)|$ is a real curve. Because C_0 is irreducible and $C_0.F = 1$ for any $F \in |\mathcal{O}(0,1)|$, the curve C_0 would intersect each real fibre $F \in |\mathcal{O}(0,1)|$ at a real point. But on $\mathbb{P}^1 \times \mathbb{P}^1$ real points do not exist. Hence $\lambda = 1$, which means $C = C_0$ is irreducible and real.
It remains to see that C must be reducible if it is not smooth. Let $x \in C$ be a singular point of C. Since C is real and $\mathbb{P}^1 \times \mathbb{P}^1$ does not contain real points, $\bar{x} \neq x$ is also a singular point on C. If we embed $\mathbb{P}^1 \times \mathbb{P}^1$ by $|\mathcal{O}(1,1)|$ as a smooth quadric into $\mathbb{P}^3$, we easily see that the linear system of curves of type $(1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ containing x and $\bar{x}$ is one–dimensional. It is cut out by the pencil of planes in $\mathbb{P}^3$ containing the line connecting x and $\bar{x}$. Therefore, any point of $\mathbb{P}^1 \times \mathbb{P}^1$ is contained in such a curve. The intersection number of C with a curve of type $(1,1)$ is four. Since x and $\bar{x}$ are singular points on C, any curve of type $(1,1)$ containing x and $\bar{x}$ and a third point of C must have a common component with C. Therefore, C cannot be irreducible and reduced. $\square$

Definition 3.5. A reduced curve C on a compact complex surface S will be called a "cycle of rational curves", if the irreducible components $C_1, \ldots, C_m$ of C are smooth rational curves with the following properties: (We use the convention $C_{m+1} = C_1$.)
$m = 2$ and C_1 intersects C_2 transversally at two distinct points, $C_1.C_2 = 2$, or
$m \geq 3$, $C_i.C_{i+1} = 1$ and $C_i \cap C_j \neq \emptyset$ implies $j \in \{i-1, i, i+1\}$.

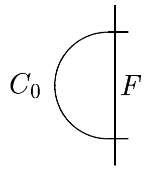
Proposition 3.6. *Assume* $\dim|-\frac{1}{2}K| \geq 1$ *and let* $S \in |-\frac{1}{2}K|$ *be smooth and real. Then there exists a blow-down* $S \to \mathbb{P}^1 \times \mathbb{P}^1$ *and a connected real member* $C \in |-K_S|$, *such that:*

- σ *is compatible with real structures, where we use the real structure of Lemma 3.4 on* $\mathbb{P}^1 \times \mathbb{P}^1$,
- *the composition* $pr_2 \circ \sigma$ *of* σ *with the second projection* $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ *is the morphism given by the linear system* $|\widetilde{F}|$, *where* $\widetilde{F} \subset S$ *is a real twistor fibre,*
- *the curve* C *is reduced and*
- *if* C *is not smooth, it is a "cycle of rational curves" and its image* C' *in* $\mathbb{P}^1 \times \mathbb{P}^1$ *has one of the following structures:*

(I) C' *has four components* $C' = F + \bar{F} + G + \bar{G}$ *where* $F \in |\mathcal{O}(0,1)|$ *is a non–real fibre and* $G \in |\mathcal{O}(1,0)|$ *is a line.*



(II) *C' has two components $C' = F + C_0$ where $F \in |\mathcal{O}(0,1)|$ is a real fibre and $C_0 \in |\mathcal{O}(2,1)|$ is real, smooth and rational.*



(III) *C' has two distinct components $C' = A' + \bar{A}'$ where $A', \bar{A}' \in |\mathcal{O}(1,1)|$.*
[By Corollary 4.3 this item can be omitted if $n = 4$ and $K^{-\frac{1}{2}}$ is not nef!]

Proof: As $\dim|-K_S| = \dim|-\frac{1}{2}K| - 1$, we have by assumption $|-K_S| \neq \emptyset$. As seen above, we can choose a real blow–down map $\sigma : S \to \mathbb{P}^1 \times \mathbb{P}^1$ such that $(\mathrm{pr}_2 \circ \sigma)^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_S(\widetilde{F})$. If $|-K_S|$ contains a smooth member, we are done. Otherwise, take a reducible real $C \in |-K_S|$ and let $C' \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the image of C. Since σ is a blow–up, C' is a real member of $|\mathcal{O}(2,2)|$. By Lemma 3.4 the components of C' are smooth and a multiple component can only be the multiple $2F$ of a real fibre $F \in |\mathcal{O}(0,1)|$. But Lemma 3.3 shows that no point on such a real fibre is blown–up. Therefore, any other member of $|2F|$ missing the $2n$ blown–up points, defines a divisor in $|-K_S|$. Choosing, for example, a conjugate pair of appropriate fibres, we obtain a real member in $|-K_S|$ whose image in $\mathbb{P}^1 \times \mathbb{P}^1$ has only reduced components.
We assume for the rest of the proof that C is chosen in this way. If C' would be irreducible, it would be smooth by Lemma 3.4. In this case C is smooth, too.
Assume C' is reducible. A component of type $(1,a)$ with $a \in \{0,1,2\}$ cannot be real, since, otherwise, it would intersect real fibres at real points. Therefore, such components appear in conjugate pairs, hence $a \leq 1$. If $a = 0$ we are in case (I). If $a \neq 0$ then $a = 1$ and $C' = A' + \bar{A}'$ with two distinct curves A', $\bar{A}'$ of type $(1,1)$. This is case (III).
Assume now that there is no component of type $(1,a)$. Then we must have a component C_0 of C' which has type $(2,a)$. If $a = 2$ it must be smooth and we are done. Therefore, $a = 1$, because $|\mathcal{O}(2,0)|$ does not contain irreducible reduced elements. Then we have $C' = C_0 + F$ with $F \in |\mathcal{O}(0,1)|$ and $C_0 \in |\mathcal{O}(2,1)|$. F and C_0 must be real, since they have different types. This is case (II) of our statement.
It remains to show that C is a "cycle of rational curves". We have seen this for the image C' in $\mathbb{P}^1 \times \mathbb{P}^1$. Exceptional components of C are always rational. Furthermore, C' has at most ordinary nodes as singularities. To obtain C from C', at every step of blowing–up, we have to subtract the exceptional locus from the total transform of C'. At every step we blow up either a conjugate pair of singular points or of smooth points. We obtain a curve which has again, at most, singularities of multiplicity two and is a "cycle of rational curves". So we obtain this property for C, too. $\square$
Using this structure result and assuming that the fundamental linear system $|-\frac{1}{2}K|$ is a pencil, we can show that the structure of its base locus is closely related to the effective divisors of degree one on Z.

Proposition 3.7. *Assume $\dim|-\frac{1}{2}K| = 1$ and denote by C the base locus of the fundamental linear system.*
If C is smooth, then Z does not contain effective divisors of degree one.
If C is not smooth, then the number of effective divisors of degree one on Z is equal to the number of components of C.

