A short note on functions of bounded semivariation and countably additive vector measures

P. Vieten

1. Introduction and an example

Let Σ denote the σ-field of Borel subsets of the interval $[0,1]$. If $\mu : \Sigma \to \mathbf{C}$ is a complex Borel measure then the function $\phi : [0,1] \to \mathbf{C}$ given by

$$\phi(t) = \begin{cases} 0 & \text{if} \quad t = 0, \\ \mu([0,t)) & \text{if} \quad 0 < t < 1, \\ \mu([0,1]) & \text{if} \quad t = 1 \end{cases} \tag{1}$$

has bounded variation, since μ has bounded variation (see e.g. [3, Theorem 6.4]. Moreover, ϕ is normalized in the sense that $\phi(0) = 0$ and $\phi(t^-) = \phi(t)$ for all $0 < t < 1$.

Conversely, if $\phi : [0,1] \to \mathbf{C}$ is a normalized function of bounded variation then there exists a unique complex Borel measure μ such that ϕ and μ are connected by (1). This can be seen as follows: For $0 \leq a < b \leq 1$ let $[a,b[$ denote the interval

$$[a,b[= \begin{cases} [a,b) & \text{if } b < 1, \\ [a,1] & \text{if } b = 1. \end{cases}$$

Let

$$\Sigma_0 = \left\{ \bigcup_{k=1}^{n} [a_k, b_k[\ : n \in \mathbf{N}, [a_k, b_k[\subseteq [0,1] \text{ pairwise disjoint} \right\}. \tag{2}$$

Then Σ_0 is a field of subsets of $[0,1]$ and we can define an additive set function μ_0 on Σ_0 by

$$\mu_0 \left(\bigcup_{k=1}^{n} [a_k, b_k[\right) = \sum_{k=1}^{n} [\phi(b_k) - \phi(a_k)].$$

Then $\phi(t) = \phi(t) - \phi(0) = \mu_0([0,t[)$ for all $0 \leq t \leq 1$. Moreover, μ_0 has a unique extension to a measure $\mu : \Sigma \to \mathbf{C}$ (see e.g. [2, Theorem 1, page 358]). Hence, there exists a one-to-one correspondence between complex Borel measures on $[0,1]$ and normalized functions of bounded variation on $[0,1]$.

In this note we want to study the connection (1) for Banach space valued measures and functions. In the sequel, X denotes a complex Banach space and X^* its dual. We recall the following definitions: Let $\widetilde{\Sigma}$ be a field of subsets of a set Ω. A function $\mu : \widetilde{\Sigma} \to X$ is called *vector measure* if it is an additive set function. A vector measure μ is called *countably additive* if

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

for all sequences (E_k) of pairwise disjoint members of $\widetilde{\Sigma}$ with $\bigcup_{k=1}^{\infty} E_k \in \widetilde{\Sigma}$. If $\sum_{k=1}^{\infty} \mu(E_k)$ converges for every sequence (E_k) of pairwise disjoint members of $\widetilde{\Sigma}$ then μ is called *strongly additive*. A function $\phi : [0,1] \to X$ has *bounded variation* if

$$\mathrm{Var}(\phi) := \sup \sum_{k=1}^{n} \|\phi(t_k) - \phi(t_{k-1})\|$$

is finite, where the supremum is taken over all finite sequences $0 \leq t_0 < t_1 < \ldots < t_n \leq 1$. The function ϕ has *finite semivariation* if $x^* \circ \phi$ has finite variation for all $x^* \in X^*$ (for equivalent formulations of finite semivariation see [4, Lemma 1.12]). We call a function ϕ of bounded semivariation to be *weakly normalized* if $x^* \circ \phi$ is normalized for all $x^* \in X^*$.

Now, let $\mu : \Sigma \to X$ be a countably additive vector measure, and define $\phi : [0,1] \to X$ by (1). Then it follows by the scalar case that ϕ is a weakly normalized function of bounded semivariation.

But, if conversely $\phi : [0,1] \to X$ is a weakly normalized function of bounded semivariation then there does not necessarily exist a countably additive vector measure μ with (1). Example 1 below shows that such a vector measure need not exist even if ϕ is a continuous function of bounded semivariation.

In the sequel we say that $\phi : [0,1] \to X$ generates the vector measure $\mu : \Sigma \to X$ if ϕ and μ are connected by (1). In section 2 we give necessary and sufficient conditions for ϕ generating a countably additive vector measure μ.

Example 1 Let c_0 be the Banach space of complex valued null-sequences. We construct a continuous function $\phi : [0,1] \to c_0$ of bounded semivariation which does not generate a countably additive vector measure.

