A DISCREPANCY PRINCIPLE FOR TIKHONOV REGULARIZATION WITH APPROXIMATELY SPECIFIED DATA

M. Thamban Nair[1] and Eberhard Schock

Abstract

Many discrepancy principles are known for choosing the parameter α in the regularized operator equation $(T^*T + \alpha I)x_\alpha^\delta = T^*y^\delta$, $\|y - y^\delta\| \leq \delta$, in order to approximate the minimal norm least-squares solution of the operator equation $Tx = y$. In this paper we consider a class of discrepancy principles for choosing the regularization parameter when T^*T and T^*y^δ are approximated by A_n and z_n^δ respectively with A_n not necessarily self–adjoint. This procedure generalizes the work of Engl and Neubauer (1985), and particular cases of the results are applicable to the regularized projection method as well as to a degenerate kernel method considered by Groetsch (1990).

1 Introduction

We are concerned with the problem of finding approximations to the minimal norm least-squares solution $\hat{x}$ of the operator equation

$$Tx = y, \tag{1.1}$$

where $T : X \to Y$ is a bounded linear operator between Hilbert spaces X and Y, and y belongs to $D(T^\dagger) := R(T) + R(T)^\perp$, the domain of the Moore-Penrose inverse $T^\dagger$ of T. It is well known [8] that if the range $R(T)$ of T is not closed, then the operator $T^\dagger$ which associates $y \in D(T^\dagger)$ to $\hat{x} := T^\dagger y$, the unique least-squares solution of minimal norm, is not continuous, and

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consequently the problem of solving (1.1) for $\hat{x}$ is ill-posed. A prototype of an ill-posed problem is the Fredholm integral equation of the first kind

$$\int_0^1 k(s,t)x(t)\,dt = y(s), \quad 0 \le s \le 1, \tag{1.2}$$

with nondegenerate kernel $k(.,.) \in L^2([0,1] \times [0,1])$, where $X = Y = L^2[0,1]$. Regularization methods are employed to find approximations to $\hat{x}$. In Tikhonov regularization one looks for the unique x_α, $\alpha > 0$, which minimizes the functional

$$x \to \|Tx - y\|^2 + \alpha\|x\|^2, \quad x \in X,$$

equivalently, one solves the well-posed equation

$$(T^*T + \alpha I)x_\alpha = T^*y \tag{1.3}$$

for each $\alpha > 0$. Since $T^*T\hat{x} = T^*y$, it follows that

$$\|\hat{x} - x_\alpha\| = \|\alpha(T^*T + \alpha I)^{-1}\hat{x}\| \le \|\hat{x}\|. \tag{1.4}$$

It is known ([8], [16]) that

$$\|\hat{x} - x_\alpha\| \to 0 \quad \text{as} \quad \alpha \to 0 \tag{1.5}$$

and

$$\hat{x} \in R((T^*T)^\nu), \quad 0 \le \nu \le 1, \quad \text{implies} \quad \|\hat{x} - x_\alpha\| = O(\alpha^\nu). \tag{1.6}$$

In practical applications the data y may not be available exactly, instead one may have an approximation y^δ with say $\|y - y^\delta\| \le \delta$, $\delta > 0$. Then one solves the equation

$$(T^*T + \alpha I)x_\alpha^\delta = T^*y^\delta \tag{1.7}$$

instead of (1.3) and requires $\|\hat{x} - x_\alpha^\delta\| \to 0$ as $\alpha \to 0$ and $\delta \to 0$. It follows from (1.3) and (1.7) that

$$\begin{aligned}
\|x_\alpha - x_\alpha^\delta\|^2 &= \|(T^*T + \alpha I)^{-1}T^*(y - y^\delta)\|^2 \\
&= \langle (T^*T + \alpha I)^{-1}T^*(y - y^\delta), (T^*T + \alpha I)^{-1}T^*(y - y^\delta)\rangle \\
&= \langle (TT^* + \alpha I)^{-2}TT^*(y - y^\delta), (y - y^\delta)\rangle \\
&\le \|(TT^* + \alpha I)^{-2}TT^*\|\ \|(y - y^\delta\|^2 \\
&\le \frac{\delta^2}{\alpha},
\end{aligned}$$

so that

$$\|\hat{x} - x_\alpha^\delta\| \leq \|\hat{x} - x_\alpha\| + \delta/\sqrt{\alpha}. \tag{1.8}$$

