Some formulae with logarithmic derivatives, related to a quantization of some infinite-dimensional Hamiltonian systems

Heinrich von Weizsäcker, Oleg G. Smolyanov

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Some formulae, containing logarithmic derivatives of (smooth) measures on infinite-dimensional spaces, arise in quite different situations. In particular, logarithmic derivatives of a measure are inserted in the Schrödinger equastion in the space consisting of functions that are square integrable with respect to this measure, what allows us to describe very simply a procedure of (canonical) quantization of infinite-dimensional Hamiltonian systems with the linear phase space. Further, the problem of reconstructing of a measure by its logarithmic derivative (that was posed in [1] independently of any applications) can be equivalent either to the problem of finding the "ground state" (considered as some measure) for infinite-dimensional Schrödinger equation, or to the problem of finding an invariant measure for a stochastic differential equation (that is a central question of so-called stochastic quantization), or to the problem of recenstructing "Gibbsian measure by its specification" (i.e. by a collection of finite-dimensional conditional distributions). Logarithmic derivatives of some measure appear in Cameron-Martin-Girsanov-Maruyama formulae and in its generalizations related to arbitrary smooth measures; they allow also to connect these formulae and the Feynman-Kac formulae. This note discusses all these topics. Of course due to its shortness the presentation is formal in main, and precise analitical assumptions are usually absent. Actually only a list of formulae with small comments is given. Let us mention also that we do not consider at all so-called Dirichlet forms to which a great deal of literature is devoted (cf. [3] and references therein to the works of S. Alberion and others).

1. Logarithmic derivatives, logarithmic densities and generalized densities of smooth measures. Further all the vector spaces are supposed to be real. For each locally convex space (LCS) E the symbol $\mathcal{B}(E)$ denotes the σ-algebra of its Borel subsets and the symbol $\mathcal{M}(E)$ denotes the vector space of all σ-additive realvalued measures on $\mathcal{B}(E)$. If h is a vector field on a LCS E (i.e. a mapping from E into E) then by T_h one denotes the mapping of E into E defined as follows: $T_h(x) = x + h(x)$. If moreover $\nu \in \mathcal{M}(E)$ then by ν_h one denotes the image of ν with respect to T_h. A measure ν is called (Fomin-)differentiable along a vector field h if the function $t_h : R^1 \longrightarrow \mathcal{M}(E),\ t \mapsto \nu_{th}$ is differentiable at 0 when the space $\mathcal{M}(E)$ is equipped with the topology of setwise convergence; in this case the measure $(-f_h^\nu)'(0)$ is called (Fomin-)derivative of the measure ν along the vector field h and is denoted by d_{h^ν} (cf. [2]). It is possible to prove that d_{h^ν} is absolutely continuous w.r.t. ν ([1], [2]), and hence $d_{h^\nu} = \beta_h^\nu(\ \cdot\ )\ \cdot \nu$; the function β_h^ν is called logarithmic derivative of ν along the vector field h. If, in particular, $h(x) = k(\in E)$ for each $x \in E$ then instead of the symbol β_h^ν we use the symbol $\beta^\nu(h,\cdot)$. It is worth noticing that, if $D(\nu) = \{k \in E : \exists\, \beta^\nu(h,\cdot)\}$ then the mapping $k \mapsto \eta^\nu(k,\cdot),\ D(\nu) \longrightarrow L_1(E,\nu)$ is linear [1]. If H is a Hilbert subspace of the space E (this means that H is a vector subspace and that H is equipped with the structure of a Hilbert space with respect to which the canonical embedding $H \longrightarrow E$ is continuous) and if $H \subset D(\nu)$ and for each $x \in E$ $h(x) \in H$ (and some other assumptions are satisfied) then

$$\beta_h^\nu(x) = \beta^\nu(h(x), x) + tr\ h'(x)$$

see [2] and references therein).

Proposition 1: *If the mapping $\beta^\nu : H \times H \longrightarrow R^1$ is continuously differentiable then there exists a function $\sigma_\nu : H \longrightarrow R^1$ for which $\sigma'_\nu(x)h = \beta^\nu(h, x)$ for each $h, x \in H$. The function σ_ν can be called the logarithmic density of ν; let us stress that this function is defined only on H and cannot be extended "in natural way" to the whole E (if dim $E = \infty$ and ν is σ-additive). The function $H \ni x \mapsto \exp \sigma_\nu(x)$ can be called the generalized density of ν ([3], [4]).*

Proposition 2: *If there exists a continuous function $\Lambda_\nu : H \times E \longrightarrow R^1$, whose restriction on the subspace $H \times H$ of $H \times E$ coinsides with the function $(h, x) \mapsto \sigma_\nu(x+h) - \sigma_\nu(x)$ then for any $h \in H$ the measure ν_h is absolutely continuous w.r.t. ν and $\nu_h = \exp(\Lambda_\nu(h, \cdot) \cdot \nu)$.*