Define a sequence (x_n) in c_0 as follows:

$$\begin{array}{rlrl} x_1 &= (1,0,0,0,\ldots) & x_2 &= (-1,0,0,0,\ldots) \\ x_3 &= (0,1/2,0,\ldots) & x_4 &= (0,-1/2,0,\ldots) \\ x_5 &= (0,1/2,0,\ldots) & x_6 &= (0,-1/2,0,\ldots) \\ x_7 &= (0,0,1/4,\ldots) & x_8 &= (0,0,-1/4,\ldots) \\ x_9 &= (0,0,1/4,\ldots) & x_{10} &= (0,0,-1/4,\ldots) \\ x_{11} &= (0,0,1/4,\ldots) & x_{12} &= (0,0,-1/4,\ldots) \\ x_{13} &= (0,0,1/4,\ldots) & x_{14} &= (0,0,-1/4,\ldots), \end{array}$$

and so on. To be more precise, $x_n = (-1)^{n+1} 2^{-k} e_k$ for $n \in \{2^k - 1, \ldots, 2(2^k - 1)\}$, where e_k denotes the k-th unit vector in c_0. Then the (formal) series $\sum_{n=1}^{\infty} x_n$ has the following properties:

(i) The series converges towards zero: This is immediately clear, because $\left\| \sum_{n=1}^{N} x_n \right\| = 0$ if N is even, and $\left\| \sum_{n=1}^{N} x_n \right\| = 2^{-k}$ if N is odd and contained in $\{2^k - 1, \ldots, 2(2^k - 1)\}$.

(ii) The series is weakly unconditionally convergent: This follows from the fact that $\sum_{n=1}^{\infty} |x_n e_k| = 2$ for all $k \in \mathbf{N}$.

(iii) The series does not converge unconditionally: Let $\pi : \mathbf{N} \to \mathbf{N}$ be the permutation which acts on each of the sets $\{2^k - 1, 2(2^k - 1)\}$ in the following way: π "collects" first the 2^{k-1} odd numbers, and then the 2^{k-1} even numbers which are contained in this set. More precisely, put

$$\pi(n) = \begin{cases} 2^k - 1 + 2(n - 2^k + 1), & n \in \{2^k - 1, 3 \cdot 2^{k-1} - 2\}, \\ 2^k + 2(n - 3 \cdot 2^{k-1} + 1), & n \in \{3 \cdot 2^{k-1} - 1, 2(2^k - 1)\}. \end{cases}$$

Then

$$\sum_{n=1}^{3 \cdot 2^{k-1} - 2} x_{\pi(n)} = e_k \quad \text{for all } k \in \mathbf{N}.$$

Hence $\sum_{n=1}^{\infty} x_n$ is not unconditionally convergent.

Now, let $\phi : [0,1] \to c_0$ be defined in the following way:

$$\phi\left(\frac{1}{2} - \frac{1}{2(N+1)}\right) = \sum_{n=1}^{N} x_n \quad \text{for } N \in \mathbf{N}$$
$$\phi(0) = 0,\ \phi(1/2) = 0 \text{ and } \phi(1) = 0,$$

and let ϕ be linear in the intervals $[1/2 - 1/(2N), 1/2 - 1/(2N+2)]$ for $N \in \mathbf{N}$, and in $[1/2, 1]$. The function ϕ has the following properties:

(iv) ϕ is continuous: It is a trivial consequence of the definition that ϕ is continuous in the intervals $[0, 1/2)$ and $(1/2, 1]$. Moreover,

$$\lim_{t \to 1/2^-} \phi(t) = \lim_{N \to \infty} \sum_{n=1}^{N} x_n = 0 = \phi(1/2),$$

and $\lim_{t \to 1/2^+} \phi(t) = 0 = \phi(1/2)$. Hence ϕ is continuous in the whole interval $[0,1]$.

(v) ϕ is of bounded semivariation: For each natural number k the real valued function $\phi_k = \langle e_k, \phi \rangle$, where e_k is considered as unit vector in the dual space l_1 of c_0, is easily seen to have variation equal to 2.

(vi) ϕ does not define a countably additive vector measure $\mu : \Sigma \to c_0$: To prove this, assume the converse to be true. Then $\mu([a,b)) = \phi(b) - \phi(a)$ for all $0 \le a < b < 1$. Consequently, it follows that the series $\sum_{n=1}^{\infty} \mu([1/2 - 1/2n, 1/2 - 1/(2n+2))$ is unconditionally convergent (see [1, page 7]). But this series is equal to the series $\sum_{n=1}^{\infty} x_n$, which is not unconditionally convergent by the first step. This contradiction makes clear that such a vector measure does not exist.

2. Functions of unconditional bounded semivariation

If ϕ generates a countably additive vector measure then we see from [1, page 7] that $\sum_{k=1}^{\infty} [\phi(b_k) - \phi(a_k)]$ should converge unconditionally for every choice of countably many pairwise disjoint intervals $[a_k, b_k)$. Therefore, we have the following definition.

