Two equivalent norms for vector-valued holomorphic functions

by Peter VIETEN, Kaiserslautern

Abstract

The following two norms for holomorphic functions F, defined on the right complex half-plane $\{z \in \mathbf{C} : \Re(z) > 0\}$ with values in a Banach space X, are equivalent:

$$\begin{aligned} \|F\|_{H_p(\mathbf{C}_+)} &= \sup_{a>0} \left(\int_{-\infty}^{\infty} \|F(a+ib)\|^p \, db \right)^{1/p}, \quad \text{and} \\ \|F\|_{H_p(\Sigma_{\pi/2})} &= \sup_{|\theta|<\pi/2} \left(\int_0^{\infty} \left\| F(re^{i\theta}) \right\|^p \, dr \right)^{1/p}. \end{aligned}$$

As a consequence, we derive a description of boundary values of sectorial holomorphic functions, and a theorem of Paley-Wiener type for sectorial holomorphic functions.

1 Two equivalent norms for vector-valued holomorphic functions

In this note we study holomorphic functions F defined on the open right complex half-plane $\mathbf{C}_+ = \{z \in \mathbf{C} : \Re(z) > 0\}$ with values in a complex Banach space X. To such functions we assign the following extended real numbers:

$$\begin{aligned} \|F\|_{H_p(\mathbf{C}_+)} &= \sup_{a>0} \left(\int_{-\infty}^{\infty} \|F(a+ib)\|^p \, db \right)^{1/p}, \quad \text{and} \\ \|F\|_{H_p(\Sigma_{\pi/2})} &= \sup_{|\theta|<\pi/2} \left(\int_0^{\infty} \left\| F(re^{i\theta}) \right\|^p \, dr \right)^{1/p}. \end{aligned}$$

Let $H_p(\mathbf{C}_+, X)$ and $H_p(\Sigma_{\pi/2}, X)$ be the respective normed spaces of those functions F for which $\|F\|_{H_p(\mathbf{C}_+)}$ and $\|F\|_{H_p(\Sigma_{\pi/2})}$ are finite.

Držbašjan and Martirosjan [1] proved that, for $p = 2$, $H_p(\mathbf{C}_+, \mathbf{C})$ is isomorphic to $H_p(\Sigma_{\pi/2}, \mathbf{C})$, and Sedleckiǐ [10] showed that this result holds for all $0 < p < \infty$. Luxemburg [6] gave a new prove of Sedleckiǐ's result. Our main Theorem 1 states that, for all $1 \leq p < \infty$ and every Banach space X, the norms $\|\cdot\|_{H_p(\mathbf{C}_+)}$ and $\|\cdot\|_{H_p(\Sigma_{\pi/2})}$ are equivalent, and the spaces $H_p(\mathbf{C}_+, X)$ and $H_p(\Sigma_{\pi/2}, X)$ are isomorphic. As a consequence, we derive in the second section a description of boundary values of sectorial holomorphic functions, and we give in the third section a theorem of Paley-Wiener type for sectorial holomorphic functions.

Throughout this note we use the following notations. The letter X stands for a complex Banach space, and we write X^* for its dual. We denote by $L_p(\mathbf{R}, X)$ the usual Bochner L_p-space, and $L_p(\mathbf{R}, \mathbf{C})$ is abbreviated by $L_p(\mathbf{R})$. For $z = a + ib \in \mathbf{C}_+$ the Poisson kernel P_z is given by

$$P_z(t) = \frac{1}{\pi} \frac{a}{(b-t)^2 + a^2}, \quad -\infty < t < \infty.$$

We say that $F : \mathbf{C}_+ \to X$ is the Poisson integral of some function $f \in L_p(\mathbf{R}, X)$ if

$$F(z) = \int_{-\infty}^{\infty} P_z(t) f(t)\, dt, \quad z \in \mathbf{C}_+.$$

Theorem 1 *Let $1 \leq p \leq \infty$ and let $F : \mathbf{C}_+ \to X$ be holomorphic. Then $F \in H_p(\mathbf{C}_+, X)$ if and only if $F \in H_p(\Sigma_{\pi/2}, X)$. Moreover,*

$$2^{-1} \|F\|_{H_p(\mathbf{C}_+)} \leq \|F\|_{H_p(\Sigma_{\pi/2})} \leq 2^{-1/p} \|F\|_{H_p(\mathbf{C}_+)}.$$