Now let $R_\alpha = (T^*T + \alpha I)^{-1}T^*$ for $\alpha > 0$. Then by (1.5) we have

$$\|R_\alpha y - T^\dagger y\| \to 0 \quad \text{as} \quad \alpha \to 0$$

for $y \in D(T^\dagger)$. Therefore, if $R(T)$ is not closed, then the family $\{R_\alpha\}_{\alpha>0}$ is not uniformely bounded so that, as a consequence of Uniform Boundedness Principle, there exists $v \in Y$ such that $\{R_\alpha v\}_{\alpha>0}$ is not bounded in Y. In particular, if $y^\delta = y + \delta v/\|v\|$, then $\|y - y^\delta\| \leq \delta$ and $\{R_\alpha y^\delta\}_{\alpha>0}$ is unbounded in Y. Therefore, the problem of choosing the regularization parameter α depending on y^δ is important. Many works in the literature are devoted to this aspect (c.f. [7], [17], [1], [2], [3], [6], [14], [4]).

In order to solve (1.7) numerically, it is required to consider approximations of T^*T and of T^*y^δ. So the problem actually at hand would be of the form

$$(A_n + \alpha I)x_{\alpha,n}^\delta = z_n^\delta, \tag{1.9}$$

where (A_n) and (z_n^δ) are approximations of T^*T and of T^*y^δ respectively.

In the well known regularized projection methods (c.f. [10], [2], [3]),

$$A_n = P_n T^* T P_n \quad \text{and} \quad z_n^\delta = P_n T^* y^\delta,$$

where (P_n) is a sequence of orthogonal projections on X such that $P_n \to I$ pointwise. In this case we have

$$\|T^*T - A_n\| \to 0 \quad \text{as} \quad n \to \infty,$$

and discrepancy principles are known for choosing the regularization parameter α in (1.9) (See e.g. [2], [3], [13], [5]).

In the degenerate kernel methods for the integral equation (1.2), A_n is obtained by approximating the kernel $\tilde{k}(.,.)$ of the integral operator T^*T by a degenerate kernel $\tilde{k}_n(.,.)$ so that $\|\tilde{k} - \tilde{k}_n\|_\infty \to 0$ as $n \to \infty$. Then it follows that

$$\|T^*T - A_n\| \leq \|\tilde{k} - \tilde{k}_n\|_2 \leq \|\tilde{k} - \tilde{k}_n\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

(See [11] and [12] for a discussion on degenerate kernel methods for integral equations). In a degenerate kernel method considered by Groetsch [9] the approximation $\tilde{k}_n(.,.)$ is obtained from

$$\tilde{k}(s,t) := \int_0^1 k(\tau,s)k(\tau,t)\,dt, \quad a \leq s,t \leq b.$$

by using a convergent quadrature rule. In this case one has $\|\tilde{k}-\tilde{k}_n\|_\infty \to 0$ as $n\to\infty$ for nice enough kernels $k(.,.)$.

Moreover, for the degenerate kernel method of Groetsch [9] as well as for the regularized projection methods, the operators A_n are non-negative and self–adjoint.

