In an autonomous Markovian dynamical system a discrete initial measure could be transformed into an absolutely continuous distribution in finite time even if the system has a locally unique deterministic solution except for countably many initial states.

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In this note, answering a question of N. Maslova, we give a two-dimensional elementary example of the phenomenon indicated in the title. Perhaps this simple example may serve as an object of comparison for more refined models like in the theory of kinetic differential equations where similar questions still seem to be unsettled.

The Markov process $(X(t) : t \geq 0)$ has the following properties:
1. The state space G is the unit square. It is the disjoint union of three sets A, B and F.
a) A is open and dense. As long as particle stays in A, it moves deterministically towards the complement of A.
b) B is countable, each point of B is isolated. B is the set of branching points. If a particle is at a point of B it enters A with equal probability on one of four possible straight lines.
c) The set F is the set of steady states. F is compact and of positive two-dimensional Lebesgue measure. (Alternatively, F can be chosen to be of arbitrarily small positive Hausdorff dimension.)
2. Every particle is entering F not later than time 1. If $X(0)$ is the center of the unit square then the probability distribution of $X(1)$ is equivalent to the uniform Hausdorff measure on F (which is absolutely continuous if F has positive Lebesgue measure).
3. All paths are continuous. They could be made smooth inside A.

The construction runs as follows:

Start with a Cantor type set E in the unit interval, i.e. E is obtained from the unit interval by removing in a first step an interval in the middle and then in the $(n+1)$-th step removing in each of the intervals which remained after the n-th step another small interval. If the intervals which are removed are small enough the set E which is the total remainder after infinitely many steps may still have positive Lebesgue measure (or if these intervals are big enough the resulting set has arbitrarily small Hausdorff-dimension).
Then take F to be the product set $E \times E$. F is the intersection of a decreasing sequence of finite unions of little closed squares. For B take the set of all centers of such squares. If x in B is

the center of such a square S on level n then there are four smaller squares on the next level inside S. Each of these small squares has two edges which are inside S. The midpoint of the first (in mathematical orientation) of these two edges is the "entrance" into this small square. A particle in x moves (with equal probability each) straight to one of these four entrance points and then proceeds again on a straight line to the center of that small square. This movement is performed in such a speed that it arrives in 2^{-} time units at this next point in B. This describes those paths which visit B at some stage. It remains to verify that the remaining space can be filled by deterministic trajectories which do not cross these branching ones and which also end in F. This can be done most easily with the help of a drawing.