Definition 2 A function $\phi : [0,1] \to X$ is of *unconditional bounded semivariation* if for each choice of countably many pairwise disjoint intervals $[a_n, b_n) \subseteq [0,1]$ the series

$$\sum_{n=1}^{\infty} [\phi(b_n) - \phi(a_n)] \text{ is unconditionally convergent.}$$

In this section (see Theorem 4 below) we show that ϕ generates a countably additive vector measure μ if and only if ϕ is a weakly normalized function of unconditional bounded semivariation.

Before we prove Theorem 4 we show that the set of functions of unconditional bounded semivariation contains the functions of bounded variation and is contained in the set of functions of bounded semivariation. This will be immediately clear from the following

Proposition 3 *Let $\phi : [0,1] \to X$ be any function.*

(i) ϕ is of bounded variation if and only if for each choice of countably many pairwise disjoint intervals $[a_n, b_n) \subseteq [0,1]$ the series

$$\sum_{n=1}^{\infty} [\phi(b_n) - \phi(a_n)] \textit{ is absolutely convergent.}$$

(ii) ϕ is of bounded semivariation if and only if for each choice of countably many pairwise disjoint intervals $[a_n, b_n) \subseteq [0,1]$ the series

$$\sum_{n=1}^{\infty} [\phi(b_n) - \phi(a_n)] \textit{ is weakly unconditionally convergent.}$$

Proof. We denote by $\mathcal{I}$ the collection of all sequences $([a_n, b_n))$ of pairwise disjoint subintervals of $[0,1]$. If $I \subseteq [0,1]$ is an interval then $\mathrm{Var}_I(\phi)$ denotes the variation of ϕ on I. Note that a function ϕ has bounded variation on $[0,1]$ iff it has bounded variation on $[0,1)$.

(i) Assume that ϕ is of bounded variation. Let $([a_n, b_n)) \in \mathcal{I}$. Then, for all $N \in \mathbf{N}$, there exist $0 \leq t_0 < t_1 < \ldots < t_k < 1$ so that $\{a_n, b_n : n = 1, \ldots, N\} = \{t_j : j = 0, \ldots, k\}$. Since the $[a_k, b_k)$'s are pairwise disjoint it follows that

$$\sum_{n=1}^{N} \|\phi(b_n) - \phi(a_n)\| \leq \sum_{j=1}^{k} \|\phi(t_j) - \phi(t_{j-1})\| \leq \mathrm{Var}(\phi).$$

Assume now that ϕ does not have bounded variation. We construct first a sequence I_n of pairwise disjoint intervals. In that construction we use the following fact: If ϕ has unbounded variation on an interval K then one can find disjoint intervals $I, J \subseteq K$ with $\mathrm{Var}_I(\phi) \geq 1$ and $\mathrm{Var}_J(\phi) = \infty$.

Since ϕ has unbounded variation on $[0, 1]$ we can find disjoint intervals $I_1, J_1 \subseteq [0, 1)$ such that $\mathrm{Var}_{I_1}(\phi) \geq 1$ and $\mathrm{Var}_{J_1}(\phi) = \infty$. Assume $I_1, \ldots, I_n$ and $J_1, \ldots J_n$ are intervals with the following properties:

(a) $I_k \cap I_l = \emptyset$ for $k \neq l$.

(b) $I_{k+1} \subseteq J_k$ for $k = 1, \ldots, n-1$.

(c) $J_k \cap \bigcup_{l=1}^{k} I_k = \emptyset$ for $k = 1, \ldots, n$.

(d) $\mathrm{Var}(I_k) \geq 1$ for $k = 1, \ldots n$ and

(e) $\mathrm{Var}(J_k) = \infty$ for $k = 1, \ldots, n$.

If we take disjoint intervals

$$I_{n+1}, J_{n+1} \subseteq J_n \text{ with } \mathrm{Var}_{I_n}(\phi) \geq 1 \text{ and } \mathrm{Var}_{J_n}(\phi) = \infty$$

then it is easily verified that I_k, J_k, $k = 1, \ldots, n+1$ also satisfy (a)-(e).

We can now find intervals $[a_1^{(k)}, b_1^{(k)}), \ldots, [a_{n(k)}^{(k)}, b_{n(k)}^{(k)}) \subseteq I_k$ for each $k \in \mathbf{N}$ such that

$$\sum_{l=1}^{n(k)} \left\| \phi(b_l^{(k)}) - \phi(a_l^{(k)}) \right\| \geq \frac{1}{2}.$$

Consequently, the series

$$\sum_{k=1}^{\infty} \sum_{l=0}^{n(k)} \phi(b_l^{(k)}) - \phi(a_l^{(k)}) \text{ is not absolutely convergent.}$$