In this paper we consider the generalized form of a class of discrepancy principles in [1], namely,

$$\|A_n x^\delta_{\alpha,n} - z^\delta_n\| = \frac{\delta^p}{\alpha^q}, \quad p>0, q>0, \tag{1.10}$$

for large enough n, to choose the regularization parameter $\alpha=\alpha(n,\delta)$ in (1.9), where (A_n) is a sequence of bounded linear operators on X and (z^δ_n) in X such that

$$\|T^*T - A_n\|\to 0 \quad \text{and} \quad \|T^*y^\delta - z^\delta_n\|\to 0 \quad \text{as} \quad n\to\infty.$$

It has to be observed that we do not assume the operators A_n to be non-negative and self–adjoint. The consideration of a general A_n, as has been done recently by Nair [15], is important from a computational point of view, because even if one starts with a non-negative self–adjoint operator as approximation of T^*T, due to truncation errors etc., one actually may not be dealing with a non-negative self–adjoint operator.

With α chosen according to (1.10), we show the convergence of the solution $x^\delta_{\alpha,n}$ of (1.9) to $\hat{x}$ as $\delta\to 0$, $n\to\infty$, and also obtain estimates for the error $\|\hat{x}-x^\delta_{\alpha,n}\|$ whenever $\hat{x}\in R((T^*T)^\nu)$, $0<\nu\le 1$. Our result on error estimates shows that if ν_0 is an estimate for the possibly unknown ν, with $0<\nu\le\nu_0\le 1$, then taking $p/(q+1)=2/(2\nu_0+1)$ one obtains the rate $O(\delta^{2\nu/(2\nu_0+1)})$. In particular, prior knowledge of ν enables us to yield the *optimal* rate $O(\delta^{2\nu/(2\nu+1)})$ (c.f. Schock [16]).

If $A_n = P_nT^*TP_n$ and $z^\delta_n = P_nT^*y^\delta$ then (1.10) coincides with a discrepancy principle considered by Engl and Neubauer [2] and we recover the optimal result in [2] as a particular case. Thus this paper generalizes the type of results in [2] and [9] for projection methods and degenerate kernel method for integral equations respectively, providing also a parameter choice strategy in the latter case.

2 Approximate Solution and Convergence

Let X and Y be Hilbert spaces and $T : X \to Y$ be a bounded linear operator with its range $R(T)$ not necessarily closed in Y. Let $y \in D(T^\dagger) := R(T) + R(T)^\perp$, $y \neq 0$, so that there exists a unique $\hat{x} \in X$ of minimal norm such that

$$\|T\hat{x} - y\| = \inf\{\|Tx - y\| : x \in X\}.$$

Let (A_n) be a sequence of bounded linear operators on X and for $\delta > 0$, let $y^\delta \in Y$, (z_n^δ) in X such that

$$\|T^*T - A_n\| \le \epsilon_n,$$
$$\|y - y^\delta\| \le \delta,$$
$$\|T^*y^\delta - z_n^\delta\| \le \eta_n^\delta,$$

where (ϵ_n) and (η_n^δ) are sequences of nonnegative real numbers such that

$$\epsilon_n \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\eta_n^\delta \to 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad \delta \to 0. \tag{2.1}$$

Throughout the paper we denote the operator T^*T by A, and c, c', c_1, c_2, etc., denote positive real constants which may assume different values at different contexts.

THEOREM 2.1 *If $\epsilon_n \le c_0\alpha$ with $0 < c_0 < 1$, then $A_n + \alpha I$ is bijective and*

$$\|(A_n + \alpha I)^{-1}\| \le 1/\alpha(1 - c_0).$$

More over, if x_α^δ and $x_{\alpha,n}^\delta$ are the unique solutions of (1.7) and (1.9) respectively, then

$$\|\hat{x} - x_{\alpha,n}^\delta\| \le c\left(\|\hat{x} - x_\alpha^\delta\| + \frac{\eta_n^\delta}{\alpha} + \frac{\epsilon_n}{\alpha}\right). \tag{2.2}$$

In particular, if $\alpha := \alpha(\delta, n)$ is chosen in such a way that

$$\alpha(\delta, n) \to 0, \quad \frac{\delta}{\sqrt{\alpha(\delta, n)}} \to 0, \quad \frac{\epsilon_n}{\alpha(\delta, n)} \to 0 \quad \textit{and} \quad \frac{\eta_n^\delta}{\alpha(\delta, n)} \to 0$$

as $\delta \to 0$ and $n \to \infty$, then

$$\|\hat{x} - x_{\alpha,n}^\delta\| \to 0 \quad \textit{as} \quad \delta \to 0 \quad n \to \infty.$$