Proof. There is nothing to prove if $p = \infty$. Let $1 \leq p < \infty$, take $F \in H_p(\mathbf{C}_+, X)$, and assume that F is the Poisson integral of some function $f \in L_p(\mathbf{R}, X)$. Let $-\pi/2 < \theta < \pi/2$, $r > 0$, and take $x^* \in X^*$ with $\|x^*\| \leq 1$. From the proof of Theorem (3.3) in [6] it follows that

$$|x^* F(re^{i\theta})|^{p/2} \leq \frac{\sin(\theta)}{\pi} \int_{-\infty}^{\infty} P(t, \theta) |x^* f(rt)|^{p/2}\, dt,$$

where $P(t, \theta) = (1 + t^2 - 2t\cos(\theta))^{-1}$. Hence

$$\left\|F(re^{i\theta})\right\|^{p/2} \leq \frac{\sin(\theta)}{\pi} \int_{-\infty}^{\infty} P(t, \theta) \|f(rt)\|^{p/2}\, dt.$$

We conclude from this estimate (see the proof of Theorem (3.3) in [6])

$$\int_0^\infty \left\| F(re^{i\theta}) \right\|^p \, dr \leq \left(\frac{\sin(\theta)}{\pi} \int_{-\infty}^\infty \frac{1}{\sqrt{|t|}} P(t,\theta)\, dt \right)^2 \|f\|_{L_p}^p \leq 2 \|F\|_{H_p(\mathbf{C}_+)}^p .$$

If $F \in H_p(\mathbf{C}_+, X)$ is not the Poisson integral of a function in $L_p(\mathbf{R}, X)$ then we proceed as follows. For $a > 0$ and $z \in \mathbf{C}_+$ put $F_a(z) = F(a+z)$. Then F_a belongs to $H_p(\mathbf{C}_+, X)$ with $\|F_a\|_{H_p(\mathbf{C}_+)} \leq \|F\|_{H_p(\mathbf{C}_+)}$, and F_a is the Poisson integral of $f_a \in L_p(\mathbf{R}, X)$, where $f_a(t) = F(a+it)$. This follows from the corresponding scalar-valued result (see e.g. the proof of Theorem I.3.5. in [2]) by applying linear functionals. Now, the first part of the proof yields for all $-\pi/2 < \theta < \pi/2$

$$\int_0^\infty \left\| F_a(re^{i\theta}) \right\|^p \, dr \leq 2 \|F_a\|_{H_p(\mathbf{C}_+)}^p \leq 2 \|F\|_{H_p(\mathbf{C}_+)}^p .$$

Hence it follows for all $0 < A < B < \infty$

$$\int_A^B \left\| F(re^{i\theta}) \right\|^p \, dr = \lim_{a \to 0^+} \int_A^B \left\| F_a(re^{i\theta}) \right\|^p \, dr \leq 2 \|F\|_{H_p(\mathbf{C}_+)}^p .$$

Now, letting $A \to 0$ and $B \to \infty$ yields the second inequality.

Conversely, let $F \in H_p(\Sigma_{\pi/2}, X)$. For $\pi/2 < \alpha < \pi$ and $z \in \mathbf{C}_+$ put

$$F_\alpha(z) = \left(\frac{\pi}{2\alpha} \right)^{1/p} z^{\frac{1}{p}(\frac{\pi}{2\alpha} - 1)} F(z^{\frac{\pi}{2\alpha}}).$$

Then $F_\alpha \in H_p(\Sigma_{\pi/2}, X)$ and

$$\|F_\alpha\|_{H_p(\Sigma_{\pi/2})} \leq \|F\|_{H_p(\Sigma_{\pi/2})} .$$

On the other hand, we know from [6] that $x^* \circ F_\alpha$ belongs to $H_p(\mathbf{C}_+)$ for all $x^* \in X^*$. Hence $x^* \circ F_\alpha$ is the Poisson integral of its boundary function $x^* \circ f_\alpha \in L_p(\mathbf{R})$, where $f_\alpha(t) = F_\alpha(it)$ for $t \neq 0$. Since f_α is p-integrable we conclude

$$F_\alpha(z) = \int_{-\infty}^\infty P_z(t) f_\alpha(t)\, dt.$$

Put $g_\alpha(t) = \|f_\alpha(t)\|$. Then

$$\|g_\alpha\|_{L_p} = \|f_\alpha\|_{L_p} \leq 2 \|F_\alpha\|_{H_p(\Sigma_{\pi/2})} \leq 2 \|F\|_{H_p(\Sigma_{\pi/2})} .$$

We now distinguish the cases $p > 1$ and $p = 1$.