Proof: Let $S \in |-\frac{1}{2}K|$ be a smooth real member. Then $|-K_S| = \{C\}$ and by Proposition 3.6 C is smooth or a "cycle" of smooth rational curves. If C is smooth, there

does not exist an effective divisor D of degree one, because $D+\bar{D} \in |-\frac{1}{2}K|$ would produce a reducible member in $|-K_S|$. Let now C be singular, hence reducible.
The rest of the proof is an adaption of an idea of Pedersen and Poon [**PP2**, p. 700].
Now let $\{P, \bar{P}\}$ be any pair of singular points on C. The image C' of C in $\mathbb{P}^1 \times \mathbb{P}^1$ does not contain a real fibre, because, otherwise, by Proposition 3.6 and Lemma 3.3 the linear system $|-K_S|$ would be at least one–dimensional. Hence, the real twistor line L_P containing P and $\bar{P}$ is not contained in S. Hence, L_P meets S transversally at P and $\bar{P}$. If Q is a point on L_P distinct from P and$\bar{P}$, then there exists a divisor $S_0 \in |-\frac{1}{2}K|$ containing Q. Since S_0 contains also C it contains three points of L_P. Hence, $L_P \subset S_0$. Therefore, the real linear system of fundamental divisors containing L_P is non–empty. This implies that we can choose a real $S_0 \in |-\frac{1}{2}K|$ containing L_P. Since S_0 contains also C and P is a singular point of C, the surface S_0 contains three curves meeting at P, namely L_P and two components (call them A and B) of C. On the other hand, L_P intersects S precisely at P and $\bar{P}$ as we have seen above. From $L_P.S = 2$ we infer that this is a transversal intersection. But A and B are contained in S and are transversally there. We can conclude that the tangent space of Z at P is generated by the tangent directions of A, B and L_P at P. Hence, the real surface S_0 is singular at P. This implies that S_0 is singular along L_P (cf. [**H2**, p.141]) and by [**PP1**, Lemma 2.1] such a divisor splits into the sum of two divisors of degree one. Therefore, we have at least as many pairs of conjugate divisors of degree one as we have pairs of conjugate singular points on C. In other words, the number of distinct divisors of degree one is at least equal to the number of components of C.
Let D and $\bar{D}$ be a conjugate pair of divisors of degree one on Z. Then $C \subset D \cup \bar{D}$. $D \cap \bar{D}$ is a real twistor line (cf. [**Ku**, Prop. 2.1]), and no component of C is a real twistor line. Hence, every component of C lies on exactly one of the surfaces D and $\bar{D}$. By Lemma 3.2 $C \cap D$ is connected. The same is true for the conjugate curve $C \cap \bar{D}$. Since C is a cycle of rational curves, $(C \cap D) \cap (C \cap \bar{D})$ consists of a conjugate pair $\{P, \bar{P}\}$ of singularities of C. Since D and $\bar{D}$ are of degree one, the real twistor line L_P containing P and $\bar{P}$ must be contained in D and in $\bar{D}$. Therefore, $D \cap \bar{D} = L_P$.
Let D' be an arbitrary divisor of degree one containing L_P. Then $D' \cap \bar{D}' = L_P$ and without loss of generality we may assume $D' \cap C = D \cap C$, since the decomposition of C into two conjugate connected curves is determined by $\{P, \bar{P}\} = L_P \cap C$. By Lemma 3.1 the restriction map $\operatorname{Pic} Z \to \operatorname{Pic} S$ is injective. Since $(D' + \bar{D}') \cap S = C$ we have $D' \cap S = D' \cap C$ and $D \cap S = D \cap C$. Hence, we have $\mathcal{O}_Z(D) \cong \mathcal{O}_Z(D')$, which means that D and D' are linearly equivalent. If $D \neq D'$, then $\dim |D| \geq 1$ and Z would contain infinitely many divisors of degree one and by Kurke [**Ku**] and Poon [**Po3**] it must be a conic–bundle twistor space. But then we should have $\dim|-\frac{1}{2}K| = 3$ in contradiction to our assumption. Hence, $D = D'$ and we have exactly as many divisors of degree one on Z as C has components. □
For technical reasons we state here the following lemma needed in Section 5.

LEMMA 3.8. *Let S be a smooth complex surface and $C \subset S$ a reduced curve. Assume $C = \sum_{i=1}^m C_i$ is a "cycle of rational curves" as defined above. If $L \in \operatorname{Pic}(S)$ is a line bundle, we define $l_i := L.C_i$. Let $I_\pm := \{i | \pm l_i > 0\}$ and $C_\pm := \sum_{i \in I_\pm} C_i$. Let γ denote the number of connected components of $C \setminus C_-$. Assume $|I_-| \geq 2$ and each connected* *component of $C \setminus C_-$ contains a component of C_+. Then we have:*

$$h^0(C,L) = \sum_{i\in I_+} l_i - \gamma.$$

PROOF: Let $\eta : \tilde{C} = \sqcup_i C_i \to C$ be the normalization of C. By P_i we denote the intersection point of C_i with C_{i+1} $(1 \le i \le m)$. By assumption $m \ge 3$. Tensoring the exact sequence $0 \to \mathcal{O}_C \to \eta_* \mathcal{O}_{\tilde{C}} \to \oplus_i \mathbb{C}_{P_i} \to 0$ with L yields the exact sequence $0 \to L \otimes \mathcal{O}_C \to \eta_*\eta^*(L \otimes \mathcal{O}_C) \to \oplus_i \mathbb{C}_{P_i} \to 0$. Hence, $H^0(C, L\otimes \mathcal{O}_C) \cong \ker(\oplus_i H^0(C_i, L_i) \xrightarrow{\rho} \oplus_i \mathbb{C}_{P_i})$. Here we denote $L_i := L \otimes \mathcal{O}_{C_i} \cong \mathcal{O}_{C_i}(l_i)$. Let $P_i' \in C_i$ and $P_i'' \in C_{i+1}$ be the two points on $\tilde{C}$ lying over P_i.
To describe ρ we observe that the map η gives isomorphisms $L_i(P_i') \xrightarrow{\sim} \mathbb{C}_{P_i}$ and $L_{i+1}(P_i'') \xrightarrow{\sim} \mathbb{C}_{P_i}$. If $s_i \in H^0(C_i, L_i)$ is a section, we denote by $s_i(P_i)$ the image of s_i under the map $H^0(C_i, L_i) \to L_i(P_i') \xrightarrow{\sim} \mathbb{C}_{P_i}$. Similarly, $s_i(P_{i-1})$ is the image of s_i under $H^0(C_i, L_i) \to L_i(P_{i-1}'') \xrightarrow{\sim} \mathbb{C}_{P_{i-1}}$. With this notation we have:

$$\rho(s_1, \dots, s_m) = (s_1(P_1) - s_2(P_1), s_2(P_2) - s_3(P_2), \dots, s_m(P_m) - s_1(P_m)).$$

Since $P_i' \neq P_{i-1}''$ on $C_i \cong \mathbb{P}^1$, the restriction of ρ $H^0(C_i, L_i) \to \mathbb{C}_{P_i} \oplus \mathbb{C}_{P_{i-1}}$ is surjective if and only if $l_i > 0$. If $C_i + \cdots + C_{i+r}$ is a connected component of $C \setminus C_-$, then we obtain by induction on r that the restriction of ρ $\oplus_{\mu=0}^r H^0(C_{i+\mu}, L_{i+\mu}) \to \oplus_{\mu=-1}^r \mathbb{C}_{P_{i+\mu}}$ is surjective. Because $H^0(C_i, L_i) = 0$ if and only if $l_i < 0$, we obtain $\mathrm{im}\,(\rho) = \sum_{P_\mu \in C_0 + C_+} \mathbb{C}_{P_\mu}$. (Here, we denote $I_0 := I \setminus (I_- \cup I_+)$ and $C_0 := \sum_{\nu \in I_0} C_\nu$.) The number of points $P_\mu \in C_0 + C_+$ is equal to $|I_0| + |I_+| + \gamma$. Therefore, we obtain

$$\dim \ker(\rho) = \sum_i h^0(C_i, L_i) - (|I_0| + |I_+| + \gamma) = \sum_{l_i \ge 0} (l_i + 1) - |I_0| - |I_+| - \gamma = \sum_{l_i > 0} l_i - \gamma.$$

□

4. The nef case

For the rest of the paper we assume $n = 4$. Remember that $(-\frac{1}{2}K)^3 = 0$, $\chi(K^{-\frac{m}{2}}) = m + 1$ and $h^0(K^{-\frac{1}{2}}) \ge 2$ in this case. Remember from Mori's theory that a line bundle $L \in \mathrm{Pic}\,(Z)$ is called *nef*, if for each irreducible curve $C \subset Z$ there holds $L.C \ge 0$.

