SMOOTH PROBABILITY MEASURES AND ASSOCIATED DIFFERENTIAL OPERATORS

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Abstract We compare different notions of differentiability of a measure along a vector field on a locally convex space. We consider in the L^2-space of a differentiable measure the analoga of the classical concepts of gradient, divergence and Laplacian (which coincides with the Ornstein-Uhlenbeck operator in the Gaussian case). We use these operators for the extension of the basic results of Malliavin and Stroock on the smoothness of finite dimensional image measures under certain nonsmooth mappings to the case of non-Gaussian measures. The proof of this extension is quite direct and does not use any Chaos-decomposition. Finally, the role of this Laplacian in the procedure of quantization of anharmonic oscillators is discussed.

1 Introduction

This paper is devoted to the foundations of the 'calculus of differentiable measures'. The recent years have seen a certain revival of the theory of differentiable measures in particular because the concept of differentiation of a measure along a vector field allows to understand some basic constructions of stochastic calculus in a new and simple way.
However there are several ways to make the differentiation of a measure along vector fields precise. Thus the first aim of this paper in section 2 is to study the connections between some of these notions. In particular it turns out that the most widely used and most flexible definition based on the formula of integration by parts is equivalent under suitable regularity assumptions to the direct definitions (like those of Fomin and Skorokhod) which in turn are particular cases of more general notions of differentiability for curves of measures on abstract spaces (cf. [SW93]). In finite dimensional spaces we give a very short proof that differentiable measures are absolutely continuous with respect to Lebesgue measure (which is one possible formulation of the

socalled 'Malliavin-Lemma').
In section 3 we fix a nonnegative measure on a locally convex space E which is differentiable along a Hilbert subspace of E. We introduce the operator D as the closure in $L^2(E,\nu)$ of the gradient operator. In the case of Gaussian measures this operator is called Malliavin derivative. We extend some elementary properties of the gradient to the operator D. The adjoint of D is the divergence operator δ_ν associated with ν. The definition of differentiability of a measure via integration by parts essentially implies that for a vector field $h : E \to H$ the function $\delta_\nu h$ is the negative of the logarithmic derivative of the measure ν along this vector field. Finally the Laplacian Δ_ν corresponding to the measure ν is defined as the composition $-\delta_\nu D$. In the Gaussian case this operator is the Ornstein- Uhlenbeck operator. We give a simple rule how this operator changes if one passes from one measure to an equivalent one. This operator and the transformation rule are closely related to the procedure of the canonical quantization of classical Hamiltonian systems whose Hamiltonian function has the form $\frac{1}{2}(Ap,p)+V(q)$. This connection is explained in more detail in the last section where we also use the concept of genereralized densities of differentiable measures introduced in [Kir94] and [SW95].
Section 4 is concerned with the smoothness of an image measure of ν under some nonlinear map $u : E \to F$ whith particular emphasis on finite dimensional images in the spirit of Malliavin's approach [Mal76] to the smoothness results of Hörmander typ. Recently there is more and more interest to get analogous results for other underlying laws. One early example is the result of Bouleau and Hirsch ([BH91], Theorem 5.2.2) on the existence of a density. For general differentiable measures and strictly differentiable maps u the question of smoothness of the density has been studied by Daletski-Steblowskaja [DF92]. Because of the stronger assumptions on u, however, this does not cover the typical setting in Wiener space. We show here that the classical result of Stroock [Str81] can be extended to general differentiable measures. The proof in this more general case looks simpler than the original proof for Wiener measure. We note however that this result does not involve analoga of Meyer's inequalities.

2 Notions of differentiability of measures

The main purpose of this section is to put the (usual) definition (see Definition 1 below) of the derivative and the logarithmic derivative of a measure along a vector field into a broader context.
Below all vector spaces are real. We call a mapping from a locally convex space (LCS) E into another LCS G *smooth with respect to a subspace* $H \subset E$ if it is Gâteaux differentiable in the directions in H infinitely many times and if both the mapping and all its derivatives are continuous on E where the spaces of linear mappings from H into suitable spaces in which the derivatives take their values are equipped inductively with the topology of uniform convergence on compact subsets of H. A *vector field* in a LCS E is a mapping $h : E \to E$; one denotes by $vect(E)$ the vector space of all vector fields on E. The *derivative of a function u (on E) along the vector field $h \in vect(E)$* is the function denoted by $u'h$ and defined by $(u'h)(x) = u'(x)(h(x))$.
We need some notions of differentiability of measures. If $(\Omega, \mathcal{B})$ is a measurable space let $\mathcal{M}(\Omega)$ be the vector space of all signed σ-additive measures on $\mathcal{B}$. Every topological space E will be equipped with its Borel σ-algebra $\mathcal{B}(E)$. We call a space C of bounded Borel functions *norm-defining for $\mathcal{M}(E)$* if $\|\mu\|_1 = \sup\left\{\int u\, d\mu : u \in C, \|u\|_\infty \leq 1\right\}$ where $\|\cdot\|_1$ is the total variation norm and $\|\cdot\|_\infty$ is the sup-norm. If $\mu \in \mathcal{M}(\Omega)$ and $u \in L^1(\mu)$ we denote by $u\mu$ the measure $A \longmapsto \int_A u d\mu$.
If E is a LCS a measure $\nu \in \mathcal{M}(E)$ is called *Fomin-differentiable* along a vector $h \in E$ [ASF71] if for every set $A \in \mathcal{B}(E)$ the mapping $f_\nu^{A,h} : \mathbb{R} \ni t \mapsto \nu(A + th)$ is differentiable at 0 (and consequently everywhere). Then the map $\nu'_h : A \mapsto (f_\nu^{A,h})'(0)$ turns out to be a (signed) measure on $\mathcal{B}(E)$ which is called the (Fomin) derivative of ν along h; moreover, it is absolutely continuous with respect to ν (see [1] or Proposition 1 below). The Radon–Nikodym derivative of ν'_h with respect to ν is denoted by $\beta^\nu(h, \cdot)$ and called *logarithmic derivative of ν in the direction h*. Thus $\nu'_h = \beta^\nu(h, \cdot)\nu$.
These definitions can be extended quite naturally in various ways. The following definition was introduced in [SW93]. Denote by τ a topology on $\mathcal{M}(\Omega)$. A function $f : t \mapsto \nu_t$ from an open interval I in $\mathbb{R}$ into the space $\mathcal{M}(E)$ is called *τ-differentiable at $t \in I$* if there exists a measure ν'_t such that $\tau - \lim_{s\to 0}(\nu_{s+t} - \nu_t)/s = \nu'_t$. If ν'_t is absolutely continuous with respect to ν_t then the Radon–Nikodym derivative $\frac{d\nu'_t}{d\nu_t}$ is denoted by $\rho(t, f)$ or simply ρ_t

and it is called *logarithmic derivative of* f at the point t.
In particular let ν be a fixed measure, let $\varepsilon > 0$ and let $\mathcal{T} = (T_t)_{-\varepsilon<t<\varepsilon}$ be a family of $\mathcal{B}$-measurable (not necessarily invertible) transformations of the set Ω with $T_0 = id$. We call $\nu \in \mathcal{M}(\Omega)$ *τ-differentiable along the family* $\mathcal{T}$ iff the map $f : t \mapsto \nu_t = \nu \circ T_t^{-1}$ is τ-differentiable at $t = 0$. The derivative $f'(0) \in \mathcal{M}(\Omega)$ is denoted by $\nu'_{\mathcal{T}}$. The logarithmic derivative of f at the point 0 (if it exists) is called the (τ-) l.d. of ν along $\mathcal{T}$ and is denoted by $\beta^{\nu}_{\mathcal{T}}$. If τ is the topology τ_s of setwise convergence on $\mathcal{M}(\Omega)$ we speak of *Fomin-differentiability* along $\mathcal{T}$. Similarly, if C is the space of bounded continuous functions on Ω for some underlying topology on Ω and τ is the topology $\tau_C = \sigma(\mathcal{M}(\Omega), C)$ then one speaks of *Skorokhod-differentiability.*
The following proposition implies that in the case of Fomin-differentiablility along a family $\mathcal{T}$ the logarithmic derivative always exists. The proposition does not hold for general τ_s-differentiable functions $f : (-\varepsilon, \varepsilon) \to \mathcal{M}(\Omega)$ (see [SW93]) nor does it extend to weaker topologies than τ_s.

Proposition 1: *Let $\nu \in \mathcal{M}(\Omega)$ be Fomin-differentiable along a family $\mathcal{T} = (T_t)_{-\varepsilon<t<\varepsilon}$ of measurable transformations with $T_0 = id$. Then*
(a) $\nu'_{\mathcal{T}} \ll \nu$, i.e. the logarithmic derivative of ν along $\mathcal{T}$ exists.
(b) Let $\nu = \nu^+ - \nu^-$ be the Hahn–Jordan decomposition of ν and let $S \in \mathcal{B}$ be such that $\nu^+(A) = \nu(A \cap S)$ for every $A \in \mathcal{B}$. Then ν^+ and ν^- are also Fomin-differentiable along $\nu_{\mathcal{T}}$ and $(\nu^+)'_{\mathcal{T}} = \nu'_{\mathcal{T}}(\cdot \cap S)$.

Proof: The proof is very close to the proof of the particular case given in [ASF71] for the shifts by a constant vector field. As before let $f^A_{\mathcal{T},\nu}$ denote the map $t \mapsto \nu(T_t^{-1}A)$.
We first prove part (b). We claim that for each function $\mathbb{R}^1 \ni \mathrm{t} \mapsto \mathrm{B(t)} \in \mathcal{B}$ one has

$$\frac{\nu\left((T_t^{-1}S \setminus S) \cap B(t)\right)}{t} \to 0 \quad as\ t \to 0 \tag{1}$$

and

$$\frac{\nu\left((S \setminus T_t^{-1}S) \cap B(t)\right)}{t} \to 0 \quad as\ t \to 0 \tag{2}$$

In fact, the function $f^S_{\mathcal{T},\nu}$ has a maximum at $t = 0$. Therefore $\nu'_{\mathcal{T}}(S) = 0$ and hence

$$\frac{\nu(T_t^{-1}S \setminus S)}{t} + \left(-\frac{\nu(S \setminus T_t^{-1}S)}{t}\right)$$

$$= \frac{\nu(T_t^{-1}S \setminus S) - \nu(S \setminus T_t^{-1}S)}{t} = \frac{\nu(T_t^{-1}S) - \nu(S)}{t} \to 0 \;\; as \; t \to 0.$$

But for every $B \in \mathcal{B}$ one has $\nu(T_t^{-1}S \setminus S) \leq \nu(T_t^{-1}S \setminus S) \cap B \leq 0$ and $-\nu(S \setminus T_t^{-1}S) \leq -\nu(S \setminus T_t^{-1}S) \cap B \leq 0$. This implies the formulae (1) and (2).
Now we prove that for every $A \in \mathcal{B}$ the derivative of $f^A_{\mathcal{T},\nu^+}$ at $t = 0$ exists. As $S = ((T_t^{-1}S) \cup (S \setminus T_t^{-1}S)) \setminus (T_t^{-1}S \setminus S)$, the following identities are true:

$$\begin{aligned} \frac{\nu^+(T_t^{-1}A) - \nu^+(A)}{t} &= \frac{\nu((T_t^{-1}A) \cap S) - \nu(A \cap S)}{t} \\ = \; & \frac{\nu((T_t^{-1}A) \cap T_t^{-1}S) - \nu(A \cap S)}{t} \\ & \frac{\nu(T_t^{-1}A \cap (S \setminus T_t^{-1}S)) - \nu(T_t^{-1}A \cap (T_t^{-1}S \setminus S))}{t}. \end{aligned}$$

The relations (1) and (2) imply that the second term at the right-hand side converges to 0; the first term converges to $(f^A_\nu)'(0)(A \cap S)$. This means that $f^A_{\nu^+}$ (and hence $f^A_{\nu^-}$) is differentiable at $t = 0$ and that $(f^A_{\nu^+})'(0)(A) = (f^A_\nu)'(0)(A \cap S)$. This proves (b).
For part (a) we see that it is enough to prove it for ν^+ and for ν^-. But if $A \in \mathcal{B}$ and $\nu^+(A) = 0$, then the function $f^A_{\nu^+}$ has a minimum at $t = 0$; hence $(f^A_{\nu^+})'(0)(A) = 0$. The case of ν^- is treated similarly. The proposition is proved. ■

Part (b1) of the following observation gives a partial converse to Proposition 1. Part (a) is a sort of mean value theorem for the transport of a measure along a flow.