If $p > 1$ then, by the reflexivity of $L_p(\mathbf{R})$, there exists a sequence (α_n) in $(\pi/2, \pi)$ with $\lim_{n\to\infty} \alpha_n = \pi/2$ and a function $g \in L_p(\mathbf{R})$ so that g is the weak limit of (g_{α_n}) in $L_p(\mathbf{R})$. This implies for all $z \in \mathbf{C}_+$

$$\begin{aligned} \|F(z)\| &= \lim_{n\to\infty} \|F_{\alpha_n}(z)\| \le \limsup_{n\to\infty} \int_{-\infty}^{\infty} P_z(t) \, \|f_{\alpha_n}(t)\| \, dt \\ &= \int_{-\infty}^{\infty} P_z(t) g(t) \, dt =: G(z). \end{aligned}$$

If $p = 1$ then, by Helly's theorem, there exists a sequence (α_n) in $(\pi/2, \pi)$ with $\lim_{n\to\infty} \alpha_n = \pi/2$ and a function $\phi : \mathbf{R} \to \mathbf{R}$ of bounded variation so that ϕ is the weak* limit of (g_{α_n}) in $C_0(\mathbf{R})^*$. Consequently,

$$\begin{aligned} \|F(z)\| &= \lim_{n\to\infty} \|F_{\alpha_n}(z)\| \le \limsup_{n\to\infty} \int_{-\infty}^{\infty} P_z(t) \, \|f_{\alpha_n}(t)\| \, dt \\ &= \int_{-\infty}^{\infty} P_z(t) \, d\phi(t) =: G(z). \end{aligned}$$

It follows e.g. from [2], I.3. that, in either case, $G : \mathbf{C}_+ \to \mathbf{R}$ is a harmonic function with

$$\sup_{a>0} \int_{-\infty}^{\infty} |G(a+ib)|^p \, db \le \sup_n \|g_{\alpha_n}\|_{L_p}^p \le 2^p \, \|F\|_{H_p(\Sigma_{\pi/2})}^p .$$

We finally conclude

$$\sup_{a>0} \int_{-\infty}^{\infty} \|F(a+ib)\|^p \, db \le \sup_{a>0} \int_{-\infty}^{\infty} \|G(a+ib)\|^p \, db \le 2^p \, \|F\|_{H_p(\Sigma_{\pi/2})}^p .$$

$\equiv$

2 Boundary values of vector-valued holomorphic functions

We investigate now the boundary behavior of vector-valued functions being holomorphic in a sector

$$\Sigma_\alpha = \{re^{i\theta} : r > 0, |\theta| < \alpha\}.$$

Similar to $H_p(\Sigma_{\pi/2}, X)$ we introduce for each $0 < \alpha < \pi/2$ the space $H_p(\Sigma_\alpha, X)$ which consists of holomorphic functions $F : \Sigma_\alpha \to X$ such that

$$\|F\|_{H_p(\Sigma_\alpha)} = \sup_{|\theta|<\alpha} \left(\int_0^\infty \left\| F(re^{i\theta}) \right\|^p \, dr \right)^{1/p} < \infty$$

if $p < \infty$, and such that

$$\|F\|_{H_\infty(\Sigma_\alpha)} = \sup_{z\in\Sigma_\alpha} \|F(z)\| < \infty$$

if $p = \infty$. We start with a study of the boundary behavior of harmonic functions $F : \mathbf{C}_+ \to X$, and than use this result for a description of the boundary values of $H_p(\Sigma_\alpha, X)$-functions. For this reason, let us introduce the following spaces of functions and classes of operators: The space $h_p(\mathbf{C}_+, X)$ consists of harmonic functions $F : \mathbf{C}_+ \to X$ for which the following norm is finite:

$$\|F\|_{h_p} := \sup_{a>0} \left(\int_{-\infty}^\infty \|F(a+it)\|^p \, dt \right)^{1/p}$$

if $p < \infty$, and $\|F\|_{h_\infty} := \sup_{z\in\mathbf{C}_+} \|F(z)\|$ if $p = \infty$. We say that F has a boundary function $f \in L_p(\mathbf{R}, X)$ if F is the Poisson integral of f.

Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. Put $Y_q(\mathbf{R}) := L_q(\mathbf{R})$ if $q < \infty$, and let $Y_\infty(\mathbf{R})$ be the space of complex-valued continuous functions on $\mathbf{R}$ which vanish at infinity. An operator $T : Y_q(\mathbf{R}) \to X$ is called p-bounded, $p > 1$, if there exists a function $f \in L_p(\mathbf{R})$ such that

$$\|Tg\| \le \int_{-\infty}^\infty |g(t)| f(t) \, dt, \quad g \in Y_q(\mathbf{R}),$$

and T is called 1-bounded if there exists a function $\phi : \mathbf{R} \to \mathbf{R}$ of bounded variation such that

$$\|Tg\| \le \int_{-\infty}^\infty |g(t)| \, d\phi(t), \quad g \in Y_\infty(\mathbf{R}).$$

It follows from [3]. IV.4, Theorem 8, that for $p > 1$ an operator T is p-bounded if and only if it is order summing, and [9], Proposition 1.3.22 states that an operator $T : Y_\infty(\mathbf{R}) \to X$ is 1-bounded if and only if it is absolutely summing. We say that T is represented by $g \in L_p(\mathbf{R}, X)$ if

$$Th = \int_{-\infty}^\infty h(t) g(t) \, dt, \quad h \in Y_q(\mathbf{R}).$$

If $p > 1$ and if X has the Radon-Nikodym property then it follows from [3], IV.4, Theorem 8, that every p-bounded operator can be represented by an $L_p(\mathbf{R}, X)$-function.

THEOREM 2 *Let $1 \le p \le \infty$. A harmonic function $F : \mathbf{C}_+ \to X$ belongs to $h_p(\mathbf{C}_+, X)$ if and only if there exists a p-bounded operator $T : L_q(\mathbf{R}) \to X$ such that*

$$F(z) = T(P_z) = \lim_{a \to 0^+} \int_{-\infty}^{\infty} P_z(t) F(a + it)\, dt, \quad z \in \mathbf{C}_+.$$

If $1 < p \le \infty$ and if X has the Radon-Nikodym property then F has a boundary function $f \in L_p(\mathbf{R}, X)$.

Proof. Let $F \in H_p(\mathbf{C}_+, X)$. Then, for every $a > 0$, the function $F_a : \mathbf{C}_+ \to X$ defined by $F_a(z) = F(a + z)$ belongs to $h_p(\mathbf{C}_+, X)$, and $\|F_a\|_{h_p} \le \|F\|_{h_p}$. Since F_a is bounded and continuous in the closed right half-plane $\overline{\mathbf{C}}_+$ (this follows e.g. from [2], I.3.) F_a is the Poisson integral of its boundary function (see [2], Lemma I.3.4. for the scalar case). It follows that

$$F(z) = \lim_{a \to 0^+} F_a(z) = \lim_{a \to 0^+} \int_{-\infty}^{\infty} P_z(t) F_a(it)\, dt.$$

Collecting these informations yields:

1. For all h in the total subset $\{P_z : z \in \mathbf{C}_+\}$ of $Y_q(\mathbf{R})$ the limit of $T_a h := \int_{-\infty}^{\infty} h(t) F_a(t)\, dt$ exists, as $a \to 0^+$.
2. The family (T_a) of operators from $Y_q(\mathbf{R})$ into X is uniformly bounded by $\|T_a\| \le \|F\|_{H_p(\mathbf{C}_+)}$.

Hence, by the Banach-Steinhaus-Theorem, there exists an operator $T : L_q(\mathbf{R}) \to X$ with $\|T\| \le \|F\|_{H_p(\mathbf{C}_+)}$, and such that

$$Th = \lim_{a \to 0^+} \int_{-\infty}^{\infty} h(t) F_a(t)\, dt = \lim_{a \to 0^+} \int_{-\infty}^{\infty} h(t) F(a + it)\, dt. \tag{1}$$

Moreover, the operator T is p-bounded. To see this, consider the functions $g_a \in L_p(\mathbf{R})$ defined by $g_a(t) = \|F_a(it)\|$. Then the family $\{g_a : a > 0\}$ is bounded in $L_p(\mathbf{R})$. Consequently, if $p > 1$, there exists a positive null-sequence (a_n) so that (g_{a_n}) has a weak limit $g \in L_p(\mathbf{R})$, if $p < \infty$, and so that (g_{a_n}) has a weak* limit $g \in L_\infty(\mathbf{R})$, if $p = \infty$. If $p = 1$ then, by Helly's theorem, (a_n) can be chosen so that (g_{a_n}) has a weak* limit $\psi : \mathbf{R} \to \mathbf{R}$ of bounded variation. Consequently,

$$\begin{aligned} \|Th\| &\leq \limsup_{n\to\infty} \int_{-\infty}^{\infty} |h(t)| \, \|F_{a_n}(it)\| \, dt \\ &= \limsup_{n\to\infty} \int_{-\infty}^{\infty} |h(t)| g_{a_n}(t) \, dt = \int_{-\infty}^{\infty} |h(t)| g(t) \, dt \end{aligned}$$

if $p > 1$, and

$$\|Th\| \leq \int_{-\infty}^{\infty} |h(t)| \, d\psi(t)$$

if $p = 1$.