Remark 1: *The following equality holds: $\Lambda_\nu(h, x) = \int_0^1 \beta^\nu(F'_1(\tau, x),\ F(\delta, x))d\delta$ where F is a mapping of $R^1 \times E$ into E (subjected to some assumptions of regularity [y]) for which $F(0, x) = x,\ F(1, x) = x + h$. If $F(t, x) = x + th(x)$, where h is a vector field on E, and* det *$(Id + (h|H)'(x)) = 0$, then the formulae from propostion **??** is valid also.*

2. Quantization. Let Q and P be two copies of a Hilbert space with the scalar product $(\cdot, \cdot)$, $E = Q \times P$ and $\mathcal{H}$ be (a realvalued) function on E defined by $\mathcal{H}(q, p) = (Ap, p) + V(q)$ where A is a selfadjoint positively defined trace class operator in P_x. Moreover, let $\nu \in \mathcal{M}(Q)$, and ν be infinitely differentiable along a Hilbert subspace $H \subset Q$, ψ_ν be a generalized density of ν and $\varphi = \psi_\nu^2$; a measure η with the generalized density φ (if this measure exists) can be called (in a natural way) the square of the measure ν; hence one can write $\eta = \nu^2$. Let $G = L_2(Q, \eta)$ and define a linear operator $\mathcal{H}_\nu$ in G by $(\hat{\mathcal{H}}_\nu f)\nu = \hat{\mathcal{H}}(f\nu)$...... ($\hat{\mathcal{H}}$ is the differential operator with the symbol $\mathcal{H}$).

Theorem 1: *If $f \in dom\ \hat{\mathcal{H}}_\nu$ then $(\hat{\mathcal{H}}_\nu f)(x) = tr(A(f''(x) + \eta^\nu(\cdot, x) \otimes f'(x)) + \frac{d\hat{\mathcal{H}}_\nu}{d\nu}(\cdot)f(x)$.*

3 Adjoint operators, invariant measures and all that.

Proposition 3: *If $\hat{\mathcal{H}}_\nu = 0$ (i.e. if ν is a "ground state" for the operator $\hat{\mathcal{H}}$) then $\hat{\mathcal{H}}_\nu f = trAf''(x) + tr\ f'(x) \otimes \beta^\nu(\cdot, x)$. One assumes further that $\hat{\mathcal{H}}_\nu = 0$.*

Remark 2: *The last equality means that $tr\ A(\beta^\nu(\cdot, x) \otimes \beta^\nu(\cdot, x) + (\beta^\nu)'_2(\cdot, x)(\cdot)) = 0$. So the problem of finding a "ground state" of the operator is reduced to the problem of finding a measure by its logarithmic derivative (that satisfies the last relation).*

Proposition 4: *If $\hat{\mathcal{H}}_\nu^*$ is a linear operator (not everywhere defined) in $\mathcal{M}(Q)$ that is adjoint to $\hat{\mathcal{H}}_\nu$ w.r.t. the natural duality between $L_1(Q, \nu)$ A dom $\hat{\mathcal{H}}_\nu$ and $\mathcal{M}(Q)$ then $\hat{\mathcal{H}}_{\nu\mu}^* = tr(A\mu'' + (\beta^\nu)'_2(\cdot, \cdot)\mu + \beta^\nu(\cdot, \cdot) \otimes \mu') = tr(A\ D^* D)\mu$ where D^* is the operator in $L_2(Q, \nu)$ that is adjoint to the operator of differentiation.*

Theorem 2: *The following relations and statements are equivalent:*

(1) $\hat{\mathcal{H}}_{\nu_0} = 0$; *(2)* $\hat{\mathcal{H}}^*_\nu \nu_0^2 = 0$; *(3)* ν_0^2 *is an invariant measure for the stochastic differential equation* $dx = \beta^\nu(\cdot, x)dt + dw$; *(4)* ν_0 *is a measure with logarithmic derivative* $\beta^\nu(\cdot,\cdot)$.

Remark 3: *The operators* $\hat{\mathcal{H}}_\nu$ *and* $\hat{\mathcal{H}}^*_\nu$ *are inserted, respectively, in forward and backward Kolmogorov equations for the stochastic differential equation from the theorem ??.*

4. Feynman-Kac formulae. If ν is a probabilistic measure on E with a dense (in E) subspace of differentiability, ψ_ν is its generalized density and g is a (realvalued) function on E then the (probabilistic) measure μ with generalized density $g \cdot \psi_\nu$ (if this measure exists) is denoted by $g \cdot \nu$; in this case the integral $\int f(x)\mu(dx)$ is denoted by $\int f(x)g(x)\nu(dx)$.

Theorem 3: *The following equality holds:*

$$(e^{-t\hat{\mathcal{H}}_\nu} f)(x) = \int \exp(\int_0^t \nu(x + b(\delta))d\delta) f(x + b(t)) \cdot \Lambda_\nu(b(t), x) w(db)$$

(w is the Wiener measure on the space of continuous functions on $[0, t]$ *taking values in* Q *and vanishing at 0) (cf [5]).*

This is a consequence of (Girsanov-Maruyama) formulae from propositon 2.

References

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