Proposition 2: *Let ν be τ_C-differentiable along a measurable flow $\mathcal{T} = (T_t)$ of bijections on Ω where C is normdefining for $\mathcal{M}(\Omega)$ and $\mathcal{T}$-invariant, i.e. $C = \{v \circ T_t : v \in C\}$ for all $t \in \mathbb{R}$.*
(a) Then the map $f : t \mapsto \nu \circ T_t^{-1}$ is τ_C-differentiable for all t and Lipschitz-continuous for the norm $\| \cdot \|_1$:

$$\|\nu \circ T_t^{-1} - \nu \circ T_s^{-1}\|_1 \leq |t - s| \; \|\nu'_{\mathcal{T}}\|_1 \tag{3}$$

(b) ν is even differentiable along $\mathcal{T}$ for the norm topology $\tau_{\|\cdot\|_1}$ if either
(b1) $\nu'_{\mathcal{T}} \ll \nu$ or
(b2) ν is twice τ_C-differentiable, i.e. if $\nu'_{\mathcal{T}}$ is also τ_C-differentiable along $\mathcal{T}$.

Proof: (a). The map $f : t \mapsto \nu_t = \nu \circ T_t^{-1}$ is τ_C-differentiable at $t = 0$ by assumption. Since each T_t is a bijection and C is $\mathcal{T}$-invariant this implies that f is everywhere τ_C-differentiable. Therefore for $s < t$ by the mean value theorem the vector $\frac{f(t)-f(s)}{t-s}$ is in the τ_C-closed convex hull of the set $M = \{f'(\theta) : s \leq \theta \leq t\}$. Since C is normdefining for $\mathcal{M}(\Omega)$ this is equal to the $\|\cdot\|_1$ closed convex hull of M. On the other hand the $\mathcal{T}$-invariance of the space C implies that $f'(\theta) = f'(0) \circ T_\theta^{-1} = \nu'_\mathcal{T} \circ T_\theta^{-1}$ and hence $\|f'(\theta)\|_1 = \|\nu'_\mathcal{T}\|$ for all θ. Thus

$$\|\frac{f(t) - f(s)}{t - s}\|_1 \leq \|\nu'_\mathcal{T}\|_1$$

which yields the assertion.
(b) In the case (b1) $\nu_\mathcal{T} \ll \nu$ this is part of the theorem in section 5 of [SW93]. If (b2) holds then $\nu'_\mathcal{T}$ itself is τ_C-differentiable along $\mathcal{T}$ and the assertion follows from the following adaption of the proof of [SW93], Prop. 2.5 (d): According to part (a) the map $t \mapsto f'(t) = \nu'_\mathcal{T} \circ T_t^{-1}$ is continuous for the norm topology and hence as above by the meanvalue theorem and the fact that C is normdefining

$$\|\frac{f(t) - f(0)}{t} - \nu'_\mathcal{T}\|_1 \leq \|\nu'_\mathcal{T} \circ T_\theta^{-1} - \nu'_\mathcal{T}\|_1$$

which tends to 0 as $t \to 0$. ■

Next we discuss differentiation of measures along vector fields. If E is a LCS and $\nu \in \mathcal{M}(E)$ one can call ν $\tau-$ (resp. Fomin)-differentiable *along the vector field* $h \in vect(E)$ if it is $\tau-$ (resp. Fomin)-differentiable along the family $\mathcal{T}_h$ given by

$$T_t^h(x) = x - th(x) \tag{4}$$

In particular, if $h(x) \equiv h_0$ then this definition coincides with the definition of differentiability of ν along the vector h_0 given above. In this case $\beta^\nu_{\mathcal{T}_h}(\cdot) = \beta^\nu(h_0, \cdot)$.

However the most flexible concept of differentiability of a measure along a vector field is based on a formula of integration by parts:

Definition 1: *Let C be a vector space of smooth scalar functions on E which together with their derivatives are bounded. Suppose moreover that C is normdefining for $\mathcal{M}(E)$. The measure $\nu \in \mathcal{M}(E)$ is called* C-differentiable *along the vector field $h \in vect(E)$ if there is a measure $\nu'_h \in \mathcal{M}(E)$ such that for every $u \in C$ the following formula of integration by parts holds:*

$$\int u'h \, d\nu = -\int u \, d\nu'_h. \tag{5}$$

If $\nu'_h \ll \nu$ the corresponding Radon-Nikodym derivative is called logarithmic derivative of ν along h and is denoted by β^ν_h.

The connection between these various definitions is partially described in the next Proposition. The reader may think of the family (T_t) either as given by $T_t x = x - th(x)$ or as the integral flow of the vector field $-h$ (if this flow exists).

Proposition 3: *Let h be a vector field on E and let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be a family of vector fields such that $T_0 = id$ and the map $F : (t, x) \mapsto T_t x$ is differentiable in t with $F'(0, x) = -h(x)$ for all $x \in E$ and suppose that $\{F'_1(t, x) : t \in \mathbb{R}, \mathrm{x} \in \mathrm{E}\}$ is bounded. Let τ_C be the topology $\sigma(\mathcal{M}(E), C)$. Consider the following conditions:*
(a) The measure ν is Fomin-differentiable along h.
(a') The measure ν is Fomin-differentiable along (T_t).
(b) The measure ν is τ_C-differentiable along (T_t).
(c) The measure ν is C-differentiable along h.
Then $(a) \Longrightarrow (b) \Longleftrightarrow (c)$. Moreover $(a') \Longrightarrow (b)$. The derivatives ν'_h and (if they exist) the corresponding logarithmic derivatives coincide in all four cases. If (b) and (c) hold for one family $\mathcal{T}$ with the above properties then they hold also for all other such families $\mathcal{T}$. If (T_t) is a measurable flow of bijections of E and $\nu'_h \ll \nu$ then $(b) \Longrightarrow (a')$.

Proof: $(a) \Longrightarrow (b)$. The existence of the logarithmic derivative β^ν_h follows from proposition 1. Then for every bounded measurable function u the map $f^u_h : t \mapsto \int u \circ T^h_t \, d\nu$ is differentiable at $t = 0$ with derivative $(f^u_h)'(0) = \int u\beta^\nu_h d\nu$. This follows via uniform approximation of u by step functions. In particular (b) holds with $\beta^\nu_h = \rho_0$ and the special family $T_t = T^h_t$ defined in (4). For the other families see below.
$(b) \Longleftrightarrow (c)$. Let $u \in C$. Then

$$\lim_{t\to 0} \frac{1}{t}\Big(\int u d\nu_t - \int u d\nu\Big) = \lim_{t\to 0} \int \frac{u(T_t x) - u(x)}{t} d\nu = -\int u'(x)h(x)d\nu$$

where we have used the change of variable formula, the mean value theorem, the boundedness of u' and F_1' and dominated convergence. Now (b) holds iff the left hand side equals $\int u\rho_0 d\nu$ and (c) holds iff the right hand side equals $\int u\beta_h^\nu d\nu$. This proves $(b) \iff (c)$. On the other hand (c) depends only on h and not on the family $\mathcal{T}$. Therefore the same is true for (b). This completes also the proof of the implication $(a) \Longrightarrow$(b). The last statement is a particular case of Proposition 2 (b). ∎

The implications $(b) \Longrightarrow (a)$ and $(c) \Longrightarrow (a)$ do not hold in general even if $\nu_h' \ll \nu$. Moreover, (a) and (a') are not equivalent. The following example illustrates these statements:

Example 1: Let $E = \mathbb{R}^2$ and let ν_0 be the surface measure on the set $G = \{(t, t^2) \ : t \in \mathbb{R}^1\} \subset \mathbb{R}^2$, generated by the usual Lebesgue measure on $\mathbb{R}^2$, and let ν be the Borel measure on $\mathbb{R}^2$, defined by $\nu(A) = \int_{G\cap A} e^{-(x,x)}\nu_0(dx)$. Let h be any smooth vector field on $\mathbb{R}^2$ such that $h(x)$ is a tangent vector to G at x of unit norm, for all $x \in G$. Then ν is τ_C-differentiable along h (and $\beta_h^\nu(x) = -2(x, h(x))$) but ν is not Fomin-differentiable along h. On the other hand for h there is a C^∞-flow $\mathcal{T}$ on $\mathbb{R}^2$ which satisfies the assumptions of the proposition. For this flow (a') holds by the proposition.

Note also that the proposition does not give any sufficient condition for (a) or (a') if the family $\mathcal{T}$ is not a measurable flow. In such cases it is therefore more convenient to work with conditions (b) or (c).

Higher derivatives of a measure are introduced as follows.

Definition 2: *Let $n \geq 2$ and let $h_1, \cdots, h_n$ be a finite sequence of vector fields. We define inductively: ν is n-*times differentiable along $h_1, \cdots, h_n$ *if the measure ν is $n-1$-times differentiable along $h_1, \cdots, h_{n-1}$ and the derivative $\nu^{(n-1)}_{h_1\cdots h_{n-1}}$ is differentiable along h_n, and in this case we define the* derivative of n-th order *by*

$$\nu^{(n)}_{h_1\cdots h_n} = (\nu^{(n-1)}_{h_1\cdots h_{n-1}})'_{h_n}.$$

If this measure is absolutely continuous with respect to ν then the corresponding logarithmic derivative of n-th order *is defined by*

$$\beta^\nu_{h_1\cdots h_n} = \frac{d\nu^{(n)}_{h_1\cdots h_n}}{d\nu}.$$

If $\mathcal{H}$ is any set of vector fields we call ν n-times differentiable along $\mathcal{H}$ if it is n-times differentiable along $h_1, \cdots, h_n$ *for every choice of the $h_1, \cdots, h_n$ in $\mathcal{H}$.*

Note that as with derivatives of functions along vector fields in general $\nu^{(2)}_{h_1 h_2} \neq \nu^{(2)}_{h_2 h_1}$. On a more technical level, note that in the definition of the logarithmic derivative of, say, second order $\beta^{(2)}_{h_1,h_2}$ it is left open whether necessarily the logarithmic derivative of the derivative ν'_{h_1} exists. Actually we do not know whether this implication holds in general.