If X has the Radon-Nikodym property, then, according to [3], IV.4, Theorem 8, for $1 < p \leq \infty$ every p-bounded operator can be represented by a function $f \in L_p(\mathbf{R}, X)$. $\equiv$

The above considerations for $h_p(\mathbf{C}_+, X)$-functions clearly hold also for $H_p(\mathbf{C}_+, X)$-functions. Our next goal is to show that for $p > 1$ similar assertions are true for functions $G \in H_p(\Sigma_\alpha, X)$. Therefore, we exchange the cartesian coordinates "$a + it$" by polar coordinates "$te^{i\eta}$", and for each $0 < \eta < \alpha$ we introduce the function $G_\eta(t) = G(|t|e^{i\mathrm{sgn}(t)\eta})$, $t \in \mathbf{R}$. We say that G has a boundary function $g \in L_p(\mathbf{R}, X)$ if

$$\lim_{\eta\to\alpha^-} \int_{-\infty}^{\infty} h(t) G_\eta(t) \, dt = \int_{-\infty}^{\infty} h(t) g(t) \, dt, \quad h \in L_q(\mathbf{R}).$$

THEOREM 3 *Let $1 < p \leq \infty$ and $0 < \alpha < \pi$. For every $G \in H_p(\Sigma_\alpha, X)$ there exists a p-bounded operator $T : L_q(\mathbf{R}) \to X$ such that*

$$Th = \lim_{\eta\to\alpha^-} \int_{-\infty}^{\infty} h(t) G_\eta(t) \, dt, \quad h \in L_q(\mathbf{R}).$$

If $1 < p \leq \infty$ and if X has the Radon-Nikodym property then G has a boundary function $g \in L_p(\mathbf{R}, X)$.

Proof. The proof is divided into two steps. In the first step we consider the case $\alpha = \pi/2$, and in the second step we prove the assertion for general α.

Step 1: Let $G \in H_p(\Sigma_{\pi/2}, X)$ and for $t \in \mathbf{R}$ define $\chi_t = \chi_{[0,t)}$ if $t > 0$, $\chi_t = -\chi_{[t,0)}$ if $t < 0$ and $\chi_0 = 0$. We show first that the limit

$$T\chi_t := \lim_{\eta \to \alpha^-} \int_0^t G_\eta(u)\, du \tag{2}$$

exists for every real number t. We know by Theorem 1 that G belongs to $H_p(\mathbf{C}_+, X)$. Hence, by Theorem 2 there exists a bounded operator $T : L_q(\mathbf{R}) \to X$ with

$$Th = \lim_{c \to 0^+} \int_{-\infty}^{\infty} h(t) G(c + it)\, dt. \tag{3}$$

Let $z = a + ib \in \mathbf{C}_+$. Then there exists $t \in \mathbf{R}$ and $0 \le \eta < \frac{\pi}{2}$ so that $z = |t| e^{i\mathrm{sgn}(t)\eta}$. We define three paths $\Gamma_1(z)$, $\Gamma_2(z)$ and $\Gamma_3(z)$ in the following way:

$$\begin{aligned} \Gamma_1(z) &= (s e^{i\mathrm{sgn}(t)\eta} : 0 \le s \le |t|), \\ \Gamma_2(z) &= (s : 0 \le s \le a), \\ \Gamma_3(z) &= (a + i\mathrm{sgn}(t) s : 0 \le s \le |b|). \end{aligned}$$

Then $(\Gamma_1(z), \Gamma_2(z), \Gamma_3(z))$ is a closed path in $\overline{\mathbf{C}}_+$. Since G is holomorphic in $\mathbf{C}_+$, and since $\|G(z)\| \le K \mathrm{Re}(z)^{-1/p}$ (for the scalar case see [4], Lemma on page 149) we infer from Cauchy's theorem

$$\begin{aligned} \int_0^t G_\eta(s)\, ds &= \mathrm{sgn}(t) e^{-i\mathrm{sgn}(t)\eta} \int_{\Gamma_1(z)} G(\nu)\, d\nu \\ &= \mathrm{sgn}(t) e^{-i\mathrm{sgn}(t)\eta} \Big(\int_{\Gamma_2(z)} G(\nu)\, d\nu + \int_{\Gamma_3(z)} G(\nu)\, d\nu \Big) \\ &= \mathrm{sgn}(t) e^{-i\mathrm{sgn}(t)\eta} \Big(\int_0^a G(s)\, ds + i \int_0^b G(a + is)\, ds \Big). \end{aligned} \tag{4}$$