Proof. Since A is nonnegative and self–adjoint, it follows from spectral theory that for each $\alpha > 0$, $(A+\alpha I)^{-1}$ exists as a bounded linear operator on X and

$$\|(A+\alpha I)^{-1}\| \le \frac{1}{\alpha}.$$

Therefore, if $\|A - A_n\| < 1/\|(A+\alpha I)^{-1}\|$ then, by results on perturbation of operators, $(A_n + \alpha I)^{-1}$ exists and is a bounded operator, and

$$\begin{aligned} \|(A_n+\alpha I)^{-1}\| &\le \frac{\|(A+\alpha I)^{-1}\|}{1 - \|A-A_n\|\|(A+\alpha I)^{-1}\|} \\ &\le \frac{1/\alpha}{1-\epsilon_n/\alpha} \\ &\le \frac{1}{\alpha(1-c_0)}. \end{aligned}$$

Now let $w^\delta_{\alpha,n}$ be the unique solution of the equation (1.9) with $T^* y^\delta$ in place of z^δ_n,i.e.,

$$(A_n + \alpha I) w^\delta_{\alpha,n} = T^* y^\delta. \tag{2.3}$$

Then from (1.7), (1.9) and (2.3), we have

$$x^\delta_{\alpha,n} - w^\delta_{\alpha,n} = (A_n+\alpha I)^{-1}(z^\delta_n - T^* y^\delta)$$

and

$$w^\delta_{\alpha,n} - x^\delta_\alpha = (A_n+\alpha I)^{-1}(A - A_n) x^\delta_\alpha.$$

Since $\epsilon_n \le c_0\alpha$, it follows that

$$\|x^\delta_{\alpha,n} - w^\delta_{\alpha,n}\| \le c_1 \frac{\eta^\delta_n}{\alpha}$$

and

$$\|w^\delta_{\alpha,n} - x^\delta_\alpha\| \le c_2 \left(\|\hat{x} - x^\delta_\alpha\| + \frac{\epsilon_n}{\alpha} \right),$$

so that

$$\|\hat{x} - x^\delta_{\alpha,n}\| \le c \left(\|\hat{x} - x^\delta_\alpha\| + \frac{\eta^\delta_n}{\alpha} + \frac{\epsilon_n}{\alpha} \right).$$

Now the assumptions on $\alpha := \alpha(\delta, n)$ together with (1.6) and (1.8) imply the convergence $\|\hat{x} - x^\delta_{\alpha,n}\| \to 0$ as $\delta \to 0$ and $n \to \infty$.

$\square$

3 The Discrepancy Principle

By our assumption (2.1) on (η_n^δ) and the fact that $0 \neq y \in D(T^\dagger)$, we have

$$c_1 \leq \|z_n^\delta\| \leq c_2$$

for all large enough n, say $n \geq n_0(\delta)$ and for each $\delta \in (0, \delta_0]$ for some δ_0. Therefore by Theorem 2.1

$$\|A_n x_{\alpha,n}^\delta - z_n^\delta\| = \|\alpha x_{\alpha,n}^\delta\| = \|\alpha(A_n + \alpha I)^{-1} z_n^\delta\| \leq \gamma_1 \tag{3.1}$$

for some constant γ_1 and for all $\alpha \geq \epsilon_n / c_0$. Moreover, if

$$\alpha \geq \gamma_0 := \max\{\epsilon_n/c_0 : n = 1, 2, \ldots\} \quad \text{and} \quad \delta \leq \delta_0,$$

then

$$\|A_n x_{\alpha,n}^\delta - z_n^\delta\| \geq \|\alpha(A_n + \alpha I)^{-1} z_n^\delta\| \geq \frac{\gamma_0 \|z_n^\delta\|}{\|A_n\| + \alpha} \geq \gamma_2 \tag{3.2}$$

for some $\gamma_2 > 0$, since (A_n) is uniformly bounded.