We conclude this section with a few remarks about the finite dimensional situation. If $E = \mathbb{R}^d$ let C_c^∞ be the space of smooth functions with compact support. Let us first, following an idea in [ASF71], give a quick proof of a version of the socalled 'Malliavin lemma':

Proposition 4 *A measure $\nu \in \mathcal{M}(\mathbb{R}^d)$ is $\tau_{C_c^\infty}$-differentiable in all directions if and only if it satisfies for some constant K, ∞ the estimate*

$$|\int \frac{\partial v}{\partial x_i} \, d\nu| \leq K \|v\|_\infty \tag{6}$$

for all $i \in \{1, \cdots, d\}$ and all $v \in C_c^\infty$. In this case ν has a Lebesgue density.

Proof: Suppose first that ν is $\tau_{C_c^\infty}$-differentiable. Then for all $y \in \mathbb{R}^n$ and $v \in C_c^\infty$

$$\int v'y(x) \; \nu(dx) = - \int v(x) \; \nu'_y(dx) \tag{7}$$

follows by differentiation under the integral sign and clearly (7) implies (6) since $\frac{\partial v}{\partial x_i} = v' e_i$ where e_i is the i-th unit vector. Conversely (6) implies for every $y \in \mathbb{R}^n$ that the left hand side of (7) can be extended to a bounded linear functional on the space C_0 of continuous functions vanishing at infinity and hence by Riesz' representation theorem this functional is induced by a measure $\nu'_y \in \mathcal{M}(\mathbb{R}^d)$. Thus ν is $\tau_{C_c^\infty}$-differentiable along every vector.
Now let us prove the existence of the density, following an idea in [ASF71]. This proof is based on the classical result of Saks that a measure $\nu \in \mathcal{M}(\mathbb{R}^d)$ has a Lebesgue density if (and only if) for every Borel set A the function $y \mapsto \nu(A + y)$ is continuous on $\mathbb{R}^d$. (We need this only for nonnegative measures and in this case a very short argument can be found in [Hew79], p.

278).
For each $i \in \{1, \cdots, d\}$ we apply Proposition 2 to $C = C_c^\infty$ and the flow $\mathcal{T}$ where $T_t x = x - te_i$. Then (3) implies that $|\nu(A + te_i) - \nu(A + se_i)| \leq \|\nu'_{e_i}\||s - t|$ and hence

$$|\nu(A + y) - \nu(A + z)| \leq K\|y - z\|_1$$

for all Borel sets A where $K = \max_{1 \leq i \leq d} \|\nu'_{e_i}$. Since the maps $\mu \mapsto \mu^+$ and $\mu \mapsto \mu^-$ are contractions in the Banach space $(\mathcal{M}(E), \| \cdot \|_1)$ we infer in particular the continuity of $y \mapsto \nu^\pm(A + y)$ and hence the existence of the density by Saks' theorem.
We mention that this continuity can be also deduced directly from (6). ■

The Lebesgue density f of ν is what is called (cf. e.g. [Zie89]) a *function of bounded variation on* $\mathbb{R}^d$ i.e. a function whose partial derivatives in the distributional sense are bounded measures. The density f is *absolutely continuous*, i.e. the distributional partial derivatives of f are Lebesgue integrable functions if the measure ν is even Fomin-differentiable (cf. [ASF71] or apply Proposition 1).
In the case $d = 1$ one has for Lebesgue-a.a. $a \in \mathbb{R}$ the representation

$$f(a) = \nu'((-\infty, a]) \tag{8}$$

In fact for each n let g_n be a smooth probability density with support $[0, \frac{1}{n}]$ and let G_n be the corresponding distribution function. Then we have for $a < b$

$$\begin{aligned}
\nu'((a, b]) &= \lim_n \int G_n(z - a) - G_n(z - b) \; \nu'(dz) \\
&= \lim_n \int g_n(z - b) - g_n(z - a) \; \nu(dz) \\
&= \lim_n \int (g_n(z - b) - g_n(z - a)) f(z) \; dz = f(b) - f(a).
\end{aligned}$$

The last equality holds in $L^1(\lambda)$ but since the limit on the left hand side exists everywhere the equality holds a.e.. Thus both sides of (8) differ only by a constant a.e.. Due to the integrability of f the only possible limit value of f at $-\infty$ is 0 and hence (8) is proved.
In higher dimensions we need higher derivatives for a similar representation. Suppose $\nu \in \mathcal{M}(\mathbb{R}^d)$ is d-times $\tau_{C_c^\infty}$-differentiable along all directions (i.e.

along the space of constant vector fields). Then for any orthonormal system of coordinates and Lebesgue-a.a. vectors $a = (a_1, \cdots, a_d)$ we have

$$f(a) = \nu^{(d)}_{e_1 \cdots e_d}((-\infty, a_1] \times \cdots \times (-\infty, a_d]). \tag{9}$$

The proof is completely analogous to the proof of (8), replacing $g_n(z-b)$ by $\prod_{i=1}^d g_n(z_i - b_i)$ etc.. As a consequence we get a version of the result of [ASF71], Lemma 3.2.5:

Proposition 5: *If $n \geq d$ and $\nu \in \mathcal{M}(\mathbb{R}^{\mathrm{d}})$ is n-times Fomin-differentiable or $n+1$-times $\tau_{C_c^\infty}$-differentiable along (the set of) all constant directions in $\mathbb{R}^{\mathrm{d}}$ then ν has a Lebesgue-density for which all derivatives up to order $n-d$ are continuous and integrable over $\mathbb{R}^{\mathrm{d}}$.*

In fact from Proposition 2(b2) applied to the shifts $T^y_t(x) = x - ty$ with $y \in \mathbb{R}^{\mathrm{d}}$ we conclude inductively that if the measure ν is $n+1$-times $\tau_{C_c^\infty}$-differentiable along all constant directions then it is n-times $\tau_{\|\cdot\|_1}$-differentiable, in particular in the sense of Fomin. Assume $n = d$. Since the measure $\nu_{e_1 \ldots e_d}$ in (9) is a Fomin derivative it is absolutely continuous with respect to ν and hence with respect to Lebesgue measure. Hence the right-hand side in (9) is continuous in a, i.e. ν has a continuous density. In the case $n > d$ the same applies to the partial derivatives of f up to order $n-d$ since we can simply differentiate the identity (9) $n-d$ times to get the result.
It is possible to improve this number $n-d$ using more subtle arguments connected to Sobolev's Lemma, see e.g. the proofs in [Yos78], p. 174 and [Nua95], p. 88.

3 Logarithmic derivative as a negative divergence and the associated Lapalacian

In this section we extend for later use a couple of elementary properties of the stochastic calculus of Wiener measure to the following setting. A simple way to produce non Gaussian examples of this situation is to start with a differentiable measure $/mu$ on $\mathbb{R}$ whose logarithmic derivative is in $L^2(\mu)$. Then the product measure $\nu = \mu^{\mathbb{N}}$ on $E = \mathbb{R}^{\mathbb{N}}$ satisfies the following assumptions with respect to the Hilbert space ℓ^2 (see [SW93] and Example 4 below).

Let H be a Hilbert subspace of E, i.e. H is a vector subspace, equipped with the structure of Hilbert space and such that the canonical embedding $H \to E$ is continuous. We suppose that the measure $\nu \in \mathcal{M}_+(E)$ is (Fomin-) differentiable along every $y \in H$ and that the logarithmic derivatives $\beta^\nu(y,\cdot)$ are even in $L^2(\nu)$ for each $y \in H$. In contrast to the previous section we assume that ν is nonnegative in order to ensure that the bilinear operation $(u,v) \mapsto \int uv\, d\nu$ which appears in the formula of integration by parts defines a Hilbert space.
We introduce the spaces $L^p_H(\nu)$ of all Borel vector fields $h : E \to H$ for which

$$\|h\|_{L^p_H(\nu)} = \Big(\int_E \|h(x)\|^p_H \nu(dx) \Big)^{\frac{1}{p}} < \infty.$$

Let C be the space of all smooth cylindrical functions on E which together with their derivatives are bounded. We define an unbounded linear operator $D_0 : L^2(\nu) \supset C \to L^2_H(\nu)$ by $dom(D_0) = C$ and the equation

$$(D_0 v(x), z) = v'(x)z \ \nu - a.e.. \tag{10}$$

Thus D_0 is the gradient operator with respect to H. Since every $v \in C$ is bounded with a bounded derivative the measurable functions $x \mapsto v'(x)z$ are indeed uniformly bounded if z varies in the unit ball of H and hence $D_0 v$ is in $L^2(\nu)$.

Proposition 6: *The operator D_0 is densely defined and closable.*

Proof: That the space C is dense in $L^2(\nu)$ is clear. Thus we need to prove only the closability. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in the domain C which converges in $L^2(\nu)$ to some 0 and such that the sequence $(D_0 v_n)_{n\in\mathbb{N}}$ converges in $L^2_H(\nu)$ to some w. We have to show $w = 0$. For this consider an arbitrary element u of C and a vector $z \in H$. Then integration by parts gives

$$\begin{aligned} & \int u(x)(w(x),z)\ \nu(dx) \qquad (11) \\ = \ & \lim_{n\to\infty} \int u(x)(D_0 v_n(x),z)\ \nu(dx) = \lim_{n\to\infty} \int u(x) v_n'(x) z\ \nu(dx) \\ = \ & \lim_{n\to\infty} \int ((uv_n)'(x) - v_n(x)u'(x))z\ \nu(dx) \\ = \ & \lim_{n\to\infty} - \int (uv_n)(x)\beta^\nu(z,x) \ + v_n(x)(u'(x),z)\ \nu(dx) \end{aligned}$$

which equals 0 since u and u' are bounded and $\beta(z,\cdot) \in L^2(\nu)$. ■

Definition 3: *We call the closure of the operator D_0 the* extended derivative *and denote it by D.*

Remark 1 (a). This extended derivative can be considered as the non-Gaussian analogue of the Malliavin derivative.
(b). Note that the domain of this operator depends of course on the measure ν. So we could write D_ν instead of D. However if ν and μ are equivalent measures and some function v is in $dom(D_\nu) \cap dom(D_\mu)$ then $D_\nu v = D_\mu v$ a.e. and therefore usually no ambiguity arises if we do not indicate the measure.

The following criterion for a function v to belong to $dom(D)$ sometimes is useful. It follows from the fact that the graph of D is convex and hence also weakly closed in the product $L^2(/nu) \times L^2_H(\nu)$.