THEOREM 4.1. *The following properties are equivalent:*

(i) *$K^{-\frac{1}{2}}$ is nef;*
(ii) *for all smooth and real $S \in |-\frac{1}{2}K|$ and all $C \in |-K_S|$, every component C_0 of C has the property $C_0.(-K_S) = 0$;*
(iii) *there exists a smooth and real $S \in |-\frac{1}{2}K|$ and a divisor $C \in |-K_S|$, such that all components C_0 of C have the property $C_0.(-K_S) = 0$.*

If $K^{-\frac{1}{2}}$ is nef, then $a(Z) \le 2$ and $\dim|-\frac{1}{2}K| \le 2$.
If $K^{-\frac{1}{2}}$ is nef and $\dim|-\frac{1}{2}K| = 2$, then $a(Z) = 2$, $|-\frac{1}{2}K|$ does not have base points and for any smooth real $S \in |-\frac{1}{2}K|$ the pencil $|-K_S|$ contains a smooth real member.

PROOF: (i)⇒(ii):
Take any smooth real $S \in |-\frac{1}{2}K|$ and an arbitrary curve $C \in |-K_S|$. Since $C.(-\frac{1}{2}K) = 0$ and $K^{-\frac{1}{2}}$ is nef we obtain (ii).

(ii)$\Rightarrow$(iii) is obvious.
(iii)$\Rightarrow$(i):
If $K^{-\frac{1}{2}}$ were not nef, then there would exist an irreducible curve $C_0 \subset Z$ with $C_0.(-\frac{1}{2}K) < 0$. If $S \in |-\frac{1}{2}K|$ is smooth and real, then $C_0 \subset S$ and $C_0.(-K_S) = C_0.(-\frac{1}{2}K) < 0$. Therefore, C_0 is a component of any element of $|-K_S|$ in contradiction to (iii).
Assume for the rest of the proof that $K^{-\frac{1}{2}}$ is nef. Let $S \in |-\frac{1}{2}K|$ be smooth and real and $C \in |-K_S|$ a real member. If C is smooth, we have shown in [**CK**] that $a(Z) \leq 2$ and $\dim|-\frac{1}{2}K| \leq 2$. Assume C is not smooth. To compute the algebraic dimension consider the exact sequences

$$0 \to K^{-\frac{m-1}{2}} \to K^{-\frac{m}{2}} \to K_S^{-m} \to 0$$

and

$$0 \to K_S^{-(m-1)} \to K_S^{-m} \to N^{\otimes m} \to 0$$

with $N := K_S^{-1} \otimes \mathcal{O}_C$. Since $K^{-\frac{1}{2}}$ is nef, $(-K_S).C_i = 0$ for any component C_i of C. But $C_i \cong \mathbb{P}^1$ and so $N \otimes \mathcal{O}_{C_i} \cong \mathcal{O}_{C_i}$. This does not imply in general $N \cong \mathcal{O}_C$, because C is a "cycle" of rational curves. But we obtain $h^0(N^{\otimes m}) = 1$ if $N^{\otimes m} \cong \mathcal{O}_C$ and $h^0(N^{\otimes m}) = 0$ if $N^{\otimes m} \ncong \mathcal{O}_C$. As in [**CK**] this implies $a(Z) \leq 2$ and $h^0(K^{-\frac{1}{2}}) = h^0(\mathcal{O}_Z) + h^0(K_S^{-1}) = 1 + h^0(\mathcal{O}_S) + h^0(N) \leq 3$.
Assume now $\dim|-\frac{1}{2}K| = 2$. Hence, using the Riemann–Roch formula we have $h^1(K^{-\frac{1}{2}}) = 1$. By Proposition 2.3 this implies $a(Z) \geq 2$. From the above considerations we obtain $h^0(N) = 1$, hence $K_S^{-1} \otimes \mathcal{O}_C \cong N \cong \mathcal{O}_C$. (The same is true if C is smooth, cf. [**CK**].) The exact sequence $0 \to \mathcal{O}_S \to K_S^{-1} \to N \to 0$ and $h^1(\mathcal{O}_S) = 0$ give a surjective restriction map $H^0(K_S^{-1}) \twoheadrightarrow H^0(\mathcal{O}_C) \cong \mathbb{C}$. Because $C \in |-K_S|$, this shows that $|-K_S|$ does not have base points. Since $\dim|-K_S| = \dim|-\frac{1}{2}K| - 1 = 1$, Bertini's Theorem [**GH**, I §1] states the existence of a smooth member in $|-K_S|$. Hence, the generic divisor in $|-K_S|$ is smooth and so the generic real member, too. On the other hand, we know from $h^1(\mathcal{O}_Z) = 0$ that the restriction map $|-\frac{1}{2}K| \twoheadrightarrow |-K_S|$ is surjective. From the freeness of $|-K_S|$ we conclude that $|-\frac{1}{2}K|$ does not have base points. $\square$

Remark 4.2. If there exists a smooth real $S \in |-\frac{1}{2}K|$ and a smooth curve $C \in |-K_S|$, then $K^{-\frac{1}{2}}$ is nef. This is clear from the theorem, because $C.(-\frac{1}{2}K) = (-\frac{1}{2}K)^3 = 0$.

Corollary 4.3. *In Proposition 3.6 we can omit case (III) if $K^{-\frac{1}{2}}$ is not nef.*

Proof: Assume $C' = A' + \bar{A}'$ as in the proof of Proposition 3.6. A' and $\bar{A}'$ intersect at a pair of conjugate points, say P and $\bar{P}$.
If σ does not blow up P and $\bar{P}$, then, by reality of the blown–up set, on (the strict transforms of) A' and $\bar{A}'$ exactly four points are blown–up. If we denote by A and $\bar{A}$ the strict transforms of A' and $\bar{A}'$ in S, then we have $C = A + \bar{A}$ and $A^2 = \bar{A}^2 = -2$. Since A and $\bar{A}$ are rational we obtain, by the adjunction formula, $A.(-K_S) = \bar{A}.(-K_S) = 0$. By Theorem 4.1 this implies that $K^{-\frac{1}{2}}$ is nef.
If σ blows up P and $\bar{P}$, then we perform an elementary transform to arrive at case (I) as follows. Let $\sigma_1 : S^{(1)} \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow–up of P and $\bar{P}$, then we have an induced real structure on $S^{(1)}$. Since A' intersects any fibre at exactly one point, P and $\bar{P}$ lie on a conjugate pair of fibres. The strict transforms in $S^{(1)}$ of these fibres form a conjugate pair

of disjoint (-1)–curves. Contracting them we obtain a blow–down map $\sigma_1' : S^{(1)} \to \mathbb{P}^1 \times \mathbb{P}^1$ which is again compatible with real structures. If we denote by E and $\bar{E}$ the exceptional curves of σ_1, then the image of C in $S^{(1)}$ is $\sigma_1^*(A' + \bar{A}') - E - \bar{E} = A^{(1)} + \bar{A}^{(1)} + E + \bar{E}$. Here $A^{(1)}$ and $\bar{A}^{(1)}$ are the strict transforms of A' and $\bar{A}'$. The morphism σ_1' maps this curve onto a curve of type (I). $\square$

5. The non–nef case

Throughout this section we assume $K^{-\frac{1}{2}}$ to be not nef and $n = 4$.
By Theorem 4.1, Remark 4.2 we know that in any smooth real $S \in |-\frac{1}{2}K|$ the anticanonical system $|-K_S|$ contains only reducible elements. By Proposition 3.6 and Corollary 4.3 we can, therefore, choose a real blow–down $S \to \mathbb{P}^1 \times \mathbb{P}^1$ and a real reduced curve $C \in |-K_S|$, whose image C' in $\mathbb{P}^1 \times \mathbb{P}^1$ is of type (I) or (II) as described there. Observe that C' has type (I) if and only if C contains a real irreducible component.