Now to choose the regularization parameter α in (1.9), we consider the discrepancy principle (1.10).

For simplicity of presentation we assume that

$$\eta_n^\delta \leq c_3 \delta^r \quad \text{and} \quad \epsilon_n \leq c_4 \delta^k \tag{3.3}$$

for some positive reals r and k, and for all $n \geq n_0(\delta)$.

THEOREM 3.1 *Let p and q be positive integers. Then for each $\delta \in (0, \delta_0]$, there exists a positive integer $n_1(\delta)$ and for each $n \geq n_1(\delta)$, there exists $\alpha := \alpha(\delta, n)$ such that (1.10) is satisfied. More over,*

$$\alpha \leq c_1 \delta^{p/(q+1)} \quad \textit{and} \quad \frac{\delta^p}{\alpha^q} \leq c_2 \delta^\mu, \quad n \geq n_1(\delta), \tag{3.4}$$

where

$$\mu = \min\left\{r,\ \frac{p}{(q+1)},\ 1 + \frac{p}{2(q+1)}\right\}.$$

Proof. Let $\delta \in (0, \delta_0]$. For $\alpha \geq \epsilon_n / c_0$ and $n = 1, 2, \ldots$, define

$$f_n(\alpha) = \alpha^q \|A_n x_{\alpha,n}^\delta - z_n^\delta\|.$$

Then from (3.1) it follows that $f_n(\alpha) \leq \gamma_1 \alpha^q$ so that

$$f_n(\epsilon_n/c_0) \to 0 \quad \text{as} \quad n \to \infty.$$

Let $n_1(\delta) \geq n_0(\delta)$ be the smallest positive integer such that for all $n \geq n_1(\delta)$,

$$\epsilon_n \leq c_0 \min \left\{ \left(\frac{\delta^p}{\gamma_2}\right)^{1/q}, \left(\frac{\delta^p}{\gamma_1}\right)^{1/q} \right\}.$$

Then taking $\alpha_o = \max\{\gamma_0, (\delta^p/\gamma_2)^{1/q}\}$, we obtain

$$\epsilon_n \leq c_0 \alpha_0 \quad \text{and} \quad \alpha_0 \geq \gamma_0$$

so that by (3.1) and (3.2), we have

$$f_n(\epsilon_n/c_0) \leq \delta^p \leq f_n(\alpha_0).$$

Therefore by the Intermediate Value Theorem, there exists $\alpha := \alpha(\delta, n)$ such that

$$\frac{\epsilon_n}{c_0} \leq \alpha \leq \alpha_0 \quad \text{and} \quad \|A_n x^\delta_{\alpha,n} - z^\delta_n\| = \frac{\delta^p}{\alpha^q}$$

for all $n \geq n_1(\delta)$. We also note that

$$x^\delta_{\alpha,n} = \frac{1}{\alpha}(z^\delta_n - A_n x^\delta_{\alpha,n})$$

so that for all $n \geq n_1(\delta)$ and $\alpha = \alpha(\delta, n)$,

$$\begin{aligned} \|z^\delta_n\| - \frac{\delta^p}{\alpha^q} &= \|z^\delta_n\| - \|A_n x^\delta_{\alpha,n} - z^\delta_n\| \\ &\leq \|A_n x^\delta_{\alpha,n}\| \\ &\leq \|A_n\| \frac{\delta^p}{\alpha^{q+1}}. \end{aligned}$$