Lemma 4: *Let v be the $L^2(\nu)$-limit of a sequence $(v_n)_{n\geq 1}$ in $dom(D)$. If the sequence $(Dv_n)_{n\geq 1}$ is bounded in $L^2_H(\nu)$ then $v \in dom(D)$ and Dv_n converges weakly in $L^2_H(\nu)$ to Dv.*

In the following proposition we give several versions of the chain rule. We do not need part (c) in the sequel.

Proposition 7: *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ and $u = (u_1, \cdots, u_d)$ with $u_i \in dom(D)$ be given.*
(a) If $\varphi \in \mathcal{C}^1(\mathbb{R}^d)$ is bounded with a bunded derivative then $\varphi \circ u \in dom(D)$ and

$$D(\varphi \circ u) = \sum_{i=1}^{d} (\frac{\partial \varphi}{\partial \xi_i} \circ u) Du^i. \tag{12}$$

(b) Let $U \subset \mathbb{R}^d$ and let φ be differentiable at every point of U. Suppose that there is a sequence $(\varphi_k)_{k\geq 1}$ of bounded $\mathcal{C}^1(\mathbb{R}^d)$-functions with bounded derivatives which together with their derivatives converge pointwise on U to φ and $\nabla\varphi$ respectively in such a way that $|\varphi_k(x)| \leq |\varphi(x)|$ and $\|\nabla\varphi_k(x)\| \leq \|\nabla\varphi(x)\|$ for all k and $x \in U$. Moreover let p, q be in the (closed) interval $[2, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.
If $u(x) \in U$ ν-a.e., $\frac{\partial\varphi}{\partial\xi_i} \circ u \in L^p(\nu)$ and $Du^i \in L^q(\nu)$ for $i = 1, \cdots, d$ then

$\varphi \circ u \in dom(D)$ and (12) holds.
(c) Let φ be a Lipschitz function with Lipschitz constant K with respect to some norm $\|\cdot\|$. Then $\varphi \circ u \in dom(D)$ with

$$|D(\varphi \circ u)| \le K\|Du\|. \tag{13}$$

Proof: Part (a) is just the usual chain rule if the components of u are smooth cylindrical functions. If (u_n) is a sequence of smooth cylindrical vector functions such that $u_n \to u$ and $u_n' \to Du$ in each component in $L^2(\nu)$ then the corresponding expressions on the right hand side of (7) converge in measure and by the boundedness of φ' even in $L^2(\nu)$. Moreover for a similar reason $\varphi \circ u_n \to \varphi \circ u$ in $L^2(\nu)$. This implies (12 in this case.
Under the assumption of (b) the φ_k satisfy the chain rule according to (a), i.e.

$$D(\varphi_k \circ u) = \sum_{i=1}^{d} (\frac{\partial \varphi_k}{\partial \xi_i} \circ u) Du^i. \tag{14}$$

Moreover $\varphi_k \circ u \to \varphi \circ u$ in $L^2(\nu)$ and $(\nabla\varphi_k) \circ u \to (\nabla\varphi) \circ u$ in $L^p(\nu)$ by dominated convergence. Thus the left-hand side of (14) converges to the left-hand side of (12) in $L^2(\nu)$ and together with the assumption $Du^i \in L^q(\nu)$ and Hölder's inequality the second cenvergence allows the same passage to the limit on the right-hand side.
2. Part (c) is proved using Lemma 4 in exactly the same way as it is done for Wiener measure e.g. in [Nua95], p.33. The idea is to mollify φ by convolutions.
■

Corollary 1: *(a) (Leibniz rule) Let u, v be two elements of $dom(D)$. Then $D(uv) = uDv + vDu$ under either of the two conditions: (i) The three functions u^2, v^2, uv are also in $dom(D)$. (ii) The integrability conditions $u, v \in L^4(\nu)$ and $Du, Dv \in L^4_H\nu)$ are satisfied.*
(b) Let $w \in dom(D)$ satisfy $\nu\{w = 0\} = 0$. If $Dw \in L^4(\nu)$ and $\frac{1}{w} \in L^8(\nu)$ then $\frac{1}{w} \in dom(D)$ and $D\frac{1}{w} = \frac{-Dw}{w^2}$.

Proof: In (a), case (i) and in (b) we apply Proposition 7 (b) with $p = q = 4$. We define the function φ in the first case by $\varphi(s,t) = st$ on $U = \mathbb{R}^2$ and in the case (b) by $\varphi(t) = 1/t$ on $U = \mathbb{R} - \{0\}$. Our integrability conditions

imply those in the Proposition. The approximation of φ by functions φ_k as required there is straightforward in the first case. In the second case φ_k can be obtained by replacing φ on the interval $[-1/k, 1/k]$ by an anti-symmetric smooth function which vanishes at 0 and which is concave on the right part of this interval.
It remains to prove that the Leibniz rule also holds if we only know that u^2, v^2 and uv are in $dom(D)$. For this let us first consider the case $u = v$. We choose a sequence $(\psi_k)_{k\geq 1}$ of bounded smooth functions on $\mathbb{R}$ such that $0 \leq \psi_k'(x) \uparrow 1$ and $\psi_k(x) \to x$ for all x. We apply the chain rule of Proposition 7 (a) in two ways. If $u^2 \in dom(D)$ then $\psi_k(u^2) \in dom(D)$ and $D\psi_k(u^2) = \psi_k'(u^2)Du^2 \to Du^2$. On the other hand we can write $\psi_k(u^2)$ in the form $(\tilde{\psi}_k(u))^2$ where $\tilde{\psi}_k(z) = \sqrt{\psi_k(z^2)}$. With the usual chain rule we compute the derivatives of the functions $\tilde{\psi}_k$ and get

$$\begin{aligned} D\psi_k(u^2) &= D[(\tilde{\psi}_k(u))^2] = 2\tilde{\psi}_k(u)D\tilde{\psi}_k(u) \\ &= 2\psi_k'(u)uDu \to 2uDu. \end{aligned}$$

Comparing we arrive at $Du^2 = 2uDu$. Now let v be another function such that v^2 and uv are in $dom(D)$ as well. Then $(u+v)^2 \in dom(D)$ and substracting the quadratic terms we arrive at the desired formula. ∎

Remark 2 The previous results are of course well known for Wiener measure. One interesting property of the derivative D which holds in the gaussian case but not in general is the socalled 'locality':

$$If\ Dv = 0\ then\ v = const\ \nu - a.e.. \tag{15}$$

If the measure $\nu \in \mathcal{M}(\mathbb{R}^{\mathrm{d}})$ has a smooth Lebesgue density whose support is compact but not connected then (15) fails. However it is not difficult to verify that for a probability measure $\mu \in \mathcal{M}(\mathbb{R})$ whose logarithmic derivative is in $L^2(\mu)$ and whose density is locally bounded away from 0 the corresponding product measure $\nu = \mu^{\mathbb{N}}$ on $\mathbb{R}^{\mathbb{N}}$ has the property (15). Note that (15) has some interesting consequences like the zero-one laws (cf [IN94])

$$If\ 1_F \in dom(D)\ then\ \nu(F)\nu(F^c) = 0. \tag{16}$$

$$If\ v \in dom(D)\ and\ 1/v \in L^2(\nu)\ then\ either\ v > 0\ \nu - a.e.\ or\ v < 0\ \nu - a.e.. \tag{17}$$

The proofs are again similar to the Gaussian case: For the first note that by Crollary 1 $D(1_F) = 21_F D(1_F)$ which implies $D(1_F) = 0$ and the assertion follows by (15). For the second choose a sequence (φ_k) of smooth functions such that $1_{[1/k,\infty)} \leq \varphi_k \leq 1_{[0,\infty)}$ and $0 \leq \varphi_k' \leq 2k$. Then $|D(\varphi_k \circ v)| = |\varphi_k'(v)Dv| \leq 2k1_{\{0<v<1/k\}}Dv$. Since $1/v \in L^2(\nu)$ we have $\nu\{0 < v < 1/k\} \leq \nu\{1/v > k\} = o(\frac{1}{k^2})$. Together we conclude that $D(\varphi_k \circ v) \to 0$ in $L^2_H(\nu)$. Therefore $1_{\{v>0\}} = \lim \varphi_k \circ v \in dom(D)$ and (16) gives (17).

Moreover the rule of integration by parts can be extended from smooth test functions to elements of $dom(D)$.

Lemma 5: *(a) If ν is differentiable along $h \in L^2_H(\nu)$ with logarithmic derivative $\beta_h^\nu \in L^2(\nu)$ then $\int w\beta_h^\nu \, d\nu = -\int (Dw, h)_H \, d\nu$ for every $w \in dom(D)$.*
(b) Let ν be m-times differentiable along the vector fields $h_1, \cdots, h_m$ in $L^2_H(\nu)$. Assume the existence of the two highest logarithmic derivatives $\beta^{(m-1)}_{h_1\cdots h_{m-1}} \in L^p(\nu)$ and $\beta^{(m)}_{h_1\cdots h_m} \in L^2(\nu)$. If $2 \leq p, q \leq \infty$ satisfy $\frac{1}{p}+\frac{1}{q} = \frac{1}{2}$ and $h_m \in L^q_H(\nu)$ then for every $w \in dom(D)$ one has

$$\int (Dw, h_m)\beta^{(m-1)}_{h_1\cdots h_{m-1}} \, d\nu = -\int w\beta^{(m)}_{h_1\cdots h_m} \, d\nu. \tag{18}$$

Proof: Part (a) is the special case of (b) where $m = 1, p = 2, q = \infty$ and $\beta^{(0)} = 1$. Thus it suffices to prove (b). Let (w_n) be a sequence of smooth functions such that $w_n \to w$ in $L^2(\nu)$ and $Dw_n \to Dw$ in $L^2_H(\nu)$. Then on the right-hand side of (18) one can pass from w_n to w. Similarly, on the left-hand side we can pass to the limit using the fact that by Hölder's inequality the product $\|h_m\|\beta^{(m-1)}_{h_1,\cdots,h_{m-1}}$ is in $L^2(\nu)$. ■

This integration by parts formula has a canonical functional analytic interpretation. Proposition 6 implies that the associated adjoint operator from $L^2_H(\nu)$ into $L^2(\nu)$ exists and is densely defined. For reasons connected to the theory of differential forms and in analogy to the tradition in the context of Wiener measure we denote this adjoint operator by δ_ν rather than D^*. However in contrast to D the values of this operator depend strongly on the measure and therefore we keep the index ν.