Let $t \in \mathbf{R}$ and put $z(\eta) = |t| e^{i\mathrm{sgn}(t)\eta} = a(\eta) + ib(\eta)$. If η converges towards $\pi/2$ then $b(\eta)$ converges towards t. Consequently, equation (3) yields

$$\begin{aligned} & \lim_{\eta \to \pi/2} i\mathrm{sgn}(t) e^{-i\mathrm{sgn}(t)\eta} \int_0^{b(\eta)} G(a(\eta) + is)\, ds \\ = & \lim_{\eta \to \frac{\pi}{2}} \int_0^t G(a(\eta) + is)\, ds - \lim_{\eta \to \frac{\pi}{2}} \int_{b(\eta)}^t G(a(\eta) + is)\, ds \\ = & \; T\chi_t, \end{aligned}$$

because

$$\left\| \int_{b(\eta)}^{t} G(a(\eta)+is)\,ds \right\| \le |t-b(\eta)|^{1/q}\, \|G\|_{H_p} \text{ and } \lim_{\eta\to\pi/2} |t-b(\eta)| = 0.$$

The first part $\int_0^{a(\eta)} G(s)\,ds$ in the sum of equation (4) converges towards zero for $\eta \to \pi/2$, because $\|G(s)\| \le Ks^{-1/p}$. Hence, equation (2) is proved for $\alpha = \pi/2$. Since the set $\{\chi_t : t \in \mathbf{R}\}$ is total in $L_q(\mathbf{R})$, and since the family $\{G_\eta : -\pi/2 < \eta < \pi/2\}$ is bounded in $L_p(\mathbf{R}, X)$, there exists, by the Banach-Steinhaus-Theorem, a bounded operator $T : L_q(\mathbf{R}) \to X$ such that, for all $h \in L_q(\mathbf{R})$

$$Th = \lim_{\eta\to\pi/2-} \int_{-\infty}^{\infty} h(t)G_\eta(t)\,dt.$$

Step 2: Now let $0 < \alpha < \pi$. Let $G \in H_p(\Sigma_\alpha, X)$ and put

$$F(z) = \left(\frac{\pi}{2\alpha}\right)^{1/p} z^{\frac{1}{p}(\frac{\pi}{2\alpha}-1)} G\left(z^{\frac{\pi}{2\alpha}}\right).$$

Then, by straightforward calculations, F belongs to $H_p(\Sigma_{\pi/2}, X)$, and for $0 < \eta < \alpha$ and $t \in \mathbf{R}$ it follows that

$$\begin{aligned} G_\eta(t) &= \left(\frac{\pi}{2\alpha}\right)^{1/p} (|t|e^{i\mathrm{sgn}(t)\eta})^{\frac{1}{p}(\frac{\pi}{2\alpha}-1)} F\left((|t|e^{i\mathrm{sgn}(t)\eta})^{\frac{\pi}{2\alpha}}\right) \\ &= \left(\frac{\pi}{2\alpha}\right)^{1/p} (|t|e^{i\mathrm{sgn}(t)\eta})^{\frac{1}{p}(\frac{\pi}{2\alpha}-1)} F_{\eta\frac{\pi}{2\alpha}}(\mathrm{sgn}(t)|t|^{\frac{\pi}{2\alpha}}). \end{aligned}$$

If $h : \mathbf{R} \to Y$, Y a Banach space, is a function then define $Bh : \mathbf{R} \to Y$ by

$$Bh(t) = \left(\frac{\pi}{2\alpha}\right)^{1/p} |t|^{\frac{1}{p}(\frac{\pi}{2\alpha}-1)} h(\mathrm{sgn}(t)|t|^{\frac{\pi}{2\alpha}}),$$

and define $Ch : \mathbf{R} \to Y$ by

$$Ch(t) = \left(\frac{2\alpha}{\pi}\right)^{1/q} |t|^{\frac{1}{q}(\frac{2\alpha}{\pi}-1)} h(\mathrm{sgn}(t)|t|^{\frac{2\alpha}{\pi}}).$$