Proposition 5.1. *If there exists a real irreducible curve intersecting $|K^{-\frac{1}{2}}|$ negatively, then $h^0(K^{-\frac{1}{2}}) = 3$ and $a(Z) = 3$.*
There exists a unique irreducible curve C_0 with $C_0.(-\frac{1}{2}K) < 0$. This curve is real, smooth and rational and $C_0.(-\frac{1}{2}K) = -2$. The base locus of $|-\frac{1}{2}K|$ is exactly C_0. Z does not contain divisors of degree one.

Proof: Let $S \in |-\frac{1}{2}K|$ be smooth and real and choose $C \in |-K_S|$ and $\sigma : S \to \mathbb{P}^1 \times \mathbb{P}^1$ with the properties of Proposition 3.6. Because any irreducible curve intersecting $K^{-\frac{1}{2}}$ negatively is contained in C, this curve has a real component. Therefore, the image C' of C in $\mathbb{P}^1 \times \mathbb{P}^1$ is of type (II). Let $C' = C_0' + F'$ be the decomposition of C'. By Lemma 3.3 none of the blown–up points lie on the real fibre F'. In particular, only smooth points of C_0' are blown–up. Hence, $C = C_0 + F$ where C_0 and F are the strict transforms of C_0' and F' respectively. Therefore, the eight blown–up points lie on C_0' which implies $C_0^2 = C_0'^2 - 8 = -4$.
By adjunction formula we obtain $-2 = C_0.(-K_S) = C_0.(-\frac{1}{2}K)$. Hence, $|-K_S| = C_0 + |F|$ and we obtain: $\dim|-K_S| = 1$ and C_0 is the base locus of $|-K_S|$. Since $h^1(\mathcal{O}_Z) = 0$ the restriction map $H^0(K^{-\frac{1}{2}}) \twoheadrightarrow H^0(K_S^{-1})$ is surjective. Hence, the linear system $|-\frac{1}{2}K|$ has dimension two and its base locus is precisely C_0 (with multiplicity one). C_0 is the unique irreducible curve in Z having negative intersection number with $-\frac{1}{2}K$, since any other such curve should be contained in the base locus of $|-\frac{1}{2}K|$.
If Z contains a divisor D of degree one, then $D + \bar{D} \in |-\frac{1}{2}K|$. If D_0 ($\bar{D}_0$ respectively) denotes the restriction of D to S, then $D_0 + \bar{D}_0 \in |-K_S| = C_0 + F$. In the proof of Lemma 3.3 we have seen that the real elements of $|F|$ are irreducible. Therefore, any real element of $|-K_S|$ consists of two distinct real irreducible curves with multiplicity one. This shows that $|-K_S|$ cannot contain a member of the form $D_0 + \bar{D}_0$.
It remains to show that the **algebraic dimension** of Z must be three in this case.
Let $\sigma : \tilde{Z} \to Z$ be the blow–up of the smooth rational curve C_0. By $E \subset \tilde{Z}$ we denote the exceptional divisor. Then we obtain a morphism $\pi : \tilde{Z} \to \mathbb{P}^2$ defined by the linear system $|-\frac{1}{2}K|$ such that $\pi^*\mathcal{O}(1) \cong \sigma^* K^{-\frac{1}{2}} \otimes \mathcal{O}_{\tilde{Z}}(-E)$. Since the restriction map $|-\frac{1}{2}K| \to |-K_S|$ is surjective, the restriction $\pi_{|S}$ is given by the linear system $|-K_S| = C_0 + |F|$. This means that π exibits S as the blow–up of a ruled surface and $\pi(S)$ is a line in $\mathbb{P}^2$. Since $\pi(\tilde{Z})$ is not contained in a linear subspace, π must be surjective. If we equip $\mathbb{P}^2$ with the

usual real structure, π becomes compatible with real structures since the linear system $|-\frac{1}{2}K|$ and the blown–up curve C_0 are real.
Since Z does not contain divisors of degree one, any real fundamental divisor S is irreducible and, therefore, smooth. By $\tilde{S} \subset \tilde{Z}$ we denote the strict transform of $S \in |-\frac{1}{2}K|$. Since C_0 is a smooth curve in a smooth surface, $\sigma : \tilde{S} \to S$ is an isomorphism. Furthermore, $E \cap \tilde{S}$ will be mapped isomorphically onto $C_0 \subset S$. Since $F.C_0 = 2$ and the restriction of π onto $\tilde{S}$ is the map defined by the linear system $|F|$, the restriction of π exibits $E \cap \tilde{S}$ as a double covering over $\pi(S) \cong \mathbb{P}^1$. Since real lines cover $\mathbb{P}^2$ the morphism $\pi : E \to \mathbb{P}^2$ does not contract curves and is of degree two.
Since generic fibres of π are smooth rational curves, the line bundle $\mathcal{O}_{\tilde{Z}}(E)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(2)$ on such fibres. Hence, after replacing (if necessary) $\mathbb{P}^2$ by the open dense set U of points having smooth fibre, the adjunction morphism $\pi^*\pi_*\mathcal{O}_{\tilde{Z}}(E) \to \mathcal{O}_{\tilde{Z}}(E)$ is surjective. This defines a U–morphism $\Phi : \tilde{Z} \to \mathbb{P}(\pi_*\mathcal{O}_{\tilde{Z}}(E))$, where $\pi_*\mathcal{O}_{\tilde{Z}}(E)$ is a locally free sheaf of rank three. The restriction of Φ to smooth fibres coincides with the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ of degree two. Therefore, the image of Φ is a three–dimensional subvariety of the $\mathbb{P}^2$–bundle $\mathbb{P}(\pi_*\mathcal{O}_{\tilde{Z}}(E)) \to U$. Hence, $\tilde{Z}$ is bimeromorphically equivalent to a quasiprojective variety and has, therefore, algebraic dimension three. □
For the rest of this section we assume that there does not exist a *real* irreducible curve contained in the base locus of $|-\frac{1}{2}K|$. We keep the assumptions $n = 4$ and $K^{-\frac{1}{2}}$ is not nef. In this situation, we obtain:

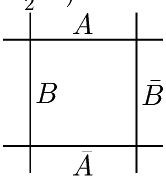
LEMMA 5.2. (a) *If $A \subset Z$ is an irreducible curve, then $A.(-\frac{1}{2}K) \geq -2$.*

(b) *If $A \subset Z$ is an irreducible curve with $A.(-\frac{1}{2}K) < 0$, then there exists at least a one–parameter family of real smooth divisors $S \in |-\frac{1}{2}K|$, containing a curve $C \in |-K_S|$ and possessing a birational morphism $\sigma : S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as in Proposition 3.6, such that moreover:*
A and $\overline{A}$ are components of C and for twistor fibres $F \subset S$ we have $F.A = F.\overline{A} = 1$. In particular, the image A' of A in $\mathbb{P}^1 \times \mathbb{P}^1$ is a "line", that means $A' \in |\mathcal{O}(1,0)|$.