Definition 6 *The symbol δ_ν denotes the closed operator whose domain $dom(\delta_\nu)$ consists of all elements h of $L^2_H(\nu)$ for which there is some constant K such that for every $v \in C$ we have*

$$|\int v'(x)h(x)\ \nu(dx)| \leq K \parallel v \parallel_{L^2(\nu)}, \tag{19}$$

and such that then $\delta_\nu h$ is the unique element of $L^2(\nu)$ satisfying

$$\int v'(x)h(x)\ \nu(dx) = \int v(x)\delta_\nu h(x)\ \nu(dx) \tag{20}$$

for all $v \in C$ and hence for all $v \in dom(D)$

$$\int (Dv(x), h(x))\ \nu(dx) = \int v(x)\delta_\nu h(x)\ \nu(dx). \tag{21}$$

Comparing (20) with the relation (5) we see that

$$\delta_\nu h = -\beta^\nu_h. \tag{22}$$

Thus the set of square integrable vector fields along which the measure ν is differentiable with a square integrable logarithmic derivative coincides with the domain of δ_ν and on this domain the equation (22) holds.

Remark 3 In the context of Wiener measure it was discovered by Gaveau and Trauber [GT82] that the Skorokhod integral is just the adjoint of the Malliavin derivative. On the other hand among others Daletskii [DM85] noted that the logarithmic derivative of a Gaussian measure along a vector field coincides with the negative of the stochastic (Ito- or even Skorokhod-) integral. The objects on both sides of (22) thus can be considered as non-Gaussian versions of the Skorokhod integral. However for fixed h as a function of the elements x of the underlying 'path space' E the random variable $\delta_\nu h$ is not ν-a.s. additive as in the Gaussian case. Thus on the first glance it does not seem have the typical bilinear structure of an integral. Nevertheless we could write symbolically $\delta_\nu h = \int h d\beta_H$ to indicate that $\delta_\nu h$ can be considered as a kind of stochastic integral of the vector field h with respect to the integrating process $(\beta(y, \cdot) : y \in H)$ on the measure space $(E, \mathcal{B}(E), \nu)$. The reader will easily verify that this is accordance to the notation for the Wiener-Itô integral.

Remark 4 If in the above construction the space C of test functions is replaced by another space C_1 of bounded smooth functions then the resulting operators D and δ_ν will not change as long as the two spaces C and C_1 have the same completion with respect to the 'graph-norm' $\|\cdot\|_{1,2}$, i.e. if the resulting domains $dom(D)$ coincide. For the proof let $v \in C_1$ be given. For the vector field $w = v'$ the first and the last member of relation (11) are equal. The same holds for the vector field $w = Dv$ and hence $Dv = v'$ for all $v \in C_1$. Thus D is also the closed extension of the restriction of the gradient to C_1. Therefore the corresponding adjoint operators coincide and the relation (22) implies that if ν is C-differentiable along a vector field $h \in L^2_H(\nu)$ with a square integrable logarithmic derivative then ν is also C_1-differentiable along h.

Remark 5 We have considered the operator δ_ν on the space $L^2_H(\nu)$ but similarly one could change the corresponding Banach spaces. For example one could consider the closure D_1^* in $L^1_H(\nu) \times L^1(\nu)$ of D^*. We have chosen the Hilbert space setting mainly for simplicity.

Sometimes one wants to change a vector field along which ν is differentiable by a scalar function. For this and many other purposes it is interesting to study $\delta_\nu h$ in the particular case where the vector field h is of 'gradient type' i.e. $h = Du$ for some $u \in dom(D)$. This leads to the following definition.

Definition 7: *We call* Laplace operator associated to ν *the operator composition* $\Delta_\nu = -\delta_\nu D$ *whose domain* $dom(\Delta_\nu)$ *consists of all* $u \in dom(D)$ *for which* $Du \in dom(\delta_\nu)$.

Since in the classical space $L^2(\mathbb{R}^d)$ the negative adjoint of the gradient is the divergence operator one can consider Δ_ν indeed as the natural analogue of the Laplace operator.
In the case of Wiener measure ν this operator is called *number* or *Ornstein-Uhlenbeck* operator (see e.g. [Nua95], p. 54 f.). In this case it is closely linked to the Wiener Chaos decomposition of $L^2(\nu)$. It is the generator of the semi-group describing the infinite-dimensional Ornstein-Uhlenbeck process. Since the use of this operator for Wiener measure is so intimately linked to these two interpretations it is somewhat surprising that for our purposes we do not need reference to any underlying stochastic process or any other additional structure of the space $L^2(\nu)$. Of course such a probabilistic interpretation

would give interesting additional insights.

The following observation describes the role of Δ_ν for the scalar modification of vector fields of differentiability and for the differentiability of measures which are absolutely continuous with respect to ν. Note that we cannot expect in general that the new logarithmic derivative is again square integrable.

Proposition 8: *Let $v \in dom(D)$ and $u \in dom(\Delta_\nu)$. Then the measure ν is differentiable along the vector field vDu with logarithmic derivative*

$$\beta^\nu_{vDu} = v\Delta_\nu u + (Dv, Du). \tag{23}$$

Moreover the measure $v\nu$ which has the Radon-Nikodym density v with respect to ν is also differentiable along the vector field Du with logarithmic derivative

$$\beta^{v\nu}_D u = \Delta_\nu u + \frac{(Du, Dv)}{v} 1_{\{|v|>0\}}. \tag{24}$$

Proof: For every smooth test function w we have by definition of δ_ν as the adjoint operator of D

$$\begin{aligned} -\int w'(vDu)\ d\nu &= -\int (vw', Du)_H\ d\nu = \int (wDv - D(vw), Du)_H\ d\nu \\ &= \int w(Dv, Du)_H\ d\nu - \int vw\ \delta_\nu Du\ d\nu \\ &= \int w(vL_\nu u + (Dv, Du)_H)\ d\nu. \end{aligned}$$

Since $v, \Delta_\nu u$ are in $L^2(\nu)$ and Dv, Du are in $L^2_H(\nu)$ the function $\beta = v\Delta_\nu u + (Dv, Du)$ is in $L^1(\nu)$. Thus the measure whose ν-density is this function is equal to ν'_{vDu}. This proves the first assertion.
Similarly this measure has the $v\nu$-density $\frac{\beta}{v} 1_{\{|v|>0\}}$ and the first integral in this calculation can also be read as $\int w'Du\ d(v\nu)$ and thus this new measure is also the derivative of $v\nu$ along the vector field Du. ■

Remark 6 If the function v is strictly positive ν-a.e.one can rewrite (24) on a symbolic level as

$$L_{v\nu} u = \Delta_\nu u + \frac{(Dv, Du)}{v}. \tag{25}$$

However the domain of these operators are quite different. Nevertheless (25) holds if u belongs to the common domain of the two operators.

A second useful fact is that like in $L^2(\mathbb{R}^d)$ the canonical bilinear operation which is defined by D can be also expressed in terms of the operator Δ_ν.

Proposition 9: *If u, v, u^2, v^2 and uv are in dom(Δ_ν) then we have the following identity of elements of $L^1(\nu)$*

$$2(Du, Dv)_H = \Delta_\nu(uv) - v\Delta_\nu(u) - u\Delta_\nu(v). \tag{26}$$

Proof: Let again w be a smooth scalar test function. Then we get

$$\begin{aligned} & \int w(\Delta_\nu(vu) - v\Delta_\nu(u) - u\Delta_\nu(v))\, d\nu \\ = \quad & -\int w\ \delta_\nu D(vu) - wv\ \delta_\nu Du - wu\ \delta_\nu Dv\ d\nu \\ = \quad & -\int DwD(vu) - (D(wv)Du + D(wu)Dv)\ d\nu \\ = \quad & \int 2wDvDu\ d\nu \end{aligned}$$

since $DwD(vu) = Dw(vDu + uDv) = D(wv)Du + D(wu)Dv - 2wDvDu$. ■

4 Images of non-Gaussian differentiable measures

Smoothness results for images of non-Gaussian measures under *smooth* maps have been obtained among others by Uglanov [Ugl81] and Daletskii and Steblow-skaya [DF92]. Here we give similar results, but also for non-smooth maps, in particular we extend the Malliavin-Stroock theorem [Str81] to non-Gaussian measures.
The main idea of the results of this section can already be seen in the following simple result for the one-dimensional case. In the Gaussian case this presumably is due to Bismut [Bis81]. See also Nualart [Nua95], p. 78 .

Proposition 10: *Let $u : E \to \mathbb{R}$ be a function in dom(D). Consider any vector field $h : E \to H$ such that $(Du(x), h(x)) = 1$ a.e., e.g. $h = \frac{Du}{||Du||_H^2}$. If ν is differentiable along h with logarithmic derivative β_h^ν then $\mu = \nu \circ u^{-1}$*

is differentiable with logarithmic derivative b where $b \circ u = E(\beta_h^\nu | u)$ and the Lebesgue density of μ is given by

$$f(z) = \int_{\{u<z\}} \beta_h^\nu \, d\nu. \tag{27}$$

Proof: Let $v : \mathbb{R} \to \mathbb{R}$ be a smooth one-dimensional test function. Let $b : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $b \circ u = E(\beta_h^\nu | u)$. Then

$$\begin{aligned} & \int_{\mathbb{R}} v(\xi) b(\xi) \, d\mu = \int (v \circ u) E(\beta_h^\nu | u) \, d\nu = \int (v \circ u) \beta_h^\nu \, d\nu \\ = \quad & - \int D(v \circ u) h \, d\nu = - \int (v' \circ u)(Du, h) \, d\nu \\ = \quad & - \int v' \circ u \, d\nu = - \int v' \, d\mu. \end{aligned}$$

This shows that the function $b \in L^1(\mu)$ is the logarithmic derivative of the measure μ. Moreover according to (8) the Lebesgue density of μ is given by

$$\begin{aligned} f(a) &= \mu'((-\infty, a)) = \int_{-\infty}^{a} b(z) \, d\mu \\ &= \int_{\{u<x\}} \beta_h^\nu \, d\nu. \end{aligned} \tag{28}$$

■

Remark 7 Of course one can go at this point into the theory of surface measures. They have been constructed for smooth measures and functions by Uglanov [Ugl81] and in the framework of infinite dimensional Sobolev spaces for Wiener measure by Airault and Malliavin [AM96]. Such a surface measure ν_s will satisfy for all sufficiently smooth vector fields h the Stokes-Green formula

$$\int_{\{u=a\}} (h, n) \, d\nu_s = \int_{\{u<x\}} \beta_h \, d\nu \tag{29}$$

where n is the normalised normal vector field of the surface. This formula has been given (without proof) in [Smo86]. In our case $n = \frac{Du}{\|Du\|}$ and hence choosing $h = \frac{Du}{\|Du\|^2}$ as in the proposition and applying (27) we have the relation

$$f(a) = \int_{\{u=a\}} \frac{\nu_s(dx)}{\|Du\|}.$$

Integrating over a we get

$$\nu(E) = \mu(\mathbb{R}) = \int_{-\infty}^{\infty} \mathrm{f(a)}\ \mathrm{da} = \int_{-\infty}^{\infty} \int_{\{\mathrm{u=a}\}} \frac{\nu_{\mathrm{s}}(\mathrm{dx})}{\|\mathrm{Du}\|}\ \mathrm{da}.$$

Applying this formula to the measures $g\nu$ instead of ν where g runs through the set of smooth cylindrical functions and using a monotone class argument we get for every Borel subset A of E the formula

$$\nu(A) = \int_{-\infty}^{\infty} \int_{\{u=a\}\cap A} \frac{\nu_s(dx)}{\|Du\|}\ da. \tag{30}$$

We would like to mention without going into details that by an induction argument for codimension larger than 1 the coarea formula

$$\nu(A) = \int_{\mathbf{R}^{\mathrm{d}}} \int_{\{u=a\}\cap A} \frac{\nu_s(dx)}{|\det Du(Du)^*|^{\frac{1}{2}}}\ da \tag{31}$$

can be connected in a similar way to the Stokes formula of finite codimension, provided the latter is also available on the manifolds given by the level sets of u.