It is easy to see that the operators B and C are isometric isomorphisms on $L_p(\mathbf{R}, Y)$. Moreover, put $d_\eta(t) = e^{i\mathrm{sgn}(t)\frac{\eta}{p}(\frac{\pi}{2\alpha}-1)}$. Then for $h \in L_q(\mathbf{R})$

$$\begin{aligned} \int_{-\infty}^{\infty} h(t)G_\eta(t)\,dt &= \int_{-\infty}^{\infty} h(t)d_\eta(t)\,(BF_{\eta\frac{\pi}{2\alpha}})(t)\,dt \\ &= \int_{-\infty}^{\infty} d_\eta(u)\,(Ch)(u)F_{\eta\frac{\pi}{2\alpha}}(u)\,du. \end{aligned}$$

By Step 1,

$$Sh := \lim_{\theta \to \pi/2-} \int_{-\infty}^{\infty} h(t) F_\theta(t)\, dt$$

defines a bounded operator $S : L_q(\mathbf{R}) \to X$. Consequently, since $\eta \frac{\pi}{2\alpha}$ converges towards $\pi/2$ from below, as $\eta \to \alpha^-$, the result of the Step 1 implies that the following limit exists:

$$\lim_{\eta \to \alpha-} \int_{-\infty}^{\infty} h(t) G_\eta(t)\, dt =: Th, \quad h \in L_q(\mathbf{R}).$$

Clearly, $T : L_q(\mathbf{R}) \to X$ is a bounded operator. In order to show that T is p-bounded, put $g_\eta(t) = \|G_\eta(t)\|$. Then there exists a sequence (η_n) converging towards α from below, so that (g_{η_n}) has a weak limit $g \in L_p(\mathbf{R})$, if $p < \infty$, and so that (g_{η_n}) has a weak* limit $g \in L_\infty(\mathbf{R})$, if $p = \infty$. Consequently, for $h \in L_q(\mathbf{R})$

$$\begin{aligned} \|Th\| &\leq \limsup_{n\to\infty} \int_{-\infty}^{\infty} |h(t)|\, \|G_{\eta_n}(it)\|\; dt \\ &= \limsup_{n\to\infty} \int_{-\infty}^{\infty} |h(t)| g_{\eta_n}(t)\, dt = \int_{-\infty}^{\infty} |h(t)| g(t)\, dt. \end{aligned}$$

If X has the Radon-Nikodym property it follows from [3], IV.4, Theorem 8, that T can be represented by a function $g \in L_p(\mathbf{R}, X)$. $\equiv$

3 A Paley-Wiener-type theorem for $H_2(\Sigma_\alpha, X)$

The classical Paley-Wiener-Theorem [7] states that the complex Laplace transform $F(z) = \int_0^\infty e^{-zt} f(t)\, dt$ is an isometric isomorphism between the spaces $L_2([0,\infty))$ and $H_2(\mathbf{C}_+, \mathbf{C})$ (see also [8], Chapter 19). The proof of this theorem uses heavily the Plancherel theorem for the Fourier transform. By Kwapien's theorem [5] a Banach space valued version of Plancherel's theorem does not exist, in general, but the Fourier transform maps $L_2(\mathbf{R}, X)$ isomorphically onto $L_2(\mathbf{R}, X)$ if and only if X is isomorphic to a Hilbert space. Hence, for Hilbert spaces X, the following vector-valued version of the Paley-Wiener-Theorem theorem can be proved (see [9], Proposition 2.2.14).

THEOREM 4 *If X is isomorphic to a Hilbert space then the X-valued complex Laplace transform*

$$F(z) = \int_0^\infty e^{-zt} f(t), \quad z \in \mathbf{C}_+,$$

is an isomorphism from $L_2([0,\infty), X)$ onto $H_2(\mathbf{C}_+, X)$.

The following corollary is an immediate consequence of this vector-valued version of the Paley-Wiener-Theorem together with Theorem 1.

COROLLARY 5 *If X is isomorphic to a Hilbert space then the vector-valued complex Laplace transform*

$$F(z) = \int_0^\infty e^{-zt} f(t), \quad z \in \mathbf{C}_+,$$

is an isomorphism from $L_2([0,\infty), X)$ onto $H_2(\Sigma_{\pi/2}, X)$.