PROOF: Our assumptions imply that C' is a curve of type (I) in Proposition 3.6. The components of C' are curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with self–intersection number zero. They are not real. Hence, after a succession of four blow–ups of a conjugate pair of points, each component A of C fulfills $A^2 \geq -4$ in S. The adjunction formula, together with the rationality of A, implies $A.(-\frac{1}{2}K) = A.(-K_S) = A^2 + 2 \geq -2$. Because a curve A with $A.(-\frac{1}{2}K) < 0$ must be a component of C, the assertion (a) is shown.
Let now $A \subset Z$ be an irreducible curve with $A.(-\frac{1}{2}K) < 0$. Then we have $A \subset C$. Let $x \in A \subset C$ be a smooth point of C and $x \in F \subset Z$ a twistor fibre. Since $|-\frac{1}{2}K|$ is at least a pencil, there exists a divisor $S \in |-\frac{1}{2}K|$ containing a given point $y \in F \setminus \{x, \overline{x}\}$. Because $F.S = 2$ and $S \cap F \supset \{y, x, \overline{x}\}$ the twistor fibre F is contained in S. So the real subsystem $|-\frac{1}{2}K|_F \subset |-\frac{1}{2}K|$ of divisors containing F is not empty. Hence, we can choose a real smooth $S \in |-\frac{1}{2}K|$ containing F. By construction, we have $F.A = F.\overline{A} \geq 1$. But $F.B \geq 0$ for any curve $B \subset S$ together with $F.(-K_S) = 2$ implies $F.A = F.\overline{A} = 1$. Because S contains only a real one–parameter family of real twistor lines, the intersection points with real twistor fibres form only a real one–dimensional subset of points z on A. Therefore, we obtain at least a one–parameter family of such surfaces S. Proposition 3.6 implies now the claim. □

PROPOSITION 5.3. *Assume the existence of an irreducible (non–real) curve $A \subset Z$ with $A.(-\frac{1}{2}K) = -2$. Then: $h^0(K^{-\frac{1}{2}}) = 4$ and $a(Z) = 3$.*
The curves A and $\bar{A}$ are disjoint smooth and rational. A and $\bar{A}$ are the unique irreducible reduced curves having negative intersection number with $-\frac{1}{2}K$. The base locus of $|-\frac{1}{2}K|$ is exactly the union of A and $\bar{A}$. Z contains infinitely many divisors of degree one and is one of the twistor spaces studied by LeBrun [**LeB1**] *and Kurke* [**Ku**].

PROOF: We choose $S \in |-\frac{1}{2}K|$ as in Lemma 5.2(b). Then we have $C \in |-K_S|$ containing A and $\overline{A}$ as smooth rational components which are mapped to "lines" A' and $\overline{A}' \in |\mathcal{O}(1,0)|$ in $\mathbb{P}^1 \times \mathbb{P}^1$. We have by the adjunction formula $A^2 = A.(-K_S) - 2 = -4$, which implies that the eight blown–up points lie on $A' + \overline{A}'$ (or the strict transforms after partial blow–ups). Hence, any member of $A' + \overline{A}' + |\mathcal{O}(0,2)|$ contains the eight blown–up points and defines, therefore, a divisor in $|-K_S|$. On the other hand, any curve in $|-K_S|$ is mapped onto a curve of type $(2,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ containing the blown–up points. Such a curve must contain A' and $\overline{A}'$ since the intersection number with A is two, but four of the blown–up points lie on A. Hence, the image of $|-K_S|$ is precisely the two–dimensional system $A' + \overline{A}' + |\mathcal{O}(0,2)|$. So we have $\dim|-K_S| = 2$. Using the exact sequence $0 \to \mathcal{O}_Z \to K^{-\frac{1}{2}} \to K_S^{-1} \to 0$, this implies $h^0(K^{-\frac{1}{2}}) = 4$. Furthermore, we see that $A + \bar{A}$ is in the base locus of $|-K_S|$ which coincides with the base locus of $|-\frac{1}{2}K|$.
To see that Z contains infinitely many divisors of degree one we modify an idea of Pedersen, Poon [**PP2**, p. 700]:
The above description of the image of $|-K_S|$ in $\mathbb{P}^1 \times \mathbb{P}^1$ shows that there exist infinitely many real curves $C \in |-K_S|$ whose image C' in $\mathbb{P}^1 \times \mathbb{P}^1$ is of type (I) and the singular points of C' are not blown–up. Then C is the strict transform of such a curve and C' consists of four irreducible components.
Let P and $\bar{P}$ denote a conjugate pair of singular points of C. Since A and $\bar{A}$ are disjoint, both points P and $\bar{P}$ are contained in $A + \bar{A}$, hence in the base locus of $|-\frac{1}{2}K|$. On the twistor space Z there exists exactly one real twistor line L_P connecting P with $\bar{P}$. Let $Q \in L_P$ be a point different from P and $\bar{P}$. The linear system $|-\frac{1}{2}K|_Q$ of all fundamental divisors containing Q has at least dimension $\dim|-\frac{1}{2}K| - 1 = \dim|-K_S| = 2$. Since fundamental divisors have degree two and any member of $|-\frac{1}{2}K|_Q$ contains P, $\bar{P}$ and Q it also contains L_P. Hence, $|-\frac{1}{2}K|_Q$ coincides with the *real* linear subsystem of divisors in $|-\frac{1}{2}K|$ containing L_P.
Choose now a point $R \in C$ which is not on $A+\bar{A}$. Then the real linear system $|-\frac{1}{2}K|_{L_P,R,\bar{R}}$ is non–empty. Let C be decomposed as $A + \bar{A} + B + \bar{B}$. Then, by our choice of C, $B.(-\frac{1}{2}K) = 2$ and $A.(-\frac{1}{2}K) = -2$. These four curves intersect as indicated in the following picture:



Any real member S_0 of $|-\frac{1}{2}K|_{L_P,R,\bar{R}}$ contains three distinct points of B, namely R and the intersection of B with $A + \bar{A}$. Hence, $B \subset S_0$. But this means that S_0 contains three curves, say A, B and L_P, which meet at $P \in S_0$. As in the proof of Proposition 3.7 we

obtain that S_0 is reducible, hence splits into the sum of two divisors of degree one. Since the intersection of a conjugate pair of divisors of degree one is a twistor line, we obtain in this way infinitely many divisors of degree one on Z.
By the result of Kurke–Poon [**Ku**], [**Po3**] we obtain that Z is a LeBrun twistor space and $A + \bar{A}$ is precisely the base locus of $|-\frac{1}{2}K|$. Hence, no other curve can have negative intersection number with $(-\frac{1}{2}K)$. This proves that really all properties of the Propositon are fulfilled. □

PROPOSITION 5.4. *Assume* $A.(-\frac{1}{2}K) \geq -1$ *for all irreducible curves* $A \subset Z$*. Then:* $h^0(K^{-\frac{1}{2}}) = 2$ *and* $a(Z) = 3$*.*
There exists a conjugate pair of irreducible curves A *and* $\bar{A}$ *which are smooth and rational and* $A.(-\frac{1}{2}K) = \bar{A}.(-\frac{1}{2}K) = -1$*. The base locus* C *of* $|-\frac{1}{2}K|$ *consists of a cycle of an even number of rational curves. The number of distinct divisors of degree one on* Z *is equal to the number of components of* C*.*