We now extend the idea of Proposition 10 to infinite dimensional image spaces. The following theorem is an extension of a result in Daletskii-Steblowskaya [DF92]. The main point is to map a vector field to the image space via the differential of the underlying function. But since the function u is typically not injective we take an average of these images with respect to the conditional law on the fibers of u. In other words we use conditional expectations also for the definition of the vector field in the image space.

Theorem 2: *Let F be a LCS and let $u : E \to F$ be a Borel function for which there is a map $D_F u : E \to L(H, F)$ which satisfies the chain rule*

$$(D(\zeta \circ u)(x), y)_H = \zeta(D_F u(x)y) \quad \nu - a.e. \tag{32}$$

for every $y \in H$ and every ζ in the dual space F'. Let $g : F \to G$ and $h : E \to H$ be two vector fields such that

$$g \circ u = E(D_F u\ h|u) \quad \nu - a.e. \tag{33}$$

where $(D_F u\ h) : E \to F$ is defined by $(D_F u\ h)(x) = D_F u(x)h(x)$ and the vector valued conditional expectation is understood via composition with *elements of the dual space F'. If ν is differentiable along h then the image measure $\mu = \nu \circ u^{-1}$ is differentiable along g and the corresponding logarithmic derivative β_g^μ is given by the relation*

$$\beta_g^\mu \circ u = E(\beta_h^\nu | u).$$

Proof: First let us extend the chain rule (32). It implies $v \circ u \in dom(D)$ and

$$(D(v \circ u)(x), y)_H = v'(u(x))(Du(x)y) \quad \nu - a.e. \tag{34}$$

whenever $v : F \to \mathbb{R}$ is a smooth bounded cylindrical function with bounded derivative. In fact we can write

$$v = \varphi(\zeta_1, \cdots, \zeta_d) \tag{35}$$

and then (34) follows from Proposition 7 (a). From (34) one gets the chain rule for differentiation along vector fields, i.e.

$$(D(v \circ u)(x), h(x))_H = v'(u(x))(Du(x)h(x)) \tag{36}$$

ν-a.e. for every measurable vector field $h : E \to H$. In fact by straightforward approximation one can reduce the proof to the case where h takes its values in a finite set.
Now we use the representation (35) once more, writing $\zeta = (\zeta_1, \cdots, \zeta_d)$. By definition of the vector valued conditional expectation in (33) we can apply (36) as follows:

$$\begin{aligned}
& -\int v(z)\ d(\nu_h' \circ u^{-1}) = -\int v \circ u\ d\nu_h' \\
= & \int (D(v \circ u)(x), h(x))_H\ d\nu = \int v'(u(x))\zeta(Du(x)h(x)))\ d\nu \\
= & \int \sum_{i=1}^d \partial_i \varphi(\zeta(u(x)))\zeta_i(Du(x)h(x)))\ d\nu \\
= & \int \sum_{i=1}^d \partial_i \varphi(\zeta(z))\zeta_i(g(z))\ d\mu = \int v'(z)g(z)\ d\mu.
\end{aligned}$$

This shows that the measure μ is differentiable along the vector field g with derivative $\mu_g' = \nu_h' \circ u^{-1}$. The statement about the logarithmic derivative follows from

$$\int v \circ u\ d\nu_h' = -\int v \circ u\ \beta_h^\nu\ d\nu = -\int (v \circ u) E(\beta_{h_g}^\nu | u)\ d\nu.$$

■

Remark 8 Let u be as in Theorem 2. Let G be a Hilbert subspace of F such that for ν-a.e. $x \in E$ the operator $D_F u(x)$ maps H continuously onto G. Then the adjoint operator $D_F u(x)^* : G \to [ker D_F u(x)]^{\perp}$ is an linear isomorphism. In this case for every vector field $g : F \to G$ one can find a particular vector field h_g which satisfies (33) by setting

$$h_g(x) = D_F u(x)^* \Big(D_F u(x) \circ D_F u(x)^* \Big)^{-1} g(u(x)). \tag{37}$$

Let us study the particular case $F = \mathbb{R}^{\mathrm{d}}$. In the remaining part of this section we consider a map $u = (u_1, \cdots, u_d) : E \to \mathbb{R}^{\mathrm{d}}$ such that $u_i \in dom(\Delta_\nu)$ for each $i \in \{1, \cdots, d\}$. Define the matrix $\sigma(x)$ by

$$\sigma(x) = \Big((Du_i(x), Du_j(x))_H \Big). \tag{38}$$

From Theorem 5.2.2 in [BH91] it follows that the image of ν under u has a Lebesgue density if $\sigma(x)$ is a.e. invertible and if in addition to our assumptions the measure ν is even quasiinvariant in all directions e_i where (e_i) is an orthonormal base of the Hilbert space H. Simple examples show that differentiability in a certain direction does not imply quasiinvariance, but we believe that the quasiinvariance assumption of Bouleau and Hirsch can be replaced by our assumptions in their result. However, very little can be said about the regularity of the density of the image measure. To prove smoothness one really seems to need the integration by parts technique. In the following approach we start from stronger assumption and get a stronger result.

We first remark that the previous result extends to higher derivatives.

Theorem 3: *Let u be as in Theorem 2. Let $g_1, \cdots, g_n$ and $h_1, \cdots, h_n \in L^4_H(\nu)$ be vector fields in F resp. E such that $E(Du\ h_i|u) = g_i \circ u$ ν-a.e.. Suppose that ν is n-times differentiable along $h_1, \cdots, h_n$ such that all higher logarithmic derivatives $\beta^{(i)}_{h_1 \cdots h_i}$ $(1 \le i \le n)$ exist and are in $L^4(\nu)$. Then $\mu = \nu \circ u^{-1}$ is n-times differentiable along $g_1, \cdots, g_n$ and the corresponding logarithmic derivatives of n-th order are related to each other by*

$$\beta^{\mu}_{g_1 \cdots g_n} \circ u = E(\beta^{\nu}_{h_1 \cdots h_n} | u).$$

Proof: The proof proceeds by induction using a similar calculation as before. Here are the main steps: If v is a smooth test function on F then by induction hypothesis and (36)

$$
\begin{aligned}
& \int v'(z)(g_n(z))\ \mu^{(n-1)}_{g_1\cdots g_{n-1}}(dz) \\
= & \int v'(u(x))(g_n(u(x)))\ \beta^{\mu}_{g_1\cdots g_{n-1}}(u(x))\ \nu(dx) \\
= & \int (D(v\circ u)(x), h_n(x))\ \beta^{\nu}_{h_1\cdots h_{n-1}}(x)\ \nu(dx) \\
= & -\int v\circ u(x)\ \beta^{\nu}_{h_1\cdots h_n}(x)\ \nu(dx).
\end{aligned}
$$

In the last equation the integration by parts is possible due to Lemma 5. As before it is now sufficient to take conditional expectations. ■

The following lemma gives under strong assumptions an elegant way to deal with higher derivatives.

Lemma 8: *Let $\mathcal{R}$ be a linear subspace of dom(Δ_ν) which contains the constant 1 and is closed both under the operator Δ_ν and under multiplication.*
(a) Let $\mathcal{M}(\nu,\mathcal{R})$ be the space of signed measures which are absolutely continuous with respect to ν with a Radon-Nikodym density in $\mathcal{R}$. Then the set $\mathcal{M}(\nu,\mathcal{R})$ is closed under differentiation along the linear hull $\mathcal{H}_{\mathcal{R}}$ of the set of vector fields $\{rDv : r,v\in\mathcal{R}\}$. In particular, the measure ν is infinitely often differentiable along $\mathcal{H}_{\mathcal{R}}$.
(b) Let $\tilde{\mathcal{R}}$ be the space of functions s/w with $s,w\in\mathcal{R}$ and $\frac{1}{w}\in\bigcap L^p(\nu)$. Then $\tilde{\mathcal{R}}$ is also a linear space closed under multiplication and under the operator Δ_ν.

Proof: (a) The relation $\|Dv\|^2=(Dv,Dv)=vL(v)-\frac{1}{2}L(v^2)$ follows from (26) in Proposition 9. It implies that

$$Dv\in\bigcap_{p\geq 1} L^p_H(\nu) \tag{39}$$

for all $v\in\mathcal{R}$. Similarly (26) implies that $(Dr,Dv)\in\mathcal{R}$ for all $r,v\in\mathcal{R}$. Consider the vector field $h=rDv$ with $r,v\in\mathcal{R}$. Proposition 8 then shows that ν is differentiable along h with $\beta^\nu_h=r\Delta v+(Dr,Dv)\in\mathcal{R}$. Consider now a measure $\lambda \in \mathcal{M}(\nu, \mathcal{R})$ with $\frac{d\lambda}{d\nu} = r_\lambda \in \mathcal{R}$. We want to differentiate λ along the same vector field $h = rDv$. Consider the new vector field $h^* = r_\lambda h \in \mathcal{H}_{\mathcal{R}}$. Let s be a smooth test function on E. We get

$$\begin{aligned}\int (t', h)\ d\lambda &= \int (t', h) r_\lambda\ d\nu = \int (t', h^*)\ d\nu \\ &= -\int t\beta^\nu_{h^*}\ d\nu \le \|t\|_\infty \|\beta^\nu_{h^*}\|_{1,\nu}.\end{aligned}$$

This show that the measure λ is differentiable along h and the measure λ'_h has the ν-density $\beta^\nu_{h^*}$ which again is in $\mathcal{R}$. So $\lambda'_h \in \mathcal{M}(\nu, \tilde{\mathcal{R}})$. Since the differentiation of a measure along a vector field is linear as a function of the vector field we can pass to the linear hull $\mathcal{H}_{\mathcal{R}}$. This proves (a).
(b) The space $\tilde{\mathcal{R}}$ is a linear space which is closed under multiplication since $\mathcal{R}$ has these properties. The space $\mathcal{R}$ is contained in $\bigcap_{p\ge 1} L^p(\nu)$ since all integer powers of all of its elements are in the domain of D and hence square integrable. The same argument then applies to $\tilde{\mathcal{R}}$. That $\mathcal{H}_{\tilde{\mathcal{R}}}$ is contained in $\bigcap_{p\ge 1} L^p_H(\nu)$ follows from (39).
From Corollary 1 we conclude that a function $r/w \in \tilde{\mathcal{R}}$ is in $dom(D)$ with

$$D(r/w) = (wDr - rDw)\frac{1}{w^2} \in \mathcal{H}_{\tilde{\mathcal{R}}}. \tag{40}$$

Let now $h = (r/w)Dv$ with $r/w \in \tilde{\mathcal{R}}$ be given. By Proposition 8 the measure ν is differentiable along h with logarithmic derivative

$$\begin{aligned}\beta^\nu_h &= \frac{r}{w}\Delta_\nu v - (D\frac{r}{w}, Dv) \\ &= \frac{r\Delta_\nu v}{w} - \frac{(Dr, Dv)}{w} + \frac{r(Dw, Dv)}{w^2}.\end{aligned} \tag{41}$$

Since all three denumerators are in $\mathcal{R}$ and $\tilde{\mathcal{R}}$ is a linear space this logarithmic derivative is again in $\tilde{\mathcal{R}}$ and in particular square integrable. Thus $h \in dom(\delta_\nu)$ and hence $\mathcal{H}_{\tilde{\mathcal{R}}} \subset dom(\delta_\nu)$ and $\delta_\nu(\mathcal{H}_{\tilde{\mathcal{R}}}) \subset \tilde{\mathcal{R}}$. Together with (40) this shows that $\tilde{\mathcal{R}} \subset dom(\Delta_\nu)$ and $\Delta_\nu(\tilde{\mathcal{R}}) \subset \tilde{\mathcal{R}}$, i.e. $\tilde{\mathcal{R}}$ is closed under the operator Δ_ν.
This completes the proof of the lemma and of the theorem. ∎

The first part of the following result extends the corresponding theorem of Stroock [Str81] to non-Gaussian measures. The statement about smooth versions of the conditional expectations shows once more the advantage of considering several measures at the same time.