We now want give the outline of a proof for a theorem of Paley-Wiener type for the complex Laplace transform of functions belonging to $H_2(\Sigma_\alpha, X)$; more precise:

THEOREM 6 *Let $0 < \alpha < \pi/2$, and let X be isomorphic to a Hilbert space Then the complex Laplace transform of any function $f \in H_2(\Sigma_\alpha, X)$ can be extended holomorphically to a unique function $F \in H_2(\Sigma_{\alpha+\pi/2}, X)$, and every $F \in H_2(\Sigma_{\alpha+\pi/2}, X)$ is the extension of the complex Laplace transform of a unique $f \in H_2(\Sigma_\alpha, X)$.*

Proof. (an outline) If f belongs to $H_2(\Sigma_\alpha, X)$ then one defines a holomorphic function $F : \Sigma_{\pi/2} \to X$ in the following way: For $-\alpha < \theta < \alpha$ define the path $\Gamma_\theta = (re^{i\theta} : 0 < r < \infty)$. Now, given $z \in e^{-i\theta}\mathbf{C}_+$ put

$$F_\theta(z) = \int_{\Gamma_\theta} e^{-z\mu} F(\mu)\, d\mu.$$

Then $F_\theta : e^{-i\theta}\mathbf{C}_+ \to X$ is a holomorphic function, and by Corollary 5

$$\sup_{|\beta|<\pi/2} \int_0^\infty \left\| F_\theta(re^{i(\theta+\beta)}) \right\|^2 dr \le M^2 \|f\|^2_{H_2(\Sigma_\alpha, X)}, \tag{5}$$

where M denotes the norm of the vector-valued complex Laplace transform from $L_2([0,\infty),X)$ into $H_2(\Sigma_{\pi/2},X)$. Moreover, one can show that $F_\theta(z) = F_\eta(z)$ whenever $-\alpha < \theta,\eta < \alpha$ and $z \in e^{-i\theta}\mathbf{C}_+ \cap e^{-i\eta}\mathbf{C}_+$. Since $\bigcup_{|\theta|<\alpha} e^{-i\theta}\mathbf{C}_+ = \Sigma_{\alpha+\pi/2}$ we can define $F : \Sigma_{\alpha+\pi/2} \to X$ by $F(z) = F_\theta(z)$, where θ is chosen so that z belongs $e^{-i\theta}\mathbf{C}_+$. Then it follows from (5) that F belongs to $H_2(\Sigma_{\alpha+\pi/2},X)$ with $\|F\|_{H_2(\Sigma_{\alpha+\pi/2})} \le M\,\|f\|_{H_2(\Sigma_\alpha)}$, and $F_{|\mathbf{C}_+}$ is the complex Laplace transform of f.

Conversely, assume F to belong to $H_2(\Sigma_{\alpha+\pi/2},X)$. Then, by [9], Satz 2.3.7, F is the complex Laplace transform of a holomorphic function $f : \Sigma_\alpha \to X$, and $F(z)$ can be evaluated by

$$F(z) = \int_{\Gamma_\theta} e^{-z\mu} f(\mu)\,d\mu, \quad z \in e^{-i\theta}\mathbf{C}_+.$$

It remains to be shown that f belongs to $H_2(\Sigma_\alpha,X)$. Let $|\theta| < \alpha$, and, for $z \in \mathbf{C}_+$, put $F_\theta(z) = F(ze^{-i\theta})$. Then, by Corollary 5, F_θ is the complex Laplace transform of a function $f_\theta \in L_2([0,\infty),X)$. Hence, for $z = re^{-i\theta}$ it follows that

$$\begin{aligned} e^{i\theta}\int_0^\infty e^{-rt} f(te^{i\theta})\,dt &= \int_{\Gamma_\theta} e^{-z\mu} f(\mu)\,d\mu \\ &= F(z) = F_\theta(r) = \int_0^\infty e^{-rt} f_\theta(t)\,dt. \end{aligned}$$

By uniqueness of the Laplace transform it follows that $e^{i\theta}f(te^{i\theta}) = f_\theta(t)$ almost everywhere in $[0,\infty)$. Consequently,

$$\int_0^\infty \left\| f(re^{i\theta}) \right\|^2 dr = \|f_\theta\|_{L_2}^2 \le L^2 \|F\|_{H_2(\Sigma_{\alpha+\pi/2})}^2,$$

where L denotes the norm of the inverse complex Laplace transform from $H_2(\mathbf{C}_+,X)$ into $L_2([0,\infty),X)$. $\equiv$

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Peter Vieten
Fachbereich Mathematik
Universität Kaiserslautern
Erwin-Schrödinger-Straße
67663 Kaiserslautern
Germany

vieten@mathematik.uni-kl.de