PROOF: By our assumption we obtain the existence of an irreducible curve $A \subset Z$ with $A.(-\frac{1}{2}K) = -1$. Now we choose $S \in |-\frac{1}{2}K|$ real and smooth and $C \in |-K_S|$ as in Lemma 5.2(b). The curve $C \in |-K_S|$ has A and $\overline{A}$ as components and the images A' and $\overline{A}'$ of A and $\overline{A}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ are members of $|\mathcal{O}(1,0)|$. But $A^2 = A.(-K_S) - 2 = -3$ implies that exactly one pair of blown–up points does not lie on $A' + \overline{A}'$. This implies that the two components of C', which are members of $|\mathcal{O}(0,1)|$, are not movable. Hence, $|-K_S| = \{C\}$ and, as above, $h^0(K^{-\frac{1}{2}}) = 2$. Because C' is of type (I), the curve C consists of 2, 3, 4, 5 or 6 pairs of conjugate rational curves.
By Proposition 3.7 we have: the number of components of C is equal to the number of effective divisors of degree one.
It remains to be seen that the **algebraic dimension** is three. Since $|-\frac{1}{2}K|$ is a pencil we study in more detail the linear system $|-K|$ on Z.
We need to investigate the structure of C before we can collect more information on the linear system $|-2K_S|$.
We know that the blow–up $\sigma : S \to S^{(0)} := \mathbb{P}^1 \times \mathbb{P}^1$ factors through a succession of four blow–ups $S = S^{(4)} \to S^{(3)} \to S^{(2)} \to S^{(1)} \to S^{(0)}$ such that at each step a conjugate pair of points is blown–up. The image of C in $S^{(i)}$ will be denoted by $C^{(i)}$. The blown–up points in $S^{(i)}$ should lie on $C^{(i)}$. If they are smooth points of $C^{(i)}$ then $C^{(i+1)} \xrightarrow{\sim} C^{(i)}$. If we blow up a conjugate pair of singular points of $C^{(i)}$, the curve $C^{(i+1)}$ has two components more than $C^{(i)}$. By assumption, $C^{(0)} = C' \subset \mathbb{P}^1 \times \mathbb{P}^1$ is of type (I). Each $C^{(i)}$ is a "cycle of rational curves". We can choose the factorization of σ in such a way that at the first k steps, only singular points of $C^{(i)}$ are blown–up and at the last $4 - k$ steps, only smooth points of $C^{(i)}$ are blown–up. Then C will have $2(2 + k)$ components, where $0 \leq k \leq 4$. If we would have a component A of C with $A^2 = 0$, then the image $A^{(0)}$ of A in $S^{(0)}$ would be a component of $C^{(0)}$ and none of the blown–up points would lie on $A^{(0)}$. But then four of the blown–up points must lie on a line or on a fibre in $S^{(0)}$, which implies that C has a component B with $B^2 = -4$. This was excluded by assumption. Therefore, for any component A of C we have $-1 \geq A^2 \geq -3$. Since A is a smooth rational curve, this means $A.(-K_S) \in \{-1, 0, +1\}$.
By assumption C is reduced. Let $C = \sum_{\nu=1}^{m} C_\nu$ be the decomposition of C into irreducible components. By Proposition 3.6 we have $C_\nu \cong \mathbb{P}^1$, $C_\nu.C_{\nu+1} = 1$ and C_ν intersects only $C_{\nu-1}$ and $C_{\nu+1}$. (This means "C is a cycle of rational curves". For convenience, we use

cyclic subscripts, that is $C_\nu = C_{\nu+m}$.) For $\varepsilon \in \{-1, 0, +1\}$ we define $I_\varepsilon := \{\nu | C_\nu.(-K_S) = \varepsilon\}$ and $C_\varepsilon := \sum_{\nu \in I_\varepsilon} C_\nu$. In this way we split C into three parts $C = C_- + C_0 + C_+$. As $C.(-K_S) = 0$ we have $|I_-| = |I_+|$. The assumption that $K^{-\frac{1}{2}}$ is not nef implies $I_- \neq \emptyset$. The curve C has no real component, hence $|I_-| = |I_+| \geq 2$.

Claim: Any two components of C_+ are disjoint.

Let C_α and C_β be two distinct components of C_+. Then $C_\alpha^2 = C_\beta^2 = -1$. If C_α and C_β are both contracted to a point on $S^{(0)}$, they are obviously disjoint. If C_α and C_β are mapped to curves in $S^{(0)}$, by our choice of S both must be members of $|\mathcal{O}(0,1)|$, because A and $\overline{A}$ are components of C_-. Finally, we have to exclude the case where C_α is mapped to a curve $C'_\alpha \in |\mathcal{O}(0,1)|$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and C_β is contracted to a point $P \in C'_\alpha$. This implies $C_\alpha \neq \overline{C}_\beta$. Since C_β is a component of the anticanonical divisor $C \subset S$, the point P must be singular on C'. Thus, we can take $S^{(1)} \longrightarrow S^{(0)} = \mathbb{P}^1 \times \mathbb{P}^1$ to be the blow–up of P and $\overline{P}$. The curve $C^{(1)}$ consists then of six (-1)–curves among which we find the images of $C_\alpha, C_\beta, \overline{C}_\alpha$ and $\overline{C}_\beta$. Because those are (-1)–curves on S, the remaining six blown–up points must lie on one pair of (-1)–curves giving rise to (-4)–curves in C contradicting our assumption.

Thus, the claim is proved and C_+ is the disjoint union of an even number of smooth rational (-1)–curves.

Since C is a cycle of rational curves, the curve $C \setminus C_+ = C_- + C_0$ has the same number of connected components as C_+.

We claim that each connected component of $C_- + C_0$ contains exactly one component of C_-. Since C_- and C_+ have the same number of components, this is equivalent to the statement that each connected component of $C_- + C_0$ contains at most one component of C_-.

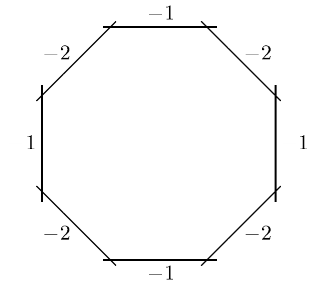
Assume the contary, that is there is a connected component of $C_- + C_0$ containing two irreducible components of C_-. Then these two components of C_- are not conjugate to each other. (Two conjugate components of C are "opposite" in the cycle C.) Therefore, C_- has at least four components and so C_+. Thus $C_- + C_0$ has at least four connected components, hence at least six irreducible ones. Therefore, C contains at least ten irreducible components, that is $k \geq 3$. Therefore, the image $C^{(3)}$ of C in $S^{(3)}$ consists of a cycle of ten rational curves with self–intersection numbers $-2, -1, -3, -1, -2, -2, -1, -3, -1, -2$ (in this order). By assumption, at the last step of blow–up, no point on a (-3)–curve is blown–up. To obtain a second pair of (-3)–curves, we have to blow up points on a conjugate pair of (-2)–curves. Such curves are not neighbours in the cycle $C^{(3)}$, they are opposite to each other. Thus, one easily sees that after the last step of blow–up, between two (-3)–curves on the cycle C we always have a (-1)–curve. This means that no connected component of $C \setminus C_+$ contains two irreducible components of C_-, as claimed.

We can now compute the dimension of $|-K|$ and $|-2K_S|$.

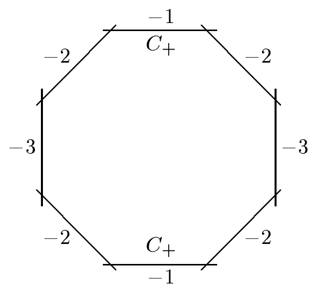
The Riemann–Roch formula and $h^0(K^{-\frac{1}{2}}) = 2$ imply $h^1(K^{-\frac{1}{2}}) = 0$. Hence, the exact sequence $0 \to K^{-\frac{1}{2}} \to K^{-1} \to K_S^{-2} \to 0$ gives $h^0(K^{-1}) = h^0(K^{-\frac{1}{2}}) + h^0(K_S^{-2}) = 2 + h^0(K_S^{-2})$ and a surjective restriction map $|-K| \twoheadrightarrow |-2K_S|$. With $N := K_S^{-1} \otimes \mathcal{O}_C$ we obtain an exact sequence $0 \to K_S^{-1} \to K_S^{-2} \to N^{\otimes 2} \to 0$. Since $h^1(K_S^{-1}) = h^1(K^{-\frac{1}{2}}) = 0$, this sequence yields $h^0(K_S^{-2}) = h^0(K_S^{-1}) + h^0(N^{\otimes 2}) = 1 + h^0(N^{\otimes 2})$. We can apply Lemma 3.8 with $L = K_S^{-2}$, because we have seen that the components of C_+ are disjoint to each other and that each connected component of $C \setminus C_+$ contains exactly one irreducible component