Therefore

$$\alpha^{q+1} \leq \delta^p(\alpha + \|A_n\|)/\|z^\delta_n\| \leq c\delta^p$$

and consequently

$$\alpha(\delta, n) \leq c_1 \delta^{p/(q+1)}, \quad n \geq n_1(\delta).$$

Now, using the estimates in (1.4), (1.8) and (2.2), we have

$$\begin{aligned}
\frac{\delta^p}{\alpha^q} &= \|A_n x^\delta_{\alpha,n} - z^\delta_n\| \\
&= \alpha\|x^\delta_{\alpha,n}\| \\
&\leq \alpha(\|\hat{x}\| + \|\hat{x} - x^\delta_{\alpha,n}\|) \\
&\leq c\alpha\left(\|\hat{x}\| + \|\hat{x} - x^\delta_\alpha\| + \frac{\eta^\delta_n}{\alpha} + \frac{\epsilon_n}{\alpha}\right) \\
&\leq c'\left(\alpha + \delta\sqrt{\alpha} + \eta^\delta_n\right) \\
&\leq c_2\delta^\mu,
\end{aligned}$$

where $\mu = \min\{r,\ p/(q+1),\ 1 + p/2(q+1)\}$. This completes the proof of the theorem.

$\square$

4 Error Estimates under the Discrepancy Principle

In order to prove the convergence of $x^\delta_{\alpha,n}$ to $\hat{x}$ and to obtain the estimates for the error $\|\hat{x} - x^\delta_{\alpha,n}\|$ under the discrepancy principle (1.10), we impose certain restrictions on the parameters p and q appearing in the discrepancy principle (1.10) in terms of the error levels η^δ_n and ϵ_n of the data A_n and z^δ_n respectively. More precisely, we assume that

$$\frac{p}{q+1} \leq \min\{2, r, k\}, \tag{4.1}$$

where r and k as in (3.3).

THEOREM 4.1 *Let $\alpha := \alpha(\delta, n)$ be chosen according to (1.10). Then we have the following.*

(i). $\|\hat{x} - x^\delta_{\alpha,n}\| \to 0$ *as $n \to \infty$ and $\delta \to 0$.*

(ii). If $\hat{x} \in R((A^\nu)$, $0 < \nu \leq 1$, then for all large enough n and small enough δ,

$$\|\hat{x} - x^\delta_{\alpha,n}\| \leq c\delta^s,$$

where

$$s = \min\left\{\frac{p\nu}{(q+1)},\ 1-\frac{p}{2(q+1)},\ r-\frac{p}{(q+1)}, k-\frac{p}{(q+1)}\right\}.$$

(iii). In particular, if

$$\min\{r,k\} \geq \frac{2\nu+2}{2\nu+1} \quad \textit{and} \quad \frac{p}{(q+1)} = \frac{2}{2\nu+1},$$

then

$$\|\hat{x} - x_{\alpha,n}^{\delta}\| \leq c\delta^{2\nu/(2\nu+1)}.$$

Proof. Using (3.4), we have

$$\frac{\delta^\ell}{\alpha^m} = \delta^{\ell-mp/q}\left(\frac{\delta^p}{\alpha^q}\right)^{m/q} \leq c\delta^{\ell-m(p-\mu)/q},$$

for every $\ell \geq 0$ and $m \geq 0$, where μ is as in Theorem 3.1. But by the assumption (4.1), $\mu = p/(q+1)$, so that it follows that

$$\frac{\delta^\ell}{\alpha^m} \leq c\delta^{\ell-mp/(q+1)}.$$

Therefore

$$\frac{\delta}{\sqrt{\alpha}} \leq c_1\delta^{1-p/2(q+1)}, \quad \frac{\eta_n^\delta}{\alpha} \leq c_2\delta^{r-p/(q+1)} \quad \text{and} \quad \frac{\epsilon_n}{\alpha} \leq c_3\delta^{k-p/(q+1)}.$$

Using this, the result in (i) follows from (1.5),(1.8) and (2.2), the estimate in (ii) follows from (1.6), (1.8) and (2.2), and that (iii) is a consequence of (ii). $\square$

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Department of Mathematics
Indian Institute of Technology Madras
Chennai 600 036, INDIA
E–Mail : **mtnair@acer.iitm.ernet.in**

Fachbereich Mathematik
Universität Kaiserslautern
Kaiserslautern, GERMANY
E–Mail : **schock@mathematik.uni-kl.de**