Theorem 4: *Let $\mathcal{R}$ be a subset of dom(L) which is closed under the operator Δ_ν and under pointwise multiplication. Let $u : E \to \mathbb{R}^d$ be a map whose components belong to $\mathcal{R}$ and such that the matrix $\sigma(x)$ defined in (38) is ν-a.e. invertible. If σ satisfies*

$$\frac{1}{\det(\sigma)} \in \bigcap_{p\geq 1} L^p(\nu) \tag{42}$$

then $\mu = \nu \circ u^{-1}$ has a Lebesgue density $f \in C^\infty(\mathbb{R}^d)$.
Moreover for every function $\psi = s/w$ with $s, w \in \mathcal{R}$ and $\frac{1}{w} \in \bigcap_{p\geq 1} L^p(\nu)$ there is a function φ on $\mathbb{R}^d$ which is C^∞ on the open set $\{f > 0\}$ such that

$$E(\psi|u) = \varphi \circ u. \tag{43}$$

Proof: Without loss of generality we may assume that $\mathcal{R}$ is a linear space and that $\mathcal{R}$ contains the constant function 1 because the common linear hull of the constants and $\mathcal{R}$ also satisfies the assumptions of the theorem. As was seen in the proof of the lemma the entries of the matrix σ are in $\mathcal{R}$. Expanding the determinant we see that $\det\sigma$ is a linear combination of products of elements of $\mathcal{R}$ and hence $\det\sigma \in \mathcal{R}$. Now by Cramer's rule the entries of the inverse matrix $\rho = \sigma^{-1}$ can be computed as $\rho_{ij} = \frac{\hat{\sigma}_{ij}}{\det\sigma}$ where the entries of the cofactor matrix $\hat{\sigma}$ are again in $\mathcal{R}$ for a similar reason. Together with our assumption (42) we see that the vector fields $h_i = \sum_{j=1}^d \rho_{ij} Du_j$ are in the space $\mathcal{H}_{\tilde{\mathcal{R}}}$ where $\tilde{\mathcal{R}}$ and $\mathcal{H}_{\tilde{\mathcal{R}}}$ are defined in Lemma 8. Then applying part (a) of the Lemma to $\tilde{\mathcal{R}}$ which is legitimate by part (b) of the Lemma, we see that ν is infinitely often differentiable along the vector fields h_i. Using the remark 8 it is easily verified that $Du(x)h_i(x) = e_i$ for each i. So by Theorem 3 the image measure ν is infinitely often differentiable along the constant directions and thus it has a C^∞ density f by Proposition 5.
In order to prove the factorization of the conditional expecations we show that even for every $\psi \in \tilde{\mathcal{R}}$ the image measure $(\psi\nu) \circ u^{-1}$ is infinitely often differentiable in the constant directions. Then by Proposition 5 also this image measure has a C^∞-density g and then the function $\varphi = \frac{g}{f} 1_{\{f>0\}}$ satisfies (43) and it is smooth on $\{f > 0\}$.
By Lemma 8 the measure ν is differentiable along the vector fields ψh_i and the corresponding logarithmic derivatives are again in $\tilde{\mathcal{R}}$. Thus for every test function $t \in C_c^\infty(\mathbb{R}^d)$ we have according to the chain rule (36)

$$\int_{\mathbb{R}^d} \partial_i t(z)\ d(\psi\nu) \circ u^{-1} = \int_E (D(t \circ u)(x), h_i(x))\psi(x)\ \nu(dx)$$

$$= \; -\int_E t \circ u \beta^{\nu}_{\psi h_i} \, d\nu = \int_{\mathrm{IR}^{\mathrm{d}}} t \; d(\beta^{\nu}_{\psi h_i} \nu) \circ u^{-1}.$$

But this means precisely that the set of images of the measures in $\mathcal{M}(\nu, \tilde{\mathcal{R}})$ under the map u is closed under differentiation along the constant vectors in IR^{d}, in particular it consists of infinitely differentiable measures, concluding the proof. ∎

Remark 9 It is natural to ask whether one can prove differentiability of finite order, say $f \in C^k(\mathrm{IR}^{\mathrm{d}})$, under weaker assumptions. A closer look at the above proof shows that this is indeed the case. The cofactors of σ are linear combinations of terms of $d-1$ factors of the type $r\Delta v$ with $r, v \in \mathcal{R}$. So if $\mathcal{R}$ is not necessarily closed under multiplication but all products of elements of $\mathcal{R}$ and $\Delta_\nu(\mathcal{R})$ with a sufficiently high bound N on the number of factors are still contained in $dom(\Delta_\nu)$ and if moreover $\det(\sigma)^{-1} \in L^q(\nu)$ for sufficently high $q > N$ one gets arbitrarily high moments of the H-norm of the vector fields h_i. Similarly using the explicit representation in (41) one can enforce arbitrarily high moments of all higher logarithmic derivatives along these vector fields of a given order. In this way one gets differentiability of ν of a given order n in the direction of the h_i and hence by Theorem 3 and Proposition 5 the differentiability of the density f of order $n - d$.

5 Remarks on the Laplacian in the context of canonical quantization

In this section we briefly indicate some of the ideas connecting the mathematical objects in the previous sections to some concepts from physics.
Let us recall the following: If ν is a (infinite dimensional) Gaussian measure the operator Δ_ν in the space $L_2(\nu)$) can be considered as the Hamiltonian of an (infinite-dimensional) harmonic oscillator; in the context of this interpretation ν describes the ground state of the oscillator, i.e. the state of minimal energy. Such an oscillator describes the 'free quantum field'; the operator Δ_ν has a discrete positive spectrum. The order number of an eigenvalue is interpreted as the number of particles in all states which correspond to elements of the associated eigenspace, i.e. the operator Δ_ν is the 'number operator'. Wiener chaos decomposition is just the decomposition of $L_2(\nu)$ into the Hilbert sum of the eigenspaces of Δ_ν. This sum is isomorphic to the Hilbert sum

of the spaces $\{\hat{\bigotimes}_{j=1}^{n} H_j\}$ $(n = 1, 2, \ldots\})$ where the n-th space is a symmetrized Hilbert tensor product of copies of H, and is interpreted as the space of n-particle states. This is actually a main point of the particle-wave dualism for quantum mechanical fields.

Now in the case of a nongaussian measure ν the Laplace operator Δ_ν can be considered as the Hamiltonian in the corresponding space $L_2(\nu)$ of an unharmonic oscillator which describes a field with selfinteraction (the measure ν depends on the potential of the interaction). So the theory of the Laplacian Δ_ν is related to the so called nonperturbative quantum field theory.

These ideas can be explained in more detail on a 'formal' (i.e. informal, omitting precise analytical assumptions) level as follows (see [SW96] [1]).

First we discuss the notion of generalized density of a differentiable measure (cf. [SW95], for a different approach see [Kir94]). Let E be a LCS, let H be a (dense) Hilbert subspace of E and let $\nu \in \mathcal{M}_+(E)$ be (Fomin-) differentiable along every $h \in H$. Actually the assumption of the following Proposition typically will be satisfied only if the Hilbert space H is strictly smaller than the 'space of differentiability', i.e. the space of directions along which the measure is differentiable. In the Gaussian case it is possible to take for H the image of the correlation operator of ν. The result follows from the Frobenius theorem.

Proposition 11: *If the mapping $\beta^\nu : H \times H \longrightarrow R^1, (h, x) \longmapsto \beta^\nu(h, x)$ is continuously differentiable then there exists a function $\sigma_\nu : H \longrightarrow R^1$ for which $\sigma_\nu^{'}(x)h = \beta^\nu(h, x)$ for each $h, x \in H$.*

The function $F_\nu : H \ni x \longmapsto \exp \sigma_\nu(x)$ is called a generalized density of ν and σ_ν is called a logarithmic density of ν. Hence the generalized density of ν is a function on H whose (usual) logarithmic derivative $(\ln F_\nu)'(= \sigma_\nu') : H \longrightarrow H'$ coincides with the function

$$H \ni x \longmapsto [H \ni h \longmapsto \beta^\nu(h, x)]$$

[1] In [SW96] unfortunately there are some of misprints. The corrected version can be gathered from the following formulas.

Roughly speaking, a generalized density of ν is a function on H whose logarithmic derivative coincides with the logarithmic derivative of ν.

Remark 10 The function σ_ν is defined on H, but if $\dim H = \infty$ (and as usual ν is σ-additive) then σ_ν cannot be extended in natural way to the whole space E. Nevertheless, in some interesting cases the function

$$\Lambda_\nu : H \times H \longrightarrow \mathbb{R}, (\mathrm{h}, \mathrm{x}) \longmapsto \sigma_\nu(\mathrm{x}+\mathrm{h}) - \sigma_\nu(\mathrm{x})$$

can be extended to the space $H \times E$ by continuity. If such an extension exists we may call it logarithmic quasiinvariant density. This terminology is justified by the following statement which was proved in [SW95].