(ii) It follows from (i) that

$$\sum_{k=1}^{\infty} \phi(b_k) - \phi(a_k) \text{ is weakly unconditionally convergent}$$

for all sequences $[a_k, b_k) \in \mathcal{I}$ if and only if $x^* \circ \phi$ is of bounded variation for all $x^* \in X^*$. $\equiv$

Theorem 4 *A function $\phi : [0,1] \to X$ generates a countably additive vector measure μ if and only if ϕ is a weakly normalized function of unconditional bounded semivariation.*

Proof. Let $\mu : \Sigma \to X$ be a countably additive vector measure which is generated by ϕ. Then it follows immediately that ϕ is weakly normalized. Now take a sequence of pairwise disjoint intervals $([a_n, b_n)) \in \mathcal{I}$. Then

$$\sum_{n=1}^{\infty}[\phi(b_n) - \phi(a_n)] = \sum_{n=1}^{\infty}\mu([a_n, b_n)) = \mu\left(\bigcup_{n=1}^{\infty}[a_n, b_n)\right)$$

converges. Consequently, ϕ is of unconditional bounded semivariation.

Conversely, assume that ϕ is a weakly normalized function of unconditional bounded semivariation. Define $\mu_0 : \Sigma_0 \to X$ by

$$\mu_0\left(\bigcup_{k=1}^{n}[a_k, b_k[\right) = \sum_{k=1}^{n}\phi(b_k) - \phi(a_k).$$

Then μ_0 is a vector measure, and $x^* \circ \mu$ is countably additive for every $x^* \in X^*$ (see the introduction).

We claim that μ_0 is strongly additive. By [1, Theorem I.1.18] it is sufficient to show that

$$\lim_{n\to\infty}\mu_0(E_n) \text{ exists}$$

for each monotone nondecreasing sequence (E_n) in Σ_0. If (E_n) is a monotone nondecreasing sequence in Σ_0 then, for each $n \in \mathbf{N}$, there exist pairwise disjoint intervals $[a_1^{(n)}, b_1^{(n)}[, \ldots, [a_{k(n)}^{(n)}, b_{k(n)}^{(n)}[$ such that

$$E_n = E_{n-1} \cup \bigcup_{k=1}^{k(n)}[a_k^{(n)}, b_k^{(n)}[\ , \text{ and } E_{n-1} \cap \bigcup_{k=1}^{k(n)}[a_k^{(n)}, b_k^{(n)}[= \emptyset,$$

where $E_0 = \emptyset$. It follows that $[a_k^{(n)}, b_k^{(n)}[$ and $[a_l^{m}, b_l^{(m)}[$ are disjoint if $(k,n) \neq (l,m)$. Consequently,

$$\mu_0(E_n) = \sum_{l=1}^{n}\sum_{k=1}^{k(l)}\mu_0\left([a_k^{(l)}, b_k^{(l)}[\right) = \sum_{l=1}^{n}\sum_{k=1}^{k(l)}\phi(b_k^{(l)}) - \phi(a_k^{(l)})$$

converges as $n \to \infty$, since ϕ is of unconditional bounded semivariation.

Now we are in the position to apply the Caratheodory-Hahn-Kluvanek extension theorem to μ_0 (see [1, Theorem I.5.2]). This theorem states that there exists a countably additive extension μ of μ_0 to the σ-field $\overline{\Sigma_0}$ generated by Σ_0. Since $\overline{\Sigma_0} = \Sigma$ we proved the existence of a countably additive vector measure $\mu : \Sigma \to X$ with $\phi(t) = \phi(t) - \phi(0) = \mu([0, t[)$. $\equiv$

Remark 5 (i) By the Bessage-Pelczinsky theorem [1, Corollary I.4.5] every weakly unconditionally convergent series in X converges unconditionally iff X does not contain an isomorphic copy of c_0. Consequently, if X does not contain c_0 every X-valued function of bounded semivariation is of unconditional bounded semivariation. Conversely, if c_0 is contained in X then it is easy to construct a function of bounded semivariation which is not of unconditional bounded semivariation. Consequently, by Theorem 4, every X-valued function of bounded semivariation generates a countably additive vector measure if and only if c_0 is not contained in X.

(ii) We finally ask the following questions: Is it true that ϕ is of

- bounded variation if the series $\sum[\phi(t_k) - \phi(t_{k-1})]$ is absolutely convergent
- unconditional bounded semivariation if the series $\sum[\phi(t_k) - \phi(t_{k-1})]$ is unconditionally convergent
- bounded semivariation if the series $\sum[\phi(t_k) - \phi(t_{k-1})]$ is weakly unconditionally convergent

for every increasing sequence (t_k) in $[0, 1]$?

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Peter Vieten
Fachbereich Mathematik
Universität Kaiserslautern
Erwin-Schrödinger-Straße
67663 Kaiserslautern

e-mail: vieten@mathematik.uni-kl.de