of C_-. We obtain $h^0(N^{\otimes 2}) = h^0(C, L) = \sum_{\nu\in I_+} C_\nu.(-2K_S) - |I_-| = 2|I_+| - |I_-| = |I_+|$. Hence, $\dim|-2K_S| = |I_+| \geq 2$ and $\dim|-K| = 2 + \dim|-2K_S| \geq 4$.
Next we study the base locus of $|-2K_S|$.
Since any component A of C_- fulfills $A.(-K_S) = -1$, C_- is in the base locus of $|-2K_S|$. Now let A be an irreducible component of C_0. Since, as we have shown above, any connected component of $C_0 + C_-$ contains a curve $B \subset C_-$ there exists a finite chain of components $A_1, \ldots, A_r = A$ of C_0 with $B \cap A_1 \neq \emptyset$ and $A_i \cap A_{i+1} \neq \emptyset \quad (1 \leq i < r)$. But for a component A_i of C_0 we have $A_i.(-2K_S) = 0$ which implies: if A_i intersects the base locus of $|-2K_S|$, it must be contained in this base locus. Hence, by induction on i, we obtain that A_i is contained in the base locus of $|-2K_S|$ for all $1 \leq i \leq r$. This shows that $C_0 + C_-$ is contained in the base locus of the linear system $|-2K_S|$. Therefore, we have $|-2K_S| = C_0 + C_- + |-K_S + C_+|$ and the map defined by $|-2K_S|$ coincides with the map given by $|-K_S + C_+|$.
We have seen above that C_+ is a disjoint union of smooth rational (-1)–curves. We can, therefore, contract C_+ to obtain a smooth surface S'. If $\sigma' : S \to S'$ is this contraction, we have $K_S^{-1} \otimes \mathcal{O}_S(C_+) \cong {\sigma'}^*(K_{S'}^{-1})$. Hence, $-K_{S'}$ is nef if and only if $-K_S + C_+$ is nef.
First we deal with the case where $C_+ - K_S$ is nef. In this case, the generic member of the moving part of $|-2K_S|$ is irreducible. This can be seen as follows:
Observe first that S' can be blown down to $\mathbb{P}^2$. This follows from [**Dem**, Prop. 3, p. 48] because $-K_{S'}$ is nef and $K_{S'}^2 = |I_+| > 0$.
Because C_+ has 2, 4 or 6 components, $S' \to \mathbb{P}^2$ is a blow–up of 7, 5 or 3 points. Therefore, we can apply a theorem of Demazure [**Dem**, p. 39 and p. 55] stating that $|-K_{S'}|$ contains a smooth irreducible member and is base point free if $|-K_{S'}|$ is nef. Hence, there exists a smooth irreducible curve in $|-K_{S'}|$ avoiding the blown–up points. Its preimage in S is a smooth irreducible member of $|\sigma^* K_{S'}^{-1}| = |C_+ - K_S| = |-2K_S - C_0 - C_-|$ which is, therefore, the moving part of $|-2K_S|$.
Since $\dim|-2K_S| \geq 2$ and the generic member of the moving part of $|-2K_S|$ is a smooth irreducible curve, the image of the map defined by $|-2K_S|$ has dimension two. We have seen before that the restriction $|-K| \twoheadrightarrow |-2K_S|$ is surjective which implies that the map $\Phi_{|-K|}$ given by $|-K|$ on Z coincides, after restriction to S, with the map given by $|-2K_S|$. Since $2S \in |-K|$, the $\Phi_{|-K|}$–image of S is contained in a hyperplane. But the $\Phi_{|-K|}$–image of Z cannot be contained in a hyperplane. Therefore, the image of $\Phi_{|-K|}$ has dimension three. This implies $a(Z) = 3$.
We are left with the case where $C_+ - K_S$ is not nef.
In this case we shall see that $\Phi_{|-K|}$ has only two–dimensional image but equips Z with a conic–bundle structure. Under the assumption that $C_+ - K_S$ is not nef we study the structure of C.
Let A be an irreducible curve in S with $A.(C_+ - K_S) < 0$. If $A \nsubseteq C$, then $A.C_+ \geq 0$. But the base locus of $|-K_S|$ is contained in C and this implies $A.(-K_S) \geq 0$. Hence, we have necessarily $A \subseteq C$. If $A \subseteq C_+$, then $A.(-K_S) = 1$ and $A.C_+ \geq A^2$, hence, $A.(C_+ - K_S) \geq A^2 + 1 = 0$. If $A \subseteq C_0$, then $A.(-K_S) = 0$ and $A.C_+ \geq 0$, hence, $A.(C_+ - K_S) \geq 0$. If, finally, $A \subseteq C_-$, then $A.(C_+ - K_S) = A.C_+ - 1 \geq -1$. So, we obtain: the irreducible curves $A \subset S$ with $A.(C_+ - K_S) < 0$ are exactly those components of C_- which are disjoint to C_+. They fulfill $A.(C_+ - K_S) = -1$.
As we have seen above, each connected component of $C \setminus C_+$ contains exactly one irreducible component of C_-. Hence, if A is a component of C_- which does not meet C_+,

then its connected component should contain at least two curves from C_0. Thus, using reality, we see that C has at least eight components. This means, using the convention introduced above, the image $C^{(2)}$ of C in $S^{(2)}$ (the surface obtained after two steps of blow–up) consists of eight curves whose self–intersection numbers are alternately -1 and -2:



One now easily sees: if we were to blow up a pair of singular points on $C^{(2)}$, in the resulting curve $C^{(3)}$ any (-2)–curve would meet a (-1)–curve and any (-3)–curve would meet two (-1)–curves. Therfore, after the last step of blow–up, no (-3)–curve is disjoint to all (-1)–curves on C. Thus, we can only blow up smooth points in the last two steps. If we were to blow up a conjugate pair of points on (-2)–curves, the resulting (-3)–curve would intersect two (-1)–curves. Then, again, in C there would be no (-3)–curve disjoint to all (-1)–curves. So we conclude that the last two pairs of conjugate blown–up points cannot lie on (-2)–components of $C^{(2)}$. If each of the four (-1)–curves in $C^{(2)}$ contains one of the blown–up points, then any component of C has zero intersection number with $-K_S$. But, then by Theorem 4.1, $K^{-\frac{1}{2}}$ would be nef which contradicts our general assumption. Thus, the four blown–up points lie on a pair of conjugate (-1)–curves. The structure of C is, therefore, the following:



In particular, we obtain: $\dim|-2K_S| = |I_+| = 2$ and $\dim|-K| = 2 + \dim|-2K_S| = 4$. Furthermore, since both components of C_- have negative intersection number with $C_+ - K_S$, the curve C_- is contained in the base locus of $|C_+ - K_S|$. This means $|-2K_S| = |2C_+ + C_0| + C_0 + 2C_-$. By our choice of S, the two components A and $\overline{A}$ of C_- are mapped onto lines A' and $\overline{A}'$ on $\mathbb{P}^1 \times \mathbb{P}^1$. The above analysis of C shows that we can decompose $\sigma : S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ into the following steps:
First we blow up a conjugate pair of singular points on the curve C' (which is of type (I)). This produces precisely two singular fibres of the ruling (whose general fibre is the image of a twistor fibre). In the second step we blow up the two singular points of these singular fibres. The exceptional curves of this blow–up form the components of C_+. Because we blow up points of multiplicity two on the fibres, the total transform of the two singular fibres contains $2C_+$. In the remaining two steps we have to blow up smooth points on $A' + \overline{A}'$. Hence, we obtain $C_0 + 2C_+ \in |2F|$. So, we can write $|-2K_S| = C_0 + 2C_- + |2F|$.