Proposition 12: *If Λ_ν is continuous on $H \times E$ and if Φ is a map from $\mathbb{R} \times E$ into E such that $\Phi(0,x) = x$ (satisfying some regularity conditions) then*

$$\Lambda_\nu(\Phi(1,x) - x, x) = \int_0^1 \beta^\nu(\Phi_1'(\tau,x), \Phi(\tau,x))d\tau.$$

Here Φ_1' denotes the derivative with respect to thefirst variable. Moreover the image measure of ν under the map $\Phi(1,\cdot)$ can be written via the Girsanov type fromula

$$\nu(\Phi(1,\cdot))^{-1}(dx) = e^{\Lambda_\nu(\Phi(1,x)-x,x)\nu(dx)}.$$

Example 2 *If* $\dim E < \infty$, $E = H$ *and* f *is the (usual) density of* ν, $f(x) > 0$ *for all* x, *then one can pose* $\sigma_\nu = \ln f$ *and* $F_\nu = f$. *In this case* $e^{\Lambda_\nu(\Phi(1,x)-x,x)} = \frac{f(\Phi(1,x))}{f(x)}$.

Example 3 *If ν is a centered Gaussian measure on E with correlation operator $B : E' \longrightarrow E$ and if $ImB \supset H$, then for all $x \in H$ and some constant* $C > 0$

$$F_\nu(x) = C \cdot e^{-\frac{1}{2}\langle B^{-1}x,x\rangle}.$$

Example 4 *If $E = \mathbb{R}^{\mathbb{N}}$ and $\nu(dx) = \bigotimes p(x_i)dx_i$ with an even probability density p satisfying $\int_{\mathbf{R}}(p'(s)^2/p(s))ds < \infty$. In this case the subspace of differentiability is the sequence space ℓ^2 (see e.g. [SW93]). Then the generalized density F_ν is given by $F_\nu(x) = \prod p(x_i)$ for those elements of ℓ^2 for which this product converges. In the particular case of a two-sided exponential distribution, i.e. $p(s) = e^{-|s|}/2$ this means that F_ν is defined on*

H whenever the Hilbert space H is contiunuously embedded in the sequence space ℓ^1. Note that in this case the usual logarithmic derivative is given by the series $\beta_h^\nu(x) = -\sum sgn(x_i)h_i$ which converges in $L^2(\nu)$ for all $h \in \ell^2$. In the case of the Cauchy distribution the product representation of F_ν makes sense on a much larger space than ℓ^1.

Using this notion of generalized density one can try to define a nonlinear function of a measure. Namely, if $\varphi : (0,\infty) \to (0,\infty)$ is a function and if F_ν is a generalized density of ν then we denote by $\varphi(\nu)$ any measure whose generalized density is the function $\varphi \circ F_\nu$. Of course, only for very special measures and functions the measure $\varphi(\nu)$ exists. Moreover the generalized density is determined at most up to a multiplicative constant the uniqueness of the measure $\varphi(\nu)$ can be hoped for only in the class of probability measures. Even this uniqueness typically is a delicate question. Not always a probability measure on an infinite dimensional space with a dense space of differentiability can be reconstructed uniquely by its logarithmic derivative. Let us illustrate the general idea for a particular function φ.

Example 5 *Let $\varphi(t) = t^2$. Then we will denote the measure $\varphi(\nu)$ by ν^2. If this measure exists then $\beta^{\nu^2}(\cdot,\cdot) = 2\beta^\nu(\cdot,\cdot)$. The latter equation can also be used to define the measure ν^2. If ν is the Gaussian measure from Example 2, then ν^2 is the Gaussian measure whose correlation operator is $B/2$; similarly one can define the measure $\nu^{\frac{1}{2}}$.*

After these preparations we come to the idea of canonical quantization: Let Q and P be two copies of a Hilbert space with a scalar product $(\cdot,\cdot)$. Let $G = Q \times P$ and let $\mathcal{H}$ be a real 'Hamiltonian' function on G given in the form

$$\mathcal{H}(q,p) = \frac{1}{2}(Ap,p) + V(q),$$

where A is a self-adjoint positive operator in P and V is some potential. If I is defined by $I : P \times Q \longrightarrow Q \times P, \ \ (p,q) \longmapsto (q,-p)$ the collection $(Q \times P, I, \mathcal{H})$ is called a (generally infinite-dimensional) classical Hamiltonian system.

If $\dim Q (= \dim P) = d < \infty$ then one can apply to this system the standard procedure of canonical quantization. According to this procedure one assigns to the function $\mathcal{H}$ the operator $\hat{\mathcal{H}}$ in $L^2(Q,\lambda^d)$ $(where \lambda^d$ is the d-dimensional Lebesgue measure) given by

$$\hat{\mathcal{H}}g = -\frac{1}{2}\Delta_A g + Vg. \tag{44}$$

Here Δ_A is the Laplace operator corresponding to the quadratic form A, i.e. $\Delta_A g = \mathrm{tr}\ Ag''$. Actually corresponding to the physical statement that the potential and thus the total energy is determined only up to an additive constant one can also add a term Cg on the right hand side of (44) where the constant C will be determined later.

Now in finite dimension it is a classical procedure to replace the space $L_2(Q,\lambda^d)$ by the space $L_2(Q,\eta)$, where η is a certain probability measure with strictly positive smooth density f_η. Then, using the natural isomorphism $\Psi : L^2(Q,\eta) \longrightarrow L^2(Q,\lambda^d)$ defined by $\Psi g = g\cdot(f_\eta)^{\frac{1}{2}}$ one can define the isomorphic image $\hat{\mathcal{H}}_\eta$ in $L^2(Q,\eta)$ of the operator $\hat{\mathcal{H}}$ in $L^2(Q,\lambda^d)$ via $(\hat{\mathcal{H}}_\eta g)(f_\eta)^{\frac{1}{2}} = \hat{\mathcal{H}}(g\cdot(f_\eta)^{\frac{1}{2}})$.

In the infinite-dimensional case Lebesgue measure no longer exists, but the space $L^2(Q,\eta)$ can be defined. Moreover, the definition of $\hat{\mathcal{H}}_\eta$ can be suitably adapted. One just needs to define the operator $\hat{\mathcal{H}}$ by the relation (44) first in a space of sufficiently smooth functions on Q without reference to any measure, rather than in the space $L^2(Q,\lambda^\infty)$ which does not exist. Instead of the density f_η of η one can use the generalized density F_η. Then one can actually use the same definition of the operator $\hat{\mathcal{H}}_\eta$ in the space $L^2(Q,\eta)$: Define $\hat{\mathcal{H}}_\eta g$ to be the function which satisfies

$$(\hat{\mathcal{H}}_\eta g)(F_\eta)^{\frac{1}{2}} = \hat{\mathcal{H}}(g(F_\eta)^{\frac{1}{2}}).$$

Then one can calculate that

$$(\hat{\mathcal{H}}_\eta g)(x) = -\frac{1}{2}\Big(\Delta_A g''(x) + \mathrm{tr}\ A\beta^\eta(\cdot,x)\otimes g'(x)\Big) + \hat{\mathcal{H}}(F_\eta^{\frac{1}{2}}(x))\ F_\eta^{-\frac{1}{2}}(x)g(x).$$

Let us remark that if one extends - in a natural way - the operator $\hat{\mathcal{H}}$ to the space $\mathcal{M}(E)$ then the function $\dfrac{\hat{\mathcal{H}}F_\eta}{F_\eta}$ is just the Radon-Nikodym density of the measure $\hat{\mathcal{H}}\eta$ with respect to η.

Now assume that $F_\eta^{\frac{1}{2}}$ is an eigenfunction of $\hat{\mathcal{H}}$ for the lowest eigenvalue. Then the measure $\eta^{\frac{1}{2}}$ can be called a 'ground state' of the operator and the constant above can be chosen in such a way that $\hat{\mathcal{H}}F_\eta^{\frac{1}{2}} = 0$. In this case the above formula simplifies to

$$(\hat{\mathcal{H}}_\eta g)(x) = -\frac{1}{2}(trAg''(x) + \mathrm{tr}\ A\beta^\eta(\cdot, x) \otimes g'(x)).$$

There is of course the question of convergence of the traces. Either one assumes that A is of trace class or one has to restrict the domain. Sometimes the sum of the two traces may exist even if they do not exist separately, in other words one may extend the domain of $\hat{\mathcal{H}}_\eta$ by rewriting it as:

$$(\hat{\mathcal{H}}_\eta g)(x) = -\frac{1}{2}\mathrm{tr}\ A((g''(x) + \beta^\eta(\cdot, x) \otimes g'(x))).$$

If in the Gaussian case $A = A_0^2$ where A_0 is a positive definite self-adjoint trace class operator then

$$V(x) = \frac{(x,x)}{2} - trA_0$$

and η is the measure whose correlation operator is A_0 and $\beta^\eta(h, x) = -(A_0^{-1}h, x)$. So the Gaussian case corresponds to the harmonic oscillator and the Hamiltonian function $\mathcal{H}(q,p) = \frac{(Ap,p)}{2} + \frac{(q,q)}{2}$): the more general function $\mathcal{H}(q,p) = \frac{(Ap,p)}{2} + V(q)$ can be considered as describing an "unharmonic oscillator" (of course such an interpretation is very wide). In this view the difference between a free field and the field with selfinteraction is just the difference between harmonic and unharmonic oscillators.

The connection of these objects to the Laplacian and the divergence operators introduced in the previous sections is given by the following observation:

Proposition 13: *$\hat{\mathcal{H}}_\eta = \delta_\eta AD$; if $A = Id$ then $\hat{\mathcal{H}}_\eta = \frac{1}{2}\delta_\eta D = -\frac{1}{2}\Delta_\eta$.*

In this way the study of the Laplace operators Δ_η is connected to the study of a wide class of quantum systems. The following result concerns the action of the associated dual operator on the measures.

Proposition 14: *Let $\hat{\mathcal{H}}_\eta^*$ be the operator in the spaces of measures which is adjoint to the operator $\hat{\mathcal{H}}_\eta$ with respect to the natural duality between the spaces of functions and of measures. Then under the above assumptions*

$$\mathcal{H}_\eta^* \mu = -\frac{1}{2} tr\ A(\mu'' - \beta^\eta \otimes \beta^\mu - (\beta^\eta)_2' \otimes \beta^\mu)$$

which implies the 'stationarity' result $\mathcal{H}_\eta^* \eta = 0$.

We conjecture that conversely $\mathcal{H}_\eta^* \nu = 0$ only if $\hat{\mathcal{H}}_\eta F_\nu^{\frac{1}{2}} = 0$. In any case it would be interesting to know under which conditions this equivalence holds.

Remark 11 A measure ν satisfying the equation $\hat{\mathcal{H}}_\eta^* \nu = 0$ is an invariant measure for the following stochastic differential equation (in Q)

$$dx = \frac{1}{2} \beta^\eta(\cdot, x) dt + dW$$

where W is Q-valued Wiener process generated by the Gaussian measure with the correlation operator A.

This means that the following problems are closely related (actually almost equivalent); the problem of finding a ground state for a quantum mechanical system; the problem of finding an invariant measure for a diffusion process and the problem of reconstructing a measure given its generalized density. As it was shown in Proposition 12, the generalized densities arise in Girsanov-Maruyama type formulae. These densities can be used also in a martingale approach to Feynman-Kac type formulae.

Hence on can expect that logarithmic densities of measures can play, in the calculus of smooth measures, sometimes an even more important role than the logarithmic derivatives.

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