Since we have $\dim|-\frac{1}{2}K| = 1$, this is true for the generic real surface $S \in |-\frac{1}{2}K|$ by Lemma 5.2 (b). Let us denote by $\Phi = \Phi_{|-K|}$ the meromorphic map $Z \dashrightarrow \mathbb{P}^4$ defined by $|-K|$. If $\varphi : S \to \mathbb{P}^2$ is the restriction of Φ to a generic smooth real $S \in |-\frac{1}{2}K|$, then the image of φ is a conic in $\mathbb{P}^2$. The general fibre of φ is a twistor fibre, hence a smooth rational curve intersecting C transversally at two points lying on A and $\overline{A}$ respectively. Let $\tilde{Z} \to Z$ be a modification such that Φ becomes a morphism $\tilde{\Phi} : \tilde{Z} \to \mathbb{P}^4$. Because the smooth real fundamental divisors S sweep out a Zariski dense subset of Z, the image of this set is also Zariski dense in $\tilde{\Phi}(\tilde{Z}) \subset \mathbb{P}^4$. As the general fibre of Φ, restricted to such surfaces S, is one–dimensional, we obtain $\dim \tilde{\Phi}(\tilde{Z}) = 2$. Since $\tilde{Z} \to Z$ is a modification, there exists an open Zariski dense subset $U \subset \tilde{\Phi}(\tilde{Z})$ such that the fibres of $\tilde{\Phi}$ are irreducible curves. Moreover, we can choose U such that the fibres of $\tilde{\Phi}$ over U are isomorphic to $\mathbb{P}^1$,because this is true over a Zariski dense subset of $\tilde{\Phi}(\tilde{Z})$. Let $\tilde{\Phi} : \tilde{Z}_U \to U$ denote the restriction of Φ over U. Then the preimage in $\tilde{Z}$ of the two components of C_- defines a pair of divisors Σ and $\bar{\Sigma}$ in $\tilde{Z}_U$ which are sections of $\tilde{Z}_U \to U$. Therefore, $\mathcal{E} := \tilde{\Phi}_* \mathcal{O}_{\tilde{Z}_U}(\Sigma + \bar{\Sigma})$ is a vector bundle of rank three on U and the canonical morphism $\tilde{\Phi}^*\mathcal{E} \to \mathcal{O}(\Sigma + \bar{\Sigma})$ is surjective. This means that we obtain a morphism $\tilde{Z}_U \to \mathbb{P}(\mathcal{E})$ which is compatible with the projections to U. Restricted to each fibre this morphism is the Veronese embedding of degree two $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$. Hence, the image of $\tilde{Z}_U$ in the quasi–projective variety $\mathbb{P}(\mathcal{E})$ is three–dimensional. This implies $a(Z) = a(\tilde{Z}_U) = 3$, which completes the proof. □

6. Conclusions

In this section we collect the results of this paper to obtain a clear picture of the situation considered. By Z we always denote a simply connected compact twistor space of positive type over $4\mathbb{CP}^2$.
We call $\mathcal{N} := \{C \subset Z \mid C \text{ irreducible curve}, C.(-\frac{1}{2}K) < 0\}$ the set of negative curves. By definition, $K^{-\frac{1}{2}}$ is nef if and only if $\mathcal{N} \neq \emptyset$. The structure of $\mathcal{N}$ is described by the following

THEOREM 6.1. *If $\mathcal{N} \neq \emptyset$ this set consists of a finite number of smooth rational curves. More precisely, only the following cases are possible:*

(a) *$\mathcal{N}$ contains a real member C_0. Then: $\mathcal{N} = \{C_0\}$ and $C_0(-\frac{1}{2}K) = -2$, $\dim|-\frac{1}{2}K| = 2$ and $a(Z) = 3$.*
(b) *$\mathcal{N}$ contains a non–real member A with $A.(-\frac{1}{2}K) = -2$. Then, $\mathcal{N} = \{A, \overline{A}\}$, $\dim|-\frac{1}{2}K| = 3, a(Z) = 3$ and Z is a LeBrun twistor space.*
(c) *Each member $A \in \mathcal{N}$ fulfills $A.(-\frac{1}{2}K) = -1$. Then $|\mathcal{N}| \in \{2, 4, 6\}$, $\dim|-\frac{1}{2}K| = 1$ and $a(Z) = 3$.*

PROOF: We have only to collect the results of Section 5. □
We can compute the algebraic dimension in the following way:

THEOREM 6.2. *$a(Z) = 3 \iff K^{-\frac{1}{2}}$ is not nef;*
$a(Z) = 2 \iff K^{-\frac{1}{2}}$ is nef and $\exists m \geq 1 : h^1(K^{-\frac{m}{2}}) \neq 0$;
$a(Z) = 1 \iff \forall m \geq 1 : h^1(K^{-\frac{m}{2}}) = 0$.

PROOF: This results from Proposition 2.3 and Theorems 4.1 and 6.1. □
We can characterize Moishezon twistor spaces as follows:

Theorem 6.3. *The following conditions are equivalent:*
(i) $a(Z)=3$;
(ii) $K^{-\frac{1}{2}}$ *is not nef;*
(iii) *there exists a smooth rational curve* $C \subset Z$ *with* $C.(-\frac{1}{2}K) < 0$.

Proof: Apply Theorems 6.1 and 6.2. □

Remark 6.4. Remembering that, by Poon's theorem, $K^{-\frac{1}{2}}$ is big if and only if Z is Moishezon, we obtain from the preceding theorem: the line bundle $K^{-\frac{1}{2}}$ is never nef and big (under our special assumptions).

LeBrun twistor spaces are characterized (see [**Ku**], [**Po3**]) by the property to contain a pencil of divisors of degree one. We can give (for the case $n = 4$) two further characterizations:

Theorem 6.5. *The following properties are equivalent:*
(i) *Z contains a pencil of divisors of degree one;*
(ii) $\dim|-\frac{1}{2}K| = 3$;
(iii) *there exists a smooth rational curve* $A \subset Z$ *with* $A \neq \bar{A}$ *and* $A.(-\frac{1}{2}K) = -2$.

Proof: The implications (i)⇒(ii) and (i)⇒(iii) follow from the Kurke–Poon theorem. The reverse implications follow from Theorem 6.1. □

Theorem 6.6. $a(Z) \geq \dim|-\frac{1}{2}K|$.

Proof: This follows directly from Proposition 2.3 and Theorems 4.1 and 6.1. □
If $|-\frac{1}{2}K|$ is not a pencil, we obtain the following nice result:

Theorem 6.7. *If* $\dim|-\frac{1}{2}K| \geq 2$, *then:*
$a(Z) = 2 \iff K^{-\frac{1}{2}}$ *is nef* $\iff |-\frac{1}{2}K|$ *does not have base points.*

Proof: The first equivalence results from the previous theorem and Theorem 6.2. If $|-\frac{1}{2}K|$ does not have base points, $K^{-\frac{1}{2}}$ is necessarily nef. If $K^{-\frac{1}{2}}$ is nef and $\dim|-\frac{1}{2}K| \geq 2$ we have seen in Theorem 4.1 that $|-\frac{1}{2}K|$ is base point free. □

Corollary 6.8. $|-\frac{1}{2}K|$ *is base point free* $\Rightarrow a(Z) = 2$.

Proof: This is immediate from the previous theorem, because a pencil $|-\frac{1}{2}K|$ has always base points. □

Remark 6.9. The reverse implication is not true, which follows from the existence theorem in [**CK**]. There, twistor spaces with $a(Z) = 2$ and $\dim|-\frac{1}{2}K| = 1$ over $4\mathbb{CP}^2$ were constructed.

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